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Authors

Ortobelli, Sergio
Rachev, Svetlozar
Schwartz, Eduardo

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THE PROBLEM OF OPTIMAL ASSET ALLOCATION WITH STABLE DISTRIBUTED RETURNS

SERGIO ORTOBELLI

University of Calabria, Italy

SVETLOZAR RACHEV

University of Karlsruhe, Germany and University of California at Santa Barbara

EDUARDO SCHWARTZ

Anderson School of Management, University of California at Los Angeles

This paper discusses two optimal allocation problems. We consider different hypotheses of portfolio selection with stable distributed returns for each of them. In particular, we study the optimal allocation between a riskless return and risky stable distributed returns. Furthermore, we examine and compare the optimal allocation obtained with the Gaussian and the stable non-Gaussian distributional assumption for the risky return.

KEY WORDS: optimal allocation, stochastic dominance, risk aversion, measure of risk, α stable distribution, domain of attraction, sub-Gaussian stable distributed, fund separation, normal distribution, mean variance analysis, safety-first analysis.

1. INTRODUCTION

This paper serves a twofold objective: to compare the normal with the stable non-Gaussian distributional assumption when the optimal portfolio is to be chosen and to propose stable models for the optimal portfolio selection according to the utility theory under uncertainty.

It is well-known that asset returns are not normally distributed, but many of the concepts in theoretical and empirical finance developed over the past decades rest upon the assumption that asset returns follow a normal distribution. The fundamental work of Mandelbrot (1963a-b, 1967a-b) and Fama (1963,1965a-b) has sparked considerable interest in studying the empirical distribution of financial assets. The excess kurtosis found in Mandelbrot's and Fama's investigations led them to reject the normal assumption and to propose the stable Paretian distribution as a statistical model for asset returns. The Fama and Mandelbrot's conjecture was supported by numerous empirical investigations in the subsequent years, (see Mittnik, Rachev and Paoletta (1997) and Rachev and Mittnik (2000)).

The practical and theoretical appeal of the stable non-Gaussian approach is given by its attractive properties that are almost the same as the normal one. A relevant desirable property of stable distributional assumption is that stable distributions have domain of attraction. The Central Limit Theorem for normalized sums of i.i.d. random variables determines the domain of attraction of each stable law. Therefore, any distribution in the domain of attraction of a specified stable distribution will have properties close to those of the stable distribution. Another attractive aspect of the stable Paretian assumption is the stability property, i.e. stable distributions are stable with respect to summation of i.i.d. random stable variables. Hence, the stability governs the main properties of the underlying distribution. Detailed accounts of theoretical aspects of stable distributed random variables can be found in Samorodnitsky and Taquq (1994) and Janicki and Weron (1994).

In our work, we analyze two investment allocation problems. By comparing the normal distribution with the stable law one, it has occurred that the results performed under the examined optimal allocation problems are generally different. They consist of the maximization of the mean minus a measure of portfolio risk. We propose a mean risk analysis that facilitates the interpretation of the results. In the first allocation problem, we consider as the risk measure the expected value of a power absolute deviation. When the power is equal to two, we obtain the classical quadratic utility functional.

The second allocation problem is a typical problem of the safety-first analysis where we assume as the risk measure, the risk of the portfolio loss, (i.e. the probability that the portfolio return is under a fixed threshold value). We examine the optimal allocation between a riskless return and a risky stable distributed return, then we compare the allocation obtained with the Gaussian and the stable non-Gaussian distributional assumption for the risky return. We chose the three months LIBOR 6% annual rate as from December 1, 1999 for riskless return. As a possible risky asset, we consider the stock indexes S&P500, DAX30 and CAC40. The models' parameters are estimated in Khindanova, Rachev and Schwartz (1999). We show that there are significant differences in the allocation when the data fit the stable non-Gaussian or the normal distributions.

Following the two empirical asset analysis, we propose symmetric and asymmetric stable models to study the multivariate portfolio selection and to analyze the two proposed optimal allocation problems. We develop three alternative stable models under the assumption that investors allocate their wealth across the available assets in order to maximize their expected utility of final wealth. We first consider the portfolio allocation between n sub-Gaussian symmetrical α -stable distributed risky assets (with $\alpha > 1$), and the riskless one. The joint sub-Gaussian α stable family is an elliptical family. Hence, as argued by Owen and Rabinovitch (1984), we can use the mean dispersion analysis in this case. The resulting efficient frontier is formally the same as Markowitz-Tobin's mean-variance analysis, but instead of variance as a risk parameter, we have to consider the scale parameter of the stable distributions. Unlike Owen and Rabinovitch, we propose a method based on the moments to estimate all stable parameters. The calculating efficient frontier exhibits the two-fund separation property and in equilibrium, we obtain an α -stable version of the Sharpe-Lintner-Mossin's CAPM. Using the Ross' necessary and sufficient conditions to the two-fund separation, we can link this stable version of asset pricing to that in Gamrowski and Rachev (1999). In order to consider the possible asymmetry of asset returns, we describe a three-fund separation model for returns in the domain of attraction of a stable law. In case of asymmetry, the model results from a new stable version of the Simaan's model, see Simaan (1993). In case of symmetry of returns, we obtain a version of a model recently studied by Götzenberger, Rachev and Schwartz (1999), that can also be viewed as a particular version of the two-fund separation of Fama's (1965b) model. Our model distinguishes itself from Götzenberger, Rachev and Schwartz's, as well as from the Simaan and Fama's models because we consider the empirical hypothesis of fat tails of return distributions together with the asymmetric distributional components.

Using the stochastic dominance rules, (see the references in Levy (1992)), we show how we determine the efficient frontier for risk averse investors. Similarly to the sub-Gaussian approach, it is possible to estimate all parameters with a maximum likelihood method and to compare this model with the Simaan's one. One of the most severe restrictions in performance measurement and asset pricing in the stable case is the assumption of a common index of stability for all assets. Hence, the last model we propose deals with the case of optimal allocation between stable distributed portfolios with different indexes of stability. In order to overcome the difficulties of the most general case of the stable law, we introduce a $k+1$ fund separation model. Then we show how to express the model's multi-parameter efficient frontier.

In Section 1, we introduce the first allocation problem. In Section 2, we compare the stable non-Gaussian with the normal distributional hypothesis for the first allocation problem. Section 3 introduces the multivariate models and their application to the proposed allocation model. In Section 4, we introduce the second allocation problem. In Section 5 we compare stable non-Gaussian and normal distributional hypothesis for the second allocation problem. In Section 6 we consider the multivariate models for the second allocation problem. In the last section, we briefly summarize the results.

2. AN OPTIMAL ALLOCATION PROBLEM WITH STABLE DISTRIBUTED RETURNS

Consider the problem of finding the optimal allocation I in an investment consisting of two positions: a risky asset, which is assumed to be stable distributed and a risk-free asset. The investor wishes to maximize the utility functional

$$(2.1) \quad U(W) = E(W) - cE(|W - E(W)|^r),$$

where c and r are positive real numbers, $W = I z_0 + (1-I)z$ is the return on the portfolio, z_0 is the risk-free asset return, and z is the risky asset return. We also assume that no short sales are allowed (i.e. $I \in [0,1]$).

We choose to study the allocation problem (2.1) because:

1) The optimal allocation we get that solves problem (2.1), is equivalent to the following maximization of the utility functional

$$(2.2) \quad aE(W) - bE\left(|W - E(W)|^r\right),$$

assuming $c = \frac{b}{a}$ in (2.1) for every $a, b > 0$. If we assume $E\left(|W - E(W)|^r\right)$ as the particular risk measure of portfolio loss, then applying the optimal allocation problem (2.1), we implicitly maximize the expected mean of the increment wealth aW , as well as minimize the individual risk $bE\left(|W - E(W)|^r\right)$.

2) Furthermore, when we assume $r=2$, the maximization of utility functional (2.1) also motivates the mean variance approach in terms of preference relations.

We assume that z is \mathbf{a} -stable distributed, with $\mathbf{a} > 1$ (which implies the existence of the first moment), that is:

$$z = S_{\mathbf{a}}^d(\mathbf{s}_z, \mathbf{b}_z, m_z),$$

where \mathbf{a} is the index of stability, \mathbf{s}_z is the scale (dispersion) parameter, \mathbf{b}_z is the skewness parameter, and m_z is the mean of z .

We know that for $I \neq 1$, all the portfolio returns $W = I z_0 + (1 - I)z$ admits stable distribution $S_{\mathbf{a}}(|1 - I|\mathbf{s}_z, \text{sign}(1 - I)\mathbf{b}_z, I z_0 + (1 - I)m_z)$. Because $I \in [0, 1]$, then

$$(2.3) \quad W = S_{\mathbf{a}}^d((1 - I)\mathbf{s}_z, \mathbf{b}_z, I z_0 + (1 - I)m_z) \text{ for } I \in [0, 1) \text{ and } W = z_0 \text{ when } I = 1.$$

From (2.3) we see that the portfolio mean is given by $m_W = I z_0 + (1 - I)m_z$ and the portfolio scale parameter is given by $\mathbf{s}_W = (1 - I)\mathbf{s}_z$, hence, since the portfolio skewness parameter is fixed, all the solutions to the problem (2.1) can be represented in the mean-dispersion plane by

$$m_W = z_0 + \frac{m_z - z_0}{\mathbf{s}_z} \mathbf{s}_W,$$

representing the efficient frontier for our optimization problem.

Recall that given two random variables X and Y , X dominates Y in the sense of Rothschild-Stiglitz (R-S), if and only if every risk averse investor prefers X to Y , that is if and only if for every concave utility function u , we have $E(u(X)) \geq E(u(Y))$, or alternatively, if and only if $E(X) = E(Y)$ and

$$\int_{-\infty}^t F_X(u) du \leq \int_{-\infty}^t F_Y(u) du \text{ for every real } t, \text{ where } F_X \text{ is the cumulative distribution function of } X, \text{ (for$$

details on stochastic dominance rules see Lèvy (1992), Rothschild and Stiglitz (1970), Hanoch and Lèvy (1969)).

Suppose X dominates Y in the sense of R-S. Because $E(X) = E(Y)$ and $f(x) = -c|x - E(X)|^r$ is a concave utility function, for every $r \in [1, \mathbf{a}]$, it follows that:

$$U(X) := E(X) - cE(|X - E(X)|^r) \geq U(Y) \quad \forall r \in [1, \mathbf{a}].$$

The above inequality implies that every risk averse investor with utility functional (2.2) should choose a portfolio $W = \mathbf{I} z_0 + (1 - \mathbf{I})z$ that maximizes the utility functional (2.1) for some $r \in [1, \mathbf{a}]$ and $\mathbf{I} \in [0, 1]$. Now, to solve the asset allocation problem

$$(2.4) \quad \max_{\mathbf{I}} E(W) - cE(|W - E(W)|^r),$$

notice first that, for all $r \in [1, \mathbf{a}]$ and $1 < \mathbf{a} < 2$, we get

$$U(W) = E(W) - cE(|W - E(W)|^r) = \mathbf{I} z_0 + (1 - \mathbf{I})m_z - E(W) - c(H(\mathbf{a}, \mathbf{b}, r))^r (1 - \mathbf{I})^r \mathbf{s}_z^r$$

where

$$(H(\mathbf{a}, \mathbf{b}, r))^r = \frac{2^{r-1} \Gamma\left(1 - \frac{r}{\mathbf{a}}\right)}{r \int_0^\infty u^{-r-1} \sin^2 u du} \left(1 + \mathbf{b}^2 \left(\tan^2\left(\frac{\mathbf{a}p}{2}\right)\right)\right)^{r/2\mathbf{a}} \cos\left(\frac{r}{\mathbf{a}} \arctan\left(\mathbf{b} \tan\left(\frac{\mathbf{a}p}{2}\right)\right)\right)$$

(see Samorodnitsky and Taquq (1994), Hardin, Jr. (1984)). The above relation analyzes the stable non-Gaussian case. When z admits normal distribution (i.e. $\mathbf{a} = 2$), then for all $r > 0$,

$$U(W) = E(W) - cE(|W - E(W)|^r) = \mathbf{I} z_0 + (1 - \mathbf{I})m_z - c \frac{2^{\frac{r}{2}} \Gamma\left(\frac{r+1}{2}\right)}{\sqrt{p}} (1 - \mathbf{I})^r \mathbf{s}_z^r.$$

Of particular interest is the above Gaussian case for $r=2$. In fact, when $r=2$, the optimization problem

$$\max E(W) - cE(|W - E(W)|^2) = \max E(-cW^2 + (2cE(W) + 1)W - c(E(W))^2),$$

reduces to a maximization of the expected quadratic utility function

$$(2.5) \quad u(W) = -cW^2 + bW - \frac{(b-1)^2}{4c},$$

where $b = 2cE(W) + 1$. The allocation that maximizes the above problem is the same as the one that maximizes the utility function $au(W) + d$ for some real constant d and $a > 0$. Optimizing the expected value of the utility function

$$(2.6) \quad v(W) = -eW^2 + aW$$

for every possible real $(\text{sign}(b))a > 0$ and $e > 0$, we obtain the same allocation by maximizing the expected utility function (2.5) with $c = \frac{e}{a - 2eE(W)}$.

