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by

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Abstract

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In chapter 1, I study the experimentation dynamics of a decision maker (DM) in a two-armed bandit setup ([5]), where the agent holds *ambiguous* beliefs regarding the distribution of the return process of one arm and is certain about the other one. The DM entertains *Multiplier preferences* à la [27], thus I frame the decision making environment as a two-player differential game against *nature* in continuous time. I characterize the DM's value function and her optimal experimentation strategy that turns out to follow a cut-off rule with respect to her belief process. The belief threshold for exploring the ambiguous arm is found in closed form and is shown to be increasing with respect to the ambiguity aversion index. I then study the effect of provision of an unambiguous information source about the ambiguous arm. Interestingly, I show that the exploration threshold rises unambiguously as a result of this new information source, thereby leading to more *conservatism*. This analysis also sheds light on the efficient time to reach for an expert opinion. The results of this chapter has been recently published in [61].

In chapter 2, I introduce a dynamic model of random search where ex ante heterogeneous venture capitalists (investors) with unknown abilities match with a variety of startups (projects). There is incomplete yet symmetric information about investors' types, whereas the projects' types are publicly observable to all investors. In the unique stationary equilibrium, the matching sets, value functions and steady state distributions are endogenously determined. Interpreting the market posterior belief about the venture capitalists' ability as their rep-

utation, I study the outcomes of the economy when the success or failure of the projects create feedback effects: innovation spillovers and reputational externalities. When there are positive spillovers from successful early stage projects to late stage business opportunities, I show increased levels of search frictions could save the market from breakdown caused by the neglect of spillover effect. When the reputational externality is at play, namely when the deal flow of each investor is inversely impacted by the distribution of other investors' reputation, I show the proportion of the high ability inactive investors is inefficiently high, and the projects suffer from early termination.

To my parents, Shokofeh and Ali, and my sister Bahar.

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Chapter 1

Robust Experimentation

1.1 Introduction

There are natural cases where the experimentation shall be performed in ambiguous environments, where the distribution of future shocks is unknown. For example, consider a diagnostician who has two treatments for a particular set of symptoms. One is the conventional treatment that has been widely tested and has a known success rate. Alternatively, there is a second treatment that is recently discovered and is due to further study. The diagnostician shall perform a sequence of experiments on patients to figure out the success/failure rate of the new treatment. However, the adversarial effects of the mistreatment on certain types of patients are fatal, thus the medics must consider the *worst-case* scenario on the patients while evaluating the new treatment. As another case, consider the R&D example of [75], where the research department of an organization is assigned with the task of selecting one of the two technologies producing the same commodity. The research division holds a prior on the generated saving of each technology, but the observations of each alternative during the experimentation stage is obfuscated by ambiguous sources such as the quality of researchers and managerial biases toward one choice. Therefore, the technology that is selected and sent to the development stage must be robust against these sources, because once developed it will be then used in mass production, thus even minor miscalculations in the research stage can lead to huge losses in the sales stage relative to what could have been possibly achieved.

At the core of our analysis is an experimentation process between two projects framed as a two-armed bandit problem. The return rate to one arm is known to be r, whereas the return rate of the second arm is a binary random variable $\theta \in \{\underline{\theta}, \overline{\theta}\}$, such that $\underline{\theta} \leq r \leq \overline{\theta}$.

The decision maker (henceforth DM) holds an initial prior $p_0 = \mathsf{P}\left(\theta = \overline{\theta}\right)$, that can be updated when she invests in the second project and learns its output. At the outset, she has to sequentially choose arms to learn about the unknown return rate while maximizing her net experimentation payoff. Specifically in our model, the observations of the second arm are obfuscated by Wiener process whose distribution, from the perspective of the DM, is unknown and therefore is called the *ambiguous* arm. Central to the agent's decision making problem is her preference for robustness against a candidate set of future shocks' distribution which are concealing the ambiguous arm's return rate. Our investigation of the multiplicity of shocks' distribution is motivated both from the subjective and objective perspectives. Subjectively, the DM might be ambiguity averse and the multiple prior set (for the shock distribution) would be part of her axiomatic utility representation ([18]). Alternatively, the DM might be subject to an experimentation setup where the results are objectively drawn from a family of distributions, and she wants to maintain a form of robustness against this multiplicity; this is along the lines of *model-uncertainty* pioneered by [27] and [29].

Summary of results

We frame the decision making environment in which the DM has Multiplier preference, à la [27], as a two-player continuous time differential game against nature — second player. The DM's goal is to find an allocation strategy between two arms that maximizes her payoff under the distribution picked by the nature. We express the (first player's) payoff function with respect to two control processes: (i) DM's allocation choice process between the two arms, and (ii) the nature's adversarial choice of underlying distribution. The DM follows the *max-min* strategy, namely at every point in time she chooses her allocation weights between two arms, and then the nature picks the shock distribution that minimizes the DM's continuation payoff. We then characterize the value function (to the DM) as a solution to a certain HJBI (Hamilton-Jacobi-Bellman-Isaac) equation.

In this game, the nature's move, i.e choice of the shock distribution, would have two important impacts (with opposite forces) on the DM. First, it affects the current flow payoff of experimentation, and secondly it distorts the DM's posterior formation and consequently her continuation strategy. In the equilibrium the DM knows the nature's best-response strategy, therefore, when she Bayes-updates her belief about θ, she is no longer concerned about *all* possible distributions of shocks. This gives rise to a unique law of motion for the posterior process, and reduces the HJBI equation to a second order HJB equation.

We derive a closed-form expression for the DM's value function with respect to her posterior, i.e $v(p)$, and characterize her robust optimal experimentation strategy. It turns out in the equilibrium her strategy follows a cut-off rule with respect to her belief. Specifically, she switches to the safe arm from the ambiguous arm whenever her posterior drops below a certain threshold $\bar{p}$. We also find a closed-form equation for the cut-off value that allows us to perform a number of comparative statics. In particular, the threshold for selecting the ambiguous arm unambiguously rises as the DM's ambiguity aversion index increases.[1] Also, we establish that the marginal value of receiving *good news* about θ is increasing, namely $v''(p) \geq 0$.

We then explore the effect of an additional unambiguous information source. In particular, we are interested to know what happens when for e.g the experimentation unit hires an expert to release risky but unambiguous information about θ. The new value function $\tilde{v}(p)$ is obtained in closed-form, and the DM's optimal strategy again turns out to follow a cut-off rule (with a different threshold $\tilde{p}$). Interestingly, we show that under any circumstances, compared to the previous case the value of cut-off rises as a result of the extra information, i.e $\tilde{p} \geq \bar{p}$. Therefore, it is interpreted as though the DM becomes more conservative against choosing the second arm when offered with such information. Lastly, we show the surplus $\tilde{v}(p) - v(p)$ generated by the expert attains its maximum at the range of beliefs where the experimentation unit would otherwise select the ambiguous arm but do not have strong enough feeling and evidence in favor of this decision. Therefore, our model sheds light on the time that is best to reach an expert opinion.

Related literature

The literature on robust bandit problem is limited, but recently there have been some attempts to bring several aspects of robustness into play. Specifically in the works done by [7] and [38] the discrete-time multi-armed bandit problem is studied while the state transition probabilities are drawn from an *ambiguous* set of conditional distributions. In [7] the set of multiple transition probabilities at every period is constrained through a relative entropy condition, whereas [38] chooses to impose an entropic penalty cost directly in the objective function of the DM rather than hard thresholding it as a constraint. In a different work [44] studies the multi-armed bandit in which the DM entertains max-min utility and follows a prior-by-prior Bayes updating from her initial *rectangular* multiple prior set, where each candidate distribution in this set is identified by the i.i.d shocks it generates in the future.

[1] The direction of such a response is intuitive, however, the sharp characterization of the threshold via the means of continuous time techniques provides us with the extent of this response.

Our work is different from these treatments in the following aspects: (i) contrary to the first two works the Brownian diffusion treatment of the Markov transitions allows for a richer set of perturbations around benchmark model which extends the scope of robustness that the DM demands; (ii) the continuous time framework lets us to obtain sharp and closed form results on the value function and the optimal experimentation policy that in turn renders the comparative static with regard to parameters of the model and importantly the ambiguity aversion index; (iii) we are explicit about the state variable in our setup, and specifically we characterize it as the DM's posterior process regarding the second arm's return rate; (iv) our setup is flexible enough that can address distinct informational environments such as the effect of the provision of an expert opinion.

In the economic literature, after the seminal work of [19], the continuous time problem of optimal experimentation in a noisy environment, where the payoff to the unexplored arm[2] is subject to a Brownian motion is studied in [5] and [36]. Aside from these works, there is a growing literature on experimentation in a multiple agent environment where the free-riding issues arise.[3]

Our treatment of robust preferences in continuous time relies heavily on the fundamental works by [27], [29] and [28].[4] Our analysis is also related to the literature studying the effects of robustness and ambiguity in different decision making frameworks such as [62], [9], [54], [76] and [48]. Also, it is related to the relatively understudied topic of learning under ambiguity.[5] Finally in a set of experimental works with adopting different notions of ambiguity aversion, it has been tested that the ambiguous arm of the experiment has a lower Gittins index that prompts the DM to undervalue the information from exploration. To name a few we can point to [3] and [53] in the context of airline choice and [74] in the investment choice.

The remainder of this chapter is organized as follows. To build intuition, in section 1.2 we present some of the forces behind the model in a two-period example. Next, in section 1.3 the full features of experimentation setup and payoff function are explained in a continuous time framework. In section 1.4, we apply the dynamic programming analysis and present variational characterizations of the value function. Section 1.5 offers the closed-form expression for value function, properties of the optimal experimentation strategy, and some comparative static results. In section 1.6, we extend our setup to capture the effect of an

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[2] Often the second arm is referred as the *unexplored* one.

[3] A nonexhaustive list includes [37], [30] and [6].

[4] In a closely related discrete-time framework [15] and [50] present recursive utility representation aimed to capture the preference for robustness.

[5] For example see [52], [14] and [13].

additional unambiguous information source. Finally the proofs of all results are expressed in the appendix A.

1.2 Two-period example

Our goal in this example is to highlight the main trade-offs that the DM and her opponent *nature* face in their dynamic interaction. Let $t \in \{1, 2\}$ and at each period the DM allocates her resources between two available choices, namely the safe and the ambiguous project. The time t incremental returns to each arm when she allocates $\mu_t \in [0, 1]$ of her resources to the safe (first) arm and $1 - \mu_t$ to the ambiguous (second) arm are

$$\begin{aligned} \Delta y_{1,t} &= (1 - \mu_t) r \\ \Delta y_{2,t} &= \mu_t \theta + \sqrt{\mu_t} \varepsilon_t. \end{aligned} \tag{1.1}$$

In that $r = 1$ is the return rate of the safe project, and $\theta \in \{0, 2\}$ is the unknown return to the second arm. The DM's prior on this set at period one is given by $p_1 = \mathsf{P}(\theta = 2)$, which is not subject to any ambiguity. However, at each period the return to the second arm is obfuscated by an independent[6] Gaussian shock that could possibly be drawn from two distributions, namely for each t the law of ε_t belongs to the set $\{\mathcal{N}(-0.5, 1), \mathcal{N}(0.5, 1)\}$.[7] We take no stance on whether this multiple prior set is the subjective belief of the DM or literally the objective moves that nature takes against the DM. Our solution concept for both cases is the the so-called *max-min*. However, the first situation reflects a decision theoretic choice of an ambiguity averse agent with a subjective multiple prior set, whereas the second interpretation is more in line with the notion of robust decision making.

The timing of this example is as follows. At the beginning of period one DM chooses μ_1. Then, nature *responds* by picking $h_1 \in \{-0.5, 0.5\}$ as the mean of ε_1. The returns to both arms, i.e $\{\Delta y_{1,1}, \Delta y_{2,1}\}$ are realized. DM forms the family of beliefs $\{p_2^{h_1} : h_1\}$ at the beginning of period two, and takes the *appropriate* action μ_2. The nature chooses h_2 as the mean of second period's shock. Subsequently the game ends and second period's returns are realized.

What happens at the sub-game perfect equilibrium of this game? For this we need to look at the sub-game starting at $t = 2$. Regardless of DM's action μ_2, the nature always picks $h_2 = -0.5$, because the game ends at this period and $h_2 = -0.5$ is the worst case

[6] For simplicity assume ε_1, ε_2 and the period one belief on θ are independent from each other.

[7] This set clearly doesn't satisfy the rectangularity condition nor the convexity property of [18], however it serves only for expositional purposes.

distribution from the DM's perspective. Because of this triviality of the nature's choice at period two, we drop the index one from h_1 and henceforth denote it by h, which is the only non-trivial choice of the nature in this example. The DM's posterior beliefs after the realizations of first period returns are

$$p_2^h = \left(1 + \frac{1-p_1}{p_1} \exp\left\{2\left(\sqrt{\mu_1}h + \mu_1 - \Delta y_{2,1}\right)\right\}\right)^{-1} 1_{\{\mu_1>0\}} + p_1 1_{\{\mu_1=0\}}, \quad h \in \{-0.5, 0.5\}. \tag{1.2}$$

It is important to note that the posterior probability is no longer unique, and DM faces a set of posteriors for each choice of nature in period one. Even though that we face a two-player game where the nature's actions are not observable to the DM, but at the equilibrium DM knows the *minimizing* choice of the nature, thereby her family of posteriors effectively reduces to a single posterior induced by the worst case action of the nature say h^*. This point becomes more clear as we proceed through the equilibrium analysis. For every member p_2 of the posterior set, the DM's optimal action at period two (anticipating that nature will choose $h_2 = -0.5$) is $\mu_2(p_2) = 1_{\{2p_2-0.5>1\}}$, that leads to the expected payoff of $v_2(p_2) = \max\{1, 2p_2 - 0.5\}$. Note that this expectation is with respect to the equilibrium distribution choice of the nature that is $h_2 = -0.5$. Assume the experimenter's intertemporal discount rate is $\delta \in (0, 1]$. Further, let P^h denote the probability measure induced by the independent product of $\varepsilon_1 \sim \mathcal{N}(h, 1)$ and $\theta \sim p_1$. Therefore, the DM's value function as of beginning of period one is

$$v_1(p_1) = \max_{\mu_1\in[0,1]} \min_{h\in\{-0.5,0.5\}} \left\{\left[(1-\mu_1) + 2\mu_1 p_1 + \sqrt{\mu_1}h)\right] + \delta \mathsf{E}^h\left[v_2(p_2^h)\right]\right\}. \tag{1.3}$$

Below we point out to some of the underlying equilibrium forces that will show up in this two period example.

(i) The nature's first period action, or alternatively, the most pessimistic perception of the DM in regard to shock distribution ε_1^h, plays two roles. **Current payoff channel**, in that the nature's choice of h affects the current payoff of the DM by changing the mean return of the ambiguous arm, i.e $\left[(1-\mu_1) + 2\mu_1 p_1 + \sqrt{\mu_1}h)\right]$. In particular, this is a *positive* force, as higher h's correspond to higher mean flow payoff. **Informational channel**, where the shock distribution ε_1^h affects the next period belief of the DM, hence changes her course of action and thereby the continuation payoff. This has a *negative* effect, because as h increases, the distribution of $\Delta y_{2,1}$ shifts to the right in the FOSD sense and for a fixed $\Delta y_{2,1}$ lowers the likelihood ratio in (1.2) that in turns depresses the continuation payoff $\mathsf{E}^h\left[v_2(p_2^h)\right]$. At the equilibrium, nature counteracts these forces and picks the one that its negative effect outweighs the positive one, and thus reduces the DM's payoff more. However, it can not completely balance out the marginal impact of these forces, mainly because we assumed the multiple prior set consists of only two distributions. When the complete mode is laid out in section 1.3, we allow for quite general multiple prior set, thus nature can precisely cancel out the marginal effects, thereby lowering the DM's payoff as much as possible.

(ii) From the point of view of the DM, there is an option value of experimentation. Specifically, in the first period she selects the ambiguous arm (even partially $0 < \mu < 1$) only to observe the payout of second arm, and then may decide to abandon the ambiguous project depending on the outcome of the first period. In this example, the DM switches back to the safe arm in the second period if her posterior in the equilibrium, i.e $p_2^{h^*}$, drops below a certain threshold, which in this case is 0.75.

(iii) The DM's value function is unambiguously increasing in her initial belief p_1 (as can be confirmed from (1.3)), but the marginal value of good news need not be increasing (meaning v'' is not always positive). This is mainly due to the finite-horizon setup of the two-period model, which is relaxed in later sections.

(iv) The value function in (1.3) refers to the max-min value of the game, which is associated to the strategic order of actions in which the DM takes her action first and then the nature responds in every period. This is the same approach that we pursue when we present the complete model. However, one might wonder when does this max-min value coincide with the min-max one? Or in the other words, when does the strategic order of players' actions become irrelevant? In this example the max-min value is strictly less than min-max. Although not related to the study of this chapter, but we confirm that with compact and convex action spaces of both players, the von-Neumann minimax theorem could be applied and therefore one can conceive the unique value of the zero-sum game between DM and the nature.

We do not intend to delve deeper into this example and express more specific results and comparative statics, mainly because such analysis will be carried out for the complete model later in the chapter.

1.3 Experimentation model

Time horizon is infinite and $t \in \mathbb{R}_+$. There are two projects available to experiment by the DM. Her choice at time t is thus to allocate her resources between two alternatives, namely μ_t to the ambiguous arm and $1-\mu_t$ to the safe arm. The return process of the projects are[8]

$$\begin{aligned} \mathrm{d}y_{1,t} &= (1-\mu_t) r \mathrm{d}t \\ \mathrm{d}y_{2,t} &= \mu_t \theta \mathrm{d}t + \sigma\sqrt{\mu_t}\mathrm{d}B_t. \end{aligned} \tag{1.4}$$

Here B is a Brownian motion relative to some underlying stochastic basis[9], that represents the shock process, and θ is unknown to the DM but belongs to the binary set $\{\bar{\theta}, \underline{\theta}\}$, where $\underline{\theta} \leq r \leq \bar{\theta}$. The DM has an initial belief $p_0 = \mathsf{P}\left(\theta = \bar{\theta}\right)$ about θ which is independent from B. The form of return processes in (1.4) follows [5], but we let the DM to associate multiple distributions to the shock process. Specifically, the DM holds a single belief over θ — so that this represents the uncertainty due to *risk* — but has multiple beliefs regarding the shock distribution B — so this represents the uncertainty due to *ambiguity*.[10]

A framework for modelling ambiguity

Our take of ambiguity or model uncertainty is similar to [29] and [28]. In particular, we assume there is a family of pairs $\left\{(\mathsf{P}^h, B^h) : h \in \mathcal{H}\right\}$ such that for each $h \in \mathcal{H}$, B^h is a Brownian motion under P^h, and DM views this as her multiple prior set. We think of $\mathcal{H}$ – which thus far has not been defined – as the nature's action space, and each $h \in \mathcal{H}$ is deemed as a possible nature's move. We assume there exists a *benchmark* probability specification P that is *equivalent* (mutually absolutely continuous with respect) to each member of $\mathcal{P} := \{\mathsf{P}^h : h \in \mathcal{H}\}$. The benchmark measure P and the set $\mathcal{P}$ are interpreted differently based on the context. For example, DM might believe that P is the underlying probability measure, but considers $\mathcal{P}$ as the approximations of the true distribution because she has preference for robustness. Alternatively, $\mathcal{P}$ could be conceived as the multiple prior set for the ambiguity averse DM in the axiomatic treatment of [18].

[8] The goal of this section is to study the interplay between ambiguity regarding the new arm and optimal experimentation, thus for simplicity we assume that the conventional arm has a sure return rate of r and is not subject to any source of randomness. Therefore, it is only the second arm that carries the Brownian motion term.

[9] The description of the underlying stochastic basis and the joint structure of processes are explained in the subsection devoted to the *weak formulation*.

[10] This type of uncertainty is sometimes referred to as *model uncertainty* in the literature.

DM has *Multiplier preference* and maximizes the following payoff over an *admissible* set of experimentation strategies $\mathcal{U}$ — with some technical considerations that are elaborated later in the chapter:

$$\inf_{h\in\mathcal{H}}\left\{\mathsf{E}^{\mathsf{P}^h}\left[\delta\int_0^\infty e^{-\delta t}\mathrm{d}\left(y_{1,t}+y_{2,t}\right)\right]+\alpha H\left(\mathsf{P}^h;\mathsf{P}\right)\right\} \tag{1.5}$$

Here δ is the time discount rate. The first term in the DM's utility is simply the expected discounted payoff from both projects taken with respect to the measure P^h, and the second term penalizes the belief misspecification using the relative discounted entropy to measure the discrepancy between P and P^h. Parameter α captures the extent of this penalization, where its larger values associate to smaller penalty. We shall also interpret α as the inverse of ambiguity aversion and relate (1.5) to the *dynamic variational utility representation* of [49] and [50]. A large α means that the DM does not suffer a lot from ambiguity aversion. In contrast as $\alpha \to 0$, the DM experiences larger utility loss due to severe penalization.

In the next subsection we use the *weak-formulation* approach from the theory of stochastic processes to elaborate and simplify DM's utility function (1.5).

Weak formulation

In this part we present a sound foundation for the joint structure of all the stochastic processes in the model[11]. Let $\left(\Omega,\mathcal{F}=\mathcal{F}_\infty,\mathbf{F}=\{\mathcal{F}_t\}_{t\in\mathbb{R}_+},\mathsf{P}\right)$ be the stochastic basis, where the filtration satisfies the *usual conditions.*[12] The average rate of return to the ambiguous project θ is a binary $\mathcal{F}_0$-measurable random variable.

Definition 1 (Strategy spaces)**.** *The DM's strategy space $\mathcal{U}$ — with a representative point $\mu\in\mathcal{U}$ — is the set of all $\mathbf{F}$-progressive processes*[13] *taking value in* $[0,1]$*. The nature's strategy space $\mathcal{H}$ — with a representative point $h\in\mathcal{H}$ — is the space of all bounded $\mathbf{F}$-progressive processes.*

Definition 2 (Integral forms)**.** *For any pair of processes $\{f,g\}$ where f is g-integrable*[14] *we use the alternative notation for integration: $(f\cdot g)_t := \int_0^t f_s\mathrm{d}g_s$. Further, the symbol $\imath$ refers*

[11] The materials in this subsection might look somewhat technical and unnecessary to some readers, but are essential for rigorous development of the model.

[12] It is right-continuous and P-complete.

[13] We refer to [33] for the definition of progressive processes.

[14] The notion of integral depends on the context that could either be the path-wise Stieltjes integral or stochastic Itô integral.

to identity mapping $t \mapsto t$ on $\mathbb{R}_+$. Then the differential return expressions in (1.4) *can be represented in the integral form $y_1 = (1-\mu)r \cdot \imath$ and $y_2 = \mu\theta \cdot \imath + \sigma\sqrt{\mu} \cdot B$.*

To model the ambiguity we appeal to the weak formulation. In particular, we think of ambiguity as the source that changes the distribution of return process $\{y_1, y_2\}$, but not its sample paths. For this on every finite interval $[0, T]$ we define the probability measure P_T^h with the following Radon-Nikodym derivative process:

$$\left.\frac{\mathrm{d}\mathsf{P}_T^h}{\mathrm{d}\mathsf{P}}\right|_{\mathcal{F}_t} := L_{t,T}^h = \exp\left\{(h \cdot B)_t - \frac{1}{2}\left(h^2 \cdot \imath\right)_t\right\}, \quad \forall t \leq T \tag{1.6}$$

This relation explains how nature with its choice of $h \in \mathcal{H}$ could induce a new probability measure. The Girsanov's theorem implies that P_T^h is mutually absolutely continuous with respect to P — that is often called *equivalent* measure and denoted by $\mathsf{P}_T^h \sim \mathsf{P}$ on $\mathcal{F}_T$. It also implies that the *mean-shifted* process $B^h := B - (h \cdot \imath)$ is a $\mathbf{F}$-Brownian motion under P_T^h over the interval $[0, T]$. The main catch here is that we can only characterize the perturbations of benchmark probability model P over finite intervals, that is for example we know how P^h looks like on $\mathcal{F}_T$ for any finite T. However, what is needed for the utility representation in (1.5) is a specification of P^h on the terminal σ-field $\mathcal{F}_\infty$. For this we need to use a limiting argument to consistently send $T \to \infty$ and obtain (P^h, L^h, B^h) as an appropriate limit of $(\mathsf{P}_T^h, L_T^h, B_T^h)$. Our proposal for this is as follows. For any process $h \in \mathcal{H}$ and an increasing sequence of finite times $\{T_n\}_{n\in\mathbb{N}}$, we repeatedly apply the Girsanov's theorem to obtain a family of consistent probability measures $\left\{\mathsf{P}_{T_n}^h, \mathcal{F}_{T_n} : n \in \mathbb{N}\right\}$, where $\mathsf{P}_{T_n}^h \sim \mathsf{P}$ on $\mathcal{F}_{T_n}$ for every $n \in \mathbb{N}$. In a similar vein we obtain the likelihood ratio process $\left\{L_{t,T_n}^h : t \leq T_n\right\}$ and the Brownian motion $\left\{B_{t,T_n}^h : t \leq T_n\right\}$ for every $n \in \mathbb{N}$. Next, we explain how to naturally define the limit of each three components.

(i) **Likelihood process limit:** Expression (1.6) implies that the sequence of likelihood processes are path-wise consistent with each other, i.e $L_{t,T_m}^h = L_{t,T_n}^h$ for every $t \leq T_m \leq T_n$. Therefore, one can define the process L^h on $[0, \infty)$ in a meaningful sense, such that its restriction to any finite interval coincides with the sequence of likelihood processes. This concludes the construction of the limit likelihood process. Importantly, this construction suggests that L^h must be a martingale process with respect to P on $\mathbb{R}_+$. To see this, note that a bounded h causes the *Novikov's* condition to hold, thereby $L_{T_n}^h$ would be an uniformly integrable martingale — on $[0, T_n]$ — with respect to P for every $n \in \mathbb{N}$. Because of the path-wise equivalence, this would immediately establish the martingale property of L^h on $\mathbb{R}_+$.

(ii) **Probability measure limit:** First, recall that for every $n \in \mathbb{N}$, $\mathsf{P}^h_{T_n}$ is a probability measure on $\mathcal{F}_{T_n}$. Then, the path-wise consistency resulted from (1.6) implies that these measures indeed match each other, namely $\mathsf{P}^h_{T_n}(A) = \mathsf{P}^h_{T_m}(A)$ for every $A \in \mathcal{F}_{T_m}$ where $m \leq n$. Thus, we can apply theorem 4.2 in [58] that guarantees the existence of a *closing* probability measure P^h on $\mathcal{F}_\infty$ such that its restrictions to finite intervals coincide with the above sequence of probability measures, yet it need not be equivalent to P on $\mathcal{F}_\infty$. That is restricted to every finite T, $\mathsf{P}^h \sim \mathsf{P}$ on $\mathcal{F}_T$, but this may not be true on $\mathcal{F}_\infty$.

(iii) **Brownian motion limit:** Applying Girsanov's theorem lets us to deduce that $B^h_{T_n} := \{B^h_{t,T_n} : t \leq T_n\}$ is a Brownian motion under $\mathsf{P}^h_{T_n}$ on $[0, T_n]$ for every $n \in \mathbb{N}$. Since $\mathsf{P}^h \equiv \mathsf{P}^h_{T_n}$ on $[0, T_n]$, then it turns out that $B^h_{T_n}$ is also a Brownian motion under P^h. Also note that the path-wise consistency holds for the sequence of Brownian motions, namely $B^h_{t,T_n} = B^h_{t,T_m}$ for all $t \leq T_m \leq T_n$. Therefore, in the same manner that we defined L^h from $\{L^h_{T_n} : n \in \mathbb{N}\}$, we can define B^h as the process on $\mathbb{R}_+$ such that its restrictions to any finite interval satisfy the properties of P^h Brownian motions.

