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Relativistic electron beam neutralization in a dense magnetized plasma

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For the case of a longitudinal magnetic guide field this problem has been treated neglecting ion dynamics [R. Lee and R. N. Sudan, Phys. Fluids **14**, 1213 (1971)]. Under the assumptions $\Omega_e \gg \omega_{pe}, \Omega_i \gg \omega_{pi}$, where ω_p, Ω are plasma and gyrofrequencies, there is no neutralization because both electrons and ions are tied to field lines. When $\Omega_e \gg \omega_{pe}$ and $\Omega_i < \omega_{pi}$, it is essential to consider ion dynamics since the ions play the dominant role in neutralization. In the final state, the beam is essentially neutralized both electrically and magnetically and the energy associated with plasma oscillations is a small fraction of the beam energy except when the radius of the beam is small compared with the plasma skin depth.

I. INTRODUCTION

Recently, several papers[1–4] have been devoted to the theory of the generation of the counter current in a dense plasma penetrated by a high current relativistic electron beam. The common result of these papers is that, under certain conditions (to be specified later), the magnitude of the counter current approximately equals that of the beam current, a result in good agreement with the experimental observations.

The physical mechanism for the return current generation can be most conveniently understood in the beam frame, wherein the beam is at rest in a back streaming plasma whose velocity v_0 is equal in magnitude to the beam velocity in the laboratory frame. The negatively charged beam tends to expel the plasma electrons out of and to draw ions into the beam, resulting in a perturbed positive charge density $\delta\rho$ inside the beam. Let $\rho_t = \delta\rho - n_b e$ and $\rho_t' = \delta\rho' - n_b' e$ be the total charge densities as seen from the laboratory frame and beam frame, respectively. To compare ρ_t and ρ_t', we express the beam frame quantities (denoted by primes) in terms of the laboratory frame quantities by use of the Lorentz transformations to obtain $\rho_t' = \gamma_0(\delta\rho - n_b e/\gamma_0^2)$, where $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. It is clear from the expression of ρ_t' that owing to the presence of the $1/\gamma_0^2$ factor, even before the beam is fully charge neutralized in the laboratory frame, the total charge density as seen from the beam frame would have already turned positive. Since there is no magnetically induced electric field in the beam frame, the electrostatic field due to this positive ρ_t' is thus responsible for the acceleration of the plasma electrons in building up the counter current.

The electrical and magnetic neutralizations are closely related in that the magnitude of the driving field of the counter current depends on the degree of charge neutralization. Therefore, unless noted otherwise, the term "neutralization" will imply both electrical and magnetic neutralization.

Using a cold electron, immobile ion, and rigid beam model, Hammer and Rostoker[2] showed that the beam will be fully neutralized provided its radius is much larger than the skin depth c/ω_{pe} of the plasma, where ω_{pe} is the electron plasma frequency. Their model was subsequently extended by Lee and Sudan[3] to include a uniform external magnetic field. In their paper it was found that the neutralization could be significantly reduced if the axial external magnetic field is strong for plasma electrons (i.e., $\Omega_e \gg \omega_{pe}$, where Ω_e is the electron cyclotron frequency in the external magnetic field). Both papers neglected the ion dynamics. While the immobile ion model is justifiable for a field-free or weakly magnetized plasma, it is not valid for a particular magnetic field regime, namely, $\Omega_e \gg \omega_{pe}$ but $\Omega_i \ll \omega_{pi}$, in which the electrons, tied to field lines, are transversely immobile, but the ions essentially still see no magnetic field. Therefore ions, instead of electrons, are expected to assume the dominant role by moving radially into the beam region in response to the negatively charged beam.

The two-fluid approach to be used in this paper is motivated by the foregoing considerations.

II. FORMULATION

We adopt essentially the same model developed by Hammer and Rostoker[2] (Fig. 1), namely, at $t=0$, a rigid cylindrical relativistic electron beam of uniform density stretching from $z=-\infty$ to $z=0$ along the z axis is propagating with constant velocity $v_0\hat{e}_z$ in an infinite and uniform plasma. The plasma is magnitized by a constant field $B_0\hat{e}_z$ and its density is much larger than the beam density so that linear perturbation theory can be used. However, instead of turning on the beam at $t=0$ and looking for solutions at $t=\infty$ as in the original model, we assume that an equilibrium condition is already reached at $t=0$. In other words, the z and t dependence of all quantities are now integrated into one variable $z-v_0t$. Unless noted otherwise, the lab frame is used throughout.

According to this model, the beam current density is

$$\mathbf{J}_b = \begin{cases} -n_b e v_0 \hat{e}_z & r \le a, \quad z - v_0 t < 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where n_b is the beam electron density, and a is the beam radius.

In reality, the beam always has a tail; however, in our semi-infinite beam model, only the front of the beam will be considered. We note that there will be no loss of generality in this model since a finite length beam can easily be modeled by superimposing two semi-infinite beams of opposite charges. In other words, the physics of an electron beam tail is simply that of a positron beam front.

