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Author
Luce, R. Duncan

Publication Date
2002-10-01

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R. Duncan Luce
University of California, Irvine

October 21, 2002

Running title: Increasing Increment Generalizations

Address: Social Science Plaza
University of California
Irvine, CA 92697-5100
Phone: 949 824-6239
FAX: 949 824-3733
e-mail rdluce@uci.edu

File: Nonzero23.tex

Abstract

Empirical evidence from both utility and psychophysical experiments suggests that people respond quite differently–perhaps discontinuously–to stimulus pairs when one consequence or signal is set to "zero." Such stimuli are called unitary. The author's earlier theories assumed otherwise. In particular, the key property of segregation relating gambles and joint receipts (or presentations) involves unitary stimuli. Also, the representation of unitary stimuli was assumed to be separable (i.e., multiplicative). The theories developed here do not invoke separability and segregation simultaneously. In the commutative case with identity e, which is relevant to utility, a class of representations more general than rank-dependent utility (RDU) is found when V is an additive representation of joint receipt, namely,

$$V(x, C; y) = V(y) + M_C \left[V(x) - V(y) \right] \quad (x \succsim y \succ e),$$

where $M_C(0) = 0$ and $M_C(R)$ is strictly increasing in R. This form and its natural generalization to gambles of order $n > 2$, which is also axiomatized, appear to encompass models of configural weights and decision affect. For joint receipts that either are non-commutative with a one-sided identity or are idempotent with no identity, which are the cases relevant to psychophysics, the results for non-unitary stimuli include the prediction of a conjoint additive representation and a prediction of constant bias independent of signal intensity.

Keywords: distributivity, generalized RDU, increasing increments, segregation, unitary stimuli

In various closely related theoretical developments both in utility theory (Aczél, et al. 2002; Luce, 2000) and in psychophysics (Luce, 2002a,b) rank-dependent representations play a key role. In the notation of utility theory, binary rank-dependent utility (RDU) has the form

$$U(x,C;y) = \begin{cases} U(x)W(C) + U(y)[1-W(C)], & x \succ y \succsim e \\ U(x), & x \sim y \succsim e \\ U(x)[1-W(\overline{C})] + U(y)W(\overline{C})], & y \succ x \succsim e \end{cases}, \tag{1}$$

where $\succsim$ denotes weak preference; e is no change from the status quo; x, y are from a set $\mathcal{D}_0$ of valued consequences; C is a chance event from an algebra $\mathcal{E}$ of events and $\overline{C}$ is its complement; $(x,C;y)$ denotes a gamble in the case of utility in which, when the underlying chance experiment is run and the event C occurs, then the consequence is x, whereas when $\overline{C}$ occurs, then the consequence is y. The utility function U is an order-preserving mapping from the domain $\mathcal{D}_1$ of binary gambles and their joint receipts (see Section 1.1 below) either onto $\mathbb{R}_+ = [0,\infty[$ or, in the bounded case, onto $[0,k[$, where $k > 0$, with $U(e) = 0$. The function W maps $\mathcal{E}$ into $\mathbb{R}_+$ (and usually onto $[0,1]$ in the utility interpretation and onto $\mathbb{R}_+$ in the psychophysical interpretation, see below) with $W(\emptyset) = 0$.

Although the present formulation is cast in the symbols of utility theory, using the reinterpretation of Luce (2002a,b) makes it also relevant to psychophysics. In that case, the symbols x, y refer to physical intensities measured as differences between the actual intensity and that of the threshold, so 0 corresponds to the threshold intensity and plays the role of e. (Observational error of whatever sort is not modelled.) The event C is replaced by the real interval $[0,p]$, $p > 0$, and the respondent is asked to report the intensity z for which the intensity "interval" $[y,z]$ stands subjectively in the ratio p to the interval $[y,x]$. This experimental method is a generalization of the well known method of magnitude production, which is the case of $y = 0$. The notation $x \circ_p y := z$ is used, which is analogous to the alternative gamble notation $x \circ_C y := (x,C;y)$. (This notation proves useful in formulating Theorems 3 and 4.)

Note that if RDU, (1), holds and preserves the order $\succsim$, then several behavioral properties are necessarily satisfied. First, $\succsim$ must be a weak order and so $\sim := \succsim \cap \precsim$ is an equivalence relation. So we assume:

Assumption 1. $\succsim$ *is a weak order.*

Assumption 2. *Complementarity: For all* $x, y \in \mathcal{D}_0, x \prec y, C \in \mathcal{E}$,

$$(x,C;y) \sim (y,\overline{C};x). \tag{2}$$

Given this assumption, it is sufficient to state things just for the case $x \succsim y \succsim e$, i.e., the case of gains.

Note that RDU also implies *idempotence:*

$$(x, C; x) \sim x \quad (x \in \mathcal{D}_0, C \in \mathcal{E}). \tag{3}$$

See Luce and Marley (2000) for some of what happens when (3) is violated. I do not, however, invoke it as an underlying assumption because of the psychophysical interpretation of the primitives in which idempotence is not necessarily true.

Because both the utility and psychophysical interpretations are possible, from here on the more neutral term "stimulus," rather than gamble, is used for $(x, C; y)$. Moreover, treating C as a parameter, these are referred to as binary stimuli, where the binary refers to the x, y pair.

For $x \succ y \succsim e$, (1) can be rewritten

$$\frac{U(x, C; y) - U(y)}{U(x) - U(y)} = W(C),$$

which makes clear that under the mappings of U and W a certain simple proportionality exists between the "interval" from the lesser consequence to the gamble and the "interval" between the two consequences.

For utility, the theory for strict losses is formally identical to that for gains. The case of mixed gambles, a gain and a loss, is discussed briefly in Section 2.5. For psychophysics, only the gains case arises. So I focus primarily on gains.

RDU also implies that the binary stimulus $(x, C; y)$ is strictly monotonic increasing in the two consequences, i.e., for events C with $0 < W(C) < 1$,

$$x \succ x' \quad \Leftrightarrow \quad (x, C; y) \succ (x', C; y) \tag{4}$$

$$y \succ y' \quad \Leftrightarrow \quad (x, C; y) \succ (x, C; y'). \tag{5}$$

For greater detail, see Luce (2000). However, because in the psychophysical interpretation p is not restricted to $[0, 1]$, one is forced to replace right strictly increasing monotonicity, (5), by the much weaker *right substitutability*,

$$y \sim y' \Rightarrow (x, C; y) \sim (x, C; y'). \tag{6}$$

For greater detail, see Aczél, et al. (2002). In addition it is plausible to suppose that $(x, C; y)$ is not a constant over any non-trivial interval of y values. In the functional equation literature, such non-constancy is termed *philandering.*

As a result of these considerations:

Assumption 3. *Stimuli satisfy left strictly increasing monotonicity and right substitutability and right philandering for* $y \succ e$,.

Magnitude production data on auditory signals, collected by Steingrimsson (2002), suggest that respondents may deal with $x \circ_p 0$ differently from $x \circ_p y$, $y > 0$. And considerable data on gambles reported by Michael Birnbaum and his colleagues (see especially Mellers, et al. 1992, and see Birnbaum, 1997, for a summary) make clear that respondents respond quite differently when $y = e$ from when $y \succ e$—in particular, for money lotteries with probabilities p near

1, consequence monotonicity, (5), is violated in the sense that for some p_0 and some $y \succ e$,

$$(x, p; e) \succ (x, p; y) \quad (p_0 \le p < 1).$$

I suspect that subjects shift their strategy when they encounter unitary elements, e.g., by calculating an approximation to expected value which is far easier to do mentally for unitary binary gambles than for non-unitary ones. In any event, there are good empirical reasons to avoid mixing *unitary* stimuli of the form $(x, C; e)$ and non-unitary ones in developing the theory for general binary stimuli $(x, C; y)$ and to have a distinct theory for $(x, C; e)$. My previous work has depended on examining the relations among unitary and non-unitary stimuli and so must be modified to accommodate these findings.

Note that by RDU, (1),

$$U(x, C; e) = U(x)W(C). \tag{7}$$

This representation of unitary stimuli is called *separable*. An obvious accommodation to the behavioral discontinuity at e is to change the representation of separability to a different weight function, and so (1) becomes

$$U(x, C; y) = \begin{cases} U(x)W(C) + U(y)[1 - W(C)] & (x \succsim y, y \succ e) \\ U(x)W_0(C) & (x \succsim y, y = e) \end{cases}, \tag{8}$$

where the function $W_0 \ne W$. However, the change to (8) is, by itself, not adequate as will be shown in Proposition 3 in Section 2.4.

This article investigates what happens when two important properties of the earlier theories are dropped. First, we do not impose separability of unitary gambles but do continue to assume segregation (defined in Section 1.2). Next, we drop segregation, which involves mixed unitary and non-unitary stimuli, and replace it by the closely related distributivity property (defined in Section 1.3) that avoids the unitary ones. There are six substantive sections plus three appendices. Section 1 introduces additional major ideas of the earlier model: joint receipt, segregation, distributivity, simple decomposability, and two new types of rank-dependent representations. Sections 2-3 explore what can happen by simply dropping separability from the structure and retaining segregation. These are, respectively, concerned with the case of commutative joint receipts (defined in Section 1.1) which is relevant in a utility context and with the non-commutative case in either a utility or a psychophysical context. Two important results are applications of Theorem 2 to induced operations (see Assumption 5) in Sections 3.2 and 3.3. Sections 4-5 do the comparable thing when separability plays no role and segregation is replaced by distributivity. The binary generalization of rank-dependent utility is extended to general gambles in Section 4.3. The concluding Section 6 has three parts. The first concerns important predictions about the representation. The theories of Theorems 1 and 5 and the corollaries of Theorems 3 and 4 all predict that the Thomsen condition of additive conjoint measurement holds for joint receipt. It should be checked experimentally. And those of non-commutative case, corollaries of Theorems 3 and

4, predict in some cases a constant bias independent of signal level. Second, with a separable representation of unitary stimuli, we consider the condition needed to be sure that the same utility function is involved for both unitary and non-unitary stimuli. And third, the results are summarized. Appendix A proves the results involving segregation. Appendix B lists two results of Lundberg (2002) concerning the distributivity and a related functional equation. These are used in the proofs of Theorems 5 and 7 in Appendix C.

1 Additional Important Concepts

1.1 Joint receipt

In addition to the binary stimuli, the theory also involves the concept of *joint receipt*, i.e., of having or receiving two valued things at once, denoted $f \oplus g$, where f, g are stimuli. We make the following general assumptions about $\oplus$:

Assumption 4. $\oplus$ *is strictly monotonically increasing in each variable.*

It is not, however, assumed that $\oplus$ is necessarily a closed operation. For the utility context, closure may be plausible, but for the psychophysical one, it may not be in a direct sense. For example, suppose that $x \oplus y$ means that intensity x is presented to the left ear and intensity y to the right, then $(x \oplus y) \oplus z$ has no natural meaning. However, certain closely related defined operations are closed and permit the construction of a representation of $\oplus$. To that end, we introduce:

Assumption 5. *For each x, y, the following three indifferences can all be solved for the z terms:*

$$x \oplus y \sim z_l \oplus e \sim e \oplus z_r \sim z_s \oplus z_s. \tag{9}$$

This solvability assumption permits the following definitions:

$$x \oplus_l y := z_l,\ x \oplus_r y := z_r,\ x \oplus_s y := z_s. \tag{10}$$

It is easy to verify the following facts: each $\oplus_i, i = l, r, s$, is a closed operation; each $\oplus_i$ is strictly monotonic increasing in each variable (because $\oplus$ is); e is a right identity of $\oplus_l$, i.e.,

$$x \oplus_l e \sim x \quad (x \in \mathcal{D}), \tag{11}$$

and a left identity of of $\oplus_r$, i.e.,

$$e \oplus_r y \sim y \quad (y \in \mathcal{D}). \tag{12}$$

The operation $\oplus_s$ is idempotent, i.e.,

$$x \oplus_s x \sim x \quad (x \in \mathcal{D}). \tag{13}$$

We assume that in addition to pure consequences and first and second order binary stimuli we also have the comparable joint receipts. This extended domain is denoted $\mathcal{D}$ and Assumptions 1-5 apply to it.

1.2 Segregation

In the earlier work the property of *right segregation*

$$(x, C; e) \oplus y \sim (x \oplus y, C; e \oplus y) \quad (x \succsim e, y \succsim e) \tag{14}$$

was assumed to hold but with the added assumption that e is a left identity of $\oplus$ and so $e \oplus y = y$. (A parallel theory holds if e is a right identity and left segregation holds in which y appears on the left.) Note that segregation involves a unitary stimulus on the left and a non-unitary one on the right when $y \succ e$.

In the case where $\oplus$ is commutative and associative, e is a two-sided identity of $\oplus$, and RDU and segregation hold, then U over $\oplus$ turns out to have the following *p-additive* form:

$$U(f \oplus g) = U(f) + U(g) + \delta U(f)U(g). \tag{15}$$

(For a proof see Luce, 2000, Theorem 4.4.4, pp. 151-152). It is often convenient to work with the representation onto $\mathbb{R}_+ := [0, \infty[$ that is defined by the non-linear transformation

$$V(x) := \begin{cases} sgn(\delta)\ln[1 + \delta U(x)], & (\delta \neq 0) \\ U(x), & (\delta = 0) \end{cases}, \tag{16}$$

which, when U is p-additive, is easily shown to be additive over $\oplus$:

$$V(x \oplus y) = V(x) + V(y) \quad (x \succsim e, y \succsim e). \tag{17}$$

Note that if $\oplus$ is a closed operation, then V is unbounded even though the p-additive form for U is bounded for any $\delta < 0$. Note, also, that $V(e) = 0$. Conversely, assume (17), and define U over $\oplus$ by

$$U(x) := \begin{cases} \frac{1}{\delta}[\exp(sgn(\delta)V(x))] - 1, & (\delta \neq 0) \\ U(x), & (\delta = 0) \end{cases}, \tag{18}$$

Then U is p-additive, (15).

