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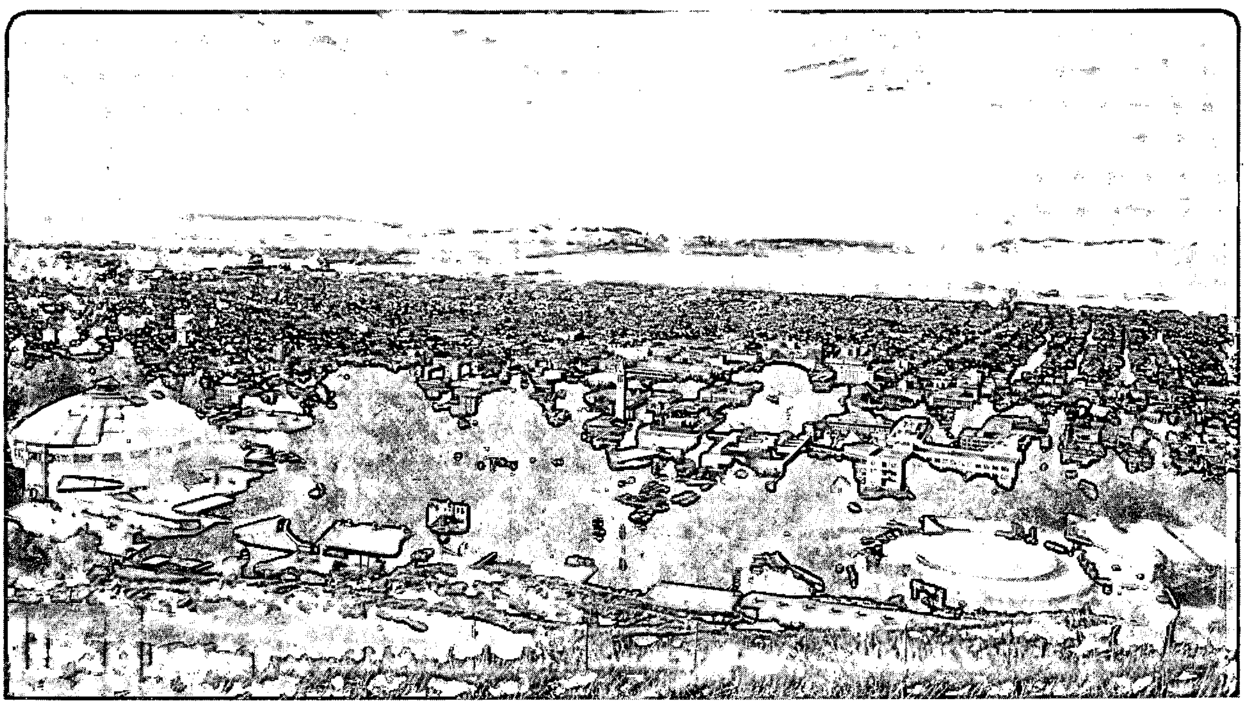

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Integral Method Solutions to Flow Problems in Unsaturated Porous Media

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Abstract. The integral method is used to derive approximate solutions for the problem of absorption of water into an initially unsaturated porous medium. This problem is governed by a nonlinear diffusion equation, for which exact solutions are generally not obtainable. Approximate solutions are obtained for media with two different commonly-used sets of characteristic curves, those of Brooks-Corey and van Genuchten-Mualem. The approximate solutions compare reasonably well with numerical results, and also have the advantage of clearly displaying the manner in which the solutions depend on the hydrological parameters of the problem.

Introduction. The integral method for deriving approximate solutions to nonlinear partial differential equations that arise in engineering and the physical sciences was introduced by Pohlhausen [1] to treat the problem of laminar flow over a flat plate. Pohlhausen approximated the velocity distribution through the boundary layer by a low-order polynomial whose coefficients depended on the unknown boundary-layer thickness. Although the polynomial did not satisfy the governing momentum equation exactly, it was forced to satisfy the integral of this equation over the boundary layer thickness. This led to a simple ordinary differential equation that governed the thickness of the boundary layer along the length of the plate. The approximate solution thus derived, using only a quartic profile, compared reasonably well with the exact numerical solution of Blasius [2]. In particular, the approximate solution predicted certain properties of interest, such as the skin friction, to within 3% of the exact value.

