Title

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Summary: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a known continuous distribution function. Robbins' problem is to find a sequential stopping rule without recall which minimizes the expected rank of the selected observation. An upper bound (obtained by memoryless threshold rules) and a procedure to obtain lower bounds of the value are known, but the essence of the problem is still unsolved. The difficulty is that the optimal strategy depends for all $n > 2$ in an intractable way on the whole history of preceding observations. The goal of this article is to understand better the structure of both optimal memoryless threshold rules and the (overall) optimal rule. We prove that the optimal rule is a "stepwise" monotone increasing threshold-function rule and then study its property of, what we call, full history-dependence. For each n, we describe a tractable statistic of preceding observations which is sufficient for optimal decisions of decision makers with half-prophetical abilities who can do generally better than we. It is shown that their advice can always be used to improve strictly on memoryless rules, and we determine such an improved rule for all n. The essence of Robbins' problem would be to prove or disprove the existence of *asymptotically* (as $n \to \infty$) relevant improvements.

Keywords: Sequential selection - Full information - Memoryless threshold rules - "Stepwise" monotonicity - prophets - Order statistics.

AMS 1991 Subject Classification: Primary 60G40, Secondary 62L15.

§1. Introduction.

1.1 The Problem. Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with c.d.f. F. We assume F to be continuous so that the X_k's are uniquely rankable (a.s.). Since the payoffs we

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consider depend only on the ranks of the X_i, we may and do assume, w.l.o.g., that F is uniform on $[0,1]$. The relative rank of observation X_k is defined as

$$r_k = \sum_{j=1}^{k} I(X_j \le X_k), \qquad 1 \le k \le n, \tag{1.1}$$

where $I(A)$ denotes the indicator of the event A. Note that we define the smaller observations to be the better ones, which proves here to be more convenient. Let

$$R_k^* = r_k + \sum_{j=k+1}^{n} I(X_j \le X_k) \tag{1.2}$$

denote the absolute rank of X_k, and let T denote the set of all stopping rules,

$$T = \{\tau : \{\tau = k\} \in \mathcal{F}_k, \, \forall k = 1, 2, \ldots, n\}, \tag{1.3}$$

where $\mathcal{F}_k$ denotes the σ-field generated by $X_1, X_2, \ldots, X_k$. R_k^* is not $\mathcal{F}_k$-measurable, and so we replace it by its conditional expectation given $\mathcal{F}_k$, $R_k = E(R_k^* | \mathcal{F}_k) = r_k + (n-k)X_k$ (see also Assaf & Samuel-Cahn (A&S-C, 1992)). Robbins' problem is then to find

$$V(n) = \inf_{\tau \in T} E(R_\tau) \tag{1.4}$$

and the stopping rule τ which attains this value. This is the full-information version of the problem studied by Chow et al. (1964).

1.2 Motivation. The motivation to study Robbins' problem goes beyond the ambition to solve the *4th* secretary problem (see Bruss and Ferguson (B&F),1993). Once we think about it, we see that this problem stands indeed for a whole class of problems, about which little is known. The central question is what to do if the optimal rule is seemingly dependent on the whole history (in a sense which we will make precise) and if we do *not* have an idea about the value. Replacing at each stage the history by some adequately looking summary of the available information, on which we would base the decisions, is of little help, because if we cannot assess the error, how can we evaluate the trial? In Robbins' problem, it turns out, that we have reasonable bounds to the value, and with a lower bound of 1.908 and an upper bound of 2.3232, we have indeed little to worry about. But what to do if the bounds are not close at all? - The problem would stay the same, and any improvement would be desirable.

1.3 Organization of the Paper. The paper is organized as follows: In Section 2 we review memoryless threshold rules, prove the uniqueness of the optimal rule in this class and briefly discuss computational methods. Section 3, which prepares the study of the overall optimal rule, introduces the notion of a half-prophet. This is a decision maker who turns into a prophet *provided* he decides not to choose the present observation and to go on. Thus a half-prophet has more power than we have, and it will be shown that he needs all preceding information to realize the optimal value. Section 4 contains the main results of the paper. We show in Lemma 4.1 that the overall optimal rule is a "stepwise" monotone threshold function rule which contrasts the lack of monotonicity with respect to preceding observations. We then discuss the notion of *full history-dependence* of a stopping rule and show that the overall optimal rule for Robbins' problem has this property. In Section 5 we describe a rule which strictly improves on the optimal memoryless rule for finite n.

§2. Memoryless Threshold Strategies.

By a threshold strategy or rule we mean a stopping rule, where the decision whether or not to select the observation X_k (i.e. stop at k) depends only on whether or not X_k is smaller (respectively larger, depending on the problem) than some real (threshold) value p_k. The p_k's are often thought of as being constants (see also Kennedy & Kertz (1990)). We must classify here threshold rules more precisely and prefer to call these *memoryless* (in short m^0-) threshold rules to distinguish them from those threshold rules where each p_k may itself be a function of some or all preceding observations. Thus a m^0-threshold rule is defined by

$$N(\mathbf{p}) = \inf\{k \geq 1 : X_k \leq p_k\}, \tag{2.1}$$

where $\mathbf{p}$ is a pre-determined sequence $(p_1, p_2, \ldots, p_n)$ with $0 \leq p_k \leq 1$, $k = 1, 2, \ldots, n-1$ and $p_n = 1$. In the case of Robbins' problem we know (see also A&S-C, (1992)) that if we use m^0-threshold rules we can confine our interest to the class of monotone rules $N(\mathbf{p})$, with $\mathbf{p}$ satisfying

$$p_1 \leq p_2 \leq \ldots \leq p_n = 1. \tag{2.2}$$

The expected rank obtained by the $N(\mathbf{p})$-rule is given by formula (2.1) of B&F (1993). The optimal p_k's, denoted by p_1^*, $p_2^*, \ldots,$ p_{n-1}^* depend themselves on n, but we use the additional indexation by n only where necessary to prevent confusion. For small n

the optimal p_k^*'s can be obtained from by solving the corresponding system of partial derivatives equations. We have

Lemma 2.1: The system of partial differential equations

$$\left\{\frac{\partial(E(R_{N(\mathbf{p})}))}{\partial p_k} = 0\right\}, \quad k = 1, 2, \ldots, n-1, \tag{2.3}$$

has, for all $n \geq 2$, a unique solution $\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_{n-1}^*, 1)$ minimizing $E(R_{N(\mathbf{p})})$.

