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Linearly Sofic Lie Algebras

Author
Cinel, Cameron

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Linearly Sofic Lie Algebras

A dissertation submitted in partial satisfaction of the
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by

Cameron Cinel

Committee in charge:

Professor Daniel Rogalski, Chair
Professor Alireza Golsefidy
Professor Amir Mohammadi
Professor Vitali Nesterenko
Professor Efim Zelmanov

2024

The Dissertation of Cameron Cinel is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2024

DEDICATION

To my mother and father.

EPIGRAPH

Truth... is much too complicated to allow anything but approximations.

John von Neumann

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VITA

2019	Bachelor of Science, University of Southern California
2019–2024	Teaching Assistant, Department of Mathematics University of California San Diego
2024	Doctor of Philosophy, University of California San Diego

ABSTRACT OF THE DISSERTATION

Linearly Sofic Lie Algebras

by

Cameron Cinel

Doctor of Philosophy in Mathematics

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Professor Daniel Rogalski, Chair

Metric approximable groups have been studied since the introduction of sofic groups by Gromov [13]. Since then, further classes of metric approximable groups have been studied, such as hyperlinear [24], linearly sofic [4], and weakly sofic groups [11]. Due to their connection with many open problems such as Gottschalk's Surjunctivity Conjecture [12], Connes' Embedding Problem [7], and Kaplansky's Direct Finiteness Conjecture [18], metric approximable groups have generated much interest.

Recently, metric approximability has been extended from groups to associative algebras through linearly sofic associative algebras [4]. These algebras were shown to have many similar properties as linearly sofic groups, such as equivalent characterization through metric

ultraproducts and almost representations. Additionally, non-linearly sofic algebras proved easier to find as compared to the case of linearly sofic groups. However, certain properties that hold for groups have not been shown to hold for associative algebras, such as preservation of linear soficity through certain extensions.

In this dissertation, we continue the work in [4] by extending the definition of linear soficity further to Lie algebras. Lie algebras are a natural object to study in this area, as they have many similarities to both groups and associative algebras. In §3, we define linear soficity for Lie algebras using metric ultraproducts, as well as give an equivalent characterization through the use of almost representations. We also give some examples of linearly sofic Lie algebras. In §4, we show the connection between linear soficity in Lie and associative algebras by showing that, over fields of characteristic 0, a Lie algebra is linearly sofic if and only if its universal enveloping algebra is.

In §5, we look at extensions of linearly sofic Lie algebras. We show that any extension of a linearly sofic Lie algebra by a Lie algebra with an amenable universal enveloping algebra is linearly sofic, as in the case of groups. In §6, we use wreath products to show that a countable metric approximable group is embeddable in a finitely generated metric approximable group of the same kind. In addition, we use a similar argument using wreath products to show that a countable dimensional linearly sofic Lie algebra is embeddable in a finitely generated linearly sofic Lie algebra.

Introduction

Metric approximations of algebraic objects first appeared with the introduction of sofic groups by Gromov [13]. Since then, various different forms of metric approximations of groups have appeared, from hyperlinear groups [24], linearly sofic groups [4], and weakly sofic groups [11]. All these classes of groups are similarly defined as subgroups of metric ultraproducts of certain classes of groups or, equivalently, groups admitting uniformly injective almost homomorphisms into the same class of groups. These groups have generated interest as many of them are the largest classes of groups satisfying many open conjectures such as Gottschalk's Surjuncitivity Conjecture [12], Connes' Embedding Problem [7], or Kaplansky's Direct Finiteness Conjecture [18]. An important open problem in this area is the existence of any non-metric approximable group for any of the studied categories. We discuss metric approximable groups further in §1.

The construction of metric approximable groups was extended by Arzhantseva and Păunescu [4] to linearly sofic associative algebras. These algebras were shown to have many of the same properties as other metric approximable groups, such as generalizing the concepts of amenability or locally embeddable in finiteness. Additionally, non-linearly sofic associative algebras were easier to find, unlike the case of groups. However, certain properties that hold for linearly sofic groups were not able to be shown for linearly sofic associative algebras, such as linear soficity being preserved under extension by amenable objects. We discuss linearly sofic algebras further in §2.

We build upon the work of [4] by extending the definition of linear soficity further to Lie algebras. Lie algebras are a natural object to study due to their similarities with both groups and associative algebras. In §3, we give the definition of linearly sofic Lie algebras through the use of metric ultraproducts. We give an equivalent characterization through the use of families of almost representations. We additionally define a related version of linear soficity in the case of restricted Lie algebras. In §4 we show the connection of linear soficity in Lie and associative algebras by showing that, under certain conditions, a Lie algebra is linearly sofic if and only if its universal enveloping algebra is.

In §5, we study extensions of linearly sofic Lie algebras. Through the use of wreath products, we show that extensions of linearly sofic Lie algebras by Lie algebras whose universal enveloping algebras are amenable are linearly sofic, similar to the case of sofic, hyperlinear, and linearly sofic groups. In §6, we look at embeddings of metric approximable groups and linearly sofic Lie algebras. Using a result of Neumann and Neumann [22], we show that any countable metric approximable group is embeddable in a finitely generated group that is metric approximable of the same type. We then prove an analog of the Neumann and Neumann theorem for Lie algebras and show that any countable dimensional linearly sofic Lie algebra is embeddable in a finitely generated linearly sofic Lie algebra.

0.1 Notation

We will use the following notation throughout this thesis, unless otherwise stated:

- For a set X, $\mathscr{P}(X)$ will denote the set of subsets of X.
- For any subset $A \subset X$, A^C will denote the complement of A (relative to X). That is $A^C = \{x \in X \mid x \notin A\}$.
- $M_n(F)$ is the associative algebra of $n \times n$ matrices over F.
- $GL_n(F)$ is the group of $n \times n$ invertible matrices over F.
- $\mathfrak{gl}_n(F)$ is the Lie algebra of $n \times n$ matrices over F.

- $\mathfrak{sl}_n(F) \subset \mathfrak{gl}_n(F)$ is the Lie subalgebra of trace 0 matrices.

- $E_{i,j} \in M_n(F)$ is the matrix whose i,j-th entry is 1 and all other entries are 0.

- For matrices $A = [a_{ij}] \in M_n(F)$ and $B \in M_m(F)$, we define

$$A \oplus B = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \in M_{n+m}(F)$$

  and

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \in M_{n\cdot m}(F)$$

- If V is a F-vector space, $\mathrm{End}_F(V)$ will denote the algebra of linear transformations from V to itself. If there is no confusion over the base field F, we will write $\mathrm{End}(V)$.

- If X is a group (resp. associate algebra, Lie algebra) and $x \in X$, then $\langle x \rangle \subset X$ is the subgroup (resp. ideal, ideal) generated by x. Similarly, if $A \subset X$, then $\langle A \rangle \subset X$ is the subgroup (resp. ideal, ideal) generated by the elements of A.

0.2 Background

0.2.1 Ultraproducts and Ultrafilters

Both soficity for groups and linear soficity algebras can be formulated in terms of metric ultraproducts. While we will discuss metric ultraproducts in §1.1, we will define general ultrafilters and ultraproducts here.

Definition 0.2.1. For a set X, a filter is a subset $\mathscr{F} \subset \mathscr{P}(X)$ such that

1. $\emptyset \notin \mathscr{F}$;

2. if $A \in \mathscr{F}$ and $B \supset A$, then $B \in \mathscr{F}$; and,

3. if $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$.

An ultrafilter on X is a filter $\mathscr{U}$ such that for every $A \in \mathscr{P}(X)$, either $A \in \mathscr{U}$ or $A^C \in \mathscr{U}$.

Remark 0.2.1. The condition for a filter $\mathscr{U} \subset \mathscr{P}(X)$ is equivalent to $\mathscr{U}$ being a maximal filter with respect to inclusion. That is $\mathscr{U}$ is an ultrafilter if and only if there does not exist another filter $\mathscr{F} \subset \mathscr{P}(X)$ such that $\mathscr{U} \subsetneq \mathscr{F}$.

Generally, the indexing set over which we are considering ultrafilters in this work will be $\mathbb{N}$. In addition, we draw the distinction between two types of ultrafilters: principal and non-principal (or free). An ultrafilter $\mathscr{U} \subset \mathscr{P}(X)$ is called principal if it contains a finite subset (and thus is of the form $\{A \subset X \mid a \in A\}$ for some $a \in X$). While most of the results involving ultrafilters do not dependent on whether the ultrafilter is principal, principal ultrafilters will generally result in trivial cases. Hence, we will generally focus on non-principal ultrafilters.

Jerzy Łoś introduced ultraproducts in their full generality in 1955 [19]. We review their definition here, at least in the restricted case of the indexing set being $\mathbb{N}$:

Definition 0.2.2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of sets and ω an ultrafilter on $\mathbb{N}$. The ultraproduct of the X_n is the quotient

$$\prod_{n\to\omega} X_n := \prod_{n\in\mathbb{N}} X_n / \sim$$

where $\sim$ is the equivalence relation on the Cartesian product given by

$$(a_n) \sim (b_n) \iff \{n \in \mathbb{N} \mid a_n = b_n\} \in \omega$$

Remark 0.2.2. Note that if ω is a principal ultrafilter, say $\omega = \{A \subset \mathbb{N} \mid a \in A\}$, then the ultraproduct is isomorphic to just X_a. Hence, principal ultrafilters result in trivial ultraproducts, which is why we will generally not consider them.

One additional concept related to ultrafilters and ultraproducts that we need to discuss is ultralimits, as they are particularly relevant to metric ultraproducts.

Definition 0.2.3. Let ω be an ultrafilter on $\mathbb{N}$, let (X,d) be a metric space, and let (x_n) be a sequence in X. The element $x \in X$ is the ultralimit of the x_n, denoted as

$$\lim_{n\to\omega} x_n = x$$

if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} \mid d(x_n, x) \leq \varepsilon\} \in \omega$$

Remark 0.2.3. If an ultralimit exists, it is unique. Moreover, if ω is a non-principal ultrafilter, then ultralimits agree with regular limits. That is, if $\lim_{n\to\infty} x_n = x$ in the traditional sense, $\lim_{n\to\omega} x_n = x$ in the ultralimit sense.

0.2.2 Amenable Groups and Algebras

Soficity (and other metric approximations) for groups and linear soficity for associative algebras can be thought of as a generalization of amenability. We recall the definitions of amenability for groups and algebras in this section here, for use later.

While there are many equivalent definitions of amenability for a group, we will focus on the Følner set definition:

Definition 0.2.4. A group G is amenable if for every finite subset $F \subset G$ and $\varepsilon > 0$, there exists a finite subset $\Phi \subset G$ such that

$$|g\Phi \,\Delta\, \Phi| < \varepsilon|\Phi|$$

for every $g \in F$. Here $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of the sets A and B.

Equivalently, if G is countable, we consider instead a Følner sequence of finite subsets $F_1 \subset F_2 \subset \cdots \subset G$ that cover G such that

$$\lim_{n\to\infty} \frac{|gF_n \,\Delta\, F_n|}{|F_n|} = 0$$

for every $g \in G$.

As the Følner sequence definition of amenability is a purely algebraic one, we can define amenability of associative algebras in an analogous way. We give the definition here:

Definition 0.2.5 ([8]). An associative algebra A is amenable if there exists a sequence of finite dimensional subspaces $W_1 \subset W_2 \subset \cdots \subset A$ that cover A such that

$$\lim_{n\to\infty} \frac{\dim(aW_n + W_n)}{\dim(W_n)} = 1$$

for every $a \in A$.

Chapter 1

Metric Approximations of Groups

Sofic groups were first introduced (though not termed as such) by Gromov [13] in relation to Gottschalk's Surjunctivity Conjecture (1976). The term sofic was later introduced by Weiss [28]. The primary purpose of this chapter is to introduce the basic notions of sofic and other metric approximable groups, as well as some of the initial results related to them.

1.1 Metric Ultraproducts

We first begin by discussing metric ultraproducts, extending the concepts introduced in §0.2.1.

Let $\{(G_n, d_n)\}_{n\in\mathbb{N}}$ be a collection of metric groups and let ω be an ultrafilter on $\mathbb{N}$. We want to quotient the Cartesian product

$$\mathscr{G} = \prod_{n\in\mathbb{N}} G_n$$

in such as way as to preserve metrics of each G_n. That is, we want sequences that become arbitrarily small on sets in ultrafilter to become the identity in the ultraproduct.

For each element $(g_n) \in \mathscr{G}$, we can get a sequence (r_n) in $\mathbb{R}$ given by

$$r_n = d_n(g_n, e_n)$$

where $e_n \in G_n$ is the identity element. If the sequence is bounded, then it is guaranteed to have an ultralimit by the Heine Borel theorem. Hence, we can consider the subgroup

$$\mathscr{G}' = \{(g_n) \in \mathscr{G} \mid \sup_n d_n(g_n, e_n) < \infty\}$$

Note that if the metrics are all bounded, then $\mathscr{G}' = \mathscr{G}$. Moreover, we want to quotient out $\mathscr{G}'$ by sequences that become arbitrarily small, i.e. we want to quotient $\mathscr{G}'$ by the subset

$$\mathscr{N} = \{(g_n) \in \mathscr{G}' \mid \lim_{n \to \omega} d_n(g_n, e_n) = 0\}$$

A priori, the subset $\mathbb{N}$ is not necessarily a subgroup, much less a normal subgroup. First note that if each metric d_n is left invariant, that is

$$d_n(ab, ac) = d_n(b, c)$$

for every $a, b, c \in G_n$, then the subset $\mathscr{N}$ is in fact a subgroup. Indeed, for a pair of sequences $(g_n), (h_n) \in \mathscr{G}'$, we have that

$$d_n(g_n h_n, e_n) = d_n(h_n, g_n^{-1}) \leq d_n(h_n, e_n) + d_n(e_n, g_n^{-1}) = d_n(h_n, e_n) + d_n(g_n, e_n)$$

and the rest follows from properties of ultralimits.

If in addition each metric is also right invariant (making each metric thus bi-invariant), then our subgroup $\mathscr{N}$ is normal in $\mathscr{G}'$. Thus, from here we get our definition:

Definition 1.1.1. Let $\{(G_n, d_n)\}_{n \in \mathbb{N}}$ be a family of metric groups with bi-invariant metric, and let ω be an ultrafilter on $\mathbb{N}$. The metric ultraproduct of the G_n is

$$\prod_{n \to \omega} (G_n, d_n) = \mathscr{G}' / \mathscr{N}$$

where $\mathscr{G}'$ and $\mathscr{N}$ retain their definitions from this section.

When the metrics for each group are understood, we will denote the metric ultraproduct implicitly as $\prod_{n\to\omega} G_n$.

Remark 1.1.1. Our metric ultraproduct comes equipped with its own bi-invariant metric

$$d([(g_n)],[(h_n)]) = \lim_{n\to\omega} d_n(g_n,h_n)$$

where $[(a_n)]$ is the equivalence class of $(a_n) \in \mathscr{G}'$. While not relevant to our results, it is worth noting anyway.

1.2 Metric Approximations of Groups

In this section, we will discuss a few of the more relevant metric approximations of groups that appear in the literature. Each of these metric approximations involved embedding a given group into a metric ultraproduct coming from a specific family of bi-invariant metric groups. We give a general definition of this concept here:

Definition 1.2.1. A group H is metric approximable by a family of bi-invariant metric groups $\mathscr{F}$ if there exists a countable collection $\{G_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ and ultraproduct ω on $\mathbb{N}$ such that there exists an embedding

$$\varphi : H \hookrightarrow \prod_{n\to\omega} G_n$$

1.2.1 Sofic Groups

As mentioned previously, sofic groups were first introduced by Gromov [13]. Sofic groups were introduced as a class of groups satisfying Gottschalk's Surjunctivity Conjecture [12].

Let S_n denote the symmetric group on n letters. The normalized Hamming distance d_n^{Ham}

is a bi-invariant metric on S_n given by

$$d_n^{Ham}(\sigma,\tau) = \frac{1}{n}|\{1 \leq i \leq n \mid \sigma(i) \neq \tau(i)\}|$$

Definition 1.2.2. A group G is sofic if it is metric approximable by $\mathscr{F} = \{(S_n, d_n^{Ham})\}_{n\in\mathbb{N}}$.

1.2.2 Hyperlinear Groups

Hyperlinear groups were named by Rădelescu [24], though similar notions of group existed before. Just as sofic groups are related to the Surjunctivity Conjecture, hyperlinear groups are a class of groups for which Connes' Embedding Problem [7] holds.

Let $U(n)$ denoted the $n\times n$ complex unitary matrices, namely

$$U(n) = \{A \in M_n(\mathbb{C}) \mid AA^* = I_n\}$$

The normalized Hilbert-Schmidt metric on $U(n)$ is defined as

$$d_n^{HS}(A,B) = \frac{1}{\sqrt{n}}\sqrt{\operatorname{tr}((A-B)^*(A-B))}$$

Note that this metric is bi-invariant.

Definition 1.2.3. A group G is hyperlinear is it is metric approximable by $\mathscr{F} = \{(U(n), d_n^{HS}\}_{n\in\mathbb{N}}$.