The function U associated with the RDU representation is called a *utility function* whereas V is called a *value function*.

The commutative case was generalized to the non-commutative one in Luce (2002a,b) in order to accommodate a psychophysical interpretation of the primitives, and so I treat that case here as well. In the non-commutative case the additive value function is replaced by a form of additive conjoint representation.

1.3 Distributivity

A very natural generalization of right segregation is:

$$(x, C; y) \oplus z \sim (x \oplus z, C; y \oplus z) \quad (x \succsim y \succ e, z \succsim e). \tag{19}$$

Property (19), which avoids unitary stimuli, is called *right distributivity.* (*Left distributivity* inverts the order of z under the operation $\oplus$.

As we will see in Theorem 1, in a fairly general context, including the earlier RDU theory, right distributivity is implied by right segregation and not conversely because right distributivity is restricted to $y \succ e$. In that sense, it is a weaker property. But three independent variables are involved rather than the two of segregation. In that sense it is a stronger property. The balance between the two is not a priori clear, but it turns out (Theorem 5 and 7) that it yields the same representations as that of Theorems 1 and 2, but without forcing separability. The proof, unfortunately, is a great deal more complex.

1.4 Event commutativity

The property

$$((x,C;y),D;y) \sim ((x,D;y),C;y), \tag{20}$$

where the stimuli are independently realized, is called *event commutativity.* It has played a fairly significant role in earlier versions of the theory, in large part, because RDU satisfies it and because it has received some empirical support in utility theory (Luce, 2000) and in psychophysics for $y = e$ (Ellermeier & Faulhammer, 2000). We will investigate it in the present context.[1]

1.5 Simple decomposability

For both stimuli and their joint receipt, a simplifying assumption is made that is called *simple decomposability.* This may be stated in terms of either U or V. Here we state it for V which has the advantage that the range of V is $[0,\infty[$. Informally, it is the assumption that $V(x \oplus y)$ and $V(x,C;y)$ depend on x, y only via $V(x), V(y)$. Formally, there exists a function $F : [0,\infty[\times[0,\infty[\overset{onto}{\longrightarrow} [0,\infty[$ and, for each $C \in \mathcal{E}$, an algebra of events, there exists a function $G_C :]0,\infty[\times]0,\infty[\overset{onto}{\longrightarrow}]0,\infty[$, such that

$$V(x \oplus y) = F[V(x),V(y)], \qquad (x \succsim e, y \succsim e) \tag{21}$$

$$V(x,C;y) = \begin{cases} G_C[V(x),V(y)] & (x \succsim y \succ e) \\ G_{\overline{C}}[V(y),V(x)] & (y \succsim x \succ e) \end{cases}, \tag{22}$$

where F is strictly increasing in both variables because $\oplus$ is (Assumption 5), and G_C is strictly increasing in the first variable because $(x,C;y)$ is, and we postulate that it is continuous and philandering in the second (Assumption 4).

Equation (18) for $\delta \neq 0$ together with (21) yields

$$\begin{aligned} U(x \oplus y) &= \frac{1}{\delta}[\exp sgn(\delta)V(x \oplus y) - 1] \\ &= \frac{1}{\delta}(\exp sgn(\delta)[V(x),V(y)] - 1) \\ &= \frac{1}{\delta}(\exp F[sgn(\delta)\ln[1+\delta U(x)], sgn(\delta)\ln[1+\delta U(y)] - 1) \\ &= F^*[U(x),U(y)], \end{aligned}$$

where F^* is defined in the obvious way in terms of F and δ. A similar calculation shows that $U(x, C; y) = G^*_C[U(x), U(y)]$. In the case of non-commutative joint receipt, I invoke decomposability in terms of U.

Note that if e is a left identity of $\oplus$, then $F(0, Y) = Y$, if e is a right identity, then $F(X, 0) = X$, and if $\oplus$ is idempotent $F(X, X) = X$. If the stimuli are idempotent then, $G_C(X, X) = X$. Further, if U is p-additive, then it is decomposable.

1.6 Increasing Utility and Value Increments

The following decomposable representations of gambles, which generalize RDU, will play a substantial role in what follows.

An *increasing value increments representation* (abbreviated *IVI*) holds iff there is a strictly increasing function $L_C : \mathbb{R}_+ \overset{into}{\longrightarrow} \mathbb{R}_+$, such that, for $x \succsim y$,

$$V(x, C; y) = L_C[V(x) - V(y)] + V(y), \tag{23}$$

where $L_C(Z) = V[V^{-1}(Z), C; e]$.

In parallel, we say *increasing utility increments (IUI)* holds if there is a strictly increasing function $M_C : [0, k[\overset{into}{\longrightarrow} [0, k[$, such that, for $x \succsim y$,

$$U(x, C; y) = M_C[U(x) - U(y)] + U(y), \tag{24}$$

with $M_C(Z) = U[U^{-1}(Z), C; e]$.

An important special case arises when L_C is decomposable in the following sense: there exists a strictly increasing function $W : \mathcal{E} \overset{onto}{\longrightarrow} W(\mathcal{E})$, where $W(\mathcal{E}) = [0, 1]$ in the utility interpretation and $= \mathbb{R}_+$ in the utility one, and a function $L : \mathbb{R}_+ \times W(\mathcal{E}) \overset{onto}{\longrightarrow} \mathbb{R}_+$ such that

$$L_C(R) = L[R, W(C)],$$

and there exists a strictly increasing function $l : \mathbb{R}_+ \overset{into}{\longrightarrow} \mathbb{R}_+$ such that

$$l[V(x, C; y) - V(y)] = l[V(x) - V(y)]W(C), \tag{25}$$

where $L_C(R) = l^{-1}[l(R)W(C)]$. The form (25) is called *proportional increasing value increments (PIVI).* If the parallel form holds for U, i.e.,

$$m[U(x, C; y) - U(y)] = m[U(x) - U(y)]W(C), \tag{26}$$

we call it *PIUI.* Note that, with $Z = U(x) - U(y), W = W(c)$, (24)) with (26) gives

$$M_C(Z) = m^{-1}(ZW). \tag{27}$$

Two related observations about the PIVI (PIUI) representation are important. First, for non-linear l (m) unitary gambles are not separable, (7), in the representation V (U) but are in lV (mU). Second, if l (m) is a power function with exponent $\beta_l > 0$ ($\beta_m > 0$), then the representation separable in V with the weighting function W^{1/β_l} (W^{1/β_m}). Thus, in these cases we may well have the more general form of RDU, (8).

2 IVI, Segregation, and Additive Joint Receipt

2.1 The basic result

We begin by exploring several aspects of the interplay of p-additivity, (15) and segregation, (14), and in Corollary 2 the corresponding impact of separability, (7), on the results.

Theorem 1. *Suppose that a structure of binary stimuli and joint receipts satisfies the following: Assumptions 1-4; there exists an additive representation V over $\oplus$ [and a p-additive representation U defined in terms of V by (18)]; and the representation of stimuli is decomposable, (22).*

(1) *The following three statements are equivalent:*

- **(a)** *Segregation holds.*
- **(b)** *The representation V is an IVI one, (23).*
- **(c)** *There is a family of functions $l_C : [1, \infty[\overset{into}{\longrightarrow} [1, \infty[$ when $\delta > 0$ and $[0, 1[\overset{into}{\longrightarrow} [0, 1[$ when $\delta < 0$ such that*

$$\delta U(x, C; y) + 1 = l_C \left(\frac{\delta U(x) + 1}{\delta U(y) + 1} \right) [\delta U(y) + 1] \tag{28}$$

(2) *The IVI representation, (23), implies that right and left distributivity both hold.*

(3) *The following three statements are equivalent:*

- **(a)** *IVI, (23), and event commutativity, (20), both hold.*
- **(b)** *The functions of the IVI representation, (23), satisfy*

$$L_C (L_D) = L_D (L_C) . \tag{29}$$

- **(c)** *The PIVI representation, (25), holds.*

The proofs of this Theorem, along with four related propositions, and of Theorems 3 and 4 are given in Appendix A. The proofs of Theorems 5-7 are in Appendix C.

2.2 Qualitative version of L_C

A. A. J. Marley suggested that I investigate the following qualitative formulation. Assume that for $x \succ y$, there exists a solution z to the indifference $x \sim z \oplus y$. Define the "subtraction" operation $\ominus$ by

$$x \ominus y \sim z \Leftrightarrow x \sim z \oplus y \quad (x \succsim y). \tag{30}$$

With that, suppose that for each $C \in \mathcal{E}$, there exists a function $\Phi_C : \mathcal{D}_0 \xrightarrow{into} \mathcal{D}_0$ that is strictly increasing in the first argument such that for all $x \succsim y \succ e$

$$(x, C; y) \sim y \oplus \Phi_C (x \ominus y) . \tag{31}$$

The function Φ_C plays a qualitative role that is close to the numerical one M_C, as the following result shows.

Proposition 1. *Suppose that the assumptions of Theorem 1 hold and that* $\ominus$*, (30), is defined.*

(1) *Then the property (31) implies right distributivity, (19).*

(2) *Any two of IVI, i.e., (23), (31), and*

$$L_C [V(z)] = V [\Phi_C(z)] \tag{32}$$

imply the third.

(3) *If segregation, (14), holds, then (31) implies for all* $z \in \mathcal{D}$

$$\Phi_C (z) \sim (z, C; e).$$

The latter assertion means that Φ_C generalizes the concept of unitary stimuli.

2.3 The IUI representation

Proposition 2. *Suppose that the conditions of Theorem 1 and IVI, (23) hold. Let* U *be related to* V *by (16). Then IUI holds with a function* $L_C : [0, k[\xrightarrow{into} [0, k[$ *iff either*

(1) $\delta = 0$, *in which case* $U = V$ *and* $M_C = L_C$, *and RDU holds over stimuli with* $x \succsim y \succ e$ *iff*

$$L_C(R) = RW(C) \quad (R \geq 0), \tag{33}$$

or

(2) $\delta \neq 0$ *in which case*

$$M_C(R) = RW(C) \tag{34}$$

$$\Leftrightarrow \quad L_C(R) = \ln [1 + W(C) (\exp R - 1)] \quad (R \geq 0), \tag{35}$$

which is RDU for (U, W).

2.4 Forcing the RDU Representation

Proposition 3. *Suppose that the assumptions of Theorem 1 and IVI, (23) hold.*

(1) *The following three statements are equivalent:*

- **(a)** *U satisfies separability.*
- **(b)** *The stronger form of RDU, (1), holds.*
- **(c)** *The function l_C is of the form: $l_C(R) = (R-1)W(C) + 1$.*

(2) *Suppose that IUI holds. Then segregation iff RDU, (1), holds.*

This means that under the assumptions of Theorem 1 we cannot have the desired representation (U, W, W_0) of (8) with $W \neq W_0$. Therefore, to avoid that unwanted conclusion, we must either not invoke separability or abandon segregation and, in general, anything that invokes *unitary stimuli* of the form $(x, C; e)$, and instead develop the theory using only strict gains, $(x, C; y)$, $x \succsim y \succ e$. Note that the restriction $y \succ e$ pertains to $(x, C; y)$ but definitely not to $x \oplus y$. A theory for $(x, C; e)$ must be dealt with separately, for instance, by invoking conjoint measurement, but then steps must be taken to insure that the same utility function works for unitary and non-unitary stimuli (see Section 6.2).

Consider the following second form of separability that arises when RDU holds for non-unitary stimuli, where the limit on the left exists by continuity:

$$\lim_{y \searrow e} U(x, C; y) = U(x)W(C), \tag{36}$$

and $W \neq W_0$.

Proposition 4. *Suppose that the following are satisfied: the assumptions of Theorem 1 and IVI, (23); V is additive; separability, (7); and the separable limit, (36), exists. Then there exists a representation (U, W, W_0) such that U over $\oplus$ is p-additive and the representation over stimuli is an RDU one in the sense of (8).*

Note that each of the conditions *(7), (23), and (36)* is necessary if RDU in the form of (8) is satisfied.

Given this result, the important open question is how to get the IVI representation without invoking segregation. It turns out the distributivity is just what is needed (Sections 4 and 5).

2.5 Mixed binary gambles

Luce (2000) worked out two possible theories for binary gambles where $x \succ e \succ y$, which are called mixed. One assumption was an generalization of right segregation, namely,

$$(x, C; y) \sim \begin{cases} (x \ominus y, C; e) \oplus y, & (x, C; y) \succsim e \\ (e, C; y \ominus x) \oplus x, & (x, C; y) \prec e \end{cases}.$$

The other, called duplex decomposition, is non-rational but has found some experimental support (see Luce, 2000)

$$(x, C; y) \sim (x, C; e) \oplus (e, C; y).$$

Later direct study of these properties by Cho, et al. (2002) make clear that between them they account for, at most, 3/4 of the data and perhaps as little as 1/2, depending upon what criterion one uses for accepting noisy data as supporting an indifference.