Proof: Since $E(R_{N(\mathbf{p})})$ is a continuous function of the p_k's, it must take its minimum somewhere on $[0,1]^{n-1}$. Let $V^{m^0}(n)$ denote this minimum value and let

$$\mathbf{p}^* = (p_1^*(n), p_2^*(n), \cdots, p_{n-1}^*(n), 1) \tag{2.4}$$

be a sequence of m^0-thresholds which achieves it. Note that we do not imply so far uniqueness of these thresholds. Clearly

$$\forall n = 2, 3, \cdots : V^{m^0}(n) > 1, \tag{2.5}$$

because, for $n > 1$, no sequential rule selects rank 1 with probability one.

We first show that $p_1^*(n)$ must be greater than 0 for all n. This is intuitively clear, since we feel it cannot be optimal to refuse almost surely *any* first observation. Formally: The statement is true for $n = 1$, since $p_1^*(1) = 1$. Suppose now that $p_1^*(n+1) = 0$ for some $n \geq 1$. Then the observation X_1 will be almost surely refused, so that memoryless optimal play on the $X_2, ..., X_{n+1}$ yields an expected total loss

$$e_{n+1}(0) = V^{m^0}_{p_1(n+1)=0}(n+1) = W^{m^0}(n) + P(X_{\tau_n} \geq X_1), \tag{2.6}$$

where τ_n denotes the stopping time induced by $(p_2^*(n+1), p_3^*(n+1), \cdots, p_n^*(n+1), 1)$ and $W^{m^0}(n)$ the corresponding expected loss for the remaining n observations only, i.e. disregarding the contribution of X_1. Clearly $W^{m^0}(n) \geq V^{m^0}(n) > 1$. Now, if we use instead of 0 the threshold p_1, say, with $1 > p_1 > 0$, then we obtain similarly by conditioning on X_1

$$e_{n+1}(p_1) = p_1(1 + np_1/2) + (1-p_1)[W^{m^0}(n) + P(X_{\tau_n} \geq X_1 | X_1 > p_1)]. \tag{2.7}$$

Clearly, the conditional probability term in (2.7) is non-increasing in p_1 on $[0,1]$ so that using (2.6) in (2.7) implies

$$e_{n+1}(p_1) \le p_1(1 + np_1/2) + (1 - p_1)e_{n+1}(0). \tag{2.8}$$

Note that the last RHS of (2.8) can be made smaller than $e_{n+1}(0)$ by choosing for p_1 any value such that $1 < 1 + np_1/2 < e_{n+1}(0)$ or equivalently $0 < p_1 < 2(e_{n+1}(0) - 1)/n$. This is possible as $e_{n+1}(0) \ge W^{m^0} > 1$ for all $n > 1$. Thus the choice $p_1^*(n+1) = 0$ would be suboptimal for any $n \ge 1$, i.e. $p_1^*(n+1) > 0$ for all $n \ge 1$.

Secondly, since $p_n^*(n) \equiv 1$, it is easy to see that we must have $p_{n-1}^*(n) < 1$. Indeed, if we used the threshold value $p_{n-1} = 1$, then we would select $\min\{X_{n-1}, X_n\}$ with probability $1/2$. Any threshold value p_{n-1} with $0 < p_{n-1} < 1$ however would do this with a higher probability and therefore yield a smaller rank, because this probability $q_{n-1}(p_{n-1})$, say, equals

$$\begin{aligned} q_{n-1}(p_{n-1}) &= \int_0^1 \{(1-x)I(x \le p_{n-1}) + xI(x > p_{n-1})\}\, dx \\ &= 1/2 + p_{n-1}(1 - p_{n-1}) > 1/2. \end{aligned} \tag{2.9}$$

Thus, in particular, $p_{n-1}^*(n) < 1$. By the monotonicity of the optimal p_k^*'s with $1 \le k \le n-1$ we can thus assure that no local or global minimum of $E(R_{N(\mathbf{p})})$ can lie on the boundary of the set $[0,1]^{n-1}$. Since all partial derivatives of $E(R_{N(\mathbf{p})})$ exist on $(0,1)^{n-1}$ and the minimum (or minima) must lie in this open set $(0,1)^{n-1}$, (2.3) must have at least one solution.

Finally, looking at the version (3.1) of B&F (1993) of the formula for $E(R_{N(\mathbf{p})})$ we see that it is a multiquadratic function of the p_k's, i.e. it is quadratic in each p_k (with positive sign) by holding the other p_j's constant. Each partial differential equation of the system (2.3) has thus at most one solution. Therefore the system (2.3) itself can have at most one solution. Since it has at least one, it must have a unique solution, which must be, accordingly, the minimum. ■

Remark 2.2. It is tempting to maximize q_{n-1} in (2.9) with respect to p_{n-1}, i.e. to choose $p_{n-1} = 1/2$ with $q_{n-1}(1/2) = 3/4$, but this does not give the optimal threshold $p_{n-1}^*(n)$ (see for instance Table 1), because this simple probability argument neglects relative ranks and the fact that, given X_{n-2} has been passed over, $X_1, X_2, ..., X_{n-2}$ are no longer i.i.d U[0,1] random variables. We shall see however later that $p^*_{n-1}(n) \to 1/2$ as $n \to \infty$.