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The integral method seems to have been first brought to bear on diffusion problems by Landahl [3], who in the succeeding years derived approximate solutions to many diffusion problems arising in biophysics. The method has since been widely used in heat conduction problems (see [4], and references therein). Since linear diffusion equations can usually be solved by classical methods such as separation of variables or Green's functions, the integral method is most useful in deriving approximate solutions to nonlinear problems for which closed-form solutions are not obtainable. One problem in the earth sciences which leads to a highly nonlinear diffusion equation is that of fluid flow in partially saturated (also called "unsaturated") porous media. In such problems, the nonlinearities are usually fairly strong, thus limiting the usefulness of perturbation methods. Of course, numerical solutions can always be obtained for such problems [5]. Numerical methods, however, have the disadvantage of not clearly showing the manner in which the solution depends on the various parameters of the problem. The integral method leads to closed-form (albeit approximate) solutions which do give insight into the effect of the various boundary conditions and constitutive parameters of the problem. The purpose of this paper is to illustrate how the integral approach can lead to simple, but relatively accurate, solutions to otherwise intractable unsaturated flow problems.

Formulation of the Problem. Horizontal flow of water in an unsaturated medium is usually thought to be governed by Richards' partial differential equation [6]:

$$\frac{\partial}{\partial x}\left[\frac{kk_r(\psi)}{\mu\phi}\frac{\partial\psi}{\partial x}\right] = \frac{\partial S}{\partial t} . \tag{1}$$

In this equation, S is the liquid saturation, which is equal to the fraction of the pore space that is filled with water. ψ is the potential, or capillary pressure, and is related to the saturation through a capillary pressure function, $S = S(\psi)$. In regions of partial liquid saturation, ψ will be negative. k is the absolute (i.e., fully-saturated) permeability of the medium, ϕ is its porosity (assumed constant), and μ is the viscosity of water. k_r is the dimensionless relative permeability function, which measures the decrease in the permeability of the medium to the presence of air in some of the pores. If hysteretic effects are neglected, k_r and S are single-valued functions of ψ. Equation (1) can also be used for the initial stages of vertical infiltration, when gravitational forces are still negligible.

Equation (1) embodies the principal of conservation of mass for the water, along with Darcy's law to relate the volumetric flux to the potential gradient. Although the physical problem of flow in an unsaturated medium actually involves both the water and the air phases, it is conventional to ignore the air by implicitly assuming it to be infinitely mobile, and at a fixed pressure of one atmosphere. Since the relative permeability and capillary pressure functions are always strongly varying functions of S, Equation (1) is highly nonlinear. Note that Equation (1) can be put into the form of a standard nonlinear diffusion equation as follows:

$$\frac{\partial}{\partial x}\left[\frac{kk_r(S)}{\mu\phi}\frac{d\psi}{dS}\frac{\partial S}{\partial x}\right] = \frac{\partial}{\partial x}\left[D(S)\frac{\partial S}{\partial x}\right] = \frac{\partial S}{\partial t} , \tag{2}$$

where $D(S) = kk_r(S)\psi'(S)/\mu\phi$. Although this form of the governing equation is frequently used in soil physics (e.g., [7]), we will find it convenient to use the form given in Equation (1).

A basic problem in the field of unsaturated flow is that of absorption from a saturated boundary at some fixed potential ψ_w into a half-space that is initially at some uniform saturation ψ_i. Without loss of generality, we can assume that $\psi_w = 0$, since the solution for $\psi_w > 0$ is related in a simple way [8] to the solution for the case $\psi_w = 0$. The boundary and initial conditions for this problem are

$$\psi(0,t) = 0 \ , \tag{3}$$

$$\psi(x,0) = \psi_i \ , \tag{4}$$

$$\lim_{x \to \infty} \psi(x,t) = \psi_i \ . \tag{5}$$

The last condition reflects the fact that, at any finite time, the wetting front moving in from the saturated boundary cannot have penetrated infinitely far into the medium. Equations (1,3-5), along with expressions for the saturation and the relative permeability as functions of ψ, completely specify the problem of horizontal one-dimensional absorption.

The capillary pressure and relative permeability functions depend on the pore geometry of the medium (see [9]), and have different forms for different media. Two of the more widely-used forms for these "characteristic equations" are those of Brooks and Corey [10], and van Genuchten-Mualem [11,12]. Neither of these sets of functions are analytic, since they typically involve fractional powers of the saturation. Furthermore, the Brooks-Corey capillary pressure function is given by different algebraic expressions in different capillary pressure regimes. While these peculiarities hinder attempts to analytical solutions, that they pose no particular difficulty for the integral method.