We use the two-fluid equations for a cold plasma to describe the dynamics of both species

$$\frac{\partial \mathbf{v}_e}{\partial t}+(\mathbf{v}_e\cdot\nabla)\mathbf{v}_e=-\frac{e}{m}\mathbf{E}-\frac{e}{mc}\mathbf{v}_e\times\mathbf{B}-\nu\mathbf{v}_e, \quad (2)$$

$$\frac{\partial \mathbf{v}_i}{\partial t}+(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i=\frac{e}{M}\mathbf{E}+\frac{e}{Mc}\mathbf{v}_i\times\mathbf{B}-\nu\mathbf{v}_i, \quad (3)$$

where m is the electron mass, and M is the ion mass.

In the above equations, a phenomenological momentum relaxation term is introduced. In the next section, it will be estimated that for the parameters typical of a relativistic electron beam experiment, the momentum transfer collision frequency ν is several orders of magnitude smaller than the plasma frequency. The duration of an electron beam is typically several tens of nsec (or several hundreds of plasma oscillations in most cases). On this time scale, the collisions will have very little damping effect, still, the introduction of ν is necessary in order to obtain an equilibrium solution in which a static beam electric field may exist. As it will turn out, ν serves to properly locate the poles when the contour integration method is used. For this purpose, we can let ν be the same for both species of fluids without qualitatively affecting the results.

The equilibrium assumption enables us to eliminate the time variable by defining $Z=z-v_0t$; thus,

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial Z}, \qquad \frac{\partial}{\partial t}=-v_0\frac{\partial}{\partial Z}. \quad (4)$$

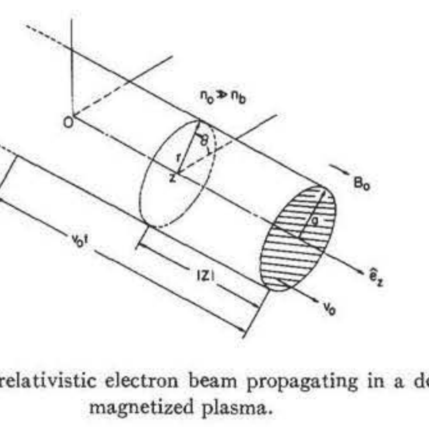


FIG. 1. A relativistic electron beam propagating in a dense magnetized plasma.

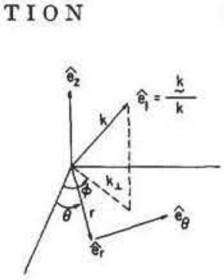


FIG. 2. $\mathbf{x}$ space and $\mathbf{k}$ space.

Substituting Eq. (4) into Eqs. (2) and (3), linearizing the resulting equations and Fourier transforming the space variables gives

$$(-ik_zv_0+\nu)\delta\bar{\mathbf{v}}_e=-(e/m)\delta\bar{\mathbf{E}}-(e/mc)\delta\bar{\mathbf{v}}_e\times\mathbf{B}_0, \quad (5)$$

$$(-ik_zv_0+\nu)\delta\bar{\mathbf{v}}_i=(e/M)\delta\bar{\mathbf{E}}+(e/Mc)\delta\bar{\mathbf{v}}_i\times\mathbf{B}_0, \quad (6)$$

where quantities in $\mathbf{k}$ space are distinguished by a bar on top and $\delta\mathbf{v}_e$, $\delta\bar{\mathbf{v}}_i$, $\delta\bar{\mathbf{E}}$, etc., are first-order perturbed quantities.

For later convenience, we introduce a new frame defined by the following unit vectors (see Fig. 2):

$$\hat{e}_1=\mathbf{k}/k, \qquad \hat{e}_3=k_\perp^{-1}\mathbf{k}\times\hat{e}_z, \qquad \hat{e}_2=\hat{e}_3\times\hat{e}_1.$$

After some straightforward manipulations of Eqs. (5) and (6) in this new frame, we obtain Ohm's law for the plasma medium

$$\delta\bar{\mathbf{J}}=n_0e(\delta\bar{\mathbf{v}}_i-\delta\bar{\mathbf{v}}_e)=\boldsymbol{\sigma}\cdot\delta\bar{\mathbf{E}}, \quad (7)$$

where

$$\sigma_{11}=\Delta^{-1}(s)\{-k^2(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2] +v_0^{-2}\omega_{pe}^2\Omega_e^2s^2[(s-\nu)^2+\Omega_i^2]\}$$

$$\sigma_{22}=\Delta^{-1}(s)\{-k^2(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2] -k_\perp^2\omega_{pe}^2\Omega_e^2[(s-\nu)^2+\Omega_i^2]\},$$

$$\sigma_{33}=-\Delta^{-1}(s)k^2(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2],$$

$$\sigma_{12}=\sigma_{21}=iv_0^{-1}\Delta^{-1}(s)k_\perp\omega_{pe}^2\Omega_e^2s\,|\,(s-\nu)^2+\Omega_i^2\,|,$$

$$\sigma_{23}=-\sigma_{32}=iv_0^{-1}\Delta^{-1}(s)s(s-\nu)^3\omega_{pe}^2\Omega_e,$$

$$\sigma_{31}=-\sigma_{31}=-\Delta^{-1}(s)k\,k_\perp\omega_{pe}^2\Omega_e(s-\nu)^3,$$

and by definition

$$\Delta(s)=4\pi k^2(s-\nu)[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2],$$

$$s=ik_zv_0, \qquad k^2=k_\perp^2+k_z^2=k_\perp^2-s^2/v_0^2.$$

Combining Ampere's and Faraday's law yields

$$\nabla\times\nabla\times\mathbf{E}=-\frac{4\pi}{c^2}\frac{\partial\mathbf{J}}{\partial t}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \quad (8)$$