Computational aspects. Due to the mentioned multiquadratic property, the computation of the p^*_k's is numerically (componentwise iteration) still tractable for larger n. Table 1a displays the sequences of thresholds for $n = 1, 2, \ldots, 12$, which we will use for examples in Section 5. The corresponding values are denoted by $V_n^{m^0}$ and displayed in Table 1b.

n	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}
1	1										
2	.5	1									
3	.3603	.5285	1								
4	.2858	.3825	.5351	1							
5	.2385	.3021	.3918	.5359	1						
6	.2052	.2509	.3110	.3954	.5349	1					
7	.1809	.2148	.2582	.3156	.3977	.5333	1				
8	.1619	.1881	.2211	.2628	.3183	.3983	.5315	1			
9	.1465	.1679	.1936	.2254	.2659	.3200	.3982	.5298	1		
10	.1341	.1516	.1724	.1974	.2284	.2679	.3208	.3978	.5283	1	
11	.1244	.1383	.1547	.1758	.2002	.2304	.2691	.3213	.3973	.5 265	1
12	.1125	.1288	.1423	.1582	.1779	.2040	.2322	.2702	.3142	.3 967	.5256

Table 1a

m^0-threshold values

n	1	2	3	4	5	6
$V_n^{m^0}$	1	1.25	1.4009	1.5606	1.5861	1.6490

n	7	8	9	10	11	12
$V_n^{m^0}$	1.7002	1.7430	1.7794	1.8109	1.8384	1.8627

Table 1b

Expected loss for the m^0-rules

The following lemma shows that "most" of the $p^*_j(n)$'s tend to 0 as $n \to \infty$. This will prove to be an essential tool in Section 5.

Lemma 2.3: Let $\mathbf{p}^*(n) = (p_1^*(n), p_2^*(n), \ldots, p_{n-1}^*(n), 1)$ be the sequence of m^o-optimal thresholds for Robbins' problem with n observations. Further let α be a constant with $0 < \alpha < 1$ and $\alpha_n = [\alpha n]$, where $[x]$ denotes the floor of x. Then, as $n \to \infty$,

$$\forall 0 < \alpha < 1\,, \forall k = k(n) \leq \alpha_n \,: p_{k(n)}(n) \to 0.$$

Proof: Suppose the contrary, i.e. there exists $\epsilon > 0$ such that for all n and some $0 < \alpha < 1$ there would exist an integer $k = k(n) < \alpha_n$ with $p_k^*(n) > \epsilon$. Since $p_k^*(n)$ is m^0-optimal this would then imply that the observation $X_k = \epsilon$, say, must be accepted. However, we shall see that we can construct another m^0-rule yielding a strictly smaller expected loss. We first note that,

$$E(R_k|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) = r_k(\epsilon) + (n-k)\epsilon, \tag{2.10}$$

where $r_k(\epsilon)$ denotes the relative rank of X_k for $X_k = \epsilon$. Let δ be fixed but arbitrarily chosen with $0 < \delta < \epsilon$, and let τ_δ be the stopping time defined by

$$\tau_\delta = \min\{n\,,\, \inf\{j > k : X_j < \delta\}\}. \tag{2.11}$$

Then

$$\begin{aligned} & E(R_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) \qquad (2.12) \\ &= E(r_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) + E((n-\tau_\delta)X_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) \\ &< E(r_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) + (n-k)E(X_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon). \end{aligned}$$

Conditioning on the outcome of X_{τ_δ} we obtain

$$\begin{aligned} & E(R_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon) \qquad (2.13) \\ &= E[E(R_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon, X_{\tau_\delta})|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon] \\ &< (1-(1-\delta)^{n-k-1})\, E(R_{\tau_\delta}|X_1, X_2, \cdots, X_{k-1}, X_k = \epsilon, X_{\tau_\delta} < \delta) + n(1-\delta)^{n-k-1}, \end{aligned}$$

where we used for $\tau_\delta = n$ the worst-case bound n as an upper bound for $R_n \leq n$.

As $n \to \infty$, the first factor of the first term in the last line of (2.13) clearly tends to 1. The second summand in this line tends to 0 since $n - k - 1 \geq n - \alpha_k - 1 \to \infty$ more quickly than some $(1-\alpha)n$. Therefore, the RHS of (2.13) is close to $E(R_{\tau_\delta}|X_{\tau_\delta} < \delta)$ as $n \to \infty$. Since $E(R_{\tau_\delta}|X_{\tau_\delta} < \delta)$ is clearly a strictly increasing function of δ on $(0, \epsilon]$, the inequality

$$E(R_{\tau_\delta}|X_k = \epsilon) < E(R_{\tau_\epsilon}|X_{\tau_\epsilon} < \epsilon) \text{ a.s.}$$

must hold for all n sufficiently large. Now, since $E(R_{\tau_\epsilon}|X_{\tau_\epsilon} < \epsilon) < E(R_k|X_k = \epsilon)$ a.s. it follows in particular that, for all n sufficiently large,

$$E(R_{\tau_\delta}|X_k = \epsilon) < E(R_k|X_k = \epsilon) \text{ a.s.}$$

This means that, for all n sufficiently large, the rule τ_δ defined by (2.11) yields a smaller expected loss than accepting some $X_k \geq \epsilon$. However, this contradicts the hypothesis of $p_k^*(n) > \epsilon$ for all n, and thus proves the lemma. ■

Now let $V(n)$ denote the overall optimal value of Robbins' problem for n variables. Clearly $V(n) \leq V^{m^0}(n)$ for all n so that

$$V = \lim_{n\to\infty} V(n) \leq \lim_{n\to\infty} V^{m^0}(n) = V^{m^0}. \tag{2.14}$$

We know (see B&F (1993)) that $V^{m^0} = 2.326....$ and that $V > 1.908$, where the latter lower bound was obtained by computing the optimal strategy for a truncated modification of the expected loss. We could also show that this procedure would converge to the correct limiting value V, but the computation times increase exponentially. The optimal strategy is very "sensitive" to the past, and no statistic is known, which summarizes enough of the information of the past to allow for a ϵ-optimal strategy while still being sufficiently tractable. We will introduce the notion of *full history dependence* to describe the mentioned sensitivity.