Solution for Brooks-Corey Media. The Brooks-Corey characteristic functions are

$$\hat{S}(\psi) = \frac{S(\psi) - S_r}{S_s - S_r} = 1 \quad \text{if} \quad |\alpha\psi| \le 1 \ ,$$

$$= |\alpha\psi|^{-n} \quad \text{if} \quad |\alpha\psi| > 1 \ ; \tag{6}$$

$$k_r(\psi) = 1 \quad \text{if} \quad |\alpha\psi| \le 1 \ ,$$

$$= |\alpha\psi|^{-(3n+2)} \quad \text{if} \quad |\alpha\psi| > 1 \ , \tag{7}$$

where S_r is the residual saturation, S_s is the saturation at zero potential, $\hat{S}$ is the normalized saturation, and α is a scaling parameter that is inversely proportional to the mean pore diameter. The parameter n must satisfy the inequality $n \geq 1$, but is not necessarily an integer [13]. The normalized saturation equals 1 for all $\psi > -1/\alpha$, after which it drops off to zero as ψ decreases, according to a power law. Since no air can enter the medium unless $|\alpha\psi| > 1$, $1/\alpha$ is often called the "air-entry pressure". The relative permeability monotonically decreases from 1 to 0 as the normalized saturation decreases from 1 to 0.

Before attempting to solve this problem, it is convenient to normalize all the variables, and transform the governing partial differential equation into an ordinary differential equation by applying a Boltzmann-type transformation [14]. If we define a normalized potential as $\hat{\psi} = \alpha\psi$, and a similarity variable η as

$$\eta = \left[\frac{\alpha\mu\phi(S_s - S_r)x^2}{kt} \right]^{1/2} , \tag{8}$$

then Equation (1) is transformed into

$$\frac{d}{d\eta}\left[k_r(\hat{S}) \frac{d\hat{\psi}}{d\eta} \right] + \frac{\eta}{2}\frac{d\hat{S}}{d\eta} = 0 , \tag{9}$$

and the three boundary/initial conditions (3-5) are transformed into the two conditions

$$\hat{\psi}(0) = 0 , \tag{10}$$

$$\lim_{\eta \to \infty} \hat{\psi}(\eta) = \hat{\psi}_i . \tag{11}$$

The above transformation has the effect of reducing the problem to a two-point ODE boundary-value problem, given by Equations (9-11).

The basic idea behind the integral method is to approximate the solution with some simple function that contains an adjustable parameter, and then fix the value of this parameter by requiring the solution to satisfy the differential equation in an integrated sense. An important fact about the use of the integral method is that reasonable forms for the solution can often be obtained merely by consideration of the boundary conditions and various simple properties of the governing equation. For example, note that since $\psi = 0$ at $\eta = 0$, by continuity there will be a region near the boundary where $\psi > -1/\alpha$. Equation (6) then shows that $\hat{S} = 1$ in this region, which implies that the term $d\hat{S}/d\eta$ in Equation (9) will be zero. Equation (9) then implies that $d\hat{\psi}/d\eta$ is a constant, and so ψ will drop off linearly from 0 to $-1/\alpha$. The value of η at which $\hat{\psi}$ reaches $-1/\alpha$ will be denoted by λ. The capillary pressure will continue to decrease as η increases, reaching its initial value $\hat{\psi}_i$ at some point $\eta = \lambda + \delta$. These considerations suggest the following trial profile (Fig. 1):

$$0 < \eta < \lambda: \qquad \hat{\psi} = -\eta/\lambda , \quad \hat{S} = 1 ;$$

$$\lambda < \eta < \lambda + \delta: \qquad \hat{S} = 1 - (1 - \hat{S}_i)\frac{\eta - \lambda}{\delta} , \quad \hat{\psi} = -\hat{S}^{-1/n} ;$$

$$\lambda + \delta < \eta < \infty: \qquad \hat{S} = \hat{S}_i \ , \quad \hat{\psi} = -\hat{S}_i^{-1/n} . \tag{12}$$

The profiles chosen for the first and third regions follow from the considerations discussed above, while a linear saturation profile is chosen in the second region merely for its simplicity.

A relationship between the parameters λ and δ can be found by requiring continuity of the capillary pressure gradient at $\eta = \lambda$. We first calculate the capillary pressure gradient in the two regions as follows:

$$\left.\frac{\partial \hat{\psi}}{\partial \eta}\right]_{\lambda^-} = \frac{1}{\lambda} , \tag{13a}$$

$$\left.\frac{\partial \hat{\psi}}{\partial \eta}\right]_{\lambda^+} = \left.\frac{d\hat{\psi}}{d\hat{S}}\right]_{\hat{S}=1} \left.\frac{\partial \hat{S}}{\partial \eta}\right]_{\eta=\lambda}$$

$$= \left[\frac{-1}{n}\hat{S}^{-(1+1/n)}\right]_{\hat{S}=1} \left[\frac{-(1-\hat{S}_i)}{\delta}\right]_{\eta=\lambda} = \frac{(1-\hat{S}_i)}{n\delta} . \tag{13b}$$