Following the same procedures as those which led to Eqs. (5) and (6), we obtain, from Eq. (8),

$$-\mathbf{k}(\mathbf{k}\cdot\delta\bar{\mathbf{E}})+[k^2+(s^2/c^2)]\delta\bar{\mathbf{E}}-(4\pi s/c^2)\delta\bar{\mathbf{J}} =(4\pi s/c^2)\bar{\mathbf{J}}_b, \quad (9)$$

where $\bar{\mathbf{J}}_b$ is the Fourier transform of $\mathbf{J}_b$ in Eq. (1):

$$\bar{\mathbf{J}}_b=[2\pi n_b e a v_0^2 J_1(k_\perp a)/k_\perp(s-\delta)]\hat{e}_z$$

and δ is an infinitesimally positive number.

Substituting Eq. (7) for $\delta\bar{\mathbf{J}}$ in Eq. (9) results in

$$\boldsymbol{\alpha}\cdot\delta\bar{\mathbf{E}}=s\Delta(s)\bar{\mathbf{J}}_b,$$

or

$$\delta\bar{\mathbf{E}}=s\Delta(s)\boldsymbol{\alpha}^{-1}\cdot\bar{\mathbf{J}}_b, \tag{10}$$

where

$$\alpha_{11}=s\{k^2(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2]-v_0^{-2}\omega_{pe}^2\Omega_e^2s^2[(s-\nu)^2+\Omega_i^2]\}+k^2s^2(s-\nu)[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2],$$

$$\alpha_{22}=s\{k^2(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2]+k_\perp^2\omega_{pe}^2\Omega_e^2[(s-\nu)^2+\Omega_i^2]\}+k^2c^2(s-\nu)\times(k_\perp^2-s^2\gamma_0^{-2}v_0^{-2})[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2],$$

$$\sigma_{33}=k^2s(s-\nu)^2[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2]+k^2c^2(s-\nu)\times(k_\perp^2-s^2\gamma_0^{-2}v_0^{-2})[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2],$$

$$\alpha_{12}=\alpha_{21}=-iv_0^{-1}k_\perp\omega_{pe}^2\Omega_e^2s^2[(s-\nu)^2+\Omega_i^2],$$

$$\alpha_{32}=-\alpha_{23}=iv_0^{-1}k\omega_{pe}^2\Omega_e s^2(s-\nu)^3,$$

$$\alpha_{13}=-\alpha_{31}=-k\,k_\perp\omega_{pe}^2\Omega_e s(s-\nu)^3.$$

Equations (7) and (10) give

$$\delta\bar{\mathbf{J}}=s\Delta(s)\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}^{-1}\cdot\bar{\mathbf{J}}_b=\frac{s\Delta(s)}{D(k_\perp,s)}\boldsymbol{\sigma}\cdot\boldsymbol{\kappa}\cdot\bar{\mathbf{J}}_b, \tag{11}$$

where $D(k_\perp,s)=\det|\boldsymbol{\alpha}|$ and $\boldsymbol{\kappa}$ is the adjoint matrix of $\boldsymbol{\alpha}$.

From Eq. (11), $\delta\bar{\mathbf{J}}_1$ and $\delta\bar{\mathbf{J}}_2$ can be calculated:

$$\delta\bar{\mathbf{J}}_1=\frac{s\bar{\mathbf{J}}_b\Delta(s)}{kD(k_\perp,s)}[-v_0^{-1}s(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{11}+k_\perp(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{12}],$$

$$\delta\bar{J}_2=\frac{s\bar{J}_b\Delta(s)}{kD(k_\perp,s)}\left(\frac{-i}{v_0}s(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{21}+k_\perp(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{22}\right).$$

The axial current density is

$$\delta\bar{J}_z=\delta\bar{\mathbf{J}}\cdot\hat{e}_z=k^{-1}[(-i/v_0)s\delta\bar{J}_1+k_\perp\delta\bar{J}_2]=\frac{s\bar{J}_b}{(k_\perp^2-s^2/v_0^2)D(k_\perp,s)}W(k_\perp,s),$$

where

$$W(k_\perp,s)=\Delta(s)\{k_\perp^2(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{22}-s^2v_0^{-2}(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{11}-isv_0^{-1}k_\perp[(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{12}+(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{21}]\}.$$

The perturbed charge density is readily found by Fourier transforming the continuity equation

$$\frac{\partial\delta\rho}{\partial t}+\boldsymbol{\nabla}\cdot\delta\mathbf{J}=0;$$

thus,

$$\delta\bar{\rho}=\frac{i}{s}k\delta\bar{J}_1=\frac{-\bar{J}_b}{v_0D(k_\perp,s)}Y(k_\perp,s),$$

where

$$Y(k_\perp,s)=-\Delta(s)[s(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{11}+ik_\perp v_0(\boldsymbol{\sigma}\cdot\boldsymbol{\kappa})_{12}].$$

Note that $D(k_\perp,s)$, $W(k_\perp,s)$, and $Y(k_\perp,s)$ are all polynomials of $k_\perp$ and s.