We recall that for arbitrary distributions with the finite variance, the Markowitz-Tobin mean-variance model is based on the quadratic utility (2.6). As a matter of fact, when expected returns and variances are finite, the quadratic utility is sufficient for asset choice to be completely described in terms of a preference relation defined over the mean and the variance of expected returns. However, quadratic utility displays the undesirable properties of satiation and increasing absolute risk aversion. Thus, economic conclusions based on the assumption of quadratic utility function are often counter intuitive and are not applicable to individuals who always prefer more wealth to less, and who treat risky investments as normal goods. For this reason, when we assume $r=2$, even if our optimal location problem (2.1) motivates the mean variance analysis in terms of preference relations, we prefer to consider different models motivated by their distributional form of returns. In appendix A, we include a table that summarizes the distributional assumption used in this paper and also describes the mean-variance and the stable mean-dispersion frontiers.

Next, we consider the optimal allocation problem for any choice of $r \in [1, \mathbf{a}]$.

2.1 Case $r=1$: Find $\max_{\mathbf{I}} E(W) - cE(|W - E(W)|)$

In this case, the first order condition of problem (2.4) shows that the optimal portfolio consists of full investment in the riskless asset z_0 (i.e. $\mathbf{I} = 1$) when

$$\frac{\partial U}{\partial \mathbf{I}} = z_0 - m_z + c(V(\mathbf{a}, \mathbf{b}, 1))\mathbf{s}_z > 0,$$

where

$$V(\mathbf{a}, \mathbf{b}, 1) = \begin{cases} H(\mathbf{a}, \mathbf{b}, 1) & \text{in the stable case } (1 < \mathbf{a} < 2) \\ \sqrt{\frac{2}{\mathbf{p}}} & \text{in the normal case } (\mathbf{a} = 2) \end{cases}.$$

In the case $\frac{\partial U}{\partial I} = z_0 - m_z + c(V(\mathbf{a}, \mathbf{b}, 1)\mathbf{s}_z) < 0$, then $I = 0$, i.e. the investor should invest everything

in the risky asset. Finally, when $\frac{\partial U}{\partial I} = 0$, then for every $0 \leq I \leq 1$, the portfolio is optimal, because in

this case $E(W) - cE(|W - E(W)|) = m_z - cV(\mathbf{a}, \mathbf{b}, 1)\mathbf{s}_z = z_0$, i.e. it is a constant for every I . Hence,

we found that when $r=1$, solving the optimization problem (2.4), we get only trivial solutions.

2.2 Case $r \in (1, \mathbf{a})$: Find $\max_I E(W) - cE(|W - E(W)|^r)$.

In this case, the first order condition of problem (2.3) shows that the optimal allocation parameter is given by

$$\bar{I} = 1 - \left(\frac{m_z - z_0}{rc\mathbf{s}_z^r V(\mathbf{a}, \mathbf{b}, r)} \right)^{\frac{1}{r-1}} \text{ if } m_z > z_0 \text{ and } \bar{I} \in [0,1],$$

where

$$V(\mathbf{a}, \mathbf{b}, r) = \begin{cases} (H(\mathbf{a}, \mathbf{b}, r))^r & \text{in the stable case } (1 < \mathbf{a} < 2) \\ \frac{r}{2^2 \Gamma\left(\frac{r+1}{2}\right)} & \text{in the normal case } (\mathbf{a} = 2) \\ \sqrt{\mathbf{p}} & \end{cases}.$$

Otherwise, the optimal allocation is given by

$$\bar{I} = 1 \text{ if } T = U(z_0) \text{ or } \bar{I} = 0 \text{ if } T = U(z),$$

where $T = \max(U(z_0), U(z))$.

3. STABLE VERSUS NORMAL OPTIMAL ALLOCATION: A COMPARISON

In this section, we analyze the differences in optimal allocations when the investor chooses:

(i) Normal distribution,

or,

(ii) Stable (non-Gaussian) distribution, as a model for the asset returns in his/her portfolio.

We examine index-daily returns for the S&P 500, DAX30 and the CAC40 (see Table I reporting the same data used by Khindanova, Rachev and Schwartz (1999)). The riskless return is 6% p.a. Using the

estimated daily index parameters (see Table I) we can compute the coefficient

$$\bar{c} = \frac{m_z - z_0}{V(\mathbf{a}, \mathbf{b}, 1)\mathbf{s}_z} .$$

Thus every investor that maximizes the utility functional $E(W) - cE(|W - E(W)|)$,

1) invests all in the risky asset if $c < \bar{c}$,

2) invests all in the riskless asset when $c > \bar{c}$.

In Table II we report the threshold value: \bar{c}_1 for the normal case and \bar{c}_2 for the stable case. It follows that for the three indexes considered, all investors that maximize the utility functional

$$E(W) - cE(|W - E(W)|) \text{ with } \bar{c}_1 < c < \bar{c}_2$$

two situations can be considered:

- a) if the data fit the normal distribution, consequently the riskless asset is chosen;
- b) if the data fit the stable distribution, therefore the risky asset is chosen.

Instead, for the three indexes considered, all investors that maximize the utility functional

$$E(W) - cE(|W - E(W)|) \text{ with } c < \bar{c}_1 \text{ or } c > \bar{c}_2$$

two further situations can be examined:

- c) if $c < \bar{c}_1$ fit for the normal or the stable, the risky asset is chosen;
- d) if $c > \bar{c}_2$ fit for the normal or the stable, the riskless asset is chosen.

When $\mathbf{a} > r > 1$, we then obtain the following optimal allocation for the problem

$$\max_I E(W) - cE(|W - E(W)|^r) .$$

In table III, we listed the optimal allocation \bar{I} for the normal and the stable fit. Recall that \bar{I} is the optimal proportion of funds invested in the risk free asset. We have chosen $r = 1.5$ and $r = 1.35$ so that r is strictly less than all indexes of stability in the data set. On the other hand, we want r to be large, far away from 1, because for $r = 1$, we obtain the trivial allocation computed previously.

The analysis of table III shows that the optimal allocation in the normal and in the stable case is more sensitive to smaller risk aversion coefficient c . In particular, the optimal allocation can be up to 40%, (see S&P 500). Our results also show that in the stable non-Gaussian case, the riskless asset allocation is greater than the normal one (except for DAX30 when $r = 1.35$). This fact is indeed due to the fat tails of the stable distribution. Recall that tail behavior of every stable non Gaussian

distribution $X = S_a^d(\mathbf{s}, \mathbf{b}, m)$, with $1 < \mathbf{a} < 2$, is given by

$$\lim_{x \rightarrow +\infty} x^{\mathbf{a}} P(\pm X > x) = C_{\mathbf{a}} \frac{1 \pm \mathbf{b}}{2} \mathbf{s}^{\mathbf{a}},$$

where $C_{\mathbf{a}} = \frac{1 - \mathbf{a}}{\Gamma(2 - \mathbf{a}) \cos(\mathbf{p}\mathbf{a}/2)}$. In particular, $E(|X|^r) = \int_0^{+\infty} P(|X|^r > x) dx$, it follows that

$$E(|X - E(X)|^r) < \infty \text{ for } r < \mathbf{a},$$

$$\text{and } E(|X - E(X)|^r) = \infty \text{ for } r \geq \mathbf{a}.$$

Hence, the weight of the risk measure $E(|W - E(W)|^r)$ for $r \in [1, \mathbf{a})$, is generally greater for the investors who use stable laws for asset returns. This also implies that when investors fit normal distributions for return assets, they miss an important component of portfolio risk. On the contrary, the investor who fits stable distributions for return assets, implicitly tries to approximate the additional component of risk related to the heavy fat tailedness as returns distributions. We also observe another consequence of the above relation, (see for example the DAX30 index). When r is more distant from the stability parameter, we have to expect that the max difference in the allocation is lower (about 10% in DAX30) and is more influenced by the differences in the trivial allocation. This fact can easily be confirmed in all the above indexes considering the lower $r = 1.35$, in the allocation problem. In this sense, the stability index plays a strategic role in the optimal portfolio selection and for this reason, it becomes very significant as an accurate estimation of this parameter. Conversely, r in the above optimization problem can be an opportune measure of the power to be given to the component of risk due to the heavy-tailedness of asset returns. The importance given to r is intuitively linked to the conditions of the market in which the investor operates.

Hence, this empirical analysis shows that the component of risk due to heavy-tail distributions and the stability property can be extremely important in the choice of the optimal portfolio.

4. THE MULTIVARIATE EXTENSION

In this section we consider different multivariate estimable stable models in the study of the portfolio selection. In particular, we analyze the problem of optimal allocation $(\mathbf{I}, x) \in R_+^{n+1}$, $x \in R_+^n$,

$\mathbf{I} = 1 - x^I e$, among $n+1$ assets: n of those assets are stable distributed risky assets with returns $z = [z_1, \dots, z_n]^I$, and the $(n+1)$ th asset is risk-free. No short selling is allowed, (i.e. $\mathbf{I} \in [0,1]$, $x_i \geq 0$, this also implies that the $x_i \leq 1$). Therefore, when the investor wishes to maximize the utility functional $U(W) = E(W) - cE\left(|W - E(W)|^r\right)$, where c and r are positive real numbers, $W = \mathbf{I} z_0 + x^I z$, z_0 is the risk free asset return, and $x^I z$ is the risky portfolio asset return, we study and analyze the possible optimal allocations for the following stable models of asset returns.

Let us assume that the vector $z = [z_1, \dots, z_n]^I$ is sub-Gaussian \mathbf{a} -stable distributed ($1 < \mathbf{a} < 2$), whose characteristic function has the following form

$$(4.1) \quad \Phi_z(t) = E(\exp(it^I z)) = \exp\left(-\left(t^I Q t\right)^{\mathbf{a}/2} + it^I \mathbf{m}\right) = \exp\left(-\int_{S_n} |t^I s|^{\mathbf{a}} \mathbf{g}(ds) + it^I \mathbf{m}\right),$$

where $Q = \left[\frac{R_{ij}}{2}\right]$ is a positive definite $(n \times n)$ -matrix, \mathbf{m} is the mean vector, and $\mathbf{g}(ds)$ is the spectral measure with support concentrated on $S_n = \{s \in R^n : \|s\| = 1\}$. The term R_{ij} is defined by

$$\frac{R_{ij}}{2} = [\tilde{z}_i, \tilde{z}_j]_{\mathbf{a}} \|\tilde{z}_j\|_{\mathbf{a}}^{2-\mathbf{a}},$$

where the covariation $[\tilde{z}_i, \tilde{z}_j]_{\mathbf{a}}$ between two jointly symmetric \mathbf{a} stable random variables $\tilde{z}_i := z_i - \mathbf{m}_i$ and \tilde{z}_j is given by

$$[\tilde{z}_i, \tilde{z}_j]_{\mathbf{a}} = \int_{S_2} s_i |s_j|^{\mathbf{a}-1} \text{sgn}(s_j) \mathbf{g}(ds),$$

in particular, $\|\tilde{z}_i\|_{\mathbf{a}} = \left(\int_{S_2} |s_i|^{\mathbf{a}} \mathbf{g}(ds)\right)^{\frac{1}{\mathbf{a}}} = ([\tilde{z}_i, \tilde{z}_i]_{\mathbf{a}})^{\frac{1}{\mathbf{a}}}$. Here the spectral measure $\mathbf{g}(ds)$ has support on the unit circle S_2 .

This model can be considered as a special case of the Owen-Rabinovitch's elliptical model (see Owen and Rabinovitch (1984)). However, no estimation procedure of the model parameters is given in the elliptical models with non-finite variance. In our approach we use (4.1) to provide the statistical estimator of the stable efficient frontier. To estimate the efficient frontier for returns given by (4.1) we need to consider one estimator for the mean vector \mathbf{m} and one estimator for the dispersion matrix Q .

The estimator of \mathbf{m} is given by the vector $\hat{\mathbf{m}}$ of sample averages. Using lemma 2.7.16 in Samorodnitsky and Taqu (1994) we can write for every p such that $1 < p < \mathbf{a}$

$$(4.2) \quad \frac{[\tilde{z}_i, \tilde{z}_j]_{\mathbf{a}}}{\|\tilde{z}_j\|_{\mathbf{a}}^{\mathbf{a}}} = \frac{E(\tilde{z}_i \tilde{z}_j^{<p-1>})}{E(|\tilde{z}_j|^p)}$$

where the scale parameter \mathbf{s}_j can be written $\mathbf{s}_j = \|\tilde{z}_j\|_{\mathbf{a}}$. Then \mathbf{s}_j can be valued with the moment method suggested by Samorodnitsky and Taqu (1994) (property 1.2.17) in the case $\mathbf{b} = 0$

$$\mathbf{s}_j^p = \|\tilde{z}_j\|_{\mathbf{a}}^p = \frac{E(|z_j - \mathbf{m}_j|^p) p \int_0^{\infty} u^{-p-1} \sin^2 u du}{2^{p-1} \Gamma(1 - p/\mathbf{a})}$$

It follows from (4.2)

$$\frac{R_{ij}}{2} = \mathbf{s}_j^2 \frac{E(\tilde{z}_i \tilde{z}_j^{<p-1>})}{E(|\tilde{z}_j|^p)}$$

The above suggests the following estimator $\hat{Q} = \left[\frac{\hat{R}_{ij}}{2} \right]$ for the entries of the unknown covariation matrix Q

$$\frac{\hat{R}_{ij}}{2} = \hat{\mathbf{s}}_j^2 \frac{\sum_{k=1}^N \tilde{z}_i^{(k)} |\tilde{z}_j^{(k)}|^{<p-1>}}{\sum_{k=1}^N |\tilde{z}_j^{(k)}|^p},$$

where the \mathbf{s}_j^2 is estimated as follows

$$\hat{\mathbf{s}}_j^2 = \frac{\hat{R}_{jj}}{2} = \left(\frac{\frac{1}{N} \sum_{k=1}^N |\tilde{z}_j^{(k)}|^p p \int_0^{\infty} u^{-p-1} \sin^2 u du}{2^{p-1} \Gamma(1 - p/\mathbf{a})} \right)^{2/p}$$

The moments estimator is meaningful mostly for each fixed p . The rate of convergence of the empirical matrix $\hat{Q} = \left[\frac{\hat{R}_{ij}}{2} \right]$ to the unknown (to-be-estimated) matrix Q , will be faster if p is as large as possible, see Rachev (1991).