The illustrated construction of $\left(\mathsf{P}^h, B^h\right)$ allows us to express the return process of the ambiguous project in term of h-Brownian motion:

$$\mathrm{d}y_{2,t} = [\mu_t \theta + \sigma \sqrt{\mu_t} h_t] \, \mathrm{d}t + \sigma \sqrt{\mu_t} \mathrm{d}B^h_t \tag{1.7}$$

The merit of weak formulation now becomes clear, where for every $\mu \in \mathcal{U}$ the return processes $\{y_1, y_2\}$ are essentially fixed, but the probability distribution that assigns weights to the subsets of sample paths is controlled by the choice of $h \in \mathcal{H}$. So in a sense the nature's move is to select the return's distribution not its sample paths.

Now that we know what is meant by P^h on $\mathcal{F}_\infty$ we can analyze both terms of (1.5) which are expectations under P^h, and this will be the goal of next subsection.

Unravelling the payoff function

We begin the simplification of (1.5) by elaborating the second term, that is the entropy cost of ambiguity aversion. Recall that P and P^h need not necessarily be equivalent measures on $\mathcal{F}_\infty$, yet their restrictions $\{\mathsf{P}_t, \mathsf{P}^h_t\}$ are indeed equivalent probability measures on $\mathcal{F}_t$. Having that said, the relative discounted entropy is defined as

$$H\left(\mathsf{P}^h; \mathsf{P}\right) := \lim_{T \to \infty} \delta \int_0^T e^{-\delta t} H\left(\mathsf{P}^h_t; \mathsf{P}_t\right) \mathrm{d}t, \tag{1.8}$$

where $H\left(\mathsf{P}_t^h;\mathsf{P}_t\right) := \mathsf{E}^h\left[\log L_t^h\right]$. Expression (1.8), which is proposed in [29], presents a proxy for the discrepancy between two measures that are not necessarily equivalent on the terminal σ-field, and hence their relative entropy $H\left(\mathsf{P}_\infty^h;\mathsf{P}_\infty\right)$ could be infinite, but on each finite interval say $[0,t]$ they are equivalent $\mathsf{P}_t^h \sim \mathsf{P}_t$ and have finite relative entropy. Therefore, one shall hope that relation (1.8) is well-defined.

Lemma 1. *The discounted relative entropy in* (1.8) *is well-defined, namely for every $h \in \mathcal{H}$ it is finite and satisfies*

$$H\left(\mathsf{P}^h;\mathsf{P}\right) = \frac{1}{2}\mathsf{E}^h\left[\int_0^\infty e^{-\delta t} h_t^2 \mathrm{d}t\right] < \infty. \tag{1.9}$$

Roughly speaking, for the first component of the payoff function we need to take the expectation of $\mathrm{d}y_2$ under the measure P^h. This is in our reach because we stated the dynamics of y_2 in terms of B^h in (1.7). However, the drift term in $\mathrm{d}y_2$ contains the random variable θ, that needs to be learned and projected onto the DM's information set. For this we present an optimal filtering result under each measure P^h.

Remark 1. *The DM's initial prior $p_0 = \mathsf{P}\left(\theta = \overline{\theta}\right)$ is unaffected under different probability distributions $\mathcal{P} = \left\{\mathsf{P}^h : h \in \mathcal{H}\right\}$. This is because the benchmark measure P and all its variations $\mathcal{P}$ agree on $\mathcal{F}_0$, resulted from $L_0^h = 1$ for every $h \in \mathcal{H}$.*

In light of this remark, we want to continuously estimate and update the DM's posterior on θ based on her available information at every point in time. Her information set at time t contains the path of output from each project $\{(y_{1,s}, y_{2,s}) : s \leq t\}$, the history of her allocation process $\{\mu_s : s \leq t\}$ and importantly the nature's moves up until time t, i.e $\{h_s : s \leq t\}$. Note that at each time t, the DM's ambiguity is with regard to the future path of h, and she has no uncertainty about the history of nature's moves in the past. Some might not be willing to make this assumption about the ex-post observability of nature's moves to the DM. However, this is not an important assumption for two reasons. First, on the equilibrium path the DM knows the history of nature's past moves. Secondly, in theory we can find the filtering equation under every possible history of nature's actions and then let the DM to pessimistically choose from this family of posteriors. In summary, the filtering problem that the DM faces at time t is to update her posterior based on the available information set $\mathcal{F}_t^{y_1,y_2,\mu,h}$. Of secondary importance is to note that y_1 conveys no information about θ, thus can be dropped out of the information set.

Definition 3. *For every $t \in (0,\infty)$, define $p_t^h := \mathsf{P}^h\left(\theta = \bar{\theta} \middle| \mathcal{F}_t^{y_2,\mu,h}\right)$ as the posterior probability and $m(p_t^h) = p_t^h\bar{\theta} + (1-p_t^h)\underline{\theta}$ as the conditional mean. At $t=0$, let $p_t^h = p_0$ and $m(p_0^h) = m(p_0)$.*

Lemma 2 ([47] theorem 8.1)**.** *The conditional probability of the event $\{\theta = \bar{\theta}\}$ given the filtration $\mathbf{F}^{y_2,\mu,h}$ evolves according to the following stochastic differential equation:*

$$\mathrm{d}p_t^h = \frac{(\bar{\theta}-\underline{\theta})\sqrt{\mu_t}}{\sigma} p_t^h \left(1-p_t^h\right)\mathrm{d}\bar{B}_t^h \tag{1.10}$$

Here $\left\{\bar{B}_t^h, \mathcal{F}_t^{y_2,\mu,h} : t \in \mathbb{R}_+\right\}$ is called the innovation process which is a Brownian motion under P^h, and is characterized by $\mathrm{d}\bar{B}_t^h = \sigma^{-1}\sqrt{\mu_t}\left[\theta - m(p_t^h)\right]\mathrm{d}t + \mathrm{d}B_t^h$. As a result of this, the law of motion for y_2 would be

$$\mathrm{d}y_{2,t} = \left[\mu_t m(p_t^h) + \sqrt{\mu_t}h_t\right]\mathrm{d}t + \sigma\sqrt{\mu_t}\mathrm{d}\bar{B}_t^h. \tag{1.11}$$

Sketch of the proof. First note that from the filtering point of view the process y_2 contains the same information as $\tilde{y}_2 := (\sqrt{\mu}\theta + \sigma h)\cdot \imath + \sigma \cdot B^h$. Therefore, on the region $\mu > 0$, we have $\mathsf{E}^h\left[\theta \middle| \mathcal{F}_t^{\tilde{y}_2,\mu,h}\right] = \mathsf{E}^h\left[\theta \middle| \mathcal{F}_t^{y_2,\mu,h}\right]$ for every $h \in \mathcal{H}$ and $t \in \mathbb{R}_+$. Next, applying theorem 8.1 of [47] and taking $\tilde{y}_2$ as the *observable* process and θ as the subject of filtering imply that:

$$\begin{aligned}\mathsf{E}^h\left[\theta \middle| \mathcal{F}_t^{\tilde{y}_2,\mu,h}\right] &= \mathsf{E}^h\left[\theta \middle| \mathcal{F}_0^{\tilde{y}_2,\mu,h}\right] \\ &\quad + \sigma^{-1}\int_0^t \left(\mathsf{E}^h\left[\theta\left(\sqrt{\mu}_s\theta + h_s\right)\middle| \mathcal{F}_s^{\tilde{y}_2,\mu,h}\right] - \mathsf{E}^h\left[\theta \middle| \mathcal{F}_s^{\tilde{y}_2,\mu,h}\right]\mathsf{E}^h\left[\sqrt{\mu}_s\theta + h_s \middle| \mathcal{F}_s^{\tilde{y}_2,\mu,h}\right]\right)\mathrm{d}\bar{B}_s^h \\ &= \mathsf{E}^h\left[\theta \middle| \mathcal{F}_0^{\tilde{y}_2,\mu,h}\right] + \sigma^{-1}\int_0^t \sqrt{\mu_s}\left(\mathsf{E}^h\left[\theta^2 \middle| \mathcal{F}_s^{\tilde{y}_2,\mu,h}\right] - \mathsf{E}^h\left[\theta \middle| \mathcal{F}_s^{\tilde{y}_2,\mu,h}\right]^2\right)\mathrm{d}\bar{B}_s^h\end{aligned} \tag{1.12}$$

This expression underlies the filtering equation for the posterior process p^h, as it readily amounts to

$$p_t^h = p_0 + \sigma^{-1}(\bar{\theta}-\underline{\theta})\int_0^t \sqrt{\mu_s}p_s^h(1-p_s^h)\mathrm{d}\bar{B}_s^h, \tag{1.13}$$

and thus verifies equation (1.10). It is worth mentioning here that since there is no ambiguity about θ at time 0 w.r.t the distribution of θ, the first term in the *rhs* of (1.12) is independent of h. □

At this stage we have developed all the required tools to present the utility function in (1.5) in terms of initial belief and the players' actions. For this we define the infinite horizon payoff as the limit of finite horizon counterparts. The reason is that the constructed process B^h is only Brownian motion over finite intervals, and we can not extend it to entire $\mathbb{R}_+$, unless we impose further restrictions on $\mathcal{H}$ and $\mathcal{U}$ to obtain the uniform integrability of likelihood processes, which we refrain to do. Therefore, inspired by (1.5) we define the utility of DM from taking action μ while nature chooses h by

$$V(p;\mu,h) := \lim_{T\to\infty} \mathsf{E}^h\left[\delta \int_0^T e^{-\delta t}\left(\mathrm{d}y_{1,t} + \mathrm{d}y_{2,t} + \alpha H\left(P_t^h; P_t\right)\mathrm{d}t\right)\right]. \tag{1.14}$$

Proposition 1. *For every choice of $\mu \in \mathcal{U}$ and $h \in \mathcal{H}$, the net discounted average payoff defined in* (1.14) *can be expressed as:*

$$V(p;\mu,h) = \mathsf{E}^h\left[\delta \int_0^\infty e^{-\delta t}\left((1-\mu_t)r + \mu_t m(p_t^h) + \sigma\sqrt{\mu_t}h_t + \frac{\alpha}{2\delta}h_t^2\right)\mathrm{d}t\right] \tag{1.15}$$

This proposition serves us well, because the integrand is now $\mathbf{F}^{y_2,\mu,h}$-progressively measurable, that in turn allows us to perform a dynamic programming scheme to express the *value function* in terms of the current belief, and this will be the goal of next section.

1.4 Dynamic programming analysis

Our analysis so far offers expression (1.15) as the DM's payoff in the two-player differential game against the nature. For any point of time, say $t \in \mathbb{R}_+$, define the expected continuation value conditioned on $\mathcal{G}_t := \mathcal{F}_t^{y_2,\mu,h}$ as

$$J(p,t;\mu,h) := \mathsf{E}^h\left[\delta \int_t^\infty e^{-\delta s}\left((1-\mu_s)r + \mu_s m(p_s^h) + \sigma\sqrt{\mu_s}h_s + \frac{\alpha}{2\delta}h_s^2\right)\mathrm{d}s\,\middle|\,\mathcal{G}_t\right]. \tag{1.16}$$

In that p is the time t value of the state process p_t^h. For every $h \in \mathcal{H}$ the process $\bar{B}^h$ as well as p^h are time *homogeneous* Markov diffusions. Furthermore, the players' action spaces at the time t sub-game — $\mathcal{U}_t$ and $\mathcal{H}_t$ resp. for the DM and the nature — are essentially isomorphic to $\mathcal{U}$ and $\mathcal{H}$. These two premises imply that the max-min value of the game for the DM, i.e $\sup_{\mu\in\mathcal{U}_t}\inf_{h\in\mathcal{H}_t} J(p,t;\mu,h)$, is time homogeneous. Specifically, there exists a value function $v(p)$ such that

$$\sup_{\mu\in\mathcal{U}_t}\inf_{h\in\mathcal{H}_t} J(p,t;\mu,h) = e^{-\delta t}v(p) \tag{1.17}$$

Our goal in the next theorem is to present a *verification* result for the value function. For this we need to appeal to the theory of viscosity solution [11] that provides the appropriate setting for Bellman equations. The reason for this is that as it turns out the value function $v(p)$ is not twice continuously differentiable everywhere, therefore classical verification techniques relying on Ito's lemma would not apply. We offer some preliminary definitions that are linked to the work of [77][15], thereby setting the groundwork for the viscosity solution concept.

Definition 4. *Let $w \in C([0,1])$. The superdifferential of w at $x_0 \in [0,1)$ is denoted by $D_+w(x_0)$:*

$$D_+w(x_0) = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \limsup_{x \to x_0} \frac{w(x) - w(x_0) - (x - x_0)\xi_1 - \frac{1}{2}(x - x_0)^2 \xi_2}{(x - x_0)^2} \leq 0 \right\} \tag{1.18}$$

A generic member of this set is referred by $\left(\partial_+ w(x_0), \partial_+^2 w(x_0)\right)$. And the subdifferential, denoted by $D_-w(x_0)$ is defined as

$$D_-w(x_0) = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \liminf_{x \to x_0} \frac{w(x) - w(x_0) - (x - x_0)\xi_1 - \frac{1}{2}(x - x_0)^2 \xi_2}{(x - x_0)^2} \geq 0 \right\}. \tag{1.19}$$

A generic member of this set is referred by $\left(\partial_- w(x_0), \partial_-^2 w(x_0)\right)$.

Notice that a continuous function may not be once or twice continuously differentiable but it always has non-empty super(sub)-differential sets on a dense subset of $[0,1]$ [46].

In the *verification* theorem that follows we show that the value function $v(\cdot)$ in (1.17) is the viscosity solution to a certain HJBI equation with the following form

$$w(p) = \sup_{\mu \in [0,1]} \inf_{h \in \mathbb{R}} \left\{ g(p, \mu, h) + \mathcal{K}(p, w'(p), w''(p), \mu, h) \right\}^{16}, \tag{1.20}$$

where the specific form of the coefficients g and $\mathcal{K}$ will be given in the theorem's statement. As a last step before presenting the therorem, we express what is meant by being a viscosity solution to a HJBI equation.

Definition 5. *A function $w \in C([0,1])$ is called a viscosity solution of (1.20) if it is both a viscosity subsolution and a viscosity supersolution that are respectively equivalent to:*

$$-w(p) + \sup_{\mu \in [0,1]} \inf_{h \in \mathbb{R}} \left\{ g(p, \mu, h) + \mathcal{K}(p, \xi_1, \xi_2, \mu, h) \right\} \leq 0, \ \forall (\xi_1, \xi_2) \in D_+w(p), \tag{1.21a}$$

$$-w(p) + \sup_{\mu \in [0,1]} \inf_{h \in \mathbb{R}} \left\{ g(p, \mu, h) + \mathcal{K}(p, \xi_1, \xi_2, \mu, h) \right\} \geq 0, \ \forall (\xi_1, \xi_2) \in D_-w(p). \tag{1.21b}$$

[15] There were some technical gaps in the proof of the verification theorem in this chapter, that are addressed and corrected in the follow up papers [24] and [25]; thanks to the anonymous referee for bringing this up to the author's attention.

[16] Notice that w' and w'' should not be confused with the first and second derivatives as they may not exist for a continuous function. This form is just a *representation* of the HJBI equation that has a viscosity solution in the sense of definition 5, and may not hold a *smooth* classical solution.

Theorem 1. *Suppose $w \in C([0,1])$ is Lipschitz and a viscosity solution to the following HJBI equation:*

$$w(p) = \sup_{\mu\in[0,1]} \inf_{h\in\mathbb{R}} \left\{(1-\mu)r + \mu m(p) + \sigma\sqrt{\mu}h + \frac{\alpha}{2\delta}h^2 + \frac{\mu}{2\delta}\Phi(p)w''(p)\right\}, \tag{1.22}$$

where $\Phi(p) := \sigma^{-2}(\bar{\theta} - \underline{\theta})^2 p^2(1-p)^2$. Then, w equals v, the value function in (1.17). *In the equilibrium, the worst-case density generator is $h^* = -\alpha^{-1}\sigma\delta\sqrt{\mu^*}$, where μ^* is the DM's best response in*

$$w(p) = \sup_{\mu\in[0,1]} \left\{(1-\mu)r + \mu m(p) - \frac{\sigma^2\delta}{2\alpha}\mu + \frac{\mu}{2\delta}\Phi(p)w''(p)\right\}^{17}. \tag{1.23}$$

As stated in previous theorem, on the equilibrium path of the game, DM knows the best response of the nature, that is $h(\mu) = -\alpha^{-1}\sigma\delta\sqrt{\mu}$. Therefore, her posterior process follows that of (1.10) for the prescribed $h(\mu)$. Importantly, this means at the equilibrium the DM is no longer concerned about all possible distributions of past shocks. The one that has been picked by the nature is known to the DM on the equilibrium path, which gives rise to the unique law of motion for the posterior belief. Note that, this does not mean that ambiguity is mitigated on the equilibrium path. However, it simply means that similar to the static decision making, where the ambiguity averse agent first perceives the worst case distribution from her multiple prior set, and then responds back, here also she forms her belief and react based on the worst case distribution choice by the nature. Henceforth, by p in (1.23) and in the rest of the chapter we mean the equilibrium posterior value, or often for brevity is simply referred as *belief*.

Note that the *rhs* of (1.23) is linear in μ. This is in part due to the effect of $\sqrt{\mu}$ as the volatility term in the ambiguous arm. Consequently, the DM's optimal strategy at every point in time is to either *explore* the ambiguous arm or *exploit* the safe arm[18]. As a result, the DM's value function satisfies the following variational relation:

$$v(p) = \max\left\{r, m(p) - \frac{\sigma^2\delta}{2\alpha} + \frac{1}{2\delta}\Phi(p)v''(p)\right\} \tag{1.24}$$

In the economic terms, r is the DM's reservation value, which can always be achieved regardless of her experimentation strategy. The term $m(p)$ is the expected rate of return from pulling the second arm when the current belief on θ is p. The important term in expression

[17] This equation should also be interpreted in the viscosity sense, by dropping the infimum in defnition 5.

[18] The trade-off between exploration vs. exploitation has studied in different context. For one we can point to [51] that explains such a trade-off for the financial incentives in entrepreneurship.

(1.24) is $\sigma^2\delta/2\alpha$, which we call it *ambiguity cost*. Higher ambiguity aversion, translated to lower α, implies higher incurred cost upon pulling the ambiguous arm. Lastly, $\frac{1}{2\delta}\Phi(p)v''(p)$ is the continuation payoff that the DM could expect by holding on to the second arm. We postpone a more elaborate set of analytical results on the value function to the next subsection and instead present the intuition behind the DM's optimal strategy.

Lemma 3. *The DM's optimal allocation choice with ambiguity aversion α admits the following representation:*

$$\mu^*(p) = \begin{cases} 1 & \text{if } \frac{1}{2\delta}\Phi(p)v''(p) - \frac{\sigma^2\delta}{2\alpha} > r - m(p) \\ \in [0,1] & \text{if } \frac{1}{2\delta}\Phi(p)v''(p) - \frac{\sigma^2\delta}{2\alpha} = r - m(p) \\ 0 & \text{otherwise} \end{cases} \tag{1.25}$$

This result is the analogue of lemma 4 in [5] tailored to capture the ambiguity aversion. One shall think of $r - m(p)$ as the opportunity cost of experimentation that the DM incurs by not choosing the safe arm. Therefore, she only selects the second project when the continuation value of experimentation adjusted by the ambiguity price exceeds its opportunity cost. Particularly, whenever the two values match, the DM can pursue a *mixed strategy*, in that she can allocate her resources between two arms in any arbitrary proportions. However, the Lebesgue measure of the time duration on which she chooses the mixed strategy is zero, precisely because p follows a diffusion process and the middle case in (1.25) never happens $\mathsf{dP} \times \mathsf{Leb}$-a.e. The ambiguity aversion essentially creates a situation in that the DM thinks that upon the continuation she will have to face with the most destructive types of shock distribution, and this already lowers the value of experimentation. Importantly, this loss is independent of the current belief level, and shall be viewed as a fixed cost that ambiguity averse agent must be compensated for to undertake the second project.

1.5 Properties of the value function and comparative statics

In this section we propose closed-form expression for the value function and present sharp comparative statics with respect to ambiguity aversion index α.

Theorem 2. *On the equilibrium path the DM's follows a cut-off experimentation strategy. In particular, there exists $\bar{p} \in [0,1]$ such she selects the safe arm if and only if her posterior belief drops below $\bar{p}$. Further, the value function v is convex on $[0,1]$.*

A substantive result of convexity is that even in the presence of ambiguity aversion the marginal value of *good news* about the second project is increasing.

Next, we want to find a closed-form expression for the value function and particularly the cut-off probability $\bar{p}$. For this we make a technical assumption that turns out to be necessary and sufficient for existence of $\bar{p}$ in $(0,1)$. Namely, we exclude the case $\bar{p}=0$ where DM always pulls the second arm, and $\bar{p}=1$ where she never does.

Assumption 1. *Define* $\eta := \frac{r-\underline{\theta}}{\bar{\theta}-\underline{\theta}} + \frac{\sigma^2\delta}{2\alpha(\bar{\theta}-\underline{\theta})}$. *Then we assume* $\eta < 1$.

As becomes clear later, one can think of η as a lower bound on $\bar{p}$. Therefore $\eta > 1$ essentially means that DM never selects the ambiguous arm. This is due to a combination of two forces, namely a large ratio of safe to ambiguous return — that is the first term in η — and high normalized ambiguity cost — that is the second term in η — which prevents the DM from exploring the second arm. Assumption 1 not only ensures that $\bar{p}<1$, but as it will turn out it implies $\bar{p}>0$. Having made this assumption, on *exploration region* $(\bar{p},1]$ the following differential equation holds:

$$v(p) = m(p) - \frac{\sigma^2\delta}{2\alpha} + \frac{1}{2\delta}\Phi(p)v''(p) \tag{1.26}$$

That has a general solution form[19]

$$v(p) = m(p) - \frac{\sigma^2\delta}{2\alpha} + cp^{1-\lambda}(1-p)^{\lambda}, \quad \text{on } p \in (\bar{p},1]. \tag{1.27}$$

Here c is a constant determined from the boundary condition and $\lambda = \frac{1+\sqrt{1+4\delta\varphi^{-2}}}{2}$, where $\varphi := (\bar{\theta}-\underline{\theta})/\sigma\sqrt{2}$. The *value-matching* (or equivalently *no-arbitrage*) condition implies that the DM should be indifferent between choosing any of the two arms at $p=\bar{p}$. Therefore, $v(\bar{p}) = r$ that yields to

$$v(p) = m(p) - \frac{\sigma^2\delta}{2\alpha} + \left(r - m(\bar{p}) + \frac{\sigma^2\delta}{2\alpha}\right)\frac{p^{1-\lambda}(1-p)^{\lambda}}{\bar{p}^{1-\lambda}(1-\bar{p})^{\lambda}}, \quad \forall p \in [\bar{p},1]. \tag{1.28}$$

The DM faces a *free-boundary* problem, namely she needs to find the optimal cut-off $\bar{p}$. For that we need to apply the *smooth-pasting*[20] condition that imposes the continuity of directional derivatives at $\bar{p}$, i.e $v'(\bar{p}^-) = v'(\bar{p}^+)$. Assumption 1 with some amount of algebra yields to the following expression for the cut-off probability:

$$\bar{p} = \frac{(\lambda-1)\eta}{\lambda-\eta} \tag{1.29}$$

[19] [60] page 547.
[20] [12].

It is positive because $\eta < 1 \leq \lambda$, and is less than one again because $\eta < 1$. This observation now supports making assumption 1.

Remark 2. *The value function in* (1.28) *with the prescribed* $\bar{p}$ *is continuous, increasing and convex. Therefore, its maximum derivative is attained at* $p = 1$*, that is bounded above because* $\lambda > 1$*, thereby satisfying the Lipschitz continuity. Hence,* v *owns all the properties of the verification theorem 1.*

Some comparative statics. The cut-off value is lower-bounded by η. Further, it is increasing in η. Expression (1.29) provides us with a sharp characterization of the cut-off value, and one could perform a number of comparative statics on $\bar{p}$ with respect to the parameters of the model. Here, we only point to two interesting ones. First, and more important is the effect of ambiguity on cut-off value. As DM becomes more ambiguity averse, namely as α becomes smaller, the value of $\bar{p}$ increases unambiguously. This confirms our intuition that a more ambiguity averse DM is more conservative and explores less. Expression (1.29) offers a fine indicator on the *extent* of this under-exploration. The second channel is the effect of $\overline{\theta} - \underline{\theta}$, that represents the *range* of possible return rates under the second arm. As this range shrinks to zero, the ambiguity cost is amplified more intensely, and DM will have less incentive to pick the second project.

As a last note in this section we point out to a concern on the entangled effects of σ and α. One might wonder that what we refer as the ambiguity aversion parameter, i.e α^{-1}, can be dissolved in volatility σ, and thus can never be identified separately even with infinite amount of data. However, this is not true, as we can offer an identification scheme that disentangles α from σ. Suppose that all other parameters are identified, namely r, δ and $\{\bar{\theta}, \underline{\theta}\}$. Then, a continuous stream of agent's belief process would let us to compute the quadratic variation $\langle p, p \rangle = (\bar{\theta} - \underline{\theta})^2 p^2 (1-p)^2/\sigma^2$ from (1.10). Further, by spotting the point where she stops the exploration and pulls the safe arm we can back out $\bar{p}$. These two equations can lead us to uniquely identify σ and α.

1.6 Value of unambiguous information

In this section we aim to study the value of information with respect to which the DM holds no ambiguity. Practically, one can think of a scenario in which the experimentation unit hires an expert to continuously provide her opinion about the *true* rate of return of the ambiguous arm. Some questions naturally arise in this context. For example what is the fair price of such service? Or, how much must the expert be compensated for providing such information? When should the experimentation unit who faces ambiguity hire this expert?

To answer such questions, let x_t be the information that the expert releases at time t about θ, which in its simplest case can be thought as the noisy signal of θ, namely:

$$\mathrm{d}x_t = \theta \mathrm{d}t + \gamma \mathrm{d}W_t \tag{1.30}$$

In this expression W is a $\mathbf{F}$-Brownian motion under the benchmark measure P and is independent of B and θ. Further, γ is the constant volatility that represents the level of DM's confidence in the expert's information. Therefore, the DM can use this signal in addition to the second arm's payoff process to update her belief about θ. Obviously, this new source of information improves the precision of the filtering process, in the sense that it lowers the conditional variance of estimated θ at every point in time. The law of motion for the new posterior process with the presence of unambiguous information source follows the logic of lemma 2:

$$\mathrm{d}p_t^h = p_t^h(1-p_t^h)\left(\bar{\theta}-\underline{\theta}\right)\left[\frac{\sqrt{\mu_t}}{\sigma}\mathrm{d}\bar{B}_t^h + \frac{1}{\gamma}\mathrm{d}\overline{W}_t\right] \tag{1.31}$$

Here $\bar{B}$ and $\overline{W}$ are independent $\mathbf{F}^{y_2,\mu,h,x}$-Brownian motions under P^h. Now we can state the counterpart of theorem 1 in this case, however its proof is easier as the candidate solution belongs to the space of $C^2([0,1])$ thus we do not need the viscosity solution concept. This is owed to the fact that the diffusion coefficient for $\overline{W}$ is independent of μ, thereby relaxing the degeneracy that appears when $\mu = 0$. As a result of restriction to the space $C^2([0,1])$, Ito's lemma can be applied directly on the candidate value function and one can apply the idea of the proof in theorem 1, bypassing the steps dealing with viscosity super(sub)-solution and replacing them with Ito's rule.