We now inverse Fourier transform $\delta\bar{J}_z$ and $\delta\bar{\rho}$ into the $\mathbf{x}$ space.

$$\delta J_z=(2\pi)^{-3}\int d^3k\,\delta\bar{J}_z\exp(i\mathbf{k}\cdot\mathbf{x})=\frac{-i}{v_0(2\pi)^3}\int_0^\infty k_\perp\,dk_\perp\int_0^{2\pi}d\phi\exp[ik_\perp r\cos(\phi-\theta)]\times\int_{-i\infty}^{i\infty}ds\frac{s\bar{J}_bW(k_\perp,s)\exp(sZ/v_0)}{(k_\perp^2-s^2/v_0^2)D(k_\perp s)}. \tag{12}$$

The ϕ integration (see Fig. 2) can immediately be carried out if one writes

$$\exp[ik_\perp r\cos(\phi-\theta)]=\sum_{n=-\infty}^{\infty}\exp[i(\phi-\theta+\tfrac{1}{2}\pi)]J_n(k_\perp r).$$

So, with $\bar{J}_b$ replaced by its explicit expression, Eq. (12) becomes

$$\delta J_z=\frac{i}{2\pi}n_beav_0\int_0^\infty dk_\perp J_0(k_\perp r)J_1(k_\perp a)\times\int_{-i\infty}^{i\infty}ds\frac{W(k_\perp,s)\exp(sZ/v_0)}{(k_\perp^2-s^2/v_0^2)D(k_\perp,s)}. \tag{13}$$

In the same manner, we get

$$\delta\rho=\frac{i}{2\pi}n_bea\int_0^\infty dk_\perp J_0(k_\perp r)J_1(k_\perp a)\times\int_{-i\infty}^{i\infty}ds\frac{Y(k_\perp,s)\exp(sZ/v_0)}{(s-\delta)D(k_\perp,s)}. \tag{14}$$

The s integration can be done using the method of contour integration. For $Z<0$, we close the contour by drawing an infinite half circle on the right half s plane. Let s_j's be the poles inside the half circle, then Eqs. (13)

and (14) can be written

$$\delta J_z = n_b e a\, v_0 \int_0^\infty dk_\perp J_0(k_\perp r) J_1(k_\perp a) \times \sum_j \frac{(s_j - s_j) W(k_\perp, s_j) \exp(s_j Z/v_0)}{(k_\perp^2 - s_j^2/v_0^2) D(k_\perp, s_j)}, \quad (15)$$

$$\delta\rho = n_b e a \int_0^\infty dk_\perp J_0(k_\perp r) J_1(k_\perp a) \times \sum_j \frac{(s_j - s_j) Y(k_\perp, s_j) \exp(s_j Z/v_0)}{D(k_\perp, s_j)}. \quad (16)$$

Although, owing to the complicated form of $D(k_\perp, s)$, an exact determination of the poles is not possible, enough information is obtainable through appropriate approximations and numerical computations. Physically, the poles can be put into two categories, those with zero imaginary part and those with finite imaginary part. The former gives rise to nonoscillatory solutions and the latter to oscillatory ones.

III. NONOSCILLATORY SOLUTIONS

In this section we intend to find the nonoscillatory terms in Eqs. (15) and (16). As noted by Lee and Sudan,[3] only those nonoscillatory terms with a decay length longer than that of the plasma oscillations can be physically regarded as the neutralizing current or charge. Consequently, the poles we are looking for should be real and satisfy $0 < s_j \leq \nu$.

In cgs units, we have

$$\omega_{pe} \approx 6\times10^4 n_0^{1/2}, \qquad \nu \leq 10^{-5} n_0 \theta_e^{-3/2},$$

if ν is given by its classical value[5] and θ_e is the average plasma electron kinetic energy in electron volts.

As the plasma is disturbed by an intense electron beam, a lower estimate of θ_e would be on the order of 1 eV. Assuming $n_0 \approx 10^{12}$, then

$$s_j/\omega_{pe} \leq \nu/\omega_{pe} \leq 1.7\times10^{-4}.$$

In a turbulent plasma, the fluctuations can produce an effective collision frequency much larger than the classical value; nevertheless, it will still fall far below the plasma frequency.

We can now expand $W(k_\perp, s)$ $Y(k_\perp, s)$, and $D(k_\perp, s)$ in terms of s and ν. Up to the lowest significant order, we obtain

$$\delta I_z^{(n)} = i n_b e a^2 v_0 \int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp} \int_{-i\infty}^{i\infty} ds \frac{[s(\omega_{pi}^2 - \Omega_i^2 \gamma_0^{-2} \beta_0^{-2}) - \nu \omega_{pi}^2] \exp(sZ/v_0)}{D^{(n)}(k_\perp, s)} \quad (17)$$

$$\delta Q^{(n)} = i n_b e a^2 \int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp} \int_{-i\infty}^{i\infty} ds \frac{[s^2(k_\perp^2\lambda^2 + \omega_{pi}^2 - \Omega_i^2\gamma_0^{-2}\beta_0^{-2}) - s\nu\omega_{pi}^2(2k_\perp^2\lambda^2+1) + \nu^2 k_\perp^2 \lambda^2 \omega_{pi}^2] \exp(sZ/v_0)}{(s-\delta) D^{(n)}(k_\perp, s)}, \quad (18)$$