See footnotes 1 and 2.

Consider two \mathbf{a} stable distributed random variables W_1 , and W_2 with $\mathbf{a} > 1$, equal skewness parameter $\mathbf{b} = \mathbf{b}_{W_1} = \mathbf{b}_{W_2}$, and equal mean $E(W_1) = E(W_2) = m$. However, suppose that $\mathbf{s}_{W_1} > \mathbf{s}_{W_2}$. Then, $W_1 \stackrel{d}{=} \mathbf{s}_{W_1} X + m$ and $W_2 \stackrel{d}{=} \mathbf{s}_{W_2} X + m$, where $X = S_{\mathbf{a}}(1, \mathbf{b}, 0)$, and thus

$$G(u) = \int_{-\infty}^u (P(W_2 \leq t) - P(W_1 \leq t)) dt = \int_{-\infty}^u (P(X \leq \frac{t-m}{\mathbf{s}_{W_2}}) - P(X \leq \frac{t-m}{\mathbf{s}_{W_1}})) dt.$$

Then, $G(u)$ is the negative decreasing function for every $u \leq m$, and $f(u) = G(u) - G(m)$ is the positive increasing function for every $u > m$. As W_1 , and W_2 are integrable random variables

($\mathbf{a} > 1$), integrating by part $G(u)$, we get $G(+\infty) = \lim_{u \rightarrow \infty} u(F_{W_2}(u) - F_{W_1}(u)) - \int_{-\infty}^u t d(F_{W_2}(t) - F_{W_1}(t)) = 0$.

Then, for every $u > m$, $f(u)$ cannot be greater than $-G(m)$ and $G(u) \leq 0$ for every real u . That is:

$$(4.3) \quad \int_{-\infty}^u (P(W_2 \leq t) - P(W_1 \leq t)) dt \leq 0 \quad \forall u \in R.$$

Hence, W_2 dominates W_1 in the sense of R-S.

In view of what stated before, when the returns $z = [z_1, \dots, z_n]^T$ are jointly \mathbf{a} stable distributed, (non necessarily sub-Gaussian), we obtain the efficient frontier of the risk averse investors as a solution of the following optimization problem:

$$(4.4) \quad \begin{cases} \min_x \mathbf{s}_W \\ 0 \leq x^T e \leq 1, \text{ and } x_i \geq 0 \quad i = 1, \dots, n, \\ x^T \mathbf{m} + (1 - x^T e) z_0 = m_W, \\ \mathbf{b}_W = \mathbf{b}^*, \end{cases}$$

where $W = (1 - x^T e) z_0 + x^T z$. Hence, every risk averse investor will choose an optimal portfolio among all portfolios described by the above three parameters efficient frontier; the three parameters are the mean m_W , the scale parameter \mathbf{s}_W , and the skewness parameter \mathbf{b}_W .

Now, in view of our optimization problem, let us recall that in the sub-Gaussian case every portfolio satisfies the relation

$$x^T z \stackrel{d}{=} S_{\mathbf{a}}(\mathbf{s}_{x^T z}, \mathbf{b}_{x^T z}, m_{x^T z})$$

and furthermore, $W = z_0$ when $x = 0$, otherwise, $W = (1 - x^T e) z_0 + x^T z \stackrel{d}{=} S_{\mathbf{a}}(\mathbf{s}_{x^T z}, \mathbf{b}_{x^T z}, (1 - x^T e) z_0 + m_{x^T z})$,

where \mathbf{a} is the index of stability, $\mathbf{s}_{x^T z} = \sqrt{x^T Q x}$ is the scale (dispersion) parameter, $\mathbf{b}_{x^T z} = 0$ is the

skewness parameter, and $m_{x'z} = x' \mathbf{m}$ is the mean of $x'z$.

Furthermore, in the sub-Gaussian case, all solutions to the optimization problem

$$\max_x E(W) - cE\left(|W - E(W)|^r\right)$$

for some $r \in [1, \mathbf{a}]$, belong to the mean-dispersion frontier

$$(4.5) \quad \mathbf{s} = \begin{cases} \frac{m - z_0}{\sqrt{A - 2Bz_0 + Cz_0^2}} & \text{if } m \geq z_0 \\ -\frac{m - z_0}{\sqrt{A - 2Bz_0 + Cz_0^2}} & \text{if } m < z_0 \end{cases},$$

where $\mathbf{m} = E(z)$; $m = x' \mathbf{m} + \mathbf{1}' z_0$; $\mathbf{1} = 1 - x' e$ is the proportion of funds invested in the riskless asset;

$e = [1, \dots, 1]'$; $A = \mathbf{m}' Q^{-1} \mathbf{m}$; $B = e' Q^{-1} \mathbf{m}$; $C = e' Q^{-1} e$, and $\mathbf{s}^2 = x' Q x$. Besides, the optimal portfolio weights x takes the following form:

$$(4.6) \quad x = Q^{-1} (\mathbf{m} - z_0 e) \frac{m - z_0}{A - 2Bz_0 + Cz_0^2}.$$

Note that (4.5) and (4.6) have the same forms of mean-variance efficient frontier. In particular, (4.6) exhibits the two-fund separation property for both the stable and the normal case, but the matrix Q and the parameter \mathbf{s} have different meaning. In the normal case, Q is the variance-covariance matrix and \mathbf{s} is the standard deviation, while in the stable case Q is a dispersion matrix (see (4.1)) and \mathbf{s} is the scale (dispersion) parameter, $\mathbf{s} = \sqrt{x' Q x}$. From the two-fund separation property of the sub-Gaussian \mathbf{a} -stable approach, we can assume the market portfolio equal to the risky tangent portfolio under the equilibrium conditions (as in the classic-variance Capital Asset Pricing Model (CAPM)). Therefore, every optimal portfolio can be seen as the linear combination between the market portfolio

$$(4.7) \quad \bar{x}' z = \frac{(\mathbf{m} - z_0 e)' Q^{-1} z}{B - Cz_0},$$

and the riskless asset return z_0 . Following the same arguments as in Sharpe-Lintner-Mossin's mean-variance equilibrium model, the return of asset i is given by:

$$(4.8) \quad E(z_i) = z_0 + \mathbf{b}_{i,m} (E(\bar{x}' z) - z_0),$$

where $\mathbf{b}_{i,m} = \frac{\bar{x}' Q e^i}{\bar{x}' Q \bar{x}}$, with e^i , the vector with 1 in the i -th component and zero in all the other

components. As a consequence of Ross' necessary and sufficient conditions of two-fund separation

(see Ross (1978)), the above model admits the form

$$z_i = \mathbf{m}_i + b_i Y + \mathbf{e}_i, \quad i = 1, \dots, n,$$

where $E(\mathbf{e} / Y) = 0$, and the vector $bY + \mathbf{e}$ is sub-Gaussian \mathbf{a} -stable distributed with zero mean.

Hence, our sub-Gaussian \mathbf{a} -stable version of CAPM, is not much different from the Gamrowski-Rachev (1999) version of the two-fund separation \mathbf{a} -stable model. As a matter of fact, Gamrowski and Rachev (1999) propose a generalization of Fama's \mathbf{a} -stable model assuming $z_i = \mathbf{m}_i + b_i Y + \mathbf{e}_i$, for $i = 1, \dots, n$, where \mathbf{e}_i and Y are \mathbf{a} -stable distributed and $E(\mathbf{e} / Y) = 0$. By their assumptions,

$$E(z_i) = z_0 + \tilde{\mathbf{b}}_{i,m} (E(\bar{x}'z) - z_0)$$

$$\text{where } \tilde{\mathbf{b}}_{i,m} = \frac{1}{\mathbf{a}[\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}}} \frac{\partial [\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}}}{\partial \bar{x}_i} = \frac{[\tilde{z}_i, \bar{x}'\tilde{z}]_{\mathbf{a}}}{[\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}}}.$$

Furthermore, the coefficient $\frac{[\tilde{z}_i, \bar{x}'\tilde{z}]_{\mathbf{a}}}{[\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}}}$ can be estimated as shown in (4.2).

Now, we see that in the above sub-Gaussian symmetric \mathbf{a} stable model $\bar{x}'Q\bar{x} = ([\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}})^{2/a}$ and

$\bar{x}'Qe^i = \frac{1}{2} \frac{\partial ([\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}})^{2/a}}{\partial \bar{x}_i}$, thus, we get the equivalence between the coefficient $\mathbf{b}_{i,m}$ of model

(4.8) and $\tilde{\mathbf{b}}_{i,m}$ of Gamrowski-Rachev's model i.e.:

$$\mathbf{b}_{i,m} = \frac{\bar{x}'Qe^i}{\bar{x}'Q\bar{x}} = \frac{1}{\mathbf{s}_{\bar{x}'z}} \frac{\partial \mathbf{s}_{\bar{x}'z}}{\partial \bar{x}_i} = \frac{[\tilde{z}_i, \bar{x}'\tilde{z}]_{\mathbf{a}}}{[\bar{x}'\tilde{z}, \bar{x}'\tilde{z}]_{\mathbf{a}}} = \tilde{\mathbf{b}}_{i,m},$$

where $\mathbf{s}_{\bar{x}'z}$ is the scale parameter of market portfolio.

In our optimal allocation problem, we suppose that $\bar{x}_i \geq 0$ because we assume that no short sale is allowed. If $\bar{x}_i < 0$ for some index i , we can exclude that asset by our allocation problem. Hence, the optimal solution of the problem in the important case $r \in (1, \mathbf{a})$, (recall that the case $r=1$ gives trivial solution: either the riskless asset or the market portfolio), is given by

$$\bar{I} = 1 - \left(\frac{m_{\bar{x}'z} - z_0}{rc \mathbf{s}_{\bar{x}'z}^r V(\mathbf{a}, 0, r)} \right)^{\frac{1}{r-1}} \text{ if } m_{\bar{x}'z} > z_0 \text{ and } \bar{I} \in (0, 1),$$

$$\text{and } x = (1 - \bar{I})\bar{x},$$

where \bar{x} is given by (4.7) and

$$V(\mathbf{a}, 0, r) = \begin{cases} (H(\mathbf{a}, 0, r))^r & \text{in the stable case } (1 < \mathbf{a} < 2) \\ \frac{2^{\frac{r}{2}} \Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\mathbf{p}}} & \text{in the normal case } (\mathbf{a} = 2) \end{cases}.$$

Furthermore, the optimal risky portfolio $\bar{x}^l z = \frac{(\mathbf{m} - z_0 e)^l Q^{-1} z}{B - Cz_0}$ has mean $m_{\bar{x}^l z} = \frac{A - Bz_0}{B - Cz_0}$ and scale

$$\text{parameter } \mathbf{s}_{\bar{x}^l z} = \frac{(A - 2Bz_0 + Cz_0^2)^{1/2}}{B - Cz_0}.$$

Again, one should expect that the optimal allocation would be different because the constant $V(\mathbf{a}, 0, r)$ and the matrix Q are different in the stable non-Gaussian and in the normal case.

Next, suppose that the returns $z = [z_1, \dots, z_n]^l$ are jointly \mathbf{a} -stable distributed with $1 < \mathbf{a} < 2$, i.e., their joint characteristic function is given by

$$\Phi_z(t) = \exp\left(-\int_{S_n} |t^l s|^{\mathbf{a}} (1 - i \operatorname{sgn}(t^l s) \tan(\mathbf{p}\mathbf{a}/2)) \mathbf{g}(ds) + it^l \mathbf{m}\right),$$

where \mathbf{a} is the index of stability, $\mathbf{g}(ds)$ is the spectral measure concentrated on S_n .