Formulas are given in Luce (2000) for the utility of these cases for both $\delta = 0$ and $\delta \neq 0$ using, of course, the utility functions U derived for both gains and losses and the corresponding weighting function W^+, W^-. Because both expressions are in terms of unitary gambles on the right, according to the present results we should merely replace the weights by W_0^+, W_0^-. However, the direct data suggests that we should probably seek a better theory.

3 IUI, Segregation, and Non-Commutative Joint Receipt

3.1 IUI and Right Segregation

We turn next to a result somewhat parallel to Theorem 1 but explicitly rejects the assumption that $\oplus$ has an additive representation. Thus, we deal only with the utility function U.

Theorem 2. *Suppose that a structure of binary gambles and joint receipts satisfy Assumptions 1-4 and there is a utility function U with a decomposable representation F for $\oplus$, and an IUI representation M_C for gambles. Define $H(X, Y) := F(X, Y) - F(0, Y)$.*

(1) *Suppose that right segregation, (14), holds. Then H and M_C satisfy the following functional equation:*

$$H[M_C(X), Y] = M_C[H(X, Y)]. \tag{37}$$

(2) *Any two of the following statements implies the third:*

- **(a)** *Right segregation, (14) holds.*
- **(b)** *PIUI, (26), holds with the functions m and W.*
- **(c)** *The operation $\oplus$ has the representation*

$$m[U(x \oplus y) - U(e \oplus y)] = m[U(x)]\sigma_r[U(y)], \tag{38}$$

where $\sigma_r[U(y)] > 0$.

(3) *If* $e \oplus e \sim e$, *then (38) becomes*

$$m[U(x \oplus y) - U(e \oplus y)] = m[U(x \oplus e)]\sigma_r^*[U(y)], \tag{39}$$

where $\sigma_r^*[U(y)] = \sigma_r[U(y)]/\sigma_r(0)$

(4) *Assume that* ***(2)(a)-(c)*** *are satisfied. Then the following statements are equivalent:*

(a) *RDU, (1), holds.*

(b) $m(X) = aX$, $a > 0$ *which is equivalent to*

$$\begin{aligned} F(X,Y) &= X\sigma_r(Y) + F(0,Y) \\ &= F(X,0)\sigma_r^*(Y) + F(0,Y) \quad [\sigma_r^*(Y) = \sigma_r(Y)/\sigma_r(0)] \\ \Leftrightarrow U(x \oplus y) &= U(x)\sigma_r[U(y)] + U(e \oplus y) \\ &= U(x \oplus e)\sigma_r^*[U(y)] + U(e \oplus y). \end{aligned} \tag{40}$$

(c) *Separability, (7), is satisfied.*

(d) *Right distributivity, (57), holds.*

(5) *If (38) holds,* $\oplus$ *is commutative, and* e *is an identity, then* m *and* σ_r *satisfy the following functional equation:*

$$m^{-1}\left[m(X)\sigma_r(Y)\right] + Y = m^{-1}\left[m(Y)\sigma_r(X)\right] + X \tag{41}$$

Note that (37) is weak functional equation that asserts that in terms of the first independent variable, H and M_C are commutative for each choice of Y. Equation (41) is a good deal stronger, but it has not been solved. One solution is

$$m(X) = \alpha X^\beta, g(Y) = (1 + \delta Y)^\beta$$

which yields the p-additive form

$$F(X,Y) = X + Y + \delta XY.$$

A second solution is

$$m(X) = a\exp(X), g(y) = b,$$

in which case it is easy to see that

$$F(X,Y) = X + Y + \ln b.$$

The assumption that e is an identity of $\oplus$ implies $\ln b = 0$, i.e., $b = 1$, which is the additive form. An important issue is whether there other solutions. I do not know the answer.

Using Parts **(2)** and **(4),** we see that right segregation can hold without right distributivity holding if we have a non-trivial PIUI generalization of RDU. This is of importance because some unpublished psychophysical data suggest that segregation may be approximately correct whereas general distributivity may not be.

There is a comparable result if left rather than right segregation holds. For example, we have in that case (38) replaced by

$$m\left[U(x \oplus y) - U(x \oplus e)\right] = \sigma_l[U(x)]m[U(y)]. \tag{42}$$

Corollary. *If (38) holds with m a power function with exponent β_m and if $\oplus$ is idempotent, (13), then*

$$U(x \oplus y) = U(x)\sigma_r[U(y)]^{1/\beta_m} + U(y)\left(1 - \sigma_r[U(y)]^{1/\beta_m}\right). \tag{43}$$

3.2 Segregation and RDU for Induced $\oplus_l, \oplus_r$

Two ear intensity data suggest that the defined operations $\oplus_l$ and $\oplus_r$, see (10), are rarely, if ever, actually commutative (Steingrimsson, 2002), which also means that $\oplus$ is not commutative. So we explore this case recalling that e is a right identity of $\oplus_l$ and a left identity of $\oplus_r$.

Theorem 3. *Suppose that Assumptions 1-5 hold for $(\mathcal{D}, \succsim, \oplus, \circ_C)$. Let $\oplus, \oplus_l, \oplus_r$ be related by (10); $\circ_{C,l}, \circ_{C,r}$ are defined in terms of $\circ_C$ by*

$$\begin{aligned} (x \circ_{C,l} y) \oplus e \quad &: \quad = (x \oplus e) \circ_C (y \oplus e) \qquad (44)\\ e \oplus (u \circ_{C,r} v) \quad &: \quad = (e \oplus u) \circ_C (e \oplus v), \qquad (45) \end{aligned}$$

and U_l, U_r are defined in terms of U by

$$\begin{aligned} U_l(x) &= U(x \oplus e) \qquad (46)\\ U_r(x) &= U(e \oplus x), \qquad (47) \end{aligned}$$

*and conversely. Then $(U_l, W, \oplus_l, \circ_{C,l})$ satisfies the properties of Part **(2)** of the left analogue of Theorem 2 and $(U_r, W, \oplus_r, \circ_{C,r})$ satisfies the properties of Part **(2)** of Theorem 2 iff the following conditions are satisfied for all $x, y, z \in \mathcal{D}_0$:*

(1) *(U, W) forms a PIUI representation of $\circ_C$ in the following sense: for $x \oplus y \succ u \oplus v$,*

$$\begin{aligned} &m\left(U[(x \oplus y) \circ_C (u \oplus v)] - U(u \oplus v)\right)\\ &\quad = m[U(x \oplus y) - U(u \oplus v)]W(C). \qquad (48) \end{aligned}$$

(2) *The representation of the operation $\oplus$ satisfies:*

$$\begin{aligned} m[U(x \oplus y) - U(e \oplus y)] &= m[U(e \oplus x)]\sigma_r[U(e \oplus y], \qquad (49)\\ m[U(x \oplus y) - U(x \oplus e)] &= m[U(y \oplus e)]\sigma_l[U(x \oplus e], \qquad (50)\\ e \oplus e &\sim e, \qquad (51) \end{aligned}$$

where σ_r and σ_l are positive and continuous.

(3) *There exists a constant* $\gamma_m > 0$ *such that*

$$\frac{m[U(x \oplus e)]}{m[U(e \oplus x)]} = \gamma_m. \tag{52}$$

Corollary. *Suppose that the properties* ***(1)-(3)*** *hold.*

(1) *If* m *is a power function with exponent* β_m*, then for some constants* $\gamma > 0$ *and* δ,

$$\begin{aligned} & U[(x \oplus y) \circ_C (u \oplus v)] - U(u \oplus v) \\ & \quad = [U(x \oplus y) - U(u \oplus v)][W(C)]^{1/\beta_m} \quad (x \oplus y \succsim u \oplus v \succ e), \quad (53) \\ & U(x \oplus y) = U(x \oplus e) + U(e \oplus y) + \delta U(x \oplus e) U(e \oplus y), \quad (54) \\ & U(x \oplus e) = \gamma U(e \oplus x) \quad (\gamma = \gamma_m^{1/\beta_m}). \quad (55) \end{aligned}$$

(2) *If* $\beta_m = 1$*, then the pair* $(\circ_C, \oplus)$ *satisfies left and right distributivity in the following sense*

$$\begin{aligned} (z \oplus x) \circ_C (z \oplus y) &\sim z \oplus (x \circ_{C,r} y), \quad (56) \\ (x \oplus z) \circ_C (y \oplus z) &\sim (x \circ_{C,l} y) \oplus z. \quad (57) \end{aligned}$$

The formulation of the results in the Corollary, which is better than my original version, was pointed out to me by E. Dzhafarov. If we adjoin

$$U(x \oplus y, C; e) = U(x \oplus y) W_0(C) \tag{58}$$

to (53) we have the general RDU representation (8).

Equation (54) is a p-additive form, but that fact does not imply that $\oplus$ is commutative unless $\gamma = 1$ nor does it imply that $\oplus$ is bisymmetric in the form

$$(x \oplus y) \oplus (u \oplus v) \sim (x \oplus u) \oplus (y \oplus v). \tag{59}$$

In the commutative case with e an identity, (54) reduces to classical p-additivity, (15). The asymmetry in the situation is caught by (55) whereas (54) is deceptively symmetric. By recasting the latter into either the left or right form, the asymmetry becomes more transparent:

$$\begin{aligned} U[e \oplus (x \oplus_r y)] &= U(x \oplus y) \\ &= U(x \oplus e) + U(e \oplus y) + \delta U(x \oplus e) U(e \oplus y) \\ &= \gamma U(e \oplus x) + U(e \oplus y) + \delta \gamma U(e \oplus x) U(e \oplus y), \end{aligned}$$

which, using (46) and (47), is equivalent to

$$U_r(x \oplus_r y) = \gamma U_r(x) + U_r(y) + \delta \gamma U_r(x) U_r(y).$$

Luce (2002a) studied only the cases where $\oplus_r$ is bisymmetric, (59), which property is satisfied iff either $\gamma = 1$ (the commutative case) or $\delta = 0$ (the weighted additive case). If U is written in terms of F, (21), and if it is assumed that F is a homogeneous function, then Aczél, et al. (2002) showed that only these two cases can arise.

3.3 Segregation and RDU for Induced $\oplus_s$

We next take up the assumption that it is the induced symmetric operation $\oplus_s$ that exhibits both right and left segregation.

Theorem 4. *Suppose that Assumptions 1-5 hold for* $(\mathcal{D}, \succsim, \oplus, \circ_C)$*. Let* $\oplus, \oplus_s$ *be related by (9) and (10);* $\circ_{C,s}$ *is defined in terms of* $\circ_C$ *by*

$$(x \circ_{C,s} y) \oplus (x \circ_{C,s} y) := (x \oplus x) \circ_C (y \oplus y); \tag{60}$$

and define U *in terms of* U_s *and conversely by*

$$U(x \oplus y) := U_s(x \oplus_s y). \tag{61}$$

Then $(U_s, W, \oplus_s, \circ_{C,s})$ *satisfies the Part* ***(2)*** *of Theorem 2 for both right and left segregation iff the following conditions are satisfied by* $(U, W, \oplus, \circ_C)$*:*

(1) (U, W) *satisfies PIUI in the sense of (48).*

(2) $(U, \oplus)$ *satisfies (38) and (42)*

(3) *Equation (51),* $e \oplus e \sim e$*, holds.*

Corollary. *Suppose that the properties of* ***(1)-(3)*** *hold.*

(1) *If* m *is a power function with exponent* β_m*, then either*

$$U(x \oplus y) = U(x \oplus e) + U(e \oplus y), \tag{62}$$

or there exists a constant $\eta \in]0, 1[$ *such that* $U(x \oplus y)$ *satisfies*

$$U(x \oplus y) = \eta U(x \oplus x) + (1 - \eta) U(y \oplus y). \tag{63}$$

(2) *Equation (63) holds iff (62) and the ratio condition (55) hold. In that case,* $\gamma = \eta/(1 - \eta)$*.]*

(3) *If* $\beta_m = 1$*, then distributivity holds in the sense that*

$$(z \oplus x) \circ_C (z \oplus y) \sim z \oplus (x \circ_{C,s} y) \tag{64}$$

$$(x \oplus z) \circ_C (y \oplus z) \sim (x \circ_{C,s} y) \oplus z. \tag{65}$$

Given the difference between (62) and the p-additive form of (54), it is clear that symmetric segregation of $\oplus_s$ neither implies nor is implied by the left segregation of $\oplus_l$ and the right segregation of $\oplus_r$. On the other hand, as is shown in Luce (2002b), one form of segregation for $\oplus_s$ and one for either $\oplus_l$ or $\oplus_r$ implies that all four hold and so (62) and (55) are satisfied.

4 IVI, Distributivity, and Commutative Joint Receipt

4.1 Forcing utility to be additive over $\oplus$

Luce (1996; see also 2000) has studied the property called *joint-receipt decomposability:* For each $x \in \mathcal{D}$ and $C \in \mathcal{E}$, there exists $D = D(x, C) \in \mathcal{E}$ such that for all $y \in \mathcal{D}$

$$(x \oplus y, C; e) \sim (x, C; e) \oplus (y, D; e). \tag{66}$$

where each of the stimuli is realized independently. Note that all three of the stimuli are unitary ones and so, at least, it does not mix unitary and non-unitary ones.