Apart from the above bounds we know little about V. We even do not know yet whether $V > 2$, although one may, in principle, be able to settle this question numerically by the truncation method proposed in B&F (1993). Gnedin (1995, INFORM Conference on Applied Probability) reported that the strict inequality, $V < V^{m^0}$, can be obtained by an adequate embedding of the $n = \infty$ case into a Poisson process. We focus on finite n. The $V(n)$ are only known for n up to 5, and computation times are prohibitive to try to do much better. This raises the question how to find tractable improvements on optimal threshold strategies, with which we deal in Section 5 and where half-prophets prove their usefulness again.

§3. Half-prophets.

The motivation to introduce the notion of a *half-prophet* (h-prophet) is to overcome one side of the deadlock resulting from the apparent history dependence, namely the future side. An h-prophet, called in at time k say, can (as we can) see the observations $X_1, X_2, \ldots, X_k$ and nothing else. However, given that he decides not to select X_k and to go on, then he foresees at time $k+1$ the whole future $X_{k+1}, X_{k+2}, \ldots, X_n$ (whereas we see of course only X_{k+1}). The value of the game for an h-prophet depends, apart from n clearly also on k, and as we shall see, on the whole past $X_1, X_2, \ldots, X_k$. We have to define more precisely:

Definition 3.1: An *h(k)-prophet* for a (discrete) sequential decision problem is a decision maker who is able to foresee the complete future if and only if he decides to enter stage $k+1$.

An *h-prophet* is a decision maker, which can be elected to be an h(k)-prophet for exactly one $k \geq 1$.

Note that an h-prophet in Robbins' problem faces a much simpler decision problem. He must only decide whether to select the present observation, or alternatively, to wait for the best later on. This "binary" feature implies that his value at time k, denoted by $h(k,n)$, say, can be computed in a straightforward manner.

Lemma 3.2: The value of an h(k)-prophet in Robbins' problem with n observations conditional on $X_1, \ldots, X_k$ is equal to

$$h(k,n) = \min\{r_k + (n-k)X_k,\, 1 + \sum_{j=1}^{k} (1 - X_j)^{n-k}\}$$

for all $k = 1, 2, \ldots, n$.

Proof: We first note that if the h(k)-prophet selects X_k then his expected loss (denoted by L_k) is equal to

$$L_k = E(R_k | X_1, X_2, \ldots, X_k) = r_k + E\Big(\sum_{j=k+1}^{n} I(X_j \leq X_k \,| X_1, X_2, \ldots, X_k)\Big). \tag{3.1}$$

By independence of X_j with $j > k$ of the past, the second term simplifies to $(n-k)\,X_k$ so that

$$L_k = r_k + (n-k)X_k. \tag{3.2}$$

Suppose now that the h(k)-prophet refuses X_k. Then he enters time $k+1$ and optimal behavior forces him to select $\iota(k) := \inf_{k<j\leq n}\{X_j\}$. The expected absolute rank of $\iota(k)$

given $X_1, X_2, \ldots, X_k$ depends on $X_1, X_2, \ldots, X_k$, but the distribution function $F_{\iota(k)}(\cdot)$ of $\iota(k)$ itself is simply the distribution function of the smallest order statistic of $(n-k)$ i.i.d. uniform r.v.'s on $[0,1]$, independent of $X_1, X_2, \ldots, X_k$. Denoting the absolute rank of $\iota(k)$ by $R^{\iota(k)}$ we obtain by conditioning on the value of $\iota(k)$ and by using its independence of $X_1, X_2, \ldots, X_k$,

$$L_k^{(+)} := E(R^{\iota(k)} \,|\, X_1, X_2, \ldots, X_k) = E\big(\, E(R^{\iota(k)} \,|\, X_1, X_2, \ldots, X_k; \iota(k))\,\big).$$

Now note that $\iota(k)$ can (by definition) only be preceded by the $X_1, X_2, \ldots, X_k$, and by none of the future values. If $\iota(k)$ is smaller than all $X_1, X_2, \ldots, X_k$, then its rank equals 1, and it moves up by 1 with each X_j it surpasses. Therefore $L_k^{(+)}$ can be written as

$$L_k^{(+)} = \int_0^1 (1 + \sum_{m=1}^{k} I(X_m \leq s))\, dF_{\iota(k)}(s) \tag{3.3}$$

where

$$dF_{\iota(k)}(s) = d(1-(1-s)^{n-k}) = (n-k)(1-s)^{n-k-1} ds.$$

Interchanging the order of integration and summation and then adjusting the boundaries of integration according to the indicators yields by straightforward calculation

$$L_k^{(+)} = 1 + \sum_{j=1}^{k} (1 - X_j)^{n-k}, \quad k = 1, 2, \ldots, n. \tag{3.4}$$

By the optimality principle,

$$h(k,n) = \min\{L_k, L_k^{(+)}\}, \tag{3.5}$$

and thus the statement of the Lemma is proved. ■

Remark 3.3: The most powerful h-prophet — apart from the h(0)-prophet, who is equivalent to a *prophet* — is, by definition, the h(1)-prophet. He can simply ignore the decision problem any later h-prophet will face. Since his probability to select rank 1 tends to one as $n \to \infty$ one feels that he is "asymptotically" as good as a prophet, and that indeed the same should be true for any h(k)-prophet for fixed k. This is true for the expected rank, but a worst case analysis shows, that all h-prophets are distinctly inferior to a prophet in the sense that $\sup_x \inf_\tau E(R_\tau | X_1 = x) > 1$. To see this, note that according to (3.2), (3.4) and (3.5)

$$\sup h(1,n) \ := \sup_x h(1,n)_{|X_1=x} \ = 1 + \sup_x \{\min((n-1)x, (1-x)^{n-1})\} = 1 + c_n, \quad (3.6)$$

say. Since $(n-1)x$ is increasing and $(1-x)^{n-1}$ decreasing on $[0,1]$, and both intersect there, c_n must be the unique solution of $(n-1)x = (1-x)^{n-1}$. Putting $x = \xi/(n-1)$ in (3.6) yields the limiting equation $\xi = e^{-\xi}$ with solution $\xi = \lim_{n\to\infty} c_n = .5671\cdots$. Thus,

$$\forall 1 \le k \le n : \sup h(k,n) \ge \sup h(1,n) \to 1.5671\cdots \quad \text{as } n \to \infty, \quad (3.7)$$

whereas a prophet can always achieve the value 1.