Under this assumption, every portfolio $x^l z$ is distributed as

$$x^l z = S_{\mathbf{a}}^d(\mathbf{s}_{x^l z}, \mathbf{b}_{x^l z}, m_{x^l z})$$

where, $\mathbf{s}_{x^l z} = \left(\int_{S_n} |x^l s|^{\mathbf{a}} \mathbf{g}(ds)\right)^{1/\mathbf{a}}$ is the scale parameter, $\mathbf{b}_{x^l z} = \frac{\int_{S_n} |x^l s|^{\mathbf{a}} \mathbf{g}(ds)}{\mathbf{s}_{x^l z}^{\mathbf{a}}}$ is the skewness

parameter, and $m_{x^l z} = x^l \mathbf{m}$ is the mean of $x^l z$. In this case, we are not able to find a closed form to the solution of the problem

$$(4.9) \quad \max_x E(W) - cE(|W - E(W)|^r) \quad \text{for } r \in [1, \mathbf{a}),$$

where $W = (1 - x^l e) z_0 + x^l z$. We can only state that the solution of (4.9) is one of the solutions of the optimization problem (4.4) varying the admissible m_W and \mathbf{b}_W (4.1). In order to find an analytical version of a three parameter stable model, we can consider the following three-fund separation model of security returns:

$$(4.10) \quad z_i = \mathbf{m}_i + b_i Y + \mathbf{e}_i, \quad i = 1, \dots, n,$$

where the random vector $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)^T$ is independent from Y and follows a joint sub-Gaussian \mathbf{a}_1 -stable distribution ($1 < \mathbf{a}_1 < 2$), with zero mean and characteristic function

$$\Phi_{\mathbf{e}}(t) = \exp\left(-|t^T Q t|^{\mathbf{a}_1/2}\right),$$

where Q is the definite positive dispersion matrix. Besides, $Y = S_{\mathbf{a}_2}^d(\mathbf{s}_Y, \mathbf{b}_Y, 0)$ is \mathbf{a}_2 -stable distributed random variable ($1 < \mathbf{a}_2 < 2$) with zero mean. A testable model, in which Y is \mathbf{a}_2 -stable symmetric distributed (i.e. $\mathbf{b}_Y = 0$), was recently studied by Rachev and Schwartz (1999). Moreover, when Y is non symmetric (i.e. $\mathbf{b}_Y \neq 0$), the model is a stable version of Simaan's one (1993). Simaan assumes that conditional on Y , the random vector \mathbf{e} follows a joint-elliptical distribution with characteristic function $\Phi_{\mathbf{e}/Y}(t) = g(t^T Q t)$, but in his considerations, he implicitly assumes the dispersion matrix Q independent of the random variable Y . Hence, Shiman's results are true if and only if $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)^T$ is independent by Y , as in our case. When $\mathbf{b}_Y = 0$ and $\mathbf{a}_1 = \mathbf{a}_2$, our model leads to the two-fund separation Fama's model. Thus, the model (4.10) distinguishes itself from the Götzenberger, Rachev and Schwartz's model, the Simaan's model, and Fama's because it incorporates asymmetric heavy-tailed distributional asset returns. Moreover under the assumptions of (4.10) all portfolios are in the domain of attraction of an $(\mathbf{a}_1, \mathbf{a}_2)$ -stable law.

The characteristic function of the vector of returns $z = [z_1, \dots, z_n]^T$ is given by:

$$(4.11) \quad \Phi_z(t) = \Phi_{\mathbf{e}}(t) \Phi_Y(t^T b) e^{it^T \mathbf{m}} = \exp\left(-|t^T Q t|^{\mathbf{a}_1/2} - |t^T b \mathbf{s}_Y|^{\mathbf{a}_2} (1 - i \mathbf{b}_Y \operatorname{sgn}(t^T b) \tan(\mathbf{p}\mathbf{a} / 2)) + it^T \mathbf{m}\right)$$

where $b = [b_1, \dots, b_n]^T$ is the coefficient vector and $\mathbf{m} = [\mathbf{m}_1, \dots, \mathbf{m}_n]^T$ is the mean vector.

Next we shall estimate the parameter in model (4.10), (4.11). First, the estimator of \mathbf{m} is given by the vector $\hat{\mathbf{m}}$ of sample averages. Then we consider as factor Y a centralized index return. Therefore, given the sequence of observations $Y^{(k)}$, we estimate its stable parameters. Observe that the random vector \mathbf{e} admits a representation as a product of two independent random variables: $\mathbf{e} = VG$. G is a Gaussian vector with null mean and variance covariance matrix Q , and $V = \sqrt{A}$, where A is an $\mathbf{a}_1/2$ -

stable subordinator, that is $A = S_{\mathbf{a}_1/2}\left(\left(\cos\left(\frac{\mathbf{p}\mathbf{a}_1}{4}\right)\right)^{2/\mathbf{a}_1}, 1, 0\right)$. Then we can generate values A_k $k=1, \dots, N$

of A independent of G . We address to Paulauskas and Rachev (1999) the problem of generating such values A_k . Regressing the centralizing returns $\tilde{z}_i = z_i - \hat{\mathbf{m}}_i$ on Y we obtain the following ML estimators for $b = [b_1, \dots, b_n]^T$ and Q :

$$\hat{b}_i = \frac{\sum_{k=1}^N \frac{1}{A_k} (Y^{(k)} \tilde{z}_i^{(k)})}{\sum_{k=1}^N \frac{1}{A_k} (Y^{(k)})^2} \quad \text{and} \quad \hat{Q} = \frac{1}{N} \sum_{k=1}^N \frac{1}{A_k} (\tilde{z}^{(k)} - \hat{b} Y^{(k)}) (\tilde{z}^{(k)} - \hat{b} Y^{(k)})^T.$$

The selection of \mathbf{a}_1 is a separate problem. A possible way to estimate \mathbf{a}_1 is to consider the *OLS*

estimator $\tilde{b}_i = \frac{\sum_{k=1}^N Y^{(k)} \tilde{z}_i^{(k)}}{\sum_{k=1}^N (Y^{(k)})^2}$ and then to evaluate the sample residuals $\tilde{\mathbf{e}}^{(k)} = \tilde{z}^{(k)} - \tilde{b} Y^{(k)}$. If these

residuals are heavy tailed, one can take the tail exponent as an estimator for \mathbf{a}_1 . The asymptotic properties of the above estimators can be derived arguing in a similar way to Paulauskas and Rachev (1999) and Götzenberger, Rachev and Schwartz (1999).

From (4.11) we see that when $\mathbf{a} := \mathbf{a}_1 = \mathbf{a}_2$, every portfolio $x^T z$ is an \mathbf{a} stable distribution and has the distributional form

$$x^T z = S_{\mathbf{a}}^d(\mathbf{s}_{x^T z}, \mathbf{b}_{x^T z}, m_{x^T z})$$

and $W = z_0$ when $x = 0$ otherwise $W = (1 - x^T e) z_0 + x^T z = S_{\mathbf{a}}^d(\mathbf{s}_{x^T z}, \mathbf{b}_{x^T z}, (1 - x^T e) z_0 + m_{x^T z})$, where

$$(4.12) \quad \mathbf{s}_{x^T z}^{\mathbf{a}} = (x^T Q x)^{a/2} + (|x^T b| \mathbf{s}_Y)^{\mathbf{a}}, \quad \mathbf{b}_{x^T z} = \frac{|x^T b \mathbf{s}_Y|^{\mathbf{a}} \operatorname{sgn}(x^T b) \mathbf{b}_Y}{\mathbf{s}_{x^T z}^{\mathbf{a}}}, \quad \text{and} \quad m_{x^T z} = x^T \mathbf{m}.$$

Consider two portfolios $x^T z$ and $y^T z$ with the same mean $x^T \mathbf{m} = y^T \mathbf{m}$, the same parameter $x^T b = y^T b$

and such that $x^T Q x > y^T Q y$. Then, $X/Y = \frac{x^T e}{\sqrt{x^T Q x}} = \frac{y^T e}{\sqrt{y^T Q y}} = S_{\mathbf{a}_1}(1, 0, 0)$ and thus, for every real u :

$$(4.13) \quad \begin{aligned} & \int_{-\infty}^u (P(y^T z \leq s) - P(x^T z \leq s)) ds = \\ & = \int_{-\infty}^u \int_{-\infty}^R (P(X \leq \frac{s - y^T \mathbf{m} - y^T b t}{\sqrt{y^T Q y}} / Y = t) - P(X \leq \frac{s - x^T \mathbf{m} - x^T b t}{\sqrt{x^T Q x}} / Y = t)) f_Y(t) dt ds = , \\ & = \int_{-\infty}^u \int_{R-\infty}^R (P(X \leq \frac{s - y^T \mathbf{m} - y^T b t}{\sqrt{y^T Q y}} / Y = t) - P(X \leq \frac{s - x^T \mathbf{m} - x^T b t}{\sqrt{x^T Q x}} / Y = t)) ds f_Y(t) dt \leq 0 \end{aligned}$$

where f_Y is the density of Y . It follows that $y^l z$ dominates $x^l z$ in the sense of R-S. In addition, when unlimited short selling is allowed, the efficient frontier of non-dominated portfolios is given by the solution of the following quadratic programming problem:

$$\left\{ \begin{array}{l} \min_x \frac{1}{2} x^l Q x \\ x^l \mathbf{m} + (1 - x^l e) z_0 = m_W . \\ x^l b = b^*, \end{array} \right.$$

We know that generally the efficient set does not present an analytical form. In particular, Dybwig [1985], Markowitz [1959], and Bawa [1976-1978] show that the mean variance efficient set for the risk averse investors and for the non-satiable investors with restrictions on short sales, consists of segments which are parabolic or horizontal line segments. In addition, kinks in the efficient sets are the rule rather than the exception. We cannot expect the multi-parameter efficient set to take a simpler form. Under our assumptions, we obtain the efficient frontier for the risk-averse investors by solving the above optimization problem that gives the following optimal portfolios

$$(4.14) \quad (1 - I_2 - I_3) z_0 + I_2 \frac{z^l Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)} + I_3 \frac{z^l Q^{-1} b}{e^l Q^{-1} b},$$

that is a linear combination of the riskless portfolio z_0 , and the risky portfolios

$$Z^{(1)} = \frac{z^l Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)} \text{ and } Z^{(2)} = \frac{z^l Q^{-1} b}{e^l Q^{-1} b}.$$

In particular, if we assume short sale restrictions in the market, then

$$1 \geq 1 - I_2 - I_3 \geq 0, \text{ and } 1 \geq x_i (I_2, I_3) \geq 0 \quad i=1, \dots, n$$

$$\text{where } x(I_2, I_3) = I_2 \frac{Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)} + I_3 \frac{Q^{-1} b}{e^l Q^{-1} b}.$$

Similar to the Simaan's model, with the three-fund separation model, we get a pricing linear relation for asset returns:

$$\mathbf{m}_i = z_0 + \tilde{b}_{i,2} \mathbf{d}_2 + \tilde{b}_{i,3} \mathbf{d}_k, \quad i = 1, 2, \dots, n,$$

where \mathbf{d}_p for $p=1,2$ are the risk premiums relative to a market factor and a skewness factor.

Note that the efficient portfolio of Simaan's three-moment model admits a similar form (4.14) of the above stable model (4.10). Whereas, the matrix Q and the parameters $b = [b_1, \dots, b_n]^l$ have different

meaning. In the Simaan's model, Q is the variance covariance matrix of z , and the parameters b_i are estimated considering the third central moment

$$b = \left(\frac{1}{E(Y^3)N} \sum_{k=1}^N (\tilde{z}^{(k)})^3 \right)^{1/3},$$

where $\tilde{z}^{(i)}$ for $i=1, \dots, N$ are centralized observations of the vector z and the random Y is a fixed non symmetric random variable.

If we try to solve analytically the optimization problem (4.9), in the stable case with $\mathbf{a} := \mathbf{a}_1 = \mathbf{a}_2$, we have to solve the following two-parameter optimal allocation problem

$$\max_{I_2, I_3} E(W) - cE(|W - E(W)|^r) = \max_{I_2, I_3} (1 - I_2 - I_3)z_0 + m_{x^I(I_2, I_3)z} - c \left(H(\mathbf{a}, \mathbf{b}_{x^I(I_2, I_3)z}, r) \right) \mathbf{s}_{x^I(I_2, I_3)z}^r$$

where,

$$\mathbf{s}_{x^I(I_2, I_3)z} = \left((x^I(I_2, I_3)Qx^I(I_2, I_3))^{a/2} + \left(|x^I(I_2, I_3)b| \mathbf{s}_Y \right)^a \right)^{1/a}, \quad m_{x^I(I_2, I_3)z} = x^I(I_2, I_3)\mathbf{m}$$

$$\text{and } \mathbf{b}_{x^I(I_2, I_3)z} = \frac{|x^I(I_2, I_3)b\mathbf{s}_Y|^a \operatorname{sgn}(x^I(I_2, I_3)b)\mathbf{b}_Y}{\mathbf{s}_{x^I(I_2, I_3)z}^a}$$

are respectively the scale parameter, the mean and the skewness parameter, of the optimal risky portfolio $x^I(I_2, I_3)z$. In order to solve the allocation problem (4.9) in the more general case when \mathbf{a}_1 is not necessarily equal to \mathbf{a}_2 , we can approximate the optimal allocation solving the following optimization problem

$$\max_{I_2, I_3} E(W) - cE(|W - E(W)|^r) = \max_{I_2, I_3} (1 - I_2 - I_3)z_0 + m_{x^I(I_2, I_3)z} - c \frac{1}{N} \sum_{k=1}^N |x^I(I_2, I_3)\tilde{z}^{(k)}|^r,$$

where $\tilde{z}^{(i)}$ for $i=1, \dots, N$ are centralized observations of the vector z . This is the typical way to approximate the solution when the efficient frontier admits the form

$$x = f(y) \text{ where } y \in B \subseteq R^k.$$

Then, according to the Law of Large Numbers, we know that an investor with utility function u can find an approximating solution to his portfolio selection problem in the portfolio weights

$$x^* = f(y^*),$$

where $y^* = \arg(\max_{y \in B} \frac{1}{s} \sum_{i=1}^s u(f(y)'z^{(i)})$ and $z^{(i)}$ $i=1, \dots, s$ are independent observations of the vector of the gross returns.