1.2.3 Linearly Sofic Groups

Linearly sofic groups were introduced by Arzhantseva and Păunescu [4]. Similar to sofic and hyperlinear groups, linearly sofic groups are related to another open problem: Kaplansky's Direct Finiteness Conjecture [18]. However, unlike sofic and hyperlinear groups, linearly sofic groups are not known to all satisfy their related conjecture.

Let F be a field. For any matrix $A \in M_n(F)$, we denote the normalized rank of A as

$$\rho_n(A) = \frac{1}{n}\operatorname{rank}(A)$$

This normalized rank function then induces a bi-invariant metric on $GL_n(F)$, which we call the normalized rank metric

$$d_n^{nr}(A,B) = \rho_n(A-B)$$

Definition 1.2.4. A group G is linearly sofic if there exists a field F such that G is metric approximable by $\mathscr{F} = \{(GL_n(\mathbb{C}), \rho_n)\}_{n\in\mathbb{N}}$.

1.2.4 Weakly Sofic Groups

One additional class of metric approximable groups that appears in the literature is weakly sofic groups, introduced by Glebsky and Rivera [11].

Definition 1.2.5. Let $\mathscr{F}$ be the collection of all finite metric groups with bi-invariant metrics. A group G is weakly sofic if it is metric approximable by $\mathscr{F}$.

1.3 Relations Between Metric Approximations

With all the different classes of metric approximable groups, it is worth asking how these classes relate to each other. One relation is immediate:

Proposition 1.3.1. *Any sofic group is weakly sofic.*

The relation between the classes of sofic, hyperlinear, and linearly sofic groups are less immediate. The following two results of this section will demonstrate that sofic groups can be thought of as the more specific class of metric approximable groups. That is, a group that is sofic will also be hyperlinear and linearly sofic.

Before turning to these two results, we recall a basic relation between the symmetric group S_n and the group of invertible matrices $GL_n(F)$, for some field F. For each permutation $\sigma \in S_n$, we can identify a matrix $A_\sigma \in GL_n(F)$ by applying σ to the rows of the identity matrix

I_n. Note that the map

$$\varphi : S_n \to GL_n(F)$$
$$\sigma \mapsto A_\sigma$$

is a group embedding. Additionally, if $F = \mathbb{C}$, then each $A_\sigma \in U(n)$.

Proposition 1.3.2 ([9])**.** *Any sofic group is hyperlinear.*

Proof. It suffices to show that for any $n \in \mathbb{N}$ and ultrafilter ω on $\mathbb{N}$ and sequence $(\sigma_k) \in \prod S_{n_k}$,

$$\lim_{k\to\omega} d_{n_k}^{Ham}(\sigma_k, e) = 0 \iff \lim_{k\to\omega} d_{n_k}^{HS}(A_{\sigma_k}, I_{n_k}) = 0$$

The forward direction of this equivalence allows us to extend the embedding φ from before to a morphism

$$\overline{\varphi} : \prod_{k\to\omega} S_{n_k} \to \prod_{k\to\omega} U(n_k)$$

and the reverse direction guarantees that this morphism is an embedding.

Suppose $\sigma \in S_n$. We calculate

$$\begin{aligned} d_n^{Ham}(\sigma, e) &= \frac{1}{n}|\{i \mid \sigma(i) \neq i\}| \\ &= \frac{1}{n}\mathrm{tr}(I_n - A_\sigma) \\ &= \frac{1}{2n}(\mathrm{tr}(I_n - A_\sigma) + \mathrm{tr}(I_n - A_{\sigma^{-1}})) \\ &= \frac{1}{2n}\mathrm{tr}((I_n - A_\sigma)(I_n - A_\sigma)^*) \\ &= \frac{1}{2}(d_n^{HS}(A_\sigma, I_n))^2 \end{aligned}$$

From this equality, equivalence of zero ultralimits for both metrics follows immediately. □

Proposition 1.3.3 ([4])**.** *Any sofic group is linearly sofic.*

Proof. We proceed as in the previous proof. Let $\sigma \in S_n$. Denote the number of fixed points of σ by $fix(\sigma)$ and the number of disjoint cycles (including one letter cycles) of σ by $cyc(\sigma)$. Thus

$$d_n^{Ham}(\sigma, e) = 1 - \frac{fix(\sigma)}{n}$$

and from Lemma 13 of [20]

$$d_n^{nr}(A_\sigma, I_n) = 1 - \frac{cyc(\sigma)}{n}$$

As $fix(\sigma) \leq cyc(\sigma)$ we have that

$$d_n^{nr}(A_\sigma, I_n) \leq d_n^{Ham}(\sigma, e)$$

As every non-fixed point cycle contains at least two elements, we have that

$$cyc(\sigma) \leq fix(\sigma) + \frac{n - fix(\sigma)}{2} \iff 1 - \frac{fix(\sigma)}{n} \leq 2\left(1 - \frac{cyc(\sigma)}{n}\right)$$

It follows that

$$d_n^{Ham}(\sigma, e) \leq 2 d_n^{nr}(A_\sigma, I_n)$$

and so the proof is complete. □

1.4 Almost Homomorphisms

Our definition for metric approximability (Definition 1.1.1) appears to depend on a choice of ultrafilter ω on $\mathbb{N}$. Namely, it might be possible for a group G to embed in some metric ultraproduct over ω but not necessarily over some other ultrafilter ω'. To what extent is this situation possible? To answer this question, it is possible to reformulate the notion of metric approximability without using metric ultraproducts. Though it is possible to do for all the metric approximable classes we have discussed previously, we will focus on sofic groups as the other classes are the same, mutatis mutandis.

For a sofic group G, the embedding $\varphi : G \to \prod_{k\to\omega} S_{n_k}$ can be thought of as approximating G by successively larger symmetric groups. That is, for every k, we can lift φ to a map from G the direct product of the symmetric groups and then project onto the k-th component of the product to give a map (which is not necessarily a morphism) $\psi_k : G \to S_{n_k}$. For two elements $g,h \in G$, in general $\psi_k(gh) \neq \psi_k(g)\psi_k(h)$. However, from the construction of the metric ultraproduct, if we pick some $\varepsilon > 0$, there exists some $S_\varepsilon \in \omega$ such that for any $m \in S_\varepsilon$,

$$d_{n_m}^{Ham}(\psi_m(gh), \psi_m(g)\psi_m(h)) < \varepsilon$$

In other words, for any pair of elements, we can guarantee that the distance between the product of their images and the image of their product in our approximations are sufficiently small by choosing sufficiently ”large” (with respect to ω) indices. If, for every finite subset $F \subset G$, we create new maps $\psi_{k,F} = \psi_k|_F : F \to S_{n_k}$, we can guarantee that for any $\varepsilon > 0$, there exist some map such that the distance between product of images and images of products are bounded by ε for any pair in the domain. We sum up these ideas in the following definition:

Definition 1.4.1. Let G be a group, $F \subset G$ a finite subset, $\varepsilon > 0$, and $n \in \mathbb{N}$. A map $\psi : F \to S_n$ is called an (F,ε)-almost homomorphism if:

1. for any pair $g,h \in F$ such that $gh \in F$, $d_n^{Ham}(\psi(gh), \psi(g)\psi(h)) < \varepsilon$; and,
2. if $e_G \in F$, then $d_n^{Ham}(\psi(e_G), e_{S_n}) < \varepsilon$

If the subset F is clear, ψ is also referred to as just an ε-almost homomorphism.

As mentioned in the previous paragraphs, the existence of a morphism from G to a metric ultraproduct of symmetric groups gives rise to an (F,ε)-almost homomorphism for every finite subset $F \subset G$ and $\varepsilon > 0$. Additionally, the converse of this statement also holds. To see this, consider a sequence of finite subsets $F_1 \subset F_2 \subset \cdots \subset G$. For each $k \in \mathbb{N}$, let $\psi_k : F_k \to S_{n_k}$ be and $(F_k, 1/k)$-almost homomorphism. We extend each ψ_k to a map $\varphi_k : G \to S_{n_k}$ via $\varphi_k(G \setminus F) = e_{S_{n_k}}$.

Finally, let ω be a non-principal ultra on $\mathbb{N}$. Then the map

$$\varphi : G \to \prod_{k \to \omega} S_{n_k}$$
$$g \mapsto [(\varphi_k(g))]$$

is in fact a group homomorphism. This is due to the fact that ω must contain all cofinite subsets of $\mathbb{N}$.

Note that the morphism constructed does not depend on the choice of ultrafilter. In fact, by the previous paragraphs, if G has a morphism to an ultraproduct of symmetric groups over ω, then it has a morphism to the ultraproduct of the same symmetric groups over any other ultrafilter of $\mathbb{N}$. Thus, the existence of our morphism does not depend on the choice of ultrafilter.

However, in the previous paragraphs, the morphism defined from the almost homomorphisms is not an embedding. There is no condition for almost homomorphisms to prevent a non-identity element in G from having a trivial image, that its for

$$\lim_{k \to \omega} d_n^{Ham}(\varphi_k(g), e_{S_{n_k}}) = 0$$

In fact, for any finite subset $F \subset G$ and $\varepsilon > 0$, the trivial map $\chi : F \to S_n$ is an (F, ε)-almost homomorphism, and taking the trivial map for each of finite subsets would induce the trivial map from G to the ultraproduct. So we need an additional condition on our almost homomorphisms to guarantee injectivity in our constructed morphism. We give this condition in the following result:

Theorem 1.4.1 ([9])**.** *A group G is sofic if and only if for every finite subset $F \subset G$ and $\varepsilon > 0$, there exists an (F, ε)-almost homomorphism $\varphi : F \to S_n$ such that for every distinct $x, y \in F$,*

$$d_n^{Ham}(\varphi(x), \varphi(y)) \geq \frac{1}{4}$$

With this additional condition on the almost homomorphisms, the constructed homomorphism from G to a metric ultraproduct maps distinct elements to values of distance at least $\frac{1}{4}$ from each other. Hence, the homomorphism maps distinct elements to distinct elements and is thus an embedding. The converse of the statement is not immediate, as the embedding into a metric ultraproduct only guarantees that the associated almost homomorphisms map distinct elements to distinct elements, not necessarily elements at distance at least $\frac{1}{4}$. However, one can use amplification from functional analysis to increase the distance between elements in the codomain. This isn't relevant to the rest of this work, so we will omit it.

From the almost homomorphism characterization of soficity, it is possible to derive classes of sofic groups. For example, the following classes of groups are sofic:

1. finite groups;
2. residually finite groups;
3. nonabelian free groups; and,
4. amenable groups.

Of special note is the family of almost homomorphisms for amenable groups. Give an amenable group G, a finite subset $F \subset G$, and $\varepsilon > 0$, there exists another finite subset $K \subset G$ such that

$$|gK \,\Delta\, K| < \varepsilon |K| \quad \forall g \in F$$

Hence we can construct an almost homomorphism $\varphi : F \to S_K$ where $\varphi(g)$ is the extension of the map $k \mapsto gk$ to a bijection from K to itself. In particular, the almost homomorphisms of G are induced by its own group action on itself. This will be relevant in the following section on permanence properties.

1.5 Permanence Properties

Theorem 1.5.1 ([10])**.** *The class of sofic groups is closed under the following constructions:*

1. *direct products, subgroups, inverse limits, and direct limits;*

2. *free products; and,*

3. *certain extensions: if $N \subset G$ is a normal subgroup such that N is sofic and G/N is amenable, then G is also sofic.*

While parts 1. and 2. are not relevant for the later work on Lie algebra, we will give a basic overview of the proof for part 3. This is to better compare with our similar result for Lie algebras (Theorem 5.3.1).

Let $\pi : G \to G/N$ be the canonical projection and let $\sigma : G/N \to G$ be a section of π. That is, $\pi(\sigma(h)) = h$ for every $h \in G/N$. Choose some finite set $F \subset G$ and $\varepsilon > 0$. We want to construct an (F, ε)-almost homomorphism satisfying the injectivity condition in Theorem 1.4.1.

As G/N is amenable, we can choose a non-empty finite subset $B \subset G/N$ with the property that

$$|\pi(g)B \setminus B| \leq \varepsilon |B|$$

for every $g \in F$. Let $A = \sigma(B)$ and define $H = N \cap (A \cdot F \cdot A^{-1})$ where

$$A^{-1} = \{a^{-1} \mid a \in A\}$$

Choose an $\left(H, \frac{\varepsilon}{3}\right)$-almost homomorphism and extend it to a map $\psi : N \to S_X$, where X is some finite set. Then we can define a map $\Phi : G \to S_{X \times A}$ via

$$\Phi(g)(x,a) = \begin{cases} (\psi(ga\,\sigma(\pi(ga))^{-1})(x), \sigma(\pi(ga))), & \text{when } \pi(ga) \in B \\ (x,a), & \text{otherwise} \end{cases}$$

Note that since σ is a section, $g\sigma(g)^{-1} \in N$ for every $g \in G$. Hence, Φ is well-defined. From here, it is possible to verify that $\Phi|_F$ is an (F, ε)-almost homomorphism that satisfies the injectivity condition for soficity. However, this verification is long and not particularly relevant to the later work, and so we shall omit it.

The importance of the amenability of G/N is that it allows us to take the almost homomorphisms of G/N as being induced by the left multiplication action of G/N on itself. This allows us to combine the almost homomorphisms coming from G/N and N by using the almost homomorphism of G/N to modify the almost homomorphism of N. When we prove a similar result for Lie algebra in a later chapter, we will use a similar construction as this almost homomorphism.

Chapter 2

Linear Soficity in Associative Algebras

While different classes of metric approximable groups have appeared throughout the literature, metric approximable algebras are much less common, at least from a purely algebraic perspective. In fact, the concept of linearly sofic associative algebras was not introduced until 2017 [4]. However, our work on metric approximable Lie algebras is built directly upon the theory of metric approximable algebras, and thus understanding them is critical to understanding the later work. To that end, the goal of this chapter is to give a basic overview of linearly sofic associative algebras, as well as results in the theory related to our results for Lie algebras.

2.1 Universal Linearly Sofic Associate Algebras

As in the case of groups, linear soficity for associative algebras can be first formulated using a similar notion of metric ultraproducts. We give the basic construction here.

Let F be a field. For $n \in \mathbb{N}$, we define the normalized rank function $\rho_n : M_n(F) \to [0,1]$ via

$$\rho_n(A) = \frac{1}{n}\text{rank}(A)$$

If there is no risk of confusion on the size of the matrices, then we will denote ρ_n by simply ρ.

Let ω be an ultrafilter on $\mathbb{N}$ and let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers such that

$\lim_{k\to\infty} n_k = \infty$. We can extend the normalized rank function ρ_n to a map

$$\rho_\omega : \prod_{k=1}^{\infty} M_{n_k}(F) \to [0,1]$$
$$(A_k) \mapsto \lim_{k\to\omega} \rho_{n_k}(A_k)$$

Using some abuse of notation, denote $\ker\rho_\omega := \rho_\omega^{-1}(\{0\})$. Then since

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B) \text{ and } \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

for every pair $A, B \in M_n(F)$, we have that $\ker\rho_\omega$ is an ideal of the full direct product. We will denote the algebra

$$\left(\prod_{k=1}^{\infty} M_{n_k}(F)\right) / \ker\rho_\omega = \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

Hence, we arrive at our metric ultraproduct:

Definition 2.1.1. A universal linearly sofic associative algebra over a field F is an algebra of the form

$$A = \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

for some ultrafilter ω on $\mathbb{N}$ and sequence of natural numbers n_k.

Just as in the case of groups, this allows us to define linear soficity.

Definition 2.1.2. An associative F-algebra A is linearly sofic if A embeds into a universal linearly sofic associative F-algebra. Moreover, if A is unital, then this embedding must also be unital.

We now go over a few properties of universal linearly sofic associative algebras that can highlight the structure of linearly sofic associative algebras. We begin with a new result:

Proposition 2.1.1. *For an algebraically closed field F, the algebra $\prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$ is simple.*

Proof. Denote

$$A := \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

Suppose $0 \neq x \in A$. Let $\delta := \rho_\omega(x)$. Since $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\delta > n^{-1}$. Hence, if $(B_k) \in \prod_k M_{n_k}(F)$ such that $[(B_k)] = x$, there exists some $S \in \omega$ such that

$$\rho_{n_k}(B_k) \geq \frac{1}{n}$$

for every $k \in S$.