Proposition 2. *Suppose $\tilde{v} \in C^2([0,1])$ is the unique solution to the following HJBI equation:*

$$\tilde{v}(p) = \sup_{\mu\in[0,1]} \inf_{h\in\mathbb{R}} \left\{(1-\mu)r + \mu m(p) + \sqrt{\mu}\sigma h + \frac{\alpha}{2\delta}h^2 + \frac{1}{2\delta}\left(\mu\Phi(p;\sigma) + \Phi(p;\gamma)\right)\tilde{v}''(p)\right\} \tag{1.32}$$

In that $\Phi(p;s) := \frac{(\bar{\theta}-\underline{\theta})^2}{s^2}p^2(1-p)^2$. Then, $\tilde{v}$ is indeed the value function in presence of unambiguous information x. In the equilibrium, the worst-case density generator is $h^* = -\alpha^{-1}\sigma\delta\sqrt{\mu^*}$, where μ^* is the DM's best response solving:

$$\tilde{v}(p) = \sup_{\mu\in[0,1]} \left\{(1-\mu)r + \mu m(p) - \frac{\sigma^2\delta}{2\alpha}\mu + \frac{1}{2\delta}\left(\mu\Phi(p;\sigma) + \Phi(p;\gamma)\right)\tilde{v}''(p)\right\} \tag{1.33}$$

Similar to the case with no source of unambiguous information, one can show that the value function is non-decreasing in p and there is a cut-off rule for the optimal experimentation strategy. Let us denote the new cut-off in the presence of unambiguous information with $\tilde{p}$. Then, the value function satisfies the following relation:

$$\tilde{v}(p) = \begin{cases} r + \delta^{-1}\varphi(\gamma)^2 p^2(1-p)^2\tilde{v}''(p) & p < \tilde{p} \\ m(p) - \frac{\sigma^2\delta}{2\alpha} + \delta^{-1}\left(\varphi(\sigma)^2 + \varphi(\gamma)^2\right)\tilde{v}''(p) & p > \tilde{p} \end{cases} \tag{1.34}$$

In that we define $\varphi(s) = \left(\bar{\theta} - \underline{\theta}\right)/s\sqrt{2}$, where $s \in \{\sigma, \gamma\}$. The top term in (1.34) relates to the region where DM selects the safe arm. Importantly, on this region her payoff is no longer r, but has a continuation component that arises from the free information x. In the case without this source, once the DM switches to the safe arm, she will never have the chance to acquire information about θ, thereby her payoff will stuck at r forever. However, in the current situation, the news about θ can still be flowing without DM pulling the second arm, and in the case of *good* news, she would expect to switch back to the second arm. This effect creates the continuation incentives for the DM on the region $(0, \tilde{p})$. At $\tilde{p}$ the continuity condition must hold so any of the two regions in (1.34) could be enclosed. The solution to this piece-wise ordinary differential equation is

$$\tilde{v}(p) = \begin{cases} r + c_1 p^{\lambda_1}(1-p)^{1-\lambda_1} & p < \tilde{p} \\ m(p) - \frac{\sigma^2\delta}{2\alpha} + c_2 p^{1-\lambda_2}(1-p)^{\lambda_2} & p > \tilde{p}, \end{cases} \tag{1.35}$$

where $\lambda_1 = \frac{1+\sqrt{1+4\delta\varphi(\gamma)^{-2}}}{2}$ and $\lambda_2 = \frac{1+\sqrt{1+4\delta(\varphi(\sigma)^2+\varphi(\gamma)^2)^{-1}}}{2}$. There are essentially three parameters to be determined, i.e $(c_1, c_2, \tilde{p})$. The optimal choice of DM is to select these constants so that the three conditions, namely *value-matching* (continuity), *smooth-pasting* (continuity of first derivative) and *super-contact* (continuity of second derivative) hold together. The derivations for this are presented in A. It turns out the new cut-off probability under unambiguous information source is

$$\tilde{p} = \frac{(\Lambda - 1)\eta}{\Lambda - \eta}, \quad \text{for } \Lambda := 1 + \lambda_1\frac{\sigma^2}{\gamma^2} + (\lambda_2 - 1)\left(1 + \frac{\sigma^2}{\gamma^2}\right). \tag{1.36}$$

Proposition 3. *The experimentation cut-off rises unambiguously when there is an unambiguous information source, namely $\tilde{p} \geq \bar{p}$ for all combinations of the variables in the model.*

The content behind this proposition is that the unambiguous source of information in effect raises the bar for exploration, that in turn means DM demands more confidence for selecting the second project. This is very much due to the free information that DM can

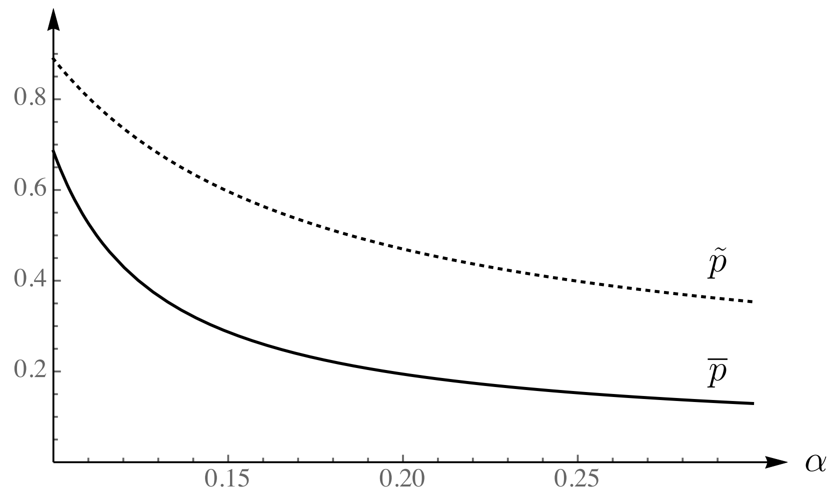


Figure 1.1: Cut-off values

$\left[r = 0.2, \underline{\theta} = 0, \overline{\theta} = 1, \delta = 0.9, \sigma = 0.4, \gamma = 0.3\right]$

acquire about θ without pulling the ambiguous arm. In the standard case, the only way to learn about the quality of the second project is to spend some time exploring that. Therefore, the DM is more willing to sacrifice the certain payoff of the first project to learn about the second one, whereas in the current case she can wait longer for the good news (and exploit the first arm meanwhile) to choose the second arm. In this spirit, as depicted in figure 1.2 the cut-off value rises unambiguously due to the provision of the new information source (i.e $\tilde{p} > \bar{p}$). Also it shows that in both environments the exploration threshold falls as the DM becomes less ambiguity averse, meaning larger values of α. One can think of a situation where the provider of this new source of information is strategic and can charge the DM for the service. Then naturally the maximum price that she can charge is $\tilde{v}(p) - v(p)$, which corresponds to extracting all the surplus from the DM. From the social welfare standpoint the p that maximizes the surplus shall be treated as a benchmark for decision to hire the expert. We refer to $\tilde{v}(p) - v(p)$ as the created surplus due the expert opinion. It is obviously positive and continuous everywhere, and is increasing over $[0, \bar{p}]$. Also as $p \to 1$ it decays to zero faster than $(1-p)^{\lambda_2}$. Therefore, the maximum created surplus occurs at a *moderate* belief value p^*, where $p^* > \bar{p}$ but is not also very close to one. Figure 1.1 presents both value functions, and the created surplus. In that the blue segment of each curve points to the region where the DM pulls the safe arm. We end this section with a remark about the most efficient time to hire an expert.

Remark 3. *The above analysis implies that it is most beneficial for the experimentation*

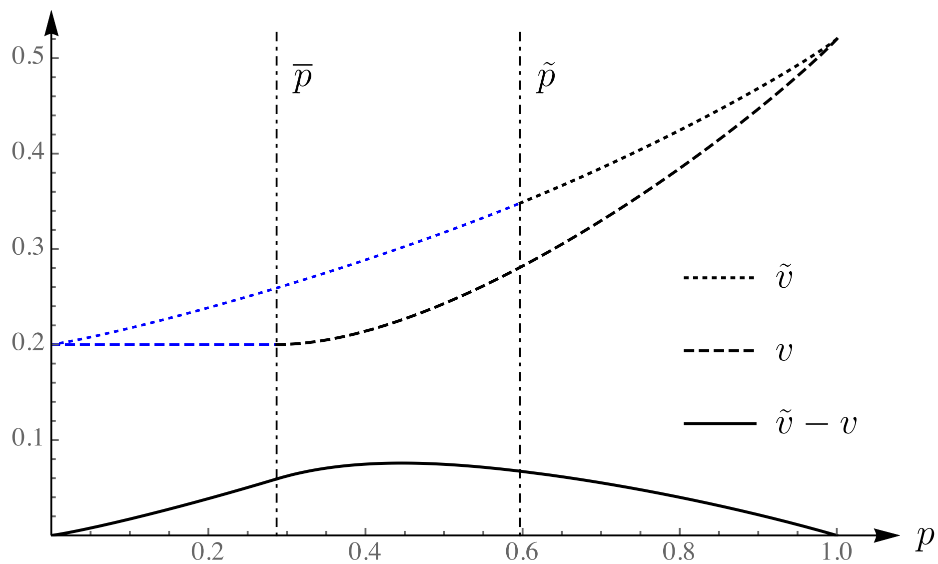


Figure 1.2: Created surplus and value functions

$\left[r = 0.2, \underline{\theta} = 0, \overline{\theta} = 1, \delta = 0.9, \sigma = 0.4, \gamma = 0.3, \alpha = 0.14\right]$

unit to hire an expert when otherwise they would select the ambiguous arm in spite of strong enough evidence and belief.

Chapter 2

Reputation, Innovation, and Externalities in Venture Capital

2.1 Introduction

There have been a number of successful and failed public attempts to promote entrepreneurship, venture capital and innovative finance over the past half-century ([40]). The fundamental rationales for such interventions have been to correct the market failures originating from the existence of information gaps between investors and businesses, and to internalize the positive externalities in the innovative sector. In this chapter, I focus on the latter reason as well as a novel one, the impact of each VC's reputation on the deal flow of the other investors. Both of these move the decentralized outcome away from the social optimum. I argue that any policy aimed to moderate the extent of such market failures in the venture capital industry should take into account two inevitable strains: the *search frictions* due to the absence of a centralized investor/investee market; and the lack of complete information about the *ability* of the investors, prompting the market participants to form rational beliefs and thus rely on the investors' *reputation*. I show that, as a result, the investors decisions are endogenously tilted along the two margins of search frictions and reputation.

On the first margin: when making investment decisions, venture capitalists rationally take into account the opportunity cost of forgoing the investment in the late stage businesses in favor of the early stage ones. Correspondingly, holding everything else constant, higher search frictions decrease this opportunity cost, and hence raise the likelihood of investment in early stage businesses. This effect becomes more prominent when the spillovers from early stage businesses to late stage counterparts are taken into account. Specifically, I show that there are regions where higher search frictions could save the market from a total breakdown created by the *individually rational neglect* of VCs to invest in early stage companies in the hope of receiving a better proposal from a late stage project.

On the second margin: financing expenses for running the startups – that translate to costly learning of investors' types – create a group of investors who have low reputation in spite of their high ability (henceforth referred to as *dormant* investors). I show when higher reputation engenders higher deal flow, the size of the dormant group is sub-optimal. This is because the more reputable investors impose a negative externality on the deal flow of the less reputable ones, which in turn pushes down the value of reputation building for the latter group, thereby increasing the size of the dormant investors. This pattern is also associated with the early termination of projects that kills the startups earlier than a constrained-efficient scheme suggests.

Next, I explain the relevant forces behind each of these two margins, and continue the introduction by shedding more light on the two points raised above.

Search frictions. The market in which venture capitalists operate as the investors and entrepreneurs as the investees is certainly far from a centralized market in which prices equilibrate the demand and supply for capital. For example, there are geographical barriers hindering the connection of remote startups to the centers of VC finance.[1] Even within the financial centers there are informational barriers vis-à-vis availability of the capital for startups and investment opportunities for investors. The VCs create and participate in syndication networks that facilitate the exchange of information about investment opportunities ([68]). However, there still remains a significant amount of unexploited partnerships between startups and VCs that would have otherwise been formed in an informationally and spatially centralized market. These observations motivate us to select the framework of dynamic matching and random search as the basis of the economy that will be studied in this chapter. Specifically, the agents of our *economy* are VCs and startups who randomly meet each other and form partnerships. The speed of such random meetings parametrizes the extent of search frictions in the economy.

Investors' ability and incomplete information. In all industries and especially in the venture capital industry, investors add value to the projects through several channels. The first and foremost one is funding the project, but the focus of this chapter is on the

[1] A testament to that is the substantial concentration of startups and VC funds in few states. According to the data released by NVCA, companies headquartered in three states of California, Massachusetts and New York collectively account for 73% and 79% of the total venture capital spending in 2019 and 2020, respectively. Also, the funds based in these three states represent 92% and 86% of the total venture capital raised in the US, respectively in 2019 and 2020.

other values they provide. A fair body of previous research has pointed out these value-added services. For example, in the surveys done by [31] and [22], VCs responded that they provide a range of post-investment services to their portfolio companies.[2] Two notable studies of [68] and [4] have gone even further by teasing out and identifying the positive treatment effect of the VCs' involvement in their portfolio companies, from the concerns regarding the sorting and selection effects. All in all, the evidence suggests the investors' ability (or lack thereof) has an impact on the success likelihood of their portfolio companies. However, one can not expect that this ability is held by all investors, and naturally some may fail to possess such qualities. Therefore, I assume that investors are endowed with types, indicating their ability, whereby a high type indicates a high-skilled investor and a low type refers to an investor who lacking the aforementioned expertise. At this stage, a crucial decision needs to be made. One needs to choose one of the two relevant information structures: (i) a *learning* model in which neither the VC nor the other market participants know the VC's type; or a (ii) *signalling* model in which the VC is aware of his type and the rest are not. Backed by the empirical validation of [20], I choose the former model, and assume the presumptive VC and the rest of the market have *incomplete yet symmetric information* about each VC's type.

The only publicly verifiable signal, resolving the uncertainty about each VC's type, is the occurrence of the successful outcome in their respective companies. Therefore, whenever a VC pairs up with a startup, a learning opportunity is created for the entire market as well as that particular VC about its type. However, learning is costly, because the associated VC must finance a startup for the learning process to take place. Therefore, at every time a partnership is formed the corresponding VC is going to solve a *stopping time* problem, by which it weighs the value of the match (as a function of its current reputation and the *type* of the partnering startup) against the reservation value – the value of holding current reputation while not being matched to a startup, that is called the reputation value function throughout the chapter. In equilibrium these two value functions are intertwined, and jointly determine the matching sets.

I show that within the space of increasing value functions (as a function of reputation), there is a unique equilibrium. The equilibrium matching sets encode the investment decisions of VCs. Namely, they specify what types of businesses (early versus late stage, or radical versus incremental ideas) an investor with a certain level of reputation chooses to invest. As

[2]The list includes strategic and operational guidance, connecting investors and customers, hiring employees, as well as supervising startup professionalization measures such as HR policies and the adoption of stock option plans. Further studies such as [23] and [41] highlight the VCs involvement and their oversight on the boards of private firms in their portfolios.

a result, I provide a framework to endogenize the so-called measure of *tolerance for failure*, that determines the extent to which VCs exhibit patience on the project they fund ([51] and [70]). Specifically, in the equilibrium, investors with higher reputation exhibit higher tolerance, as the distance of their current reputation to the endogenous separation point of the match is larger. Furthermore, when it comes to cross-company comparison, they show higher patience toward the late stage companies. Also, the equilibrium outcome predicts that, as the cost of financing the startups falls – due to the technological developments, the investors increase the *variety* of the projects they finance. Namely, they start to channel their capital to the early stage projects as well as the late stage ones. This prediction confirms the prevalence of the investment approach, "*spray and pray*", that is presented in [17].

Leveraging the baseline model, I study the outcome of the economy when there are spillovers from successful early stage projects to the late stage investment opportunities. One can alternatively interpret this in the context of knowledge spillovers from radical innovations to incremental ones. At any rate, there are empirical evidences suggesting that small innovative firms are particularly weak in protecting their intellectual property and/or extracting all of their created social rents.[3] Therefore, one should naturally expect not all of the future gains created by investing in early stage ventures are internalized in the decisions of their respective investors, and hence the decentralized outcome of the economy inevitably exhibits under-investment in this group of companies. By solving the social optimum in the planner's problem, I obtain the magnitude and the direction of the decentralizing transfers which satisfy budget neutrality and restore the market efficiency. The optimal redistribution policy features a tax on the reputation value function – the investors' valuation as a function of their reputation while not investing in projects – and a subsidy to the early stage investors.

Next, building on the baseline model, I study the outcome of the economy when there is a reputational externality at play. In particular, I study the direction along which the social optimum of the economy departs from the equilibrium outcome. I show when the matching function (between VCs and startups) accounts for *higher deal flow due to higher reputation* – via the means of a *reputation weight function* – the more reputable investors slow down the deal flow of the less reputable ones, thereby making the latter group less patient by lowering their value of reputation building. This amounts to under-learning and early termination of projects by the novice investors. Specifically, it turns out the equilibrium threshold to terminate the funding from the businesses would be tighter than what is socially optimal. Through a comparative static exercise on the choice of the reputation weight function, I

[3] The public policy report about the New Zealand government's efforts to stimulate the venture capital industry by [43] highlights many of these issues.

further show as the effect of the reputation on the deal flow *wanes*, the threshold to terminate the funding and end the partnerships tightens, while the opportunity cost of investing in early stage projects falls.

The results set forth above on the relation between reputation and deal flow are related to a substantial body of previous research that studies the *individual benefits* associated with higher reputation. The findings include the theory of grandstanding, and lower pay-for-performance for smaller and younger VC firms toward the goal of establishing a reputation and enjoying a higher deal flow ([21] and [20]); Or how VCs with higher reputation acquire startup equity at a discount ([32]). Relatedly, by dissecting investment-level data [56] finds that initial success confers preferential access to deal flow and perpetuates the early superior performances made by successful VCs. However, in contrast to what I study in this chapter, none of these previous studies investigates the social return and the aggregate outcome when the deal flow of a single investor depends on the average reputation weight of the remaining body of the investors.

Other related literature. This analysis is also related to a group of other works, [67], [16] and [64], that study the search and matching between VCs and entrepreneurs in an environment where VCs' types are perfect knowledge thus there is no room for reputation building and learning. On the macroeconomic impact of VC sector, [57] develops an endogenous growth model in which VCs' intermediation in conjunction with entrepreneurs' ideas and labor contribute to the aggregate growth. In another work, [1] develops a static equilibrium model with perfect information on agents' types that captures and estimates the positive assortative matching between entrepreneurs and VCs.

Organization of the current chapter. In section 2.2, I present the baseline model of the economy with VCs and startups as the agents and meetings subject to search frictions. The equilibrium value functions and matching sets are determined and the investors trade-offs vis-à-vis projects are explained. In section 2.3, I express the economy's social surplus and verify the constrained-efficiency. The learning outcome of the economy, namely the extent to which the decentralized outcome can uncover the venture capitalists' types is studied in section 2.4. In section 2.5, the inflow of the late stage projects in the economy are endogenized by letting them to be proportional to the mass of successful early stage projects. In addition, the first case of market failure, in which investors fail to internalize the spillovers from early to late stage projects, is shown in this section. In section 2.6, the matching function exhibits the reputational externality, accounting for the fact that higher reputation increases the deal flow. In light of this externality, I establish the theoretical grounds behind the second case of market failure, and show the direction along which the equilibrium outcome departs from the social optimum. All the proofs and verifications that are not stated in the main body are relegated to the appendix.

2.2 Equilibrium in the baseline economy

In this section, I describe the constituents of an economy populated by a unit mass of long-lived venture capitalists (investors) and a continuum of startup projects (investees).

Investors (supply side). The agents in this side of the market are the long-lived investors, who care about their reputation, which is the market posterior belief about their type $\theta \in \{L, H\}$. Throughout this chapter I take venture capitalists as the leading example for the investor side. Given the market filtration $\boldsymbol{I} = \{\mathcal{I}_t\}$, $\pi_t = \mathsf{P}(\theta = H | \mathcal{I}_t)$ refers to the time t reputation of a generic VC. The σ-field $\mathcal{I}_t$ aggregates all information that market participants hold at time $t \in \mathbb{R}_+$. The share of high-type VCs is equal to $\mathfrak{p}$, exogenously set and publicly known.

Investees (demand side). The entities on the demand side of the market are simply treated as investment opportunities that are chosen by the investors. Specifically, they have no bargaining power against investors.[4] The leading case for investees throughout the chapter are the startups. Each startup is endowed with a type $q \in \{a, b\}$, which is publicly observable. The (unnormalized) mass of type-q projects is φ_q for $q \in \{a, b\}$.

Matching and partnerships. Pairwise meetings between agents in two sides of this market take place. The meetings are subject to search frictions with the meeting rate $\kappa > 0$, and the matching technology is *quadratic* à la [8] (and the references therein), that is the probability with which a generic VC meets a type-q startup over the period $\mathrm{d}t$ is $\kappa\varphi_q\mathrm{d}t$. Furthermore, the matches are one to one, that is both parties have capacity constraint over

[4]This assumptions makes the analysis substantially simpler, yet it downplays the strategic role of startups in the equilibrium outcomes. However, given the chapter's focus on the VC side and their reputational concerns such an abstraction seems plausible. Also from the empirical standpoint there are evidences suggesting that venture-backed firms can continue their projects without their original entrepreneurs; see [72] and the references therein such as [23] and [31].

the number of partners they can match with.

Output and reputation. Given the partnership between a type-θ VC and a type-q startup, the success arrives at the rate $\lambda_q(\theta)$, where $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, with normalized payoff of one. The VC has to cover the flow cost of project $c > 0$ that is common across all matches. In return, she receives the right to terminate the funding at her will, so conceptually a stopping time problem is solved by each VC ex post to every partnership formation. I assume there is a mechanism in the market that tracks the output of each partnership and records the Bayes-updated posterior of every VC during its match. This information is reflected in the market filtration $\boldsymbol{I}$. The posterior dynamics for the reputation process thus follows

$$\mathrm{d}\pi_t = -\pi_t(1-\pi_t)\Delta_q \mathrm{d}t, \tag{2.1}$$

prior to the success, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. For the purpose of simplicity, I assume the breakthroughs are *conclusive* in the sense that $\underline{\lambda}_q = 0$, that is the success never happens to a low type VC, therefore upon the success event π_t immediately jumps up to one.[5] Further, without loss of generality it is assumed $\lambda_b := \bar{\lambda}_b > \lambda_a := \bar{\lambda}_a$. Also, I assume $\mathfrak{p} > c/\lambda_b$ throughout, because otherwise there are cases in which even the high-type projects are not worth the investment.

Figure 2.1 summarizes the dynamic timeline of the investment path for a generic venture capitalist, who starts the cycle with reputation π, and after some exponential random time meets a startup randomly drawn from the population of unmatched entrepreneurs. A decision to accept or reject the contacted startup is made by the VC; Upon rejection the VC returns to the initial node, and conditioned on acceptance an investment problem with the flow cost of c is solved by the VC. Finally, a success or a failure at the terminal node guides the entire market participants to rationally update their beliefs about the ongoing VC, and the associated VC returns back to the pool of available investors.

Value functions and matching sets

The rate of time preference for investors in this economy is $r > 0$. Let $w(\pi)$ be the value of holding reputation π, when the VC is *unmatched*. This function shall be treated as the

[5] In the supplementary appendix B.1, I relax this assumption and study the general case, where the success is not necessarily conclusive and the there is a continuum of projects with the type space $[a, b]$ distributed according to an *arbitrary* CDF function ϕ. On a farther note, the notion of *conclusive breakthroughs* is studied by [37] in the context of strategic experimentation, and in a follow-up paper [35] highlight the technical contrasts with the case of *inconclusive breakthroughs*.

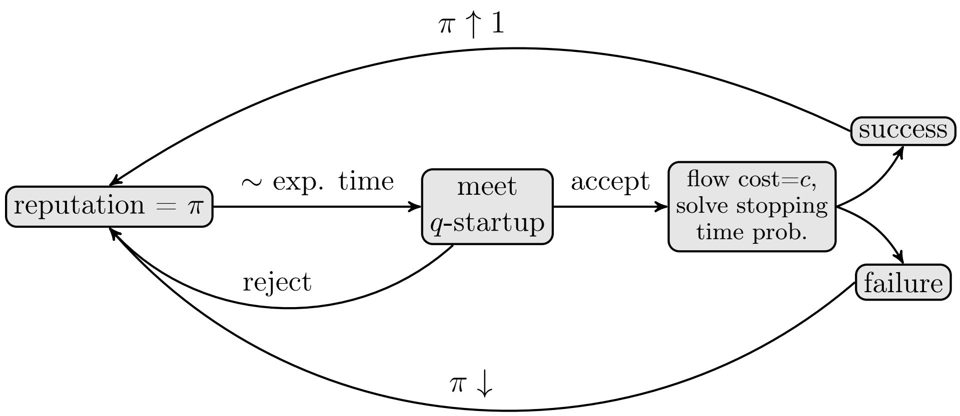


Figure 2.1: Investment timline for a generic VC

VC's outside option and is weighed against the *matching value function* upon the meetings. The matching value function when a reputation-π VC pairs up with a type-q startup is $v(\pi, q)$, that is the expected value of discounted future payoffs generated by this partnership. Therefore, a match is *profitable* if $v(\pi, q) > w(\pi)$, in that case I say $(q, \pi) \in \mathcal{M} \subseteq \{a, b\} \times [0, 1]$, where $\mathcal{M}$ is called the matching set. Also, understood from the context, $\mathcal{M}(\pi)$ (resp. $\mathcal{M}_q$) refers to the π (resp. q) *section* of this set.[6] In addition, often in the text I use the indicator function $\chi_q(\pi)$ to denote whether a reputation-π VC matches with a q-startup, that is whether $(q, \pi) \in \mathcal{M}$ or not. Recall that φ denotes the stationary mass of available startups in the economy (that are so far treated exogenously as the primitives of the model). Below, I invoke a standard dynamic programming analysis for $w(\pi)$:

$$\begin{aligned} w(\pi) \approx \kappa \sum_{q \in \mathcal{M}(\pi)} \left(w(\pi) + [v(\pi, q) - w(\pi)]\right) \varphi_q \mathrm{d}t + \kappa \sum_{q \in \{a,b\} \setminus \mathcal{M}(\pi)} w(\pi) \varphi_q \mathrm{d}t \\ + \left(1 - \kappa \varphi(\{a, b\}) \mathrm{d}t\right) (1 - r \mathrm{d}t) w(\pi) \end{aligned} \tag{2.2}$$

The first term in the *rhs* is the expected value of payoffs generated from all *acceptable* matches, taking into account that the next project with type q arrives at the rate of $\kappa\varphi_q$. The second term is the expected payoff over all *denied* partnerships, and the third term simply refers to the discounted payoff conditioned on receiving no investment proposal over the period $\mathrm{d}t$. Accounting for these three sources, the following Bellman equation for the

[6]That is for example $\mathcal{M}(\pi) = \{q : (q, \pi) \in \mathcal{M}\}$ and $\mathcal{M}_q = \{\pi : (q, \pi) \in \mathcal{M}\}$.

reputation value function w is resulted:

$$rw(\pi) = \kappa \sum_{q \in \mathcal{M}(\pi)} \left[v(\pi, q) - w(\pi) \right] \varphi_q \tag{2.3}$$

Next, I examine the matching value function $v(\pi, q)$. Imagine a partnership of a VC with initial reputation π and a type-q startup. Let σ represent the random exponential time of success with unit payoff and the arrival intensity of λ_q if $\theta = H$. Therefore, the matching value function $v(\cdot, q)$ is an endogenous outcome of a *free-boundary* problem with the outside option w. In that the VC selects an optimal stopping time τ, upon which she stops funding the project, taking into account the project's success payoff and her reputation value w:

$$\begin{aligned} v(\pi, q) = \sup_{\tau} \Bigg\{ &\mathsf{E} \left[e^{-r\sigma} - c \int_0^{\sigma} e^{-rs} \mathrm{d}s + e^{-r\sigma} w(\pi_\sigma); \sigma \leq \tau \right] \\ &+ \mathsf{E} \left[-c \int_0^{\tau} e^{-rs} \mathrm{d}s + e^{-r\tau} w(\pi_\tau); \sigma > \tau \right] \Bigg\} \end{aligned} \tag{2.4}$$

The exit option upon the stopping time τ is the VC's reservation value of holding reputation π_τ. The corresponding HJB representation for this stopping time problem is

$$rv(\pi, q) = \max \left\{ rw(\pi), -c + \lambda_q \pi \left(1 + w(1) - v(\pi, q)\right) - \lambda_q \pi (1 - \pi) v'(\pi, q) \right\}. \tag{2.5}$$

The above HJB is presented in the *variational* form, that is the first expression in the *rhs* is the value of stopping – refusing the match and holding on to the outside option w – and the second expression represents the Bellman equation over the *continuation region* $\mathcal{M}_q$, on which $v(\pi, q) > w(\pi)$. The first term in the Bellman expression is the flow cost of the project borne by the VC, the second term is the created surplus upon the success event that is the unit payout added to the value of holding the maximum reputation ($\pi = 1$) minus the current value of the match, and the last term captures the marginal reputation loss due to the lack of success. Induced by the above stopping time problem, the matching set $\mathcal{M}$ can thus be interpreted as the continuation set for the free-boundary problem (2.5), namely

$$\mathcal{M} = \left\{ (q, \pi) \in \{a, b\} \times [0, 1] : v(\pi, q) > w(\pi) \right\}, \tag{2.6}$$

and on the stopping region $\mathcal{M}^c$, the matching value function equals the VC's reputation function, i.e $v(\pi, q) = w(\pi)$.

