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ABSTRACT

The nuclear matrix elements for EO transition probabilities associated with beta-vibrational transitions of spheroidal nuclei are derived by two different models. The first model, previously studied by Peaslee and by Reiner, involves the volume-conserving quadrupole surface oscillations of a uniformly-charged spheroid about an equilibrium deformation of β. The second model sums over the coherent contributions of all the proton orbitals involved in an individual particle representation of the collective motion. The second model when corrected for volume-conservation gives about twice the ratio of EO to E2 for Sm^{152} and Pu^{238} given by the first model. A brief comparison with experiment is made, showing satisfactory agreement for Pu^{238} and serious disagreement for Sm^{152}.

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In recent years Church and Weneser have given attention to the theory of internal conversion of orbital electrons by nuclear electric monopole transitions.[1] Reiner[2] has also considered the theory in detail and has surveyed the experimental data. Striking cases of such transitions were found by Asaro and Perlman[3] to be associated with the decay of excited 0+ states in deformed even-even nuclei, and Albridge and Hollander[4] have assigned conversion electron lines in the decay of Np^{238} as 2+ to 2+ and 4+ to 4+ transitions. Reiner cites work on Sm^{152}, Gd^{154}, and Th^{232} as also indicating E0 transitions. All the above-mentioned experimental cases are classified as proceeding from the first excited beta-vibrational band to the ground band.

It is of interest first to consider the model of a uniformly charged spheroid undergoing quadrupole oscillations without change of volume. From the form of the E0 matrix element, $< r^2 >$, it can readily be appreciated that quadrupole surface oscillations about a spherical equilibrium shape have a vanishing E0 matrix element, since the excursion outward of some charge elements is always counterbalanced by inward motion of other charge. When the oscillation occurs about a deformed equilibrium shape, the cancellation is not complete and there is a certain collective contribution to the E0 matrix element. Peaslee carried out a calculation[5] of the ratio of E0 to E2 transitions depopulating a 0+ beta vibrational state. Reiner has made the same calculation but in greater detail.

It is the purpose of this paper to (a) repeat the vibrating spheroid derivation with attention to some higher order terms of significance, and (b) carry out a derivation in terms of single proton contributions in the asymptotic (anisotropic harmonic oscillator) representation. A serious failure of these models might give proof of and fundamental information on the involvement of compressional modes of oscillation.

Vibrating Spheroid Model

Church and Weneser give E0 transition rates in terms of a reduced transition probability, ρ^2, defined as

$$\rho^2 = \left| \sum_p \varphi_f^* \left(\frac{r_p^2}{R^2} - \sigma \frac{r_p^4}{R^4} + \dots \right) \varphi_i \, d\tau \right|^2 \qquad (1)$$

where R is the nuclear radius, taken as $1.20 \times 10^{-13} A^{1/3}$ cm, and the summation is taken over all protons.

Let us consider a uniformly charged spheroidal nucleus of charge Z and surface defined by the equation

$$R(\theta) = R_o \left(1 + \alpha Y_{20}(\theta) - \alpha^2/4\pi\right). \qquad (2)$$

The third term is included to make the nuclear volume independent of α to second order.

A nucleus in the spheroidally deformed region will have an energy minimum at some finite value of α, designated β. Let γ measure the excursion of the shape from its equilibrium value. That is, $\beta + \gamma = \alpha$.

We take the usual convention for the Hamiltonian for vibrational motion as a harmonic oscillator

$$H_{coll} = (1/2) B |\dot{\gamma}|^2 + (1/2) C |\gamma|^2.$$

With this definition the oscillator energy quantum is $\hbar\omega = \hbar(C/B)^{1/2}$. With this definition the matrix element of γ connecting the ground and first excited oscillator state is

$$\langle \gamma_{01} \rangle = \hbar^{1/2} 2^{-1/2} (BC)^{-1/4}. \qquad (3)$$

To evaluate ρ^2 we note that the charge density is the atomic number Z divided by the volume and carry out the integration below:

$$< \sum_p r_p^2 > = \int_o^{\pi} \int_o^{2\pi} \sin\theta \, d\theta d\varphi \int_o^{R(\theta)} r^2 \cdot \left(\frac{Z}{(4/3)\pi R_o^3} \right) r^2 \, dr$$