Let J_k be the normal form of B_k where $k \in S$. Then the J_k is row equivalent to

$$C_k^{(m)} := \sum_{i=1}^{\lfloor \frac{n_k}{n} \rfloor} E_{\lfloor \frac{mn_k}{n} \rfloor + i, \lfloor \frac{mn_k}{n} \rfloor + i}$$

for any $0 \leq m \leq n-1$. Hence, there exists some $P_k^{(m)}, Q_k^{(m)} \in M_{n_k}(F)$ such that

$$C_k^{(m)} = P_k^{(m)} B_k Q_k^{(m)}$$

Define $D_k \in M_{n_k}(F)$ via

$$D_k = \begin{cases} \sum_{m=0}^{n-1} C_k^{(m)}, & k \in S \\ 0, & k \notin S \end{cases}$$

and let $y = [(D_k)] \in A$. By construction, we have that for any $k \in S$,

$$D_k = \sum_{m=0}^{n-1} P_k^{(m)} B_k Q_k^{(m)}$$

and for $k \notin S$,

$$D_k = 0B_k$$

Hence

$$(D_k) = \sum_{m=0}^{n-1} (P_k^{(m)})(B_k)(Q_k^{(m)})$$

where we define $P_k^{(m)} = Q_k^{(m)} = 0$ for $k \notin S$. Therefore

$$y = \sum_{m=0}^{n-1} [(P_k^{(m)})]x[(Q_k^{(m)})] \in \langle x \rangle$$

For any $k \in S$, we have that

$$\operatorname{rank}(D_k - I_{n_k}) \leq n-1$$

Let $\varepsilon > 0$. Since ω is a free ultrafilter, S must be infinite. Therefore, there exists an infinite subset $S' \subset S$ such that $\rho_{n_k}(D_k - I_{n_k}) < \varepsilon$. Notice that $S' \in \omega$ since otherwise

$$(\mathbb{N} \setminus S') \cap S \in \omega$$

making ω a principal ultrafilter. Therefore $\rho_\omega(y - [(I_{n_k})]) = 0$. Thus, $y = [(I_{n_k})]$ which implies that $\langle x \rangle = A$. $\square$

A more significant result for the structure of universal linearly sofic associative algebras has to do with units in the algebra. First, we recall the definition of direct and stable finiteness.

Definition 2.1.3. A unital ring R is called directly finite if for every $x, y \in R$,

$$xy = 1 \iff yx = 1$$

If $M_n(R)$ is directly finite for every $n \in \mathbb{N}$, then R is called stably finite.

Proposition 2.1.2 ([4])**.** *Any universal linearly sofic associative algebra is stably finite.*

Proof. This follows from the fact that for any $A, B \in M_n(F)$, we have that

$$\operatorname{rank}(AB - I_n) = \operatorname{rank}(BA - I_n)$$

and for any $n, m \in \mathbb{N}$,

$$M_m(M_n(F)) \cong M_{mn}(F)$$

□

As stable finiteness is inherited by subalgebras, it follows that any linearly sofic associative algebra must be stably finite. This makes it quite simple to find non-linearly sofic associative algebras, which differs from the difficulty of finding non-examples in the case of groups. For instance, the associative F-algebra

$$A = \langle x, y \mid xy = 1 \rangle$$

is by construction not directly finite, and thus not linearly sofic. All examples of non-linearly sofic associative algebras in the literature are examples of non-stably finite associative algebras.

One additional result we will discuss in this section is the permanence properties of linearly sofic associative algebras:

Proposition 2.1.3 ([4]). *Subalgebras, direct products, and inverse limits of linearly sofic associative algebras are also linearly sofic.*

One thing to note about the permanence properties of linearly sofic associative algebras is that linear soficity is not known to be preserved under extensions. This differs from the case of groups, where soficity (as well as linear soficity) is known to be preserved under extensions by amenable groups (Theorem 1.5.1)

2.2 Connection with Linearly Sofic Groups

As the definitions of linearly sofic groups and linearly sofic associative algebras are very similar, one would hope that these two concepts coincide. In essence, a linearly sofic group should be associated with a linearly sofic associative algebra and from each linearly sofic associative algebra, one should be able to construct a corresponding linearly sofic group. We start with the associative algebra to group direction. First, for any ring R, we denote the group of units of R by $R^\times$.

Proposition 2.2.1 ([4]). *For any field F, ultrafilter ω on $\mathbb{N}$, and sequence of natural numbers (n_k), we have that*

$$\left(\prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega\right)^\times \cong \prod_{k\to\omega} GL_{n_k}(F)$$

In particular, this result shows that if A is a linearly sofic associative $\mathbb{C}$-algebra, then $A^\times$ is a linearly sofic group. Indeed, the embedding

$$\varphi : A \to \prod_{k\to\omega} M_{n_k}(\mathbb{C})/\ker\rho_\omega$$

induces a group embedding

$$\varphi^\times : A^\times \to \prod_{k\to\omega} GL_{n_k}(\mathbb{C})$$

Moreover, if B is a linearly sofic associative F-algebra for a field $F \subset \mathbb{C}$, then $B^\times$ is also linearly sofic as we have a natural embedding

$$\prod_{k\to\omega} M_{n_k}(F) \hookrightarrow \prod_{k\to\omega} M_{n_k}(\mathbb{C})$$

Thus, we have for each linearly sofic associative F-algebra for $F \subset \mathbb{C}$ a corresponding linearly sofic group. The reverse direction is not quite as clear. That is, it is not immediate that every linearly sofic group is associated to some linearly sofic associative $\mathbb{C}$-algebra. The first step to show this relation would be to specify what the associated associative algebra is for a given group. The natural choice is, of course, a group algebra. We recall its definition here.

Definition 2.2.1. Let G be a group and F a field. The group algebra of G over F is the associative algebra $F[G]$ whose basis is given by the elements of G and whose multiplication is defined as

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{\substack{k\in G \\ gh=k}} a_g b_h k$$

Remark 2.2.1. An important fact about group algebras is that if G is a group and A is an associative F-algebra, a group homomorphism

$$\Theta : G \to A^{\times}$$

induces an associative algebra homomorphism

$$\widetilde{\Theta} : F[G] \to A$$

via

$$\widetilde{\Theta}\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \Theta(g)$$

Note that even if Θ is a group embedding, it does not guarantee that the lift to $F[G]$ is also an embedding. Hence, it is not trivial to lift the embedding defining linear soficity for a group to one defining linear soficity for an associative algebra.

Now we can talk about the relation. Due to its similarity to another result we will show for Lie algebras, we will include the entire proof in order to better compare with the Lie algebra case.

Theorem 2.2.2 ([4]). *Let G be a group and let F be a field. If $\Theta : G \to \prod_{k\to\omega} GL_{n_k}(F)$ is a* *group embedding, then there exists an associative algebra embedding*

$$\Phi : F[G] \to \prod_{k\to\omega} M_{m_k}(F)/\ker\rho_\omega$$

.

Proof. Choose a lift of the map Θ to the whole direct product, namely a map

$$\hat{\Theta} : G \to \prod_{k\in\mathbb{N}} GL_{n_k}(F)$$

such that if $\pi : \prod_{k\in\mathbb{N}} GL_{n_k}(F) \to \prod_{k\to\omega} GL_{n_k}(F)$ is the natural projection, then

$$\pi \circ \hat{\Theta} = \Theta$$

For each $k \in \mathbb{N}$, let θ_k be the projection of $\hat{\Theta}$ onto the k-th component of the direct product.

For every $i \in \mathbb{N}$, define a map $\theta_k^i : G \to GL_{n_k^i}(F)$ via

$$\theta_k^i(g) = \underbrace{\theta_k(g) \otimes \cdots \otimes \theta_k(g)}_{i \text{ times}}$$

and set $\Theta^i : G \to \prod_{k\to\omega} GL_{n_k^i}$ to be

$$\Theta^i(g) = [(\theta_k^i(g))]$$

Note that for every i, the map Θ^i is a group homomorphism For $m \geq i$, define $\theta_k^{i,m} : G \to GK_{n_k^m}$ via

$$\theta_k^{i,m}(g) = \theta_k^i(g) \otimes I_{n_k^{m-i}}$$

We define maps $\varphi_k : G \to GL_{n_k^k 2^k}$ via

$$\varphi_k = (\theta_k^{1,k} \otimes I_{2^{k-1}}) \oplus (\theta_k^{2,k} \otimes I_{2^{k-2}}) \oplus \cdots \oplus (\theta_k^{k,k} \otimes I_{2^0}) \oplus (I_{n_k^k} \otimes I_{2^0})$$

Set $\Phi : G \to \prod_{k\to\omega} GL_{n_k^k 2^k}$ to be

$$\Phi(g) = [(\varphi_k(g))]$$

and let $\widetilde{\Phi}$ be its lift to $F[G]$.

Let $f \in F[G]$. We calculate:

$$\begin{aligned}
\rho_\omega(\widetilde{\Phi}(f)) &= \lim_{k\to\omega} \frac{\operatorname{rank}(\widetilde{\varphi}_k(f))}{n_k^k 2^k} \\
&= \lim_{k\to\omega} \frac{1}{n_k^k 2^k} \sum_{i=1}^{k} \operatorname{rank}(\widetilde{\theta}_k^{i,k} \otimes I_{2^{k-i}}(f)) \\
&= \lim_{k\to\omega} \sum_{i=1}^{k} \frac{1}{n_k^k 2^i} \operatorname{rank}(\widetilde{\theta}_k^{i,k}(f)) \\
&= \lim_{k\to\omega} \sum_{i=1}^{k} \frac{1}{n_k^i 2^i} \operatorname{rank}(\widetilde{\theta}_k^{i}(f)) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_\omega(\widetilde{\Theta}^i(f))
\end{aligned}$$

Hence it follows that

$$\ker \widetilde{\Phi} = \bigcap_{i=1}^{\infty} \ker \widetilde{\Theta}^i$$

Now to show injectivity of $\widetilde{\Phi}$, suppose $f \in F[G]$. Assume that f is of the form

$$f = a_1 g_1 + a_2 g_2 + a_3 g_3$$

where each $a_i \neq 0$ and the g_i's are all distinct elements of G. Suppose that $f \in \ker \widetilde{\Phi}$. Then we know that

$$a_1\Theta(g_1)+a_2\Theta(g_2)+a_3\Theta(g_3)=0$$

$$a_1\Theta(g_1)\otimes\Theta(g_1)+a_2\Theta(g_2)\otimes\Theta(g_2)+a_3\Theta(g_3)\otimes\Theta(g_3)=0$$

$$a_1\Theta(g_1)\otimes\Theta(g_1)\otimes\Theta(g_1)+a_2\Theta(g_2)\otimes\Theta(g_2)\otimes\Theta(g_2)+a_3\Theta(g_3)\otimes\Theta(g_3)\otimes\Theta(g_3)=0$$

If we amplify the first equation by $\Theta(g_3)$ and subtract it from the second equation, we get

$$a_1\Theta(g_1)\otimes(\Theta(g_1)-\Theta(g_3))+a_2\Theta(g_2)\otimes(\Theta(g_2)-\Theta(g_3))=0$$

Applying the same methods to equations 2 and 3 gives

$$a_1\Theta(g_1)\otimes\Theta(g_1)\otimes(\Theta(g_1)-\Theta(g_3))+a_2\Theta(g_2)\otimes\Theta(g_2)\otimes(\Theta(g_2)-\Theta(g_3))=0$$

Now amplifying equation 4 with $\Theta(g_2)$ inside the existing tensor product and subtracting it from equation 5 gives

$$a_1\Theta(g_1)\otimes(\Theta(g_1)-\Theta(g_2))\otimes(\Theta(g_1)-\Theta(g_3))=0$$

As $a_1\neq 0$ and $\Theta(g_1)$ is a unit and thus non-zero, we conclude that either $\Theta(g_1)=\Theta(g_2)$ or $\Theta(g_1)=\Theta(g_3)$. As the g_i are distinct, this contradicts the injectivity of Θ. Similar methods can be applied to any non-zero element of $F[G]$. Hence, $\widetilde{\Phi}$ is injective. $\square$

Of particular importance is the construction of the extension $\widetilde{\Phi}$ such that its kernel lies in the intersection of the tensor power of $\widetilde{\Theta}$. We record this as a separate result here for later use, as was done in [4].

Proposition 2.2.3 ([4, Proposition 7.6])**.** *Let $\{\Theta_i\}_i$, $\Theta_i : A\to\prod_{k\to\omega}M_{n_{i,k}}(F)/\ker\rho_\omega$ be a sequence of homomorphisms for an associative F-algebra A. Then there exists a homomorphism*

$\Phi : A \to \prod_{k\to\omega} M_{m_k}(F)/\ker\rho_\omega$ *such that*

$$\rho_\omega(\Phi(x)) = \sum_{i=1}^{\infty} \frac{1}{2^i}\rho_\omega(\Theta_i(x))$$

for any $x \in A$. *In particular,* $\Phi(x) = 0$ *if and only if* $\Theta_i(x) = 0$ *for all i. Moreover, if each* Θ_i *is a unital homomorphism, then* Φ *can be taken to be unital as well.*

An additional aspect to pay attention in this proof is the $\widetilde{\Theta}^i$ sequence from which the embedding was constructed. Essentially, each $\widetilde{\Theta}^i$ was a successive application of tensor of powers of $\widetilde{\Theta}$. Where these tensor powers come from seems to be the comultiplication of $F[G]$.

To be more specific, $F[G]$ come naturally equipped with a Hopf algebra structure, and in particular a comultiplication map

$$\Delta : F[G] \to F[G] \otimes F[G]$$

given by $\Delta(g) = g \otimes g$ for every $g \in G$. Thus, each $\widetilde{\Theta}^i$ can be thought of as successively applying the comultiplication map Δ i-times on $\widetilde{\Theta}$.

Given a Lie algebra, the natural was to construct an associative algebra is through the use of the universal enveloping algebra. Of note here is that the universal enveloping algebra also has a Hopf algebra structure, just like a group algebra, and thus comes with its own comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

for any x in the Lie algebra. We will see later when looking at the relation between linear soficity in Lie algebras and associative algebras, a similar construction of an embedding coming from a sequence of successive comultiplications will appear.

However, there will be one difficulty present in Lie algebras that does not appear in group algebras. First, we define two special cases of elements in a Hopf algebra.

Definition 2.2.2. Let H be a Hopf algebra with comultiplication Δ. A group like element is an element $g \in H$ such that

$$\Delta(g) = g \otimes g$$

A primitive element is an element $x \in H$ such that

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

We can see that in a group algebra, the elements of the group are group like, and in a universal enveloping algebra, the elements of the Lie algebra are primitive. However, the converse doesn't hold the same in both cases. For a group algebra $F[G]$, the set of group like elements is precisely the group G. However, for a universal enveloping algebra $U(L)$, the set of primitive elements is equal to L only if the characteristic of the base field is 0. This inability to move backwards from universal enveloping algebras to their corresponding Lie algebra through the Hopf structure alone will present difficulties in our similar result for Lie algebras over positive characteristic fields.

Additionally, this result gives the relation between linear soficity in groups and associative algebras we were looking for:

Corollary 2.2.3.1. *A group G is a linearly sofic group if and only if $\mathbb{C}[G]$ is a linearly sofic associative algebra.*

2.3 Almost Representations

As in the case of groups, linear soficity for associative algebras can be equivalently defined using sequences of local approximations of the algebra. As the work in this section is very similar to that of §3.3, we will omit proofs of the results for this section. To begin, we define what we mean by local approximations, using a definition by Elek:

Definition 2.3.1 ([9])**.** A unital F-algebra A has almost finite dimensional representations if for any finite dimensional subspace $1 \in E \subset A$ and $\varepsilon > 0$, there exists a finite dimensional F-vector space V with subspace $V_\varepsilon \subset V$ and linear map $\varphi : E \to \mathrm{End}_F(V)$ such that

1. $\varphi(1) = Id_V$;

2. for any $a, b \in E$ such that $ab \in E$,

$$\varphi(ab)(v) = \varphi(a)\varphi(b)(v)$$

   for every $v \in V$; and,

3. $\dim V - \dim V_\varepsilon < \varepsilon \cdot \dim V$.

The map φ in this case is called an (E, ε)-almost representation or, if there is no confusion about the subspace E, an ε-almost representation. In the case where A is not unital, we drop condition 1.

Remark 2.3.1. Unlike the case of groups, a given unital algebra A is not guaranteed to have almost finite dimensional representations. This is because groups always have the trivial map into any other group, whereas not every unital associative algebra has a unital algebra morphism into another algebra.

In essence, an almost representation of E is a linear map into $\mathrm{End}(V)$ such that on the subspace V_ε, the map ”acts like an algebra morphism,” in the sense that it respects the multiplication in E. As with the case of groups, existence of almost representations guarantees a map to a universal linearly sofic associative algebra.

Proposition 2.3.1 ([4])**.** *A unital algebra A has almost finite dimensional representations if and only if there exists a unital morphism from A to some universal linearly sofic associative algebra.*

The proof of this result is essentially the same as in groups. For a unital associative algebra A, you can take a sequence

$$\{1\} \subset E_1 \subset E_2 \subset \cdots \subset A$$

and then take a family of $\frac{1}{n}$-almost representations

$$\varphi_n : E_n \to \operatorname{End}(V_n) \cong M_{\dim V_n}(F)$$

to construct a unital morphism. The converse comes from lifting the morphism to a unital linearly sofic associative algebra to a map to the direct product and then projecting onto each component.

A priori, this unital morphism is not necessarily injective, at no condition was put on the images of elements in the almost representations to prevent them from approaching 0. Hence, we need another condition on the almost representations to prevent this condition. To this end, we introduce the sofic radical to encompass elements for which this condition is satisfied.

Definition 2.3.2 ([4])**.** Let A be a unital associative algebra. If A does not have almost finite dimensional representations, then the sofic radical of A is defined as $SR(A) = A$. Otherwise, $p \in SR(A)$ if for any $\delta > 0$, there exists a finite dimensional subspace E with $\{1, p\} \subset E$ and there exists $n_\delta > 0$ such that if $0 < \varepsilon < n_\delta$ and $\varphi : E \to \operatorname{End}(V)$ is an ε-almost representation then

$$\dim \operatorname{Im} \varphi(p) < \delta \cdot \dim V$$

Here $\operatorname{Im} \varphi(p)$ denotes the image of the map $\varphi(p) : V \to V$.