Equilibrium construction

The goal of this section is to progressively suggest the necessary conditions pinning down the equilibrium outcome and finally express the properties of the matching sets in equilibrium.

Definition 6 (Stationary equilibrium)**.** *Given the mass function φ for the unmatched startups, the tuple $\langle w, v, \mathcal{M} \rangle$ constitutes a stationary equilibrium, if (i) given v and $\mathcal{M}$, the reputation value function w satisfies* (2.3)*; (ii) Given w, the matching value function v and the matching set $\mathcal{M}$ together solve the free boundary system* (2.5) *and* (2.6)*.*

The two-way feedback between the reputation function w and the matching variables $\langle v, \mathcal{M} \rangle$ are portrayed in figure 2.2. The link connecting w to the $\langle v, \mathcal{M} \rangle$ block is upheld by the stopping time problem (2.4), and its recursive representation (2.5). The reverse link from the matching variables block to w is supported by the Bellman equation for the reputation function (2.3). Then, the stationary equilibrium is formally the fixed-point to the endogenous loops of figure 2.2.

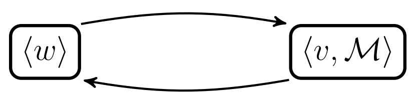


Figure 2.2: Equilibrium feedbacks

Next lemma uses (2.3) to express the reputation value function in terms of v and $\mathcal{M}$, and thereby provides a partial characterization of matching sets only in terms of the matching value functions. Its proof is offered in the supplementary material B.2.

Lemma 4. *A VC with reputation π accepts both types of companies, namely $\pi \in \mathcal{M}_a \cap \mathcal{M}_b$ iff*

$$v(\pi, a)\left(1 - \frac{1}{1 + r^{-1}\kappa\varphi_a}\right) < v(\pi, b) < v(\pi, a)\left(1 + \frac{1}{r^{-1}\kappa\varphi_b}\right). \tag{2.7}$$

In addition, $\pi \in \mathcal{M}_b \cap \mathcal{M}_a^c$ iff the upper-bound is achieved, $\pi \in \mathcal{M}_a \cap \mathcal{M}_b^c$ iff the lower bound is achieved, and $\pi \in \mathcal{M}_a^c \cap \mathcal{M}_b^c$ iff the upper and lower bounds coincide, which is only the case where all value functions are zero.

Intuitively, this lemma asserts that the ratio $v(\pi, b)/v(\pi, a)$ always lies in a bounded interval for $\pi \in \mathcal{M}_a \cup \mathcal{M}_b$. At its maximum where it reaches the upper bound, the VCs do not invest in a-projects and alternatively, when it hits the lower bound, the investors only choose the a-startups. This analysis renders much of the results in the next proposition on the equilibrium shape of the matching sets.

Throughout the analysis I seek to construct equilibria with increasing[7] value functions in π. Specifically, in the baseline model and its proceeding extensions the focus is given to increasing functions $v(\cdot, q)$ and $w(\cdot)$ in π. Toward this construction, let us rewrite equation (2.3) as

$$w(\pi) = \frac{r^{-1}\kappa\left[v(\pi,a)\varphi_a\chi_a(\pi) + v(\pi,b)\varphi_b\chi_b(\pi)\right]}{1 + r^{-1}\kappa\left[\varphi_a\chi_a(\pi) + \varphi_b\chi_b(\pi)\right]}. \tag{2.8}$$

First, note that the proof of lemma 4 (presented in supplementary section B.2) will fall out of a case by case *guess and verify* over the relative orderings of $v(\pi, a)$, $v(\pi, b)$ and $w(\pi)$. Second, this representation of $w(\pi)$ and lemma 4 allow us to express the *equilibrium* reputation value function w as the output of a maximization problem over the space of all Borel measurable indicator functions $\chi_q(\pi)$ (similar idea to lemma 1 of [66]):

$$w(\pi) = \max_{\chi}\left\{\frac{r^{-1}\kappa\left[v(\pi,a)\varphi_a\chi_a(\pi) + v(\pi,b)\varphi_b\chi_b(\pi)\right]}{1 + r^{-1}\kappa\left[\varphi_a\chi_a(\pi) + \varphi_b\chi_b(\pi)\right]}\right\} \tag{2.9}$$

An important consequence of the above representation is that if $v(\cdot, a)$ and $v(\cdot, b)$ are increasing in π, then it would be case that $w(\cdot)$ is increasing in π as well. The reverse direction is the result of the following lemma. This lemma assures us that in any equilibrium an increasing pair of matching value functions lead to an increasing reputation value function and vice-versa.

Lemma 5. *The matching value functions $\{v(\cdot, q) : q \in \{a, b\}\}$ are increasing in π if and only if $w(\cdot)$ is increasing in π.*

Continuing the path toward equilibrium construction, I would now analyze the Bellman equation for the matching value functions. In the sequel, I repeatedly use the general solution form for the Bellman equation (2.5) on the continuation region $\mathcal{M}_q$, in that $\varpi(q)$ is the constant dependent on the appropriate boundary conditions:

$$v(\pi, q) = -\frac{c}{r} + \frac{\lambda_q}{r + \lambda_q}\left(1 + w(1) + \frac{c}{r}\right)\pi + \varpi(q)\left(1 - \pi\right)^{1 + r/\lambda_q}\pi^{-r/\lambda_q} \tag{2.10}$$

To further examine the essence of the stopping time (2.5), I point out to two *necessary* conditions that the optimal matching value function and the continuation region must satisfy[8]. The dynamics of the reputation process can be compactly represented by $\mathrm{d}\pi_t =$

[7] I use the word increasing to refer to a non-decreasing function.

[8] These two conditions are standard in the literature of optimal stopping time and can be found in chapter 2 of [59].

$(1-\pi_{t^-})\left[\mathrm{d}\iota_t - \lambda_q \pi_{t^-}\mathrm{d}t\right]$, in that ι is the success indicator process, that is $\iota_t := 1_{\{t\geq\sigma\}}$. The infinitesimal generator associated to this stochastic process is $\mathcal{L}_q : C^1[0,1] \to C^1[0,1]$, where for a generic $u \in C^1[0,1]$[9]:

$$\left[\mathcal{L}_q u\right](\pi) = \lambda_q \pi \left(1 + w(1) - u(\pi)\right) - \lambda_q \pi (1-\pi) u'(\pi) \tag{2.11}$$

For every candidate equilibrium tuple $\langle w, v, \mathcal{M}\rangle$, the following two conditions must hold for all $\pi \in [0,1]$ and $q \in \{a, b\}$:

(i) *Majorant property*: $v(\pi, q) \geq w(\pi)$.

(ii) *Superhamonic property*: $[\mathcal{L}_q v](\pi, q) - rv(\pi, q) - c \leq 0$.

The first condition simply means that in every partnership the VC has the option to terminate the funding, thus enjoying her reputation value w by severing the match. The second condition means *on expectation* a generic VC *loses* if she decides to invest on the stopping region. The following proposition establishes a set of descriptive properties of equilibrium when the value functions are increasing and belong to $C^1[0,1]$. In doing so, it is important to recall that because of continuity of value functions the sections of matching sets, $\mathcal{M}_a$ and $\mathcal{M}_b$, are open subsets of $[0,1]$. So, to characterize them, it is sufficient to identify their boundary points. For this I employ lemma 4 and the above two optimality conditions in conjunction with $\lambda_b > \max\{\lambda_a, c\}$ to identify these boundary points, thereby the equilibrium shape of the matching set. As it turns out there appear two distinct equilibrium regimes, *low* and *high cost*, that respectively correspond to $\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$ and $\lambda_a - c \leq \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$.

Proposition 4 (Equilibrium shape of the matching sets)**.** *In every stationary equilibrium with increasing value functions belonging to* $C^1[0,1]$*, the following properties hold:*

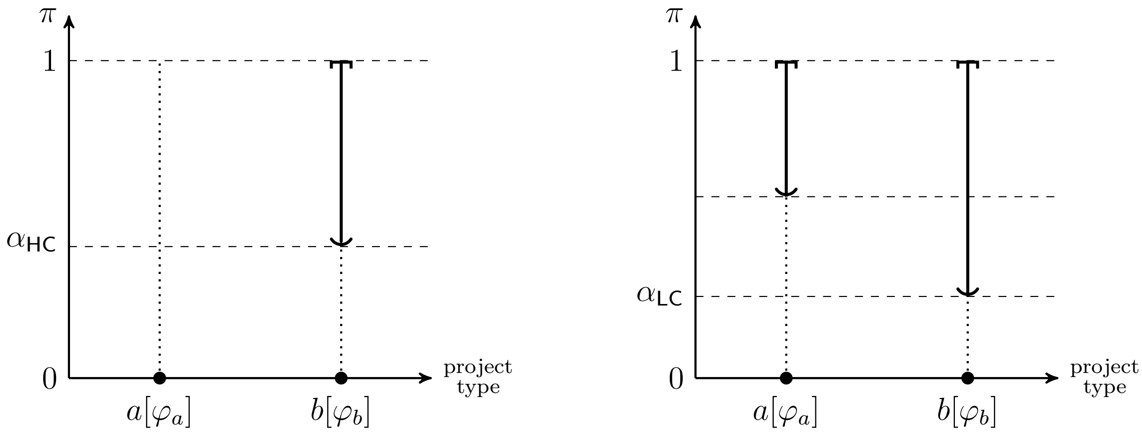
(i) (Status of $\pi = 1$*): in both regimes* $1 \in \mathcal{M}_b$*, and* $1 \in \mathcal{M}_a$ *only in the low cost regime.*

(ii) In both regimes the matching set $\mathcal{M}_b$ *is a connected subset of* $[0,1]$*.*

(iii) In the high cost regime $\mathcal{M}_a = \emptyset$ *and in the low cost regime* $\mathcal{M}_a$ *is a connected subset of* $\mathcal{M}_b$*.*

Figure 2.3 illustrates the equilibrium matching sets in both cost regimes. There are a few points related to this result that should be raised. First, it is the comparison between

[9] Space of continuously differentiable functions on $[0,1]$.



(a) high cost matching sets

(b) low cost matching sets

Figure 2.3: Equilibrium matching sets

the expected flow payoff of investing in a-projects and the opportunity cost of forgoing the wait for the next b-projects that determines the cost regime:

$$\text{low cost regime} \Leftrightarrow \lambda_a - c > \underbrace{\frac{\kappa\varphi_b(\lambda_b - c)}{r + \lambda_b + \kappa\varphi_b}}_{\substack{\text{opportunity cost of forgoing} \\ \text{the wait for a b-project}}} \tag{2.12}$$

For instance, as the share of available b-projects (φ_b) increases, the opportunity cost of investing in a-projects increases and consequently VCs become more reluctant to invest in a-companies. Second, one can verify that lowering c increases the expected flow payoff of investment in a-startups more than it does the opportunity cost component, thereby enhancing the *variety* of financed projects. Therefore, the equilibrium response observed in the matching sets confirms the prevalence of the investment approach "*spray and pray*" that arises due to the cost-reducing technological shocks, mentioned in [17]. Third, this model suggests a method to endogenize the *tolerance for failure* (see [70] and [51]) by relating it to the investor's reputation.[10] The equilibrium observation in figure 2.3 on *connectedness* of the matching sets advances the idea that VCs with higher reputation have higher tolerance for failure. In other words, the distance to the endogenous separation point α is larger for a more reputable VC than a less reputable one. Furthermore, when it comes to cross-company comparison, the investors show more patience toward b-companies – that confer faster success

[10]Specifically, in [70] VCs learn about the quality of the startup over the course of the match, whereas in my model the startup's quality is observable and the learning is about VCs' type. Consequently, the scheme here suggests one way to endogenize the *tolerance parameter* ϕ in [70].

time on average. Fourth, in light of $\mathcal{M}_a \subset \mathcal{M}_B$ the model offers the testable prediction that VCs who exit the market and do not raise subsequent funds made their last few investments in the high-growth companies (i.e b-startups). Formally, in both panels of figure 2.3 we see that the endogenous termination point α is the lower boundary point of $\mathcal{M}_b$ (not $\mathcal{M}_a$), at which the matching value function $v(\cdot, b)$ *smoothly* meets the zero function (as shown in the proof of the last proposition).[11] Also in the proof, it is established that in equilibrium

$$\alpha = \frac{c}{\lambda_b\left((1 + w(1)\right)}, \tag{2.13}$$

where $w(1)$ is the value of holding maximum reputation, i.e $\pi = 1$, in each cost regimes. In the high cost regime $w(1)$ only depends on the b-parameters, because $\mathcal{M}_a = \emptyset$, whereas in the low cost regime it takes the a-related parameters into account as well. Some easy-to-verify comparative statics (for instance in the former case) are $\frac{\partial \alpha}{\partial c} > 0$, $\frac{\partial \alpha}{\partial \lambda_b} < 0$ and $\frac{\partial \alpha}{\partial \varphi_b} < 0$.

Having known the form of the matching sets that are sustained in the equilibrium, I can now state the main theorem related to the decentralized behavior of this economy, i.e the fixed-point outcome of figure 2.2.

Theorem 3 (Stationary equilibrium, existence and uniqueness)**.** *There exists a unique stationary equilibrium in the space of continuously differentiable and increasing payoff functions in each cost regime. Further, for large values of discount rate r, this equilibrium is unique in the bigger space of continuous functions $C[0, 1]$.*

The substantial result of this theorem is that there always exists an equilibrium tuple in which the value functions are increasing and continuously differentiable in reputation. Furthermore, there is not a possibility for multiple equilibria of such kind. However, the possibility of other equilibria with non-increasing value functions can not be ruled out unless the discount rate is large enough so that a contraction theorem type method can be applied.

2.3 Social surplus

The economy as stated thus far exhibits no externality, because the investment decision made by every venture capitalist, regardless of its reputation level, has no impact on the deal flow of other investors. First, this is owed to the fact that the mass of unmatched startups are treated exogenously, and not impacted by VCs actions. Second, the matching technology

[11] It is shown in the proof of proposition 4, the smooth pasting and value matching at α is ensued in spite of the Poissonian environment and the absence of diffusion processes.

exhibits no interaction effect from one group of VCs to another. Therefore, one would expect that the equilibrium proposed in the previous section is constrained-Pareto-optimal.

To fix ideas, the planner's problem is to maximize the social welfare, taking the matching set $\mathcal{M}$ as the control variable:

$$S(\mathcal{M}) = \int_{[0,1]} w(\pi) G(\mathrm{d}\pi) + \sum_{q \in \{a,b\}} \int_{\mathcal{M}_q} v(\pi, q) F(\mathrm{d}\pi, \{q\}), \tag{2.14}$$

The measures G and F are the steady state distributions of unmatched and matched VCs, subject to $w, v \geq 0$ and be continuous increasing functions. Requiring increasing value functions means that $\mathcal{M}_a \cup \mathcal{M}_b$ must be connected and contain $\pi = 1$.[12] Connectedness also implies that in the steady state the matched distribution F only places positive mass at $\pi = 1$, and the support of the unmatched distribution G is comprised of the lowest boundary point denoted by $\alpha := \inf \mathcal{M}_a \cup \mathcal{M}_b$ and the highest boundary point 1. Note that $F(\{\alpha\}, \{a, b\})$ must be zero because the reputation process spends no time at this point, as it either immediately drops below α and hence not belongs to $\mathcal{M}$ anymore, or has already jumped up to 1 before reaching α. Therefore, $\alpha \notin \mathcal{M}_a \cup \mathcal{M}_b$, and since the VCs holding such a reputation will never be rematched again then $w(\alpha) = 0$. Consequently, the steady state population of VCs can be summarized by *four* distinct masses: $n(\alpha)$ VCs trapped at α, in addition to three other groups with maximum reputation, $n(1)$ unmatched, $m_a(1)$ matched to a-companies and $m_b(1)$ matched to b-companies. Inflow outflow equations together with the Bayesian consistency at the steady state amount to

$$n(\alpha) + n(1) + m_a(1) + m_b(1) = 1 \tag{2.15a}$$

$$\alpha n(\alpha) + n(1) + m_a(1) + m_b(1) = \mathfrak{p} \tag{2.15b}$$

$$\kappa n(1) \varphi_a \chi_a(1) = \lambda_a m_a(1) \tag{2.15c}$$

$$\kappa n(1) \varphi_b \chi_b(1) = \lambda_b m_b(1) \tag{2.15d}$$

The first equation simply says that the total mass of VCs is one. The second equation states that in the steady state the average ability of VCs must be equal to the initial average ability $\mathfrak{p}$. The third (resp. fourth) expression equates the inflow to the group of VCs investing in a-startups (resp. b-startups) to its outflow (that is the rate at which these projects experience success thus their corresponding VCs exit their position and become unmatched). These distributional results help us in the next proposition in which I analyze the constrained-efficiency of this economy. Specifically, in the next proposition I treat the choice of the

[12]Since $(\mathcal{M}_a \cup \mathcal{M}_b)^c = \{\pi : w(\pi) = 0\}$, then an increasing w means $\{\pi : w(\pi) > 0\} = \mathcal{M}_a \cup \mathcal{M}_b$ must be a connected set.

matching sets as the only instrument of a benevolent planner and prove the constrained efficiency.

The central trade-off in the choice of the matching sets in the planner's problem is between the benefits of lowering α, thus increasing the size of active investors in the economy, and its associated cost born by the VCs as a result of longer financing periods. Lowering α on one hand increases the pool of active VCs and improves the *learning* prospects of the economy by letting VCs to experiment longer, thereby ending up owning more *certain* beliefs about their skills. On the other hand, learning about their know-how abilities is costly, thus limiting the scope of perfect learning and resolution of investors' types. The welfare result in the next proposition asserts that in the absence of externalities among VCs and the presence of exogenous flow of startups, i.e φ_a and φ_b not being impacted by investors' decisions, the economy is constrained-efficient. Both of these premises are relaxed in the proceeding sections.

Proposition 5. *The equilibrium matching sets characterized in proposition 4 are constrained-Pareto-optimal.*

2.4 Imperfect learning

The welfare analysis offered above sheds some light on the close connection between the learning outcome and the social surplus. Specifically, it was shown that there exists a mass of $n(\alpha)$ of VCs who are inactive and no longer raise funds and take on projects, some of whom indeed have the expertise and the know-how yet failed to prove it in their first few investments. Henceforth, I refer to this group as *dormant* investors. In this section, I aim to study the steady state distribution of VCs reputation, its distance to the *perfect learning benchmark*, and its connection to the social surplus of the economy. The analysis in the proof of previous proposition as well as the shape of the matching sets suggest that in the stationary equilibrium there is a non-zero mass of VCs trapped at α_b, the lower-boundary point of $\mathcal{M}_b$. The main reason behind the existence of this group is that learning is costly, therefore at some point the cost does not rationalize the expected payoff and the learning stops.

Distance to perfect learning. Constrained by the search frictions the maximum created surplus, that is also achieved in the equilibrium, as found in proposition 5 follows

$$rS_{\mathsf{HC}} = \frac{\mathfrak{p} - \alpha_{\mathsf{HC}}}{1 - \alpha_{\mathsf{HC}}} \frac{\kappa\varphi_b/\lambda_b}{1 + \kappa\varphi_b/\lambda_b} (\lambda_b - c), \tag{2.16a}$$

$$rS_{\mathsf{LC}} = \frac{\mathfrak{p} - \alpha_{\mathsf{LC}}}{1 - \alpha_{\mathsf{LC}}} \frac{1}{1 + \kappa\varphi_a/\lambda_a + \kappa\varphi_b/\lambda_b} \left(\frac{\kappa\varphi_a}{\lambda_a} (\lambda_a - c) + \frac{\kappa\varphi_b}{\lambda_b} (\lambda_b - c) \right), \tag{2.16b}$$

where HC stands for the high-cost and LC for the low-cost regimes. Furthermore, α_{HC} (resp. α_{LC}) is the lower boundary point of $\mathcal{M}_b$ in the high (resp. low) cost regime, that follows equation (2.13). It is helpful to examine the distance between the steady state distribution of VCs' reputation, denoted by P_∞, and its perfect learning benchmark, denoted by $\mathsf{P}^* = (1-\mathfrak{p})\delta_0 + \mathfrak{p}\delta_1$.[13] These two probability measures assume different supports thus are not absolutely continuous with respect to each other. Therefore, I choose the *total variation* as a natural candidate for their distance. Let $\mathcal{B}[0,1]$ be the Borel σ-field on the unit interval, then

$$\mathsf{d}_{\mathsf{TV}}(\mathsf{P}_\infty, \mathsf{P}^*) = \sup \{ |\mathsf{P}_\infty(A) - \mathsf{P}^*(A)| : A \in \mathcal{B}[0,1] \}. \tag{2.17}$$

In both regimes $\mathsf{P}_\infty = \frac{1-\mathfrak{p}}{1-\alpha_b}\delta_{\alpha_b} + \frac{\mathfrak{p}-\alpha_b}{1-\alpha_b}\delta_1$, but with different α_b's resulted from distinct $w(1)$'s in (2.13). Then a simple analysis for $\alpha_b \le \mathfrak{p} < 1$ yields

$$\mathsf{d}_{\mathsf{TV}}(\mathsf{P}_\infty, \mathsf{P}^*) = (1-\mathfrak{p}) \max \left\{ \frac{\alpha_b}{1-\alpha_b}, \frac{1}{1-\alpha_b}, 1 \right\} = \frac{1-\mathfrak{p}}{1-\alpha_b}. \tag{2.18}$$

A substantial result of this analysis is that a lower distance between steady state reputation measure and the perfect learning benchmark is associated to lower values of α_b and corresponds to higher welfare outcomes followed from (2.16). Notice that one can interpret the surplus expressions in (2.16) as the product of the extensive margin and the intensive margin, for example in the high-cost regime:

$$rS_{\mathsf{HC}} = \underbrace{\frac{\mathfrak{p} - \alpha_{\mathsf{HC}}}{1 - \alpha_{\mathsf{HC}}} \frac{\kappa\varphi_b/\lambda_b}{1 + \kappa\varphi_b/\lambda_b}}_{\text{extensive margin}} \underbrace{(\lambda_b - c)}_{\text{intensive margin}} \tag{2.19}$$

In an ideal environment the type of every investor is known to herself and to the public, thus only the high-skilled VCs with the mass of $\mathfrak{p}$ would invest and the others stay inactive, corresponding to the maximum of the extensive margin with $\alpha = 0$, and reaching the maximum surplus $S_{\max}$. In figure 2.4 the ratio of the equilibrium surplus in the high cost

[13] A unit mass concentrated on x is denoted by δ_x.

regime over the maximum surplus (when the learning is perfect) is plotted as a function of α, that is clearly decreasing, supporting the fact that a closer distance to the perfect learning benchmark is associated with smaller loss of surplus. Further in this graph, I plotted the matching value function as a function of the reputation π, that is shown to smoothly meet the horizontal axis at α. Thus, any policy that is aimed to push down the termination point α toward the origin, equivalently easing up the financial costs for VCs, or encouraging higher tolerance for failure reduces the welfare gap.

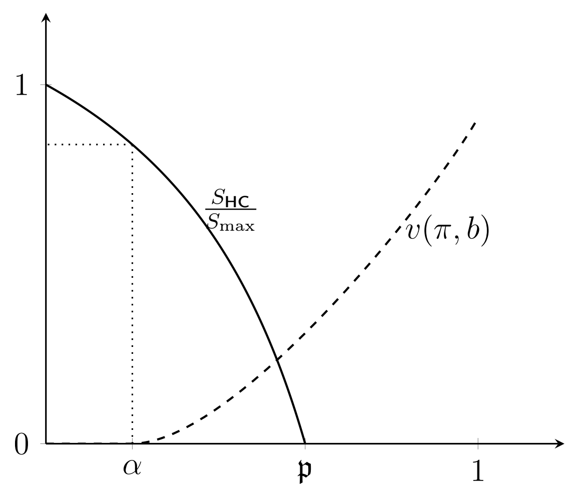


Figure 2.4: Surplus and the extent of imperfect learning

2.5 Early/Late stage and endogenous mass of projects

In the previous sections we saw that the investment decisions made by VCs are actually constraint Pareto optimal when the mass of available projects are exogenous. However, one could envision an economy where these masses depend on the past decisions of investors, so they are endogenously determined in the equilibrium. Specifically, the choice of the matching sets analyzed in previous sections could potentially have an impact on the supply side of this economy and particularly the mass of available projects (see figure 2.5). Let us interpret the two types of available projects, i.e $\{a, b\}$, as early and late stage ventures. In the venture capital industry the early stage startups are usually classified as those that are early in the fund-raising cycle (round B or earlier), and the late stage ones refer to

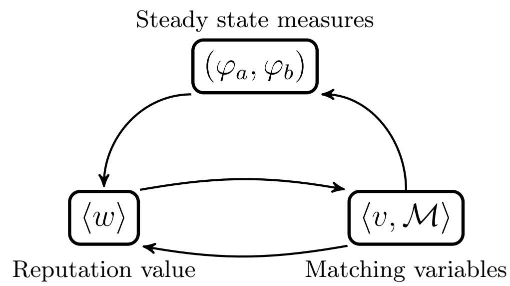


Figure 2.5: Equilibrium feedbacks with endogenous mass of projects

more mature companies that passed the round C fund-raising.[14] In a broader context of innovation literature, I shall interpret the early stage projects as the ones associated to risky radical innovations with longer average time to success and the late stage projects as the safer incremental ventures with shorter average time to success. In both readings, there is an spillover from successful early stage developments to the late stage opportunities. Formally, the stationary mass of φ_b depends on the mass of successful early stage projects. Toward this construction, suppose a fraction ζ of successful early stage projects would spill over to the rest of economy, and give rise to the creation of late stage businesses. Therefore, in any steady state outcome it must be that

$$\zeta \lambda_a m_a(1) \chi_a(1) = \kappa \varphi_b n(1). \tag{2.20}$$

So, conditioned on $\chi_a(1) = 1$, then $\varphi_b = \zeta \varphi_a$. Consequently, if

$$\lambda_a - c > \frac{\kappa \zeta \varphi_a (\lambda_b - c)}{r + \lambda_b + \kappa \zeta, \varphi_a}, \tag{2.21}$$

then VCs invest in early stage startups, of which the successful ones create the late stage opportunities. This is because the opportunity cost of forgoing the option to wait for the next late stage project is not high enough to preclude the investment in the early stage companies. Therefore, in the stationary equilibrium both types of companies coexist. I call this equilibrium the *maximum surplus equilibrium*. On the other hand, when $\lambda_a \leq c$, VC firms do not invest in any company, thus the investment activity is shut down, and it is

[14]According to NVCA 2020 yearbook based on the data provided by the PitchBook, in 2019 \$80.7B is invested in the late stage startups and \$52.8B in early and seed stage startups. The number of deals made in the former group was 2717 and in the latter one was 8642. Hence, both groups constitute a noticeable share of total investment activity.

referred as *zero surplus equilibrium*.[15] Importantly, we observe that higher search frictions – translating to lower κ – brings down the opportunity cost of forgoing the option to wait for late stage proposals, and hence increases the likelihood of investment in early stage businesses.