This concept played the following role in the earlier work. It is not difficult to show that if (U, W) is separable over unitary stimuli of the form $(x, C; e)$, i.e., UW is order preserving of these stimuli, and if U is p-additive over $\oplus$, (15), then joint-receipt decomposition is satisfied. Somewhat more difficult to prove is the fact that if U^*W^* is order preserving over unitary stimuli, if $\oplus$ has a p-additive representation U^{**}, and if joint receipt decomposability holds, then there exist (U, W) such that both UW is order preserving over unitary stimuli and U is p-additive over $\oplus$, where, of course, for some $\beta > 0$ $UW = (U^*W^*)^\beta$.

The import of the following result is that generalizing segregation is a delicate matter and without care it can lead to results that are stronger than one might wish. Indeed, the conclusion one draws is that when unitary stimuli are part of the overall structure and joint-receipt decomposability holds, then one may not assume both right segregation and right distributivity unless one is prepared to accept that the utility function satisfying RDU is also additive over joint receipt.

Because we will now avoid unitary stimuli, this means that G_C is not defined at 0. Of course, because of the assumed continuity, we may continuously extend G_C to include 0; however we may not interpret this extension as giving $V(x, C; e)$. Such an extension may seem an odd thing to do; however, some of the mathematics I use assumes that the domains are $[0, k[$, not $]0, k[$.

Proposition 5. [2] *Suppose that the following are satisfied: Assumptions 1-4; stimuli are idempotent, (3); $\oplus$ is commutative and e is its identity; an order-preserving mapping V of the domain $\mathcal{D}$ of binary stimuli and joint receipts onto $[0, \infty[$ exists; and the functions F, G_C defined by (21) and (22) exist and are continuous in each of the variables. If right distributivity, (19), right segregation, (14), and joint-receipt decomposability, (66), all hold, then V is additive over $\oplus$, (17), and there exists $W : \mathcal{E} \xrightarrow{into} [0, 1]$ such that (V, W) is an RDU representation, (8).*

Note that by Assumptions 3 and 4, F is strictly increasing in each variable, and G_C is strictly increasing in x and philandering in y.

The objection to this result is that it ties RDU together with an additive representation of $\oplus$, which for various empirical reasons is widely regarded as wrong. As in the earlier work, we would like the U of an RDU representation to be p-additive over $\oplus$, not purely additive.

So, the conclusion is that in the presence of right distributivity, we should avoid assuming both segregation and joint-receipt decomposability. Having investigated segregation in Sections 2 and 3, we now drop it in favor of distributivity.

4.2 The basic result for binary stimuli

Theorem 5. *Suppose the following are satisfied: Assumptions 1-4; $\oplus$ is commutative and e is its identity; V is an order-preserving mapping of the domain $\mathcal{D}$ onto $[0,\infty[$, and the functions F, G_C defined by (21), (22) exist and are continuous in each of the variables. Consider the following three statements:*

(1) *V is additive over $\oplus$, (17).*

(2) *Right distributivity, (19), is satisfied.*

(3) *For each $C \in \mathcal{E}$, there exists a function $L_C : \mathbb{R}_+ \xrightarrow{into} \mathbb{R}_+$, that is strictly increasing such that IVI, (23), holds.*

*Then: **(1)** and **(2)** imply **(3)**; **(1)** and **(3)** imply **(2)**; and **(2)** and **(3)** imply that there exists V^* and L^*_C such that*

$$\begin{aligned} V^*(x \oplus y) &= V^*(x) + V^*(y), \\ V^*(x, C; y) &= V^*(y) + L^*_C[V^*(x) - V^*(y)]. \end{aligned}$$

Observe that IVI was also derived in Theorem 1 except that its domain of application here is limited to $y \succ e$ whereas in Theorem 1 it holds for $y \succsim e$. Also, the proof of Theorem 5 is a good deal more complex than that of Theorem 1 in that it rests on a rather deep mathematical result. Because the resulting representation is again IVI, Propositions 1-4 apply here as well as when segregation is assumed.

Note that from statements **(2)** and **(3)** one does not claim that V itself is additive, but rather that there is a V^* that is both additive and of the same form as (23).

An example of (23) that does not reduce to U satisfying RDU is for $L_C(Z) = ZW(C)$, which means that V itself satisfies RDU.

In the next subsection we explore the generalization of (23) to stimuli of order $n > 2$.

4.3 Generalization of IVI to general gambles[3]

The form of (23) may be of considerable interest in decision making. A natural generalization of it to gambles of order n is

$$V(x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) - V(x_n) = \sum_{i=1}^{n-1} L_{\mathbf{C}}^{(i)}[V(x_i) - V(x_n)] \quad (x_n \succ e) \tag{67}$$

where $\mathbf{C} = (C_1, ..., C_i, ..., C_n)$, each $L_{\mathbf{C}}^{(i)} : \mathbb{R}_+ \overset{onto}{\longrightarrow} \mathbb{R}_+$ is strictly increasing, and $L_{\mathbf{C}}^{(i)}(0) = 0$. As is easily seen, if V is additive over $\oplus$, this form implies a natural generalization of right distributivity

$$\begin{aligned}&(x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) \oplus z \\ &\quad \sim (x_1 \oplus z, C_1; ...; x_i \oplus z, C_i; ...; x_n \oplus z, C_n) \quad (x_n \succ e).\end{aligned} \tag{68}$$

However, this generalization of distributivity does not, by itself, imply (67). So more than generalized right distributivity is required to insure that (67) holds.

We continue to suppose that gambles are decomposable in the sense that there exists a function $G_{\mathbf{C}} : \mathbb{R}_+^n \overset{onto}{\longrightarrow} \mathbb{R}_+$ such that

$$V(x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) = G_{\mathbf{C}}[V(x_1); ...; V(x_n)]. \tag{69}$$

Theorem 6. *Suppose that the following conditions are satisfied:* $\succsim$ *is a weak order on general gambles of order* $n > 2$ *and their joint receipts; gambles are idempotent and strictly monotonic increasing over each subset of consequences;* V *is additive over* $\oplus$; G_C *is decomposable in the sense of (69); and generalized right distributivity for* $x_n \succ e$*, (68). Then, the representation* V *satisfies (67).*

Note that it is perfectly reasonable to suppose that a degenerate stimulus of order n is actually one of order $n - 1$ in the following sense

$$\begin{aligned}&(x_1, C_1; ...; x_{i-1}, C_{i-1}; x_i, \emptyset; x_{i+1}, C_{i+1}; ...; x_n, C_n) \\ &\quad \sim (x_1, C_1; ...; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; ...; x_n, C_n).\end{aligned}$$

Given that, we see that

$$L^{(i)}_{\{C_1,..,C_j,\emptyset,C_{j+1},...,C_n\}} = L^{(i)}_{\{C_1,..,C_j,C_{j+1},...,C_n\}} \quad (i \neq j).$$

Thus, knowing the functions for gambles of order n means that they are determined for all orders smaller than n.

4.4 Special cases

The representation (67) is of interest because, with $U = V$, it encompasses several models in the literature including, of course, the standard RDU model.

For example, when $n = 3$, and (C, D, E) is a partition of Ω,

$$\begin{aligned}
&U(x, C; y, D; z, E) \\
&\quad = U(x)W(C) + U(y)[W(C \cup D) - W(C)] + U(z)[1 - W(C \cup D)] \\
&\quad = [U(x) - U(z)]W(C) + [U(y) - U(z)][W(C \cup D) - W(C)] + U(z).
\end{aligned}$$

More interesting is the fact that Birnbaum's (1997, and many references there) configural weight models are also examples. The general class of models was written in the following form[4] for $n = 2, 3$ by Birnbaum, et al. (1992):

$$\begin{aligned}
U(x, p; y, q) &= \frac{BU(x) + AU(y)}{A + B} \qquad\qquad (p + q = 1) \\
&= [U(x) - U(y)]\frac{B}{A + B} + U(y) \\
U(x, p; y, q; z, r) &= \frac{CU(x) + BU(y) + AU(z)}{A + B + C} \qquad (p + q + r = 1) \\
&= [U(x) - U(z)]\frac{C}{A + B + C} + [U(y) - U(z)]\frac{B}{A + B + C} + U(z),
\end{aligned}$$

where A, B, C are functions of the probability distribution (p, q, r). A case discussed by Birnbaum (1997) [see also Birbaum & Navarrete, 1998, and Birnbaum & Stegner (1979, Eq. (10)], called the RAM model, entails setting

$$A = aS(r), B = (1 - a)[1 - S(q)], C = (1 - a)[1 - S(p)],$$

where $S : [0, 1] \xrightarrow{onto} [0, 1]$ is strictly increasing. If we write $W(p) = 1 - S(1 - p)$, $\alpha = a/(1 - a)$, and define

$$W_{p,q,\alpha}(s) := \frac{W(s)}{W(p) + W(q) + \alpha[1 - W(p + q)]} \quad (s = p, q),$$

then the RAM expressions become

$$\begin{aligned}
U(x, p; y, 1 - p) &= [U(x) - U(y)]W_{p,0,\alpha}(p) + U(y), \\
U(x, p; y, q; z, r) &= [U(x) - U(z)]W_{p,q,\alpha}(p) + [U(y) - U(z)]W_{p,q,\alpha}(q) + U(z),
\end{aligned}$$

In terms of (67) we see that

$$L_{p,q}(X) = XW_{p,q,\alpha}.$$

Another class of models of this form has arisen in decision affect theory (Mellers, et al. 1997; Mellers, 2000).

Given (67), I do not yet know of specific behavioral constraints that limit it to the particular forms postulated by Birnbaum and his colleagues.

5 IVI, Distributivity, and Non-Commutative Joint Receipt

We turn to the analogue of Theorem 2 with right distributivity playing the role of right segregation.

Theorem 7. *Suppose that the following are satisfied: Assumptions 1-4; $\oplus$ is not commutative; $U : \mathcal{D} \overset{onto}{\longrightarrow} \mathbb{R}_+$ is order preserving; and the functions F, G_C defined in terms of U by (21) and (22) exist and are continuous in the x, y variables. Then any two of the following three statements imply the third:*

(1) *Right distributivity, (19), is satisfied.*

(2) *The function U forms a* generalized left-weighted representation *of the operation $\oplus$ in the sense that there is a positive, increasing, real function σ_r^* such that (40) is satisfied.*

(3) *There exists a function $W : \mathcal{E} \overset{into}{\longrightarrow} [0, \infty[$ such that (U, W) forms an RDU representation, (8), for binary stimuli with $x \succsim y \succ e$.*

The form arrived in Part **(2)** is the same as Part **(2)(c)** of Theorem 2. There is a comparable result for left distributivity. Thus, we can also get Theorems 3 and 4 and their corollaries by assuming right and left distributivity rather than right and left segregation.

6 Concluding Remarks

6.1 Joint receipt and the Thomsen condition

The following additive conjoint representation of $\oplus$ occurs in Theorems 1 and 5 and in the Corollaries to Theorems 3 and 4:

$$\Phi(x \oplus y) = \Phi_1(x) + \Phi_2(y). \tag{70}$$

It is well known that this implies the important Thomsen condition:

$$\left.\begin{array}{l} x \oplus v \sim y \oplus w \\ y \oplus u \sim z \oplus v \end{array}\right\} \Longrightarrow x \oplus u \sim z \oplus w \tag{71}$$

(Krantz, et al. 1971, Ch. 6).

Clearly, (71) is an essential property to verify experimentally. In the case of loudness, the literature is mixed on this point. Falmagne, Iverson, and Marcovici (1979) and Levelt, Riemersma, and Bunt (1972) support it, whereas Falmagne (1976) with just one respondent and Gigerenzer and Strube (1983) with 12 do not support it. More data are currently being collected.

6.2 Extensions of U to unitary stimuli

Just as in the theory based on segregation, we have the problem of knowing when it is possible to use the U constructed using distributivity (Theorems 5 and 7) to construct a conjoint measurement representation $U(x)W_0(C)$ of unitary stimuli $(x, C; e)$. Actually, nothing is changed at all from the case of segregation because all that is involved is the form of $U(x \oplus y)$, which is the same as in the earlier theories. In particular, when it is p-additive, (15), as in the case of commutative and associative $\oplus$ (Theorem 5), the necessary and sufficient condition is joint-receipt decomposability, (66) (Luce, 1996, 2000). We do not know the full answer for the generalized weighted additive representation, (40), of Theorem 2 nor for the generalized weighted average representation, (43), of the Corollary to Theorem 2. All we know are the special cases where the weighting functions, σ_r and σ_l, are constant. In those cases, the necessary and sufficient condition is simple joint-receipt decomposability which is (66) with D an independent realization of C, meaning that $W_0(D) = W_0(C)$ (Luce, 2002a, Aczél, et al. 2002). No attempt has yet been made to extend the current theory to non-idempotent stimuli as was done by Luce and Marley (2000) for RDU.