§4. Properties of the optimal rule.

We turn now to the main results of this paper. The first is Lemma (4.1) which shows a monotonic feature which contrasts many non-monotonic features of Robbins' Problem. This will then be used to show that the optimal rule depends in a certain sense on the full history of the process.

4.1 Stepwise Monotonicity of the Optimal Threshold Functions. We denote by $\tau^* = \tau^*(n)$ the (overall) optimal rule for Robbins' problem. An observation X_k, which can be selected (respectively, must be refused) under the rule $\tau^*(n)$) will be called shortly *acceptable*, (respectively, *unacceptable*).

Lemma 4.1: $\tau^* = \tau^*(n)$ is a *stepwise* monotone increasing threshold function rule for all n, i.e. it is of the form

$$\tau^*(n) = \inf\{1 \le k \le n \, : \, X_k \le p_k(X_1, X_2, \ldots, X_{k-1})\}, \quad (i)$$

where the functions $p_k(...)$ satisfy

$$p_k(X_1, X_2, \ldots, X_{k-1}) \le p_{k+1}(X_1, X_2, \ldots, X_{k-1}, X_k) \text{ a.s.} \quad (ii)$$

and

$$p_{n-1}(X_1, X_2, \cdots, X_{n-2}) < 1 \text{ a.s.} \quad (iii)$$

Proof: (i) We show first, that $\tau^*(n)$ is of the described form. For this we have to show that, at any stage k, the following is true: Whenever it is optimal to stop at X_k it would be optimal to stop for any $X'_k \leq X_k$, and whenever it is optimal to refuse X_k it is optimal to refuse any $X'_k \geq X_k$.

To see this let $L_k(x)$ be defined as in (3.2) with X_k being replaced by x. Thus

$$L_k(x) = \sum_{j=1}^{k-1} I(X_j < x) + 1 + (n-k)x \tag{4.1}$$

describes our expected loss by accepting x at stage k. Clearly, $L_k(x)$ is strictly increasing in x for all $1 \leq k < n$.

On the other hand, the expected loss by refusing x at stage k and continuing optimally thereafter equals

$$L_k^*(x) = \text{ess inf}_{\tau > k} E(R_\tau | X_1, X_2, \ldots, X_{k-1}, X_k = x), \tag{4.2}$$

where it is understood that n is arbitrarily chosen but fixed, and $1 \leq k < n$.

Let τ_x^* now be that rule which achieves $L_k^*(x)$ given in (4.2). The latter can then be written in the form

$$L_k^*(x) = \sum_{j=1}^{k-1} P(X_{\tau_x^*} > X_j) + 1 + P(X_{\tau_x^*} \geq x) + E\Big(\sum_{j=k+1, j \neq \tau_x^*}^{n} I(X_j \leq X_{\tau_x^*}) \Big). \tag{4.3}$$

Further, let $\tilde{L}_k(x,y)$ be that modification of (4.3) which replaces x (only) by y without replacing τ_x^* by τ_y^*, i.e. formally

$$\tilde{L}_k(x,y) = L_k^*(x) - P(X_{\tau_x^*} \geq x) + P(X_{\tau_x^*} \geq y). \tag{4.4}$$

$\tilde{L}_k(x,y)$ is then the conditional expected loss, given $X_1, X_2, \ldots, X_{k-1}, X_k = y$, of the value obtained by the suboptimal continuation τ_x^* (which is optimal for $x = y$). Since $P(X_{\tau_x^*} \geq y)$ is non-increasing in y, it follows that $\tilde{L}_k(x,y)$ is non-increasing in y. Thus

$$\forall y \in [x,1] : \tilde{L}_k(x,y) \leq \tilde{L}_k(x,x) = L_k^*(x) \text{ a.s.} \tag{4.5}$$

On the other hand, suboptimality of τ_x^* for the history $X_1, X_2, \ldots X_{k-1}, X_k = y$ implies

$$\tilde{L}_k(x,y) \geq \tilde{L}_k(y,y) = L_k^*(y) \text{ a.s..} \tag{4.6}$$

Combining inequalities (4.5) and (4.6) yields

$$\forall 0 \le x \le y \le 1 : L_k^*(x) \ge L_k^*(y), \text{ a.s.} \tag{4.7}$$

i.e., $L_k^*(x)$ is a.s non-increasing in x.

An increasing $L_k(x)$ faces therefore a non-increasing $L_k^*(x)$ as a function of x. (4.1) tells us further that for all $k = 1, 2, \ldots, n$, $L_k(0) = 1$ and $L_k(1) = n$ a.s., so that $L_k^*(x)$ defined in (4.2) must lie in this range for any $x \in [0, 1]$. This proves for each stage k and $1 \le j \le k$ the existence of optimal thresholds $p_j(\cdots)$ such that any $X_j > p_j(\cdots)$ must be refused whereas the first $X_k \le p_k(\cdots)$ must be accepted.

(ii) The proof of the second part (optimal thresholds are monotone increasing) becomes shorter when we note that, by definition, the $p_k(...)$'s need only be defined for those $X_1, X_2 \ldots, X_{k-1}$ which are unacceptable. Therefore it suffices to show that if we replace an acceptable $X_k = x$ by an unacceptable $X'_k > x$ then any $X_{k+1} \le x$ is again acceptable.

Let $X_k = x \le p_k(...)$. This X_k is thus , by definition, acceptable. Now, if X_k is replaced by an unacceptable X'_k then, by definition, $X'_k > x$. It follows then from (4.1) that for a fixed history $X_1, X_2, ..., X_{k-1}$ we have $L_{k+1}(x) = L_k(x) - x$, because $X'_k > x$ leaves the relative rank of x at stage $k + 1$ unchanged. Therefore $L_{k+1}(x) < L_k(x)$ a.s.