The sofic radical can be thought of as the "bad" elements of A, at least in regard to linear soficity. Essentially, the images of elements of the sofic radical become increasing small as the error bound on the almost representations become very small. Thus, in a morphism into a universal linearly sofic associative algebra, the image sequences have an ultralimit of 0. This idea is summed up in the following result:

Proposition 2.3.2 ([4]). *Let A be a unital associative F-algebra. Then $p \in SR(A)$ if and only if for any unital morphism*

$$\Theta : A \to \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

we have that $p \in \ker\Theta$.

From here we deduce a corollary which motivates calling the sofic radical a radical:

Corollary 2.3.2.1 ([4]). *For an associative algebra A, $SR(A)$ is an ideal and $SR(A/SR(A)) = \langle 0\rangle$.*

This follows from the fact that the sofic radical is the intersection of the kernels of all morphisms from A to a universal linearly sofic associative algebra. Additionally, the sofic radical allows us to characterize linear soficity without the use of ultraproducts for associative algebras.

Theorem 2.3.3 ([4]). *An algebra A is linearly sofic if and only if $SR(A) = \langle 0\rangle$.*

Remark 2.3.2. Since showing the sofic radical of an algebra is 0 is sufficient to showing an associative algebra is linearly sofic, it is useful to have the negation of the definition of a sofic radical. To that end, we provide the definition here for use later. An element $p \notin SR(A)$ if and only if there exists $\delta > 0$ such that for any finite dimensional subspace $\{1, p\} \subset E \subset A$ and $n > 0$, there exists $0 < \varepsilon < n$ and an ε-almost representation $\varphi : E \to \mathrm{End}(V)$ such that

$$\dim \mathrm{Im}\,\varphi(p) \geq \delta \cdot \dim V$$

2.4 Examples of Linearly Sofic Associative Algebras

Using the sofic radical, it is possible to show that the following classes of associative algebras are linearly sofic:

1. simple unital algebras with almost finite dimensional representations;

2. simple local embeddable in finite algebras; and,

3. amenable (Definition 0.2.5) algebras without zero divisors.

Of particular importance to our work is the linear soficity of amenable algebras. This is due to the universal enveloping algebra of any Lie algebra having no zero divisors, and thus being linearly sofic whenever it is amenable. For this reason, we reproduce the proof of this result.

Proposition 2.4.1 ([4])**.** *Any amenable associative algebra without zero divisors is linearly sofic.*

Proof. Let A be an amenable associative algebra without zero divisors. Let $\{1\} \subset E_1 \subset E_2 \subset \cdots \subset A$ be a sequence of finite dimensional subspaces covering A. Choose a sequence of positive real numbers (ε_k) such that

$$v_k := \varepsilon_k \cdot \dim E_k \overset{k\to\infty}{\to} 0$$

As A is an amenable algebra, for any k there exists a finite dimensional subspace $S_k \subset A$ such that

$$\dim(aS_k \cap S_k) > (1-\varepsilon_k) \cdot \dim S_k$$

for every $a \in E_k$. Thus, for every $a \in E_k$, we can define a linear map $\varphi_k(a) \in \mathrm{End}(S_k)$ via

$$\varphi_k(a)(s) = as$$

for any $s \in aS_k \cap S_k$ and then extending arbitrarily to a linear transformation on all of S_k. Thus $\varphi_k(a)$ acts like left multiplication by a on a subspace of S_k of dimension at least $(1-\varepsilon_k) \cdot \dim S_k$. It follows that $\varphi_k : E_k \to \mathrm{End}(S_k)$ is a v_k-almost representation. Additionally, as A has no zero divisors, left multiplication by a is injective. Hence,

$$\dim \mathrm{Im}\, \varphi_k(a) \geq (1-\varepsilon_k) \dim S_k$$

for every $a \in E_k \setminus \{0\}$. Thus, A is linearly sofic. □

Remark 2.4.1. The importance of the proof of this result is the family of almost representations that characterize soficity for A. In particular, the finite dimensional subspaces can be taken to be subspaces of A itself and the representation action is given by left multiplication on a large subspace given by amenability.

Chapter 3

Linear Soficity in Lie Algebras

3.1 Universal Linearly Sofic Lie Algebras

For a given an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$, ultrafilter ω on $\mathbb{N}$, and field F, we can construct a universal linearly sofic associative F-algebra

$$A = \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

As an associative algebra, it has an associated commutator Lie algebra $A^{(-)}$, which we denote as

$$\prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega := A^{(-)}$$

That allows us to define

Definition 3.1.1. For a field F, a universal linearly sofic Lie algebra is and algebra of the form

$$L = \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$$

for some increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and ultrafilter ω on $\mathbb{N}$.

For any sequence of subalgebras $L_{n_k} \subset \mathfrak{gl}_{n_k}(F)$, we can consider the subalgebra

$$\prod_{k\to\omega} L_{n_k}/\ker\rho_\omega \subset \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$$

that just consists of equivalence classes of elements of $\prod_k \mathfrak{gl}_{n_k}(F)$ that have representatives which are sequences in $\prod_k L_{n_k}(F)$. Due to the nature of the metric approximations, we have the following result

Proposition 3.1.1. *For any field F, increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and ultrafilter ω on $\mathbb{N}$, we have that*

$$\prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega \cong \prod_{k\to\omega} \mathfrak{sl}_{n_k}(F)/\ker\rho_\omega$$

Proof. Let $L := \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$ and $M := \prod_{k\to\omega} \mathfrak{sl}_{n_k}(F)/\ker\rho_\omega$. We have the natural inclusion $\varphi : M \hookrightarrow L$. Suppose $[(A_k)] \in L$. For each $k \in \mathbb{N}$, consider the element

$$B_k = A_k - \mathrm{tr}(A)E_{1,1} \in \mathfrak{sl}_{n_k}(F)$$

Notice that

$$\rho_\omega(A_k - B_k) = \lim_{k\to\omega} \rho_{n_k}(\mathrm{tr}(A)E_{11}) = \lim_{k\to\omega} \frac{1}{n_k} = 0$$

Thus

$$[(A_k)] = [(B_k)] = \varphi([(B_k)])$$

Hence the natural inclusion is an isomorphism of Lie algebras. $\square$

Additionally, from Proposition 2.1.1 and [14, Theorem 2], we have the following result

Proposition 3.1.2. *For an algebraically closed field F, the nontrivial proper ideals of a universal linearly sofic Lie algebra are contained in its center.*

3.2 Linearly Sofic Lie Algebras

As in the case of groups and associative algebras, we define linear soficity for Lie algebras via embeddings into universal linearly sofic objects

Definition 3.2.1. A Lie algebra is linearly sofic if it is embeddable into a universal linearly sofic Lie algebra.

Example 3.2.2. Every finite dimensional Lie algebra is linearly sofic. If L is a finite dimensional Lie F-algebra, by Ado's Theorem there exists an $n \in \mathbb{N}$ and an embedding in $\varphi : L \hookrightarrow \mathfrak{gl}_n(F)$. Thus, for any ultrafilter ω on $\mathbb{N}$, we get an embedding

$$\begin{aligned} L &\to \prod_{k\to\omega} \mathfrak{gl}_n(F)/\ker\rho_\omega \\ x &\mapsto [(\varphi(x))] \end{aligned}$$

3.3 Almost representations

As in the case of groups and associative algebras, linear soficity in Lie algebras can be reformulated via families of locally defined maps that are similar to a Lie algebra homomorphism, up to some small error. First, we must be precise in what we mean by homomorphism up to small error. This brings up to our definition:

Definition 3.3.1. Let L be a Lie algebra, $W \subset L$ a finite dimensional subspace, V a finite dimensional vector space, and $\varepsilon > 0$. A linear map $\varphi : W \to \mathfrak{gl}(V)$ is called and ε-almost representation of W if there exists a subspace $V_\varepsilon \subset V$ such that

1. for every $x, y \in W$ such that $[x, y] \in W$ and $v \in V_\varepsilon$,

$$\varphi([x,y])(v) = [\varphi(x), \varphi(y)](v)$$

2. $\dim V - \dim V_\varepsilon \leq \varepsilon \dim V$

Example 3.3.2. Given a homomorphism $\varphi : L \hookrightarrow \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$, for every finite subspace $W \subset L$ and $\varepsilon > 0$, one can construct an ε-almost representation of W. Precisely, one can construct a map $\psi_k : L \to \mathfrak{gl}_{n_k}(F)$ which is just a lift of φ to the full Cartesian product projected onto the k-th coordinate. As φ is a Lie algebra homomorphism and W is finite dimensional, there exists $k \in \mathbb{N}$ such that we can construct of ψ_k such that

$$\rho_{n_k}(\psi_k([x,y]) - [\psi_{n_k}(x), \psi_{n_k}(y)]) < \varepsilon$$

for every $x, y \in W$ such that $[x,y] \in W$. Hence, ψ_{n_k} is an ε-almost representation of W.

From the example, morphisms to universal linearly sofic Lie algebras give rise to families of almost representations up to arbitrary small errors. By a similar argument, if for every $\varepsilon > 0$ a Lie algebra has an ε-almost representation for all of its finite dimensional subspaces, then the Lie algebra has a morphism to a universal linearly sofic Lie algebra. However, there is no guarantee that this morphism is going to be an embedding. In the case of groups and associative algebras, additional conditions were required for their almost homomorphisms and representations to guarantee linear soficity.

The remainder of this section will be dedicated to giving such a condition to almost representations of Lie algebras.

Definition 3.3.3. For a Lie algebra L, the linear sofic radical of L, denoted by $SR(L)$, is defined as follows: $p \in SR(L)$ if for every $\delta > 0$, there exists a finite dimensional subspace $W \subset L$ containing p and $n_\delta > 0$ such that if $0 < \varepsilon < n_\delta$ and $\varphi : W \to \mathfrak{gl}(V)$ is an ε-almost representation, then

$$\dim \operatorname{Im} \varphi(p) < \delta \cdot \dim V$$

The linear sofic radical encapsulates the elements of our Lie algebra that are considered ”bad” as far as trying to make an embedding into a universal linearly sofic Lie algebra is concerned. The elements of the linear sofic radical are the ones whose images under any morphism to the metric ultraproduct will, once our error is forced to be sufficiently small, have vanishing normalized rank, and thus have image equal to 0. This idea is formalized in the following result.

Lemma 3.3.1. *For a Lie algebra L, $p \in SR(L)$ if and only if for every Lie algebra homomorphism of the form*

$$\Theta : L \to \prod_{k \to \omega} \mathfrak{gl}_{n_k}(F)/\ker \rho_\omega$$

we have that $\Theta(p) = 0$.

Proof. Suppose that $p \in SR(L)$ and let $\Theta : L \to \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker \rho_\omega$ be a Lie algebra homomorphism. Choose a lift $\theta : L \to \prod_k \mathfrak{gl}_{n_k}(F)$ and for each $k \in \mathbb{N}$, consider the projection $\theta_k : L \to \mathfrak{gl}_{n_k}(F)$. Fix $\delta > 0$ and choose $n_\delta > 0$ and finite dimensional subspace $p \in W \subset L$ from the definition of the linear sofic radical. Choose $0 < \varepsilon < n_\delta$. As Θ is a Lie algebra homomorphism, there exists an $S \in \omega$ such that $\theta_k|_W$ is an ε-almost representation of W for every $k \in S$. Indeed, since for every $x, y \in W$ such that $[x, y] \in W$, we have that

$$\lim_{k\to\omega} \rho_{n_k}(\theta_k([x,y]) - [\theta_k(x), \theta_k(y)]) = 0$$

Thus for every such pair, we can choose an $S_{x,y} \in \omega$ such that

$$\rho_{n_k}(\theta_k([x,y]) - [\theta_k(x), \theta_k(y)]) < \varepsilon$$

for every $k \in S$. Since W is finite dimensional and ω is closed under finite intersection, we can choose $S \in \omega$ that contains all such $S_{x,y}$.

Since $\theta_k|_W$ is an ε-almost representation

$$\dim \operatorname{Im} \theta_k(p) < \delta n_k$$

for every $k \in S$. In other words,

$$\rho_{n_k}(\theta_k(p)) < \delta$$

for every $k \in S$. Since δ was an arbitrary positive real number, we conclude that $\Theta(p) = 0$.

Now suppose that $q \in L \setminus SR(L)$. There exists $\delta > 0$ such that for any finite dimensional subspace $q \in W \subset L$ and $n > 0$, there exists $0 < \varepsilon < n$ and an ε-almost representation $\varphi : W \to \mathfrak{gl}(V)$ such that

$$\dim \operatorname{Im} \varphi(q) \geq \delta \cdot \dim V$$

Let W_k be an increasing sequence of finite dimensional subspaces of L containing p such that $\bigcup W_k = L$. Then there exists a sequence of $\varepsilon_k > 0$ that converges to 0 and ε_k-almost representations $\varphi_k : W_k \to \mathfrak{gl}(V_k)$ such that

$$\dim \operatorname{Im} \varphi_k(p) \geq \delta \cdot \dim V_k$$

We define a map $\hat{\Phi} : L \to \prod_k \mathfrak{gl}(V_k)$ by $\hat{\Phi}(x) = (\hat{\varphi}(x))$ where

$$\hat{\varphi}(x) = \begin{cases} \varphi_k(x), & x \in W_k \\ 0, & x \in L \setminus W_k \end{cases}$$

Choose an ultrafilter ω on $\mathbb{N}$ and let $\Phi : L \to \prod_{k \to \omega} \mathfrak{gl}(V_k) / \ker \rho_\omega$ be the composition of $\hat{\Phi}$ with the projection onto the quotient. Since $\varepsilon_k \to 0$, we have that Φ is a Lie algebra homomorphism and

$$\rho_\omega(\Phi(p)) \geq \delta > 0$$

□

From this result, we get the following corollary that motivates the name radical,

Corollary 3.3.1.1. *For a Lie algebra L, $SR(L)$ is an ideal. Moreover, $SR(L/SR(L)) = \langle 0 \rangle$.*

Proof. From Lemma 3.3.1, we have that

$$SR(L) = \bigcap_{\Theta \in X} \ker \Theta \tag{3.1}$$

where X is the set of all Lie algebra homomorphisms from L to a universal linearly sofic Lie algebra. Thus, $SR(L)$ is an ideal.

Now suppose $0 \neq p \in L/SR(L)$ and let $q \in L$ be a pre-image of p under the quotient map $\pi : L \to L/SR(L)$. Since $q \notin SR(L)$, there exist a Lie algebra homomorphism

$$\Theta : L \to \prod_{k \to \omega} \mathfrak{gl}_{n_k}(F)/\ker \rho_\omega$$

such that $\Theta(q) \neq 0$. Since $SR(L) \subset \ker \Theta$, we can get a map

$$\Psi : L/SR(L) \to \prod_{k \to \omega} \mathfrak{gl} n_k(F)/\ker \rho_\omega$$

such that $\Theta = \Psi \circ \pi$. We then have that $\Psi(p) = \Theta(q) \neq 0$ so $p \notin SR(L/SR(L))$ $\square$

We can now use the linearly sofic radical to characterize linear soficity in Lie algebras.

Theorem 3.3.2. *A Lie algebra L is linearly sofic if and only if* $SR(L) = \langle 0 \rangle$.

Proof. Necessity follows from Equation 3.1, so we only concern ourselves with showing sufficiency.

Let L be a Lie algebra such that $SR(L) = \langle 0 \rangle$. Then for every $p \in L \setminus \{0\}$, there exists a Lie algebra homomorphism $\Theta_p : L \to \prod_{k \to \omega} \mathfrak{gl}_{n,p}(F)/\ker \rho_\omega$ such that $\Theta_p(p) \neq 0$.

Let $\{x_i\}_{i \in \mathbb{N}} \subset L$ be a basis for L as a vector space. We shall inductively construct maps $\Psi_m : L \to \prod_{k \to \omega} \mathfrak{gl}_{n_{k,m}}(F)/\ker \rho_\omega$ such that

$$\ker \Psi_m \cap \operatorname{span}\{x_1, \ldots, x_n\} = \langle 0 \rangle$$

for every $m \in \mathbb{N}$.

Let $\Psi_1 = \Theta_{x_1}$. Now suppose that for $m \geq 2$, we have a map Ψ_{m-1} as above. Then,

$$\dim(\Psi_{m-1} \cap \operatorname{span} x_1, \ldots, x_m) \leq 1$$

If the dimension is 0, we set $\Psi_m = \Psi_{m-1}$. Otherwise, choose a non-zero element

$$y_m \in \Psi_{m-1} \cap \operatorname{span} x_1, \ldots, x_m$$

We then set $\Psi_m = \Psi_{m-1} \oplus \Theta_{y_m}$. Ir

$$z \in \Psi_m \cap \operatorname{span} x_1, \ldots, x_m$$

we have that

$$z \in \Psi_{m-1} \cap \operatorname{span} x_1, \ldots, x_m$$

Thus $z = \alpha y_m$ for some $\alpha \in F$. Since $z \in \ker \Theta_{y_m}$ as well, we must have that $\alpha = 0$ so $z = 0$. Thus, we have the desired function Ψ_m.