We draw our conclusion on the three- fund separation model, underlining that in the three-moment model, the solution of the allocation problem (4.9) depends on the choice of the non-symmetric random variable Y . Clearly, one should expect that the optimal allocation will be different assuming that asset returns are in the domain of attraction of an $(\mathbf{a}_1, \mathbf{a}_2)$ -stable law, or depending on the three moments.

As empirical studies show, one of the most severe restrictions in performance measurement and asset pricing in the stable case is the assumption of a common index of stability for all assets – individual securities and portfolio alike. It is well recognized that asset returns are not normally distributed. We also know the return distributions do not have the same index of stability. Recent studies have shown how to define the efficient frontier for risk averse investors who wish to allocate their initial wealth between n investments whose returns are in the domain of attraction of $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ -stable law, see Ortobelli and Rachev (1999). In these cases, it is not generally possible to find a closed form to the efficient frontier. Instead, generalizing the above model, we get the following $k+1$ fund separation model, (for details on k fund separation models see Ross (1978)):

$$(4.15) \quad z_i = \mathbf{m}_i + b_{i,2}Y_2 + \dots + b_{i,k}Y_k + \mathbf{e}_i, \quad i = 1, \dots, n.$$

Here $n \geq k \geq 2$, the vector $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)'$ is independent from Y_2, \dots, Y_k and follows a joint sub-Gaussian \mathbf{a}_1 -stable distribution with $1 < \mathbf{a}_1 < 2$, zero mean, and characteristic function $\Phi_{\mathbf{e}}(t) = \exp\left(-|t'Qt|^{\mathbf{a}_1/2}\right)$, and the random variables $Y_j \stackrel{d}{=} S_{\mathbf{a}_j}(\mathbf{s}_{Y_j}, \mathbf{b}_{Y_j}, 0)$, $j=2, \dots, k$ are \mathbf{a}_j -stable distributed with $1 < \mathbf{a}_j < 2$ and zero mean. If we want to insure the separation obtains in situations where the above model degenerates into a p -fund separation model with $p < k+1$, then we need the rank condition (see Ross (1978)).

Using similar arguments of (4.3) and (4.13), we can prove that $y^l z$ dominates $x^l z$ in the sense of R-S for every couple of portfolios $x^l z$ and $y^l z$ with the same mean $x^l \mathbf{m} = y^l \mathbf{m}$, and the same parameters $y^l b_{\bullet, j} = x^l b_{\bullet, j} = c_j$, $j = 2, \dots, k$, where $b_{\bullet, j} = [b_{1,j}, \dots, b_{n,j}]'$, and such that $x^l Qx > y^l Qy$. Hence, when unlimited short selling is allowed, the efficient frontier of non dominated portfolios is given by the solution of the following quadratic programming problem:

$$(4.16) \quad \begin{cases} \min_x \frac{1}{2} x' Q x \\ x' \mathbf{m} + (1 - x' \mathbf{e}) z_0 = m_W \\ x' \mathbf{b}_{\bullet, j} = c_j \quad j=2, \dots, k. \end{cases}$$

By solving the optimization problem (4.16), we obtain that the riskless portfolio and other k risky portfolios span the efficient frontier for the risk averse investors. It follows that portfolio efficient frontier is given by

$$(4.17) \quad (1 - \sum_{i=2}^{k+1} \mathbf{I}_i) z_0 + \mathbf{I}_2 \frac{z' Q^{-1} (\mathbf{m} - z_0 \mathbf{e})}{e' Q^{-1} (\mathbf{m} - z_0 \mathbf{e})} + \sum_{i=2}^k \mathbf{I}_{i+1} \frac{z' Q^{-1} \mathbf{b}_{\bullet, i}}{e' Q^{-1} \mathbf{b}_{\bullet, i}}.$$

In order to estimate the parameters, we can still use the above maximum likelihood method, but we need to know the joint distribution of the vector (Y_2, \dots, Y_k) . For example, we can simply assume independent random variables $Y_j \quad j=2, \dots, k$, then the characteristic function of the vector of returns

$$z = [z_1, \dots, z_n]' \text{ is given by } \Phi_z(t) = \Phi_e(t) \prod_{j=2}^k \Phi_{Y_j}(t' \mathbf{b}_{\bullet, j}) e^{it' \mathbf{m}}.$$

Given that $\mathbf{a} := \mathbf{a}_1 = \mathbf{a}_2 = \dots = \mathbf{a}_k$ and the random variables $Y_j \quad j=2, \dots, k$, are independent, the following k -parameter optimal allocation problem correspondent to (4.9) has to be solved:

$$\max_{\mathbf{I}_2, \mathbf{I}_3, \dots, \mathbf{I}_{k+1}} (1 - \mathbf{I}_2 - \mathbf{I}_3 - \dots - \mathbf{I}_{k+1}) z_0 + m_{x'(I_2, I_3, \dots, I_{k+1})z} - c(H(\mathbf{a}, \mathbf{b}_{x'(I_2, I_3, \dots, I_{k+1})z}, r))^r \mathbf{s}_{x'(I_2, I_3, \dots, I_{k+1})z}^r,$$

where,

$$x(\mathbf{I}_2, \dots, \mathbf{I}_{k+1}) = \mathbf{I}_2 \frac{Q^{-1} (\mathbf{m} - z_0 \mathbf{e})}{e' Q^{-1} (\mathbf{m} - z_0 \mathbf{e})} + \sum_{i=2}^k \mathbf{I}_{i+1} \frac{Q^{-1} \mathbf{b}_{\bullet, i}}{e' Q^{-1} \mathbf{b}_{\bullet, i}}$$

is the optimal risky portfolio weight; and

$$\mathbf{b}_{x'(I_2, I_3, \dots, I_{k+1})z} = \frac{\sum_{j=2}^k |x'(I_2, I_3, \dots, I_{k+1}) \mathbf{b}_{\bullet, j} \mathbf{s}_{Y_j}|^a \text{sign}(x'(I_2, I_3, \dots, I_{k+1}) \mathbf{b}_{\bullet, j}) \mathbf{b}_{Y_j}}{\mathbf{s}_{x'(I_2, I_3, \dots, I_{k+1})z}^a},$$

$$\mathbf{s}_{x'(I_2, I_3, \dots, I_{k+1})z} = \left((x'(I_2, I_3, \dots, I_{k+1}) Q x'(I_2, I_3, \dots, I_{k+1}))^{a/2} + \sum_{j=2}^k (|x'(I_2, I_3, \dots, I_{k+1}) \mathbf{b}_{\bullet, j} \mathbf{s}_{Y_j}|^a) \right)^{1/a},$$

and

$$m_{x'(I_2, I_3, \dots, I_{k+1})z} = x'(I_2, I_3, \dots, I_{k+1}) \mathbf{m}$$

are respectively the skewness parameter, the scale parameter, and the mean of the optimal risky portfolio $x'(I_2, I_3, \dots, I_{k+1})z$.

More generally, when the stability indexes $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_k$ are not necessarily equal, and the random variables Y_j $j=2, \dots, k$, are not necessarily independent, we can approximate the optimal allocation of the problem (4.9) solving the following optimization problem:

$$\max_{I_2, I_3, \dots, I_{k+1}} (1 - I_2 - I_3, \dots, I_{k+1}) z_0 + m_{x^I(I_2, I_3, \dots, I_{k+1})z} - c \frac{1}{N} \sum_{k=1}^N |x^I(I_2, I_3, \dots, I_{k+1}) \tilde{z}^{(k)}|^r$$

where $\tilde{z}^{(i)}$ for $i=1, \dots, N$ are centralized observations of the vector z .

We recall that all the above multivariate models are motivated by arbitrage considerations as in the Arbitrage Pricing Theory (APT) (see Ross (1976)). In this context we do not intend to go into details, however, it should be noted that there are two versions of the APT for \mathbf{a} -stable distributed returns, namely a so-called equilibrium (see Chen and Ingersoll (1983), Dybvig (1983), Grinblatt and Titman (1983)) and an asymptotic version (see Huberman (1982)). Besides, Connor (1984) and Milne (1988) introduced a general theory which encompassed the equilibrium APT as well as the mutual fund separation theory for returns belonging to any normed vector space (hence also \mathbf{a} -stable distributed returns). Whereas Gamrowski and Rachev (1999) provide the proof for the asymptotic version of \mathbf{a} -stable distributed returns. Thus, it follows from Connor and Milne's theory that the above random law of the returns is coherent with classic arbitrage pricing theory and the mean returns can be approximated by the linear pricing relation

$$\mathbf{m}_i \approx z_0 + b_{i,2} \mathbf{d}_2 + \dots + b_{i,k} \mathbf{d}_k, \quad i = 1, 2, \dots,$$

where \mathbf{d}_p , for $p = 1, \dots, k$, are the risk premiums relative to the different factors.

As it follows from the above discussion, all the multivariate models introduced here can be empirically tested and will be summarized in Appendix A. It must be pointed out that in this section, we proposed a maximum likelihood method to estimate the parameters of the stable models, see also Rachev and Mittnik (2000) for some alternative methods. A more general empirical analysis with further discussions, studies and comparisons of the above multivariate cases does not enter in the objective of this paper and it will be the subject of future research.

5. A SAFETY FIRST OPTIMAL ALLOCATION PROBLEM
WITH STABLE DISTRIBUTED RETURNS

In this section we analyze the optimal allocation problem of a non-satiable investor who wishes to maximize the utility functional

$$(5.1) \quad V(W) = E(W) - cP(W \leq -VaR),$$

where c is a positive real number and VaR is a real number. Many are the reasons why we propose to study the above allocation problem. However, in our opinion the most important reasons are:

a) In accordance with the above allocation problem, we are able to propose an alternative mean-risk analysis of portfolio selection. In fact, the optimal allocation that we get solving the following problem

$$(5.2) \quad \max_{W \in A} E(W) - cP(W \leq -VaR),$$

where A is the set of all the admissible returns, is equal to one in which we maximize the utility functional

$$aE(W) - bP(W \leq -VaR),$$

for every $a, b > 0$, assuming in (5.2) $c = \frac{b}{a}$. If we assume $P(W \leq -VaR)$ as the particular risk measure of portfolio loss, then using the optimal allocation problem (5.2), we can find the optimal portfolios that maximize the expected mean of the increment wealth aW as well as minimizing the individual risk of loss $bP(W \leq -VaR)$.

b) The allocation problem (5.2) can be considered a safety first optimal allocation problem. In fact, every non-satiable investor with increasing utility function

$$(5.3) \quad u(x) = x - cI_{[x \leq -VaR]}(x)$$

tends to choose portfolios that maximize the utility functional (5.1). At the same time the investor, who uses the increasing utility function (5.3), maximizes the expected value as well as the probability of survival of his portfolio, as postulated in the safety first principles (for references on the safety first principles see Roy (1952), Tesler (1955/6), Bawa (1978)).

Some recent studies proved that there are many reasons why the safety first approach should be considered as an alternative to the classic mean-volatility one in portfolio selection theory. Principally, the main motivations leading to safety first portfolio choice are the following:

- 1) we can consider a portfolio selection for returns with unknown distributions;
 - 2) it is possible to develop a multi-parameter safety first analysis of optimal choices in the market;
 - 3) safety first analysis provides a representation of the efficient frontier in function of the threshold VaR ,
 - 4) some efficient programming methods approximate optimal safety first portfolios;
 - 5) the market trend can be studied and analyzed,
- (see Ortobelli (1999a-b-c), Ortobelli and Rachev (1999-2000)). In appendix B, we include a brief summary of the basic results on safety first portfolio selection.

We first assume $W = I z_0 + (1 - I)z$, where z_0 is the risk-free asset return and z is the risky asset return, that is, \mathbf{a} -stable distributed, with $\mathbf{a} > 1$, that is: $z = S_{\mathbf{a}}^d(\mathbf{s}_z, \mathbf{b}_z, m_z)$.

