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Prevalent Behavior of Strongly Order Preserving Semiflows

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Abstract

New formulations of some of the classical theorems on genericity of quasiconvergence and convergence in the theory of monotone semiflows are obtained. Employing the measure-theoretic notion of a shy set, developed in [7, 1], we give sufficient conditions such that certain subsets of the state space are prevalent (their complement is shy), including the quasiconvergent points, the convergent points, and the set of points that converge to an equilibrium that is not linearly unstable. New sufficient conditions for the set of convergent points to be open and dense, which reduce the compactness requirements of earlier results, are also obtained.

1 Introduction

The signature results in the theory of monotone dynamics are that certain dynamic behaviors are generic, for example, convergence to equilibrium is generic under suitable conditions. In order to be more precise, some notation is useful but technical definitions will be deferred to a subsequent section. Let $\mathbb{B}$ be an ordered separable Banach space, $X \subset \mathbb{B}$, and consider a semiflow $\Phi : \mathbb{R}_+ \times X \to X$ which is strongly monotone with respect to a cone K with nonempty interior.

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Denote the sets

$$\begin{aligned} B &= \{x \in X \mid \text{the orbit } O(x) \text{ has compact closure in } X\} \\ Q &= \{x \in X \mid \omega(x) \subseteq E\} \\ C &= \{x \in X \mid \omega(x) = \{e\} \text{ for some } e \in E\}. \end{aligned}$$

The elements of C are said to be *convergent* or to have *convergent solution*, and those of Q are said to be *quasiconvergent*.

It was established in [3] that the generic element of B is quasiconvergent, where 'generic' is made specific in two different senses: the topological sense ($B\backslash Q$ is meager or Q is residual), and the measure theoretic sense ($\mu(B-Q) = 0$ for any gaussian measure μ). Later, Smith and Thieme [16], motivated by work of Poláčik, [12], provided sufficient conditions for C to contain an open and dense set.

A drawback of topological genericity is that closed, nowhere dense subsets of X may still be quite large in terms of measure. In fact, it is well known that there exists a Cantor subset of $[0, 1]$ with positive measure whose complement is open and dense in $[0, 1]$. On the other hand, asking for a set to have measure zero in an infinite dimensional space $\mathbb{B}$ is difficult to formalize, since there doesn't exist a measure with the basic properties of the Lebesgue measure in finite dimensions. A definition of 'sparseness' that turns out to be very useful in infinite dimensions is that of prevalence [7, 1]: a set $W \subseteq \mathbb{B}$ is *shy* if there exists a compactly supported Borel measure μ on $\mathbb{B}$, such that $\mu(W + x) = 0$ for every $x \in \mathbb{B}$. A set is said to be *prevalent* if its complement is shy. Given $A \subseteq \mathbb{B}$, we say here that a set W is *prevalent in* A if $A - W$ is shy. Useful properties of the idea of prevalence are given in [7]. Most importantly in the current paper, a shy set has empty interior, and in finite dimensions W is shy if and only if W has Lebesgue measure zero.

In this paper, we obtain the counterparts of the genericity results of Hirsch and Smith and Thieme with prevalence as the notion of genericity. We show that Q is prevalent in B, and that under additional hypotheses, so is C. Genericity in this measure-theoretic sense seems natural to the theory of monotone systems. The canonical family of compactly supported Borel measures for our purposes are given by the uniform measures μ_v, $v > 0$ supported on the segment $J_v = \{tv : 0 \leq t \leq 1\}$ joining zero to the positive vector. Recalling Hirsch's result [4] that, when $X = B$, a totally ordered arc J_v has the property that $J_v \setminus Q$ is countable, and therefore its translates have μ_v measure zero, one sees that prevalence is a natural notion of genericity.

An earlier result [5, 6] giving sufficient conditions that Q (and also C) contains an open and dense set is significantly improved. Roughly, this earlier result replaces certain compactness assumptions on the semiflow by order properties of the state-space, namely, omega limit sets should have infima or suprema. We improve it by requiring that some possibly larger space contains an infima or suprema of the limit sets. Our extension facilitates the application of the theory to partial differential equations on state spaces that continuously imbed in a space of continuous functions.

If Φ is C^1 and $e \in E$ an equilibrium, define $\rho(e,t)$ to be the spectral radius of the Frechet derivative $D_x\Phi(t,e)$. We say that e is *linearly stable* if $\rho(e,t) < 1$ for all $t > 0$, *linearly unstable* if $\rho(e,t) > 1,\ t > 0$, and *neutrally stable* if $\rho(e,t) = 1,\ t > 0$. Finally, define $E_s \subseteq E$ to be the set of equilibria that are either linearly stable or neutrally stable. We show that the set of initial data corresponding to trajectories that converge to a point in E_s is prevalent in X. Using this result and a well known result of Kishimoto and Weinberg [9], we conclude that for a strongly cooperative reaction diffusion system of n equations with Neumann boundary conditions on a convex domain Ω, the set of initial data in $C(\overline{\Omega},\mathbb{R}^n)$ corresponding to orbits that converge towards a constant equilibrium is prevalent.

