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Imagery and Problem Solving*

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Abstract

In this paper we discuss the role of imagery in understanding problems and the processes of using images to solve problems. On the basis of two experiments and computer simulation, we show how subjects, in solving a particular problem, form mental images to represent a changing physical state. By "running" and watching the mental image they can draw qualitative conclusions about the situation, then derive a quantitative equation to solve the problem.

1. Introduction

Imagery can integrate local information and make explicit the spatial relations among objects (Kosslyn, 1980; Larkin and Simon, 1987). Images can also be manipulated, for example, by mental rotation (Shepard and Metzler, 1971; Shepard and Cooper, 1982). In this paper we will describe two experiments, one in detail and the other briefly, that show how imagery can be used to represent a process. By "running" and watching the image, subjects draw qualitative conclusions that then allow them to derive the relevant equations that solve the problem.

2. Experiment 1

2.1 Instruction

In this experiment, the subjects were instructed as follows:

Image

This task will allow you to check your capacity for imaging.

Please try to see these events in your mind:
(1) There is a rod AB, i.e., one end of the rod is named A, the other end is named B; ok?
(2) This rod is moving, with velocity v, along its axis; ok?
(3) Now, try to see that a ray of light is emitted, with velocity c, from A towards B along the axis of the moving rod.
OK?
Based on your image, please derive the expression of the time which the light will spend when it travels from A to B. (Hint: distance = time * velocity)
[You can write down any formula while you derive the result, if you need to. BUT, please DO NOT draw any diagram. Watch the picture **in your mind!**]

2.2 Method

Fifteen subjects, members of a class in English as a Second Language, were run. The

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subjects were foreign graduate students and visiting scholars at Carnegie Mellon University. The instructions were printed at the top of a sheet of 8 1/2 x 11 paper, leaving the lower half as the answer sheet. After the subjects had received the instructions, the experimenter read them aloud, sentence by sentence. When the subjects responded "Ok" for one sentence, the experimenter read the next one.

After reading the instructions, the experimenter told the subjects that they would be given five minutes to derive the equation called for by the instructions, and emphasized: "You can write down any formula while you derive the result, if you need to. BUT, please DO NOT draw any diagram. *Watch the picture in your mind!*" When five minutes had elapsed, the experimenter asked the subjects: "Please draw the picture in your mind on the other side of the paper."

After giving the correct equation to the subjects, and explaining the purpose of the experiment, the experimenter collected the answer sheets. Ten subjects returned their answer sheets.

2.3 Results

Figure 1 shows the diagrams drawn by the subjects, all of them reduced in scale by the same amount. We will comment on these drawings presently.