where

$\lambda = c/\omega_{pe}$, is the plasma skin depth; $\delta I_z^{(n)} = \int_0^a \delta J_z^{(n)} 2\pi r\, dr$, is the total counter current flowing inside the beam, $\delta Q^{(n)} = \int_0^a \delta\rho^{(n)} 2\pi r\, dr$, is the total perturbed charge per unit length inside the beam,

$$D^{(n)}(k_\perp, s) = s^2[\omega_{pi}^2(k_\perp^2\lambda^2+1) + \Omega_i^2(k_\perp^2\lambda^2 - \gamma_0^{-2}\beta_0^{-2})] - s\nu[2k_\perp^2\lambda^2\omega_{pi}^2 + \omega_{pi}^2 + k_\perp^2\lambda^2\Omega_i^2] + \nu^2 k_\perp^2 \lambda^2 \omega_{pi}^2,$$

and the superscript n denotes the nonoscillatory solution.

The two poles given by the zeros of $D^{(n)}(k_\perp, s)$ are

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \frac{\{1 + 2k_\perp^2\lambda^2 + k_\perp^2\lambda^2(\Omega_i^2/\omega_{pi}^2) \pm [[1 + k_\perp^2\lambda^2(\Omega_i^2/\omega_{pi}^2)]^2 + (4\Omega_i^2/\gamma_0^2\beta_0^2\omega_{pi}^2) k_\perp^2\lambda^2]^{1/2}\}\nu}{2\{1 + k_\perp^2\lambda^2 + (\Omega_i^2/\omega_{pi}^2)[k_\perp^2\lambda^2 - (1/\gamma_0^2\beta_0^2)]\}}. \quad (19)$$

The pole s_2 agrees with our assumption $0 < s_2 \leq \nu$ for all $k_\perp$. When $\Omega_i^2 \ll \omega_{pi}^2$, s_1 also agrees with the same assumption for all $k_\perp$, but as Ω_i gets larger, s_1 satisfies $0 < s_1 \leq \nu$ only for $k_\perp \geq k_{\perp c}$, where $k_{\perp c}$ can be determined from Eq. (19).

With all the relevant poles known, Eqs. (17) and (18) can then be integrated over s:

$$\frac{\delta I_z^{(n)}}{|I_b|} = 2\int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp} \left\{ \frac{[s_1(1 - \gamma_0^{-2}\beta_0^{-2}\Omega_i^2\omega_{pi}^2) - \nu] \exp(s_1 Z/v_0) - [s_2(1 - \gamma_0^{-2}\beta_0^{-2}\Omega_i^2\omega_{pi}^2) - \nu] \exp(s_2 Z/v_0)}{(s_1 - s_2)[1 + k_\perp^2\lambda^2 + \Omega_i^2\omega_{pi}^{-2}(k_\perp^2\lambda^2 - \gamma_0^{-2}\beta_0^{-2})]} \right\} \quad (20)$$

$$\frac{\delta Q^{(n)}}{|Q_b|} = 2\int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp} \left\{ 1 + \frac{k_\perp^2\lambda^2\Omega_i^2\omega_{pi}^{-2}[(-s_1+\nu) \exp(s_1 Z/v_0) - (-s_2+\nu) \exp(s_2 Z/v_0)]}{(s_1 - s_2)[1 + k_\perp^2\lambda^2 + \Omega_i^2\omega_{pi}^{-2}(k_\perp^2\lambda^2 - \gamma_0^{-2}\beta_0^{-2})]} \right\}, \quad (21)$$

where $I_b = -\pi a^2 n_b e v_0$, is the total beam current, and $Q_b = -\pi a^2 n_b e$, is the total beam charge per unit length.

The $k_\perp$ integration can be carried out analytically in the following limits:

Case (i) $\Omega_i^2\lambda^2/\omega_{pi}^2 a^2 \ll 1$.

Because of the presence of $J_1^2(k_\perp a)/k_\perp$, the integrands in Eqs. (20) and (21) peak at $k_\perp \sim 1/a$, while they decay as $1/k_\perp^2$ for $k_\perp a \gg 1$, and as $k_\perp$ for $k_\perp a \ll 1$ (Fig. 3). We thus expect the dominant contribution to the integrals to be in the neighborhood of $k_\perp \sim 1/a$, provided the other factors in the integrands behave smoothly in this region. This last condition can be shown a posteriori to be true. Consequently, $k_\perp$ in Eqs. (20) and (21) may be treated as on the order of $1/a$, and s_1 and s_2 reduce to

$$s_1 \simeq \nu, \qquad s_2 \simeq k_\perp^2\lambda^2\nu/(k_\perp^2\lambda^2+1),$$

where $O(\Omega_i^2\lambda^2/\omega_{pi}^2a^2)$ terms have been neglected.