6.3 Summary

This article provides a way to accommodate the empirical findings of both utility theory and psychophysics in which respondents fail to respond to unitary elements $(x, C; e)$ as if they are limiting cases of $(x, C; y)$ as y approaches e from above. This meant either dropping separability while retaining right (left) segregation (Sections 2 and 3) or replacing right (left) segregation by the, in some ways, stronger property of right (left) distributivity (Sections 4 and 5). Right distributivity is stronger than segregation in the sense that three rather than two variables are involved, but it is weaker in that two of these three variables are required to be $\succ e$—see (19). Three cases, already treated in the literature using both separability and segregation, were explored under both of these weaker assumptions for cases where joint receipt $\oplus$ is commutative and associative and has an identity e; where $\oplus$ is non-commutative but has either a left or right identity but not both[5]; and where $\oplus$ is non-commutative and has neither a left nor a right identity but is idempotent. In the commutative case a new class of representations, (23), resulted (Theorems 1 and 5). It includes the usual rank-dependent representation as a special case (Proposition 3). For utility theory, the general form of the binary case was generalized (Theorem 6) to stimuli with more than two consequences, which generalization may prove useful in providing a theoretical background for such phenomena as regret, disappointment, and configural weighting. By contrast, except for unitary elements, $(x, C; e)$, the representations in the two non-commutative cases did not change. This means that one can simply bypass the use of 0 intensities in the psychophysical applications because, for the most part, psychophysical interpretations of $\oplus$ appear to be non-commutative. Moreover, there seemed little point in exploring a generalization of Theorems 2, 3, 4, and 7 to stimuli of order $n > 2$ which seems to lack a clear interpretation for psychophysics.

Notes

* Some of this material is based upon work supported by the National Science Foundation under Grant Number SES-0136431 to the University of California, Irvine.

Drs. János Aczél and A. A. J. Marley have both commented in detail on versions of this manuscript, correcting errors and ambiguities and making many very useful suggestions. Dr. Ehtibar Dzhafarov, in comments on a draft of Luce (2002b), has greatly improved the formulation of the results when both right and left segregation are assumed, as in Theorems 3 and 4. And thanks to Dr. Anders Lundberg who, at my urging, modified his earlier results so as to get the two propositions summarized in Appendix B; these underlie Theorems 5 and 7.

1. Doing so, as well as developing Proposition 1 and the property (31) below, was suggested by A. A. J. Marley (personal communications, December, 2001, and January 2002).

2. Conversations with A. A. J. Marley during the week of March 25, 2002, led me to prove this result. I appreciate the hospitality of the Hanse-Wissenschaftskolleg for that week.

3. Because there is no very clear psychophysical interpretation of stimuli of order $n > 2$, but of course there is a utility one, I revert in this and the next subsection to the term "gamble."

4. Because they wrote $0 < x < y < z$, their order of A, B, C was the natural one. Here, with the reverse convention, they seem to be in an unnatural order. Note in this context that these symbols represent numerical functions of the event partition, not events themselves.

5. Only the left identity case was described explicitly.

6. Note that in the following statements e denotes the exponential constant, not no change from the status quo. The two uses can be distinguished in terms of whether an expression pertains to the underlying structure or to its representation.

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Appendix A: Segregation Proofs

Theorem 1.

Proof: **(1)(a)⇔(b)** *Suppose IVI, then*

$$\begin{aligned}V[(x,C;e)\oplus y] &= V(x,C;e)+V(y)\\ &= L_C[V(x)]+V(y)\\ &= L_C[V(x)+V(y)-V(y)]+V(y)\\ &= L_C[V(x\oplus y)-V(y)]+V(y)\\ &= V(x\oplus y,C;y),\end{aligned}$$

thereby proving segregation.

Conversely, from segregation, decomposability, and additivity and setting $X=V(x), Y=V(y)$,

$$G_C(X+Y,Y)-Y=G_C(X,0),$$

or rewriting

$$G_C(R,S)=l_C(R-S)+S \quad (R\geq S),$$

where $l_C(R)=G_C(R,0)$, which is IVI

(b)⇔(c) Using (16), we see that IVI is equivalent to

$$\begin{aligned}sgn(\delta)\ln[1+\delta U(x,C;y)] &= L_C\left(\ln\frac{1+\delta U(x)}{1+\delta U(y)}\right)+sgn(\delta)\ln[1+\delta U(y)]\\ \Longleftrightarrow \frac{1+\delta U(x,C;y)}{1+\delta U(y)} &= \exp L_C\left(\ln\frac{1+\delta U(x)}{1+\delta U(y)}\right),\end{aligned}$$

which is (28) with $l_C(R)=\exp L_C(\ln R)$. Note that for $\delta>0$,

$$1\leq\frac{1+\delta U(x)}{1+\delta U(y)}<\infty,$$

and for $\delta<0$,

$$0\leq\frac{1-|\delta|\,U(x)}{1-|\delta|\,U(y)}<1.$$

(2) Suppose IVI. Then

$$\begin{aligned}V[(x,C;y)\oplus z] &= V(x,C;y)+V(z)\\ &= L_C[V(x)-V(y)]W(C)+V(y)+V(z)\\ &= L_C[V(x)+V(z)-V(y)-V(z)]W(C)+V(y\oplus z)\\ &= L_C[V(x\oplus z)-V(y\oplus z)W(C)+V(y\oplus z)\\ &= V(x\oplus z,C;y\oplus z),\end{aligned}$$

which is right distributivity. The left follows by commutativity.

(3)(a)⇔(c) Assume PIVI. It is clearly a special case of IVI, and

$$\begin{aligned}V((x,C;y),D;y)-V(y) &= l^{-1}\left(l[V(x,C;y)-V(y)]W(D)\right)\\ &= l^{-1}\left(l[V(x)-V(y)]W(D)W(C)\right),\end{aligned}$$

which is clearly commutative in C and D and so event commutativity holds. The converse is immediate.

(a)⇔(b) Applying IVI, (23), twice

$$\begin{aligned} V((x,C;y),D;y) &= V(y) + L_D\,[V(x,C;y) - V(y)] \\ &= V(y) + L_D\,(L_C[V(x) - V(y)])\,, \end{aligned}$$

and so clearly event commutativity holds iff (29). □

Proposition 1.

Proof:

(1) Because V is additive, $\oplus$ is commutative and associative which in turn is known to imply

$$x \ominus y \sim (x \oplus z) \ominus (y \oplus z).$$

Using (31) and the commutativity and associativity of $\oplus$,

$$\begin{aligned} (x \oplus z, C; y \oplus z) &\sim (y \oplus z) \oplus \Phi_C[(x \oplus z) \ominus (y \oplus z)] \\ &\sim [y \oplus \Phi_C(x \ominus y)] \oplus z \\ &\sim (x, C; y) \oplus z, \end{aligned}$$

which is right distributivity.

(2). Suppose that (23) and (31) hold. Let $z \sim x \ominus y$. Then

$$\begin{aligned} V(y) + V[\Phi_C(z)] &= V\,[y \oplus \Phi_C(x \ominus y)] \\ &= V(x, C; y) \\ &= V(y) + L_C[V(x) - V(y)] \\ &= V(y) + L_C[V(z)], \end{aligned}$$

whence (32).

Suppose that (23) and (32) hold. Then

$$\begin{aligned} V(x, C; y) &= V(y) + L_C[V(x) - V(y)] \\ &= V(y) + L_C[V(z)] \\ &= V(y) + V_C[\Phi(z)] \\ &= V[y \oplus \Phi_C(x \ominus y)], \end{aligned}$$

whence (31).

Suppose that (31) and (32) hold. Then

$$\begin{aligned} V(x, C; y) &= V(y \oplus \Phi_C[x \ominus y)] \\ &= V(y) + V[\Phi_C(z)] \\ &= V(y) + L_C[V(z)] \\ &= V(y) + L_C[V(x) - V(y)], \end{aligned}$$

whence (23).

(3) Assuming segregation and (31) we have

$$
\begin{aligned}
y \oplus \Phi(x \ominus y, C) &\sim (x, C; y) \\
&\sim (x \ominus y, C; e) \oplus y.
\end{aligned}
$$

By commutativity and monotonicity, the result follows. □

Proposition 2.

Proof: The result is obvious for $\delta = 0$. So consider $\delta \neq 0$. By the definition of V in terms of U, (16), and taking exponentials in (23), we see that

$$U(x, C; y) = M_C[U(x) - U(y)] + U(y)$$

is equivalent to

$$
\begin{aligned}
1 + \delta M_C[U(x) - U(y)] + \delta U(y) &= 1 + \delta U(x, C; y) \\
&= [1 + \delta U(y)] \exp L_C \left(\ln \frac{1 + \delta U(x)}{1 + \delta U(y)} \right).
\end{aligned}
$$

Let $X = 1 + \delta U(x), Y = 1 + \delta U(y), Z = (X - Y)/\delta$, $\widetilde{L}_C = \exp(L_C) - 1$, then with minor rearranging

$$
\begin{aligned}
\delta M_C(Z) &= Y \left[\exp L_C \left(\ln \frac{Y + \delta Z}{Y} \right) - 1 \right] \\
&= Y \widetilde{L}_C \left(1 + \frac{\delta Z}{Y} \right).
\end{aligned}
$$

Because the left side is independent of Y and the right side is differentiable, differentiate it with respect to Y,

$$
\begin{aligned}
0 &= \widetilde{L}_C \left(1 + \frac{\delta Z}{Y} \right) - Y \widetilde{L}'_C \left(1 + \frac{\delta Z}{Y} \right) \frac{\delta Z}{Y^2} \\
\Longleftrightarrow \widetilde{L}_C(R) &= \widetilde{L}'_C(R)(R - 1) \quad \left(R = 1 + \frac{\delta Z}{Y} \right) \\
\Longleftrightarrow \frac{d}{dR} \ln \widetilde{L}_C(R) &= \frac{1}{R - 1} \\
\Longleftrightarrow \widetilde{L}_C(R) &= W(C)(R - 1),
\end{aligned}
$$

where $\ln W(C)$ is the constant of integration. Returning to the l_C notation, $m_C(Z) = ZW(C)$, i.e., RDU. □.

Proposition 3.

Proof: **(1)** Assume that the IVI representation holds and set $y = e$ in (28). This with Part **(a)**, separability, implies that

$$
\begin{aligned}
\delta X W(C) + 1 &= l_C(\delta X + 1) \\
\Leftrightarrow l_C(R) &= (R - 1)W(C) + 1,
\end{aligned}
$$

which is the form of Part **(c)** of the Proposition. Part **(c)** when substituted yields RDU, which is Part **(b)**. Clearly, Part **(b),** RDU, implies separability, which is Part **(a)**.

(2) Suppose IUI. Consider the left side of the right segregation property

$$\begin{aligned} U[(x,C;e)\oplus y] &= U(x,C;e)[1+\delta U(y)]+U(y) \\ &= M_C[U(x)][1+\delta U(y)]+U(y), \end{aligned}$$

and the right side,

$$\begin{aligned} U(x\oplus y,C;y) &= M_C[U(x\oplus y)-U(y)]+U(y) \\ &= M_C[U(x)(1+\delta U(y))]+U(y). \end{aligned}$$

Setting $U(x)=1$, $W(C)=M_C(1)$, and $R=1+\delta U(y)$, we see that segregation holds iff

$$\begin{aligned} M_C(R) &= RW(C) \\ \Leftrightarrow U(x,C;y)-U(y) &= [U(x)-U(y)]W(C), \end{aligned}$$

which is RDU. □

Proposition 4.

Proof: Define V by (16). Then by (36),

$$\begin{aligned} \lim_{y\searrow e} V(x,C;y) &= \lim_{y\searrow e} sgn(\delta)\ln[1+\delta U(x,C;y)] \\ &= sgn(\delta)\ln\left[1+\delta\lim_{y\searrow e} U(x,C;y)\right] \\ &= sgn(\delta)\ln[1+\delta U(x)W(C)]. \end{aligned}$$

And by (23) and the continuity of L_C,

$$\begin{aligned} \lim_{y\searrow e} V(x,C;y) &= \lim_{y\searrow e} (V(y)+L_C[V(x)-V(y)]) \\ &= V(e)+\lim_{y\searrow e} L_C[V(x)-V(y)] \\ &= L_C[V(x)]. \end{aligned}$$

Equating these two expressions for $\lim_{y\searrow e} V(x,C;y)$, taking the exponential, and writing $X=\delta U(x)$ gives

$$1+XW(C)=\exp L_C[\ln(1+X)].$$

Returning to (23), letting $Y = \delta U(y)$, substituting the definition of U in terms of V, (18), taking the exponential, and using the above equation,

$$\begin{aligned} 1 + \delta U(x, C; y) &= (1+Y) \exp L_C \left[\ln \left(\frac{1+X}{1+Y} \right) \right] \\ &= (1+Y) \exp L_C \left[\ln \left(1 + \frac{X-Y}{1+Y} \right) \right] \\ &= (1+Y) \left[1 + \frac{X-Y}{1+Y} W(C) \right] \\ &= 1 + Y + (X-Y)W(C) \\ &= 1 + \delta U(y) + \delta[U(x) - U(y)]W(C), \end{aligned}$$

whence RDU. □

Theorem 2.

Proof: **(1)** By right segregation and IUI we have

$$\begin{aligned} F\left(M_C[U(x)], U(y)\right) &= F[U(x, C; e), U(y)] \\ &= U[(x, C; e) \oplus y] \\ &= U(x \oplus y, C; e \oplus y) \\ &= M_C[U(x \oplus y) - U(e \oplus y)] + U(e \oplus y) \\ &= M_C\left(F[U(x), U(y)] - F[0, U(y)]\right) + F[0, U(y)]. \end{aligned}$$

Set $X = U(x), Y = U(y)$, and $H(X, Y) = F(X, Y) - F(0, Y)$, and we have the functional equation (37).