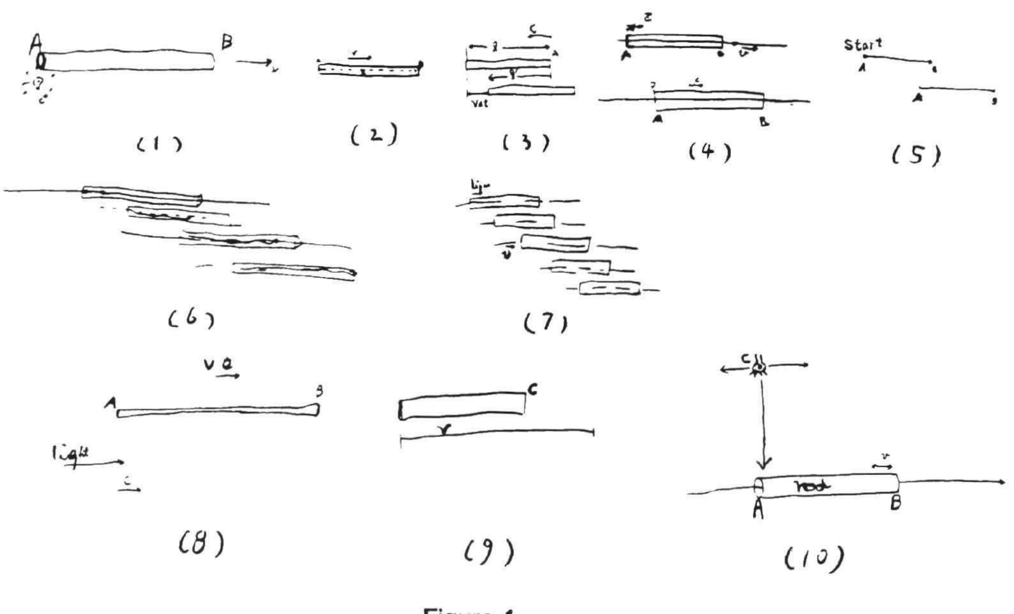

Figure 1.

The equation to be written down depends on how subjects choose the relative directions of c and v. If c and v are assumed to lie in the same direction, then the answer is $t = r_{AB} / (c-v)$, where r_{AB} is the length of the rod. If c and v are assumed to lie in opposite directions, then the answer is $t = r_{AB} / (c+v)$.Within 5 minutes, 5 of the 10 subjects, those drawing diagrams (1) to (5), obtained a correct answer. Two other subjects, those drawing diagrams (6) and (7), wrote the (incorrect) equation: $t = r_{AB} / c$. The 3 subjects who drew diagrams (8) to (10), did not write down an equation.

As we have noted, the instructions were (intentionally) ambiguous in some respects, and the values of some variables were given only implicitly. For example:

a) In sentence 1, the direction of movement of the rod is not specified, nor whether the rod lies in the horizontal, vertical, or some other direction. It is not specified which end of the rod is A, and which B. Nor is there information about the length or shape of the rod.

b) Sentence 2 provides no information about the direction of the velocity v.

c) Both v and c are, implicitly, measured relative to a stationary frame of reference -- presumably the frame of the subject, watching the moving rod.

2.4 Individual Differences

To form their mental images, the subjects had to add information to resolve the ambiguities and make implicit values explicit. Only one subject asked about the direction of v while the experimenter was reading the instructions, and was told, "It depends on your choice."

From Figure 1, we see that there are large differences in the information added by different subjects, e.g., the length and shape of the rod, and the way of representing the light (2 subjects drew electric bulbs). However, much of the information was introduced in almost the same way by nearly all the subjects. For example:

a) All of the rods in the images were laid horizontally.

b) 9 of 10 subjects used the left end of the rod as A, and the right end as B.

c) 8 of 10 subjects pointed v and c in the same direction, from A to B; only one subject, Figure 1, (3), pointed v from B to A, but c from A to B as called for by the instructions. One subject, Figure 1, (10), drew vectors v and c in both directions.

There was less information about the consistency among subjects in making implicit values explicit. The 5 subjects who solved the problem assumed that v and c were to be measured relative to the subject's stationary frame. The two subjects who obtained the answer $t = r_{AB} / c$ might have chosen c relative to the position of the rod. The frames of reference used by the 3 subjects who did not write equations cannot be determined.

We see that the subjects are relatively consistent in the way in which they supply information, although there is considerable variety in the visual appearance of their diagrams. In Figure 1, we see that 5 diagrams depict the successive locations of the rod, giving us a clue as to how the images represent a moving process, and how, on this basis, the subjects can derive the quantitative equation. Notice that none of the subjects who failed to write equations showed the rod in more than one location, in contrast with 5 of the 7 who wrote an equation. Presently, we will ask why two subjects who solved the problem did not depict the rod in more than one location, althoug.ı doing so can be important to the reasoning.

2.6 Deriving the Equation from the Image

There are at least two routes by which the equation can be derived from the image. Let us assume that c and v are chosen with the same direction, from A to B. One route to the solution begins with the inference that the *net* velocity of the light relative to the rod is c - v. The other route rests on the inference that the light, relative to the stationary frame, must travel, in order to reach B from A., the distance r_{AB} + vt -- that is the length of the rod, plus the distance (vt) the rod has traveled during the time of travel (t) of the light, so that $t = (r_{AB} + vt) / c$.