$$= \left(\frac{3Z R_o^2}{20\pi} \right) \int_o^{\pi} \int_o^{2\pi} (1 + \alpha Y_{20}(\theta) - \alpha^2/4\pi)^5 \sin\theta d\theta d\varphi$$

$$= (3/5) Z R_o^2 \left(1 + \frac{5\alpha^2}{4\pi} + \ldots \right), \qquad (4)$$

neglecting terms of higher power in α. Our result is 5/7 that of Peaslee's[4] and Reiner's,[2] since we included the second order volume-conserving term in the surface expansion in the numerator. Reiner discusses the question of the inclusion of this term and decides against it; we feel a true test of the volume-conserving oscillation model requires its inclusion as we have done. Substituting $\beta + \gamma$ for α we obtain for the linear term in γ, which is the only one connecting the nuclear states in question,

$$< \sum_p r_p^2 > = \frac{3}{2\pi} Z R_o^2 \beta\gamma + \ldots \qquad (5)$$

From (3), (5), and (1) we have the result

$$\rho^2 = \frac{9}{8\pi^2} Z^2 \beta^2 \hbar (BC)^{-1/2} \left(\frac{R_o}{R} \right)^4 . \qquad (6)$$

We may well choose our R_o the same as Church and Weneser's R, and the last factor will be unity. The contribution of the higher order term in (1) can be similarly evaluated, and its effect can be expressed by an additional factor of $[1 - 2\sigma (R_o^2/R^2)]^2$ multiplying the right hand side of Eq. (6).

The reduced E2 transition probability from the 0+ first beta-vibrational state to the 2+ state in the ground band is, according to Eqs. (V.33) and (V.34) of Alder et al.,[6]

$$B(E2;\ 0 \longrightarrow 2) = \frac{9}{32\pi^2} Z^2 e^2 R_o^4 \hbar (BC)^{-1/2}. \qquad (7)$$

We see from Eq. (6) and (7) that the dimensionless ratio

$$X = \frac{\rho^2 e^2 R_o^4}{B(E2;\ 0 \longrightarrow 2)}$$

is independent of B, C, R_o and Z. Thus, the ratio of E0 to E2 depopulation may be compared with theory, even in the absence of knowledge of B and C.

$$X = 4\beta^2. \qquad (8)$$

The quantity $(BC)^{-1/4}$ is essentially the zero-point amplitude of vibration. That the ratio in Eq. (8) is independent of it simply means that the dependence of E0 and E2 transition probabilities on the vibration amplitude is the same.

Individual Particle Description

The model of the nucleus as a uniformly charged body undergoing quadrupole oscillations neglects many significant details in describing the motion of the nuclear charge. It would be preferable to describe the motion as a coherent superposition of individual excitations of the more easily polarizable protons near the Fermi surface. For calculations of only the E2 to E0 ratios and not the absolute transition probabilities it is sufficient to evaluate only the relative weight of the various proton excitations making up the "vibrational" motion. We shall evaluate these weights by separating off one at a time each single proton in the quadrupole vibrational field provided by the other A-1 nucleons. This approach bears a close analogy to formulation of rotational properties of nuclei in terms of single nucleon excitations.[7] The calculation of absolute transition probabilities by this model would raise many fundamental questions (regarding self-consistency, etc.) not easily answered, but we believe the calculation of the relative probabilities as outlined here is valid.

Bohr and Mottelson in their comprehensive paper[8] in 1953 outlined a perturbation approximation as well as the widely-used strong coupling approximation for the coupling of individual particle and collective surface motions. The latter approximation has proved applicable in greater detail to a larger group of nuclei and in that sense has been more fruitful. We should like to apply here a hybrid of these approximations for treatment of the "vibrational

excitations" of strongly deformed nuclei about their spheroidal equilibrium shape. The basis functions of the individual nucleons are to be chosen appropriate to the spheroidal well, and for simplicity we shall use harmonic oscillator functions separated in cylindrical coordinates z, ρ, φ. (The "asymptotic representation" of Eqs. (4) and (6) of Alaga[9]). The wave functions are labelled by the familiar asymptotic quantum numbers, N, n_z, Λ, and Σ.