2 Definitions and Basic Results

Let X be an ordered metric space with metric d and *partial order* relation $\leq$. We write $x < y$ if $x \leq y$ and $x \neq y$. Given two subsets A and B of X, we write $A \leq B$ ($A < B$) when $x \leq y$ ($x < y$) holds for each choice of $x \in A$ and $y \in B$. We assume that the order relation and the topology on X are compatible in the sense that $x \leq y$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all n. For $A \subset X$ we write $\bar{A}$ for the closure of A and Int$A$ for the interior of A. A subset of an ordered space is *unordered* if it does not contain points x, y such that $x < y$. A is *order-convex* if $x \leq z \leq y$ and $x, y \in A$ implies $z \in A$.

Let $A \subset X$ and let $L = \{x \in X : x \leq A\}$ be the (possibly empty) set of lower bounds for A in X. In the usual way, we define $\inf A := u$ if $u \in L$ and $L \leq u$; u is unique if it exists. Similarly, $\sup A$ is defined.

The notation $x \ll y$ means that there are open neighborhoods U, V of x, y respectively such that $U \leq V$. Equivalently, (x, y) belongs to the interior of the order relation. The relation $\ll$ is sometimes referred to as the *strong ordering*. We write $x \geq y$ to mean $y \leq x$, and similarly for $>$ and $\gg$.

In most applications, X is a subset of an ordered Banach space $\mathbb{B}$ having an order cone $\mathcal{K}$. In this case, $x \leq y$ if and only if $y - x \in \mathcal{K}$. If $\mathcal{K}$ has nonempty interior, then $x \ll y$ if and only if $y - x \in \text{Int}\mathcal{K}$. A subset A of $\mathbb{B}$ is *p-convex* if $x < y$ and $x, y \in A$ implies $z = tx + (1-t)y \in A$ for $0 < t < 1$.

A *semiflow* on X is a continuous map $\Phi : \mathbb{R}_+ \times X \to X$, $(t, x) \mapsto \Phi_t(x)$ such that:

$$\Phi_0(x) = x, \quad (\Phi_t \circ \Phi_s)(x) = \Phi_{t+s}(x) \qquad (t, s \geq 0,\ x \in X)$$

The *orbit* of x is the set $O(x) = \{\Phi_t(x) : t \geq 0\}$. An *equilibrium* is a point x for which $O(x) = \{x\}$. The set of equilibria is denoted by E.

The *omega limit set* $\omega(x)$ of $x \in X$, defined in the usual way, is closed and positively invariant. When $\overline{O(x)}$ is compact, $\omega(x)$ is also nonempty, compact, invariant, connected, and it attracts x. A point $x \in X$ is *quasiconvergent* if $\omega(x) \subset E$. The set of such points is denoted by Q. If $\omega(x)$ is single point, necessarily an equilibrium, then x is *convergent*. The set of convergent points is denoted by C.

Let Φ denote a semiflow in an ordered space X. We call Φ *monotone* provided

$$\Phi_t(x) \leq \Phi_t(y) \text{ whenever } x \leq y \text{ and } t \geq 0.$$

Φ is *strongly monotone* if $x < y$ implies that $\Phi_t(x) \ll \Phi_t(y)$ for all $t > 0$ and *eventually strongly monotone* if it is monotone and there exists $t_0 \geq 0$ such that $x < y$ implies that $\Phi_t(x) \ll \Phi_t(y)$ for $t \geq t_0$. Φ is *strongly order-preserving*, SOP for short, if it is monotone and whenever $x < y$ there exist open subsets U, V of X with $x \in U$ and $y \in V$ and $t_0 \geq 0$ such that

$$\Phi_{t_0}(U) \leq \Phi_{t_0}(V).$$

Monotonicity of Φ then implies that $\Phi_t(U) \leq \Phi_t(V)$ for all $t \geq t_0$. Strong monotonicity implies eventual strong monotonicity which implies SOP. See e.g. [15, 14, 5].

Theorem 1 *Nonordering of Limit Sets] Let Φ be SOP and ω be an omega limit set. Then no two points of ω are related by $<$.*

Theorem 2 *Limit Set Dichotomy] Let Φ be SOP. If $x < y$ then either*

(a) $\omega(x) < \omega(y)$, or

(b) $\omega(x) = \omega(y) \subset E$.

Smith and Thieme [16] improve part (b) of the Limit Set Dichotomy to read $\omega(x) = \omega(y) = \{e\}$ for some $e \in E$ under additional smoothness and strong monotonicity conditions. For example, this Improved Limit Set Dichotomy holds if $X \subset Y$ is order convex in the ordered Banach space Y with cone Y_+ having non-empty interior, $\Phi_t(x)$ is C^1 in x and its derivative is a compact, strongly positive operator. See e.g. [16, 14, 5].