In each of the above two cases, there exists a unique stationary equilibrium, however, in the intermediate case

$$c \leq \lambda_a \leq c + \frac{\kappa\zeta\varphi_a(\lambda_b - c)}{r + \lambda_b + \kappa\zeta\varphi_a}, \tag{2.22}$$

there is no stationary equilibrium. If it were one, then VCs must invest in both groups of companies, i.e $1 \in \mathcal{M}_a \cap \mathcal{M}_b$, that is not the case because λ_a is not high enough. Assume initially $\varphi_b = 0$, then VCs only invest in early stage companies, because $\lambda_a > c$ and there is no better option available to them. As a result of subsequent spillovers, late stage opportunities start to appear, so $\varphi_b > 0$. Consequently, the VCs approached by the b-companies optimally choose to invest in their ventures, thereby reducing the net investment in the early stage companies. So, the population of successful early stage startups declines, lowering φ_b all the way down to zero again, and the economy returns to the initial point in the cycle. This mechanism essentially calls for a nonstationary equilibrium in the intermediate region (2.22).

The planner can however intervene when the economy is trapped in the zero surplus equilibrium. In particular, to shift the equilibrium, the planner can subsidize the investment in the early stage companies by taxing the output of late stage projects. That would in turn encourage the private investors to fund early stage projects, some of which turn into successful late stage companies. Once $m_b(1)$ reaches a critical mass, the planner can tax their output to finance the permanent subsidy of early stage investments, thereby sustaining the maximum surplus equilibrium on a balanced-budget forever. So effectively, by adopting the redistributive policy the planner internalizes the positive spillovers from early stage to late stage projects in the investment decisions made by investors.

Toward a better understanding of the constrained optimum and the source of externality in this economy, I express the planner's constrained optimization problem below. The maximand is the expected social surplus of the economy and the constraints are the dynamical equations for the population of VCs and startups. Let $m_q(1)$ be the mass of investors with maximum reputation connected to a q-project; $n(1)$ the mass of unmatched startups with

[15] In more developed countries the former equilibrium seems to be the prevailed outcome, in which sizable investments are made by the venture capital industry in both early as well as late stage companies. For example in the recent survey by [22] 62% of US institutional VC firms specialize in a particular stage, among them 36% indicated they invest in seed or early stage companies and 14% invest only in mid to late stage startups.

reputation 1; $m_q(\pi)$ the *density* of matched investors to a q-project, and finally $n(\pi)$ is the density of unmatched investors with reputation π. All of these measures are time-dependent (even though the time index t is suppressed). Therefore, the discounted social surplus of this economy is

$$S = \int_0^\infty e^{-rt} \left(\sum_q (\lambda_q - c) m_q(1) + \int (\lambda_q \pi - c) m_q(\pi) \mathrm{d}\pi \right) \mathrm{d}t. \tag{2.23}$$

The planner chooses the time-dependent matching indicators $\chi_q(\pi)$ to maximize S subject to the following law of motions for the population measures:

$$\dot{m}_q(1) = -\lambda_q m_q(1) + \kappa \varphi_q n(1) \chi_q(1), \text{ for } q \in \{a, b\} \tag{2.24a}$$

$$\dot{n}(1) = \sum_q \lambda_q m_q(1) - \sum_q \kappa \varphi_q n(1) \chi_q(1) + \sum_q \int \lambda_q \pi m_q(\pi) \mathrm{d}\pi \tag{2.24b}$$

$$\dot{m}_q(\pi) = -\lambda_q \pi m_q(\pi) + \kappa \varphi_q n(\pi) \chi_q(\pi) + \lambda_q \partial_\pi \left(\pi(1-\pi) m_q(\pi) \right), \text{ for } q \in \{a, b\} \tag{2.24c}$$

$$\dot{n}(\pi) = -\sum_q \kappa \varphi_q n(\pi) \chi_q(\pi) \tag{2.24d}$$

The first equation above combines the in- and out-flows from $m_q(1)$, namely the outflow of successful ventures of type q and the inflow of the matches between type-q projects with investors of reputation 1, conditioned on the matching indicator $\chi_q(1)$. The second law of motion accounts for the flows in and out of $n(1)$. The first and the last term represent the inflow from successful matches whose investors now become unmatched, while the second term captures the outflow due to currently formed partnerships between investors with maximum reputation and all admissible projects. The third forward equation captures how the population of investors with reputation π that are matched with type-q projects evolves. The first term is the outflow of those who become successful and thus leave the group; the second term is the inflow of recently formed partnership; and the third term is the net *learning inflow*: summing the inflow of the group with reputation in $(\pi, \pi + \mathrm{d}\pi)$ who experience a decline in reputation, and the outflow of the ones leaving $(\pi - \mathrm{d}\pi, \pi)$. The fourth equation expresses how the density of unmatched investors with intermediate reputation declines over time. The last state constraint that should be considered in the planner's problem is

$$\dot{\varphi}_b = \underbrace{\zeta \lambda_a \left(m_a(1) + \int \pi m_a(\pi) \mathrm{d}\pi \right)}_{\substack{\text{spillover from successful early} \\ \text{to late stage projects}}} - \underbrace{\kappa \varphi_b \left(n(1) \chi_b(1) + \int n(\pi) \chi_b(\pi) \mathrm{d}\pi \right)}_{\text{outflow due to recent partnerships}}. \tag{2.25}$$

This equation relates the rate of change of the mass of available late stage projects ($\dot{\varphi}_b$) to the inflow originated from spillovers of successful early stage ventures and the outflow of the recent partnerships made with the members of the unmatched late stage group.

Let $\{v_*(1,q), w_*(1), v_*(\pi,q), w_*(\pi)\}$ respectively be the co-state processes for equations (2.24a), (2.24b), (2.24c) and (2.24d); each of them shall be interpreted as the *social marginal value* of an additional member to its associated group. For example, $v_*(\pi,q)$ is the social marginal value of adding one more (π,q) match. Also, denote the co-state process for equation (2.25) by ρ. Packed up by these state equations, I solve for the social optimum of this economy by analyzing the current value Hamiltonian in appendix B. It is established there that from the planner's viewpoint:

$$\chi_q^*(\pi) = 1 \Leftrightarrow v_*(\pi,q) > w_*(\pi) \tag{2.26}$$

Particularly, a match between an investor with reputation π and a type-q project is socially optimal if the social marginal value of the match ($v_*(\pi,q)$) exceeds the social marginal value of holding reputation ($w_*(\pi)$).

Also shown in the appendix, in the steady state, where the time derivatives are zero, the following co-state equations are resulted for social contributions:

$$\begin{aligned}
rv_*(1,q) &= \lambda_q - c + \lambda_q\left(w_*(1) - v_*(1,q)\right) + \boxed{\rho\zeta\lambda_a 1_{\{q=a\}}} \text{ if } \chi_q^*(1) = 1 \\
rw_*(1) &= \sum_q \kappa\varphi_q\left(v_*(1,q) - w_*(1)\right) - \chi_q^*(1) - \boxed{\rho\kappa\varphi_b\chi_b^*(1)} \\
rv_*(\pi,q) &= \lambda_q\pi - c + \lambda_q\pi\left(w_*(1) - v_*(\pi,q)\right) - \lambda_q\pi(1-\pi)v_*'(\pi,q) + \boxed{\rho\zeta\lambda_a\pi 1_{\{q=a\}}} \text{ if } \chi_q^*(\pi) = 1 \\
rw_*(\pi) &= \sum_q \kappa\varphi_q\left(v_*(\pi,q) - w_*(\pi)\right)\chi_q^*(\pi) - \boxed{\rho\kappa\varphi_b\chi_b^*(\pi)}
\end{aligned} \tag{2.27}$$

The above differential characterization of the social value functions should be juxtaposed with the private valuations of (2.3) and (2.5). In particular, the terms in the boxes precisely characterize the sources of the departure of the social from private incentives. These terms can guide us about the profile of taxes that decentralizes the social optimum. When there is spillovers from early stage to late stage businesses the above expressions suggest the following redistributive schedule:

- Cost subsidization of early stage projects.
- Taxing the output of late stage businesses.

The subsidization ($\rho\zeta\lambda_a\pi$) can be made either as a flow payment that depends on the current value of investor's reputation (π), or equivalently (and much easier) as a one-off rebate that investors of early stage projects receive upon the success with the face value of $\rho\zeta$. On the other hand the tax imposed on the unmatched investors is $\rho\kappa\varphi_b$, where $\varphi_b = \zeta\varphi_a$ in the steady state level resulted from equation (2.25).[16]

In the steady state the redistributive schedule is budget neutral, so the planner runs no deficit or surplus. This is owed to the following accounting analysis:

$$\begin{aligned} \text{total subsidy} &= \rho\zeta\lambda_a m_a(1) + \rho\zeta\lambda_a \int \pi m_a(\pi)\mathrm{d}\pi \\ \text{total tax revenue} &= \rho\kappa\varphi_b \left(n(1)\chi_b(1) + \int n(\pi)\chi_b(\pi)\mathrm{d}\pi \right) \end{aligned} \tag{2.28}$$

Since in the steady state $\dot{\varphi}_b = 0$, the above two sums match each other, and the redistribution is self-financing. Furthermore, in the steady state of this economy the densities should be identically equal to zero, and all the masses concentrate discretely on the boundaries. This observation hints to the condition under which intervention, namely setting $\chi_a(1) = 1$, is justified in steady state. Particularly, $\chi_a^*(1) = 1$ iff the resulting social surplus exceeds zero, which is what economy achieves when $\chi_a(1) = 0$ and $\varphi_b = \chi_b(1) = 0$. So,

$$\chi_a^*(1) = 1 \Leftrightarrow \lambda_a m_a(1) + \lambda_b m_b(1) > 0. \tag{2.29}$$

This condition translates to

$$\chi_a^*(1) = 1 \Leftrightarrow \zeta > \frac{\lambda_b(c - \lambda_a)}{\lambda_a(\lambda_b - c)}. \tag{2.30}$$

There is a very important message behind this derivation: the centralized intervention – in form of tax and subsidy and even setting the choice of matching sets – is justified if and only if the spillovers from early to late stage ventures is large enough.[17]

[16] Even though the *direction* of the corrective tax/subsidy seems natural, there are real world examples where the implementation amendments undo the original promise of the government intervention. For example, the Finnish Industry Investment Ltd (FII), a government owned investment agency, started its operation in 1995 with the core mandate of financing and stimulating the venture capital funds investing in seed and early stage startups. However, FII was also set by the government to operate *profitably*. In the evaluation report published in [55], it is stated that this requirement "has led the organization to seek *later stage* investments in order to meet the profitability target".

[17] The failure to correctly predict the extent of such positive spillovers doomed the sizable upfront investments that the Malaysian government made to boost the biotechnological developments in BioValley [42].

Using this observation the first order condition for ρ, i.e the shadow social value of φ_b, is presented in the appendix, which in the steady state reduces to:

$$r\rho = \kappa n(1)\left(v_*(1,b) - w_*(1) - \rho\right)\chi_b^*(1) \tag{2.31}$$

This equations confirms that that $\rho \geq 0$ and therefore the *direction* of transfers explained above is indeed correct.

Next, I explain what can be achieved in the steady state of the economy if the tax/subsidy scheme can *only* depend on the type of the projects and not on whether the investors are matched or not.

Proposition 6. *If the transition probability to late stage startups is large enough, particularly $\lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c) < \zeta \leq 1$, then there exists a budget-balanced redistribution scheme that shifts the economy from zero surplus to maximum surplus equilibrium that depends on the type of projects and not on the matching status of their investors.*

Proof. In the case where VCs invest in both types of companies, the total surplus is $(\lambda_a - c)m_a(1) + (\lambda_b - c)m_b(1)$ that is equal to

$$r\ S = \underbrace{\frac{\mathfrak{p} - \alpha}{1 - \alpha}\frac{\kappa\varphi_a}{1 + \kappa\varphi_a/\lambda_a + \kappa\zeta\varphi_a/\lambda_b}}_{\text{extensive margin}}\underbrace{\left(\frac{\lambda_a - c}{\lambda_a} + \frac{\zeta(\lambda_b - c)}{\lambda_b}\right)}_{\text{intensive margin}}. \tag{2.32}$$

Therefore, the created surplus is positive iff $\zeta > \lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c)$, and only then is it optimal to intervene. The goal of any redistribution policy should be to provide enough incentives to VCs to invest in *both* types of companies, so that the intensive margin – present value of the profit from a generic investment – be nonzero, and the market does not break down.

Toward this, assume the planner collects a fraction $\mathfrak{t}_b$ of the success outcome of late stage companies, and use this revenue to subsidize the investment to early (resp. late) stage startups by a fraction $\mathfrak{s}_a$ (reps. $\mathfrak{s}_b$). Budget neutrality requires

$$m_a(1)c\mathfrak{s}_a + m_b(1)c\mathfrak{s}_b = \lambda_b m_b(1)\mathfrak{t}_b. \tag{2.33}$$

Since at steady state $\lambda_b m_b(1) = \kappa\varphi_b n(1) = \kappa\zeta\varphi_a n(1)$, then

$$\frac{c\mathfrak{s}_a}{\zeta\lambda_a} + \frac{c\mathfrak{s}_b}{\lambda_b} = \mathfrak{t}_b. \tag{2.34}$$

Further, note that the value of holding maximum reputation after redistribution when $1 \in \mathcal{M}_a \cap \mathcal{M}_b$ is

$$w_{ab}(1) := \frac{r^{-1}\kappa\zeta\varphi_a\left(\lambda_b(1-\mathfrak{t}_b)-c(1-\mathfrak{s}_b)\right)(r+\lambda_a)+r^{-1}\kappa\varphi_a\left(\lambda_a-c(1-\mathfrak{s}_a)\right)(r+\lambda_b)}{(r+\lambda_a)(r+\lambda_b)+\kappa\zeta\varphi_a(r+\lambda_a)+\kappa\varphi_a(r+\lambda_b)}. \tag{2.35}$$

The incentive constraints for $1 \in \mathcal{M}_a \cap \mathcal{M}_b$, resulted from proposition 4, are

$$\lambda_a - c(1-\mathfrak{s}_a) > rw_{ab}(1) \Leftrightarrow \lambda_a - c(1-\mathfrak{s}_a) > \frac{\kappa\zeta\varphi_a\left(\lambda_b(1-\mathfrak{t}_b)-c(1-\mathfrak{s}_b)\right)}{r+\lambda_b+\kappa\zeta\varphi_a}, \tag{2.36a}$$

$$\lambda_b(1-\mathfrak{t}_b) - c(1-\mathfrak{s}_b) > rw_{ab}(1) \Leftrightarrow \lambda_b(1-\mathfrak{t}_b) - c(1-\mathfrak{s}_b) > \frac{\kappa\varphi_a\left(\lambda_a-c(1-\mathfrak{s}_a)\right)}{r+\lambda_a+\kappa\varphi_a}. \tag{2.36b}$$

The planner must design $(\mathfrak{s}_a, \mathfrak{s}_b, \mathfrak{t}_b)$, subject to the budget-balanced condition (2.34) and the above incentive constraints. I define $\mathfrak{e}_b := \mathfrak{t}_b - \frac{c\mathfrak{s}_b}{\lambda_b}$ as the *effective* tax-rate on b-companies. Assume $\mathfrak{e}_b$ is small enough, so that the expected payoff from investment in b-startups is higher than that of a-startups and larger than zero, namely

$$\lambda_b(1-\mathfrak{t}_b) - c(1-\mathfrak{s}_b) \geq \max\left\{\lambda_a - c(1-\mathfrak{s}_a), 0\right\}. \tag{2.37}$$

This amounts to an upper-bound on the effective tax rate $\mathfrak{e}_b$, that automatically guarantees (2.36b):

$$\mathfrak{e}_b \leq \min\left\{\frac{\lambda_b-\lambda_a}{\lambda_b+\zeta\lambda_a}, \frac{\lambda_b-c}{\lambda_b}\right\} = \frac{\lambda_b-\lambda_a}{\lambda_b+\zeta\lambda_a} < 1 \tag{2.38}$$

The middle identity holds because $\zeta > \lambda_b(c-\lambda_a)/\lambda_a(\lambda_b-c)$. The incentive constraint (2.36a) boils down to

$$\mathfrak{e}_b > \frac{(c-\lambda_a)(r+\lambda_b)+\kappa\zeta\varphi_a(\lambda_b-\lambda_a)}{\zeta\left(\lambda_a(r+\lambda_b)+\kappa\varphi_a(\lambda_b+\zeta\lambda_a)\right)}. \tag{2.39}$$

Therefore, I have to show for every $\zeta \in (\lambda_b(c-\lambda_a)/\lambda_a(\lambda_b-c), 1]$, the upper bound on the effective tax rate in (2.38) is larger than the lower bound in (2.39), so that one can always find an incentive compatible redistribution. For this note that

$$\frac{(c-\lambda_a)(r+\lambda_b)+\kappa\zeta\varphi_a(\lambda_b-\lambda_a)}{\zeta\left(\lambda_a(r+\lambda_b)+\kappa\varphi_a(\lambda_b+\zeta\lambda_a)\right)} < \frac{\lambda_b-\lambda_a}{\lambda_b+\zeta\lambda_a} \Leftrightarrow \frac{\lambda_b(c-\lambda_a)}{\lambda_a(\lambda_b-c)} < \zeta. \tag{2.40}$$

Therefore, one can always find a *range* of redistributive schemes inducing the investment in both early and late stage companies incentive compatible, while being budget balanced. Specifically, any choice in this region leaves the intensive margin of the social surplus function unaffected, because after all it is a redistribution policy. In addition, for any choice in this interval, the subsidy rate for a-companies is less than one, because $\mathfrak{s}_a = \zeta\lambda_a\mathfrak{e}_b/c < 1$, due to the fact that $\lambda_a < c$, $\zeta < 1$ and $\mathfrak{e}_b \leq \frac{\lambda_b-\lambda_a}{\lambda_b+\zeta\lambda_a} < 1$. □

At the heart of both methods of interventions (namely equation 2.27 and the previous proposition) is the intuitive idea that subsidizing the *early* stage projects – financed by a tax on late stage output – prevents the market breakdown originated from investors' failure to internalize the spillovers from early to late stage projects. The spirit of this failure is somewhat reminiscent of the neglect of R&D positive spillovers by the private sector ([42] and [26]).

Lastly in this section I signify the point that the proposed interventions are only justifiable when there is large enough spillover from early to late stage projects. This was formally seen in the planner's problem and in the proof of proposition 6. There are ample examples of failed government interventions in the venture capital industry which the central planner had made upfront investments to jump-start the investment activity but these efforts were not picked up by the private sector later on.[18]

2.6 Reputation and deal flow

In this section, I aim to examine the equilibrium outcomes when there is reputational externality at play. Specifically, I ask what are the indirect impacts of a reputable actor on the *remaining* body of investors? This question stands in a contrast to what has been so far studied about the reputation effects in the venture capital industry.

Particularly, there is a considerable body of research highlighting the *individual benefits* associated with higher reputation. The findings include the lower pay-for-performance for smaller and younger firms (associated with the theory of grandstanding [21]) aimed at establishing reputation and enjoying its subsequent benefits; Or how VCs with higher reputation acquire startup equity at a discount [32]. Also noted are the preferential access to deal flow ([56]) and the ability to become more central in the VCs' syndication network and thereby receiving a larger set of proposals ([69]). However, less is known about the *social returns* to reputation. One may except inefficiencies would arise in an economy in which highly reputable VCs slow down the deal flow of less reputable ones, and hence hindering the learning and investing opportunities of the latter group, and eventually the overall economy. This force would have not been a concern if there was a *price for reputation* and a centralized market in which startups could partner with VCs. Yet, the predominant feature of this economy is the time cost of the search that underlies the VCs' investment decisions, and the *dispersed* investment opportunities.

[18]The book by [40] reviewed many of these examples.

Equilibrium outcome with long-lived investors

To capture the aforementioned interaction effect, I would propose a different matching technology. Thus far, the matching function was assumed *uniformly* quadratic. That is over every period $\mathrm{d}t$ the total mass of proposals from entrepreneurs to investors is $\kappa(\varphi_a + \varphi_b)\mathrm{d}t$, and it is uniformly distributed among the unit mass of VCs. Holding the total rate of proposals constant, now I assume this flow is not uniformly distributed among VCs, rather it contacts more (resp. less) reputable VCs with higher (resp. lower) probability, according to the *reputation-weight* function $\psi(\cdot)$. Specifically, let $\boldsymbol{\pi_\infty}$ be the stationary distribution of VCs' reputation. Then the rate with which q-companies meet a VC with reputation π is

$$\kappa\varphi_q \frac{\psi(\pi)}{\mu}, \quad \text{where} \quad \mu := \mathsf{E}\left[\psi(\boldsymbol{\pi_\infty})\right].^{19} \tag{2.41}$$

I assume $\psi \geq 0$, $\psi' \geq 0$, $\psi(0) = 0$ and $\psi(1) = 1$. One might expect that any hope to prove a uniqueness theorem such as the one in theorem 3 without having a much more restrictive assumptions on $\psi(\cdot)$ is doomed to fail. This is mainly because the analogue of proposition 4 – in which we prove the convexity of matching sets – for the general reputation weight function is very complicated and requires making a collection of assumptions on $\psi(\cdot)$ in conjunction with other primitives. However, to a large extent this analysis is futile in this context, because alternatively I propose an equilibrium that exists for every $\psi(\cdot)$ satisfying the above minimal conditions. Consequently, we can perform the comparative statics on this equilibrium with respect to the choice of $\psi(\cdot)$.

Inspired by the analysis in section 2.2, I conjecture that there exists an equilibrium featuring $\mathcal{M}_b = (\alpha_e, 1]$, $\mathcal{M}_a \subset \mathcal{M}_b$ and $1 \in \mathcal{M}_a$ iff

$$\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{\mu(r + \lambda_b) + \kappa\varphi_b}. \tag{2.42}$$

Further, in this equilibrium the value of holding the maximum reputation is

$$w(1) = \max_{\chi} \left\{ \frac{r^{-1}\kappa\left[\varphi_b\left(\lambda_b - c\right)\left(r + \lambda_a\right)\chi_b(1) + \varphi_a\left(\lambda_a - c\right)\left(r + \lambda_b\right)\chi_a(1)\right]}{\left(r + \lambda_a\right)\left(r + \lambda_b\right)\mu + \kappa\varphi_b\left(r + \lambda_a\right)\chi_b(1) + \kappa\varphi_a\left(r + \lambda_b\right)\chi_a(1)} \right\}, \tag{2.43}$$

and α_e is the fixed-point of the following system:

$$\mu = \frac{1 - \mathfrak{p}}{1 - \alpha}\psi(\alpha) + \frac{\mathfrak{p} - \alpha}{1 - \alpha} \tag{2.44a}$$

$$\alpha = \frac{c}{\lambda_b\left(1 + w(1)\right)}. \tag{2.44b}$$

[19] Notice that μ is the *steady-state* average reputation weight, and is not the current population average of reputation weights, i.e $\int_0^1 \psi(\pi_{it})\mathrm{d}i$. This assumption simplifies the equilibrium analysis, particularly by letting us to focus on the time-independent termination policies, i.e constant α over time.

Relation (2.44a) is owed to the presence of $\frac{1-\mathfrak{p}}{1-\alpha}$ VCs with reputation α and the remaining $\frac{\mathfrak{p}-\alpha}{1-\alpha}$ with reputation one in the steady state. And equation (2.44b) simply expresses the endogenous termination point in line with the analysis offered for (2.13). I refer to any equilibrium with the above features as *normal* equilibrium.

Proposition 7. *In the described economy with reputational externality,*

(i) there always exists a normal equilibrium with $\alpha_e < \mathfrak{p}$.

(ii) If $\psi'' \leq 0$ *the normal equilibria are Pareto ranked. Further, the* α_e *for the most (least) preferred equilibrium is increasing with respect to the pointwise order on* ψ.

Part (i) ensures the existence of the normal equilibrium under the new choice of the matching function that exhibits the reputational externality. In light of that, we can safely claim that the sort of matching sets depicted in figure 2.3 are applicable in this case as well. Specifically, the normal equilibria requires the matching sets to be connected and hence the outcome of learning in the economy at the steady state can be characterized by examining the masses at the endpoints, i.e $\pi \in \{1, \alpha_e\}$.

Emboldened by the existence of normal equilibria, the analogue of the results based on proposition 4 would apply in this section too, with the change of $\kappa\varphi_q$ to $\kappa\varphi_q/\mu$ in all expressions. Specifically, when it comes to cost regime determination, the characterization (2.12) changes to (2.42). In a meaningful contrast with the baseline model – where the reputational externality was absent in the matching function – the investors' equilibrium response to whether invest on a-projects depends on the *average reputation score* (μ) of the whole body of investors. Specifically, any increase in the equilibrium value of μ lowers the opportunity cost of forgoing the option to wait for b-projects, that in turn relaxes the constraint for investing in a-projects. Therefore, *softening* the extent of reputational externalities would encourage investors toward the early stage projects. To sharpen the meaning behind *softening the reputational externality*, I examine the effect of the choice of ψ as a *parameter* picked from the following family of admissible functions:

$$\Psi := \{\psi : [0,1] \to [0,1] | \, \psi(0) = 0, \psi(1) = 1, \psi' \geq 0, \psi'' \leq 0\}, \tag{2.45}$$

Endow Ψ with the pointwise order, that is $\psi_2 \succsim \psi_1$ iff $\psi_2(x) \geq \psi_1(x)$ $\forall x \in [0,1]$ (see figure 2.6). Inspired by this figure, I say ψ_2 is *softer* than ψ_1, because the marginal return to a higher reputation in ψ_2 is smaller than ψ_1. In part (ii) of the previous proposition, it is shown that the equilibrium termination point α_e is increasing w.r.t to $\succsim$ on Ψ. Therefore,

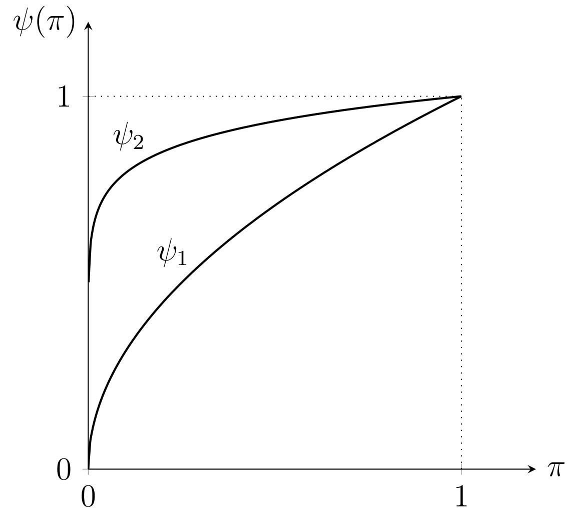


Figure 2.6: Weighting functions

softening the extent of reputational externality (i.e increasing ψ in a pointwise manner), reduces the investors' patience (i.e increases the equilibrium α_e), by lowering the equilibrium value of reputation building (i.e $w(1)$), and thereby relaxing the constraint for investing on a-projects in equation (2.42). The following line summarizes the result of this comparative static exercise:

$$\psi \uparrow \;\; \Rightarrow \;\; w(1) \downarrow, \;\; \mu_e \uparrow \;\; \text{and } \alpha_e \uparrow \tag{2.46}$$

Because of the reputational externality, one would expect *under-learning* in the equilibrium outcome relative to the social optimum. That is the reputable group of investors receive a higher than socially optimal share of investment proposals, leaving the lesser known group with fewer options, thereby lowering their reservation value w.