For description of the beta vibration the contribution of the A-1 nucleons will be considered collectively via creation and destruction of phonons. The amplitude of each individual proton excitation in the wave function of the first excited vibrational state will be proportional, by perturbation theory, to a matrix element divided by an energy denominator $(\epsilon'_p - \epsilon_p - \hbar\omega + \Delta)$, where ϵ'_p and ϵ_p are upper and lower orbital energies for the excited proton in question, $\hbar\omega$ is the phonon quantum energy which is approximately the energy of the vibrational state, and Δ is a term to represent the energy loss in breaking a proton pair. The matrix element of interest is of the form of Bohr and Mottelson's[8] Eq. (II.12), but we consider for the interaction Hamiltonian, H_{int}, not a surface interaction as in their (II.9) but a volume interaction appropriate to quadrupole motion of the harmonic oscillator potential well, i.e.

$$H_{int} = -M\omega_o^2 \gamma r^2 Y_{20}(\theta), \tag{9}$$

where γ is as defined in the first section, M is the nucleon mass, ω_o is related to the energy separation of major shells and may be taken, after Nilsson, as $\hbar\omega_o \sim 41\, A^{-1/3}$ Mev.[10] r and θ are position coordinates of the proton in the well. In cylindrical coordinates we have

$$H_{int} = -M\omega_o^2 \gamma \sqrt{5/4\pi}\ (z^2 - \rho^2/2). \tag{9a}$$

The distance coordinates r, z, ρ are to be understood as having the dimensions of distance and are not the dimensionless parameters often introduced in harmonic oscillator calculations. It could be argued that H_{int} might better use a higher power of r than the square, if the interaction is predominantly on the surface.reaction

The interaction (9a) involves a single particle operator which cannot change the projections, Σ, Λ, of the angular momenta. It does have off-diagonal elements in n_z and n_ρ. n_ρ will be defined as twice the number of cylindrical nodes in the nucleon wave function and is identically equal to $N - n_z - \Lambda$ and can have only even integral values. The selection rules of H_{int} are $\Delta n_z = 0,\pm 2$ with $\Delta n_\rho = 0$; also $\Delta n_\rho = 0,\pm 2$ with $\Delta n_z = 0$. The matrix elements of z^2 for the one-dimensional harmonic oscillator and of ρ^2 for the two-dimensional case have been taken from a paper of W. H. Shaffer[11] and are as follows:

$$< n_z \mid z^2 \mid n_z - 2 > = \frac{\hbar}{2} (M\omega_z)^{-1} [n_z(n_z-1)]^{1/2} \qquad (10a)$$

$$< n_\rho \Lambda \mid \rho^2 \mid n_\rho - 2\ 2\Lambda > = - \frac{\hbar}{2} (M\omega_\rho)^{-1} [n_\rho(n_\rho+2\Lambda)]^{1/2} \qquad (10b)$$

Thus, the relative amplitude in the first beta-vibrational state of the proton excitation leaving two odd protons in orbitals of $\mid N, n_z, n_\rho, \Lambda >$ $\mid N-2, n_z - 2, n_\rho, \Lambda >$ is as follows:

$$a_{\tau\tau'} (\Delta n_z = 2) \approx \frac{< H_{int} >}{\Delta E} \approx - \frac{\omega_o^2}{8\omega_z^2} \left[\frac{5\hbar\, n_z(n_z-1)}{2\pi\, (BC)^{1/2}} \right]^{1/2} \qquad (11a)$$

where we have simplified the energy denominator by considering $\Delta \approx \hbar\omega$, a reasonable assumption, and that $\epsilon_p' - \epsilon_p \approx 2\hbar\omega_z$. Similarly, the amplitude in the excited vibrational state of the configuration with two odd protons in orbitals $\mid N, n_z, n_\rho, \Lambda >$ and $\mid N, n_z, n_\rho - 2, \Lambda >$ is

$$a_{\tau\tau'} (\Delta n_\rho = 2) \approx \frac{\omega_o^2}{16\omega_\rho^2} \left[\frac{5\hbar\, n_\rho\, (n_\rho + 2\Lambda)}{2\pi\, (BC)^{1/2}} \right]^{1/2} \qquad (11b)$$