To solve the allocation problem (5.2) we first note that,

$$V(W) = E(W) - cP(W \leq -VaR) = I z_0 + (1 - I)m_z - cF_W(-VaR),$$

where F_W is the cumulative distribution function of W . Clearly, we are interested in the non-trivial solution of the problem $I \in (0,1)$ given by the solution of the first order condition with risk aversion parameter c . When $I \in [0,1)$, $W = S_{\mathbf{a}}^d((1 - I)\mathbf{s}_z, \mathbf{b}_z, I z_0 + (1 - I)m_z)$, then $V(W)$ is given by:

a) in the stable non-Gaussian case, for $2 > \mathbf{a} > 1$,

$$(5.4) \quad V(W) = I z_0 + (1 - I)m_z - cF_{\mathbf{a},\mathbf{b}}\left(\frac{-VaR - I z_0 - (1 - I)m_z - (1 - I)\mathbf{b} \mathbf{s}_z \tan(\mathbf{p}\mathbf{a}/2)}{(1 - I)\mathbf{s}_z}\right).$$

Here, $F_{\mathbf{a},\mathbf{b}}$ is the cumulative distribution function of the stable random variable $S_{\mathbf{a}}(1, \mathbf{b}_z, -\mathbf{b} \tan(\mathbf{p}\mathbf{a}/2))$, (see Nolan (1998) and Zolotarev (1986)) defined by:

$$F_{\mathbf{a},\mathbf{b}}(x) = \begin{cases} 1 - \frac{1}{\mathbf{p}} \int_{q_0}^{p/2} \exp(-(x - z)^{a/(a-1)} K(\mathbf{J}, \mathbf{a}, \mathbf{b})) d\mathbf{q} & \text{for } x > z \\ \frac{1}{\mathbf{p}} \left(\frac{\mathbf{p}}{2} - q_0 \right) & \text{for } x = z \\ 1 - F_{\mathbf{a},-\mathbf{b}}(-x) & \text{for } x < z \end{cases}$$

where

$$z = z(\mathbf{a}, \mathbf{b}) = -\mathbf{b} \tan\left(\frac{\mathbf{p}\mathbf{a}}{2}\right), \quad q_0 = q_0(\mathbf{a}, \mathbf{b}) = \frac{\arctan\left(\mathbf{b} \tan\left(\frac{\mathbf{p}\mathbf{a}}{2}\right)\right)}{\mathbf{a}},$$

$$\text{and } K(\mathbf{q}, \mathbf{a}, \mathbf{b}) = \cos(\mathbf{a}\mathbf{q}_0)^{1/(a-1)} \left(\frac{\cos \mathbf{q}}{\sin(\mathbf{a}(\mathbf{q} + \mathbf{q}_0))} \right)^{a/(a-1)} \frac{\cos(\mathbf{a}\mathbf{q}_0 + (\mathbf{a}-1)\mathbf{q})}{\cos \mathbf{q}};$$

b) in the normal case for $\mathbf{a} = 2$

$$(5.5) \quad V(W) = \mathbf{I} z_0 + (1-\mathbf{I})m_z - c \int_{-\infty}^{\frac{-VaR - \mathbf{I} z_0 - (1-\mathbf{I})m_z}{(1-\mathbf{I})s_z}} \frac{1}{\sqrt{2p}} \exp\left(-\frac{t^2}{2}\right) dt.$$

Thus, we can find a numerical solution of the maximization problem, see (5.4) and (5.5), as well as the stable non-Gaussian and the normal optimal allocation $\bar{\mathbf{I}} \in (0,1)$.

6. COMPARISON BETWEEN STABLE NON-GAUSSIAN AND NORMAL OPTIMAL ALLOCATIONS

In this section, we analyze the difference that occurs when the investor fits normal and stable (non-Gaussian) distributions in the above safety first optimal allocation. In Table 1, we have chosen the same empirical VaR for the risky indexes S&P 500, DAX30, CAC40 provided by Khindanova, Rachev and Schwartz (1999) for the various index-daily returns (see Table IV). This permits us not only to compare the different allocation in the stable and Gaussian case for different values of VaR , but also to see how important it is in relation to the choice of the risk aversion coefficient c in the optimal allocation $\max_{\mathbf{I}} E(W) - cP(W \leq -VaR)$. Moreover, we choose the empirical VaR (95% and 99%) for the different indexes because they are the most used in VaR analysis. In some sense, our allocation problem (5.2) can be considered the inverse problem of VaR analysis. Since in VaR analysis, we analyze the VaR of a given portfolio for a fixed probability of loss. Here we choose a portfolio that minimizes the (non-fixed) probability of loss.

In table V we have listed the optimal proportion of funds invested in the bond, $\bar{\mathbf{I}}$ for the normal and the stable fit when we use the above empirical VaR . The riskless return is given by the current value of three months LIBOR, 6% p.a. We use Mathematica (S. Wolfram) to evaluate the optimal allocation in the normal case, and we use Stable (Nolan's (1998) program) to find a numerical solution of the stable optimal allocation.

The analysis of table V shows that the differences between optimal allocation in the normal case and in the stable one can be up to 60%. In fact, we observe two significant differences.

- 1) In the case of lower risk aversion coefficient c , the “normal investor”(i.e. an investor who fits data with normal distributions) invests more in the riskless return, (this difference can be up to the 30%). This diversity is due to the “*kurtosis effect*”. In fact, the stable distributions are more peaked around the mean than the normal one. Consequently the investor, who fits the data using the stable distribution and who is not too much averse to the risk of loss, gives more importance to the mean than the “normal investor” with the same coefficient of aversion to the risk of loss. The latter prefers a lower mean for a lower risk of loss instead. Hence, the “normal investor” loses some opportunities of earning because of the kurtosis effect.
- 2) In the case of greater risk aversion coefficient c , it is the “stable non-Gaussian investor” who invests more in the riskless asset. This second difference (about the 60% in the allocation on the riskless asset) is due to the “*tail effect*”. In this context, the normal tail, the probability $P(W \leq -VaR)$ tends to zero exponentially and for higher riskless component the probability $P(W \leq -VaR)$ is almost null. Therefore, the “normal investor’s allocation” in the riskless asset improves only using a very high coefficient c . This does not happen to the “stable non Gaussian investor”. As a result, the behavior of the “normal investor” can be very dangerous. The fact of maximizing the utility functional (5.1) is equivalent to choose the return, that maximizes its expected value plus the expected value of a gamble in which the investor has to pay c if his return will be lower than a prefixed threshold $-VaR$. Now, the normal investor is not too much interested in improving the value c because he prefers the higher mean, being the probability $P(W \leq -VaR)$ almost null. Hence, because of the tail effect the “normal investor” does not consider that returns have heavy tail and so he risks more than the “stable non Gaussian investor” .

In some sense, we must expect very different optimal allocation in this problem. As a matter of fact, here we have a direct presence of the tail risk measure given by the $P(W \leq -VaR)$. Since the above safety first utility functional suits the non-satiabile, investors not necessarily risk averse, it underlines and exalts the differences, beyond the greater adherence to the data of stable non Gaussian distributional approach. This comparison highlights that the choice of stable distributions for asset returns is more recommendable in the above safety first optimization problem. As a matter of fact, the safety first utility functional, considered in the comparison, suits the non-satiabile not necessarily risk averse investors. Whereas stable distributional assumption presents a more protective behavior,

especially considering the component of risk due to heavy tails. Hence, the differences among the stable distributions having different stability indexes explain the big diversities in the allocation problem, and we see the crucial effect of the stable non-Gaussian distribution having heavy tails.

7. CONSIDERATIONS ON THE SAFETY FIRST OPTIMIZATION PROBLEM FOR THE MULTIVARIATE MODELS

In this section, we consider the problem of optimal allocation among $n+1$ assets: n of those assets are stable distributed risky assets with returns $z = [z_1, \dots, z_n]^T$ and the $(n+1)th$ asset is risk-free. Suppose the investor wishes to maximize the utility functional $V(W) = E(W) - cP(W \leq -VaR)$, where c is a positive real number, $W = (1 - x^T e)z_0 + x^T z$, z_0 is the risk-free asset return, and $x^T z$ is the risky portfolio asset return.

In the case the vector of risky returns $z = [z_1, \dots, z_n]^T$ is sub-Gaussian \mathbf{a} -stable distributed with $1 < \mathbf{a} < 2$, then the portfolio weight efficient frontier is given by (4.6) and the optimal portfolios are a linear combination between the market portfolio $\bar{x}^T z = \frac{(\mathbf{m} - z_0 e)^T Q^{-1} z}{B - Cz_0}$ and the riskless asset return

z_0 . Hence, we can find the optimal allocation (\bar{I}, x) solving

$$(7.1) \quad \max_{\bar{I}} E(W) - cP(W \leq -VaR),$$

where $W = \bar{I} z_0 + (1 - \bar{I}) \bar{x}^T z$. Thus, under the “no short sale” assumption $(\bar{I}, \bar{x}_i \geq 0)$, the optimal allocation will be given by $\bar{I} \in (0, 1)$, and $x = (1 - \bar{I}) \bar{x}$ that maximize

$$E(W) - cP(W \leq -VaR) = \bar{I} z_0 + (1 - \bar{I}) m_{\bar{x}^T z} - cF_{\mathbf{a},0} \left(\frac{-VaR - \bar{I} z_0 - (1 - \bar{I}) m_{\bar{x}^T z}}{(1 - \bar{I}) \mathbf{s}_{\bar{x}^T z}} \right).$$

Here $F_{\mathbf{a},0}$ is the above defined cumulative distribution function of the stable random variable

$$S_{\mathbf{a}}(1, 0, 0); m_{\bar{x}^T z} = \frac{A - Bz_0}{B - Cz_0}, \mathbf{s}_{\bar{x}^T z} = \sqrt{\bar{x}^T Q \bar{x}} = \frac{(A - 2Bz_0 + Cz_0^2)^{1/2}}{B - Cz_0}$$

are respectively the mean and the scale parameter of the optimal risky portfolio $\bar{x}^T z$, and $A = \mathbf{m}^T Q^{-1} \mathbf{m}$; $B = e^T Q^{-1} \mathbf{m}$; $C = e^T Q^{-1} e$.

According to safety first portfolio selection theory (see Ortobelli and Rachev (1999)), we know that all the portfolio weights of the risk averse efficient frontier (4.6) are static points of the optimization problem

$$(7.2) \quad \min_x P(W \leq t) = \min_x P((1-x^l e)z_0 + x^l z \leq t)$$

for some real t . In particular, the solutions of (7.2) for $t \geq z_0$ are all the portfolio weights belonging to the portfolio weight non-satiable risk-averse efficient frontier. Thus, for every optimal allocation (\bar{I}, x) , solving (7.1), there exists a value $\tilde{t} = -\overline{VaR} \geq z_0$ such that

$$x = (1-\bar{I})\bar{x} = \arg\left(\min_x P((1-x^l e)z_0 + x^l z \leq -\overline{VaR})\right),$$

and the portfolio weight efficient frontier for non satiable risk averse investors can be expressed as a function of \tilde{t} (the safety first representation of the efficient frontier). Moreover, when we consider short sale restrictions ($\bar{I} \in (0,1), \bar{x}_i \geq 0$), the set of optimal portfolios for non-satiable investors contains only the efficient frontier for non-satiable risk averse investors, (see Bawa (1978) and Ortobelli and Rachev (1999)).

Analogously to section 4, we can assume the three-fund separation stable model (4.10) for the risk returns. We obtain the optimal allocation solving

$$\max_{I_2, I_3} E(W) - cP(W \leq -VaR),$$

where $W = (1-I_2 - I_3)z_0 + x^l(I_2, I_3)z$, and

$$x(I_2, I_3) = I_2 \frac{Q^{-1}(\mathbf{m} - z_0 e)}{e^l Q^{-1}(\mathbf{m} - z_0 e)} + I_3 \frac{Q^{-1}b}{e^l Q^{-1}b}.$$

Assuming that no short sale is allowed (i.e. $(1-I_2 - I_3) \geq 0, x_i(I_2, I_3) \geq 0$), then the optimal allocation is given by $\bar{I} \in (0,1)$, and $x(I_2, I_3)$ that maximize $V(W) = E(W) - cP(W \leq -VaR)$. Hence, we can approximate the optimal allocation on the efficient frontier solving the following optimization problem

$$\begin{aligned} \max_{I_2, I_3} E(W) - cP(W \leq -VaR) = & \max_{I_2, I_3} (1 - I_2 - I_3)z_0 + m_{x^l(I_2, I_3)z} + \\ & - c \frac{1}{N} \sum_{i=1}^N \left((1 - I_2 - I_3)z_0 + x^l(I_2, I_3)z^{(i)} \right) \mathbb{I}_{[x^l(I_2, I_3)z^{(i)} \leq -VaR - (1 - I_2 - I_3)z_0]}, \end{aligned}$$

where $z^{(i)}$ for $i=1, \dots, N$ are observations of the vector z . In particular, when $\mathbf{a} := \mathbf{a}_1 = \mathbf{a}_2$, we can

solve numerically the following two-parameter optimal allocation problem

$$\max_{I_2, I_3} (1 - I_2 - I_3)z_0 + m_{x^l(I_2, I_3)z} - cF_{\mathbf{a}, \mathbf{b}_{x^l(I_2, I_3)z}} \left(\frac{-VaR - (1 - I_2 - I_3)z_0 - m_{x^l(I_2, I_3)z} - \mathbf{b}_{x^l(I_2, I_3)z} \mathbf{s}_{x^l(I_2, I_3)z} \tan(\mathbf{pa}/2)}{\mathbf{s}_{x^l(I_2, I_3)z}} \right)$$

where $F_{\mathbf{a}, \mathbf{b}_{x^l(I_2, I_3)z}}$ is the above definite cumulative distribution function of the stable random variable

$$S_{\mathbf{a}} \left(\mathbf{1}, \mathbf{b}_{x^l(I_2, I_3)z}, -\mathbf{b}_{x^l(I_2, I_3)z} \tan(\mathbf{pa}/2) \right), \text{ and } \mathbf{b}_{x^l(I_2, I_3)z} = \frac{|x^l(I_2, I_3)b\mathbf{s}_Y|^a \operatorname{sgn}(x^l(I_2, I_3)b)\mathbf{b}_Y}{\mathbf{s}_{x^l(I_2, I_3)z}^a},$$

$$\mathbf{s}_{x^l(I_2, I_3)z} = \left((x^l(I_2, I_3)Qx^l(I_2, I_3))^{a/2} + (|x^l(I_2, I_3)b\mathbf{s}_Y|^a)^{1/a} \right), \quad m_{x^l(I_2, I_3)z} = x^l(I_2, I_3)\mathbf{m} \quad \text{are}$$

respectively, the skewness parameter, the scale parameter, and the mean of the optimal risky portfolio

$$x^l(I_2, I_3)z.$$

As a further generalization, we can assume the $(k+1)$ fund separation stable model (4.15) as the risk returns. Thus, we obtain the optimal allocation solving

$$(7.3) \quad \max_{I_2, \dots, I_{k+1}} E(W) - cP(W \leq -VaR),$$

where $W = \bar{I}z_0 + x^l(I_2, I_3, \dots, I_{k+1})z$, with $\bar{I} = 1 - I_2 - I_3 - \dots - I_{k+1}$ and

$$x(I_2, \dots, I_{k+1}) = \mathbf{I} \frac{Q^{-1}(\mathbf{m} - z_0e)}{e^T Q^{-1}(\mathbf{m} - z_0e)} + \sum_{i=2}^k I_{i+1} \frac{Q^{-1}\mathbf{b}_{\bullet, i}}{e^T Q^{-1}\mathbf{b}_{\bullet, i}}$$