(2) (a) & (b)⇒(c) Clearly PIUI, (26), means IUI, (24), is satisfied. So, with right segregation and decomposability, we know from Part **(1)** that (37) holds. Setting $X = U(x), Y = U(y), W = W(C)$ in (37) and using the PIUI form (26) and (27), then (37) reduces to

$$H[m^{-1}(XW), Y] = m^{-1}\left(mH[m^{-1}(X), Y]W\right).$$

Let $K(X, Y) = mH[m^{-1}(X), Y]$ and that last display reduces to

$$K(XW, Y) = K(X, Y)W,$$

whence $K(X, Y) = \sigma_r(Y)X$, where the subscript r reflects the fact we are using right segregation. Note that by strict monotonic increasing in X, $\sigma_r(Y) > 0$. Substituting this expression back into the definition relating K to H, we obtain (38).

(a) & (c)⇒(b) Assume right segregation, (14,) and (38). By Part **(1),** (38) and (37) yield

$$m^{-1}[mM_C(X)\sigma_r(Y)] = M_C\left(m^{-1}[m(X)\sigma_r(Y)]\right).$$

Let $N_C(X) = m\left(M_C[m^{-1}(X)]\right)$, then

$$N_C[m(X)]\sigma_r(Y) = N_C[m(X)\sigma_r(Y)].$$

We show that $N_C(R) = RW(C)$, where $W(C) = \frac{1}{a}N_C(a)$ and a is any number such that
$$b = \inf_Y \frac{a}{\sigma_r(Y)} < 1 < \sup_Y \frac{a}{\sigma_r(Y)} = B.$$
Because σ_r is continuous, for $X \in]b, B[$ we may choose Y such that $X\sigma_r(Y) = a$. So, with $W(C) = \frac{1}{a}N_C(a)$, we have $N_C(X) = XW(C)$. Now, for $X \in]b, B[$, consider $Z = X/\sigma_r(Y)$ which lies in the interval $\left]\frac{b}{B}, \frac{B}{b}\right[$ and so

$$XW(C) = N_C(X) = N_C[Z\sigma_r(Y)] = N_C(Z)\sigma_r(Y),$$

so
$$N_C(Z) = \frac{X}{\sigma_r(Y)}W(C) = ZW(C).$$
Note that $\frac{b}{B} < b < B < \frac{B}{b}$. By induction this can be done for any $Z \in \left]\frac{b}{B^n}, \frac{B}{b^n}\right[$ which converges to $]0, \infty[$. With the form of N_C determined, we substitute back and get
$$M_C(R) = m^{-1}[m(R)W(C)],$$
which yields (26), PIUI.

(b) & (c)⇒(a) Using PIUI and (38), we calculate both sides of right segregation:

$$\begin{aligned}
U[(x, C; e) \oplus y] &= m^{-1}\left(m[U(x, C; e)]\sigma_r[U(y)]\right) + U(e \oplus y) \\
&= m^{-1}\left(m[U(x)]W(C)\sigma_r[U(y)]\right) + U(e \oplus y). \\
U(x \oplus y, C; e \oplus y) &= m^{-1}\left(m[U(x \oplus y) - U(e \oplus y)]W(C)\right) + U(e \oplus y) \\
&= m^{-1}\left(m[U(x)]\sigma_r[U(y)]W(C)\right) + U(e \oplus y),
\end{aligned}$$

which are clearly equal.

(3) Substitute $y \sim e$ and $e \oplus e \sim e$ in (38) and we see that

$$m[U(x \oplus e)] = m[U(x)]\sigma_r(0),$$

and (39) immediately follows.

(4) (a)⇔(b) Using PIUI and RDU

$$\begin{aligned}
m[U(x) - U(y)]W(C) &= m[U(x, C; y) - U(y)] \\
&= m\left([U(x) - U(y)]W(C)\right).
\end{aligned}$$

So by the usual arguments, $m(X) = aX, a > 0$. The converse is immediate.

(a)⇔(c) The implication is immediate. By PIUI and separability we see

$$m[U(x)]W(C) = m[U(x, C; e)] = m[U(x)W(C)],$$

so by the standard argument m is linear, whence RDU.

(b)⇔(d) Assuming PIUI and (38) we calculate the two sides of right distributivity:

$$
\begin{aligned}
&U[(x,C;y)\oplus z]\\
&\quad= m^{-1}\left(m[U(x,C;y)]\sigma_r[U(z)]\right)+U(e\oplus y)\\
&\quad= m^{-1}\left([m[U(x)-U(y)]W(C)+U(y)]\,\sigma_r[U(z)]\right)+U(e\oplus z).\\
&U(x\oplus z,C;y\oplus z)\\
&\quad= m^{-1}\left(m[U(x\oplus z)-U(y\oplus z)]W(C)\right)+U(y\oplus z)\\
&\quad= m^{-1}\left[m\left(m^{-1}\left(m[U(x)]\sigma_r[U(z)]\right)-m\left[m^{-1}\left(m[U(y)]\sigma_r[U(z)]\right)\right]\right)W(C)\right]\\
&\qquad+m^{-1}\left(m[U(y)]\sigma_r[U(z)]\right)+U(e\oplus z).
\end{aligned}
$$

Clearly if m is linear, the two sides are equal. Conversely, assuming that they are equal, set $x=y$ and see that

$$m^{-1}(YZ)=m^{-1}[m(Y)Z]$$

and so $m(Y)=Y$, i.e., **(b)** with $a=1$.

(5) This result is completely routine from (38). □

Corollary to Theorem 2.

Proof: If m is a power function with exponent β_m, then (38) with $x=y$ and the assumption that $\oplus$ is idempotent yields

$$U(y)=U(y\oplus y)=U(y)\sigma_r[U(y)]^{1/\beta_m}+U(e\oplus y),$$

so in (38) we may replace $U(e\oplus y)=U(y)\left(1-\sigma_r[U(y)]^{1/\beta_m}\right)$. □

Theorem 3.

Proof: We first suppose that the induced operations satisfy the properties of Part **(2)** of Theorem 2 and of its left analogue.

(1) Consider

$$
\begin{aligned}
&U[(x\oplus y)\circ_C(u\oplus v)]=U\left([e\oplus(x\oplus_r y]\circ_C[e\oplus(u\oplus_r v]\right) \quad (10)\\
&\quad= U\left(e\oplus[(x\oplus_r y)\circ_{C,r}(u\oplus_r v)]\right) \quad (45)\\
&\quad= U_r[(x\oplus_r y)\circ_{C,r}(u\oplus_r v)] \quad (47)\\
&\quad= m^{-1}\left(m[U_r(x\oplus_r y)-U_r(u\oplus_r v)]W(C)\right)+U_r(u\oplus_r v) \quad [(26)\text{ for }\circ_{C,r}]\\
&\quad= m^{-1}\left[m\left(U[e\oplus(x\oplus_r y)]-U[e\oplus(u\oplus_r v)]\right)W(C)\right]+U[e\oplus(u\oplus_r v)]\\
&\qquad\qquad (47)\\
&\quad= m^{-1}\left(m\left[U(x\oplus y)-U(u\oplus v)\right]W(C)\right)+U(u\oplus v), \quad (10)
\end{aligned}
$$

which is (48).

(2) Observe that by the definition of U in (47) and using (9), (10), and (40) for $\oplus_r$,

$$
\begin{aligned}
U(x\oplus y) &= U[e\oplus(x\oplus_r y)] \quad (9),(10)\\
&= U_r(x\oplus_r y) \quad (47)\\
&= m^{-1}\left(m[U_r(x)][\sigma_r[U_r(y)]\right)+U_r(y) \quad [(40)\text{ for}\oplus_r]\\
&= m^{-1}\left(m[U(e\oplus x)]\sigma_r[U(e\oplus y)]\right)+U(e\oplus y) \quad (47),(9),(10)
\end{aligned}
$$

which is (49). The proof for (50) is similar.

Next, set $x = y = e$ in (49),

$$\begin{aligned} U(e \oplus e) &= m^{-1}\left(m[U(e \oplus e)]\sigma_r[U(e \oplus e)]\right) + U(e \oplus e) \\ \Longleftrightarrow 0 = m(0) &= m[U(e \oplus e)]\sigma_r[U(e \oplus e)]. \end{aligned}$$

By Theorem 2(2)(c), $\sigma_r > 0$, so we conclude $U(e \oplus e) = 0 = U(e)$, whence $e \oplus e \sim e$.

(3) This follows by setting $y = e$ in (49) and use (51)

Conversely, suppose that conditions **(1)-(3)** hold. We prove, first, that $(U_r, W, \oplus_r, \circ_{c,r})$ satisfies the three conditions of Theorem 2. First, PIUI:

$$\begin{aligned} m\left[U_r(x \circ_{C,r} y) - U_r(y)\right] &= m\left(U[e \oplus (x \circ_{C,r} y)] - U(e \oplus y)\right) \quad (47) \\ &= m\left(U[(e \oplus x) \circ_C (e \oplus y)] - U(e \oplus y)\right) \quad (45) \\ &= m[U(e \oplus x) - U(e \oplus y)]W(C) \quad (53) \\ &= m[U_r(x) - U_r(y)]W(C) + U_r(y) \quad (47), \end{aligned}$$

which is PIUI for U_r.

Next, we show that U_r satisfies (49):

$$\begin{aligned} m\left[U_r(x \oplus_r y) - U_r(y)\right] &= m\left(U[e \oplus (x \oplus_r y)] - U(e \oplus y)\right) \quad (47) \\ &= m\left[U(x \oplus y) - U(e \oplus y)\right] \quad (10) \\ &= m[U(e \oplus x)]\sigma_r[U(e \oplus y)] \quad (49) \\ &= m[U_r(x)]\sigma_r[U_r(y)] \quad (47) \\ &= m[U_r(e \oplus_r x)][U_r(y)] \quad (12), \end{aligned}$$

which was to be shown.

Right segregation follows from what has just been shown and Theorem 2.

By (52),

$$\gamma = \frac{m[U(x \oplus e)]}{m[U(e \oplus x)]} = \frac{m[U_l(x)]}{m[U_r(x)]} = \frac{m[U_l(x \oplus_l e)]}{m[U_r(e \oplus_r x)]},$$

which is the ratio condition, (55), for the induced operations.

A parallel proof holds for $(U_l, W, \oplus_l, \circ_{c,l})$. □

Corollary to Theorem 3.

Proof: **(1)** Assume m is a power function with exponent β_m. Equation (53) follows immediately from (48) and (55) follows immediately from (52). To show (54), consider

$$\begin{aligned} U(x \oplus y) &= \sigma_r[U(e \oplus y)]^{1/\beta_m} U(e \oplus x) + U(e \oplus y) \quad (49) \qquad (72) \\ &= U(x \oplus e) + \sigma_l[U(x \oplus e)]^{1/\beta_m} U(y \oplus e) \quad (50) \\ &= \gamma U(e \oplus x) + \sigma_l[\gamma U(e \oplus x)]^{1/\beta_m} U(e \oplus y), \quad [(52) \text{ with } \gamma = \gamma_m^{1/\beta_m}] \end{aligned}$$

whence

$$0 = \left(\gamma - \sigma_r[U(e \oplus y)]^{1/\beta_m}\right) U(e \oplus x) + \left(\sigma_l[\gamma U(e \oplus x)]^{1/\beta_m} - 1\right) U(e \oplus y).$$

Choose $e \oplus x \succ e, e \oplus y \succ e$,

$$\frac{\gamma - \sigma_r[U(e \oplus y)]^{1/\beta_m}}{U(e \oplus y)} = \frac{\sigma_l[\gamma U(e \oplus x)]^{1/\beta_m} - 1}{U(e \oplus x)} = -\gamma\delta,$$

a constant defining δ. Observe that the equality of the first and third expression gives

$$\sigma_r[U(e \oplus y)]^{1/\beta_m} = \gamma[1 + \delta U(e \oplus y)].$$

Substituting this into (72),

$$\begin{aligned} U(x \oplus y) &= \sigma_r[U(e \oplus y)]^{1/\beta_m} U(e \oplus x) + U(e \oplus y) \\ &= \gamma[1 + \delta U(e \oplus y)]U(e \oplus x) + U(e \oplus y) \\ &= U(x \oplus e) + U(e \oplus y) + \delta U(x \oplus e)U(e \oplus y), \end{aligned}$$

which is (54).

(2) Assume $\beta_m = 1$. We show right distributivity:

$$\begin{aligned} &U[(x \oplus z) \circ_C (y \oplus z)] = [U(x \oplus z) - U(y \oplus z)]W(C) + U(y \oplus z) \ (53) \\ &\quad = ([U(x \oplus e) - U(y \oplus e)]W(C) + U(y \oplus e))\,[1 + \delta U(e \oplus z)] \\ &\qquad + U(e \oplus z) \quad (54) \\ &\quad = U[(x \oplus e) \circ_C (y \oplus e)][1 + \delta U(e \oplus z)] + U(e \oplus z) \qquad (53) \\ &\quad = U[(x \circ_{C,l} y) \oplus e][1 + \delta U(e \oplus z)] + U(e \oplus z) \qquad (44) \\ &\quad = U[(x \circ_{C,l} y) \oplus z] \qquad (54). \end{aligned}$$

Left distributivity is similar. □

Theorem 4.

Proof: We first show that if the induced operations $\circ_{C,s}$ and $\oplus_s$ satisfy the three properties of Theorem **2**, **(2)**, then the above assertions **(1)-(3)** of Theorem 4 hold. Recall that it follows immediately from its definition that $\oplus_s$ is idempotent, (13).