The comparison of the steady state equilibrium surplus with the steady state social optimum in the current environment of *long-lived* agents *ignores* the previous costs born by investors on the investment path (starting from $\mathfrak{p}$ and ending at α_e). Particularly, the steady state social surplus is maximized at $\alpha = 0$, because it fails to take into account the cost of pushing α down to zero. This is owed to the fact that in the steady state there will be no investors with reputation in $(\alpha_e, \mathfrak{p}]$. Therefore, in the next subsection, I will allow for exogenous birth and death of investors to obtain a *non-degenerate* stationary economy, justifying the comparison of the steady state equilibrium outcome with the steady state social optimum, by the means of having a continuous distribution of investors on $(\alpha_e, \mathfrak{p}]$.

This tweak helps us to understand the spirit of the reputational externality and the extent to which the decentralized outcome under-appreciates the gains from more patience.

Short-lived investors

The nature of reputational externality can be easily described if we focus only on one group of projects, say the b-startups and henceforth in this section I drop the b-index from variables. Since the focus of the forthcoming analysis is the stationary distribution of VCs' reputation and its impact on the investment pattern, and not the spillovers between different types of projects, this assumption is largely innocuous.

The investors are short-lived. Specifically, they leave the economy exogenously at the rate of δ, and are born with the same rate and the initial reputation $\mathfrak{p}$. The matching function is quadratic and exhibits reputational externality normalized by the steady state reputation score $\mu = \mathsf{E}\left[\psi(\boldsymbol{\pi}_{\boldsymbol{\infty}})\right]$.[20] I assume minimal structure on ψ by letting it be only increasing and concave, and fixing $\psi(1) = 1$. I conjecture (and prove) that there exists a symmetric stationary equilibrium in which all investors terminate their matches at a common α. In light of this conjecture, denote the *cross-sectional* density function of the matched VCs by $m(\pi)$ supported on $[\alpha, \mathfrak{p}]$. Let $m(1)$ and $n(1)$ be the discrete measures of the matched and unmatched VCs with maximum reputation, respectively, and finally $n(\alpha)$ and $n(\mathfrak{p})$ are the discrete measures of unmatched VCs at α and $\mathfrak{p}$. Figure 2.7 plots all pieces of the cross-sectional steady state distribution of investors' reputations.

The inflow outflow equations at the discrete masses are

$$\dot{m}(1) = -\lambda m(1) + \kappa\varphi\frac{n(1)}{\mu} - \delta m(1) \tag{2.47a}$$

$$\dot{n}(1) = \lambda m(1) - \kappa\varphi\frac{n(1)}{\mu} - \delta n(1) + \int_\alpha^{\mathfrak{p}} \lambda\pi m(\pi)\mathrm{d}\pi \tag{2.47b}$$

$$\dot{n}(\mathfrak{p}) = -\kappa\varphi\frac{\psi(\mathfrak{p})}{\mu}n(\mathfrak{p}) - \delta n(\mathfrak{p}) + \delta \tag{2.47c}$$

Notice that $n(\alpha)$ is determined through population conditions such as the conservation of first and second moment. The forward equation for $m(\pi)$ is

$$\dot{m}(\pi) = -\underbrace{\lambda\pi m(\pi)}_{\substack{\text{outflow of}\\ \text{successful investors}}} + \underbrace{\lambda\partial_\pi\left(\pi(1-\pi)m(\pi)\right)}_{\text{net learning inflow}} \underbrace{-\delta m(\pi)}_{\text{exogenous exits}}. \tag{2.48}$$

[20]Recall from the previous section that taking μ as the *steady state* average of reputation weights supports a time-invariant termination point α in the equilibrium.

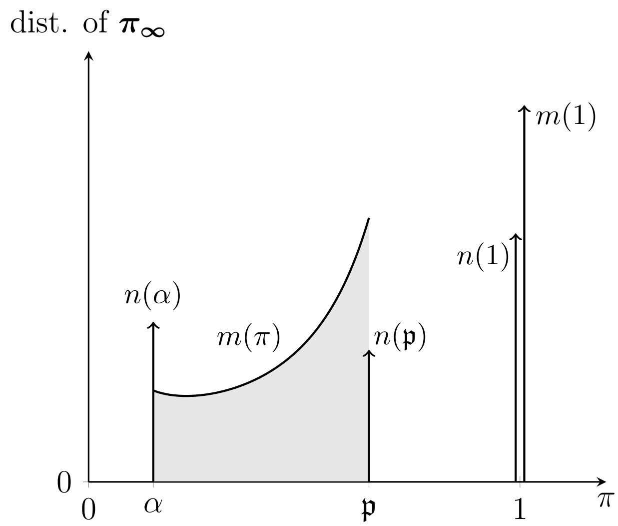


Figure 2.7: Steady state cross-sectional distribution of π_∞

The first component in the *rhs* is the outflow from $m(\pi)$ (due to the recent success events) to $n(1)$. The second term captures the net learning effect, by factoring the inflow of investors whose reputation is in $(\pi, \pi + \mathrm{d}\pi)$ and thus falling due to the lack of success and the outflow of the unsuccessful group with reputation in $(\pi - \mathrm{d}\pi, \pi)$.[21] Finally, the third term is associated to the exogenous departures. In the steady state $\dot{m}(\pi) = 0$ thus raising a differential equation for the density function whose solution is

$$m(\pi) = m(\alpha) \left(\frac{\pi}{\alpha}\right)^{\delta/\lambda - 1} \left(\frac{1-\pi}{1-\alpha}\right)^{-(\delta/\lambda+2)}, \quad \forall \pi \in [\alpha, \mathfrak{p}]. \tag{2.49}$$

The group of VCs with minimum reputation at $\pi = \alpha$ are subject to two flows: the inflow from the matched ones in $(\alpha, \mathfrak{p}]$ and the outflow due to the exogenous exits at the rate of $\delta n(\alpha)$. Therefore, in the steady state it must be that the inflow equals $\delta n(\alpha)$.

Lastly, the net inflow to the matched VCs on the region $(\alpha, \mathfrak{p}]$ must match the net outflow in the steady state, that is

$$\underbrace{\kappa\varphi\frac{\psi(\mathfrak{p})}{\mu}n(\mathfrak{p})}_{\substack{\text{new matches}\\ \text{originating from } \mathfrak{p}}} = \underbrace{\lambda\int_\alpha^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi}_{\substack{\text{outflow of}\\ \text{successful investors}}} + \underbrace{\delta\int_\alpha^{\mathfrak{p}} m(\pi)\mathrm{d}\pi}_{\text{exogenous departure}} + \underbrace{\delta n(\alpha)}_{\substack{\text{endogenously}\\ \text{separated matches}}}. \tag{2.50}$$

[21] The first two terms can also be understood in the context of Kolmogorov Forward equation (see theorem 17.4.14 of [10]) related to the density function of the reputation process $\mathrm{d}\pi_t = (1 - \pi_{t^-})\,[\mathrm{d}\iota_t - \lambda\pi_{t^-}\mathrm{d}t]$.

Lemma 6. *In the steady state of the above economy,*

$$\int_{\alpha}^{\mathfrak{p}} m(\pi)\mathrm{d}\pi = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\mathfrak{p}-\alpha}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}\left(\Upsilon_1(\alpha)-\frac{\lambda}{\delta+\lambda}\Upsilon_2(\alpha)\right), \tag{2.51a}$$

$$\int_{\alpha}^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\delta}{\delta+\lambda}\frac{(\mathfrak{p}-\alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{2.51b}$$

$$m(1) = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\kappa\varphi/\mu}{\delta+\lambda+\kappa\varphi/\mu}\frac{\lambda}{\delta+\lambda}\frac{(\mathfrak{p}-\alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{2.51c}$$

$$n(\alpha) = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\Upsilon_2(\alpha)-\mathfrak{p}\Upsilon_1(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{2.51d}$$

where

$$\Upsilon_i(\alpha) := \left(\frac{\mathfrak{p}}{\alpha}\right)^{\delta/\lambda-1}\left(\frac{1-\mathfrak{p}}{1-\alpha}\right)^{-(\delta/\lambda+2)}\mathfrak{p}^i(1-\mathfrak{p})-\alpha^i(1-\alpha), \quad \textit{for } i\in\{1,2\}. \tag{2.52}$$

Given the results found in this lemma one can examine the limits when the VCs become long-lived agents, that is as $\delta\to 0$. It is easy to verify that for both $i\in\{1,2\}$:

$$\Upsilon_i(\alpha)\to\alpha(1-\alpha)\frac{\mathfrak{p}-\alpha}{1-\mathfrak{p}} \tag{2.53}$$

And accordingly $\int_{\alpha}^{\mathfrak{p}} m(\pi)\mathrm{d}\pi\to 0$, $\int_{\alpha}^{\mathfrak{p}}\pi m(\pi)\mathrm{d}\pi\to 0$, $m(1)\to\frac{\kappa\varphi/\mu}{\lambda+\kappa\varphi/\mu}\frac{\mathfrak{p}-\alpha}{1-\alpha}$, and $n(\alpha)\to\frac{1-\mathfrak{p}}{1-\alpha}$ as $\delta\to 0$; confirming the previous results on the economy with long-lived investors.

Toward the equilibrium analysis, each investor stipulates the population average for ψ, say μ, and accordingly specifies the maximum attainable reputation value via the mapping $\mathsf{W}:[0,1]\to\mathbb{R}_+$:

$$\mathsf{W}(\mu) := \frac{(r+\delta)^{-1}\kappa\varphi/\mu}{r+\delta+\lambda+\kappa\varphi/\mu}(\lambda-c) \tag{2.54}$$

Then, following the Bellman equation on the continuation region induced by $w(1)=\mathsf{W}(\mu)$, namely

$$rv(\pi) := \lambda-c+\lambda\left(w(1)-v(\pi)\right)-\lambda\pi(1-\pi)v'(\pi)-\delta v(\pi), \tag{2.55}$$

the investor terminates the project at $\alpha=\mathsf{A}(w(1))$, where $\mathsf{A}:\mathbb{R}_+\to[0,1]$ and

$$\mathsf{A}(w) := \frac{c}{\lambda(1+w)}. \tag{2.56}$$

In the symmetric stationary equilibrium the initial stipulation about μ is self-fulfilling that is $\mu=\mathsf{M}\left(\mu,\mathsf{A}\circ\mathsf{W}(\mu)\right)$, where $\mathsf{M}:[0,1]^2\to\mathbb{R}_+$ returns the population average of reputation weights:

$$\mathsf{M}(\mu,\alpha)=\mathsf{E}\left[\psi(\boldsymbol{\pi_\infty})\right]=m(1)+n(1)+\psi(\mathfrak{p})n(\mathfrak{p})+\int_{\alpha}^{\mathfrak{p}}\psi(\pi)m(\pi)\mathrm{d}\pi+\psi(\alpha)n(\alpha) \tag{2.57}$$

Definition 7 (Symmetric stationary equilibrium)**.** *The symmetric stationary equilibrium in this economy with reputational externality is the set of all fixed-points of the mapping* $\mathsf{M}(\cdot, \mathsf{A} \circ \mathsf{W}(\cdot))$ *on the unit interval; A generic member is denoted by* μ_e*. Associated to the equilibrium outcome* μ_e *is the equilibrium termination point* $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$*.*

In the appendix B, I show that an increase in α or μ, holding the other variable constant, *positively* shifts the steady state distribution of $\boldsymbol{\pi}_\infty$ in the sense of *second-order stochastic dominance*. So, assuming a concave increasing form for $\psi(\cdot)$ one can deduce that $\mathsf{M}(\mu, \alpha)$ is an increasing function in each argument. In addition to that, the composition map $\mathsf{A} \circ \mathsf{W}$ is increasing, therefore the mapping $\mu \mapsto \mathsf{M}(\mu, \mathsf{A} \circ \mathsf{W}(\mu))$ is a continuous increasing function from the unit interval to itself.[22] Hence, a fixed-point μ_e and $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$ exist, establishing the existence of a symmetric stationary equilibrium.

To contrast the equilibrium outcome with the socially optimal choice, I express the steady state flow surplus of the economy in terms of the measures found in lemma 6:

$$
\begin{aligned}
rS(\mu, \alpha) &= (\lambda - c)m(1) + \int_\alpha^{\mathfrak{p}} (\lambda\pi - c)m(\pi)\mathrm{d}\pi \\
&= \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta + \kappa\varphi\psi(\mathfrak{p})/\mu} \frac{(\mathfrak{p} - \alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha) - \alpha\Upsilon_1(\alpha)} \times \\
&\left\{\lambda\left(\frac{\delta}{\delta + \lambda} + \frac{\kappa\varphi/\mu}{\delta + \lambda + \kappa\varphi/\mu}\frac{\lambda}{\delta + \lambda}\right) - c\left(\frac{\Upsilon_1(\alpha)}{\Upsilon_2(\alpha)} - \frac{\lambda}{\delta + \lambda} + \frac{\kappa\varphi/\mu}{\delta + \lambda + \kappa\varphi/\mu}\frac{\lambda}{\delta + \lambda}\right)\right\}
\end{aligned} \tag{2.58}
$$

A benevolent social planner selects an α so that jointly with its induced μ, that is the fixed-point of $\mathsf{M}(\cdot, \alpha)$, maximize the social surplus $S(\mu, \alpha)$.

Definition 8 (Planner's problem)**.** *The planner's problem is*

$$\max S(\mu, \alpha) \text{ subject to } \mu = \mathsf{M}(\mu, \alpha) \tag{2.59}$$

Remember the externality failed to be internalized in the investors' decision is originated from the impact of their choices on μ. Therefore, it is essential to incorporate $\mu = \mathsf{M}(\mu, \alpha)$ as the constraint of the planner's problem.

Next proposition explains why the equilibrium outcome is not socially efficient, and highlights the direction along which the social surplus increases.

[22] It is clearly continuous on $(0, 1]$, and it is made continuous at $\mu = 0$ by letting $\mathsf{W}(0) := \lim_{\mu\to 0} \mathsf{W}(\mu)$ and $\mathsf{M}(0, \alpha) := \lim_{\mu\to 0} \mathsf{M}(\mu, \alpha)$, where both limits exist in light of the expression (2.54) and lemma 6.

Proposition 8. *Every symmetric stationary equilibrium of the economy with reputational externality is not constrained-efficient. In particular, a local reduction in the termination point α_e increases the social surplus.*

Proof. Every symmetric equilibrium is characterized by its associated pair of termination policy α_e and the population average of reputation weights μ_e, in which $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$ and $\mu_e = \mathsf{M}(\mu_e, \alpha_e)$. It is further a stable equilibrium if $\partial_\mu \mathsf{M}(\mu_e, \alpha_e) < 1$. From the expression for the social surplus in (2.58) one can see that S is decreasing in μ, therefore, if $\mathsf{M}(\cdot, \alpha)$ has multiple fixed-points for a given α the one with the smallest μ is the most efficient one. Furthermore, this equilibrium (with the smallest μ) is stable because $\mathsf{M}(0, \alpha) > 0$, and $\mathsf{M}(\cdot, \alpha)$ *downcrosses* the 45-degree line in its first intersection.

Toward proving the constrained inefficiency, I employ a variational approach in the neighborhood of α_e. Suppose the economy is in a stable pair (α_e, μ_e), and the planner moves α_e by $\Delta\alpha$. The new smallest fixed-point $\mu_e + \Delta\mu$ satisfies

$$\mu_e + \Delta\mu = \mathsf{M}(\mu_e + \Delta\mu, \alpha_e + \Delta\alpha) \approx \mathsf{M}(\mu_e, \alpha_e) + \partial_\mu \mathsf{M}\Delta\mu + \partial_\alpha \mathsf{M}\Delta\alpha, \tag{2.60}$$

hence $\Delta\mu = \frac{\partial_\alpha \mathsf{M}}{1-\partial_\mu \mathsf{M}}\Delta\alpha$. Consequently, the change in the social surplus function would be

$$r\Delta S = r\left(\frac{\partial_\alpha \mathsf{M}}{1-\partial_\mu \mathsf{M}}\partial_\mu S + \partial_\alpha S\right)\Delta\alpha. \tag{2.61}$$

Note that in every stable fixed-point of $\mathsf{M}(\cdot, \alpha_e)$, $\frac{\partial_\alpha \mathsf{M}}{1-\partial_\mu \mathsf{M}} > 0$, because M is shown to be increasing in α and due to the stability $\partial_\mu \mathsf{M} < 1$. Further, $\partial_\mu S < 0$ as can readily be verified from (2.58). Therefore, lowering α_e, i.e $\Delta\alpha < 0$, leads to a strict improvement in the social surplus if $\partial_\alpha S < 0$. Relying on (2.58) and applying some rearrangements lead to

$$\begin{aligned} r\partial_\alpha S(\mu_e, \alpha_e) &= (\lambda - c)\partial_\alpha m(1) - (\lambda\alpha - c)m(\alpha) \\ &= -\underbrace{\frac{\kappa\varphi\psi(\mathfrak{p})/\mu_e}{\delta + \kappa\varphi\psi(\mathfrak{p})/\mu_e}\frac{1-\mathfrak{p}}{(1-\alpha_e)^2}\left(\frac{\mathfrak{p}}{1-\mathfrak{p}}\right)^{-\delta/\lambda}\left(\frac{\alpha_e}{1-\alpha_e}\right)^{\delta/\lambda}}_{>0} \times \\ &\qquad \left[\frac{\delta(\lambda\alpha_e - c)}{\lambda\alpha_e} + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e}\right]. \end{aligned} \tag{2.62}$$

Therefore, the sign of $\partial_\alpha S(\mu_e, \alpha_e)$ is opposite of the sign of the expression in the bracket. Recalling that in the equilibrium $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$, so

$$\begin{aligned} \frac{\delta(\lambda\alpha_e - c)}{\lambda\alpha_e} + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e} &= -\delta\mathsf{W}(\mu_e) + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e} \\ &= -\delta\mathsf{W}(\mu_e) + \delta\lim_{r\to 0}\mathsf{W}(\mu_e) \geq 0, \end{aligned} \tag{2.63}$$

where the last inequality holds because $\mathsf{W}(\mu_e)$ is decreasing in r. This concludes that $\partial_\alpha S(\mu_e, \alpha_e) < 0$, and hence a small reduction of equilibrium α_e leads to a strict improvement of the social surplus function. $\square$

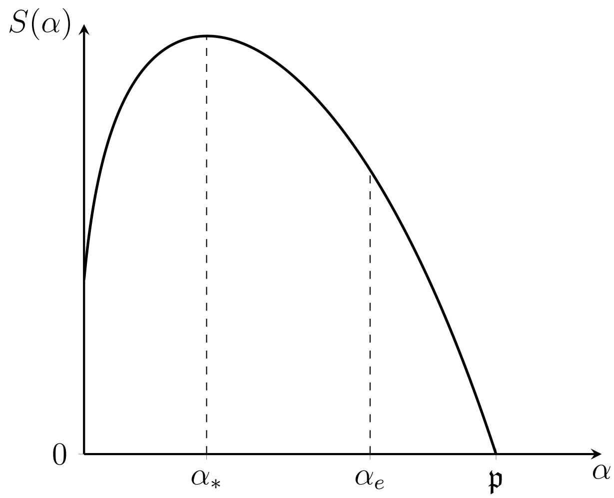


Figure 2.8: Social surplus with reputational externality

Figure 2.8 is the result of a simulation that plots the social surplus as a function of α, while implicitly satisfying $\mu = \mathsf{M}(\mu, \alpha)$ at every $\alpha \in [0, \mathfrak{p}]$. As it is expressed in this plot, the equilibrium termination point α_e is greater than the socially optimal point α_*. Hence, the equilibrium outcome is associated with early termination of projects, and predicts a lower tolerance for failure than what is socially efficient.

Chapter 3

Dissertation Conclusion

In chapter one of the dissertation, I ask this question that, how does a decision maker who is *uncertain* about the payoff distribution of two alternative choices operate the dynamics of experimentation? Understanding how an ambiguity averse agent values a project and determining the *price* of ambiguity are particularly important when the experimentation task is delegated to such agent. In this chapter, we develop a dynamic decision making framework that offers closed-form characterizations for the agent's optimal strategy as well as her valuation. Specifically, we assumed the DM has Multiplier preferences, that consists of two components. The discounted expected future return from both arms, and a penalty term that captures the extent of perturbation of probability specification relative to the benchmark model. We framed the decision making environment as a two-player differential game that DM plays against the nature, and found a closed-form expression for DM's value function in terms of her belief. Also, we have shown that in the equilibrium her optimal strategy is to select the safe arm of the project whenever her belief drops below a certain threshold, the value of which is controlled by all the parameters of the model and specifically the ambiguity aversion index. Our analysis offers sharp results on how much an ambiguity averse DM must be compensated to act as if she is not subject to ambiguity. In particular, one can send $\alpha \to \infty$ in the results of section 1.5 to predict the behavior of an ambiguity neutral agent. Finally, we explored the effect of an unambiguous constantly flowing information source in the dynamics of experimentation. It turned out that the exploration cut-off rises as a result of such provision, namely the DM waits longer to receive good news about the ambiguous arm of the project. We investigated the generated surplus due to this additional source and offered policy analysis on the efficient time to recruit an external expert to guide the experimentation process.

In chapter two of dissertation, I study the decentralized outcome of a dynamic economy populated by venture capitalists with unknown abilities and projects with observable qualities, where individuals randomly meet each other subject to search frictions. Since the venture capitalists fund their portfolio startups, the path to build a reputation is going to be costly for them. Therefore, the combination of costly learning and search frictions create a group of investors with high ability yet low reputation who rationally choose to stop investing (that is referred to as *dormant* investors in the text). In addition to this, the equilibrium shape of the matching sets between investors and startups rationalize a number of empirical findings in other papers, such as the relation between the tolerance for failure and investors' reputation as well as the prevalence of the investment approach, "*spray and pray*", as a consequence of cost reducing positive technological shocks.

I extend the baseline model to capture two sources of market failure: missing to internalize the innovation spillovers on the projects' side, and under-investment as a result of the reputational externality. In the former case, when there is positive spillovers from successful early stage projects to late stage businesses and the institutions are weak to protect the property rights and intangible assets of small young firms, the decentralized outcome of the economy could feature a complete market breakdown, caused by the under-investment of venture capitalists in early stage businesses and consequently ending up with sub-optimal levels of late stage companies and social surplus. Importantly, I show there are regions where higher search frictions could save the market from breakdown, as it reduces the opportunity cost of investing in early stage startups. In the latter case, when the deal flow of a single VC is *inversely* impacted by the reputation of other investors, the decentralized outcome of the economy features an inefficiently small size of high ability active investors and early termination of projects. A comparative static analysis on the equilibrium outcome suggests softening the extent of reputational externality has two distinct impacts: (i) Overall, VCs become less patient, and the proportion of high ability active investors falls; (ii) The equilibrium value of reputation building falls, thereby relaxing the constraint for investing in early stage startups.

As a possible future step, one could extend the introduced model of this chapter to an economy where there is *two-sided* incomplete information and hence two-sided learning, that is the projects' types as well as the investors' types are unknown. This is a challenging question because now both sides of the economy will have long-run reputational concerns. It naturally finds its applications in other contexts that feature two sided learning: for example in the labor market, where employers and employees jointly learn about their type as well as their partner's; Or in the educational systems, where there are incomplete information about the qualities of schools as well as students.

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Appendix A

Appendix to chapter 1

Proof of lemma 1

For every finite T and $h \in \mathcal{H}$ the integral can be simplified as:

$$\begin{aligned} \delta \int_0^T e^{-\delta t} H\left(\mathsf{P}_t^h; \mathsf{P}_t\right) \mathrm{d}t &= \delta \int_0^T e^{-\delta t} \mathsf{E}^h \left[\log L_t^h\right] \mathrm{d}t \\ &= \delta \int_0^T e^{-\delta t} \mathsf{E}^h \left[(h \cdot B)_t - \frac{1}{2}\left(h^2 \cdot \imath\right)_t\right] \mathrm{d}t \\ &= \delta \int_0^T e^{-\delta t} \mathsf{E}^h \left[\left(h \cdot B^h\right)_t + \frac{1}{2}\left(h^2 \cdot \imath\right)_t\right] \mathrm{d}t \end{aligned} \tag{A.1}$$

Since $\{B_t^h : t \leq T\}$ is P_T^h-Brownian motion and h is bounded, the first term in above has zero expectation, leaving us only with the second term, for which integration by part yields

$$\mathsf{E}^h \left[\frac{\delta}{2} \int_0^T e^{-\delta t} \left(h^2 \cdot \imath\right)_t \mathrm{d}t\right] = \mathsf{E}^h \left[-\frac{1}{2} e^{-\delta T} (h^2 \cdot \imath)_T + \frac{1}{2} \int_0^T e^{-\delta t} h_t^2 \mathrm{d}t\right]. \tag{A.2}$$

The first term inside expectation is uniformly bounded over $\Omega \times \mathbb{R}_+$ and goes to zero in a point-wise sense as $T \to \infty$. Therefore,

$$H\left(\mathsf{P}^h; \mathsf{P}\right) = \frac{1}{2} \lim_{T \to \infty} \mathsf{E}^h \left[\int_0^T e^{-\delta t} h_t^2 \mathrm{d}t\right] = \frac{1}{2} \mathsf{E}^h \left[\int_0^\infty e^{-\delta t} h_t^2 \mathrm{d}t\right], \tag{A.3}$$

where in the last relation we used the monotone convergence theorem. The limit is finite due to the boundedness of $h \in \mathcal{H}$. It is worthwhile to point out that for integrals with finite upper limit T we appeal to P_T^h, and for the infinite time integral we use P^h. This replacement does not cause any problem because of the consistency of $\{\mathsf{P}_T^h : T \in \mathbb{R}_+\}$ with P^h as explained in item (ii) of subsection 1.3. □

Proof of proposition 1

Let us define

$$V^T(p;\mu,h) := \mathsf{E}^h\left[\delta\int_0^T e^{-\delta t}\left(\mathrm{d}y_{1,t}+\mathrm{d}y_{2,t}+\alpha H\left(P_t^h;P_t\right)\mathrm{d}t\right)\right]. \tag{A.4}$$

For the first two components of (A.4), one just need to recall that over every finite interval $[0,T]$, the pair $\left\{B_t^h,\mathcal{F}_t : t\leq T\right\}$ is a Brownian motion under P^h. Consequently, stochastic integrals of bounded processes with respect to that are martingales and hence average out to zero. Using equation (1.11) yields to

$$\begin{aligned}\mathsf{E}^h\left[\delta\int_0^T e^{-\delta t}\left(\mathrm{d}y_{1,t}+\mathrm{d}y_{2,t}\right)\right] &= \mathsf{E}^h\left[\delta\int_0^T e^{-\delta t}\left\{\left((1-\mu_t)r+\mu_t m(p_t^h)+\sigma\sqrt{\mu_t}h_t\right)\mathrm{d}t+\sigma\sqrt{\mu_t}\;d\bar{B}_t^h\right\}\right]\\ &= \mathsf{E}^h\left[\delta\int_0^T e^{-\delta t}\left((1-\mu_t)r+\mu_t m(p_t^h)+\sigma\sqrt{\mu_t}h_t\right)\mathrm{d}t\right]\end{aligned} \tag{A.5}$$

The entropy component of the integrand in (A.4) has already been analyzed in the proof of lemma 1 and specifically in equation (A.2). Combining that analysis with (A.5) leads to

$$\begin{aligned}V^T(p;\mu,h) = \mathsf{E}^h&\left[\delta\int_0^T e^{-\delta t}\left((1-\mu_t)r+\mu_t m(p_t^h)+\sigma\sqrt{\mu_t}h_t+\frac{\alpha}{2\delta}h_t^2\right)\mathrm{d}t\right]\\ &-\frac{1}{2}e^{-\delta T}\mathsf{E}^h\left[(h^2\cdot\imath)_T\right]\end{aligned} \tag{A.6}$$

Since $h\in\mathcal{H}$ is bounded, the second term in (A.6) vanishes as $T\to\infty$. For the first term in (A.6) we apply the dominated convergence theorem and use the fact that $\left\{\mathsf{P}_T^h : T\in\mathbb{R}_+\right\}$ is consistent with $\mathsf{P}^h\in\Delta\left(\Omega,\mathcal{F}_\infty\right)$[1] — as explained in (ii) of subsection 1.3 — while sending $T\to\infty$. This concludes the proof of $\lim_{T\to\infty}V^T(p;\mu,h)=V(p;\mu,h)$, thereby leading to (1.15). □

Proof of theorem 1

For the proof of this theorem we need the following lemma proved in [77] using the Mollification method.