It suffices now to show that $L_k^*(x)$ (see (4.2)) is non-decreasing in k, because then we have

$$\forall k = 1, \ldots, n-1 : L_k(x) \le L_k^*(x) \text{ a.s.} \Rightarrow L_{k+1}(x) \le L_{k+1}^*(x) \text{ a.s,} \tag{4.8}$$

i.e. $X_{k+1} = x$ is then acceptable, so that $p_{k+1}(...) \ge p_k(...)$ a.s. for any k and any history $X_1, X_2, \cdots, X_{k-1}$.

To see that $L_k^*(x)$ is non-decreasing in k a.s., we first note that $L_k^*(x)$ is invariant a.s. with respect to any permutation of the history. Indeed, if the stopping rule τ_x^* achieves $L_k^*(x)$ in (4.3) then τ_x^* achieves the same value for the history $(X_{i_1}, X_{i_2}, \cdots, X_{i_{k-1}}, X_{i_k} = x)$, because such a permutation only changes the order of summation of the first k terms on the RHS and since $(X_{k+1}, \cdots, X_n)$ is independent of the past. Now

$$L_{k+1}^*(x) = \text{ ess inf}_{\{k+1<\tau\le n\}} E(R_\tau | X_1, \ldots, X_{k-1}, X_k, X_{k+1} = x)$$

$$= \text{ess inf}_{\{k+1<\tau\le n\}} E(R_\tau | X_1, x_2, \ldots, X_{k-1}, X_k = x, X_{k+1}) =: \tilde{L}_{k+1}^*(x, X_{k+1}), \tag{4.9}$$

say, since $L^*_{k+1}(x)$ has the above described invariance property, and since X_k and X_{k+1} are (unconditioned) i.i.d. r.v.'s. But then

$$L^*_{k+1}(x, X_{k+1}) \geq \text{ess}\inf_{\{k<\tau\leq n\}} E(R_\tau | X_1, \cdots, X_{k-1}, X_k = x), \tag{4.10}$$

because optimal behavior on the set $\{k < \tau \leq n\}$ allows for stopping at stage $k+1$, or to continue optimally otherwise. Thus, from (4.9) and (4.10), $L^*_k(x) \leq L^*_{k+1}(x)$ a.s. so that (ii) is proved.

(iii) Finally, the proof of $p_{n-1}(X_1, X_2, \cdots, X_{n-2}) < 1$ a.s. follows immediately from $p_n(X_1, X_2, \cdots, X_{n-1}) \equiv 1$ and the argument we used in (2.9), because otherwise *any* memoryless threshold value p_{n-1} would do strictly better in expectation. ■

4.2 Full History Dependence of the Optimal Rule. In this section we will see that the optimal rule has the property of being fully history dependent, that is at no time may one discard past information. At each stage $k \leq n-1$, the exact values of all past observations may play a role in future optimal decisions. Clearly X_k could not be accepted if X_j for some $1 \leq j < k$ would have been accepted so that the events $\{\tau = k\}$ and $\{\tau > k\}$ always depend on all preceding observations in this sense. However, this does not imply that all preceding observations are relevant for future optimal decisions. A fully history dependent rule allows for no fading of memory at all. Varying some X_j (in certain cases even *any* X_j) by an arbitrary little amount may keep $X_j, \ldots, X_{k-1}$ unacceptable but be decisive for X_k to be acceptable or not.

To see that τ^* is fully history-dependent, we show that for each stage $s \leq n-1$, τ^* leads with positive probability to a situation where all preceding information $X_1, X_2, \cdots,$ X_{s-1} is needed to decide whether to stop and accept or, alternatively, to go on. Note that no information at all is needed at stage $s = n$, since $p_n(X_1, X_2, \cdots, X_{n-1}) \equiv 1$. Therefore we first look at $s = n-1$. The probability to reach stage $n-1$ equals

$$P(\tau^* > n-2) = E(\prod_{j=1}^{n-2} (1 - p_j(X_1, X_2, \cdots, X_{j-1})) \text{ a.s.}, \tag{4.11}$$

where we used the definition of optimal threshold functions as given in Lemma 4.1 and the definition (convention) $p_1(\cdot) := p_1$. According to Lemma 4.1 we have $p_j(X_1, X_2, \cdots, X_{j-1})$ $\leq p_{j+1}(X_1, X_2, \cdots, X_j) < 1$ a.s. for all $1 \leq j \leq n-2$, so that $P(\tau^* > n-2) > 0$. Similarly we obtain $P(\tau^* > n-2 | \tau^* \geq k) > 0$ for all $1 \leq k \leq n-2$.

Suppose now that $\tau^* I(\tau^* > n-2)$ depends on all preceding information, i.e. that τ^* is fully history-dependent on the set $\{\tau^* > n-2\}$. At stage k, the decision to go on depends by the principle of optimality on the optimal (future) expected loss. This expected loss involves, with $P(\tau^* > n-2 | \tau^* \geq k)$ being a.s. strictly positive, by definition the expected loss on the set $\{\tau^* > n-2\}$. In return, the latter depends according to our hypothesis on $X_1, X_2, \cdots, X_{n-2}$, i.e. in particular on $X_1, X_2, \cdots, X_{k-1}$.

Therefore, if all preceding information is needed at stage $s = n-1$, then all preceding information is needed at each stage $2 \leq k \leq n-2$, and thus it suffices to show that τ^* is fully history-dependent on $\{\tau^* > n-2\}$.

To see the latter we recall first that, at stage $s = n-1$, we have the same capacity as a half-prophet. According to Lemma 3.2 we therefore accept X_{n-1} if

$$r_{n-1} + X_{n-1} \leq n - \sum_{j=1}^{n-1} X_j \tag{4.12}$$

and accept X_n otherwise. If we write $r_{n-1}(x)$ for the rank of X_{n-1} given $X_{n-1} = x$, then the decision criterion becomes

$$r_{n-1}(x) + 2x \leq \; n - \sum_{j=1}^{n-2} X_j, \tag{4.13}$$

where the RHS does not depend on x. We shall now distinguish between two cases.