The notion of a shy set was introduced by Yorke et al and Christensen [7, 1]. Let A be a subset of the Banach space $\mathbb{B}$. A is said to be *shy* if there exists a compactly supported Borel measure μ and a Borel set A' such that $A \subset A'$ and $\mu(A' + x) = 0$ for every $x \in \mathbb{B}$. The complement of a shy set is called a *prevalent* set. The notion of a shy set is a natural generalization to infinite dimensional spaces of a (Lebesgue) measure zero subset of $\mathbb{R}^n$ in the sense that a subset of $\mathbb{R}^n$ has measure zero if and only if it is shy. A countable union of shy sets is shy. Moreover, built into the definition is translation invariance: if A is shy then so is $A + x$. Prevalent sets are dense. These and other properties can be found in [7].

3 C is Prevalent in B

Consider an SOP semiflow Φ defined on a subset X of the separable Banach space B, ordered with respect to a cone $\mathcal{K}$. Given $v \in B, v \neq 0$, define the Borel measure μ_v on B to be the uniform measure supported in the set $S_v := \{tv \,|0 \leq t \leq 1\}$. That is, $\mu_v(A) = m\{t \in [0,1] \,|tv \in A\}$, where m is the Lebesgue measure in $[0,1]$.

Lemma 1 *Let $W \subseteq X$ be such that $L \cap W$ is countable, for every straight line L parallel to a positive vector $v > 0$. Then W is shy.*

Proof. Consider $v > 0$ and the uniform measure μ_v. Let $L = \mathbb{R}v - x$ for an arbitrary $x \in \mathbb{B}$. Then

$$(W + x) \cap S_v \subseteq (W + x) \cap \mathbb{R}v = (W \cap L) + x.$$

Therefore clearly $\mu_v(W + x) = 0$, and W is shy with respect to μ_v. ∎

The proof of Hirsch's generic convergence theorem as stated in terms of prevalence becomes clear at this point. See Theorem 4.4 of [3], and [4].

Theorem 3 *Let $\mathbb{B}$ be a separable Banach space, and consider a strongly monotone system defined on $X \subseteq \mathbb{B}$. Then Q is prevalent in B.*

If the Improved Limit Set Dichotomy holds for Φ, then C is prevalent in B.

Proof. Let $N = B - Q$ be the set of states $x \in B$ such that $\omega(x) \not\subseteq E$ (see Section 6 for a proof that this set is measurable). Let $L \subseteq X$ be a straight line parallel to a vector $v > 0$. Note that if $x, y \in L \cap N$, $x < y$, then $\omega(x) < \omega(y)$, since otherwise $\omega(x) = \omega(y) \subseteq E$ by the Limit Set Dichotomy.

We can apply an argument as in Theorem 7.3 c) of Hirsch [4] to conclude that $N \cap L$ is countable: consider the set $Y = \cup_{x \in N \cap L} \omega(x)$ with the topology inherited by $\mathbb{B}$. Since no point in $\omega(x)$ can bound $\omega(x)$ from below or above (Theorem 1), no point in $\omega(x)$ can be the limit of elements in $\omega(y)$, $y \neq x$. Therefore $\omega(x)$ is open in Y, for every $x \in N \cap L$. The countability of $N \cap L$ follows by the separability of Y.

Since $N \cap L$ is countable for every strongly ordered line L, N must be shy by Lemma 1.

If the Improved Limit Set Dichotomy holds for Φ, then we can argue exactly as above to show that $N = B \setminus C$ is shy. ∎

Define for any set $A \subset X$ the *strict basin of attraction*

$$SB(A) := \{x \in X \,|\, \omega(x) = A \text{ and } x \in B\}.$$

Note the difference with the usual basin of attraction of A, $\mathcal{B}(A) = \{x \in X \,|\, \omega(x) \subseteq A\}$.

Theorem 4 *Let $\mathbb{B}$ be a separable Banach space, and let X be p-convex in B. If C is dense in B, then C is prevalent in B.*

Proof. Let $K = B - C$ be the set of the states $x \in B$ such that $\omega(x)$ is not a singleton (see Section 6 for a proof that this set is measurable). We will show that K is shy with respect to the measure μ_v, for every $v > 0$. From the assumption that C is dense in B, it holds that $SB(\omega(x))$ has empty interior for every $x \in K$.

Let L be a straight line in X parallel to a positive vector $v > 0$, $\mathcal{P}(X)$ denote the set of all subsets of X, and consider the function $\gamma : L \cap K \to \mathcal{P}(X)$ defined by $\gamma(x) = \omega(x)$. Then this function is injective. Indeed, if $x, y \in L \cap K$, $x < y$, were such that $\gamma(x) = \gamma(y) = W$, then the strong order preserving property implies that for any point $u = sx + (1-s)y$, $0 < s < 1$ there is a neighborhood U of u and $t_0 \geq 0$ such that $\Phi_t(x) \leq \Phi_t(U) \leq \Phi_t(y)$ for $t \geq t_0$ and therefore $\omega(v) = W$ for every $v \in U$ by the Limit Set Dichotomy. As U is a nonempty open subset of X, this implies that $SB(W)$ has nonempty interior in X, a contradiction to the fact that x belongs to K.