Substituting s_1 and s_2 into Eqs. (20) and (21), we obtain

$$\begin{aligned}\frac{\delta I_z^{(n)}}{|I_b|} &= 2\int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp}\left\{\frac{\exp[k_\perp^2\lambda^2\nu Z/(k_\perp^2\lambda^2+1)v_0]}{k_\perp^2\lambda^2+1} + O\left(\frac{\Omega_i^2\lambda^2}{\omega_{pi}^2a^2}\right)\right\}\\ &= [1-2I_1(a/\lambda)K_1(a/\lambda)]\exp(Z/l) + O(\Omega_i^2\lambda^2/\omega_{pi}^2a^2)\\ &= \begin{cases}[1-O(\lambda/a)]\exp(Z/l)+O(\Omega_i^2\lambda^2/\omega_{pi}^2a^2) & \text{if } a\gg\lambda,\\ O(\Omega_i^2\lambda^2/\omega_{pi}^2a^2) & \text{if } a\ll\lambda,\end{cases}\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial Q^{(n)}}{|Q_b|} &= 2\int_0^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp}\left[1+O\left(\frac{\Omega_i^2\lambda^2}{\omega_{pi}^2a^2}\right)\right]\\ &\simeq 1+O(\Omega_i^2\lambda^2/\omega_{pi}^2a^2),\end{aligned} \tag{23}$$

where I_1 and K_1 are modified Bessel functions of the first and second kind, respectively, and

$$l = (a^2+\lambda^2)v_0/\lambda^2\nu,$$

is the decay length of the counter current. Note that, in carrying out the $k_\perp$ integrations, we have replaced $k_\perp$ in the exponential by its dominant value a^{-1}.

In the above equations, we see that the beam will be charge neutralized for any beam radius, and current neutralized if $a \gg \lambda$.

To sum up, we obtain the following criteria for neutralization:

(a) Charge neutralization criterion:

$$\lambda^2\Omega_i^2/a^2\omega_{pi}^2 \ll 1.$$

(b) Current neutralization criterion:

$$(\lambda^2/a^2)[(\Omega_i^2/\omega_{pi}^2)+1] \ll 1.$$

This leads to the conclusion that the neutralization criteria derived under the immobile ion model[3] should be modified by a factor m/M.

Despite the important roles played by the ion dynamics and the longitudinal guide field in the induction of the counter current, neither of them could alter the decay rate of the counter current, which is due to the flow of plasma electrons along the magnetic field lines. As a result, the decay length l derived above agrees with the results of earlier authors.[3,4]

Case (ii) $\Omega_i^2\lambda^2/\omega_{pi}^2a^2 \gg 1$

By the same consideration if we neglect $O(\omega_{pi}^2a^2/\Omega_i^2\lambda^2)$ terms, s_1 and s_2 reduce to

$$s_1 \simeq k_\perp^2\lambda^2\nu/(k_\perp^2\lambda^2 - \gamma_0^{-2}\beta_0^{-2}), \qquad s_2 \simeq 0+,$$

hence,

$$\frac{\delta I_z^{(n)}}{|I_b|} = \frac{\delta Q^{(n)}}{|Q_b|} = 2\int_{k_{\perp c}}^\infty dk_\perp \frac{J_1^2(k_\perp a)}{k_\perp}\left\{\frac{\exp[k_\perp^2\lambda^2\nu Z/(k_\perp^2\lambda^2-\gamma_0^{-2}\beta_0^{-2})v_0]}{1-k_\perp^2\lambda^2\gamma_0^2\beta_0^2} + O\left(\frac{\omega_{pi}^2a^2}{\Omega_i^2\lambda^2}\right)\right\}, \tag{24}$$

where

$$k_{\perp c} \simeq \sqrt{2}/\lambda\gamma_0\beta_0.$$

It is interesting to note that, aside from the $O(\omega_{pi}^2a^2/\Omega_i^2\lambda^2)$ terms, the results are identical to the infinite field limit of the immobile ion model. The reason is that for such a strong field, both electrons and ions are transversely immobile. But electrons have a greater axial mobility, hence dominate the situation.

It can be shown that the integral in Eq. (24) is negative and vanishingly small. Physically, owing to the lack of charge neutralization, the plasma electrons are pushed ahead with the beam electrons, thus resulting in a total charge and current slightly in excess of those of the beam.

The general dependence of neutralization on the magnetic field is computed numerically from Eqs. (20) and (21). The results shown in Fig. 4 conform to our analytic analysis for the two limiting cases.

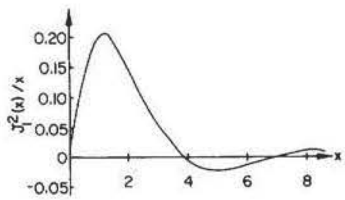
It should be pointed out that case (i) virtually covers all the practical experimental conditions. For instance, with a background plasma density near 10^{12} particles/cm^3, it will take a magnetic field over 1000 kG to break away from case (i).

In Fig. 5, charge and current neutralization are again plotted vs the absolute strength of B_0 and in comparison with the stationary ion model. It is observed that, in the practically achievable magnetic field range, (i) charge neutralization is almost total under all conditions, (ii) current neutralization is more sensitive to a/λ than to B_0; and (iii) the effects of ion dynamics are important even for a few kG, and become progressively so for larger B_0.

It is known that the electrostatic potential well created by an unneutralized intense relativistic electron beam could be as high as many MV. Its application for fast electrostatic heating of ions to thermonuclear fusion temperature is apparent.

It was believed that this purpose could be achieved through the simplest scheme of injecting the beam parallel into a magnetized plasma. However, the charge neutralization criterion derived above indicates that realization of this scheme requires an impractically strong magnetic field. On the other hand, a potential well of the order of MV is not really most favorable for fusion purposes considering the sudden burst of over energetic ions it will generate. We are thus led to the investigation of other schemes which will provide more favorable, as well as realizable conditions, for electrostatic heating of ions. Results of this study will be reported shortly.