Of the 5 subjects who solved the problem, 2 used the former (relative velocity) route, while the other 3 used the latter (distance) route. In Figure 1, 1) and 2) are the drawings of the subjects in the velocity group, while 3), 4), and 5) are the drawings of the subjects in the distance group. The velocity group did not, but the distance group did, depict the rod in different locations. On the basis of the diagrams, we can infer that all the subjects noticed that while the light traveled from A to B the rod also moved. The subjects in the velocity group noted this by recording the velocities; the subjects in the distance group by recording the initial and final positions of the rod.

3. Discussion

From the fact that the velocity and distance groups of subjects recorded different kinds of images, and that at the same time, they used different lines of inference to the conclusion, we conclude that there is a close correspondence here between imagery and reasoning processes. If we assume that the reasoning takes place in a representation defined by the imagery, we see that the imagery determines what reasoning steps are available. (Cf. Paige and Simon, 1966.)

When the experimenter explained the answer, and tried to show the subjects how to use the relative velocities (the algebraically simpler procedure) to solve the problem, some subjects found it difficult to understand the reasoning. We, the authors, also find that it is easier to image the distance than the relative velocity.

The experimental evidence does not prove definitively that subjects chose the velocity route *because* they saw the relative velocities in their mental imagery. However that may be, using the relative velocities to solve the problem (perhaps because they had learned that concept in previous physics training) would direct their attention to the velocities, and would make the successive positions of the rod irrelevant. In contrast, the subjects in the distance group did form an image that depicted the traveling rod. However, they did not try to show motion, as such, but simply the positions of the rod at the two critical moments -- the time when the light was emitted, and the time it reached the end of the rod. In this as in many other problems (Paige and Simon, 1966), change can be represented, not as a continuous process, but simply in terms of the "before" and "after" states of the system.

The process we have sketched out above for the distance group in solving the problem, which seems consistent with the data, can be outlined as follows:

a) Setting the goal: to find the time required for the light to travel from A to B.

b) Forming and running the mental image, i.e., doing the mental experiment.

c) Observing the changed states of rod and light.

d) Noting that the distance the light travels is greater (or less) than r_{AB}.

e) Deriving the qualitative result that the time required is greater (or less) than r_{AB}/c.

f) Determining the quantitative value of the difference (vt) between r_{AB} and the distance the light traveled.

g) Deriving the equation: $t = r_{AB} / (c-v)$ (or, $t = r_{AB} / (c+v)$).

h) Checking the quantitative with the qualitative result.

A corresponding model, going from image to qualitative inferences and from qualitative inferences to quantitative equation, could be described for the subjects in the relative velocity group. We can gain further insight into the model for the distance group from the protocol of a subject taken in an earlier experience. We turn to this protocol next.

4. Experiment 2

This experiment was carried out in our laboratory. Subjects were asked to understand, with the help of their mental images, the first, kinematical, part of Einstein's 1905 paper: "On the Electrodynamics of Moving Bodies" (Einstein, 1905). This part is focused on the basic concepts and the deriving of the Lorentz transformation equations. Einstein derives one of the key equations as follows (with questions inserted by the experimenters in the reading material in bold face):

> We imagine further that at the two ends A and B of the rod, clocks are placed which synchronize with the clocks of the stationary system, that is to say that their indications correspond at any instant to the "time of the stationary system" at the place where they happen to be. These clocks are therefore "synchronous in the stationary system."
>
> **[Q : Following the instruction given in this part, try to form an image in your mind, describe this image to us, and then draw it.]**
>
> We imagine further that with each clock there is a moving observer, and that these observers apply to both clocks the criterion established in $1 for the synchronization of two clocks. Let a ray of light depart from A at the time t_A, let it be reflected at B at the time t_B, and reach A again at the time t'_A.
>
> **[Q : Following the instruction given in this part, try to form an image in your mind, describe this image to us, and then draw it.]**
>
> Taking into consideration the principle of the constancy of the velocity of light we find that
>
> $$t_B - t_A = r_{AB}/(c-v)$$
>
> and
>
> $$t'_A - t_B = r_{AB}/(c+v)$$
>
> where r_{AB} denotes the length of the moving rod -- measured in the stationary system.