Equating these two gradients yields

$$\lambda = \frac{n\delta}{(1-\hat{S}_i)} . \tag{13c}$$

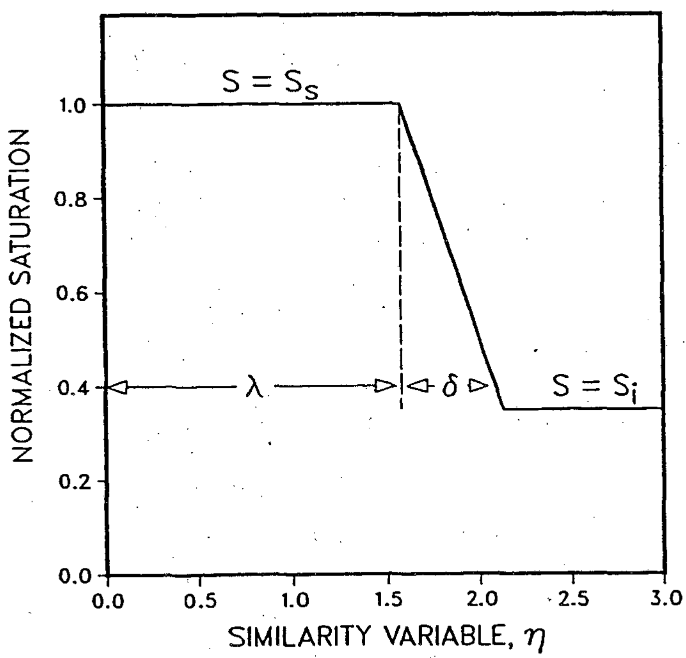

Figure 1. Assumed saturation profile for absorption into a Brooks-Corey medium, as given by Equation (12).

Note that we do not require the capillary pressure gradient to be continuous at $\eta=\lambda+\delta$; imposition of this condition leads to only slight increases in accuracy, at the expense of much additional algebraic complexity. If desired, one can imagine that there is a small "tail" on the saturation profile at $\eta = \lambda+\delta$ that smoothly connects the two piecewise-linear parts of the profile (see Fig. 1), but which is sufficiently localized so as to have no appreciable effect on the required integrals (see Equation (15)).

To find an expression for δ, we integrate Equation (9) from $\eta=0$ to $\eta=\infty$. The first term in Equation (9) integrates out to

$$k_r(\hat{S})\frac{d\hat{\psi}}{d\eta}\Big]_0^{\infty} = \frac{-1}{\lambda} = \frac{-(1-\hat{S}_i)}{n\delta} . \tag{14}$$

Since the term $d\hat{S}/d\eta$ is non-zero only in the range $\lambda<\eta<\lambda+\delta$, the second term integrates out to

$$\int_0^{\infty}\frac{\eta}{2}\frac{d\hat{S}}{d\eta}d\eta = -\int_{\lambda}^{\lambda+\delta}\frac{\eta}{2}\frac{(1-\hat{S}_i)}{\delta}d\eta$$

$$= \frac{-(1-\hat{S}_i)}{2\delta}\frac{\eta^2}{2}\Big]_{\lambda}^{\lambda+\delta} = \frac{-(1-\hat{S}_i)}{4\delta}[(\lambda+\delta)^2-\lambda^2] . \tag{15}$$

Combining Equations (14) and (15) gives

$$\frac{4}{n} = \delta^2 + 2\lambda\delta . \tag{16}$$

Using Equation (13c) to eliminate λ from Equation (16) leads to

$$\frac{4}{n} = \delta^2\left[1 + \frac{2n}{(1-\hat{S}_i)}\right] , \tag{17}$$

which can be solved for

$$\delta = 2\left[n\left[1 + \frac{2n}{(1-\hat{S}_i)}\right]\right]^{-1/2} . \tag{18}$$

Equations (12,13,18) specify the approximate solution to the problem.