(1) Let r, t be defined by $x \oplus y \sim r \oplus r, u \oplus v \sim t \oplus t$. Then

$$\begin{aligned} &m\,(U\,[(x \oplus y) \circ_C (u \oplus v)] - U(u \oplus v)) \\ &\quad = m\,(U\,[(r \oplus r) \circ_C (t \oplus t)] - U(t \oplus t)) \qquad (\text{def.}\, r, t) \\ &\quad = m\,(U\,[(r \circ_{C,s} t) \oplus (r \circ_{C,s} t)] - U(t \oplus t)) \qquad (60) \\ &\quad = m\,(U_s[(r \circ_{C,s} t) \oplus_s (r \circ_{C,s} t)] - U_s(t \oplus_s t)) \qquad (61) \\ &\quad = m\,[U_s(r \circ_{C,s} t) - U_s(t)] \qquad (13) \\ &\quad = m[U_s(r) - U_s(t)]W(C) \qquad (\text{PIUI of } \circ_{C,s}) \\ &\quad = m\,[U(r \oplus r) - U(s \oplus s)]\,W(C)) \qquad (61) \\ &\quad = m\,[U(x \oplus y) - U(u \oplus v)]\,W(C) \qquad (\text{def.}\, r, t) \end{aligned}$$

PIUI as defined by (48). The converse is immediate.

(2) Given (61) it is immediate that (38) holds for $\oplus_s$ iff it holds for $\oplus$. A similar remark holds for (42).

(3) First, observe that by idempotence of $\oplus_s$

$$U(e \oplus e) = U_s(e \oplus e) = U_s(e) = 0 = U(e),$$

so $e \oplus e \sim e$. By Part **(3)** of Theorem 2 we have (52). The converse is immediate. □

Corollary to Theorem 4.
Proof: **(1)** By Parts **(2)** and **(3)** of Theorem 4, we have by Theorem 2 that (39) is satisfied. If $\sigma_r^*[U(e \oplus y)] \equiv 1$ in that equation, then (62) is satisfied. So, suppose that is not the case. Applying the Corollary to Theorem 2 to $\oplus_s$ and also the left analogue of that Corollary, yields

$$\begin{aligned} U(x \oplus y) &= U_s(x \oplus_s y) \\ &= U_s(x)\sigma_r[U_s(y)]^{1/\beta_m} + U_s(y)\left[1 - \sigma_r[U_s(y)]^{1/\beta_m}\right] \\ &= U_s(x)\left[1 - \sigma_l[U_s(x)]^{1/\beta_m}\right] + U_s(y)\sigma_l[U_s(x)]^{1/\beta_m}. \end{aligned}$$

Thus, from the equality of the last two expressions,

$$0 = [U_s(x) - U_s(y)]\left[\sigma_r[U_s(y)]^{1/\beta_m} + \sigma_l[U_s(x)]^{1/\beta_m} - 1\right].$$

For x not equivalent to y, the first factor is non-zero, so the right one, which depends on both x and y, must be 0, in which case it follows that $\eta = \sigma_r[U_s(y)]^{1/\beta_m}$ is a constant and so using the idempotence of $\oplus_s$,

$$\begin{aligned} U(x \oplus y) &= \eta U_s(x \oplus_s x) + (1-\eta)U_s(y \oplus_s y) \\ &= \eta U(x \oplus x) + (1-\eta)U(y \oplus y). \end{aligned}$$

(2) Using this result,

$$\begin{aligned} U(x \oplus e) &= \eta U(x \oplus x) + 0, \\ U(e \oplus x) &= 0 + (1-\eta)U(x \oplus x), \end{aligned}$$

whence both (62) and the ratio condition, (55), with $\gamma = \eta/(1-\eta)$.

(3) Assume $\beta_m = 1$.We show right distributivity:

$$U[(x \oplus z) \circ_C (y \oplus z)] = [U(x \oplus z) - U(y \oplus z)]W(C) + U(y \oplus z) \qquad (53)$$
$$= ([U(x \oplus e) - U(y \oplus e)]W(C) + U(y \oplus e))\,[1 + \delta U(e \oplus z)]$$
$$+U(e \oplus z) \quad (54)$$
$$= U[(x \oplus e) \circ_C (y \oplus e)][1 + \delta U(e \oplus z)] + U(e \oplus z) \qquad (53)$$
$$= U[(x \circ_{C,l} y) \oplus e][1 + \delta U(e \oplus z)] + U(e \oplus z) \qquad (44)$$
$$= U[(x \circ_{C,l} y) \oplus z], \qquad (54)$$

Left distributivity is similar. □

Appendix B: Auxiliary Results

Using the notation of (21) and (22), right distributivity, (19), yields

$$F[G_C(X,Y),Z] = G_C[F(X,Z),F(Y,Z)] \tag{73}$$

where $X,Y \in]0,\infty[\,, X \geq Y, Z \in [0,\infty[$,which when C is held fixed is the (right) distributivity equation on p. 341 of Aczél (1966). The results reported there are largely due to Hosszú (1953, 1959) and rested on the assumption that the unknown functions are twice continously differentable. Lundberg (1982,1985) proved similar results without differentiability assumptions under monotonicity conditions; however, these were somewhat stronger than is appropriate in the current applications. Lundberg (2002) has proved the result (Proposition B-1) under our assumptions.

Assuming continuity, as we do, (73) can be extended to the domain $\mathbb{R}_+$. By the monotonicity assumptions about stimuli and joint receipt, both F and G_C are strictly increasing in both variables and are onto $\mathbb{R}_+$ when C is fixed.

Proposition B-1. (Lundberg, 2002) *Assuming that (73) holds on the non-negative real numbers, that F, G_C both exist, F is strictly increasing in both variables, G_C is strictly increasing in the first variable and continuous and philandering in the second, then there are two classes of solutions*:

(a)

$$F(X,Y) = h^{-1}[h(X) + \alpha(Y)] \tag{74}$$

$$G_C(X,Y) = h^{-1}\left(h(Y) + \psi_C[h(X) - h(Y)]\right). \tag{75}$$

(b)

$$F(X,Y) = h^{-1}[h(X)g(Y) + \alpha(Y)] \tag{76}$$

$$G_C(X,Y) = h^{-1}\left(h(Y) + a(C)[h(X) - h(Y)]\right), \tag{77}$$

where h, α, and ψ are strictly increasing functions, $h(0) = 0, \alpha(0) = 0$, and for some $p > 0$ we have $\psi_C(pu) = p\psi_C(u)$.

Corollary to Proposition B-1. *Suppose that the conditions of Proposition B-1 are satisfied. Then,*

(1) *If $F(0,Y) = Y$ (which is equivalent to e* being *a left identity of $\oplus$), then (74) simplifies to:*

$$F(X,Y) = h^{-1}[h(X) + h(Y)], \tag{78}$$

and so is F symmetric and so $\oplus$ is commutative.

(2). *If $\oplus$ is commutative and has an identity e, then (76) can be sharpened to*

$$h[F(X,Y)] = h(X) + h(Y) + \rho h(X)h(Y), \tag{79}$$

where ρ is a constant.

Proof: **(1)** By $h(0) = 0$,

$$h(Y) = h[F(0,Y)] = h(0) + \alpha(Y) = \alpha(Y),$$

and so substitute into (74).

(2) Using $h(0) = 0$ and $F(0,Y) = Y$, which is assumed, in the (b) solution, we see that $\alpha(Y) = h(Y)$, so

$$h[F(X,Y)] = h(X)g(Y) + h(Y).$$

By the symmetry of F,

$$\frac{g(X)-1}{h(X)} = \frac{g(Y)-1}{h(Y)},$$

which must be a constant ρ, whence $g(X) = 1 + \rho h(X)$ and so (79). □

The following proposition, which improves Theorem 2 of Aczél and Luce (2002), is needed to prove Theorems 5 and 7. It is stated as is needed, which is slightly less general than what Lundberg (2002) has proved.

Proposition B-2. (Lundberg, 2002) *Suppose the functional equation*

$$h[Y + f(X-Y)] = h(Y) + g[h(X) - h(Y)] \quad (X \geq Y \geq 0) \tag{80}$$

holds under the assumptions that, for $k \in]0,\infty]$, h from $[0,k[$ onto a non-negative real interval and f and g from $[0,k[$ onto $[0,k[$ are all strictly increasing and continuous. Then, one of the following cases obtain:

(1)

$$h(X) = cX + d, \; f(X) = \frac{1}{c}g(cX) \quad (c > 0), \tag{81}$$

and if $k < \infty$, then $0 < c \leq 1$.

(2) *There exist constants a, b, c, q, r such that*[6]

$$h(X) = a\ln(re^{bX} + q), \tag{82}$$

$$f(Z) = \frac{1}{b}\ln(ce^{bZ} + 1 - c), \tag{83}$$

$$g(Z) = a\ln(ce^{Z/a} + 1 - c). \tag{84}$$

(3) *There exist constants a, b, c, q such that*

$$h(X) = ae^{bX} + q, \tag{85}$$

$$f(Z) = \frac{1}{b}\ln(ce^{bZ} + 1 - c), \tag{86}$$

$$g(Z) = cZ. \tag{87}$$

(4) *There exist constants* a, c, p, q *such that*

$$\begin{aligned} h(X) &= a\ln(pX+q), \quad (88)\\ f(Z) &= cZ, \quad (89)\\ g(Z) &= a\ln(ce^{Z/a}+1-c). \quad (90)\end{aligned}$$

In all cases of the constants must be such that the functions f, g, h *are non-negative and strictly increasing. When* $k < \infty$, $0 < c \leq 1$ *in cases 2-4 and, in addition, in* ***(2)*** $b > 0, ar > 0$, *in* ***(3)*** $b > 0, a > 0$, *and in* ***(4)*** $ap > 0$.

Corollary to Proposition B-2. *Assuming that (80) varies with both* X, Y, *it is impossible for* $f(Z) = \tilde{c}Z$ *in cases* ***(2)*** *and* ***(3)***, *and it is impossible for* $g(X) = \bar{c}X$ *in* ***(2)*** *and* ***(4)***.

Proof: Consider, first, (83). Then,

$$\begin{aligned} &\tilde{c}Z = f(Z) = \frac{1}{b}\ln[c\exp(bZ)+1-c]\\ \Leftrightarrow\quad &e^{b\tilde{c}Z} = ce^{bZ}+1-c.\end{aligned}$$

Taking the derivative yields $e^{b\tilde{c}Z} = \frac{c}{\tilde{c}}e^{bZ}$, so

$$ce^{bZ}\left(\frac{1}{\tilde{c}}-1\right) = 1-c,$$

whence $\tilde{c} = 1$. which is impossible because it eliminates the dependence on Y in (80). The cases of (84) and (90) are similar. □

In the following proofs, either F or G_C or both are constrained, and we seek solutions under those constraints.

Appendix C: Distributivity Proofs

Proposition 5:

Proof: Assuming both right distributivity and right segregation means that we are allowing the right variable of G_C to have the value 0. Rewrite joint receipt decomposability, (66), in terms of simple decomposability, (21) and (22) to obtain

$$G_C[F(X,Y),0] = F[G_C(X,0), G_D(Y,0)]. \quad (91)$$

By Proposition B-1, we know that the solutions to the distributivity equation when $\oplus$ is commutative take one of two forms:

(a)

$$\begin{aligned} F(X,Y) &= h^{-1}[h(X)+h(Y)], \quad (92)\\ G_C(X,Y) &= h^{-1}\left(h(Y)+\psi_C[h(X)-h(Y)]\right). \quad (93)\end{aligned}$$

(b)

$$h[F(X,Y)] = h(X)+h(Y)+\rho h(X)h(Y) \quad (94)$$
$$G_C(X,Y) = h^{-1}\left(h(Y)+a(C)[h(X)-h(Y)]\right). \quad (95)$$

First, using case (a) in (91):

$$\begin{aligned} h^{-1}\left(\psi_C[h(X)+h(Y)]\right) &= h^{-1}\left(\psi_C[hF(x,y)]\right) \quad (92) \\ &= h^{-1}[h(0)+\psi_C[hF(X,Y)-h(0)] \quad (h(0)=0) \\ &= G_C[F(X,Y),0] \quad (93) \\ &= F[G_C(X,0)+G_D(Y,0)] \quad (91) \\ &= h^{-1}[hG_C(X,0)+hG_D(Y,0)] \quad (92) \\ &= h^{-1}\left(\psi_C[h(X)]+\psi_D[h(Y)]\right) \quad (93). \end{aligned}$$

Applying h and setting $R=h(X), S=h(Y)$, we have

$$\psi_C(R+S)=\psi_C(R)+\psi_D(S).$$

This is a Pexider equation with the solutions

$$\psi_C(R)=\alpha R+\beta_C, \quad \psi_D(R)=\alpha R.$$

Substituting and taking into account $\psi_C(0)=0$, which follows from idempotence, (3) of gambles, we see that $\psi_C(R)=\alpha R$. This is the RDU, and, of course, the F expression is additive.

Next, consider case (b). By applying (95) and (94) to (91) yields

$$a(C)[h(X)+h(Y)+\rho h(X)h(Y) = a(C)h(X)+a(D)h(Y)+\rho a(C)h(X)a(D)h(Y),$$

whence

$$a(C)-a(D)+\rho a(C)h(X)[1-a(D)]=0.$$

Thus, either $a(C)=a(D)=1$ or $h(X)=$ a constant, both of which are impossible, ruling out case (b). □

Theorem 5:

Proof:

(1) and **(2)** imply **(3)**: We are in the situation of the right distributivity functional equation. Thus, there are the two subcases of Proposition B-1.