Lemma 7. *Let $w\in C([0,1])$ be a given Lipschitz function. For every $x_0\in[0,1)$, with $(\xi_1,\xi_2)\in D_+w(x_0)$ (resp. $(\xi_1,\xi_2)\in D_-w(x_0)$), there exists a twice-continuously differentiable function $\psi\in C^2([0,1])$, that satisfies:*

[1] $\Delta\left(\Omega,\mathcal{F}\right)$ denotes the set of all probability measures on the measure space $(\Omega,\mathcal{F})$.

(i) $\psi(x_0) = w(x_0)$ *and* $\psi(x) > w(x)$ *(resp.* $\psi(x) < w(x)$*) everywhere else.*

(ii) $\psi'(x_0) = \xi_1$ *and* $\psi''(x_0) = \xi_2$.

To continue the proof of the theorem assume $\exists w \in C([0,1])$ satisfying all presumptions of the theorem. We also use the following notation throughout the proof:

$$g(p,\mu,h) := (1-\mu)r + \mu m(p) + \sigma\sqrt{\mu}h + \frac{\alpha}{2\delta}h^2 \tag{A.7}$$

Recall that $\{p_t^h\}$ follows the diffusion process $\mathrm{d}p_t^h = \sqrt{\mu_t \Phi(p_t^h)}\mathrm{d}\bar{B}_t^h$, where $\bar{B}^h$ is a $\mathcal{G}$-Brownian motion under P^h over any finite horizon. Suppose at $t = 0$, $p_t^h = p$, and $\left(\partial_- w(p), \partial_-^2 w(p)\right) \in D_- w(p)$, then from lemma 7 one can find $\psi \in C^2([0,1])$ such that $\psi(p) = w(p)$, $\psi(x) < w(x)$ elsewhere, $\psi'(p) = \partial_- w(p)$ and $\psi''(p) = \partial_-^2 w(p)$. Then, for every $t > 0$:

$$\begin{aligned} e^{-\delta t} w(p_t^h) - w(p) &\geq e^{-\delta t}\psi(p_t^h) - \psi(p) \\ &= \int_0^t e^{-\delta s}\left[\sqrt{\mu_s \Phi(p_s^h)}\psi'(p_s^h)\mathrm{d}\bar{B}_s^h + \left(\frac{1}{2}\mu_s\Phi(p_s^h)\psi''(p_s^h) - \delta\psi(p_s^h)\right)\mathrm{d}s\right] \end{aligned} \tag{A.8}$$

Therefore, taking the expectation w.r.t P^h on both sides and using the martingale property of $\bar{B}^h$ lead to:

$$\frac{1}{t}\left(\mathsf{E}^h\left[e^{-\delta t}w(p_t^h)\right] - w(p)\right) \geq \frac{1}{t}\int_0^t e^{-\delta s}\mathsf{E}^h\left[\frac{1}{2}\mu_s\Phi(p_s^h)\psi''(p_s^h) - \delta\psi(p_s^h)\right]\mathrm{d}s \tag{A.9}$$

Since $p_s^h \to p$, P^h-almost surely as $s \to 0$, and hence in distribution, then taking the limit on both sides as $t \to 0$ yields:

$$\begin{aligned} \liminf_{t\to 0}\frac{1}{t}\left(\mathsf{E}^h\left[e^{-\delta t}w(p_t^h)\right] - w(p)\right) &\geq \frac{1}{2}\mu\Phi(p)\psi''(p) - \delta\psi(p) \\ &= \frac{1}{2}\mu\Phi(p)\partial_-^2 w(p) - \delta w(p) \end{aligned} \tag{A.10}$$

Let $\mu = \mu^*$ in the above expression. Since w is the viscosity solution for (1.22), then from the *supersolution* property (1.21b) it holds that $\inf_h\left\{g(p,\mu^*,h) + \frac{\mu^*}{2\delta}\Phi(p)\partial_-^2 w(p) - w(p)\right\} \geq 0$, therefore for every h:

$$\frac{1}{2}\mu^*\Phi(p)\partial_-^2 w(p) - \delta w(p) \geq -\delta g(p,\mu^*,h). \tag{A.11}$$

Combining the last two equations amounts to

$$\liminf_{t\to 0}\frac{1}{t}\left(\mathsf{E}^h\left[e^{-\delta t}w(p_t^h)\right] - w(p)\right) \geq -\delta g(p,\mu^*,h). \tag{A.12}$$

This is a fundamental implication that one would have obtained much easier under Ito's lemma if w was twice continuously differentiable.

Next, for every $t > 0$

$$\begin{aligned}
\mathsf{E}^h\left[e^{-\delta t} w(p_t^h)\right] - w(p) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_t^{t+\varepsilon} \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right] \mathrm{d}s - \int_0^{\varepsilon} \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right] \mathrm{d}s\right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_\varepsilon^{t+\varepsilon} \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right] \mathrm{d}s - \int_0^{t} \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right] \mathrm{d}s\right) \\
&= \lim_{\varepsilon \to 0} \int_0^t \frac{\mathsf{E}^h\left[e^{-\delta(s+\varepsilon)} w(p_{s+\varepsilon}^h)\right] - \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right]}{\varepsilon} \mathrm{d}s \qquad \text{(A.13)} \\
&\geq \int_0^t \liminf_{\varepsilon \to 0} \frac{\mathsf{E}^h\left[e^{-\delta(s+\varepsilon)} w(p_{s+\varepsilon}^h)\right] - \mathsf{E}^h\left[e^{-\delta s} w(p_s^h)\right]}{\varepsilon} \mathrm{d}s \\
&\geq -\delta \int_0^t \mathsf{E}^h\left[e^{-\delta s} g(p_s^h, \mu_s^*, h_s)\right] \mathrm{d}s,
\end{aligned}$$

where in the second last inequality we used the Fatou's lemma, given that w is bounded from above, and in the last inequality we used the inequality (A.12). Rearranging the above terms implies that for every $t > 0$

$$w(p) \leq \mathsf{E}^h\left[\int_0^t \delta e^{-\delta s} g(p_s^h, \mu_s^*, h_s) \mathrm{d}s\right] + \mathsf{E}^h\left[e^{-\delta t} w(p_t^h)\right] \qquad \text{(A.14)}$$

Since every $h \in \mathcal{H}$ is assumed bounded, then one can use dominated convergence theorem and send $t \to \infty$ to obtain

$$w(p) \leq \mathsf{E}^h\left[\int_0^\infty \delta e^{-\delta s} g(p_s^h, \mu_s^*, h_s) \mathrm{d}s\right]. \qquad \text{(A.15)}$$

Taking the infimum over all $h \in \mathcal{H}$ thus implies

$$w(p) \leq \inf_{h \in \mathcal{H}} \mathsf{E}^h\left[\int_0^\infty \delta e^{-\delta s} g(p_s^h, \mu_s^*, h_s) \mathrm{d}s\right], \qquad \text{(A.16)}$$

and consequently

$$w(p) \leq \sup_{\mu} \inf_{h \in \mathcal{H}} \mathsf{E}^h\left[\int_0^\infty \delta e^{-\delta s} g(p_s^h, \mu_s, h_s) \mathrm{d}s\right]. \qquad \text{(A.17)}$$

For the reverse direction of the above inequality we shall use the superdifferentials of w at p and the viscosity subsolution inequality (1.21a). Let $\left(\partial_+ w(p), \partial_+^2 w(p)\right) \in D_+ w(p)$. Using an analogous argument as above we reach:

$$\limsup_{t \to 0} \frac{1}{t}\left(\mathsf{E}^h\left[e^{-\delta t} w(p_t^h)\right] - w(p)\right) \leq \frac{1}{2}\mu \Phi(p) \partial_+^2 w(p) - \delta w(p) \qquad \text{(A.18)}$$

Choose an arbitrary $\mu \in \mathcal{U}$ and set $h = \tilde{h} = -\alpha^{-1}\sigma\delta\sqrt{\mu}$ — the point achieving the infimum in the HJBI equation. Because of the subsolution property of w, it holds that $g(p,\mu,\tilde{h}) + \frac{\mu}{2\delta}\Phi(p)\partial_+^2 w(p) - w(p) \leq 0$, therefore

$$\limsup_{t\to 0} \frac{1}{t}\left(\mathsf{E}^h\left[e^{-\delta t}w(p_t^h)\right] - w(p)\right) \leq -\delta g(p,\mu,\tilde{h}). \tag{A.19}$$

Employing the same recipe of (A.13) – this time with limsup instead of liminf in the Fatou's lemma and using the lower bound instead of upper bound on w – we get

$$w(p) \geq \mathsf{E}^{\tilde{h}}\left[\int_0^t \delta e^{-\delta s} g(p_s^{\tilde{h}},\mu_s,\tilde{h}_s)\mathrm{d}s\right] + \mathsf{E}^{\tilde{h}}\left[e^{-\delta t}w(p_t^{\tilde{h}})\right]. \tag{A.20}$$

Using the dominated convergence theorem to send $t \to \infty$ implies

$$w(p) \geq \mathsf{E}^{\tilde{h}}\left[\int_0^\infty \delta e^{-\delta s} g(p_s^{\tilde{h}},\mu_s,\tilde{h}_s)\mathrm{d}s\right] \Rightarrow w(p) \geq \inf_{h\in\mathcal{H}} \mathsf{E}^h\left[\int_0^\infty \delta e^{-\delta s} g(p_s^h,\mu_s,h_s)\mathrm{d}s\right]. \tag{A.21}$$

Taking the supremum over all $\mu \in \mathcal{U}$ yields

$$w(p) \geq \sup_\mu \inf_{h\in\mathcal{H}} \mathsf{E}^h\left[\int_0^\infty \delta e^{-\delta s} g(p_s^h,\mu_s,h_s)\mathrm{d}s\right]. \tag{A.22}$$

Equations (A.17) and (A.22) together imply that $w = v$, that concludes the verification proof. □

Proof of theorem 2

For the proof of this proposition we need few lemmas.

Lemma 8. *For any $p \in (0,1)$ the value function is lower bounded by* $\max\left\{r, m(p) - \frac{\sigma^2\delta}{2\alpha}\right\}$.

Proof. By replacing nature's best response $h = -\alpha^{-1}\sigma\delta\sqrt{\mu}$ in (1.15), one gets the following payoff representation:

$$v(p) = \sup_\mu \mathsf{E}^\mu\left[\delta\int_0^\infty e^{-\delta t}\left((1-\mu_t)r + \mu_t m(p_t^\mu) - \mu_t\frac{\sigma^2\delta}{2\alpha}\right)\mathrm{d}t\right], \tag{A.23}$$

where E^μ and p_t^μ are resp. the probability measure and the posterior probability obtained from $h = -\alpha^{-1}\sigma\delta\sqrt{\mu}$. Furthermore, using the local-martingale property of $m(p_t^h)$, we get the following inequality for every $\mathcal{G}_0$-measurable control process, i.e $\mu_t \in \mathcal{G}_0$ for all $t \in \mathbb{R}_+$, and hence $\mu_t \equiv \mu$ (up to evanescence):

$$\begin{aligned} v(p) &= \sup_{\mu \in \mathcal{U}} \mathsf{E}^{\mu} \left[\delta \int_0^{\infty} e^{-\delta t} \left((1-\mu_t) r + \mu_t m(p_t) - \mu_t \frac{\sigma^2 \delta}{2\alpha} \right) \mathrm{d}t \right] \\ &\geq \sup_{\mu \in \mathcal{G}_0} \left\{ \mathsf{E}^{\mu} \left[\delta \int_0^{\infty} e^{-\delta t} \left((1-\mu) r + \mu m(p_0) - \mu \frac{\sigma^2 \delta}{2\alpha} \right) \mathrm{d}t \right] + \mathsf{E}^{\mu} \left[\delta \int_0^{\infty} e^{-\delta t} \mu x_t^h \mathrm{d}t \right] \right\} \end{aligned} \tag{A.24}$$

Here x^h is the local-martingale part of $m(p^h)$ resulted from lemma 1.10. Having set $\mu \in \mathcal{G}_0$, the expectation of the second term vanishes due to the P^h-local-martingale property of x^h and using the dominated convergence theorem for approximating the infinite horizon integral with finite counterparts. This proves the lower bound on $v(p)$.‖

Lemma 9. *Let $\mathcal{S}_i$ be subset of $[0,1]$ where the DM optimally chooses the i-th project if $p \in \mathcal{S}_i$, where $i \in \{1,2\}$. Then the value function is convex restricted to each of these subsets.*

Proof. On $\mathcal{S}_1$ the value function is identical to r, and hence is convex. On $\mathcal{S}_2$ the DM chooses the second arm and $\mu = 1$, hence

$$v(p) = m(p) - \frac{\sigma^2 \delta}{2\alpha} + \frac{1}{2\delta} \Phi(p) v''(p), \tag{A.25}$$

which implies that

$$\frac{1}{2\delta} \Phi(p) v''(p) = v(p) - m(p) + \frac{\sigma^2 \delta}{2\alpha} \geq \left(m(p) - \frac{\sigma^2 \delta}{2\alpha} \right) - m(p) + \frac{\sigma^2 \delta}{2\alpha} = 0. \tag{A.26}$$

Therefore, $v''(p) \geq 0$ and hence the restriction of v onto $\mathcal{S}_2$ is also convex.‖

Lemma 10. *The subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected subsets of $[0,1]$.*

Proof. First note that $[0,1] = \mathcal{S}_1 \cup \mathcal{S}_2$, therefore the case of one subset being the empty set and the other being the whole unit interval trivially passes the lemma. Now assume both subsets are non-empty, and suppose $\mathcal{S}_1$ is not connected. Therefore, it must contain two disjoint open intervals, say (a_1, b_1) and (a_2, b_2), such that $b_1 < a_2$. This means that $[b_1, a_2] \subset \mathcal{S}_2$. The continuity must holds at the boundaries, namely $v(b_1) = v(a_2) = r$, otherwise there appears an *arbitrage* opportunity for the DM and she could improve her strategy subsets, $\mathcal{S}_1$ and $\mathcal{S}_2$, so as to strictly be better off. Also, one can easily confirm from (A.23) that $v(\cdot)$ is a non-decreasing function in p. Since $v(\cdot)$ is always greater than or equal to r, then $v \equiv r$ on the entire $[b_1, a_2]$. This means essentially $[b_1, a_2] \subset \mathcal{S}_1$ that violates the initial assumption on $\mathcal{S}_1$. Therefore, $\mathcal{S}_1$ must be a connected subset of $[0, 1]$. We use the proof by contradiction again to show $\mathcal{S}_2$ is connected as well. Suppose it is not, then it contains two disjoint open sets, say (c_1, d_1) and (c_2, d_2) such that $d_1 < c_2$. Note that at the boundary points the continuity must hold — precisely to rule out the arbitrage — that means $v(d_1) = v(c_2) = r$. This means either $v \equiv r$ on (c_1, d_1), which then one should include this interval in $\mathcal{S}_1$, or there exists some point $z \in (c_1, d_1)$ such that $v(z) > r$. This violates the non-decreasingness of v, and hence concludes the proof.‖

The existence of cut-off strategy now falls out of the connectedness of $\mathcal{S}_1$ and $\mathcal{S}_2$ from previous lemma and monotonicity of $v(\cdot)$. It is thus left to prove the global convexity of $v(\cdot)$. For this denote the cut-off point by $\bar{p}$, and note that $\mathcal{S}_1 = [0, \bar{p}]$ and $\mathcal{S}_2 = (\bar{p}, 1]$.[2] So far, we know that v is separately convex on $\mathcal{S}_1$ and $\mathcal{S}_2$. To show that convexity is preserved on the whole region $[0, 1]$, we pick the arbitrary points $p_1 \in \mathcal{S}_1$ and $p_2 \in \mathcal{S}_2$ and an arbitrary mixing weight $\xi \in (0, 1)$. Define $p_\xi := \xi p_2 + (1-\xi)p_1$. If $p_\xi \in \mathcal{S}_1$, then $\xi v(p_2) + (1-\xi)v(p_1)$ is clearly greater than or equal to $v(p_\xi) = r$. Now suppose $p_\xi \in \mathcal{S}_2$, then

$$\xi v(p_2) + (1-\xi)v(p_1) = \xi v(p_2) + (1-\xi)v(\bar{p}) \geq v\left(\xi p_2 + (1-\xi)\bar{p}\right) \geq v(p_\xi) \tag{A.27}$$

where for the first inequality we used the convexity of v on $\mathcal{S}_2$, and for the second one we used the monotonicity of v and the fact that $p_1 \leq \bar{p}$. This concludes the global convexity of v, and hence the proof the theorem. □

Optimal constants for value function with unambiguous information source

The following list is the set of all boundary conditions required for the DM's best-responding:

$$\begin{aligned}
&(\text{value-matching}) : r + c_1\tilde{p}^{\lambda_1}(1-\tilde{p})^{1-\lambda_1} = m(\tilde{p}) - \frac{\sigma^2\delta}{2\alpha} + c_2\tilde{p}^{1-\lambda_2}(1-\tilde{p})^{\lambda_2} \\
&(\text{smooth-pasting}) : c_1\left(\frac{\lambda_1}{\tilde{p}} - \frac{1-\lambda_1}{1-\tilde{p}}\right)\tilde{p}^{\lambda_1}(1-\tilde{p})^{1-\lambda_1} \\
&\qquad\qquad = \left(\overline{\theta} - \underline{\theta}\right) + c_2\left(\frac{1-\lambda_2}{\tilde{p}} - \frac{\lambda_2}{1-\tilde{p}}\right)\tilde{p}^{1-\lambda_2}(1-\tilde{p})^{\lambda_2}
\end{aligned} \tag{A.28}$$

[2] It is not important whether $\bar{p}$ belongs to $\mathcal{S}_1$ or $\mathcal{S}_2$, since essentially the DM is indifferent between two arms when her belief is $\bar{p}$. However, since we laid out the HJB equation on $\mathcal{S}_2$, it is preferred to have an open set as the domain of the differential equation.

We still need a third condition to determine all unknown variables. Note that,

$$\mathrm{d}p = \begin{cases} \sqrt{2}p(1-p)\left[\varphi(\sigma)\mathrm{d}\overline{B} + \varphi(\gamma)\mathrm{d}\overline{W}\right] & \text{for } p \geq \tilde{p} \\ \sqrt{2}p(1-p)\varphi(\gamma)\mathrm{d}\overline{W} & \text{for } p < \tilde{p} \end{cases} \tag{A.29}$$

Let us call the conjectured value function $\tilde{v}$ on $[0, \tilde{p})$ by $\tilde{v}_-$ and on $(\tilde{p}, 1]$ by $\tilde{v}_+$. The threshold argument means to experiment for $p > \tilde{p}$ and to stop on $p < \tilde{p}$. Now suppose the agent instead of experimenting at $\tilde{p}$ stops for a period of Δt, in which $\Delta p \approx \sqrt{2}p(1-p)\varphi(\gamma)\sqrt{\Delta t}$, because the Bayesian formulae in the experimentation regime no longer applies. Then, the net gain from this deviation at $p = \tilde{p}$ must be negative if $\tilde{p}$ is the optimal cut-off point. The following variational analysis implies the gain for such a deviation:

$$\begin{aligned}
&\delta r\Delta t + (1-\delta\Delta t)\left[\frac{1}{2}v_-(p-\Delta p) + \frac{1}{2}v_+(p+\Delta p)\right] - v(p) \\
&\approx \delta r\Delta t - v(p) \\
&+ (1-\delta\Delta t)\left\{\frac{1}{2}\left[v_-(p) - (\Delta p)v_-'(p) + \frac{1}{2}(\Delta p)^2 v_-''(p)\right]\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{2}\left[v_+(p) + (\Delta p)v_+'(p) + \frac{1}{2}(\Delta p)^2 v_+''(p)\right]\right\} \\
&= \frac{1}{2}(1-\tilde{p})^2\tilde{p}^2\varphi(\gamma)^2\left(\tilde{v}_-''(\tilde{p}) + v_+''(\tilde{p})\right)\Delta t - \delta\left(\tilde{v}() - r\right)\Delta t \leq 0,
\end{aligned} \tag{A.30}$$

therefore,

$$\begin{aligned}
&\frac{1}{2}(1-\tilde{p})^2\tilde{p}^2\varphi(\gamma)^2\left(\tilde{v}_-''(\tilde{p}) + v_+''(\tilde{p})\right) \leq \delta\left(\tilde{v}() - r\right) = \varphi(\gamma)^2\tilde{p}^2(1-\tilde{p})^2\tilde{v}_-''(\tilde{p}) \\
&\qquad \Rightarrow \frac{1}{2}\left(\tilde{v}_-''(\tilde{p}) + v_+''(\tilde{p})\right) \leq \tilde{v}_-''(\tilde{p}).
\end{aligned} \tag{A.31}$$

By a mirror argument one can see

$$\frac{1}{2}\left(\tilde{v}_-''(\tilde{p}) + v_+''(\tilde{p})\right) \leq \tilde{v}_+''(\tilde{p}). \tag{A.32}$$

Consequently it holds that $\tilde{v}_-''(\tilde{p}) = \tilde{v}_+''(\tilde{p})$, leading to the super-contact condition:

$$\frac{c_1\delta}{\tilde{p}^2(1-\tilde{p})^2\varphi(\gamma)^2}\tilde{p}^{\lambda_1}(1-\tilde{p})^{1-\lambda_1} = \frac{c_2\delta}{\tilde{p}^2(1-\tilde{p})^2\left(\varphi(\sigma)^2 + \varphi(\gamma)^2\right)}\tilde{p}^{1-\lambda_2}(1-\tilde{p})^{\lambda_2} \tag{A.33}$$

Therefore, the value of constants (c_1, c_2) are determined in terms of the cut-off point $\tilde{p}$:

$$c_1 = \frac{\frac{\sigma^2}{\gamma^2}\left(r - m(\tilde{p}) + \frac{\sigma^2\delta}{2\alpha}\right)}{\tilde{p}^{\lambda_1}(1-\tilde{p})^{1-\lambda_1}}, \; c_2 = \frac{\left(1 + \frac{\sigma^2}{\gamma^2}\right)\left(r - m(\tilde{p}) + \frac{\sigma^2\delta}{2\alpha}\right)}{\tilde{p}^{1-\lambda_2}(1-\tilde{p})^{\lambda_2}} \tag{A.34}$$

Proof of proposition 3

The associated equations for the cut-off probabilities in each case are expressed in (1.29) and (1.36). First, we show that $\Lambda \geq \lambda$ for any combination of variables. One can easily check from definition of Λ and λ that $\Lambda \geq \lambda$ iff

$$\frac{\sigma^2}{\gamma^2}\sqrt{1+\beta\gamma^2} + \left(1+\frac{\sigma^2}{\gamma^2}\right)\sqrt{1+\beta\frac{\sigma^2\gamma^2}{\sigma^2+\gamma^2}} \geq \sqrt{1+\beta\sigma^2}, \tag{A.35}$$

in that we denote $\beta := 8\delta/(\overline{\theta}-\underline{\theta})^2$. Because $\gamma^2 \geq \sigma^2\gamma^2/(\sigma^2+\gamma^2)$ the *lhs* is larger than

$$\left(1+\frac{2\sigma^2}{\gamma^2}\right)\sqrt{1+\beta\frac{\sigma^2\gamma^2}{\sigma^2+\gamma^2}}. \tag{A.36}$$

Therefore, a sufficient condition for (A.35) to hold is $\left(1+\frac{2\sigma^2}{\gamma^2}\right)^2 \geq \frac{\left(1+\frac{\sigma^2}{\gamma^2}\right)\left(1+\beta\sigma^2\right)}{1+\frac{\sigma^2}{\gamma^2}+\beta\sigma^2}$, which holds because

$$\left(1+\frac{2\sigma^2}{\gamma^2}\right) \geq 1 \geq \frac{1+\beta\sigma^2}{1+\frac{\sigma^2}{\gamma^2}+\beta\sigma^2}. \tag{A.37}$$

Now we verify that $\tilde{p} \geq \bar{p}$. For this, note that $\tilde{p} = 0$ if $\Lambda \leq \eta$, in that case $\lambda \leq \eta$ which implies $\bar{p} = 0$. For the region $\lambda > \eta$ both cut-offs are positive. They are equal to one if $\eta \geq 1$ and are strictly smaller than one if $\eta < 1$, in that case $\tilde{p} \geq \bar{p}$ because $\Lambda \geq \lambda$. □

Appendix B

Appendix to chapter 2

Proof of lemma 5

Suppose both matching value functions, i.e $v(\cdot,a)$ and $v(\cdot,b)$, are increasing in π. Then, the representation (2.9) implies that $w(\cdot)$ should be increasing in π as well. Conversely, assume $w(\cdot)$ is increasing in π, and hence almost everywhere differentiable on $[0,1]$[1], and recall that $v(\cdot,q)$ is the solution to the optimal stopping time problem (2.4). In that τ is the stopping time adapted to all possible future information. However, note that no information is released until the breakthrough time σ, hence τ only uses the current information. This means that I can restrict the optimization space to the set of all deterministic times:

$$\begin{aligned} v(\pi,q) &= \sup_{\tau\in\mathbb{R}_+} V(\pi,q;\tau) \\ V(\pi,q;\tau) &:= \int_0^\tau \left[r^{-1}c\left(e^{-rt}-1\right)+e^{-rt}\left(1+w(1)\right)\right]\lambda_q\pi e^{-\lambda_q t}\mathrm{d}t \\ &\quad + \left(1-\pi+\pi e^{-\lambda_q\tau}\right)\left[r^{-1}c\left(e^{-r\tau}-1\right)+e^{-r\tau}w(\pi_\tau)\right]. \end{aligned} \tag{B.1}$$

Since w is almost everywhere differentiable, then $V(\cdot,q;\tau)$ inherits this property too. Let us now define $\frac{\partial V}{\partial\pi}(\pi,q;\tau) := I_1+I_2+I_3$, where

$$\begin{aligned} I_1 &:= r^{-1}c\left[\frac{\lambda_q}{r+\lambda_q}\left(1-e^{-(r+\lambda_q)\tau}\right)-e^{-r\tau}\left(1-e^{-\lambda_q\tau}\right)\right], \\ I_2 &:= \frac{\left(1+w(1)\right)\lambda_q}{r+\lambda_q}\left(1-e^{-(r+\lambda_q)\tau}\right)-\left(1-e^{-\lambda_q\tau}\right)e^{-r\tau}w(\pi_\tau), \\ I_3 &:= e^{-r\tau}\left(1-\pi+\pi e^{-\lambda_q\tau}\right)w'(\pi_\tau)\frac{\partial\pi_\tau}{\partial\pi}. \end{aligned} \tag{B.2}$$

[1]This is due the seminal Lebesgue theorem; see chapter 6 of [63].