FIG. 3. Plot of $J_1^2(x)/x$ vs x.



IV. OSCILLATORY SOLUTIONS

These solutions consist of the various modes of cold plasma oscillations. The fact that the plasma is anisotropic in the presence of an external magnetic field greatly complicates the analysis. However, as a result of the anisotropy, the imaginary parts of the oscillatory poles are functions of $k_\perp$. When integrated over $k_\perp$, those oscillatory terms become negligible owing to phase and frequency mixing. The only appreciable patterns of oscillations occur in the following limits.

Case (i) $a \gg \lambda$.

As before, the dominant contribution to all integrals comes from the region $k_\perp \approx 1/a$, or for this case, $k_\perp\lambda \ll 1$. Accordingly, $k_\perp\lambda$ can be treated as small parameters in terms of which $W(k_\perp, s)$, $Y(k_\perp, s)$, and $D(k_\perp, s)$ can be expanded. Keeping only the zero-order terms, we obtain these quantities of interest from Eqs. (10) and (14)

$$\delta\rho^{(0)} = \frac{i}{2\pi} n_b e a \int_0^\infty dk_\perp\, J_0(k_\perp r) J_1(k_\perp a) \int_{-i\infty}^{i\infty} ds \frac{\omega_{pe}^2 \exp(sZ/v_0)}{(s-\delta)[(s-\nu)^2+\omega_{pe}^2]}$$

$$= \begin{cases} -n_b e \exp[(\nu/v_0)(z-v_0t)] \cos[(\omega_{pe}/v_0)(z-v_0t)] & \text{for } r\le a,\ z-v_0t<0 \\ 0, & \text{otherwise} \end{cases}$$

$$\delta E_z^{(0)} = -i2n_b e a v_0 \int_0^\infty dk_\perp J_0(k_\perp r) J_1(k_\perp a) \int_{-i\infty}^{i\infty} ds \frac{\exp(sZ/v_0)}{(s-\nu)^2+\omega_{pe}^2}$$

$$= \begin{cases} -4\pi n_b e (v_0/\omega_{pe}) \exp[(\nu/v_0)(z-v_0t)] \sin[(\omega_{pe}/v_0)(z-v_0t)] & \text{for } r\le a,\ z-v_0t<0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta E_r^{(0)} \simeq \delta E_\theta^{(0)} = 0$$

where the superscript 0 denotes the oscillatory solution.

By virtue of the equipartition law, the average kinetic energy of the plasma equals the average oscillatory electric field energy; thus, for a unit length of plasma inside the beam,

$$\langle \text{K.E.} \rangle_{\text{av}} = \int_0^a dr\, 2\pi r \left\langle \frac{|\delta\mathbf{E}|^2}{8\pi} \right\rangle_{\text{av}}$$

$$= N_b^2 e^2 (v_0^2/\omega_{pe}^2 a^2) \exp[(2\nu/v_0)(z-v_0t)],$$

where $N_b = \pi a^2 n_b$.

This is two orders of magnitude smaller than the electrostatic field energy inside an unneutralized electron beam ($\approx N_b^2 e^2/4$), which, for an intense relativistic electron beam, is comparable to the beam kinetic energy.

Case (ii) $a \ll \lambda$

In this limit, the opposite of case (i) is true, namely, the dominant contribution to each integral is at $k_\perp\lambda \gg 1$. So treating $(k_\perp\lambda)^{-1}$ as a small parameter in the expansion and keeping only the zero-order terms, we obtain

$$\delta\rho^{(0)}=\frac{i}{2\pi}n_bea\int_0^\infty dk_\perp J_0(k_\perp r)J_1(k_\perp a)\int_{-i\infty}^{i\infty}ds\frac{[\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2]\exp(sZ/v_0)}{(s-\nu)\{[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2]+\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2\}}$$

$$=\begin{cases}-n_be\left[\dfrac{\omega_{pi}^2\Omega_e^2\exp[(\nu/v_0)(z-v_0t)]}{(\omega_{pe}^2+\Omega_e^2)(\omega_{pi}^2+\Omega_i^2)}\cos\dfrac{\omega_{lh}(z-v_0t)}{v_0}+\dfrac{\omega_{pe}^2\exp[(\nu/v_0)(z-v_0t)]}{\omega_{pe}^2+\Omega_e^2}\cos\dfrac{\omega_{uh}(z-v_0t)}{v_0}\right] & \text{for}\quad r\leq a,\ z-v_0t<0\\ 0 & \text{otherwise}\end{cases}$$

$$\delta E_e^{(0)}=\frac{-n_bea}{\pi}\int_0^\infty dk_\perp\frac{J_1(k_\perp a)}{k_\perp}\int_0^{2\pi}d\phi\exp[ik_\perp r\cos(\phi-\theta)]\cos(\phi-\theta)$$

$$\int_{-i\infty}^{i\infty}ds\ \frac{[(s-\nu)^2+\Omega_e^2][(s-\nu)+\Omega_i^2]\exp(sZ/v_0)}{(s-\delta)\{[(s-\nu)^2+\Omega_e^2][(s-\nu)^2+\Omega_i^2]+\omega_{pe}^2(s-\nu)^2+\omega_{pi}^2\Omega_e^2\}}$$