We wish to calculate the ratio of E0 to E2 transition probability for this model, hence the ratio, X, of Church and Weneser's ρ^2 to the dimensionless quantity $B(E2)/e^2R^4$, where R is the nuclear radius. The contribution do E0 of a particular proton excitation with orbitals specified by $\tau\tau'$ will be

$$a_{\tau\tau'} \left\langle \frac{r^2}{R^2} \right\rangle_{\tau\tau'} = \frac{a_{\tau\tau'}}{R^2} \left\langle z^2 + \rho^2 \right\rangle_{\tau\tau'} ,$$

and its contribution to the corresponding E2 transition probability will be

$$\frac{a_{\tau\tau'}}{R^2} \left\langle r^2 Y_{20} \right\rangle_{\tau\tau'} = \frac{a_{\tau\tau'}}{R^2} \sqrt{\frac{5}{4\pi}} \left\langle z^2 - \rho^2/2 \right\rangle_{\tau\tau'}$$

The contributions of these components of the total wave function will add coherently, since they are coupled in phase to the collective shape oscillation of the well. Contributions arise only from pairs of orbitals one of which is occupied and one unoccupied in the ground configuration.

$$X = \frac{4\pi}{5} \frac{\left| \sum_{\tau\tau'} \frac{n_z(n_z-1)}{\omega_z^3} \delta_{n_z,n_z'+2} \delta_{n_\rho,n_\rho'} - \frac{n_\rho(n_\rho+2\Lambda)}{2\omega_\rho^3} \delta_{n_z,n_z'} \delta_{n_\rho,n_\rho'+2} \right|^2}{\left| \sum_{\tau\tau'} \frac{n_z(n_z-1)}{\omega_z^3} \delta_{n_z,n_z'+2} \delta_{n_\rho,n_\rho'} + \frac{n_\rho(n_\rho+2\Lambda)}{4\omega_\rho^3} \delta_{n_z,n_z'} \delta_{n_\rho,n_\rho'+2} \right|^2} \qquad (12)$$

It can be verified that the numerator vanishes for summation over all transitions from a closed oscillator shell where $\omega_z = \omega_\rho$, the spherical limit. As one goes beyond the closed shell, filling successive levels in a spheroidal well, the numerator will increase in value but will again decrease as the next closed shell is approached.

In any practical example there will be a great number of terms in the summations in Eq. (12). We have evaluated the expression numerically for Pu^{238} and Sm^{152}. In Pu^{238} we have taken the deformation parameter δ as 0.24, and have used the following equation from Nilsson.[10]

$$\frac{\omega_z}{\omega_\rho} = \sqrt{(1 - 4\delta/3)/(1 + 2\delta/3)} \qquad (13)$$

Taking the level schemes from Nilsson and Mottelson's Figs. 3 and 5,[7] there are three occupied proton levels with N = 6 and nine with N = 5. Lower shells are completely occupied. There are thus six terms between N = 6 and N = 8, eighteen terms between N = 5 and N = 7. There are twelve terms between the completely filled N = 3 and the partially unoccupied N = 5. There are 27 terms between N = 4 and N = 6. Using Eq. (12) we calculate for Pu^{238} the ratio X = 0.86.

Similarly we calculate for Sm^{152} at the same deformation the value $X = 0.98$.

These values are to be compared with a value of 0.23 from Eq. (8) for the vibrating spheroid model. How is this considerable difference in predictions to be understood? It seems most likely that the protons in only slightly filled shells are contributing especially heavily to the E0 transition probability, because they lie predominantly in the tips of the nuclear well.

Some additional insight into this individual particle model is afforded by examining the contributions from a completely filled oscillator shell N to a completely empty shell N + 2. It can be shown in such a case that the numerator of Eq. (12) vanishes when $\omega_z = \omega_\rho$, the case of no deformation. Thus, in terms of ω_z and ω_ρ

$$X_{\text{closed shell}} = \frac{4\pi}{5} \frac{\left(\frac{1}{\omega_z^3} - \frac{1}{\omega_\rho^3}\right)^2}{\left(\frac{1}{\omega_z^3} + \frac{1}{2\omega_\rho^3}\right)^2}$$

Substituting from Eq. (13) and expanding for small δ in powers of δ we have to lowest order

$$X = \frac{16\pi}{5} \delta^2 + \ldots$$

Using from Nilsson[10] the relation $\delta = (3/2)\sqrt{5/4\pi} \cdot \beta$

$$X = 9 \beta^2 + \ldots \qquad (14)$$

That this result is over twice that for the vibrating spheroid model in Eq. (8) probably reflects the fact that the volume-conserving second order term was specifically included there and was not taken into account in the individual particle model here. It is possible to include the volume-conserving term in the derivation of Eq. (12) by reducing the matrix elements of $\langle z^2 \rangle$ by the factor $\left(1 - \frac{\beta}{\sqrt{5\pi}}\right)$ and increasing the matrix elements of $\langle \rho^2 \rangle$ by the factor $\left(1 + \frac{2\beta}{\sqrt{5\pi}}\right)$. This alteration reduces the Pu^{238} calculation of X to 0.50 and

the Sm^{152} value of X to 0.61. Thus, the individual particle model here predicts X values about twice that of Eq. (8).