Case 1: Suppose first that there exists x^*, say, such that in (4.13) equality holds for both sides. Then, by definition, $X_{n-1} = x^*$ is acceptable. But since all $X_1, X_2, \cdots, X_{n-2}$ are a.s. different, their nearest neighbor distance ϵ, say, is a.s. positive.

Now move an arbitrarily chosen X_j on the RHS to $X_j + \epsilon/2$, say. This leaves by construction the relative ranks of $X_1, X_2, \cdots, X_{n-2}$ unchanged and the chosen jth observation stays unacceptable (becomes less desirable) by Lemma 4.1. Now $X_1, X_2, \cdots X_{j-1}$ clearly stay unacceptable because they precede X_j chronologically. On the other hand $X_{j+1}, X_{j+2}, \cdots, X_{n-2}$ are unacceptable as well, because they *were* unacceptable before the change and face with the change from X_j to $X_j + \epsilon/2$ a reduced optimal expected loss by being passed over. Therefore the modified history $X_1, X_2, \cdots, X_{j-1}, X_j + \epsilon/2, X_{j+1}, \ldots,$ X_{n-2} stays within the set $\{\tau^* > n-2\}$. However, the RHS of the criterion (4.13) decreases by $\epsilon/2$, so that now $X_{n-1} = x^*$ is unacceptable.

Case 2: Suppose now that (for a given history) there is no x^* to yield equality of both sides of (4.13). Then the critical value must coincide with one of $X_1, X_2, \cdots, X_{n-2}$. Suppose it coincides with X_k. For $\epsilon > 0$ chosen as the distance to the nearest neighbor let $X_{n-1} := x \in (X_k, X_k + \epsilon/2)$. Then X_{n-1} lies to the right of the critical value and is therefore unacceptable. But now move X_k into the interval $(x, x+\delta)$ with $0 < \delta < \epsilon/2$. This leaves the relative ranks of the new X_k and all other X_j's with $1 \leq j \leq n-2$ unchanged whereas the relative rank of X_{n-1} drops by one. The LHS of the criterion is therefore reduced by $1-2\delta$, whereas the RHS decreases by δ. But since $r_{n-1}(X_k)$ differs from $n - (X_1 + X_2, + \cdots + X_{n-2})$ by less than one, there must exist a sufficiently small δ such that this move renders $X_{n-1} = x$ acceptable.

However, as in case 1, for such a δ the observations $X_k + \delta, X_{k+1}, \cdots, X_{n-2}$ stay unacceptable. This means that the modified history stays also within the set $\{\tau^* > n-2\}$, and that X_k determines whether $\tau^* = n-1$, or, alternatively $\tau^* = n$.

Remark 4.3: Note that in case 2 (when a jump of the LHS occurs in X_k) the influence of X_k on the decision to accept or to refuse happens to be particularly strong, but that the inequalities in the criterion may be reversible by keeping X_k and X_{n-1} fixed and only varying the other preceding observations.

§5. Improving on Optimal Memoryless Rules.

In this section we shall describe a strategy which is superior to the optimal m^0-rule and (still) tractable. The construction depends on the following property of m^0-optimal thresholds $p^*_{n-1}(n)$ (see Section 2):

Lemma 5.1: The m^0-optimal threshold value $p^*_{n-1}(n)$ satisfies: $p^*_{n-1}(n) \to 1/2$ as $n \to \infty$.

Proof: Suppose we have observed $X_1, \ldots, X_{n-1}$ and are wondering whether to stop with $X_{n-1} = x$. The memoryless rule may not use any of the past information involving $X_1, \ldots, X_{n-2}$, except that we know $X_j > p_j(n)$ for $j = 1, \ldots, n-2$. If we stop with $X_{n-1} = x$ we expect to pay

$$\begin{aligned} &E\Big(\sum_{j=1}^{n-2} I(X_j < x) | X_1 > p_1(n), \ldots, X_{n-2} > p_{n-2}(n)\Big) + 1 + x \\ &\qquad = 1 + x + \sum_{j=1}^{n-2} \frac{(x - p_j(n))^+}{1 - p_j(n)}. \end{aligned} \tag{5.1}$$

If we continue with $X_{n-1} = x$, we must stop at the next observation and so expect to pay

$$
\begin{aligned}
E\Big(\sum_{j=1}^{n-2} I(X_j < X_n) | X_1 > p_1(n), \ldots, X_{n-2} > p_{n-2}(n)\Big) + (1-x) + 1 \\
= 2 - x + \sum_{j=1}^{n-2} \frac{(1-p_j(n))^2}{2},
\end{aligned} \tag{5.2}
$$

since under the conditioning, $X_1, \ldots, X_{n-2}, X_n$ are independent with X_j uniform on $(p_j(n), 1)$ for $j = 1, \ldots, n-2$ and X_n uniform on $(0,1)$. Therefore the critical value for X_{n-1} is that value of $x = p_{n-1}(n)$ at which (5.1) is equal to (5.2), namely,

$$
2p_{n-1}(n) + \sum_{j=1}^{n-2} \frac{(p_{n-1}(n) - p_j(n))^+}{1 - p_j(n)} = 1 + \sum_{j=1}^{n-2} \frac{(1-p_j(n))^2}{2}. \tag{5.3}
$$

Recall now Lemma 2.3 which states that for every $\alpha \in (0,1)$ and every sequence $k(n) \le n\alpha$, we have $p_{k(n)}(n) \to 0$ as $n \to \infty$. This implies that the average of the $p_j(n)$'s must converge to zero, because

$$
\begin{aligned}
\frac{1}{n-2} \sum_{j=1}^{n-2} p_j(n) &\le \frac{1}{n-2} \sum_{j=1}^{[n\alpha]} p_j(n) + 1 - \alpha \\
&\le \frac{1}{n-2} p_{[n\alpha]}(n) + 1 - \alpha \to 1 - \alpha.
\end{aligned} \tag{5.4}
$$

Since this is true for all $\alpha < 1$, we have $\sum_{j=1}^{n-2} p_j(n)/(n-2) \to 0$. This implies that the right side of (5.3), when divided by $n-2$, converges to 1/2.