The instantaneous liquid flux into the medium, per unit surface area, can be found from Darcy's law as follows:

$$q = \frac{-kk_r}{\mu}\frac{\partial\psi}{\partial x}\bigg]_{x=0} = \frac{-k}{\mu}\frac{d\psi}{d\hat{\psi}}\frac{\partial\hat{\psi}}{\partial\eta}\bigg]_{\eta=0}\frac{\partial\eta}{\partial x}\bigg]_{x=0}$$

$$= \frac{-k}{\mu}\frac{1}{\alpha}\frac{-1}{\lambda}\left[\frac{\alpha\mu\phi(S_s - S_r)}{kt}\right]^{1/2}$$

$$= \frac{k}{\mu}\frac{(1-\hat{S}_i)}{\alpha n\delta}\left[\frac{\alpha\mu\phi(S_s - S_r)}{kt}\right]^{1/2}$$

$$= \left[\frac{k\phi(S_s - S_i)}{2\alpha\mu t}\left[1 + \frac{(S_s - S_i)}{2n(S_s - S_r)}\right]\right]^{1/2}. \tag{19}$$

If we write the instantaneous flux $q(t)$ as $\mathbf{S}/2\sqrt{t}$, then the cumulative flux up to some time t is equal to

$$Q(t) = \int_0^t q(\tau)\,d\tau = \int_0^t \frac{\mathbf{S}}{2}\tau^{-1/2}\,d\tau = \mathbf{S}t^{1/2}. \tag{20}$$

The constant **S** is often referred to as the sorptivity [7,15]. From Equation (19) and (20), **S** can be expressed as

$$\mathbf{S} = \left[\frac{2k\phi(S_s - S_i)}{\alpha\mu}\left[1 + \frac{(S_s - S_i)}{2n(S_s - S_r)}\right]\right]^{1/2}. \tag{21}$$

In order to judge the accuracy of the approximate solution, we can compare it to the results of numerical solutions to Equations (9-11). These equations represent a two-point (ordinary differential equation) boundary-value problem, which can be solved using the shooting method [16]. Figs. 2 and 3 compare the capillary pressure and saturation profiles of the approximate solution and the numerical solution for a Brooks-Corey medium with $n=2$, for two different initial saturations. This value of n is close to the values that have been estimated [17] for the welded tuffs at Yucca Mountain, Nevada, a potential site of an underground nuclear waste repository. The approximate solution very accurately predicts λ, the length of the fully saturated zone, but slightly overpredicts δ, the width of the partially-saturated zone. Since the sorptivity is proportional to the slope of the capillary pressure profile at $\eta=0$, and to the area bounded by the saturation profile and the line $S=S_i$, it is clear that the approximate solution estimates **S** very accurately. Fig. 4 shows the normalized sorptivity $\mathbf{S}/[k\phi/\alpha\mu]^{1/2}$ plotted against the initial saturation, for a few different values of n. For simplicity, S_r is taken to be 0, and S_s is taken to be 1. The approximate solution is seen to estimate the sorptivity very accurately, over the entire range of initial saturations. Although the accuracy is higher for larger values of n, and for higher values of the initial saturation, in all cases the approximate sorptivity is correct to within 10%.

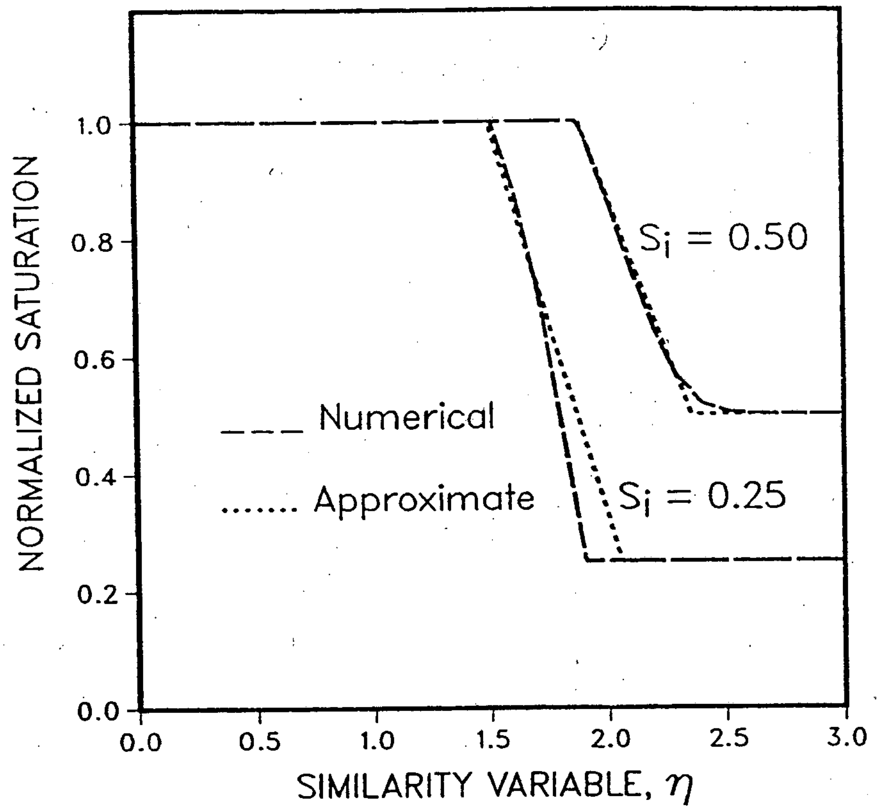

Figure 2. Saturation profiles for absorption into a Brooks-Corey medium, according to the approximate and numerical solutions.