From a theoretical standpoint it certainly seems proper to introduce the higher order volume-conserving factors discussed above; at least the formulation is appropriate for testing what we set out to test, whether volume-conserving collective motion can explain the observed E0 to E2 ratios without explicitly involving compressional modes of vibration.

To the extent that the surface interaction has anharmonic terms such as $r^4 Y_{20}$ there will be an involvement of proton excitations within the same oscillator shell (i.e. via $\langle z^2 \rho^2 \rangle$). These excitations will contribute to the second term of Eq. (1) and further decrease the individual particle estimate of X. The matrix elements will be much smaller than the harmonic terms previously considered, but the energy denominators will be an order of magnitude smaller, being about equal to $2(\hbar\omega_\rho - \hbar\omega_z)$. The model also suffers from the use of pure cylindrical wave functions and the neglect of the pairing interaction, which blurs the Fermi surface. Corrections for the effects mentioned in this paragraph would probably not be very large but would be in the direction of further reducing the relative strength of the monopole transitions.

Comparison with Experiment

Only a preliminary comparison with experiment is appropriate here, since the experimental work is still largely unpublished and results are not in final form.

Asaro, Stephens, and Perlman[12] have worked out the ratio of E0 to E2 de-excitation of the 935 kev 0+ state in Pu^{238} and find the ratio between K electrons (E0) to ground and E2 photons to the 44 kev 2+ state of 0.5. Choosing $R_o = 1.20\ A^{1/3}$ fermis the dimensionless ratio $X = \frac{\rho^2 e^2 R_o^4}{B(E2)} = 0.14$.

Nathan[13] has measured the similar ratio in Sm^{152} following electron capture of Eu^{152m} and finds the K/photon ratio from the 687 kev 0+ state to be 0.013. Recently Graham, Ewan, and Geiger[14] have beta spectrometrically resolved the E0 K line from a very near-lying line, finding the two lines to be of about equal intensity. Thus, the K/photon ratio of Nathan's report should be reduced a factor of 2, to .0065, yielding the ratio X = 0.016.

Table I summarizes the results of this paper.

Table I

Summary of theoretical and experimental reduced E0:E2 ratios, X

Nucleus	Assumed deformation δ	Uniform spheroid model		Individual proton model		Experiment
		Without vol. conserv.	With vol. conserv.	Without vol. conserv.	With vol. conserv.	
Pu^{238}	.24	.32	.23	.86	.50	.14
Sm^{152}	.24	.32	.23	.98	.61	.016

The agreement of theory with experiment is satisfactory in the Pu^{238} case, and we may conclude that volume-conserving quadrupole surface oscillations are of predominant importance in the E0 transition here and that monopole compressional modes certainly must make a considerably lesser contribution.

The bad disagreement in the Sm^{152} case is not easy to explain. We note that Sm^{152} is on the very borderline of the region of strongly deformed nuclei, and the model may be breaking down in ways which would either depress the E0 transition or enhance the competing E2 transition. One such effect, capable of treatment, would enhance the E2 transition in the following way: The nuclear spheroid for the first excited state (2+) will be somewhat more distended than in the ground state by virtue of centrifugal effects (vibration-rotation interaction). In other words the vibration-rotation interaction causes configuration mixing between the 2+ state in the ground rotational band and the 2+ state in the beta-vibrational band. This mixing causes an enhancement of the E2 matrix element between the excited 0+ and the 2+ state of the ground band. Our preliminary estimates[15] set this enhancement of the order of a factor of 5, not sufficient to explain the observed E0 to E2 ratio. This effect is quite small in Pu^{238} with its considerably larger moment of inertia. It will be of interest to Coulombic excitation considerations from ground to the β-vibrational band that this configuration-mixing effect depresses the E2 transition probability here below expectation from the simple vibrating spheroid formulas.

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