We now construct a morphism $\Phi : L \to \prod_{k \to \omega} \mathfrak{gl}_{n_k}(F) / \ker \rho_\omega$ such that

$$\ker \Phi \subset \bigcap_{m=1}^{\infty} \ker \Psi_m$$

Let

$$n_k = \prod_{i=1}^{k} n_{k,i}$$

Each component of Ψ_m tensored with an appropriately sized identity matrix gives us maps $\hat{\Psi}_m : L \to \prod_{k \to \omega} \mathfrak{gl}_{\hat{n}_{k,m}}(F) / \rho_\omega$ where $\hat{n}_{k,m} = n_k$ if $m \leq k$ and $n_{k,m}$ otherwise. Let $\hat{\psi}_{k,m} : L \to \mathfrak{gl}_{n_k}(F)$ be a lift of $\hat{\Psi}_m$ for $m \leq k$.

Define a map $\varphi_k : L \to \mathfrak{gl}_{2^k n_k}$ via

$$\varphi_k(x) = (\hat{\psi}_{k,1}(x) \otimes \mathrm{Id}_{2^{k-1}}) \oplus (\hat{\psi}_{k,2}(x) \otimes \mathrm{Id}_{2^{k-2}}) \oplus \cdots \oplus (\hat{\psi}_{k,k}(x) \otimes \mathrm{Id}_1) \oplus \mathrm{Id}_{n_k}$$

We also define $\Phi : L \to \prod_{k \to \omega} \mathfrak{gl}_{2^k n_k}(F)/\ker \rho_\omega$ via $\Phi(x) = [(\varphi_k(x))]$. A direct calculation gives that for any $x \in L$,

$$\begin{aligned}
\rho_\omega(\Psi(x)) &= \lim_{k\to\omega} \frac{\mathrm{rank}(\varphi_k(x))}{2^k n_k} \\
&= \lim_{k\to\omega} \frac{1}{2^k n_k} \sum_{i=1}^{k} \mathrm{rank}(\hat{\psi}_{k,1}(x) \otimes \mathrm{Id}_{2^{k-1}}) \\
&= \lim_{k\to\omega} \frac{1}{n_k} \sum_{i=1}^{k} \frac{\mathrm{rank}(\hat{\psi}_{k,1}(x))}{2^i} \\
&= \lim_{k\to\omega} \sum_{i=1}^{k} \frac{\mathrm{rank}(\psi_{k,1}(x))}{2^i n_{k,i}} \\
&= \sum_{i=1}^{k} \frac{1}{2^i} \rho_\omega(\Psi_i(x))
\end{aligned}$$

where $\psi_{k,m} : L \to \mathfrak{gl}_{n_{k,m}}(F)$ is a lift of Ψ_m. It follows that $\ker \Phi \subset \bigcap \ker \Psi_m = \langle 0 \rangle$, so Φ is an embedding. $\square$

3.4 Examples

We now give examples of Lie algebras which are linearly sofic.

Proposition 3.4.1. *Every abelian Lie algebra is linearly sofic.*

Proof. Suppose that L is an abelian Lie algebra over a field F and suppose that $0 \neq p \in L$. For any finite dimensional subspace $p \in W \subset L$ and $\varepsilon > 0$, we can consider the dual element $p^* : L \to F \cong \mathfrak{gl}_1(F)$. As L and F are both abelian, we have that p^* is a Lie algebra homomorphism and thus $p^*|_W$ is an ε-almost representation of W. Since

$$\dim \mathrm{Im}\, p^*(p) = 1 = \dim F$$

we have that $p \notin SR(L)$. $\square$

In the case of groups, we have that subexponential growth implies soficity (and thus linear soficity). We show that the same holds in the case of Lie algebras:

Theorem 3.4.2. *Any Lie algebra of locally subexponential growth is linearly sofic.*

Proof. Let L be a Lie algebra of locally subexponential growth. In order to show that $SR(L) = \langle 0 \rangle$, we only need to show that we can find appropriate ε-almost representations of finite dimensional subspaces containing $0 \neq p \in L$. Since every finite dimensional subspace is contained in some finitely generated subalgebra of L, we can assume without loss of generality that L is finitely generated. Hence, L is of subexponential growth.

Let $X \subset L$ be a finite generating set. Let $V_n \subset L$ denoted the subspace spanned by words from X of length at most n. We inductively create a basis for L as follows. Choose any basis

$$\{x_1, x_2, \ldots, x_{\dim V_1}\} \subset V_1$$

for V_1. Now suppose that $n \geq 2$ and that we have a basis $B_{n-1} = \{x_1, \ldots, x_{\dim V_{n-1}}\} \subset V_{n-1}$ of V_{n-1}. Let $B = \{x_{\dim V_{n-1}+1}, \ldots, \dim V_n\} \subset L$ be a pre-image of a basis for V_n / V_{n-1}. We then define a basis $B_n = B_{n-1} \cup B$ for V_n. We also define $W_n \subset U(L)$ to be the subspace spanned by words of length at most n from the image of X in $U(L)$ and let $\gamma(n) = \dim W_n$.

For $m > n$, consider the linear map $\varphi_{n,m} : V_n \to \mathfrak{gl}(W_m)$ where

$$\varphi_{n,m}(x_i)(x_{j_1} \cdots x_{j_k}) = \begin{cases} x_i x_{j_1} \cdots x_{j_k}, & x_i x_{j_1} \cdots x_{j_k} \in W_m \\ 0, & \text{otherwise} \end{cases}$$

Notice that for every $v \in W_{m-n}$ and $x, y \in V_n$, we have that

$$\varphi_{n,m}([x,y])(v) = [\varphi_{n,m}(x), \varphi_{n,m}(y)](v)$$

Hence $\varphi_{n,m}$ is an ε-almost homomorphism of V_n if

$$\frac{\gamma(m)-\gamma(m-n)}{\gamma(m)} = \frac{\dim W_m - \dim W_{m-n}}{\dim W_m} < \varepsilon$$

As L is of subexponential growth, we have from [26] that γ is a function of subexponential growth. This implies that

$$\lim_{m\to\infty} \frac{\gamma(m-n)}{\gamma(m)} = 1$$

Indeed, if $f:\mathbb{R}\to\mathbb{R}$ is a function of exponential growth and $d>0$, we have that if

$$\lim_{x\to\infty} \frac{f(x)}{f(x+d)} = \frac{1}{L} < 1$$

then there exists $x_0 \in \mathbb{R}$ such that

$$f(x_0+nd) \geq L^n f(x_0)$$

contradicting the subexponential growth of L. Therefore, for a fixed $n\in\mathbb{N}$ and $\varepsilon>0$, we can choose m sufficiently large such that

$$\frac{\gamma(m)-\gamma(m-n)}{\gamma(m)} < 1-(1-\varepsilon) = \varepsilon$$

and so $\varphi_{m,n}$ is an ε-almost homomorphism of V_n.

Now suppose that $p\in L\setminus\{0\}$ and $V\subset L$ is a finite dimensional subspace containing p. Fix $\varepsilon>0$ and choose $n\in\mathbb{N}$ such that $V\subset V_n$. We can choose a sufficiently large $m\in\mathbb{N}$ such that $\psi=\varphi_{n,m}|_V$ is an ε-almost representation of V. On the subspace W_{m-n}, we have that $\psi(p)$ works like left multiplication by p, when considered as an element of $U(L)$. Hence

$$\dim \operatorname{Im} \psi(p) \geq \dim W_{m-n}$$

By our choice of m, we have that

$$\dim W_{m-n} > (1-\varepsilon)\dim W_m$$

By perhaps enlarging m, we have that

$$\dim \operatorname{Im} \psi(p) \geq \frac{1}{2}\dim W_m$$

Therefore, $p \notin SR(L)$. $\square$

Remark 3.4.1. Note that for the family of finite dimensional subspaces defining linear soficity in the subexponential case, the finite dimensional subspaces can be taken to be subspaces of the universal enveloping algebra. Moreover, the action of the Lie algebra on these finite dimensional subspaces is given by left multiplication in the universal enveloping algebra. This is similar to the case of amenable associative algebras without zero divisors (Proposition 2.4.1). Note that every associative algebra of subexponential growth is amenable [8].

Due to the almost representations being defined locally, we obtain the following corollary:

Corollary 3.4.2.1. *Any Lie algebra of subexponential growth is linearly sofic.*

This result gives an interesting example of an associative algebra that is not linearly sofic, but whose associated Lie algebra is linearly sofic.

Corollary 3.4.2.2. *The algebra $A = \langle x, y \mid xy = 1 \rangle$ is not linearly sofic, but the Lie algebra $A^{(-)}$ is linearly sofic.*

Proof. As the algebra A is not directly finite (since $yx \neq 1$) we that from [4, Proposition 2.8] that A is not linearly sofic. However, the algebra A is of quadratic growth, and thus the Lie algebra $A^{(-)}$ is of subexponential growth. Therefore, $A^{(-)}$ is linearly sofic. $\square$

The remainder of this section is dedicated to providing explicit families of almost representations to show linear soficity for two Lie algebras. While both Lie algebras are of subexponential growth, the constructed families of almost representations are not based on their words subspaces of words of particular lengths as in Theorem 3.4.2.

3.4.1 Witt Algebra

The Witt algebra L_W is the Lie algebra of derivations of the algebra $\mathbb{C}[t,t^{-1}]$. It has a $\mathbb{Z}$-index basis given by

$$x_i = -t^{i+1}\frac{d}{dt}$$

We consider the finite dimensional subspaces

$$V_n = \operatorname{span}\{x_i\}_{i=-n}^{n} \subset L_W$$

and

$$W_n = \operatorname{span}\{t^i\}_{i=-n}^{n} \subset \mathbb{C}[t,t^{-1}]$$

For $m \geq n$, define a linear map $\varphi_{n,m} : V \to \mathfrak{gl}(W_m)$ via

$$\varphi_{n,m}(x_i)(t^j) = \begin{cases} -jt^{i+j}, & j \leq m-i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi_{n,m}(x_{-i})(t^j) = \begin{cases} -jt^{-i+j}, & j \geq i-m \\ 0, & \text{otherwise} \end{cases}$$

for $i \geq 0$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $\varphi_{n,m}$ is an ε-almost representation of V_n for any $m \geq M$. Indeed, we have that on the subspace $W_{m-n} \subset W_m$, $\varphi_{n,m}(x_i)$ acts like derivation by the element x_i. Therefore, for any $x, y \in V_n$ such that $[x, y] \in V_n$ and $w \in W_{m-n}$,

$$\varphi_{n,m}([x,y])(w) = [\varphi_{n,m}(x), \varphi_{n,m}(y)](w)$$

Since

$$\lim_{m\to\infty} \frac{\dim W_m - \dim W_{m-n}}{\dim W_m} = \lim_{m\to\infty} \frac{2m+1-(2m-2n+1)}{2m+1} = 0$$

we have the desired result.

Suppose $0 \neq p \in L_W$, $V \subset L_W$ is a finite dimensional subspace containing p, and $\varepsilon > 0$. We can choose $n \in \mathbb{N}$ such that $V \subset V_n$. Then there exists $m \in \mathbb{N}$ such that $\psi = \varphi_{n,m}|_V$ is an ε-almost representation of V. As $\psi(p)$ acts like derivation by p on W_{m-n}, we have that

$$\dim \operatorname{Im} \psi(p) \geq \dim W_{m-n} - 1$$

By the previous argument, we have that

$$\dim W_{m-n} - 1 > (1-\varepsilon) \dim W_m$$

for sufficiently large m. Hence, by perhaps enlarging m, we can choose ψ such that

$$\dim \operatorname{Im} \psi(p) \geq \frac{1}{2} \dim W_m$$

and so $p \notin SR(L_W)$. Thus $SR(L_W) = \langle 0 \rangle$ and so the Witt algebra is linearly sofic.

3.4.2 Virasoro Algebra

The Virasoro algebra is the unique central extension of the Witt algebra. Explicitly, as a vector space it is of the form

$$L_V = L_W \oplus \mathbb{C}c$$

with the relations

$$[x_i, c] = 0$$
$$[x_i, x_j] = (i-j)x_{i+j} + \frac{1}{12}(i^3 - i)\delta_{m+n,0}c$$

where $\delta_{a,b}$ is the Kronecker delta. For our families of almost representations of L_V, we are going to use approximations of Verma modules of L_V. For more information on Verma modules, see [27].

For an element $0 \neq p \in L_V$, if $p \in L_W \subset L_V$, then from our previous example, for any finite dimensional subspace $p \in V \subset L_V$, we can find an ε-almost representation $\psi : V \to \mathfrak{gl}(W)$ by extending the ε-almost representation $\varphi : V \cap L_W \to \mathfrak{gl}(W)$ via $\psi(x_i) = \varphi(x_i)$ and $\psi(c) = 0$. We then have that

$$\dim \operatorname{Im} \psi(p) = \dim \operatorname{Im} \varphi(x)$$

which gives us that $p \notin SR(L_V)$. So we assume $0 \neq p \in L_V \setminus L_W$.

We consider two subspaces of L_V,

$$\mathfrak{h} = \mathbb{C}x_0 + \mathbb{C}c$$

$$\mathfrak{n}_+ = \sum_{k=1}^{\infty} \mathbb{C}x_i$$

Given $\lambda \in \mathfrak{h}^*$, we define the Verma module to be the space

$$M(\lambda) = U(L_V) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}$$

where the action of $\mathfrak{h} \oplus \mathfrak{n}_+$ on $\mathbb{C}$ is given by

$$(h+x) \cdot \alpha = \lambda(h)\alpha$$

Let $\lambda = c^* \in \mathfrak{h}^*$ For $m, d \in \mathbb{N}$, we consider the subspaces

$$M(\lambda)_{m,d} = \operatorname{span}\{x_{-i_1}x_{-i_2}\cdots x_{-i_k} \otimes 1 \mid 0 \leq k \leq d,\ 0 \leq i_k \leq \cdots \leq i_1 \leq m\}$$

For $m \geq 2n > 0$ and $d > 2$, we define linear maps $\varphi_{n,m,d} V_n + \mathbb{C}c \to \mathfrak{gl}(M(\lambda)_{m,d})$ via

$$\varphi_{n,m,d}(x_r)(x_{-i_1}\cdots x_{-i_r} \otimes 1) = \begin{cases} 0, & r = d \text{ or } k \geq i_1 - m \\ x_k x_{-i_1}\cdots x_{-i_r} \otimes 1, & \text{otherwise} \end{cases}$$

and $\varphi_{n,m,d}(c) = \mathrm{id}_{M(\lambda)_{m,d}}$. Since

$$\dim \operatorname{Im} \varphi_{n,m,d}(c) = \dim M(\lambda)_{m,d}$$

and $\varphi_{n,m,d}(x_i)$ works like the action of L_V on $M(\lambda)$ on a sufficiently large subspace, we only need to show that $\varphi_{n,m,d}$ can be made into an ε-almost homomorphism for some m and d.

Fix $\varepsilon > 0$. Note that on $M(\lambda)_{m+k,d-1} \cap M(\lambda)_{m,d}$, we have that $\varphi_{n,m,d}(x_k)$ acts like $x_k \in L_V$ on $M(\lambda)$. Thus for $v \in M(\lambda)_{m-2n,d-2}$, we have that

$$\begin{aligned} [\varphi_{n,m,d}(x_{k_1}), \varphi_{n,m,d}(x_{k_2})](v) &= x_{k_1}x_{k_2} \cdot v - x_{k_2}x_{k_1} \cdot v \\ &= [x_{k_1}, x_{k_2}] \cdot v \\ &= \varphi_{n,m,d}([x_{k_1}, x_{k_2}])(v) \end{aligned}$$

Moreover, if $v' \in M(\lambda)_{m-2n,d}$ is a monomial of degree d, then both sides of the above equality would be 0. Additionally, if v' is a monomial of degree $d-1$, then the action of x_k would produce a monomial of degree d and the action of an additional $x_{k'}$ would give 0. Thus, the equality holds on $M(\lambda)_{m-2n,d}$.

Fix $n \geq 1$ and $d > 2$. Note that

$$\frac{\dim M(\lambda)_{m,d} - \dim M(\lambda)_{m-2n,d}}{\dim M(\lambda)_{m,d}} = 1 - \frac{\binom{m-2n+d}{d}}{\binom{m+d}{d}} = 1 - \frac{m^d + f_{n,d}(m)}{m^d + g_{n,d}(m)}$$

where $f_{n,d}, g_{n,d} \in \mathbb{Z}[x]$ are polynomials of degree at most $d-1$. Thus,

$$\lim_{m\to\infty} \frac{\dim M(\lambda)_{m,d} - \dim M(\lambda)_{m-2n,d}}{\dim M(\lambda)_{m,d}} = 0$$

Hence, for every $n \geq 1$, by choosing m sufficiently large, we can make $\varphi_{n,m,d}$ an ε-almost homomorphism. So $SR(L_V) = \langle 0 \rangle$ and thus L_V is linearly sofic.