Note also that, by the Limit Set Dichotomy, the image of γ is an ordered collection of sets: $x < y$ then $\gamma(x) < \gamma(y)$. Following the same argument as in the proof of Theorem 3, it follows that $L \cap K$ is countable. By Lemma 1, K is shy with respect to μ_v, for $v > 0$. ∎

4 C_s is Prevalent in C for smooth Φ

In this section we assume that $\mathbb{B}$ is a separable Banach space ordered by a cone $\mathcal{K}$ with nonempty interior and Φ is a strongly order preserving semiflow on the p-convex subset $X \subset \mathbb{B}$. We assume also that for every $t \geq t_0$ the time evolution operators Φ_t are compact, (Frechet) C^1 and have compact derivatives (for some fixed $t_0 \geq 0$).

We say that an equilibrium point $e \in E$ is *irreducible* if for some $t = t_e > 0$, $\Phi'_t(e)$ is a strongly positive operator (i.e. $x > 0$ implies $\Phi'_t(e)x \gg 0$). Observe that if $\Phi'_t(e)$ is strongly positive, so is $\Phi'_s(e),\ s \geq t$. The point e is said to be *non-irreducible* otherwise. Denote by $\rho(A)$ the spectral radius of a bounded linear operator A on $\mathbb{B}$. By the well-known Krein-Rutman theorem [13], if A is compact and strongly positive then its spectral radius is a simple eigenvalue with eigenspace spanned by a positive vector $v \gg 0$; moreover v is the unique eigenvector belonging to $\mathcal{K}$, up to scalar multiple.

Let

$$E_s = \{e \in E : \rho(\Phi'_{t_0}(e)) \leq 1\}$$

denote the set of "neutrally stable" equilibria and

$$C_s = \{x \in X : \omega(x) = \{e\},\ e \in E_s\}$$

the set of points convergent to a neutrally stable equilibrium. An equilibrium not in E_s will be called linearly unstable; this implies that it is unstable in the sense of Lyapunov.

The aim of this section is to provide sufficient conditions for C_s to be prevalent in C and in X.

The following result is well-known.

Lemma 2 *Let $T : X \to X$ be a continuous (nonlinear) operator. Let $e \in X$ be a fixed point of T, and assume that the Frechet derivative $T'(e) : T \to T$ exists and is compact. Assume also that there exists a sequence $e_1,\ e_2,\ \ldots$ of fixed points of T, $e_k \neq e$, such that $e_k \to e$ as $n \to \infty$.*

Then the unit vectors $v_k := (e_k - e)/\left|e_k - e\right|$ have a subsequence that converges towards a unit vector $w \in \mathbb{B}$, and $T'(0)w = w$.

Lemma 3 *If $(e_k)_{k\in\mathbb{N}}$ is a sequence of equilibria of the semiflow Φ such that $e_k < e_{k+1}$ ($e_k < e_{k+1}$) for all k, and if the sequence (e_k) converges towards a irreducible equilibrium $e \in E$, then $e \in E_s$.*

Proof. Let $\tau \geq t_0$ be such that $\Phi'_\tau(e)$ is strongly positive, and let $T := \Phi_\tau$. Then T satisfies the hypotheses of Lemma 2, so that defining $v_k = (e_k -$

$e)/|e_k - e|$, there exists a subsequence v_{k_i} which converges to a unit vector $w \in \mathbb{B}$. Furthermore, $T'(e)w = w$. From the fact that $e < e_k$ for every k, we conclude that $v_k > 0$ and consequently that the unit vector $w > 0$ so $w \gg 0$.

By the Krein Rutman theorem, the fact that $T'(e)$ has a positive eigenvector with eigenvalue 1 implies that in fact $\rho(T'(e)) = 1$. Therefore $e \in E_s$, and this concludes the proof.

The case $e_{k+1} < e_k$ for every $k \in \mathbb{N}$ can be treated similarly. ∎

We introduce the property (P):

(P) Every set of equilibria $\hat{E} \subseteq E$ which is totally ordered by $<$ has at most enumerably many non-irreducible points.

For instance, this condition holds if all equilibria in X are irreducible (see condition (S) in [14], p. 19). It also holds if every totally ordered subset of E has at most enumerably many points.

Lemma 4 *Let property (P) be satisfied. If $\hat{E} \subseteq E$ is totally ordered by $<$, and if every element of $\hat{E}$ is linearly unstable, then $\hat{E}$ is countable.*

Proof. Suppose that $\hat{E}$ is not countable. Then the set $\tilde{E} \subseteq \hat{E}$ of irreducible elements in $\hat{E}$ is also uncountable, by property (P). Let $e \in \tilde{E}$ be an accumulation point of $\tilde{E}$, which exists by the separability of the Banach space $\mathbb{B}$ (otherwise $\mathbb{B}$ would contain an uncountable set of pairwise disjoint open balls). Then there exists a monotone sequence of elements in $\tilde{E}$ which converges towards e. By the previous lemma it holds that $e \in E_s$, contradicting $e \in \tilde{E}$. ∎

Lemma 5 *If $e \in E \setminus E_s$, then $SB(e)$ is unordered and hence shy.*

Proof. The same argument as Lemma 2.1 in [17] shows that $SB(e)$ is unordered. This implies that it is shy with respect to μ_v for any $v > 0$, by Lemma 1, since any line parallel to v meets $SB(e)$ at most once. ∎

Our next result is similar to Theorem 4.4 in [3] in finite dimensions, and to a lesser extent to Theorem 10.1 in [4] but it drops the assumptions of finiteness or discreteness for the set E.