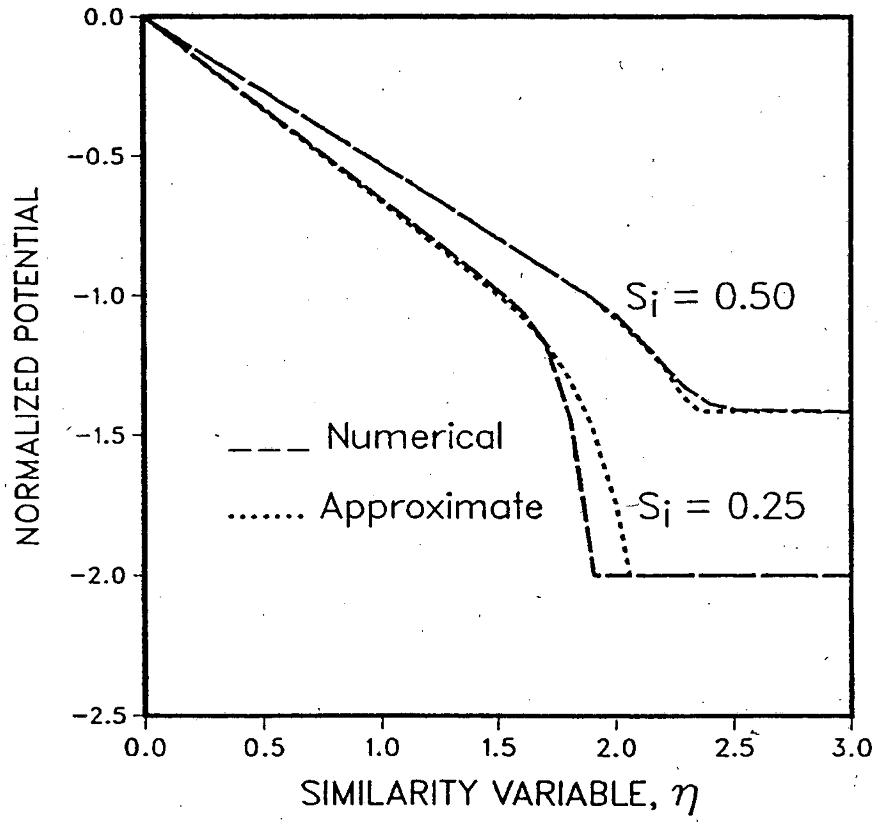

Figure 3. Potential profiles for absorption into a Brooks-Corey medium, according to the approximate and numerical solutions.