[Q: Here are the equations about the characteristics of the image described in the reading material. Tell us how you would use the image to explain and justify these equations.]

We can see that the instructions of our Experiment 1 are a simplification of this part of the reading material from Experiment 2. The major difference is that in Experiment 1 the subjects were not given the final equation, but had to find it, while in Experiment 2, the equations were given to the subject. However, in the latter experiment, the subjects were also asked to explain the equations in terms of their mental images.

Protocols were obtained from 7 subjects, most of them undergraduate or graduate students in Electrical and Computer Engineering or in the School of Computer Science at Carnegie Mellon University. We will examine a single, particularly informative, protocol here, The protocol of S_g . S_g tried both the velocity and distance routes to a solution.

Form image

S_g read the material, formed an image and drew it , as shown in Figure 2.
[After he had finished his diagram, the diagram was taken away by the experimenter]

Read equations

S read the equations.

Examine image

"the length of that ray and--Uhmm--we see that is the sum of observe the length of the rod and the amount that it moved because--Ahmm--the ray left--Ahaa--left from-Ahaa--the origin with respect to our stationary system and it moved along at a constant speed C-Ahmm"..."So that --Ahaa--that that is the--okay--and then you take take the length of that ray you divide that by c minus v and Oh boy! This is hard to visualize Uhmm."

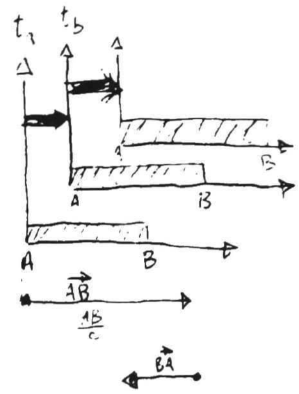

Figure 2.

Set goal.

"Uhmm--We said we want to find this quantity t_B - t_A "

Find starting point, image and qualitative relation

"Uhmm and what do we know--Ahmm--about that?--Uhmm-- We know that it took okay the entire distance" "it had to travel the length of the rod--Ahmm--which is -- r_{AB}. Uhmm--let me see that it is the same but that's not enough because its moved--Ahaa--the, the rod has moved you can see that by looking at the t_B frame and see that--Uhmm--its moved--during that that time that it traveled "

Gesture

His left hand and right hand moved toward right in parallel

Transfer equations, guided by the image

"Should I try and do this symbolically? Should I--Uhmm should I think of my picture here and--Uhmm--try and show how I can justify these equations? "
S copied the equations from the reading material and transferred them as follows:

$r_{AB} = (t_B - t_A)(c - v)$

$r_{AB} / (t_B - t_A) = c - v$

$r_{AB} = (t'_A - t_B)(c + v)$

$r_{AB} / (t'_A - t_B) = c + v$

Consider relative velocity

"Uhmm--and--Ahaa--ray travels at a speed c in respect to the stationery frame but the whole rod is moving with a positive velocity v so the ray is actually traveling with

> respect to the moving frame at a velocity c - v --Uhmm--so--Uhmm--if we want to derive the time that it took for this ray to get from t_B or t_A to t_B when--Ahaa--let's see what this is parts of r_{AB} over c-v."

[E: Why are these c-v and c+v ?]

Do semi-quantitative reasoning	Total time times light velocity = length of the rod + some component, because the rod is moving. $c(t_B \quad t_A) = r'_{AB}$
Do quantitative reasoning	$r_{AB} = r'_{AB} - v(t_B - t_A)$ $r_{AB} = c(t_B - t_A) \quad v(t_B - t_A)$ $r_{AB} / (c - v) = (t_B - t_A)$

From S_g 's protocol, we can see that his process for deriving the equations was not as "neat" as our model. He used means-ends analysis, trying different methods. There was frequent alternation between forming and examming the mental image and engaging in reasoning. S obtained some clues from the form of the given equation, and was able to derive it by means of the relative velocities. But he could not explain clearly what he had done until he shifted to the distance route. Then he could explain more clearly, following closely the path of our model except for the insertion of a "semi-quantitative reasoning" episode in his protocol.