Theorem 5 *In addition to the assumptions of this section, let property (P) be satisfied. Then C_s is prevalent in C.*

Proof. We follow a very similar argument as in the proof of Theorem 3. Let $N = C - C_s$ be the set of $x \in C$ such that $\omega(x)$ is a linearly unstable equilibrium. It will be shown in Section 6 that this set is Borel. Let $v > 0$ and let L be a line parallel to v. Then we can define the function $\sigma : L \cap N \to X$ by $\sigma(x) = \lim_{t\to\infty} \Phi(t, x)$. If $x_1, x_2 \in L \cap N$, $x_1 < x_2$, then necessarily $\sigma(x_1) < \sigma(x_2)$ by the Limit Set Dichotomy since $SB(\omega(x_1))$ is unordered by Lemma 5. Thus σ is injective. As $\hat{E} = \text{range } \sigma$ is totally ordered, it is countable by Lemma 4, and so is $L \cap N$ by injectivity. By Lemma 1, N is shy. ∎

See also Theorem 4.4 and Theorem 4.1 of [3].

Theorem 6 *In addition to the assumptions of this section, suppose $B = X$, X is order convex, every equilibrium is irreducible, and that Φ is eventually strongly monotone. Then C_s is prevalent in X.*

Proof. In this case, the Improved Limit Set Dichotomy holds so C is prevalent in X by Theorem 3. As C_s is prevalent in C by Theorem 5, the result follows since $X \setminus C_s = (X \setminus C) \cup (C \setminus C_s)$ is the union of two subsets, each shy relative to the same μ_v, $v > 0$. ∎

5 Q contains an Open and Dense set

Let Φ be an SOP semiflow on the ordered metric space X, having compact orbit closures. In this section we improve a result in [6] by weakening the conditions for Q to contain an open and dense set. We introduce the following hypothesis:

(K) $X \subset Z$ where Z is an ordered metric space with order relation $\leq_Z$, the inclusion $i : X \to Z$ is continuous and X inherits its order relation from Z. Φ extends to a mapping $\Psi : \mathbb{R}^+ \times Z \to Z$, not necessarily continuous, where

(a) $\Psi|_{\mathbb{R}^+ \times X} = \Phi$,

(b) Ψ is monotone on Z.

(c) For every $z \in Z$, there exists $t_z \geq 0$ such that $\Psi_{t_z}(z) \in X$.

Observe that if $z \in Z$, then $\Psi_t(z) \in X$ and $\Psi_t(z) = \Phi_t(z)$ for $t \geq t_z$. Consequently, the omega limit set of the orbit through z exists in both X and in Z and they agree.

A point $x \in X$ is *doubly accessible from below* (respectively, above) if in every neighborhood of x there exist f, g with $f < g < x$ (respectively, $x < f < g$).

For $p \in E$ define $C(p) := \{z \in X : \omega(z) = \{p\}\}$. Note that $C = \bigcup_{p \in E} C(p)$. All topological properties used hereafter are relative to the space X.

Lemma 6 *Let (K) hold. Suppose $x \in X \setminus Q$ and $a = \inf \omega(x) \in Z$ exists. Then $\omega(a) = \{p\}$ where $p \in X$ satisfies $p < \omega(x)$, and $x \in \overline{\mathrm{Int}\, C(p)}$ provided x is doubly accessible from below.*

Proof. Fix an arbitrary neighborhood M of x. Note that $a <_Z \omega(x)$ because $\omega(x) \subset X$ is unordered (Theorem 1). By invariance of $\omega(x)$ and (K) we have $\Psi_t a \leq_Z \Psi_t \omega(x) = \Phi_t \omega(x) = \omega(x)$, hence $\Psi_t a \leq_Z a, t \geq 0$. It follows from (K) that $\Phi_t(w) \leq w := \Psi_{t_a}(a)$ for $t \geq 0$ and therefore the Convergence Criterion Theorem **??** implies that $\omega(a)$ is an equilibrium $p \in X$ with $p \leq a$. Because $p < \omega(x)$, SOP yields a neighborhood N of $\omega(x)$ and $s \geq 0$ such that $p \leq \Phi_t N$ for all $t \geq s$. Choose $r \geq 0$ with $\Phi_t x \in N$ for $t \geq r$. Then $p \leq \Phi_t x$ if $t \geq r + s$. The set $V := (\Phi_{r+s})^{-1}(N) \cap M$ is a neighborhood of x in M with the property that $p \leq \Phi_t V$ for all $t \geq r + 2s$. Hence:

$$u \in V \Rightarrow p \leq \omega(u) \tag{1}$$

Now assume x doubly accessible from below and fix $y_1, y \in V$ with $y_1 < y < x$. By the Limit Set Dichotomy $\omega(y) < \omega(x)$, because $\omega(x) \not\subset E$. By SOP we fix a neighborhood $U \subset V$ of y_1 and $t_0 > 0$ such that $\Phi_{t_0} u \leq \Phi_{t_0} y$ for all $u \in U$. The Limit Set Dichotomy implies $\omega(u) = \omega(y)$ or $\omega(u) < \omega(y)$; as $\omega(y) < \omega(x)$, we therefore have:

$$u \in U \Rightarrow \omega(u) < \omega(x) \tag{2}$$

For all $u \in U$, (2) implies $\omega(u) \leq \omega(a) = \{p\}$, while (1) entails $p \leq \omega(u)$. Hence $U \subset C(p) \cap M$, and the conclusion follows. ∎

An analogous result holds if "$a = \inf \omega(x) \in Z$ exists" is replaced by "$b = \sup \omega(x) \in Z$ exists", in which case $\omega(b) = \{q\}$ where $q > \omega(x)$. Furthermore, the conclusion $x \in \overline{Int\, C(p)}$ holds provided x is doubly accessible from above.

We introduce an additional condition on the semiflow Φ:

(L) Either every omega limit set $\omega(x)$, $x \in X$, has an infimum in Z and the set of points that are doubly accessible from below has dense interior in X, or every omega limit set has a supremum in Z and the set of points that are doubly accessible from above has dense interior in X.

Theorem 7 *Let Φ be an SOP semiflow on the ordered metric space X, having compact orbit closures, and satisfying axioms* (L) *and* (K). *Then $X \setminus Q \subset \overline{\text{Int}\, C}$, and* $\text{Int}\, Q$ *is dense.*

Proof. To fix ideas we assume the first alternative in (L), the other case being similar. Let X_0 denote a dense open set of points doubly accessible from below. Lemma 6 implies $X_0 \subset Q \cup \overline{\text{Int}\, C} \subset Q \cup \overline{\text{Int}\, Q}$, hence the open set $X_0 \setminus \overline{\text{Int}\, Q}$ lies in Q. This prove $X_0 \setminus \overline{\text{Int}\, Q} \subset \text{Int}\, Q$, so $X_0 \setminus \overline{\text{Int}\, Q} = \emptyset$. Therefore $\overline{\text{Int}\, Q} \supset X_0$, hence $\overline{\text{Int}\, Q} \supset \overline{X_0} = X$. ∎

Axiom (L) is a restriction on both the space X (order and topology) and the semiflow (limit sets). If X continuously embeds in $Z = C(A, \mathbb{R})$, the Banach space of continuous functions on a compact set A with the usual ordering, Φ extends to a monotone mapping on Z with the smoothing property (c), then axiom (L) holds. This is true because every compact subset of $C(A, \mathbb{R})$ has a supremum and infimum (see Schaefer [13], Chapt. II, Prop. 7.6). In particular, X may be a Euclidean space $\mathbb{R}^n$, a Hölder space $C^{k+\alpha}(\Omega, \mathbb{R}^n)$, $0 \leq \alpha < 1, k = 0, 1, 2, \cdots$, for Ω a compact smooth domain in R^m, or a Sobolev space $H^{k,p}(\Omega)$ for $k - \frac{n}{p} \geq 0$ where the usual functional ordering is assumed. These cases cover ordinary, delay, and parabolic partial differential equations under suitable hypotheses.

Theorem 7 extends the corresponding result in [6], where it was assumed that $Z = X$, by allowing X to be imbedded in a larger space Z in which it is more likely that omega limit sets have infima and suprema. This extension is important for partial differential equations for the reasons mentioned above.

5.1 Reaction-Diffusion Systems

Consider a reaction-diffusion system

$$
\begin{aligned}
u_t &= D\Delta u + f(u), \ x \in \Omega \\
\frac{\partial u}{\partial n} &= 0, \ x \in \partial\Omega \\
u(0,x) &= u_0(x), \ x \in \overline{\Omega}
\end{aligned}
\tag{3}
$$

We assume that the state space $C(\overline{\Omega}, \mathbb{R}^n)$. Kishimoto and Weinberger [9] showed that if Ω is a convex domain, and assuming that $\partial f_i / \partial u_j > 0$ for all $i \neq j$, then any nonconstant equilibrium $\overline{u}$ is linearly unstable. A careful reading of the proof in that paper will show that in fact it is sufficient that $\partial f_i / \partial u_j \geq 0$ for all $i \neq j$ and that the Jacobian matrix is irreducible. We call the vector field f cooperative and irreducible if this is the case. By making a linear change of variables, we may extend the following result to any system which is monotone with respect to one of the other orthants $\mathcal{K}$. See [14].