Assuming that no short sale is allowed (i.e. $\bar{I}, x_i(I_2, I_3, \dots, I_k) \geq 0$), then the optimal allocation is given by $\bar{I} \in (0, 1)$, and $x(I_2, \dots, I_{k+1})$, that maximize $V(W) = E(W) - cP(W \leq -VaR)$. Therefore, we can approximate the optimal allocation on the efficient frontier by solving the following optimization problem:

$$\begin{aligned} \max_{I_2, \dots, I_{k+1}} E(W) - cP(W \leq -VaR) &= \max_{I_2, \dots, I_{k+1}} (1 - I_2 - \dots - I_{k+1})z_0 + m_{x^l(I_2, \dots, I_{k+1})z} + \\ &- c \frac{1}{N} \sum_{i=1}^N \left[(1 - I_2 - \dots - I_{k+1})z_0 + x^l(I_2, \dots, I_{k+1})z^{(i)} \right] \mathbb{I}_{[x^l(I_2, \dots, I_{k+1})z^{(i)} \leq -VaR - (1 - I_2 - \dots - I_{k+1})z_0]} \end{aligned}$$

Similarly to the three fund separation model, when $\mathbf{a} := \mathbf{a}_1 = \dots = \mathbf{a}_k$, we can solve numerically the allocation problem using the analytical formulation of the cumulative distribution function.

Under the assumptions of the model (4.15), given two portfolios $x^l z$ and $y^l z$ with the same mean

$$x^l \mathbf{m} = y^l \mathbf{m} \text{ and the same parameters } \frac{y^l \mathbf{b}_{\bullet,j}}{\sqrt{y^l Qy}} = \frac{x^l \mathbf{b}_{\bullet,j}}{\sqrt{x^l Qx}} = \tilde{c}_j, \quad j = 2, \dots, k, \text{ where } \mathbf{b}_{\bullet,j} = [b_{1,j}, \dots, b_{n,j}]^l,$$

and such that $x^l Qx > y^l Qy$. Thus, for every real u :

$$\begin{aligned} & \int_{-\infty}^u (P(y^l z \leq s) - P(x^l z \leq s)) ds = \\ &= \int_{-\infty}^u \int_{R^{k-1}} P(X \leq \frac{s - y^l \mathbf{m}}{\sqrt{y^l Qy}} - \tilde{c}_2 t_1 - \dots - \tilde{c}_k t_k \mid Y_2 = t_2, \dots, Y_k = t_k) f_{Y_2, \dots, Y_k}(t) dt ds + \\ & - \int_{-\infty}^u \int_{R^{k-1}} P(X \leq \frac{s - x^l \mathbf{m}}{\sqrt{x^l Qx}} - \tilde{c}_2 t_1 - \dots - \tilde{c}_k t_k \mid Y_2 = t_2, \dots, Y_k = t_k) f_{Y_2, \dots, Y_k}(t) dt ds = \\ &= \int_{R^{k-1}} \int_{-\infty}^u (P(X \leq \frac{s - y^l \mathbf{m}}{\sqrt{y^l Qy}} - \tilde{c}_2 t_1 - \dots - \tilde{c}_k t_k \mid Y_2 = t_2, \dots, Y_k = t_k) + \\ & \quad - P(X \leq \frac{s - x^l \mathbf{m}}{\sqrt{x^l Qx}} - \tilde{c}_2 t_1 - \dots - \tilde{c}_k t_k \mid Y_2 = t_2, \dots, Y_k = t_k)) ds f_{Y_2, \dots, Y_k}(t) dt \leq 0 \end{aligned}$$

where $X /_{Y_2, \dots, Y_k} = \frac{x^l \mathbf{e}}{\sqrt{x^l Qx}} = S_{a_1}(1, 0, 0)$, f_{Y_2, \dots, Y_k} is the joint density of (Y_2, \dots, Y_k) , the last inequality

can be proved using similar arguments as in (4.3). It follows that $y^l z$ dominates $x^l z$ in the sense of R-S. Next, we can obtain the efficient frontier for the risk averse investors when unlimited short selling is allowed, as the solution of the alternative following the quadratic programming problem:

$$\begin{cases} \min_x \frac{1}{2} x^l Qx \\ x^l \mathbf{m} + (1 - x^l e) z_0 = m_w, \\ \frac{x^l \mathbf{b}_{\bullet,j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k, \end{cases}$$

Moreover, for every fixed parameters $\frac{x^l \mathbf{b}_{\bullet,j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k$, it also follows:

$$\begin{aligned} P(x^l z + (1 - x^l e) z_0 \leq s) &= \int_{R^{k-1}} P(X \leq \frac{s - x^l \mathbf{m} - (1 - x^l e) z_0}{\sqrt{x^l Qx}} - \tilde{c}_2 t_1 - \dots - \tilde{c}_k t_k \mid Y_2 = t_2, \dots, Y_k = t_k) f_{Y_2, \dots, Y_k}(t_2, \dots, t_k) dt_2 \dots dt_k = \\ &= g\left(\frac{s - x^l \mathbf{m} - (1 - x^l e) z_0}{\sqrt{x^l Qx}}\right), \end{aligned}$$

where $X/Y_2, \dots, Y_k = \frac{x^l \mathbf{e}}{\sqrt{x^l Qx}} = S_{a_1}(1, 0, 0)$, f_{Y_2, \dots, Y_k} is the joint density of (Y_2, \dots, Y_k) and g is an increasing

function. As a consequence, if the portfolio $x^l z + (1 - x^l e)z_0$ maximizes the probability of survival

for a fixed threshold s and fixed parameters $\frac{x^l b_{\bullet, j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k$, then $s \leq x^l \mathbf{m} + (1 - x^l e)z_0$, and

there is no portfolio $y^l z + (1 - y^l e)z_0$ such that

$$x^l \mathbf{m} + (1 - x^l e)z_0 = y^l \mathbf{m} + (1 - y^l e)z_0 \text{ and } y^l Qy < x^l Qx.$$

With similar arguments, we can prove that every portfolio that minimizes the probability of survival for

a fixed threshold s and fixed parameters $\frac{x^l b_{\bullet, j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k$, is a portfolio that minimizes the

volatility for fixed mean and fixed parameters $\frac{x^l b_{\bullet, j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k$, (i. e. it is a non-dominated

portfolio in the sense of R-S). However, the opposite is also true, (comparing the solutions of (4.16)

with the static points of the following optimization problem (7.4) proves it). As in the sub-Gaussian

stable case, when unlimited short selling is allowed, all the portfolios of the risk averse efficient

frontier (4.17), are static points of the optimization problem

$$(7.4) \quad \begin{cases} \min_x P((1 - x^l e)z_0 + x^l z \leq s) = \min_x \frac{s - x^l \mathbf{m} - (1 - x^l e)z_0}{\sqrt{x^l Qx}} \\ s = s^*, \\ \frac{x^l b_{\bullet, j}}{\sqrt{x^l Qx}} = \tilde{c}_j \quad j = 2, \dots, k \end{cases}$$

varying s^* , and $\tilde{c}_j \quad j = 2, \dots, k$. In particular, the solutions of (7.4) for $s \geq z_0$ are all the portfolio

weights belonging to the portfolio weight efficient frontier for the non-satiable risk-averse investors.

Thus, for every optimal allocation $(\bar{I}, x(I_2, \dots, I_{k+1}))$, solving (7.3), there exist $\tilde{s} = -\overline{VaR} \geq z_0$ and

$c_j^*, \quad j = 2, \dots, k$, such that

$$x(I_2, \dots, I_{k+1}) = \arg \left(\min_x P((1 - x^l e)z_0 + x^l z \leq -\overline{VaR}) \right) \left. \begin{array}{l} \frac{x^l b_{\bullet, j}}{\sqrt{x^l Qx}} = c_j^* \quad j = 2, \dots, k \end{array} \right\}$$

Furthermore, the portfolio weight efficient frontier for the non-satiable risk averse investors can be

expressed as a function of $\tilde{\sigma}$ and c_j^* $j=2,\dots,k$, (the safety first representation of the multi-parameter efficient frontier). Also in this case, when we consider short sale restrictions, the set of optimal portfolios for the non-satiable investors generally contains (in the strict sense) the efficient frontier for the non-satiable risk averse investors.

8. CONCLUSIONS

In first analysis, the comparison made between the stable non-Gaussian and the normal approach to the allocation problems has indicated that the stable non-Gaussian allocation is more risk preserving than the normal one. Precisely the stable approach, differently from the normal one, considers the component of risk due to the fat tails. Other differences can be seen in the allocation as due to the different kurtosis in the stable and normal distributions. In this case a “stable investor”, not too averse to the risk of loss, invests more in the risky assets than a “normal investor”. Therefore, we found that the two main differences (tails and kurtosis) between Gaussian and stable non Gaussian approaches imply important differences in the allocation problems. Taken into account that the stable approach is more adherent to the reality of the market, then, as argued by Götezenberger, Rachev and Schwartz (1999), we can obtain models that improve the performance measurements with the stable distributional assumption.

In second analysis, we study, analyze and discuss portfolio choice models considering returns with heavy tailed distributions. The first distributional model considered: the case of sub Gaussian \mathbf{a} stable distributed returns permits a mean risk analysis pretty similar to the Markowitz-Tobin mean variance one. As a matter of fact, this model admits the same analytical form for the efficient frontier, but the parameters in the two models have a different meaning . Therefore, the most important difference is given by the way of estimating the parameters. In order to present heavy tailed models that consider the asymmetry of the returns, we study a three fund separation model where the portfolios are in the domain of attraction of an $(\mathbf{a}_1, \mathbf{a}_2)$ stable law. Finally, we analyze the case of $k+1$ fund separation model with portfolios in the domain of attraction of an $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)$ stable law. For all models we explicate the efficient frontier for the risk averse investors. Then, we show how to estimate all parameters and to determine the safety first representation of the multi-parameter efficient frontier. In

this context, we have shown that if the stable optimal portfolio analysis is stable, our approach is theoretically and empirically possible. Indeed, this work should be viewed only as a starting point for new empirical and theoretical studies on the topic of optimal allocation. In order to know which stable model is preferable, the market conditions in which the investors operate must be analyzed. Furthermore, the numbers of parameters can be increased in the model so to achieve a better approximation, although it can be far more “expensive” in the calculation time. Finally, as far as the choice of the best indexes to choose in the $k+1$ fund separation model, we can refer to Ross’ analysis on this issue.

APPENDIX A

In table VI we summarize the different models used in this paper.

APPENDIX B: ON SAFETY FIRST PORTFOLIO CHOICE

Roy [1952], Bawa [1976, 1978], Pyle and Turnovsky [1970, 1971] suggest the safety first rules as a criterion for decision making under uncertainty. The practical appeal of the generalized safety first rules is demonstrated by Bawa [1979]. In the context of realistic portfolio selection problems where the distributions of portfolio returns are unknown, safety first rules can be used by applying them to the empirical distribution of portfolio returns used in lieu of the true, but unknown, distributions. More recently, Ortobelli and Rachev (2000) also showed that the Young (1998) minimax principle is a particular case of the more general safety first principle. Moreover, the safety first portfolio selection rules suggest consistent statistics to approximate optimal portfolios for the non-satiable investors and the non-satiable risk averse investors. These criteria maintain almost the same advantages of the minimax method. Hence, the developed programming methods to determine safety first portfolio selection have the potential to make portfolio optimization a tool very accessible to any financial manager.