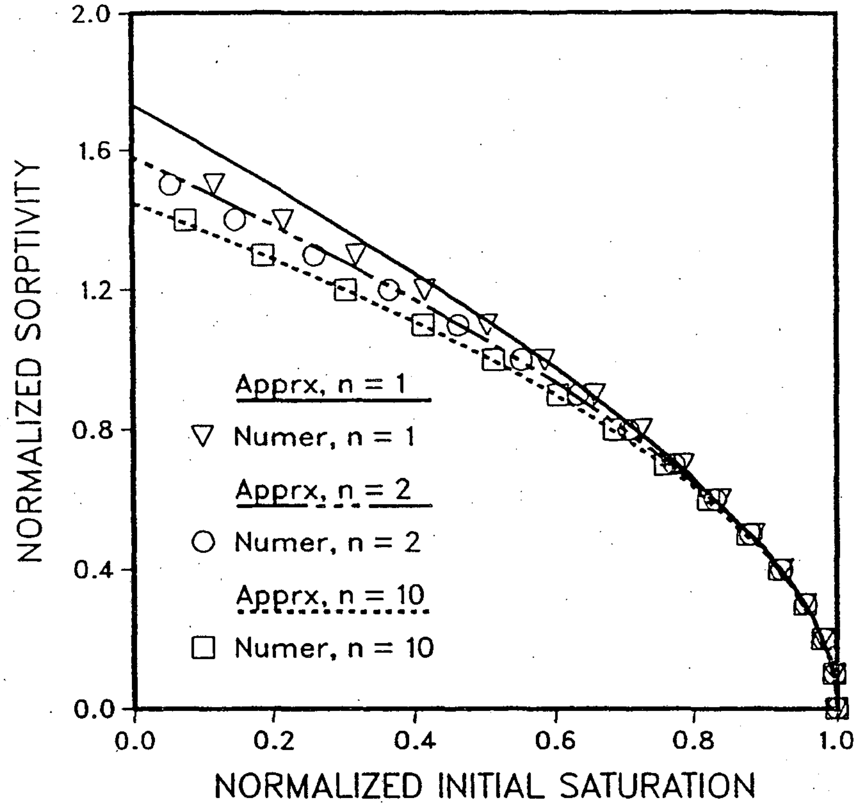

Figure 4. Sorptivity of a Brooks-Corey medium as a function of the initial saturation, for various values of the parameter n.

Solution for van Genuchten Media. Another commonly-used form for the characteristic curves of a porous rock or soil are those of van Genuchten [11] and Mualem [12]:

$$\hat{S}(\psi) = \frac{S(\psi) - S_r}{S_s - S_r} = [1+(\alpha|\psi|)^n]^{-m} \ , \tag{22}$$

$$k_r(\psi) = \frac{\{1-(\alpha|\psi|)^{n-1}[1+(\alpha|\psi|)^n]^{-m}\}^2}{[1+(\alpha|\psi|)^n]^{m/2}} \ . \tag{23}$$

The parameters S_s, S_r, and α have similar interpretations as they do in the Brooks-Corey expressions (6) and (7), while the parameter m is merely a shorthand expression for $1-1/n$. The parameter n should satisfy the inequality $n \geq 2$ [13]. The main qualitative difference between the Brooks-Corey and van Genuchten characteristic equations is that, in the latter case, the capillary pressure is a single-valued and continuous function of saturation in the region near $\hat{S}=0$.

If the similarity transformation (8) is used on the governing equations, they again reduce to the two-point boundary-value problem given by Equations (9-11). In order to arrive at an appropriate form for the trial saturation profile, we first note that, for the flux of liquid into the medium to be finite, the slope $d\hat{\psi}/d\eta$ must be finite at $\eta=0$. Since $\hat{\psi}(0)=0$, this implies that $\hat{\psi} = -a\eta + \cdots$ for small values of η, where a is some constant. Substitution of this into Equation (22) shows that, to first order, $\hat{S} = 1-ma^n\eta^n$. If we denote the length of the wetted zone by δ, a saturation profile of the form $\hat{S} = 1 - ma^n\eta^n$ will only satisfy the condition $\hat{S}(\delta) = \hat{S}_i$ if $ma^n = (1-\hat{S}_i)/\delta^n$, so that

$$\hat{S}(\eta) = 1 - (1-\hat{S}_i)\left[\frac{\eta}{\delta}\right]^n , \quad \text{for} \quad 0<\eta<\delta ,$$

$$\hat{S}(\eta) = \hat{S}_i , \quad \text{for} \quad \delta<\eta<\infty . \tag{24}$$

Note that, as for the Brooks-Corey medium, we are using the simplest profile that is consistent with the boundary conditions and continuity requirements.

The parameter δ is found by integrating the governing equation (9) from $\eta=0$ to $\eta=\infty$, with the saturation profile (24) substituted for $\hat{S}(\eta)$. The first term in Equation (9) integrates to

$$k_r(\hat{S})\frac{d\hat{\psi}}{d\eta}\bigg]_0^\infty = a = \frac{(1-\hat{S}_i)^{1/n}}{m^{1/n}\delta} . \tag{25}$$

The second term in Equation (9) integrates to

$$\int_0^\infty \frac{\eta}{2}\frac{d\hat{S}}{d\eta}d\eta = -\int_0^\delta \frac{\eta}{2}\frac{(1-\hat{S}_i)n\eta^{n-1}}{\delta^n}d\eta = \frac{-(1-\hat{S}_i)n}{2\delta^n}\int_0^\delta \eta^n d\eta$$

$$= \frac{-(1-\hat{S}_i)n\delta}{2(n+1)} . \tag{26}$$

Combining Equations (25) and (26) leads to the following expression for δ:

$$\delta = \left[\frac{2(n+1)(1-\hat{S}_i)^{-m}}{n(m^{1/n})}\right]^{1/2} . \tag{27}$$