When a second subject, S_j, tried the relative velocity route for solving the problem, he imaged two trucks running in the same or opposite directions as an analog, a method of doing qualitative reasoning. Although there were many individual differences among the protocols in Experiment 2, the general processes revealed in them are essentially those of our models.

5. Computer Simulation

We are building a program in OPS5 to simulate this process, and have finished the qualitative and quantitative reasoning parts. The input of this program is information derived from the image. The major qualitative reasoning production rules are the following:

If the direction of the light is the same as that of the rod then the distance the light travels will be larger than the length of the rod.
If the direction of the light is opposite to the rod then the distance the light travels will be less than the length of the rod.
If the direction of the light is the same as that of the rod then the distance the light travels will be the length of the rod plus the distance the rod moved.
If the direction of the light is opposite to the rod then the distance the light travels will be the length of the rod minus the distance the rod moved.
If the direction of the light is the same as that of the rod then the velocity of the light relative to the rod is less than that relative to the stationary system.
If the direction of the light is opposite to the rod then the velocity of the light relative to the rod is larger than that relative to the stationary system.

Figure 3 gives two examples of the input and output of the program. The "IMAGE" part is the input of the program. The remainder is the output. Part (1) is the simulation of the distance group , and part (2) is the simulation of the velocity group. The difference between the input of these two examples is only in the relative system for the velocity of the light.

```
** IMAGE **
Rod:
    Relative system      S_system
    Measuring system     S_system
    Length               rAB
    Direction            right
    Velocity             v
    Distance             nil
Light:
    Emitted system       M_system
    Relative system      S_system
    Measuring system     S_system
    Direction            right
    Velocity             nil
    Distance             nil
** REASONING**
Time is tB-tA
[Quality:  ] Call the principle of constancy of velocity of light
[Quantity:] The velocity of light is c
[Quality:  ] Distance traveled by light is larger than length of rod
[Quantity:] Distance moved by rod is v * tB-tA
[Quantity:] Distance traveled by light is rAB + v * tB-tA
** MATH_REASONING**
RESULT
tB-tA = rAB / ( c - v )
```

(1)

```
** IMAGE **
Rod:
    Relative system      S_system
    Measuring system     S_system
    Length               rAB
    Direction            right
    Velocity             v
    Distance             nil
Light:
    Emitted system       M_system
    Relative system      M_system
    Measuring system     S_system
    Direction            right
    Velocity             nil
    Distance             nil
** REASONING**
Time is tB-tA
[Quality:  ] Consider light velocity relative to rod
[Quality:  ] Light velocity relative to rod less than
                velocity relative to stationary system
[Quantity:] Velocity of light is c - v
[Quantity:] Distance traveled by light is rAB
** MATH_REASONING**
RESULT
tB-tA = rAB / ( c - v )
```

(2)

Figure 3.

6. Conclusion

From our two experiments, we can conclude:

1. For a simple physics problem involving relative motion, subjects can form mental images to represent a process of change in the physical state of a system. They can "run" the image to carry out mental experiments and watch the change of the physical states.

2. By watching the process, subjects can infer the qualitative relations among the physical variables, and draw qualitative conclusions.

3. Based on previous knowledge (e.g., distance equals rate times time), subjects can derive quantitative equations from the qualitative relations.

4. The process is not generally linear. There is a good deal of interaction among the processes of forming and observing the mental image and reasoning from it.

In the simple example of relative motion, we have seen that there may be more than one route to the solution, and that different routes may correspond to different mental images. In the example discussed here, the subjects did not have to change the concepts they already held, or construct new concepts. In other protocols we have observed such changes in concepts, and will discuss them in subsequent papers.

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