Equilibria of (3), solutions of the associated elliptic boundary value problem, are known for having multiple and sometimes unexpected solutions. Not only is it possible for a strongly monotone reaction diffusion system to have several spatially nonhomogeneous equilibria, but it is in fact possible that there is a continuum of them. The following application of Theorem 6 shows that the generic solution converges to a uniform (constant) solution.

Theorem 8 *Consider a C^1 finite dimensional system*

$$\dot{x} = f(x) \tag{4}$$

which is cooperative and irreducible and assume that all initial value problems have bounded solutions for $t \geq 0$. If Ω is convex, then the set of initial conditions $u_0 \in C(\overline{\Omega}, \mathbb{R}^n)$ corresponding to solutions of (3) that converge towards a uniform equilibrium is prevalent in $C(\overline{\Omega}, \mathbb{R}^n)$.

Proof. We need to show that all the general assumptions of the previous section are satisfied, as well as the hypotheses of Theorem 6. Clearly $X = C(\overline{\Omega}, \mathbb{R}^n)$ is a separable Banach space under the uniform norm with cone given by $C(\overline{\Omega}, \mathbb{R}^n_+)$. The fact that the time evolution operators generate a semiflow of compact operators with compact derivatives on X is well known in the literature; see for instance [10, 11, 5]. The fact that $B = X$ follows from comparison with solutions of the ordinary differential equations (4); see e.g. Theorem 7.3.1 in [14]. To see that the system (3) has no non-irreducible equilibria, let $\hat{u}$ be an equilibrium of the system, and recall that the linearization around $\hat{u}$ is of the form

$$u_t = D\Delta u + M(x)u,$$

together with Neumann boundary conditions, where $M(x) = \partial f/\partial u(\hat{u}(x))$. According to Theorem 7.4.1 of [14], to prove that this system is strongly monotone it is enough to verify that the associated finite-dimensional system with no diffusion is monotone for every fixed value of $x \in \Omega$, and strongly monotone for at least one value of x. This therefore follows from the irreducibility assumption on the linearizations of (4).

By Theorem 6, C_s is prevalent in $B = X$. But by the main theorem in [9], any initial condition in C_s has a solution which converges towards an equilibrium which is uniform in space. This completes the proof. ∎

6 Appendix-Measurability

It is important to observe that in order to apply measure-theoretic arguments on Theorems 3, 4, and 5, one needs to prove first that the sets involved in each result are Borel measurable. This is carried out in the present section. We assume throughout this section that X is a Borel subset of $\mathbb{B}$, a *separable* Banach space, that $\Phi : \text{Dom } \Phi \to X$ is a continuous local semiflow defined on the open subset Dom Φ of $R_+ \times X$ containing $\{0\} \times X$. For each $x \in X$, $\{t \geq 0 : (t, x) \in \text{Dom } \Phi\} = [0, \sigma_x)$. The set

$$Ext = \{x \in X : \sigma_x = +\infty\} = \cap_q \{x \in X : \sigma_x > q\},$$

where the intersection is taken over all positive rational q, is Borel since it is the countable intersection of open subsets of X. Therefore, we may as well rename $X = Ext$ and consider the global semiflow $\Phi : \mathbb{R}_+ \times X \to X$ where X is Borel. We assume there exists $t_0 \geq 0$ such that Φ_t is completely continuous for $t \geq t_0$. Given $p \in \mathbb{B}$ and $r > 0$, let $B_r(p) = \{x \in \mathbb{B} : |x - p| < r\}$.

Let $D \subseteq X$ be a closed set in X and $r \in R_+$, and consider the set

$$W(D, r) = \{x \in X \mid \Phi_t(x) \in D, \text{ for all } t \geq r\} = \bigcap_{q \in Rat,\ q > r} \Phi_q^{-1}(D),$$

where Rat denotes the rationals. The equality holds from the continuity of Φ. Since each operator Φ_q is continuous, $W(D, r)$ is a Borel measurable set.

Lemma 7 *Assume there exists $t_0 \geq 0$ such that Φ_t is completely continuous for $t \geq t_0$. Then the set B of the elements $x \in X$ with precompact orbit is Borel measurable.*

Proof. If $x \in B$, then clearly $O(x)$ is bounded. If $x \in X$ is such that $O(x)$ is bounded, then $O(\Phi_{t_0}(x)) = \Phi_{t_0}(O(x))$ is precompact since Φ_{t_0} is completely continuous, and clearly $x \in B$. We conclude that B is the set of $x \in X$ with bounded orbit, that is

$$B = \bigcup_{n \in \mathcal{N}} W(B_n(0), n).$$

∎

In the following we assume only that Φ is a continuous semiflow on the closed or open set X.

Lemma 8 *If X is Borel in $\mathbb{B}$, then the set B of the elements $x \in X$ with precompact orbit is Borel measurable.*

Proof. B is the set of $x \in X$ such that $O(x)$ is totally bounded. We can express this as follows. Let $\{p_i\}_{i \in N}$ be a countable dense set in $\mathbb{B}$ and let $\mathfrak{F}$ be the family of all finite subsets of N. Then $\mathfrak{F}$ is countable and

$$B = \cap_{n \in N} \left[\cup_{F \in \mathfrak{F}} \left(\cap_{q \in Rat_+} \Phi_q^{-1} (\cup_{i \in F} B_{1/n}(p_i)) \right) \right]$$

As $\Phi_q^{-1}(\cup_{i \in F} B_{1/n}(p_i))$ is Borel, it follows that B is Borel.