Case (a). Let $X=V(x), Y=V(y)$. By (21) with $\oplus$ having an additive representation V, $F(X,Y)=X+Y$, and then by (74) and $h(0)=0$,we conclude that $h(X)=cX$, $c>0$, and so by (75)

$$G_C(X,Y)=Y+\frac{1}{c}\psi_C[c(X-Y)]=Y+L_C(X-Y),$$

where $L_C(X):=\frac{1}{c}\psi_C(cX)$. This is an acceptable solution.

Case (b). For (79), following from (76), define

$$m(x) := \rho h(X) + 1 > 0, \tag{96}$$

which when substituted into that equation yields

$$m[F(X,Y)] = m(X)m(Y). \tag{97}$$

Because $F(X,Y) = X + Y$, it follows immediately from (97) that $m(X) = \exp(X)$, whence

$$h(X) = \frac{1}{\rho}\left[\exp(X) - 1\right]. \tag{98}$$

Substituting this into (77) yields

$$G_C(X,Y) = \ln\left(\exp(Y) + a(C)[\exp(X) - \exp(Y)]\right). \tag{99}$$

Substituting the definition of V in terms of U yields that U satisfies RDU, which is the special case of (23) with L_C given by (35).

(1) and **(3)** imply **(2)**: From V satisfies (23) and, as in Case (a), $F(X,Y) = X + Y$, then

$$\begin{aligned} V[(x,C;y)\oplus z] &= V(x,C;y) + V(z) \\ &= V(y) + L_C[V(x) - V(y)] + V(z) \\ &= V(y\oplus z) + L_C[V(x\oplus z) - V(y\oplus z)] \\ &= V(x\oplus z, C; y\oplus z), \end{aligned}$$

whence right distributivity.

Assume that **(2)** and **(3)** hold. Then using Case (a) of the right distributivity equation, i.e., (78) and (75), we have

$$\begin{aligned} h[F(X,Y)] &= h(X) + h(Y) \\ h[Y + L_C(X-Y)] &= h[G_C(X,Y)] = h(Y) + \psi_C[h(X) - h(Y)]. \end{aligned} \tag{100}$$

So Proposition B-2 applies, which has four subcases:

Case (1). h is linear and so F is additive.

Case (2). For $h(x) = a\ln\left(re^{bx} + q\right)$, a routine calculation using (100) with $h = f, \psi = g$ shows that U is p-additive over $\oplus$ and so V, defined in terms of U by (16), is additive.

Case (3). For $h(x) = ae^{bx} + q$ we see

$$e^{bF(X,Y)} = e^{bX} + e^{bY} - 1$$

from which $U(x\oplus y)$ is additive over $\oplus$ and so, by (16), is V.

Case (4). For $h(x) = a\ln(rx + q)$, a routine calculation shows that V is p-additive over $\oplus$. Thus, there is a transformation to V^* that is additive and so with right distributivity holding there exist L_C^* such that (V^*, L_C^*) satisfies (23) or $U^* = V$ satisfies RDU which is a special case of (23).

Case (b) of the right distributivity equation holds, i.e.,

$$\begin{aligned} h[F(X,Y)] &= h(X)g(Y)+\alpha(Y) \\ h[G_C(X,Y)] &= h(Y)+a(C)[h(X)-h(Y)]. \end{aligned}$$

By the Corollary to Proposition B-1, (79) holds. Thus, Proposition B-2 again applies with its four subcases:

Case (1). A routine substitution of the linear h into (79) leads to F being p-additive, and so we proceed as in Case (a)(4).

Case (2-4). The Corollary to Proposition B-2 rules out these possibilities.□

Theorem 6:

Proof: First, we establish that there is an additive decomposition of gambles of the form $V(x_1) \geq ... \geq V(x_n) > 0$.

$$f\left(G_{\mathbf{C}}[V(x_1);...;V(x_n)]\right) = \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}[V(x_i)], \tag{101}$$

where f and each $L_{\mathbf{C}}^{(i)}$ is strictly increasing and unique up to positive affine transformations with a common unit. To do so, we invoke Theorem 4 of Wakker (1991) which says that in the ranked context the following conditions are sufficient: weak order, essentialness of each coordinate, general strict monotonicity, solvability, Archimedeanness, and no maximal or minimal consequences. We have assumed weak ordering, essentialness, and general strict monotonic increase for each subset of alternatives. The minimal element e corresponding to $V(e) = 0$ is excluded and V is unbounded. Solvability holds because V is onto $\mathbb{R}_+$. To show Archimedeanness, let $\overrightarrow{Y}_{\bar{i}}$ and $\overrightarrow{Z}_{\bar{i}}$, $\overrightarrow{Y}_i \prec \overrightarrow{Z}_i$ denote choices of vectors on all alternatives except for the i^{th}, i.e., $\overrightarrow{Y}_i, \overrightarrow{Z}_i \in \mathbb{R}_+^{n-1}$, and suppose that

$$\min(Y_{i-1}, Z_{i-1}) > \max(Y_{i+1}, Z_{i+1}).$$

For some X_i in that interval, the vectors $(X_i, \overrightarrow{Y}_{\bar{i}})$ and $(X_i, \overrightarrow{Z}_{\bar{i}})$ are appropriately rank ordered and, by strict monotonicity, $(X_i, \overrightarrow{Y}_{\bar{i}}) \prec (X_i, \overrightarrow{Z}_{\bar{i}})$. So, a standard sequence $\{X_j\}$ is defined by

$$(X_{i,j+1}, \overrightarrow{Y}_i) \sim (X_{i,j}, \overrightarrow{Z}_i).$$

Such a sequence is certainly bounded by the fact $X_{i,j} \leq \min(Y_{i-1}, Z_{i-1})$. If it is not finite, the by the fact it is an increasing bounded sequence, it converges to a value, say A_i, and so

$$(A_i, \overrightarrow{Y}_i) \sim (A_i, \overrightarrow{Z}_i).$$

But this contradicts strict monotonicity. So Archimedeanness is satisfied, and therefore (101) is satisfied. As noted earlier, the choice of f and $L_{\mathbf{C}}^{(i)}$ is unique up to positive affine transformations with a common unit. So with no loss of generality, we may choose the unit to be 1 and the additive constants so that $\lim_{X\to 0} L_C^{(i)}(X) = 0$ and so define $L_{\mathbf{C}}^{(i)}(0) = 0$.

By the additivity of V and generalized right distributivity,

$$\begin{aligned} & V(x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) + V(z) \\ & \quad = V[(x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) \oplus z] \\ & \quad = V(x_1 \oplus z, C_1; ...; x_i \oplus z, C_i; ...; x_n \oplus z, C_n). \end{aligned}$$

Invoking the decomposability assumption

$$G_{\mathbf{C}}[V(x_1); ...; V(x_n)] + V(z) = G_{\mathbf{C}}[V(x_1) + V(z); ...; V(x_n) + V(z)], \quad (102)$$

and substituting (101) into (102) yields

$$f^{-1}\left(\sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}[V(x_i)]\right) + V(z) = f^{-1}\left(\sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}[V(x_i) + V(z)]\right).$$

Because gambles are idempotent, if we set $x_1 \sim ... \sim x_n \sim x$ and $X = V(x), Z = V(z)$, then we have from (102) that

$$f^{-1}(X) + Z = f^{-1}(X + Z).$$

Let $Y = X + Z$, then

$$f^{-1}(Y) = f^{-1}(Y - Z) + Z.$$

If $Y = Z$, then

$$f^{-1}(Z) = f^{-1}(0) + Z.$$

Thus, the last functional equation reduces to

$$\sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(X_i) + Z = \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(X_i + Z).$$

If we let $X_i = V(x_i)$, $Y_i = X_i + Z$, and $y_i = V^{-1}(Y_i)$, this equation reduces to

$$\begin{aligned} V(y_1, C_1; ...; y_i, C_i; ...; y_n, C_n) &= \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(Y_i) \\ &= \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(X_i + Z) \\ &= \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(X_i) + Z \\ &= \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}(Y_i - Z) + Z \\ &= \sum_{i=1}^{n} L_{\mathbf{C}}^{(i)}[V(y_i) - V(z)] + V(z). \end{aligned}$$

Let $z \sim y_n$ and recall that $L_{\mathbf{C}}^{(n)}(0) = 0$, we end up with (67). □

Theorem 7:

Proof: Throughout, we have (21) and (22) holding.

(1) and **(2)** imply **(3)**: So we assume right distributivity, (19), and for $U(x \oplus y)$ the form (40). So, as usual, there are the two cases of Proposition B-1, (a) and (b):

Case (a). By (74) and (40),

$$h[F(X,Y)] = h(X) + \alpha(Y) = F(X,0)\sigma_r^*(Y) + F(0,Y).$$

From the first equality, setting $X = 0$ yields $\alpha(Y) = h[F(0,Y)]$ and setting $Y = 0$ yields $h[F(X,0)] = h(X)$. So, if we set $R = F(X,0), S = F(0,Y), f(S) = \sigma_r^*[F^{-1}(S)]$,

$$h[Rf(S) + S] = h(R) + h(S) = h[Sf(R) + R],$$

whence

$$\frac{f(S) - 1}{S} = \frac{f(R) - 1}{R} = \rho,$$

and so $h[F(X,Y)] = h(X) + h(Y) + \rho h(X)h(Y)$, which is the commutative case, which has been ruled out.

Case (b). Using (40) and (94),

$$h[F(X,0)\sigma_r(Y) + F(0,Y)] = h[F(X,Y)] = h(X)g(Y) + \alpha(Y).$$

Set $X = 0$ to get $\alpha(Y) = h[F(0,Y)]$ and $Y = 0$ to get $h[F(X,0)] = h(X)g(0)$, where $g(0) > 0$ since $F(X,0)$ is strictly increasing in X. Thus,

$$h[F(X,0)\sigma_r(Y) + F(0,Y)] = h[F(X,0)]g^*(Y) + h[F(0,Y)],$$

where $g^*(Y) = g(Y)/g(0)$. Set $R = F(X,0)$, $S = F(0,Y)$, $f(S) = \sigma_r^*[F^{-1}(S)]$, $\varphi(S) = g^*[F^{-1}(S)]$, where $F^{-1}(S) = Y$ iff $S = F(0,Y)$,

$$h[Rf(S) + S] = h(R)\varphi(S) + h(S) \quad [f(0) = 1, \varphi(0) = 1, h(0) = 0].$$

This equation is a special case of the class of functional equations solved by Lundberg and Ng (1975). Of their 6 subcases, our assumptions rule out all but Cases 1 and 6. In the former, $h(Z) = \alpha Z + \beta, \alpha > 0$. Because $h(0) = 0$, $\beta = 0$. Substituting this into (77) yields RDU. Case 6 reduces to

$$f(S) = aS + b, \varphi(S) = a'h(S) + 1, h(Z) = \alpha Z^c \quad (c \neq 0).$$

Substituting,

$$(bR + S + aRS)^c = R^c + S^c + a'\alpha R^c S^c.$$

Differentiating first with respect to R and then with respect to S, dividing, and simplifying yields

$$\frac{1 + a'\alpha S^c}{S^{c-1}(b + \alpha S)} = \frac{1 + a'\alpha R^c}{R^{c-1}(1 + \alpha R)}.$$

Because R and S may be chosen independently, the common value must be a constant, which is possible only if $c = 1$, which is as in Case 1 of Proposition B-2.

(2) and **(3)** imply **(1)**: By **(2)** we have (40) in the version

$$U(x \oplus y) = U(x)\sigma_r(y) + U(e \oplus y).$$

Using that and the RDU form assumed in **(3)**, we have for the left side of distributivity

$$\begin{aligned} U[(x, C; y) \oplus z] &= U(x, C; y)\sigma_r(z) + U(e \oplus z) \\ &= ([U(x) - U(y)]W(C) + U(y))\,\sigma_r(z) + U(e \oplus z), \end{aligned}$$

and for the right side

$$\begin{aligned} U(x \oplus z, C; y \oplus z) &= [U(x \oplus z) - U(y \oplus z)]\,W(C) + U(y \oplus z) \\ &= [U(x) - U(y)]\,\sigma_r(z)W(C) + U(y)\sigma_r(z) + U(y \oplus z), \end{aligned}$$

and these are therefore equal. Thus, **(1)** holds.

(1) and **(3)** imply **(2)**: By the argument at the beginning of the proof, we need only consider case (b) of the right distributivity condition. Using that plus the assumption **(3)** that RDU holds, we have

$$h[(X - Y)W(C) + Y] = [h(X) - h(Y)]a(C) + h(Y).$$

This is (80) of Proposition B-2 with

$$f(Z) = ZW(C), g(Z) = za(C).$$

By the Corollary to B-2, Cases **(2)**-**(4)** of Proposition B-2 are ruled out. So by Case **(1)** h is linear and by Part (b) of Proposition B-1 and using the usual trick of setting $X = 0$ and then $Y = 0$,

$$\begin{aligned} F(X, Y) &= Xg(Y) + Y \\ &= F(X, 0)g(Y) + F(0, Y), \end{aligned}$$

which is Statement **(2)** of the theorem. □