A similar argument shows that the left side of (5.3), when divided by $n-2$, must eventually for large n be within a preassigned $\epsilon > 0$ of $p_{n-1}(n)$. We may conclude that $p_{n-1}(n) \to 1/2$ as $n \to \infty$. ■

Combining m^0-rules with half-prophet rules. Let $\tau(n)$ be the optimal m^0-rule defined by $p_1^*(n), p_2^*(n), \ldots, p_{n-1}^*(n), 1$, i.e.

$$
\tau(n) = \inf\{1 \le k \le n : X_k \le p_k^*(n)\}. \tag{5.5}
$$

Remember that it is unique. Further let

$$
\tau_h(n) := \min\{1 \le k \le n : L_k = h(k,n)\}, \tag{5.6}
$$

i.e. $\tau_h(n)$ denotes the earliest time k, at which a h(k)-prophet would stop. Finally, let

$$\sigma^*(n) := \min\{\tau(n), \tau_h(n)\}. \tag{5.7}$$

Theorem 5.2 For all sufficiently large n,

$$E(R_{\sigma^*(n)}) < E(R_{\tau(n)}).$$

Proof: We first note that $\tau_h(n) \leq n$ since $h(n,n) \leq n$. Also, if we stop at stage k with $\tau_h(n) = k$, whereas $\tau(n) > k$, then we act optimally since, according to (3.3), (3.4) and (3.5),

$$L_k^{(+)} = \inf_{\tau > k}(\int_0^1 (1 + \sum_{m=1}^{n} I(X_m \leq x))dF_\tau(x)) \leq L_k^*, \tag{5.8}$$

and therefore $L_k \leq L_k^*$. Thus $E(R_{\sigma^*(n)}) \leq E(R_{\tau(n)}|\tau(n) > t_h(n))$ for all n. Moreover, the preceding inequality becomes strict for $n > 2$, because if there are at least two more observations to come, any strategy will miss the smallest of these with a positive probability. It is therefore sufficient to show that for all sufficiently large n,

$$P(\sigma^*(n) < \tau(n)) = P(\tau_h(n) < \tau(n)) > 0. \tag{5.9}$$

Now let $a_n = p_{n-1}^*(n)$, $b_n = (1 + a_n)/2$ and $A_n = \{x_k \in (a_n, b_n), 1 \leq k \leq n\}$. Since $a_n < b_n < 1$ for all $n \geq 2$ we have $P(A_n) > 0$ for all $n \geq 2$. Moreover, since on A_n no X_k with $1 \leq k \leq n-1$ is acceptable under the optimal m^0-rule, we obtain $P(\tau(n) = n) \geq P(A_n) > 0$. Now

$$P(\tau_h(n) < \tau(n)) \geq P(\tau_h(n) \leq n-1|A_n)P(A_n)$$

so that it suffices to show that $P(\tau_h(n) \leq n-1|A_n) > 0$. Note that

$$P(\tau_h(n) \leq n-1|A_n) \geq P(\tau_h(n) \leq n-1|A_n, r_{n-1} = 1)P(r_{n-1} = 1|A_n).$$

By Renyi's record value theorem (Renyi (1962)) for i.i.d. r.v's we have $P(r_{n-1} = 1|A_n) = 1/(n-1) > 0$, so that it suffices again to show that $P(\tau_h(n) \leq n-1|A_n, r_{n-1} = 1) > 0$, for all n sufficiently large. This latter term equals indeed one for almost all n, since

$$P(\tau_h(n) \leq n-1|A_n, r_{n-1} = 1) = P(L_{n-1} \leq L_{n-1}^{(+)}|A_n, r_{n-1} = 1)$$

$$= P(1 + X_{n-1} \le 1 + \sum_{j=1}^{n-1}(1 - X_j)|A_n) \ge P(2X_{n-1} \le 1 + (n-2)(1-b_n)),$$

where we used $X_1, X_2, \cdots, X_{n-2} < b_n$ on A_n in the inequality. Using it again for X_{n-1} yields

$$P(2X_{n-1} \le 1 + (n-2)(1-b_n)) \ge P(0 \le n - 1 - nb_n) = 1 \text{ for almost all } n,$$

because, by Lemma 5.1, $b_n \to 3/4$ as $n \to \infty$, and this completes the proof. ■

Remark 5.3: Table 1a of Section 2 strongly suggests that $p_{n-1}(n)$ increases for $n = 2$ up to $n = 5$ with value $.5359\cdots$ and decreases thereafter, converging (as we know) to $1/2$, but the proof of this does not seem to be straightforward. One can easily verify that it would imply strict improvement of the rule $\sigma^*(n)$ for all $n \ge 3$.

More important than this observation however is the fact (see e.g. Example 5.4) that the strict improvement is not only due to what the proof is based on, namely to detecting late *record* values.

Example 5.4: Let $n = 12$ and $X_1 = 0.12, X_2 = 0.13$. Further let $X_3, X_4, \cdots, X_{10}$ all be approximately $.6$, and finally let $X_{11} = 0.55$. Then all X_k for $1 \le k \le 11$ are unacceptable under the optimal m^0-rule (see Table 1a), but (5.10) and Lemma 3.2 show that $\sigma^*(12) = \tau_h(12) = 11$ and stops with a 3-record only. Nevertheless,the expected loss by stopping at stage 11 equals 3.55, whereas stopping at stage 12 yields a much higher expected loss of $12 - 0.12 - 0.13 - 0.55 - (X_3 + \cdots + X_{10}) \approx 6.4$, so that even for non-record values the difference can be quite large. Clearly it would go up, if we moved X_1 and/or X_2 to the right of 0.6. ■

We could also give examples to show that the improvement can become effective earlier than at stage $n-1$. $\sigma^*(n)$ is so far the "uniformly" best strategy we know.

The half-prophet rule $\tau_h(n)$ *alone* would not do well for larger n; it succeeds more often than the optimal m^0-rule to select the smallest or second smallest observation but allows for worse outliers than the optimal m^0-rule. Lemma 3.2 shows the influence of the powers $n - k$, which let $L_k^{(+)}$ grow only slowly as k increases.

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