∎

Lemma 9 *Let $D \subseteq X$ be a closed set, and let $C(D) = \{x \in B \,|\, \omega(x) \subseteq D\}$. Then $C(D)$ is Borel measurable.*

Proof. Given a set $A \subseteq X$ and $\epsilon > 0$, let

$$A_\epsilon = \{x \in X \,|\, d(A, x) \leq \epsilon\},$$

which is a closed set by continuity of the function $d(\cdot, A)$. Then we can write

$$\{x \in B \,|\, \lim_{t \to \infty} d(\Phi_t(x), A) = 0\} = \bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} W(A_{\frac{1}{m}}, k).$$

Finally, note that for any closed set $D \subseteq X$ and for any $x \in B$, it holds that

$$\omega(x) \subseteq D \Leftrightarrow \lim_{t \to \infty} d(\Phi_t(x), D) = 0.$$

The first statement follows. ∎

Corollary 1 *The set Q of quasiconvergent elements is Borel measurable.*

Proof. The proof follows immediately from the above result by noting that E is a closed set. ∎

It follows that the set $N = B - Q$ involved in the proof of Theorem 3 is measurable, by the previous Corollary and Lemma 8.

Lemma 10 *The set C of convergent elements is Borel measurable.*

Proof. Let (b_i) be a dense enumerable collection of elements of X. Then the statement follows from the equation

$$C = \bigcap_{k=1}^{\infty} \bigcup_{i=1, r=1}^{\infty} W(\overline{B}_{\frac{1}{k}}(b_i), r).$$

To see this, let first $x \in C$. Note that for any fixed k, there exists some b_i within $1/(2k)$ of $\omega(x)$, and that therefore $x \in W(\overline{B}_{\frac{1}{k}}(b_i), r)$ for some large enough r. Therefore for every fixed k, C is contained in the union of the RHS, and thus one direction is proven. Conversely, let x be in the RHS term. For every k, let a_k, r_k be such that $x \in W(\overline{B}_{\frac{1}{k}}(a_k), r_k)$; such sequences exist by hypothesis. Note that for $k_1 \neq k_2$, it must hold

$$W(\overline{B}_{\frac{1}{k_1}}(a_{k_1}), r_{k_1}) \cap W(\overline{B}_{\frac{1}{k_2}}(a_{k_2}), r_{k_2}) \neq \emptyset$$

In particular the sequence (a_k) is Cauchy, and it therefore converges towards a point $a \in \mathbb{B}$. Given $\epsilon > 0$, let k large enough that $1/k < \epsilon/2$ and that a_k lies within $\epsilon/2$ of a. By definition, $|x(t) - a| < \epsilon$ for $t \geq r_k$; we conclude that $x(t) \to a$, and therefore that $a \in X$ and $x \in C$. ∎

Lemma 11 *Assume that the time evolution operators is continuously differentiable. Then the set C_s is Borel measurable.*

Proof. Note first that the spectral radius function $\rho(T)$, though not a continuous function of the linear bounded operator $T : L(\mathbb{B}, \mathbb{B}) \to \mathbb{R}$ (see Kato [8]), is nevertheless a measurable function. To see this, simply write it as the pointwise limit of continuous functions as $\rho(T) = \lim_n \|T^n\|^{1/n}$. Fix now $t > t_0$, and define $\beta : X \to \mathbb{R}$, $\beta(z) := \rho(\Phi_t'(z))$. Since $z \to \Phi_t'(z)$ is a continuous function by hypothesis, it follows that β is measurable.

The next step is to note that the function $z \to \omega(z)$ (defined on C) is also measurable. To see this, write it as the pointwise limit of the continuous functions $\omega(z) = \lim_n \Phi_n(z)$. Thus, the function $z \to \beta(\omega(z))$ is itself measurable. But

$$C_s = \{z \in C \,|\, \beta(\omega(z)) \leq 1\},$$

and the proof is complete. ∎

Note that the continuous differentiability of the time evolution operators was only used to show that β is measurable; it would be sufficient to assume $z \to \Phi_t'(z)$ to be measurable, which should be satisfied in very large generality.

Lemma 12 *Let $A \subseteq B$ be compact. Then $SB(A)$ is Borel measurable.*

Proof. For every $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ which cover A. Let $\{R_i\}_{i \in I}$ be the union of such collections, for $\epsilon = 1/n, n = 1, 2, 3 \ldots$. Then the set $\bigcup_{i \in I} W(R_i^C, 0)$ consists of the vectors $x \in X$ such that $a \notin \omega(x)$ for some $a \in A$. Consequently,

$$SB(A) = C(A) - \bigcup_{i \in I} W(R_i^C, 0),$$

and this set is also Borel measurable. ∎

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