3.5 Restricted Linear Soficity

3.5.1 Restricted Lie Algebras

As linear soficity is defined solely by existence of embeddings into universal linearly sofic algebras, the definition can be extended to similar algebraic objects. In particular, linear soficity can analogously be defined for restricted Lie algebras. The primary motivation for this is to generalize the extension theorem (Corollary 4.1.1.1) to Lie algebras over fields of positive characteristic. As the proof relies on the primitive elements of universal enveloping algebras, the positive characteristic case cannot be proved similarly. Hence, in order to state a similar result for positive characteristics, we must consider restricted universal enveloping algebras.

We start by recalling the definition of a restricted Lie algebra.

Definition 3.5.1. Let L be a Lie algebra over a field F of characteristic p. We say L is a restricted Lie algebra if there exists a map $\cdot^{[p]} : L \to L$ such that for any $\alpha \in F$ and $x, y \in L$,

1. $(\alpha x)^{[p]} = \alpha^p x^{[p]}$;

2. $\mathrm{ad}_{x^{[p]}} = \mathrm{ad}_x^p$; and,

3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$ where $s_i(x,y)$ is the coefficient of t^{i-1} in the expression $\mathrm{ad}_{tx+y}^{p-1}(x)$.

We call such a map a p-operation on L.

Remark 3.5.1. Note that any Lie algebra over a field of characteristic 0 is a restricted Lie algebra, with the p-operation being the identity map.

Remark 3.5.2. Any associative algebra A over a field of characteristic $p > 0$ can be made into a restricted Lie algebra, with the commutator bracket $[x,y] = xy - yx$ and the p-operation being $x^{[p]} = x^p$.

We also recall the definition of a restricted Lie algebra homomorphism and the restricted universal enveloping algebra.

Definition 3.5.2. Let L and M be two restricted Lie algebras. A Lie algebra homomorphism $\varphi : L \to M$ is a restricted Lie algebra homomorphism if for every $x \in L$

$$\varphi(x^{[p]_L}) = \varphi(x)^{[p]_M}$$

Definition 3.5.3. Let L be a restricted Lie algebra. Let $I \subset U(L)$ be the two-sided ideal generated by $\{x^p - x^{[p]}\}_{x \in L}$. Then the restricted universal enveloping algebra of L is the associative algebra

$$U_p(L) = U(L)/I$$

Remark 3.5.3. By the Poincaré-Birkhoff-Witt Theorem, $U_p(L)$ has a basis given by ordered monomials of basis elements of L of the form $x_1^{n_1} \cdots x_k^{n_k}$ such that $0 \leq n_i < p$ for every i.

As we are primarily concerned with restricted Lie algebras to lift linear soficity to their associated restricted universal enveloping algebra, we require that restricted Lie algebra homomorphisms can be lifted to associative algebra homomorphisms on the restricted universal enveloping algebra. To that end, we use the following result

Theorem 3.5.1 ([16, Section 7, Theorem 12])**.** *Let L be a restricted Lie algebra and A an associative algebra. If $\varphi : L \to A$ is a restricted Lie algebra homomorphism, then there exists a unique lift of φ to an associative algebra homomorphism $\hat{\varphi} : U_p(L) \to A$.*

3.5.2 Linear Soficity of Restricted Lie Algebras

Since the universal linearly sofic Lie algebra is a Lie algebra given by the commutators of an associative algebra, as in Remark 3.5.2, we can turn it into a restricted Lie algebra with the p-map $x^{[p]} = x^p$ for fields of characteristic $p > 0$. Hence, we can essentially restate Definition 3.2.1 in the restricted Lie algebra case.

Definition 3.5.4. A restricted Lie algebra L is restricted linearly sofic if there exists a restricted Lie algebra homomorphism

$$\varphi : L \to \prod_{k \to \omega} \mathfrak{gl}_{n_k}(F) / \ker \rho_\omega$$

Just as in regular linear soficity, restricted linear soficity can be equivalently defined by using almost homomorphisms, just as in §3.3. We state this as a single result, instead of consider the entire sofic radical. First, we define what an almost representation is for a restricted Lie algebra.

Definition 3.5.5. Let L be a restricted Lie algebra, $W \subset L$ a finite dimensional subspace, and $\varepsilon > 0$. A restricted ε-almost representation is an ε-almost representation $\varphi : W \to \mathrm{End}(V)$ such that

$$\varphi(x^{[p]})(v) = \varphi(x)^p(v)$$

for every $v \in V_\varepsilon$.

We now state our result for restricted Lie algebras.

Theorem 3.5.2. *A restricted Lie algebra L is restricted linearly sofic if and only if for every $0 \neq p \in L$, there exists $\delta > 0$ such that for any finite dimensional $p \in W \subset L$ and $\varepsilon > 0$, there* *exists a restricted ε-almost representation $\varphi : W \to \mathrm{End}(V)$ such that*

$$\dim \mathrm{Im}\, \varphi(p) \geq \delta \cdot \dim V$$

The proof of this result is essentially the same as the proof of Lemma 3.3.1 and Theorem 3.3.2. We see that elements $p \in L$ of the above theorem are not in the kernel of some Lie algebra homomorphism from L to a universal linearly sofic Lie algebra and vice versa. With the additional property of the restricted almost representations, we can additionally assume that our Lie algebra homomorphism is a restricted Lie algebra homomorphism, since the restricted almost representations carry the additional information of the p-operation. Thus, just as in the proof of Theorem 3.3.2, we derive the above result.

The work in this chapter is based on the article *Linearly Sofic Lie Algebras*, Journal of Algebra and its Applications (2023). The dissertation author was the primary investigator and sole author of this paper.

Chapter 4

Linear Soficity of Universal Enveloping Algebras

The purpose of this section is to demonstrate that linear soficity for Lie algebras and linear soficity for associative algebras coincide. In particular, we will show that a Lie algebra is linearly sofic if and only if its universal enveloping algebra is, at least over fields of characteristic 0. For fields of positive characteristic, our proof requires an additional assumption on how our Lie algebra homomorphism to a universal linearly sofic Lie algebra acts on primitive elements of our universal enveloping algebra. We are able to obtain a similar result for the positive characteristic case when considering restricted universal enveloping algebras instead.

4.1 Universal Enveloping Algebras

Theorem 4.1.1. *Let L be a Lie F-algebra and $P(U(L)) \subset U(L)$ be the Lie subalgebra of primitive elements. Suppose $\Theta : L \to \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$ is a Lie algebra homomorphism and suppose that $\hat{\Theta}$ is the extension of Θ to $U(L)$. If the restriction $\hat{\Theta}|_{P(U(L))}$ is injective, then there exists an injective associative algebra homomorphism $\Psi : U(L) \to \prod_{k\to\omega} M_{m_k}(F)/\ker\rho_\omega$.*

Proof. Let

$$A = \prod_{k\to\omega} M_{n_k}(F)/\ker\rho_\omega$$

Define $\Theta_1 = \Theta$ and $\Theta_j = 1 \otimes \Theta_{j-1}$ for $j \geq 2$. We define new Lie algebra homomorphisms

$$\Theta^i : L \to \bigotimes_{j=1}^{i} A$$

via $\Theta^0(x) = 0$ for all $x \in L$, $\Theta^1 = \Theta$ and

$$\Theta^i = \Theta^{i-1} \otimes 1 + 1 \otimes \Theta_{i-1}$$

for $i \geq 2$.

We can extend each Θ^i to a map

$$\widetilde{\Theta}^i : U(L) \to \bigotimes_{j=1}^{i} A$$

As tensor products of matrix rings are isomorphic to matrix rings, we can view each $\widetilde{\Theta}^i$ as a map from $U(L)$ to some universal linearly sofic associative algebra. Thus, by Proposition 2.2.3, we can construct a map

$$\Phi : U(L) \to \prod_{k \to \omega} M_{m_k}(F) / \ker \rho_\omega$$

such that

$$\ker \Phi = \bigcap_{i=1}^{\infty} \ker \widetilde{\Theta}^i$$

We claim that Φ is injective.

Let $\{x_j\}_{j \in J} \subset L$ be a basis. Suppose that $x_{j_1} x_{j_2} \cdots x_{j_d} \in \ker \Phi$ is a monomial of degree $d \geq 2$. Then, by construction,

$$\widetilde{\Theta}^i(x_{j_1} x_{j_2} \cdots x_{j_d}) = 0$$

for every $i \geq 1$. As $\widetilde{\Theta}^i$ is an algebra homomorphism for every i, we have that $\Theta^i(x_{j_1} \cdots x_{j_d})$ is a sum of pure tensors consisting of 1's and products of the $\Theta^i(x_{j_k})$'s. In particular, for $\widetilde{\Theta}^d(x_{j_1} \cdots x_{j_d})$, there are two sets of pure tensors in the summation: one set consisting of pure tensors with at least one 1 and at least one product of two or more $\Theta^d(x_{j_k})$'s and the other set consisting of pure tensors with only terms of the form $\Theta^d(x_{j_k})$, i.e. pure tensors of the form

$$v_\sigma = \Theta^d(x_{j_{\sigma(1)}}) \otimes \Theta^d(x_{j_{\sigma(2)}}) \otimes \cdots \otimes \Theta^d(x_{j_{\sigma(d)}})$$

for some $\sigma \in S_d$. In fact, we get exactly one v_σ for every $\sigma \in S_d$. We will call the first set the low order tensors and the second set the higher order tensors.

Notice that the pure tensors in the summation of $\widetilde{\Theta}^{d-1}(x_{j_1} \cdots x_{j_d})$ are the lower order tensors of $\widetilde{\Theta}^d(x_{j_1} \cdots x_{j_d})$ with one of the 1's removed. Thus, we can tensor the equation

$$\widetilde{\Theta}^{d-1}(x_{j_1} \cdots x_{j_d}) = \underbrace{0 \otimes \cdots \otimes 0}_{d-1 \text{ times}}$$

by 1 in the d possible spots to get that the sum of the lower order tensors of $\widetilde{\Theta}^d(x_{j_1} \cdots x_{j_d})$ is 0. Hence, we derive that

$$\sum_{\sigma \in S_d} v_\sigma = 0$$

Note that if $x_{j_{k_1}} = x_{j_{k_2}}$, then

$$v_\sigma = v_{\sigma(k_1\ k_2)}$$

where $(k_1\ k_2)$ is the transposition that swaps k_1 and k_2. Let $H \subset S_d$ be a subset such that $v_\sigma \neq v_\tau$ for distinct $\sigma, \tau \in H$. Then there exists $n_\sigma \in \mathbb{N}$ for each $\sigma \in H$ such that our previous summation can be written as

$$\sum_{\sigma \in S_d} v_\sigma = \sum_{\sigma \in H} n_\sigma v_\sigma$$

Now suppose that $i > d$. In the sum of pure tensors of $\widetilde{\Theta}^i(x_{j_1} \cdots x_{j_d})$, we see that every single pure tensor is a lower order tensor. Hence, repeating the previous process using

$$\widetilde{\Theta}^{i-1}(x_{j_1} \cdots x_{j_d}) = \underbrace{0 \otimes \cdots \otimes 0}_{i-1 \text{ times}}$$

we get back to the equation that $\widetilde{\Theta}^i(x_{j_1}\cdots x_{j_d})=0$.

Now we are ready to prove the injectivity of Φ. We split our proof into two cases: characteristic 0 and positive characteristic.

Case 1: Characteristic 0

Assume that F is of characteristic 0. Suppose that $a\in\ker\Phi$ is a non-zero element. Assume we have an ordering on the index set J of the basis for L. By the Poincaré-Birkoff-Witt Theorem [16], we can uniquely write a as

$$a=\sum_{i=1}^{n}\alpha_i x_{j_1^{(i)}}^{e_1^{(i)}}\cdots x_{j_{d_i}^{(i)}}^{e_{d_i}^{(i)}}+\alpha_0$$

where

1. $\alpha_i\in F$;
2. $\alpha_i\neq 0$ for every $i\geq 1$;
3. $e_j^{(i)}\geq 1$ for every j and $i\geq 1$;
4. $d_i\geq 1$ for every $i\geq 1$; and,
5. $j_k^{(i)}\leq j_{k+1}^{(i)}$ for every $i\geq 1$ and $1\leq k\leq d_i-1$.

From the equation $\widetilde{\Theta}^0(a)=0$, we have that $\alpha_0=0$. Let

$$D_i=\sum_{k=1}^{d_i}e_k^{(i)}$$

be the degree of i-th monomial and let $D=\max\{D_i\}_{i=1}^{n}$. If $D=1$, we have that $a\in L$ and since $\Theta(a)=0$, we already have that $a=0$ by assumption of injectivity for Θ. Hence, we can assume that $D\geq 2$.

Repeating the same process we did for monomials, take the equation

$$\widetilde{\Theta}^{D-1}(a) = \underbrace{0 \otimes \cdots \otimes 0}_{D-1 \text{ times}},$$

tensor it with 1 in the D possible, and add up all the resulting equations. We then subtract this resulting equation from

$$\widetilde{\Theta}^{D}(a) = \underbrace{0 \otimes \cdots \otimes 0}_{D \text{ times}}$$

For every i such that $D_i < D$, the terms coming from the i-th monomial will vanish in the resulting equation. Hence, we are left with only terms coming from monomials of a with degree exactly D.

As in the case of the monomials, we are left with terms of the form v_σ for some $\sigma \in S_D$. Since our original monomials are ordered, the only way for two such pure tensors to be equal is if they came from the same monomial. Therefore, we have a non-trivial linear combination of elements of the form v_σ. As with monomials, we can group together identical v_σ's. After this reduction, the weights in this linear combination must be of the form $n\alpha_i$ for some $n \in \mathbb{N}$ and $i \geq 1$.

As Θ is an injective linear map, the set $\{\Theta(x_j)\}_{j\in J} \subset A$ is linearly independent. Thus, all the weights of our linear combination of the v_σ's must be 0. As F is of characteristic 0, it follows that $\alpha_i = 0$ for every i such that $D_i = D$. This is a contradiction to our assumption that $\alpha_i \neq 0$ for every $i \geq 1$. Hence, no such a exists and $\ker\Psi = \{0\}$.

Case 2: Positive Characteristic

Now we consider the case when the characteristic of F is $p > 0$. Suppose that there exists $x_j \in L$ such that the power of x_j in each monomial of a of degree D is of the form p^n for some $n \geq 1$. Then each of the weights in the linear combination of the v_σ would be divisible by p and thus 0 in our field, giving us a trivial linear combination. Thus, we must take a different approach in the positive characteristic case.

Suppose that

$$y = x_{j_1}^{e_1} \cdots x_{j_n}^{e_n} \in U(L)$$

is a monomial. We can decompose each e_m uniquely as

$$e_m = \sum_{i=0}^{k_m} b_{i,m} p^i$$

where $0 \leq b_{i,m} < p$ for every i and m. We can then rewrite y as

$$y = \left(\prod_{i=0}^{k_m} (x_{j_1}^{p^i})^{b_{i,1}} \right) \cdots \left(\prod_{i=0}^{k_m} (x_{j_n}^{p^i})^{b_{i,n}} \right)$$

Consider the natural number

$$d = \sum_{m=1}^{n} \sum_{i=0}^{k_m} b_{i,m}$$

We repeat the same process of subtracting the equation $\widetilde{\Theta}^{d-1}(y) = 0$ tensored with 1's from the equation $\widetilde{\Theta}^{d}(y) = 0$, as in the characteristic 0 case. Now, we get a sum of pure tensors consisting of elements of the form $\Theta(x_j)$ and $\widetilde{\Theta}(x_j^{p^i})$ for $1 \leq j \leq n$ and $i \geq 1$ that is equal to 0. Moreover, since $0 \leq b_{j,m} < p-1$ for every j and m, we have that each pure tensor can at most $(p-1)!$ times in the sum.

Now suppose that $0 \neq q \in \ker \Psi$. We can write

$$q = \sum_{i=0}^{t} \alpha_i y_i$$

for some $\alpha_i \in F$ and monomials $y_i \in U(L)$ such that $y_0 = 1$. Assume that this linear combination is minimal in the sense that $\alpha_i \neq 0$ for $i \geq 1$. As in the characteristic 0 case, since $\widetilde{\Theta}^0(q) = 0$, we have that $\alpha_0 = 0$.

For each y_i, we can find a natural number d_i in the same manner as we found d previously.

Let $D = \max\{d_i\}_{i=1}^n$. As in the characteristic 0 case, we consider the difference of the equations $\widetilde{\Theta}^{D-1}(q) = 0$, tensored with 1's, and $\widetilde{\Theta}^D(q) = 0$. This gives us a linear combination of pure tensors consisting of elements of the form $\Theta(x_j)$ and $\widetilde{\Theta}(x_j^{p_i})$ for $i \geq 1$ that is equal to 0. Moreover, by the Poincaré-Birkhoff-Witt Theorem, two tensors can be equal if and only if they come from the same monomial. As each pure tensor can occur at most $(p-1)!$ times, we have a non-trivial linear combination that is equal to 0.

Note that in the characteristic p case, the elements of the form $x^{p^i} \in U(L)$ are primitive for any $x \in L$ and $i \geq 0$. As our extension $\widetilde{\Theta}$ is injective on $P(U(L))$, we have that our linear combination consists of linearly independent elements. Hence, all the weights of our linear combination must be 0 and so $q = 0$. This is a contraction to assuming that $\alpha_i \neq 0$ for $i \geq 1$. Thus, $\ker \Psi = \{0\}$ and Ψ is injective.