Application of Darcy's law, as in Equation (19), along with the use of Equations (8) and (20), leads to the sorptivity in the form

$$\mathbf{S} = \left[\frac{2nk\phi(S_s - S_i)^{1+1/n}}{\alpha\mu(n+1)[m(S_s - S_r)]^{1/n}}\right]^{1/2} . \tag{28}$$

As an example of the accuracy of this approximate solution, consider the problem of one-dimensional absorption from, say, a saturated fracture into the adjacent rock in the Topopah Spring unit at Yucca Mountain, Nevada, a potential site of an underground repository for high-level radioactive waste. This unit is a welded volcanic tuff whose hydraulic properties have been estimated [17] to be $\phi = 0.14$, $k = 3.9\times10^{-18}\,\mathrm{m}^2$, $S_s = 0.984$, $S_r = 0.318$, $n = 3.04$, $m = 0.671$, and $\alpha = 1.147\times10^{-5}\,\mathrm{Pa}^{-1}$. Consider the problem of absorption of water into a block of Topopah Spring welded tuff that is initially at a capillary pressure of -1 bar, which corresponds to an initial liquid saturation of 0.6765. If the temperature is taken to be 20 °C, then the viscosity of the water will be 0.001 Pa s (1 cp). The saturation profiles of the approximate and essentially exact (numerical) solutions after 1×10^7 s (116 days) of infiltration are shown in Fig. 5. Note that the approximate solution predicts the location of the wetting front extremely accurately, while the sorptivity, which is proportional to the area under the saturation curve, is overpredicted by a few percent. This is due to the fact that the while saturation follows a power-law profile near the boundary, this simple one-parameter expression (Equation (24)) does not represent the actual profile throughout the entire wetted zone with complete accuracy. However, the remarkable fact remains that a reasonably accurate approximate solution has been obtained, requiring neither extensive mathematical manipulations, nor any particular "physical insight" in order to arrive at the proper form for the saturation profile.

Conclusions. The integral method has been used to develop closed-form approximate solutions to the problem of water absorption into porous rock or soil. Solutions were developed for two widely-used forms of the capillary pressure and relative permeability equations, those of Brooks and Corey and van Genuchten-Mualem. The method requires only elementary integrations and differentiations, and leads to sorptivity predictions that are typically accurate to within better than 10%. In contrast to numerical solutions, the results of the integral method clearly display the manner in which the parameters of the problem affect the solution. Another point which was illustrated by these examples is that acceptable profiles can be found merely from consideration of boundary and continuity conditions. Other examples of the use of the integral method to find approximate solutions of porous media flow problems can be found in [18-20].

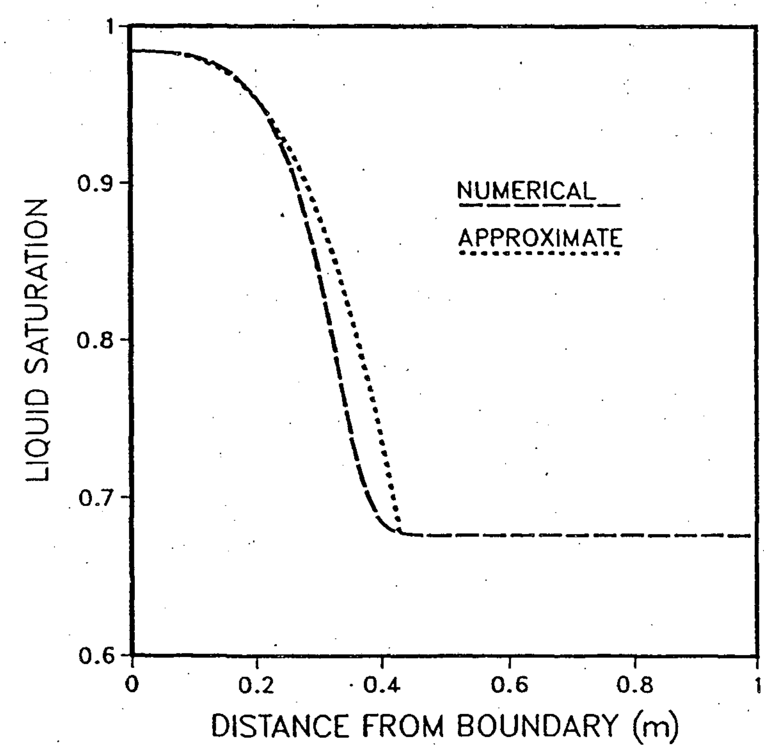

Figure 5. Saturation profile after 116 days of absorption into Topopah Spring welded tuff. The tuff is modeled as a van Genuchten medium; hydrological parameters are listed in text.

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