$$=-2\pi n_be\left(\frac{\Omega_e^2\omega_{pi}^2\exp[(\nu/v_0)(z-v_0t)]}{(\omega_{pe}^2+\Omega_e^2)(\omega_{pi}^2+\Omega_i^2)}\cos\frac{\omega_{lh}(z-v_0t)}{v_0}+\frac{\omega_{pe}^2\exp[(\nu/v_0)(z-v_0t)]}{\omega_{pe}^2+\Omega_e^2}\cos\frac{\omega_{uh}(z-v_0t)}{v_0}\right)\cdot\begin{cases}r & r\leq a,\\ a^2/r & r>a\end{cases}$$

$$\delta E_z^{(0)}\simeq\delta E_\theta^{(0)}\simeq 0,$$

where

$$\omega_{lh}=[\Omega_e^2(\omega_{pi}^2+\Omega_i^2)/(\omega_{pe}^2+\Omega_e^2)]^{1/2},\quad \omega_{uh}=(\omega_{pe}^2+\Omega_e^2)^{1/2},$$

are the lower and upper hybrid mode, respectively.

For the same reason as before, the average plasma kinetic energy per unit length inside the beam is

$$\langle\text{K.E.}\rangle_{\text{av}}=\int_0^a\left\langle\frac{|\delta\mathbf{E}|^2}{8\pi}\right\rangle_{\text{av}}2\pi r\,dr$$

$$\simeq\tfrac{1}{8}(N_b^2e^2)\left[\left(\frac{\Omega_e^2\omega_{pi}^2}{(\omega_{pe}^2+\Omega_e^2)(\omega_{pi}^2+\Omega_i^2)}\right)^2+\left(\frac{\omega_{pe}^2}{\omega_{pe}^2+\Omega_e^2}\right)^2\right].$$

In contrast to case (i), this energy amounts up to half of the nonneutral beam electrostatic field energy depending on the external magnetic field strength. Note that when $\Omega_e\gg\omega_{pe}$ and $\Omega_i\ll\omega_{pi}$, not only the plasma energy is at its maximum value, but it is mostly shared by the ions owing to the dominance of the lower hybrid mode, a fact again conforming to the relative activity of the ions in this magnetic field regime.

Although this situation is difficult to realize experimentally because of the requirement of an extremely thin beam, it points to an interesting as well as explorable heating mechanism for the ions.

All along we have adopted a model under which the beam has zero rise time. What if the beam has a finite rise time as it always does? As shown by Lee,[6] the expressions corresponding to a beam with finite rise time are readily obtainable by an integration of their respective zero rise time expressions (remember we are in the linear theory regime). For our purposes, it sufficies to make a general comment on the finite rise time effects instead of carrying out the detailed calculation for a specific case. Let τ_r be the rise time of the beam, and τ_c be the characteristic time of a particular quantity, such as the decay time of the counter current or the period of an oscillatory mode. Then, to account for the effects of τ_r, one may roughly add a factor g to the quantities under the zero rise time model, where

$$g=(1+\tau_r/\tau_c)^{-1}.$$

For all intense relativistic electron beam experiments, τ_r is between a few to a few tens of nanoseconds, while the counter current decay time is at least of the order of microseconds. Hence, under general experi-

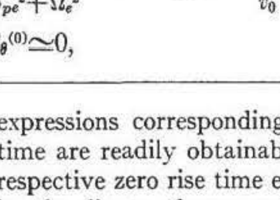


FIG. 4. Upper: Charge neutralization vs Ω_i/ω_{pi}. Lower: Current neutralization (at the beam front) vs Ω_i/ω_{pi}.

mental conditions, $g \simeq 1$, so that the finite rise time has little effect on the current neutralization phenomena.

On the other hand, the finite rise time may cause strong phase mixing of the oscillatory modes and thereby reduce their amplitudes considerably. To see this numerically, we assume τ_r to be in the same range as before and $n_0 \gtrsim 10^{12}$ cm^{-3}; thus,

$$\tau_{cu} \equiv 2\pi/\omega_{uh} \lesssim \tau_{ep} \equiv 2\pi/\omega_{pe} \lesssim 10^{-1} \text{ nsec},$$

$$\tau_{cl} \equiv 2\pi/\omega_{lh} \lesssim 5A^{1/2} \text{ nsec},$$

where A is the mass number of the ions. Thus, for the longitudinal plasma oscillations and the upper hybrid mode, $g \ll 1$, while for the lower hybrid mode, $g \lesssim 0.5$. Therefore, only the lower hybrid mode can survive the finite rise time effects. However, as pointed out before, it is also this mode that appears most interesting for the heating of plasma ions.

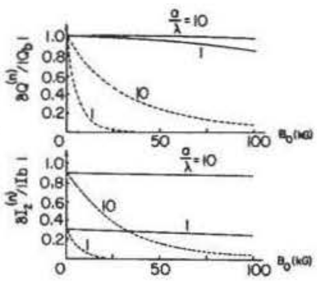


FIG. 5. Charge and current (at the beam front) neutralization vs B_0. Solid curves: $M=2$; dashed curves: $M=\infty$.

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