□

Remark 4.1.1. Note that there does exist a Lie algebra L over a positive characteristic field F such that an injective homomorphism $\Theta : L \to \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega$ can be extended to an injective homomorphism on the primitive elements of $U(L)$. Let $\mathbb{F}_p$ be the finite field of order p and let $F = \mathbb{F}_p(x)$ be the field of rational functions over $\mathbb{F}_p$. Consider $L = Fy$, the one dimensional abelian Lie algebra over F. We have an injective homomorphism

$$\begin{aligned} \Theta : L &\to \prod_{k\to\omega} \mathfrak{gl}_k(F)/\ker\rho_\omega \\ y &\mapsto [(xI_k)] \end{aligned}$$

As $x^{p^n} \neq x^{p^m}$ for $n \neq m$, we have that the extension

$$\begin{aligned} \Theta : P(U(L)) &\to \prod_{k\to\omega} \mathfrak{gl}_k(F)/\ker\rho_\omega \\ y^{p^n} &\mapsto [(x^{p^n} I_k)] \end{aligned}$$

is also an injective Lie algebra homomorphism.

Remark 4.1.2. It is worth comparing this result to the similar result for linearly sofic groups and associative algebras, Theorem 2.2.2. Both results made use of the Hopf structure of their respective objects (universal enveloping algebras and group algebras), specifically the comultiplication. Through repeated applications of the comultiplication on the original injective homomorphism $\widetilde{\Theta}$, an injective homomorphism on the larger structure was obtained. However, the results differ in the necessary conditions for their conclusions, likely stemming from the Hopf structure of the larger objects. In particular, over any field, the group like elements of a group algebra is the original group. However, the primitive elements of a universal enveloping algebra is the original Lie algebra only over fields of zero characteristic. Thus, for this result, we additionally require that the original embedding remains an embedding when extended to the primitive elements of the universal enveloping algebra.

For any Lie algebra L over a field of characteristic 0, we have that $P(U(L)) = L$. Hence, we have the following corollary:

Corollary 4.1.1.1. *Let L be a Lie algebra over a field of characteristic 0. Then L is linearly sofic if and only if $U(L)$ is linearly sofic.*

4.2 Restricted Universal Enveloping Algebras

For a Lie algebra L over a field of positive characteristic, our proof of Theorem 4.1.1 encountered difficulties with elements of the form $x^{p^i} \in U(L)$ where $x \in L$ and $i \geq 1$. We circumvented this problem in our proof by requiring that the extension of our embedding L to $U(L)$ was an embedding when restricted to the Lie algebra $P(U(L))$. An alternative method would be to instead of extending our embedding to $U(L)$, to instead extend it to a quotient of $U(L)$ where we no longer have elements of the form x^{p^i} for $x \in L$ and $i \geq 1$. We formalize this approach in the following result.

Theorem 4.2.1. *A restricted Lie algebra L is restricted linearly sofic if and only if its restricted universal enveloping algebra $U_p(L)$ is linear sofic.*

Proof. Let L be a restricted Lie algebra over a field F of characteristic $p > 0$. Suppose that we have a restricted Lie algebra embedding

$$\Theta : L \to \prod_{k\to\omega} \mathfrak{gl}_{n_k}(F)/\ker\rho_\omega =: A$$

As in the proof of Theorem 4.1.1, construct new Lie algebra homomorphisms

$$\Theta^j : L \to \bigotimes_{i=1}^{j} A$$

for $j \geq 1$ and $\Theta^0(x) = 0$ for all $x \in L$. Note that for every j, we have that Θ^j is a restricted Lie algebra homomorphism. Indeed, if $f : L \to B$ and $g : L \to C$ are restricted Lie algebra homomorphism where B and C are associative F-algebras, then for any $x \in L$,

$$f(x^{[p]}) \otimes 1 + 1 \otimes g(x^{[p]}) = (f(x) \otimes 1)^p + (1 \otimes g(x))^p = (f(x) \otimes 1 + 1 \otimes g(x))^p$$

As each Θ^j is of the form $f \otimes 1 + 1 \otimes g$ for some restricted Lie algebra homomorphisms f and g, we get that Θ^j is a restricted Lie algebra homomorphism.

Again, we lift each Θ^j to a map $\widetilde{\Theta}^j$ on $U(L)$ and construct a new map

$$\Psi : U(L) \to \prod_{k\to\omega} M_{m_k}(F)/\ker\rho_\omega$$

such that

$$\ker\Psi = \bigcap_{j=0}^{\infty} \widetilde{\Theta}^j$$

We claim that Ψ is injective.

Fix an ordered basis $\{x_i\}_{i\in I}$ of L. As in Remark 3.5.3, we have a basis of $U_p(L)$ consisting of ordered monomials of the x_i's such that power of any factor is at most $p-1$. For any monomial $y = x_{i_1}^{e_1} \cdots x_{i_n}^{e_n} \in U(L)$, let

$$d = \sum_{i=1}^{n} e_i$$

be the degree of y. As in the proof of Theorem 4.1.1, we consider the difference of the equations $\widetilde{\Theta}^{d-1}(y) = 0$, tensored with 1's, and $\widetilde{\Theta}^{d}(y) = 0$. This difference gives us a linear combination of pure tensors of elements of the form $\Theta(x_i)$ that is equal to 0. Moreover, as the largest power of x_i in y is $p-1$, we have that each pure tensor occurs at most $(p-1)!$ times.

Suppose that $q \in \ker \Psi$. We can write

$$q = \sum_{k=0}^{m} \alpha_k y_k$$

for some $\alpha_k \in F$ and monomials y_k such that $y_0 = 1$. Assume this linear combination is minimal in the sense that $\alpha_i \neq 0$ for $i \geq 1$. As $\widetilde{\Theta}^0(q) = 0$, we have that $\alpha_0 = 0$.

Let d_k be the degree of y_k and let $D = \max\{d_k\}_{k=1}^{m}$. We consider the difference of the equations $\widetilde{\Theta}^{D-1}(y) = 0$, tensored with 1's, and $\widetilde{\Theta}^{D}(y) = 0$. This gives us a linear combination of pure tensors of elements of the form $\Theta(x_i)$ that is equal to 0. As each monomial is ordered and each pure tensor from a monomial can occur at most $(p-1)!$ times, we have that our linear combination is non-trivial. As Θ is injective, we thus have a non-trivial linear combination of linearly independent elements that is equal to 0. This is a contradiction, and thus no such q exists. Hence $\ker \Psi = \{0\}$ and so Ψ is injective. $\square$

The work in this chapter is based on the article *Linearly Sofic Lie Algebras*, Journal of Algebra and its Applications (2023). The dissertation author was the primary investigator and sole author of this paper.

Chapter 5

Extensions of Metric Approximable Objects

5.1 Wreath Products

Our results on extensions of metric approximable objects rely on a construction for groups and Lie algebras known as a wreath product. We recall their definitions here:

Definition 5.1.1. For groups G and H, the (unrestricted) wreath product of G by H is the semidirect product

$$G \wr H := \left(\prod_{h \in H} G \right) \rtimes H$$

where H acts on the direct product via left shifts. Explicitly, the group operation of $G \wr H$ is

$$((g_x)_{x\in H}, h) \cdot ((g'_x)_{x\in H}, h') = ((g_{h'x} g'_x)_{x\in H}, hh')$$

Definition 5.1.2 ([23])**.** For Lie algebra L and M, the wreath product of L by M, denoted by $M \wr L$, is the Lie algebra whose underlying vector space is $\mathrm{Hom}(U(L), M) \times L$ and whose Lie bracket is given by

$$[(f,\ell),(f',\ell') = ([f,f'] + \ell \cdot f' - \ell' \cdot f, [\ell,\ell'])$$

where

$$(\ell \cdot f)(x) = -f(\ell x)$$

and

$$[f,f'](x) = \sum [f(x_{(1)}), f'(x_{(2)})]$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ is the coproduct of $U(L)$ in Sweedler notation. That is, $L \wr M$ is the semi-direct product of $\mathrm{Hom}(U(L), M)$ by L where L acts via left shifts.

An important result of wreath products that is of interest to this work is the Kaljounine-Krasner Theorem [17], sometimes called the universal embedding theorem:

Theorem 5.1.1 (Kaljounine-Krasner, [17])**.** *Let*

$$1 \to G \to K \to H \to 1$$

be a short exact sequence of groups. Then there exists a group embedding $K \hookrightarrow G \wr H$.

An analogous result holds for wreath products of Lie algebras:

Theorem 5.1.2 ([23])**.** *Let*

$$0 \to M \to N \to L \to 0$$

be a short exact sequence of Lie algebras. Then there exists a Lie algebra embedding $N \hookrightarrow M \wr L$.

The importance of these results for our work is that they allow us to completely characterize metric approximability of extensions of objects simply by whether their wreath product is metric approximable. In addition, in §5, wreath products are a useful tool for construction specific extensions of metric approximable objects.

5.2 Extensions of Metric Approximable Groups

The first result for preservation of metric approximability of groups under extensions was the Elek and Szabó's Extension Theorem (Theorem 1.5.1), which stated that any extension of a sofic group by an amenable group is itself sofic. While Elek and Szabó's proof did not use wreath products, Arzhantseva et al. [3] later showed that (unrestricted) wreath products of sofic groups by amenable groups was itself sofic. Additionally, it was shown that the same result held for hyperlinear groups. Finally, this result was further extended by Brude and Sasyk [6] to additionally hold for weakly sofic and linearly sofic groups as well. We restate their result here:

Theorem 5.2.1 ([6])**.** *Let G be a group and let H be an amenable group.*

1. *If G is weakly sofic, $G \wr H$ is weakly sofic.*
2. *If G is sofic, $G \wr H$ is sofic.*
3. *If G is linearly sofic, $G \wr H$ is linearly sofic.*
4. *If G is hyperlinear, $G \wr H$ is hyperlinear.*

From this result and Theorem 5.1.1, we immediately obtain

Corollary 5.2.1.1. *Let G be a sofic (resp. weakly sofic, linearly sofic, hyperlinear) group and let H be an amenable group. Then any extension of G by H is sofic (resp. weakly sofic, linearly sofic, hyperlinear).*

5.3 Extensions of Linearly Sofic Lie Algebras

The goal of this section is to prove the following theorem:

Theorem 5.3.1. *Let L be a Lie algebra such that $U(L)$ is amenable. Then for any linearly sofic Lie algebra M, the wreath product $M \wr L$ is linearly sofic.*

Proof. Suppose that $E \subset M \wr L$ is a non-zero, finite dimensional subspace. We have two linear projections of $M \wr L$, namely $\pi_M : M \wr L \to \mathrm{Hom}(U(L), M)$ and $\pi_L : M \wr L \to L$. Let $E_M = \pi_M(E)$ and let $E_L = \pi_L(E)$.

As $U(L)$ is an amenable algebra without zero divisors, from Proposition 11.16 of [4], $U(L)$ is linearly sofic. Hence, for any $\varepsilon > 0$, there exists a finite dimensional subspace V and an

ε-almost representation $\varphi_L : E_L \to \mathrm{End}(V)$. Moreover, following the proof of Proposition 11.16, we can choose $V \subset U(L)$ and φ_L to be such that for any $y \in E_L$, $\varphi_L(y)$ is multiplication by y on a subspace $V_\varepsilon \subset V$ where $\dim V_\varepsilon > (1-\varepsilon)\dim V$. Furthermore, we can choose V and φ_L such that $\Delta(V) \subset V \otimes V$ and $\varphi_L(z)(x) = 0$ for any $z \in E_L$ and $x \in V \setminus V_\varepsilon$.

Let $E_M = \sum_{f \in E_H} f(V)$. Then E_M is a finite dimensional subspace of M and thus for any $\varepsilon > 0$, there exists a finite dimensional vector space W and an ε-almost representation $\varphi_M : E_M \to \mathfrak{gl}(W)$. We now construct a new linear map $\psi : E_H \to \mathrm{End}(\mathrm{Hom}(V,W))$ where

$$\psi(f)(\tau)(v) = \varphi_M(f(v_{(1)})(\tau(v_{(2)})$$

where $\Delta(v) = v_{(1)} \otimes v_{(2)}$ is the coproduct of $U(L)$ in sumless Sweedler notation. We also define an action E_L on $\mathrm{Hom}(V,W)$, where

$$\ell \cdot \tau(v) = -\tau(\varphi_L(\ell)(v))$$

Note that when $v \in V_\varepsilon$ that the action is just a left shift.

We now define a map $\Phi : E \to \mathfrak{gl}(\mathrm{Hom}(V,W) \oplus V)$ via

$$\Phi(f,\ell)(\tau,v) = (\psi(f)(\tau) + l \cdot \tau, \varphi_L(\ell)(v))$$

We first check that Φ is an almost representation.

Notice that

$$\begin{aligned}\Phi([(f_1,\ell_1),(f_2,\ell_2)]) =& \varphi([f_1,f_2] + \ell_1 \cdot f_2 - \ell_2 \cdot f_1, [\ell_1,\ell_2])(\tau,v) \\ =& (\psi([f_1,f_2])(\tau) + \psi(\ell_1 \cdot f_2)(\tau) + \psi(\ell_2 \cdot f_1)(\tau) + [\ell_1,\ell_2] \cdot \tau, \\ & \varphi_L([\ell_1,\ell_2])(v))\end{aligned}$$

Since $\ell_i \in L$, we have that $\Delta(\ell_i) = \ell_i \otimes 1 + 1 \otimes \ell_i$. Therefore,

$$\Delta(\ell_i v) = \ell_i v_{(1)} \otimes v_{(2)} + v_{(1)} \otimes \ell_i v_{(2)}$$

Hence

$$\begin{aligned}\Phi(f_1,\ell_1)\Psi(f_2,\ell_2)(\tau,v) =& \Psi(f_1,\ell_1)(\psi(f_2)(\tau) + \ell_2 \cdot \tau, \varphi_L(\ell_2)(v)) \\ =& (\psi(f_1)\psi(f_2)(\tau) + \psi(f_1)(\ell_2 \cdot \tau) + \ell_1 \cdot \psi(f_2)(\tau) + \ell_1 \cdot \ell_2 \cdot \tau, \\ & \varphi_L(\ell_1)\varphi_L(\ell_2)(v))\end{aligned}$$

So if

$$[\Phi(f_1,\ell_1), \Phi(f_2,\ell_2)](\tau,v) = (\widetilde{\tau}, [\varphi_L(\ell_1), \varphi_L(\ell_2)](v))$$

then since $U(L)$ is cocommutative,

$$\begin{aligned}\widetilde{\tau}(v) =& [\varphi_M(f_1(v_{(1)})), \varphi_M(f_2(v_{(2)}))](\tau(v_{(3)})) - \varphi_M(f_2(\ell_1 v_{(1)}))(\tau(v_{(2)})) \\ &+ \varphi_M(f_1(\ell_2 v_{(1)}))(\tau(v_{(2)})) - \tau([\varphi_L(\ell_1), \varphi_L(\ell_2)](v))\end{aligned}$$

By comparing $\Phi([(f_1,\ell_1),(f_2,\ell_2)]$ and $[\Phi(f_1,\ell_1), \Phi(f_2,\ell_2)]$, we get that if

$$(\Phi([(f_1,\ell_1),(f_2,\ell_2)] - [\Phi(f_1,\ell_1), \Phi(f_2,\ell_2)])(\tau,v) = (\overline{\tau}, \overline{v})$$

then

$$\begin{aligned}\overline{\tau}(v) =& (\varphi_M([f_1(v_{(1)}), f_2(v_{(2)})]) - [\varphi_M(f_1(v_{(1)})), \varphi_M(f_2(v_{(2)}))])\tau(v_{(3)}) \\ &- \tau((\varphi_L([\ell_1,\ell_2]) - [\varphi_L(\ell_1), \varphi_L(\ell_2)])(v))\end{aligned}$$

and

$$\overline{v} = ([\varphi_L(\ell_1), \varphi_L(\ell_2)] - [\varphi_L(\ell_1), \varphi_L(\ell_2)])(v)$$

Since φ_M and φ_L are both ε-almost homomorphisms, we have that $\overline{\tau}$ and $\overline{v}$ are both 0 if $v \in V_\varepsilon$ and $\operatorname{Im}\tau \subset W_\varepsilon$. Thus Φ operates like a Lie algebra representation the subspace $\operatorname{Hom}(V, W_\varepsilon) \oplus V_\varepsilon$. That is, $\Phi([(f_1, \ell_1), (f_2, \ell_2)]$ and $[\Phi(f_1, \ell_1), \Phi(f_2, \ell_2)]$ are equal on $\operatorname{Hom}(V, W_\varepsilon) \oplus V_\varepsilon$.

Since

$$\frac{\dim(\operatorname{Hom}(V, W_\varepsilon) \oplus V_\varepsilon)}{\dim(\operatorname{Hom}(V, W) \oplus V)} = \frac{\dim V \cdot \dim W_\varepsilon + \dim V_\varepsilon}{\dim V \cdot \dim W + \dim V} > (1 - \varepsilon)\frac{\dim V(\dim W + 1)}{\dim V(\dim W + 1)} = 1 - \varepsilon$$

we have that Φ is an ε-almost representation.