Pyle and Turnovsky [1970, 1971], Bawa [1976, 1978] and more recently Ortobelli [1999b-c], Ortobelli and Rachev (1999-2000) showed that when the returns belong to an elliptical family of distributions, then safety first analysis provides a representation of the mean dispersion efficient frontier in function of the threshold VaR . In this case it appears more realistic to assume sub-Gaussian

stable distributions of the returns in safety first analysis. Moreover, the equivalence of the efficient frontiers is not only a consequence of the elliptical distributional approach, in fact, Ortobelli (1999b-c), Ortobelli and Rachev (1999-2000) realize the Pyle and Turnowsky's conjecture and generalize safety first analysis to a multi-parameter portfolio selection. Hence, under some regularity conditions on the distributions of returns, we have equivalence rules between the non stochastically dominated sets, the sets of safety first portfolios and the moments efficient frontiers. These correspondences also imply that safety first analysis is an alternative and can be more general than moments analysis because it does not necessarily require distributional restrictions. In Ortobelli and Rachev (2000), the concept of stochastic bounds of the market is introduced, where cumulative distributions can be obtained as envelope of optimal portfolio cumulative distributions. The studies on the stochastic bounds permit to give a modern interpretation of equilibrium and to analyze the trend of a complete market with short sale restrictions.

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FOOTNOTES

1) We recall that Chamberlein (1983) shows that the families of elliptical distributions with finite variance are necessary and sufficient for the expected utility of final wealth to be a function only of the mean and the variance.

2) $x^{(t)} := (\operatorname{sgn} x)|x|^t$.

TABLES

Table I. Estimated daily index parameters

The following table was obtained by Khindanova, Rachev and Schwartz (1999). It summarizes the estimated parameters of the normal and the stable fit the sample distribution of z when z is either the index S&P 500 or DAX30 or CAC40.

Series	Normal		Method	Stable			
	Mean	Standard Deviation		a	b	m	s
S&P 500	0.032	0.930	ML	1.708	0.004	0.036	0.512
DAX30	0.026	1.002	ML	1.823	-0.084	0.027	0.592
CAC40	0.028	1.198	ML	1.784	-0.153	0.027	0.698

Table II: Coefficient \bar{c} for the normal and the stable case in the allocation problem

$$\max_I E(W) - cE(|W - E(W)|) .$$

This table computes the coefficient

$$\bar{c} = \frac{m_z - z_0}{V(\mathbf{a}, \mathbf{b}, 1) \mathbf{s}_z}$$

where $V(\mathbf{a}, \mathbf{b}, 1) = \begin{cases} H(\mathbf{a}, \mathbf{b}, 1) & \text{in the stable case } (1 < \mathbf{a} < 2) \\ \sqrt{\frac{2}{p}} & \text{in the normal case } (\mathbf{a} = 2) \end{cases}$, z_0 is the riskless rate three months

LIBOR 6% annual rate (daily $z_0 = 0.0001 \bar{6} = \frac{0.06}{360}$) and z is either the index S&P 500 or DAX30 or

CAC40. In particular, we point out with \bar{c}_1 and \bar{c}_2 the coefficient \bar{c} respectively for the normal and the stable case.

Series	Normal case coefficient \bar{c}_1	Stable case coefficient \bar{c}_2
S&P 500	0.043	0.051
DAX30	0.032	0.036
CAC40	0.029	0.030

Table III: Optimal allocation for the normal and the stable fit the optimization problem

$$\max_I E(W) - cE(|W - E(W)|^r) \quad \text{when } r=1.5 \text{ or } r=1.35.$$

This table computes the optimal allocation \bar{I} in the riskless return three months LIBOR 6% annual rate (daily $z_0 = 0.0001 \bar{6} = \frac{0.06}{360}$) for different risk aversion coefficient c of the optimization problem

$$\max_I E(W) - cE(|W - E(W)|^r)$$

where $W = I z_0 + (1-I)z$ and z is either the index S&P 500 or DAX30 or CAC40. In particular, we analyse the normal and the stable when $r=1.5$ or $r=1.35$.

SERIES	Coefficient "c" of the optimization problem	Normal		Stable	
		Optimal Allocation \bar{I}		Optimal Allocation \bar{I}	
		With r=1.35	With r=1.5	With r=1.35	With r=1.5
S&P500	c=0.0276	0.000	0.006	0.000	0.415
	c=0.03	0.000	0.159	0.000	0.505
	c=0.032	0.069	0.261	0.096	0.565
	c=0.033	0.148	0.305	0.172	0.591
	c=0.034	0.217	0.345	0.240	0.615
	c=0.036	0.335	0.416	0.356	0.656
	c=0.038	0.430	0.476	0.447	0.691
	c=0.04	0.508	0.527	0.522	0.721
	c=0.045	0.649	0.626	0.659	0.780
	c=0.05	0.740	0.697	0.748	0.822
	c=0.055	0.802	0.750	0.808	0.853
	c=0.065	0.877	0.821	0.881	0.895
	c=0.1	0.964	0.924	0.965	0.955
	DAX30	c=0.021	0.000	0.096	0.000
c=0.022		0.000	0.176	0.000	0.265
c=0.023		0.012	0.247	0.000	0.327
c=0.024		0.126	0.308	0.021	0.382
c=0.025		0.222	0.362	0.129	0.431
c=0.027		0.375	0.453	0.301	0.512
c=0.0285		0.465	0.509	0.401	0.562
c=0.03		0.538	0.557	0.482	0.605
c=0.033		0.648	0.634	0.606	0.673
c=0.035		0.702	0.675	0.667	0.709
c=0.04		0.797	0.751	0.772	0.778
c=0.05		0.893	0.841	0.880	0.858
c=0.1		0.985	0.960	0.983	0.964
CAC40		c=0.017	0.000	0.063	0.000
	c=0.018	0.000	0.164	0.029	0.466
	c=0.019	0.000	0.250	0.168	0.520
	c=0.02	0.085	0.323	0.281	0.567
	c=0.0205	0.148	0.356	0.330	0.588
	c=0.0215	0.256	0.414	0.416	0.625
	c=0.023	0.387	0.488	0.518	0.673
	c=0.024	0.457	0.530	0.573	0.699
	c=0.025	0.517	0.567	0.620	0.723
	c=0.028	0.650	0.655	0.725	0.779
	c=0.033	0.781	0.751	0.828	0.841
	c=0.04	0.874	0.831	0.901	0.892
	c=0.1	0.991	0.973	0.993	0.983

Table IV: Estimated empirical VaR of risky indexes

The following table was obtained by Khindanova, Rachev and Schwartz (1999). It summarizes the estimated empirical VaR (95% and 99%) for the daily risky indexes S&P 500, DAX30 and CAC40.

Series	Empirical VaR 99%	Empirical VaR 95%
S&P 500	2.293	1.384
DAX30	2.564	1.508
CAC40	3.068	1.819

Table V: Safety first optimal allocation for the normal and the stable fit using riskless rate three

months LIBOR 6% annual rate (daily $z_0 = 0.0001 \bar{r} = \frac{0.06}{360}$).

This table computes the optimal allocation \bar{I} in the riskless return three months LIBOR 6% annual rate (daily $z_0 = 0.0001 \bar{r} = \frac{0.06}{360}$) for different risk aversion coefficient c of the optimization problem

$$\max_{W \in \mathcal{A}} E(W) - cP(W \leq -VaR)$$

where $W = I z_0 + (1-I)z$ and z is either the index S&P 500 or DAX30 or CAC40. In particular, we distinguish the normal and the stable fit when we have the empirical VaR (95% and 99%) estimated in table IV.

SERIES	Coefficient “c” of the optimization problem	Normal		Stable	
		Optimal Allocation \bar{I} WITH		Optimal Allocation \bar{I} WITH	
		VaR=1.384 ;	VaR=2.293;	VaR=1.384 ;	VaR=2.293
S&P500	$c=0.173$	0.035	0.000	0.000	0.000
	$c=0.2$	0.182	0.000	0.000	0.000
	$c=0.25$	0.277	0.000	0.000	0.000
	$c=0.3$	0.325	0.000	0.000	0.000
	$c=1$	0.483	0.064	0.496	0.000
	$c=2$	0.530	0.168	0.724	0.267
	$c=7$	0.588	0.284	0.946	0.819
	$c=15$	0.613	0.332	0.981	0.937
	$c=150$	0.667	0.432	0.999	0.998
	$c=5000$	0.717	0.521	1.000	1.000
	$c=5 \cdot 10^{10}$	0.810	0.682	1.000	1.000
		VaR=1.508 ;	VaR=2.564;	VaR=1.508 ;	VaR=2.564
DAX30	$c=0.14$	0.022	0.000	0.000	0.000
	$c=0.16$	0.167	0.000	0.000	0.000
	$c=0.2$	0.266	0.000	0.000	0.000
	$c=0.25$	0.325	0.000	0.118	0.000
	$c=0.5$	0.432	0.000	0.355	0.000
	$c=1$	0.495	0.064	0.514	0.000
	$c=1.5$	0.521	0.125	0.589	0.126
	$c=7$	0.591	0.271	0.917	0.736
	$c=15$	0.616	0.319	0.967	0.894
	$c=1700$	0.703	0.484	1.000	1.000
	$c=170000$	0.748	0.564	1.000	1.000
	$c=5 \cdot 10^{10}$	0.809	0.672	1.000	1.000
			VaR=1.819 ;	VaR=3.068;	VaR=1.819 ;
CAC40	$c=0.16$	0.095	0.000	0.000	0.000
	$c=0.17$	0.147	0.000	0.000	0.000
	$c=0.2$	0.231	0.000	0.000	0.000
	$c=0.3$	0.341	0.000	0.187	0.000
	$c=0.5$	0.419	0.000	0.375	0.000
	$c=1$	0.484	0.052	0.562	0.017
	$c=1.5$	0.512	0.115	0.679	0.205
	$c=7$	0.585	0.266	0.923	0.819
	$c=15$	0.610	0.314	0.978	0.930
	$c=150$	0.663	0.415	0.999	0.996
	$c=5000$	0.713	0.505	1.000	1.000
	$c=5 \cdot 10^{10}$	0.806	0.671	1.000	1.000

Table VI:A summary of the models considered in our discussion.

This table summarizes the different models used in this paper.

	Portfolio	Model	On the random parts of the model	Efficient frontier	Estimator for the parameters
Two Assets Approach	$W = I z_0 + (1 - I) z$	z_0 is the riskless return,	z is a Normal distributed return	Mean-variance frontier $m_W = z_0 + \frac{m_z - z_0}{s_z} s_W$	Classic moment estimators of mean and variance
		z is the risky return	z is an \mathbf{a} stable distributed return $1 < \mathbf{a} < 2$	Mean-dispersion Frontier $m_W = z_0 + \frac{m_z - z_0}{s_z} s_W$	Maximum Likelihood Method or Fourier Transform method
MULTIVARIABLE ARCH	$W = (1 - x^l e) z_0 + x^l z$ z_0 is the riskless return	Two fund separation model	z is the vector of jointly elliptically distributed returns with finite variance	Mean-variance frontier with optimal portfolio weights $x = Q^{-1}(\mathbf{m} - z_0 e) \frac{m - z_0}{A - 2Bz_0 + Cz_0^2}$	Classic moment estimators of mean and variance
		z is the vector of jointly elliptically distributed returns	z is the vector of jointly sub-Gaussian symmetrically \mathbf{a} stable distributed returns with $1 < \mathbf{a} < 2$	Mean-dispersion Frontier with optimal portfolio weights: $x = Q^{-1}(\mathbf{m} - z_0 e) \frac{m - z_0}{A - 2Bz_0 + Cz_0^2}$	Method based on the moments
		Three fund separation model $z_i = \mathbf{m}_i + b_i Y + \mathbf{e}_i$	\mathbf{e} is vector jointly elliptically distributed. Y is a non symmetric random variable. $b_i Y + \mathbf{e}_i \quad \forall i$ admit finite third moment	Mean-variance-skewness frontier with optimal portfolios: $(1 - I_2 - I_3) z_0 + I_3 \frac{z^l Q^{-1} b}{e^l Q^{-1} b} + I_2 \frac{z^l Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)}$	Using moment estimators of mean, variance and skewness
		\mathbf{e} vector jointly elliptically distributed and independent from Y	\mathbf{e} is vector jointly sub-Gaussian symmetrically \mathbf{a}_1 stable distributed. Y is \mathbf{a}_2 stable distributed. Besides $1 < \mathbf{a}_{1,2} < 2$	Mean-dispersion-skewness frontier with optimal portfolios: $(1 - I_2 - I_3) z_0 + I_3 \frac{z^l Q^{-1} b}{e^l Q^{-1} b} + I_2 \frac{z^l Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)}$	Maximum Likelihood method or OLS method
		k+1-fund separation stable model $z_i = \mathbf{m}_i + b_{i,2} Y_2 + \dots + b_{i,k} Y_k + \mathbf{e}_i$, \mathbf{e} vector independent from Y_2, \dots, Y_k	\mathbf{e} is vector jointly sub-Gaussian symmetrically \mathbf{a}_1 stable distributed. $Y_j, j=2, \dots, k$ are \mathbf{a}_j -stable distributed with $1 < \mathbf{a}_j < 2$.	Multi-parameter efficient frontier with optimal portfolios: $(1 - \sum_{i=2}^{k+1} I_i) z_0 + \sum_{i=2}^k I_{i+1} \frac{z^l Q^{-1} b_{i,1}}{e^l Q^{-1} b_{i,1}} + I_2 \frac{z^l Q^{-1} (\mathbf{m} - z_0 e)}{e^l Q^{-1} (\mathbf{m} - z_0 e)}$	Maximum Likelihood method or OLS method