Suppose that $(g, \ell) \in E$ is a non-zero element. In order to show that $M \wr L$ is linearly sofic, we need to show that there exists $\delta > 0$ such that for any finite dimensional subspace of $M \wr L$ containing (g, ℓ), there exists an ε-almost representation Φ_ε such that

$$\rho(\Phi_\varepsilon(g, \ell)) \geq \delta$$

That is, we want to find a δ independent of our choice of E.

In the first case, suppose that $g \neq 0$. Then there exists some $i \in U(L)$ such that $g(i) \neq 0$. By enlarging our subspace V if necessary, we can assume that $i \in V$ and $\dim(iV \cap V) > (1 - \varepsilon)\dim V$. Hence, there exists some $\delta \in (0, 1)$ such that for every $\varepsilon^* > 0$, there exists some ε^*-almost representation φ_M on E_M such that

$$\rho(\varphi_M(g(i))) \geq \delta$$

Moreover, as M is linearly sofic, δ can be chosen independent of E_M.

For any $ix \in iV \cap V$, the coproduct $\Delta(ix)$ contains a term of the form $i \otimes x$. Moreover, if

$$\Delta(ix) = i \otimes x + \sum_{j=1}^{n} i_j \otimes x_j$$

then the set $\{x, x_1, \ldots, x_n\}$ is linearly independent. Thus for any $w \in W$, we can choose $\tau \in \mathrm{Hom}(V,W)$ such that $\tau = x^* \cdot w$, where $x^* \in V^*$ is the dual element corresponding to x. For such a τ, we have that

$$\psi(g)(\tau)(ix) = \varphi_M(g(i))(w)$$

It follows that $\dim \mathrm{Hom}(iV \cap V, \mathrm{Im}\, \varphi_M(g(i))) \leq \dim \mathrm{Im}\, \psi(g)$

If $\ell = 0$, then

$$\rho(\Phi(g,0)) \geq \frac{(1-\varepsilon)\dim V \cdot \delta \dim W}{\dim V \cdot \dim W + \dim V} \geq (1-\varepsilon)\frac{\delta}{2}$$

If $\ell \neq 0$, then the set $\{x, \ell ix, x_1, \ldots, x_n\}$ is also linearly independent, so

$$\psi(g)(\tau)(ix) + \ell \cdot \tau(ix) = \psi(g)(\tau)(ix)$$

Therefore

$$\rho(\Phi(g,\ell)) \geq \frac{(1-\varepsilon)\dim V \cdot \delta \dim W + (1-\varepsilon)\dim V}{\dim V \cdot \dim W + \dim V} \geq (1-\varepsilon)\delta$$

Now suppose that $g = 0$. Since $(g,\ell) \neq 0$, we must have that $\ell \neq 0$. Hence, we have that the restriction of $\varphi_L(\ell)$ to $V_\varepsilon \subset V$ is an injection, where $\dim V_\varepsilon > (1-\varepsilon)\dim V$. Hence, for any $\sigma \in \mathrm{Hom}(V_\varepsilon, W)$, we can choose $\tau \in \mathrm{Hom}(V,W)$ such that

$$-\tau(\varphi_L(\ell)(v)) = \sigma(v)$$

for every $v \in V_\varepsilon$. Therefore

$$\rho(\Phi(0,\ell)) = \frac{(1-\varepsilon)\dim V \cdot \dim W + (1-\varepsilon)\dim V}{\dim V \cdot \dim W + \dim V} = (1-\varepsilon)$$

Thus, there exists $\delta > 0$ such that for any $\varepsilon > 0$, there exists an ε-almost homomorphism $\Phi_\varepsilon : E \to \mathfrak{gl}(U)$ such that

$$\rho(\varphi_\varepsilon(g,\ell)) \geq \frac{\delta}{4}$$

Moreover, this δ can be chosen independently of E and depends only on our element (g,ℓ). Hence, by Theorem 3.3.2, $M \wr L$ is linearly sofic. $\square$

From this result and Theorem 5.1.2, we immediately obtain the following corollary:

Corollary 5.3.1.1. *Let L be a Lie algebra such that $U(L)$ is amenable, and let M be a linearly sofic Lie algebra. Then any extension of M by L is linearly sofic.*

This result shows that linear soficity in Lie algebras is similar to same the property in groups. That is, extensions of linearly sofic Lie algebras by Lie algebras satisfying an amenability like condition are themselves linearly sofic, just as in the case with groups. However, in the case of linear soficity for associative algebras, the preservation of linear soficity under extensions is not as clear. These leads to the following open question:

Question 5.3.1. If A is a linearly sofic associative algebra and B is an amenable algebra, is an extension of A by B also linearly sofic?

As associative algebras do not have their own form of wreath products with a corresponding Kaljounine-Krasner Theorem (Theorem 5.1.1), our methods for Lie algebras do not immediately generalize.

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Chapter 6

Embeddings of Metric Approximable Objects

6.1 Embeddings of Metric Approximable Groups

The existence of embeddings for any countable group into a finitely generated group was first shown in [15]. Later, a different construction, using wreath products, was shown in [22]. Later work further showed that if the countable group held certain properties, then the embedding could be constructed such that the finitely generated group also held the same properties. In particular, in [5], it was shown that a countable group of subexponential growth is embeddable in a finitely generated group of subexponential growth.

We proceed in this section that a similar result holds for metric approximable groups. In order to do so, we make use of the embedding construction in [22], which we review in the following.

Theorem 6.1.1 ([22]). *Every countable group G can be embedded into a finitely generated subgroup of $(G \wr \mathbb{Z}) \wr \mathbb{Z}$.*

We provide an abbreviated version of the proof of this result for comparison for the later theorem of Lie algebras.

Proof. Let $G = \{g_i\}_{i=1}^{\infty}$ be a countable group. Consider the infinite cyclic group $Z = \langle z \rangle$. We consider a family of functions $f_i : Z \to G$ where

$$f_i(z^t) = g_i^{-t}$$

Now we consider the elements $k_i = (f_i^{-1}(z \cdot f_i), 1) \in G \wr Z$. Notice that

$$\begin{aligned}
[(f_i, 1), (\mathrm{id}, z)] &= (f_i^{-1}, 1)(\mathrm{id}, z^{-1})(f_i, 1)(\mathrm{id}, z) \\
&= (f_i^{-1}, 1)(\mathrm{id}, z^{-1})(z \cdot f_i, z) \\
&= (f_i^{-1}, 1)(z \cdot f_i, 1) \\
&= (f_i^{-1}(z \cdot f_i), 1) = k_i
\end{aligned}$$

Furthermore, we have that

$$(f_i^{-1}(z \cdot f_i))(z^t) = g_i^t g_i^{-t+1} = g_i$$

and so we have an embedding

$$\begin{aligned}
G &\to G \wr Z \\
g_i &\mapsto k_i
\end{aligned}$$

Now consider another infinite cyclic group $D = \langle d \rangle$ and let $G' = G \wr Z$. Consider the family of elements $d_k = d^{2^k - 1} \in D$ for $k \geq 1$. Note that $d_{k_1} d_{k_2} \neq d_{k_3}$ for any triplets k_1, k_2, k_3. Define a function $q : D \to G'$ via

$$\begin{aligned}
q(1) &= z \\
q(d_k^{-1}) &= f_i \\
q(x) &= \mathrm{id}_P
\end{aligned}$$

for all other values $x \in D$.

Consider the function $h_i = [d_i \cdot q, q]$. Notice that

$$h_i(1) = k_i$$
$$h_i(d^k) = \mathrm{id}_P, \quad k \neq 0$$

Hence the subgroup generated by the h_i's is isomorphic to that generated by the k_i's, which is in turn isomorphic to G. Since each h_i is a product of powers of d and q, we have that this subgroup generated by the h_i's is contained in a finitely generated subgroup of $G' \wr D$. □

From this result, we deduce the following corollary.

Corollary 6.1.1.1. *Let G be a countable group. If G is sofic (resp. linearly sofic, weakly sofic, hyperlinear), then G can be embedded into a finitely generated sofic (resp. linearly sofic, weakly sofic, hyperlinear) group.*

Proof. This follows immediately from Theorems 5.2.1 and 6.1.1, using the fact that the group $\mathbb{Z}$ is amenable. □

6.2 Embeddings of Linearly Sofic Lie Algebras

The goal of this section is to construct an embedding for countable dimensional linearly sofic Lie algebras similar to our embedding for countable metric approximable groups. That is, the embedding will be into a finitely generated subalgebra of an iterated wreath product of the linearly sofic algebra with two Lie algebras with amenable universal enveloping algebra.

Similar to the case of groups, the existence of an embedding from a countable dimension Lie algebras into finitely generated Lie algebras was shown in [25]. Additionally, in the case where the countable dimensional algebra is of subexponential growth, it was shown in [1] that the embedding can be constructed such that the finitely generated algebra is also of subexponential growth.

We proceed in this section to prove an analogous result for countable dimensional linearly sofic Lie algebras. However, the embeddings in both [25] and [1] did not make use of wreath products of Lie algebras (Definition 5.1.2). We proceed first to construct an embedding from a countable dimensional Lie algebra into a finitely generated algebra using wreath products, in a method similar to Theorem 6.1.1:

Proposition 6.2.1. *Let L be a countable dimensional Lie algebra over a field of characteristic 0. Then L embeds into a finitely generated subalgebra of $(L \wr M_1) \wr M_2$, where M_1 is a one dimensional Lie algebra and M_2 is the Witt algebra.*

Proof. Let L be a countable dimensional Lie algebra over a field F with basis $X = \{x_i\}_{i=1}^{\infty}$. Consider the one-dimensional Lie algebra $M_1 = Fy$. We consider two families of functions $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \subset \mathrm{End}(U(M_1), L)$ defined as

$$f_i(y^t) = \begin{cases} x_i, & t = 0 \\ 0, & t > 0 \end{cases}$$

$$g_i(y^t) = -tx_i$$

Notice that in the wreath product $L' = L \wr M_1$, we have that L is isomorphic to the subspace spanned by the f_i's. Additionally, we have that

$$[g_i, y](y^t) = -(y \cdot g_i)(y^t) = g_i(y^{t+1}) = -(t+1)x_i$$

Therefore

$$g_i - [g_i, y] = f_i$$

Let M_2 be the Witt algebra (see §3.4.1). Consider the standard basis $Z = \{z_i\}_{i \in \mathbb{Z}}$ with order $z_i < z_j$ if $i < j$. Define $d_k = z_{2^k-1}$ for $k \geq 1$. Note that for $i \neq j$,

$$[d_i, d_j] = (2^i - 2^j) z_{2^i-2^j-2} \notin \operatorname{span}\{d_k\}_{k=1}^{\infty}$$

as the base field has characteristic 0. Thus, we can get a basis of $U(W)$ that consists of ordered monomials of the z_i's, except for those of the form $z_{2^i+2^j-2}$ for $i \neq j$, with possibly pairs of d_k's out of order.

We partially define a function $q \in \operatorname{End}(U(M_2), L')$ initially as follows:

$$q(1) = y$$
$$q(d_k) = g_k$$
$$q(d_k^n) = 0, \; n \geq 2$$

We also define a function

$$h_k = -d_k \cdot q + [d_k \cdot q, q]$$

Note that

$$h_k(1) = f_k$$
$$h_k(d_k) = 0$$

We also have that for $m \neq k$,

$$h_k(d_m) = q(d_k d_m) - [q(d_k d_m), q(1)] - [q(d_k), q(d_m)] = q(d_k d_m) + y \cdot q(d_k d_m) - [g_k, g_m]$$

Notice that

$$[g_k, g_m](y^t) = \sum_{i=0}^{t} t(t-i) \binom{t}{i} [x_k, x_m]$$

Hence, we define $q(d_k d_m)(y^t) = \alpha(t)[x_k, x_m]$ where $\alpha : \mathbb{N} \to F$ is defined recursively such that

$\alpha(0) = 1$ and

$$\alpha(t+1) = \alpha(t) - \sum_{i=0}^{t} t(t-i)\binom{t}{i}$$

This definition guarantees that $h_k(d_m) = 0$ for $m \neq k$.

We now proceed to inductively define our function q on monomials of $U(W)$ of arbitrary degree. Suppose that for some $N \in \mathbb{N}$ with $N \geq 3$, we have defined q on all monomials of degree strictly less than N. Suppose further that for any monomial p of degree strictly between 1 and $N-1$, we have that

$$q(p)(y^t) = \sum_i \alpha_p^{(i)}(t) v_p^{(i)}$$

for some functions $\alpha_p^{(i)} : \mathbb{N} \to F$ and elements $v_p^{(i)} \in L$. Now suppose that $a \in U(M_2)$ is an ordered monomial of degree N. If we write

$$a = z_{i_1} z_{i_2} \cdots z_{i_N}$$

we define the first of a to be the element z_{i_1}. If the first factor of a is not a scalar multiple of an element of $\{d_k\}_{k=1}^{\infty} \cup \{[d_k, d_m]\}_{k \neq m}$, we define $q(a) = 0$.

Now suppose that the first of a is in $\{d_k\}_{k=1}^{\infty}$, say d_i. We consider the monomial

$$b = z_{i_2} \cdots z_{i_N} \in U(M_2)$$

if $N \geq 2$, otherwise $b = 1$. We want that $h_i(b) = 0$. That is, we want that

$$q(a) - [q(a), y] = -[d_i \cdot q, q](b)$$

Note that

$$-[d_i \cdot q, q](b)(y^t) = \sum_{j=0}^{n} \beta_j(t) w_j$$

for some functions $\beta_j : \mathbb{N} \to F$ and elements $w_j \in L$. Thus, we can let

$$q(a)(y^t) = \sum_{j=0}^{n} \alpha_a^{(j)}(t) v_a^{(j)}$$

where $\alpha_a^{(j)} : \mathbb{N} \to F$ is defined recursively with $\alpha_a^{(j)}(0) = 1$ and

$$\alpha_a^{(j)}(t+1) = \alpha_a^{(j)}(t) + \beta_j(t)$$

for $t \geq 0$. We also define $v_a^{(j)} = w_j$.

Now suppose that the first factor is a scalar multiple of some $[d_k, d_m]$ with $k \neq m$ and let $b \in U(M_2)$ have the same definition as before. By repeating the previous argument of the previous step twice, we can define the values $q(d_k d_m b)$ and $q(d_m d_k m)$. Thus, by linearity, we can define

$$q(a) = q(d_k d_m b) - q(d_m d_k b)$$

Thus, by construction, our family of functions $\{h_k\}_{k=1}^{\infty} \subset \mathrm{End}(U(M_2), L')$ are such that $h_k(x) = \varepsilon(x) f_k$ where $\varepsilon : U(M_2) \to F$ is the co-unit map. Hence, the subalgebra spanned by the h_k's in $L' \wr M_2$ is isomorphic as a Lie algebra to the subspace spanned by f_i's in $L \wr M_1$, which in turn is isomorphic as a Lie algebra to L. Since M_2 is a finitely generated Lie algebra, the subspace spanned by the h_k's is contained in a finitely generated subalgebra of $L' \wr M_2$, namely the subalgebra generated by q and the generators of M_2. Therefore, the isomorphism

$$\begin{aligned} &L \to L \wr M_1 \to (L \wr M_1) \wr M_2 \\ &x_i \mapsto f_i \mapsto h_i \end{aligned}$$

has its image contained in a finitely generated subalgebra of $(L \wr M_1) \wr M_2$, completing the proof. $\square$

From this, we deduce our main result of this section:

Theorem 6.2.2. *Every countable dimensional, linearly sofic Lie algebra can be embedded into a finitely generated, linearly sofic Lie algebra.*

Proof. Let L be a linearly sofic Lie algebra. From the previous theorem, we can embed L into a finitely generated subalgebra of $(L \wr M_1) \wr M_2$, where M_1 is a one-dimensional Lie algebra and M_2 is the Witt algebra. As $U(M) \cong F[y]$, it is amenable and so the Lie algebra $L \wr M_1$ is linearly sofic. Furthermore, M_2 is a Lie algebra of subexponential growth and so, by [26], $U(M_2)$ is an associative algebra of subexponential growth. Hence, by Proposition 3.1 of [8], we have that $U(M_2)$ is amenable and so $(L \wr M_1) \wr M_2$ is linearly sofic. As subalgebras of linearly sofic algebras are themselves linearly sofic, the proof is finished. $\square$

As with the preservation of linear soficity under extensions, the ability to embed countable dimensional linearly sofic associative algebras into finitely generated linearly sofic is not known. Again, the lack of a similar wreath product construction for associative algebras prevents our method from generalizing to this case. We pose this open question here:

Question 6.2.1. If A is a countable dimensional linearly sofic associative algebra, does there exist an embedding of A into a finitely generated linearly sofic associative algebra?

As in the case of groups and Lie algebras, it is known that it is possible to embed a countable dimensional associative algebra into a finitely generated associative algebra [21]. Additionally, if the countable dimensional algebra is of subexponential growth, then the embedding can be constructed such that the finitely generated algebra is also of subexponential growth [2]. Hence, it seems likely that a similar result should hold in the case of linearly sofic countable dimensional associative algebras.

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