The expression for I_1 is zero when $\tau = 0$, and has positive derivative w.r.t τ, therefore, it is non-negative for all $\tau \geq 0$. The third term I_3 is obviously non-negative, because w is assumed increasing and due to the Bayes law $\partial \pi_\tau / \partial \pi > 0$. In regard to the second term:

$$\begin{aligned} I_2 &\geq \frac{(1 + w(1))\, \lambda_q}{r + \lambda_q} \left(1 - e^{-(r+\lambda_q)}\right) - \left(1 - e^{-\lambda_q \tau}\right) e^{-r\tau} w(1) \\ &\geq w(1) \left[\frac{\lambda_q}{\lambda_q + r} \left(1 - e^{-(r+\lambda_q)\tau}\right) - e^{-r\tau} \left(1 - e^{-\lambda_q \tau}\right) \right] \end{aligned} \tag{B.3}$$

The term in the bracket above is increasing in τ and equals zero at $\tau = 0$, therefore, it is always non-negative. To sum, $\partial V / \partial \pi \geq 0$ almost everywhere, and therefore V becomes increasing in π. Since $v(\pi, q) = \sup_\tau V(\pi, q; \tau)$, the matching value function $v(\cdot, q)$ must be increasing too. $\square$

Proof of proposition 4

Proof of part (i): At $\pi = 1$ the following fixed-point system falls out of (2.9) and the rearranged version of (2.5):

$$w(1) = \max_\chi \left\{ \frac{r^{-1}\kappa \left[v(1,a)\varphi_a \chi_a(1) + v(1,b)\varphi_b \chi_b(1) \right]}{1 + r^{-1}\kappa \left[\varphi_a \chi_a(1) + \varphi_b \chi_b(1) \right]} \right\} \tag{B.4a}$$

$$v(1,q) = \max \left\{ w(1), \frac{\lambda_q - c}{r + \lambda_q} + \frac{\lambda_q}{r + \lambda_q} w(1) \right\} \quad \text{for} \quad q \in \{a, b\} \tag{B.4b}$$

From (B.4b) it follows that

$$\chi_a(1) = 1 \Leftrightarrow r w(1) < \lambda_a - c, \tag{B.5a}$$

$$\chi_b(1) = 1 \Leftrightarrow r w(1) < \lambda_b - c. \tag{B.5b}$$

So there are three cases that could possibly arise from (B.5):

(a) $1 \notin \mathcal{M}_b \cup \mathcal{M}_a \Rightarrow w(1) = 0$, yet this never happens because $\lambda_b > c$ implies $v(1, b) > 0$ and hence $w(1) > 0$.

(b) $1 \in \mathcal{M}_b \cap \mathcal{M}_a^c$ so

$$w(1) = \frac{r^{-1}\kappa\varphi_b \left(\lambda_b - c\right)}{r + \lambda_b + \kappa\varphi_b}. \tag{B.6}$$

The pair $v(1, a) = w(1)$ and $v(1, b) = (1 + r/\kappa\varphi_b)\, w(1)$ satisfy (and is the only solution of) the fixed-point system (B.4) if $\lambda_a - c \leq \frac{\kappa\varphi_b(\lambda_b - c)}{r + \lambda_b + \kappa\varphi_b}$.

(c) $1 \in \mathcal{M}_b \cap \mathcal{M}_a$ so

$$w(1) = \frac{r^{-1}\kappa\varphi_b\left(\lambda_b - c\right)\left(r+\lambda_a\right) + r^{-1}\kappa\varphi_a\left(\lambda_a - c\right)\left(r+\lambda_b\right)}{\left(r+\lambda_a\right)\left(r+\lambda_b\right) + \kappa\varphi_b\left(r+\lambda_a\right) + \kappa\varphi_a\left(r+\lambda_b\right)}. \tag{B.7}$$

If $\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$ the above $w(1)$ satisfies (B.5). Moreover, the obtained $v(1,a)$ and $v(1,b)$ from (B.4b) once replaced as the optimization input in the *rhs* of (B.4a) confirms the $w(1)$ in (B.7), thereby closing the equilibrium loop.‖

Proof of part (ii): In the sequel I use the symbol $\underline{\partial} A$ to denote the lower boundary of the subset $A \subset [0,1]$. To establish the convexity of $\mathcal{M}_b$, I first derive a useful identity for any strictly positive point $x \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$. Since x is a *lower-boundary* point for $\mathcal{M}_b$, then a generic VC finds it optimal to terminate the funding when π approaches down to x. Importantly, at this point the principles of continuous and smooth fit ([12]) must hold. The VC's outside option just below x is equal to $w(x)$ that is supported by the option value of meeting an a-type startup because $x \in \mathcal{M}_a$, so

$$v(x,b) = w(x) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a} v(x,a) \quad \text{and} \quad v'(x,b) = w'(x) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a} v'(x,a). \tag{B.8}$$

Now let $\Omega(x,q) := -c + \lambda_q x\left(1 + w(1)\right)$ and $\Gamma(x,q) := r + \lambda_q x$. Then, employing the HJB equations on the continuation region leads to

$$\frac{v'(x,b)}{v'(x,a)} = \frac{\lambda_a}{\lambda_b}\frac{\Omega(x,b) - \Gamma(x,b)v(x,b)}{\Omega(x,a) - \Gamma(x,a)v(x,a)}. \tag{B.9}$$

The previous two systems of equations give rise to

$$\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}\Gamma(x,a) - \Gamma(x,b)\right) v(x,a) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a}\Omega(x,a) - \Omega(x,b) \tag{B.10a}$$

$$\Rightarrow \frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a} - 1\right) r v(x,a) = -c\left(\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a} - 1\right) - \frac{rx\lambda_b\left(1+w(1)\right)}{r+\kappa\varphi_a} \tag{B.10b}$$

Now assume to the contrary that $\mathcal{M}_b$ is not connected, hence, it contains at least two separate open sets, say (x_0, x_1) and (x_2, x_3). This implies that $[x_1, x_2] \subset \mathcal{M}_a$, because otherwise w assumes zero at some point in this interval which violates the monotonicity of w. Therefore, $x_2 \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$, and (B.10b) holds at x_2. I claim that $x_0 \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$ too, because otherwise x_0 would be the lower boundary point at which $v(\cdot, b)$ smoothly meets the zero function, hence applying continuous and smooth fit to equation (2.10) yields

$$x_0 = \frac{c}{\lambda_b\left(1+w(1)\right)}. \tag{B.11}$$

This expression for x_0 leads to an upper-bound for $v(x_2, a)$ using (B.10b):

$$\begin{aligned} \frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}-1\right) rv(x_2,a) &\leq -c\left(\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a}-1\right)-\frac{rx_0\lambda_b\left(1+w(1)\right)}{r+\kappa\varphi_a} \\ &= \frac{c\kappa\varphi_a}{r+\kappa\varphi_a}\left(1-\frac{\lambda_b}{\lambda_a}\right)<0. \end{aligned} \tag{B.12}$$

This means that $v(x_2, a) < 0$, hence a contradiction results. Therefore, x_0 and x_2 both belong to $\mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$. One can now apply (B.10b) at these two points and subtract their corresponding sides from each other:

$$\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}-1\right) r\left[v(x_2,a)-v(x_0,a)\right] = -r\left(x_2-x_0\right)\frac{\lambda_b\left(1+w(1)\right)}{r+\kappa\varphi_a} \tag{B.13}$$

The *lhs* to this equation is positive because of the monotonicity of $v(\cdot, a)$, but the *rhs* is negative, hence a contradiction is resulted, thereby proving the connectedness of $\mathcal{M}_b$.‖

Proof of part (iii):
High cost regime: First, I show in this regime $\mathcal{M}_a$ cannot have a lower boundary point in $\mathcal{M}_b$, that is $\underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b = \emptyset$. Toward the contradiction assume $\exists y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$. Then, a similar analysis to the previous part yields

$$\left(\frac{\lambda_b}{\lambda_a}-1\right) rv(y,b) = -c\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a}-1\right)+\frac{ry\lambda_b\left(1+w(1)\right)}{\kappa\varphi_b}. \tag{B.14}$$

In light of lemma 4, such a y is a global maximum for $v(\cdot, b)/v(\cdot, a)$ on the region $w > 0$, therefore, conditioned on the existence of the second derivative, it must be non-positive at y^+, so

$$\frac{v''(y,b)}{v(y,b)} \leq \frac{v''(y,a)}{v(y,a)} \Rightarrow v''(y,b) \leq \frac{r+\kappa\varphi_b}{\kappa\varphi_b}v''(y,a). \tag{B.15}$$

Next, I find an equation for the second derivative by differentiating the HJB equation (2.5) on the continuation region:

$$\begin{aligned} rv'(y,q) &= \lambda_q\left(1+w(1)-v(y,q)\right)-\lambda_q yv'(y,q) \\ &\quad -\lambda_q(1-2y)v'(y,q)-\lambda_q y(1-y)v''(y,q) \end{aligned} \tag{B.16}$$

Substituting $v'(\cdot, q)$ from the HJB in the above equation leads to

$$\begin{aligned} \lambda_q y(1-y)v''(y,q) &= \lambda_q\left(1+w(1)-v(y,q)\right)-\frac{\left(r+\lambda_q(1-y)\right)}{\lambda_q y(1-y)}\times ... \\ &\quad ...\left[-c+\lambda_q y\left(1+w(1)\right)-\left(r+\lambda_q y\right)v(y,q)\right] \\ &= -\frac{r\left(1+w(1)\right)}{1-y}+\frac{r+\lambda_q}{\lambda_q y(1-y)}rv(y,q)+\frac{c\left(r+\lambda_q(1-y)\right)}{\lambda_q y(1-y)}. \end{aligned} \tag{B.17}$$

Plugging the second derivates from above into (B.15) and applying some rearrangements yield the following *equivalent* relation

$$\begin{aligned} rv(y,b)\left(\frac{\lambda_b}{\lambda_a}-1\right)\left(1+\frac{r}{\lambda_a}+\frac{r}{\lambda_b}\right) &\geq \left[ry\left(1+w(1)\right)-c(1-y)\right]\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a}-1\right) \\ &\quad -\frac{cr}{\lambda_b}\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b^2}{\lambda_a^2}-1\right). \end{aligned} \tag{B.18}$$

Then, one can substitute (B.14) in above and apply several regroupings to obtain:

$$y\left\{(1+w(1))\left[\lambda_a\left(r+\lambda_b\right)-\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\right]-c\left(\lambda_b+r^{-1}\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\right)\right\}\geq cr \tag{B.19}$$

I would then substitute $w(1)$ from (B.6) in above and get an equivalent conditions to (B.15) that is *only* in terms of primitives:

$$\begin{aligned} \frac{cr^2}{r+\kappa\varphi_b}\left(1+\frac{\kappa\varphi_b}{r+\lambda_b}\right)+cy\lambda_b\left(1+\frac{r}{r+\lambda_b}\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\right) \\ \leq y\left[\lambda_a\left(r+\lambda_b\right)-\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\right] \end{aligned} \tag{B.20}$$

Then, I am going to show that the *lhs* above is always greater than the *rhs* thus there is no $y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$. Obviously at $y = 0$ the *lhs* is greater than the *rhs*. At $y = 1$, the *rhs* is increasing in λ_a, so can be upper-bounded when λ_a assumes its maximum level in the high-cost regime, i.e $c+\frac{\kappa\varphi_b(\lambda_b-c)}{r+\lambda_b+\kappa\varphi_b}$. Therefore the *rhs* of (B.20) at $y = 1$ is upper-bounded as

$$\lambda_a\left(r+\lambda_b\right)-\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\leq c\left(r+\lambda_b\right). \tag{B.21}$$

However, the *lhs* of (B.20) takes $c(r+\lambda_b)$ at $y = 1$. So (B.20) can never be satisfied, therefore in the high cost regime $\mathcal{M}_a$ can not have a lower boundary point in $\mathcal{M}_b$. Given $1 \notin \mathcal{M}_a$ and the monotonicity of w on $\mathcal{M}_b^c$, the only possible candidate for a non-empty $\mathcal{M}_a$ is (α_a, β_a) such that $\alpha_a < \alpha_b := \inf \mathcal{M}_b$. Because of optimality, $v(\cdot, a)$ must smoothly meet the zero function at α_a, so similar analysis to (B.11) would imply $\alpha_a = c/\lambda_a(1 + w(1))$, in that $w(1)$ follows (B.6). Further, the superharmonic condition for $v(\cdot, b)$ requires that at $\pi = \alpha_a$

$$0\geq[\mathcal{L}_bv](\alpha_a,b)-rv(\alpha_a,b)-c=\lambda_b\alpha_a\left(1+w(1)\right)-c=\left(\frac{\lambda_b}{\lambda_a}-1\right). \tag{B.22}$$

However, this never holds, because the rightmost side above is positive. So the only continuation set that survives the high-cost regime is $\mathcal{M}_a = \emptyset$.

Low cost regime: Note that in this regime $w(1)$ follows (B.7). I first prove in equilibrium it must be that $\mathcal{M}_a \subset \mathcal{M}_b$. We have seen in the part (i) that $1 \in \mathcal{M}_a \cap \mathcal{M}_b$ in this regime. To show the above set inclusion, I prove $\alpha_a := \inf \mathcal{M}_a \in \mathcal{M}_b$, that is the lowest boundary point of $\mathcal{M}_a$ denoted by α_a is contained in $\mathcal{M}_b$. Toward contradiction assume $\alpha_a < \alpha_b$, where $\alpha_b = \inf \mathcal{M}_b$. Examining the superharmonicity of $v(\cdot, b)$ on $[0, \alpha_a]$ leads to

$$\begin{aligned}\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c = \lambda_b \pi \left(1 + w(1)\right) - c &= \frac{\lambda_b}{\lambda_a}\lambda_a \pi \left(1 + w(1)\right) - c \\ &= \frac{\lambda_b}{\lambda_a}\lambda_a \left(\pi - \alpha_a\right)\left(1 + w(1)\right) + \left(\frac{\lambda_b}{\lambda_a} - 1\right) c.\end{aligned} \tag{B.23}$$

As π approaches α_a from below, the first term above converges to zero while the second term remains a positive constant. Therefore, $\exists\ \pi_0 < \alpha_a$ such that $\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c > 0$ for all $\pi_0 < \pi \leq \alpha_a$. This violates the superharmonicity of $v(\cdot, b)$, so there can be no equilibrium in which the lowest boundary point $\alpha_a \notin \mathcal{M}_b$. Next, I show having $\mathcal{M}_a \subset \mathcal{M}_b$ leads us to the connectedness of $\mathcal{M}_a$. Because of optimality of $v(\cdot, b)$ the principles of continuous and smooth fit hold at $\pi = \alpha_b$ with the zero outside option. Combining this with (2.10) implies the following expression for $v(\cdot, b)$:

$$\begin{aligned}v(\pi, b) = &-\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\pi \\ &+ \left\{\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right\}\left(\frac{1 - \pi}{1 - \alpha_b}\right)^{1 + r/\lambda_b}\left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b},\end{aligned} \tag{B.24}$$

with α_b following (B.11). Furthermore, the above value function is convex if and only if

$$\left\{\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right\} \geq 0. \tag{B.25}$$

Substituting α_b in this leads to an equivalent condition for convexity:

$$\frac{c}{r} - \frac{c}{r + \lambda} - \frac{c^2}{r(r + \lambda_b)(1 + w(1))} = \frac{c}{r(r + \lambda_b)}\left(\lambda_b - \frac{c}{1 + w(1)}\right) \geq 0. \tag{B.26}$$

The above condition always holds because $\lambda_b > c$ and $w(1) > 0$, therefore $v(\cdot, b)$ followed from (B.24) is a convex function. Now define $[\mathcal{D}_a v](\pi, a) := [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c$, and note that from the HJB equation

$$[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b}{r + \kappa\varphi_b}(\lambda_b - \lambda_a)\frac{rv(\pi, b) + c}{\lambda_b} + \frac{r\lambda_a\pi(1 + w(1)) - cr}{r + \kappa\varphi_b}. \tag{B.27}$$

Consequently, convexity of $v(\cdot, b)$ implies

$$\frac{\partial^2}{\partial \pi^2}[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b(\lambda_b - \lambda_a)}{(r + \kappa\varphi_b)\,\lambda_b}v''(\pi, b) < 0. \tag{B.28}$$

Therefore, $[\mathcal{D}_a v](\cdot, a)$ is a concave function in π. Were $\mathcal{M}_a$ not be connected then at least it has two disjoint components, say (x_1, x_2) and (x_3, x_4) where $x_2 < x_3$. Superharmonicity jointly with the satisfaction of Bellman equation on the continuation region require that $[\mathcal{D}_a v](\cdot, a)$ is negative just below x_1, is zero on $[x_1, x_2]$, becomes negative again on (x_2, x_3), followed by being zero on (x_3, x_4). This pattern is not consistent with the concavity of $[\mathcal{D}_a v](\cdot, a)$, therefore $\mathcal{M}_a$ must be connected. □

Proof of theorem 3

I prove the assertion only for the high-cost regime, as the proof of other case follows the same logic, but is just lengthier. From proposition 4, we know in this regime the only matching sets that survive in the equilibrium are $\mathcal{M}_a = \emptyset$ and $\mathcal{M}_b = (\alpha_b, 1]$, where α_b is found via the continuous and smooth fit principles as

$$\alpha_b = \frac{c}{\lambda_b\,(1 + w(1))}. \tag{B.29}$$

Also, from the construction of that proposition we know that the following profile embodies the only candidate for an equilibrium with increasing $C^1[0, 1]$ value functions on $(\alpha_b, 1]$:

$$w(\pi) = \frac{\kappa\varphi_b}{r + \kappa\varphi_b} v(\pi, b) \tag{B.30a}$$

$$v(\pi, a) = w(\pi) \tag{B.30b}$$

$$\begin{aligned} v(\pi, b) = {} & -\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\pi \\ & + \left\{\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right\}\left(\frac{1-\pi}{1-\alpha_b}\right)^{1+r/\lambda_b}\left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b} \end{aligned} \tag{B.30c}$$

And all equal to zero on $[0, \alpha_b]$. Therefore, our only task here is to employ a *verification* scheme to show that the above value functions are indeed the optimal equilibrium values. I divide the proof into three steps: (a) verifying the majorizing and superharmonicity conditions; (b) using these two and applying a Martingale method argument to establish the *optimality* of the above profile of the value functions; (c) for large r the Banach fixed point theorem is applied and proves the uniqueness of the identified equilibrium in the larger space of bounded continuous functions.

Step (a):

Majorizing. This step is quite straightforward because in (B.30) $w = v(\cdot, a)$ and $v(\cdot, b) \geq$

$w = \frac{\kappa\varphi_b}{r+\kappa\varphi_b} v(\cdot, b)$.

Superharmonicity of $v(\cdot, b)$. Obviously the superharmonic condition holds with equality on $(\alpha_b, 1]$ because of the Bellman equation. However, it needs to be checked on $[0, \alpha_b]$ as it carried out below:

$$[\mathcal{L}_b v](\pi, b) - rv(\pi, b) - c = \lambda_b \pi \left(1 + w(1)\right) - c \leq \lambda_b \alpha_b \left(1 + w(1)\right) - c = 0. \tag{B.31}$$

Superharmonicity of $v(\cdot, a)$. Remember that in the high cost regime $\mathcal{M}_a = \emptyset$, thus $v(\cdot, a) = w(\cdot)$. So on $[0, \alpha_b]$:

$$\begin{aligned} [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c &= \lambda_a \pi \left(1 + w(1)\right) - c \\ &\leq \lambda_b \alpha_b \left(1 + w(1)\right) - c \leq 0, \end{aligned} \tag{B.32}$$

where in the last inequality I used the expression (B.29) for α_b, that consequently verifies the superharmonicity on $[0, \alpha_b]$. The analysis of the superharmonicity of $v(\cdot, a)$ on $(\alpha_b, 1]$ however needs a little more work:

$$\begin{aligned} [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c &= \left[\mathcal{L}_a \left(\frac{\kappa\varphi_b}{r + \kappa\varphi_b} v\right)\right](\pi, b) - \frac{r\kappa\varphi_b}{r + \kappa\varphi_b} v(\pi, b) - c \\ &= \frac{\kappa\varphi_b}{r + \kappa\varphi_b} \left([\mathcal{L}_a v](\pi, b) - rv(\pi, b) - c\right) \\ &\qquad + \frac{r\lambda_a \pi}{r + \kappa\varphi_b} \left(1 + w(1)\right) - \frac{cr}{r + \kappa\varphi_b} \\ &= -\frac{\kappa\varphi_b}{r + \kappa\varphi_b} \left[(\mathcal{L}_b - \mathcal{L}_a) v\right](\pi, b) + \frac{r\lambda_a \pi}{r + \kappa\varphi_b} \left(1 + w(1)\right) - \frac{cr}{r + \kappa\varphi_b} \\ &= -\frac{\kappa\varphi_b}{r + \kappa\varphi_b} \left(\lambda_b - \lambda_a\right) \pi \left(1 + w(1) - v(\pi, b) - (1 - \pi) v'(\pi, b)\right) \\ &\quad + \frac{r\lambda_a \pi}{r + \kappa\varphi_b} \left(1 + w(1)\right) - \frac{cr}{r + \kappa\varphi_b} \end{aligned} \tag{B.33}$$

Some straightforward manipulations analogous to equation (B.25) implies the candidate $v(\cdot, b)$ in (B.30) is also convex, therefore, $v(\pi, b) + (1 - \pi)v'(\pi, b) \leq v(1, b)$ that yields an upper bound on the above relation:

$$\begin{aligned} [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c &\leq -\frac{\kappa\varphi_b}{r + \kappa\varphi_b} \frac{r\left(\lambda_b - \lambda_a\right)\pi}{r + \lambda_b} \left(1 + w(1) + \frac{c}{r}\right) + \frac{r\lambda_a \pi \left(1 + w(1)\right) - cr}{r + \kappa\varphi_b} \\ &\leq \left(-\frac{\kappa\varphi_b}{r + \kappa\varphi_b} \frac{r\left(\lambda_b - \lambda_a\right)}{r + \lambda_b} \left(1 + w(1) + \frac{c}{r}\right) + \frac{r\lambda_a \left(1 + w(1)\right) - cr}{r + \kappa\varphi_b}\right)^+ \end{aligned} \tag{B.34}$$

In the second inequality above I used the fact that the *rhs* of the first inequality is negative at $\pi = 0$. Now denote the argument of $(\cdot)^+$ by $\mathfrak{z}$. It is increasing in λ_a, hence can be bounded above when λ_a is replaced with $c + rw(1)$ (its maximum value in the high-cost regime):

$$\begin{aligned}\mathfrak{z} &\leq -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b - c - rw(1)\right)}{r+\lambda_b}\left(1 + w(1) + \frac{c}{r}\right) + \frac{r\left(c + rw(1)\right)\left(1 + w(1)\right) - cr}{r+\kappa\varphi_b} \\ &= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b - c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b + c\right)}{r\left(\kappa\varphi_b + r + \lambda_b\right)^2} + \frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b - c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b + c\right)}{r\left(\kappa\varphi_b + r + \lambda_b\right)^2} = 0,\end{aligned} \tag{B.35}$$

where in the second line $w(1)$ is replaced from (B.6). This concludes the superharmonicity of $v(\cdot, a)$ w.r.t $\mathcal{L}_a$ on $(\alpha_b, 1]$, and hence on the entire unit interval.

Step (b): Define $\mathbf{v}(\iota, \pi, q) := v(\pi, q)1_{\{\iota=0\}} + (\iota + w(\pi))\,1_{\{\iota=1\}}$, where ι is the success indicator process. Since $\mathbf{v}$ is a bounded function, for each $q \in \{a, b\}$, one can find a bounded (and hence uniformly integrable) Martingale process M^q such that:

$$e^{-rt}\mathbf{v}(\iota_t, \pi_t, q) = \mathbf{v}(\iota, \pi, q) + \int_0^t e^{-rs}\left[\mathbb{L}_q\mathbf{v}(\cdot,\cdot,q) - r\mathbf{v}(\cdot,\cdot,q)\right](\iota_{s^-}, \pi_{s^-})\mathrm{d}s + M_t^q \tag{B.36}$$

In that $\mathbb{L}_q\mathbf{v}(\iota, \pi, q) = \left(\mathcal{L}_q v(\pi, q)\right)1_{\{\iota=0\}}$. From the majorant condition, for every stopping time τ, we have $\mathbf{v}(\iota_\tau, \pi_\tau, q) \geq \iota_\tau + w(\pi_\tau)$, therefore

$$\begin{aligned}e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) &\leq \mathbf{v}(\iota, \pi, q) + \int_0^\tau e^{-rs}\left[\mathbb{L}_q\mathbf{v}(\cdot,\cdot,q) - r\mathbf{v}(\cdot,\cdot,q)\right](\iota_{s^-}, \pi_{s^-})\mathrm{d}s + M_\tau^q \\ &\leq \mathbf{v}(\iota, \pi, q) + \int_0^\tau c e^{-rs}\mathrm{d}s + M_\tau^q,\end{aligned} \tag{B.37}$$

where in the second inequality I used the superharmonic property proven before. Applying Doob's optional stopping theorem yields $\mathsf{E}M_\tau^q = 0$, hence for every stopping time τ:

$$\mathbf{v}(\iota, \pi, q) \geq \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right] \tag{B.38}$$

That in turn implies

$$v(\pi, q) \geq \sup_\tau \mathsf{E}_{\pi,q,\iota=0}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right]. \tag{B.39}$$

Now for each q, let $\tau(q) := \inf\{t \geq 0 : \pi_t \notin \mathcal{M}_q \text{ or } \iota_t = 1\}$ that is the optimal stopping policy. Using this in (B.36) yields

$$\begin{aligned} e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)})\right) &= e^{-r\tau(q)}\mathbf{v}(\iota_{\tau(q)}, \pi_{\tau(q)}, q) \\ &= \mathbf{v}(\iota, \pi, q) + \int_0^{\tau(q)} e^{-rs}\left[\mathbb{L}_q\mathbf{v}(\cdot,\cdot,q) - r\mathbf{v}(\cdot,\cdot,q)\right](\iota_{s^-}, \pi_{s^-})\mathrm{d}s + M^q_{\tau(q)} \\ &= \mathbf{v}(\iota, \pi, q) - \int_0^{\tau(q)} ce^{-rs}\mathrm{d}s + M^q_{\tau(q)}, \end{aligned} \tag{B.40}$$

which after taking expectations of both sides amounts to

$$\mathbf{v}(\iota, \pi, q) = \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)})\right) - c\int_0^{\tau(q)} e^{-rs}\mathrm{d}s\right], \tag{B.41}$$

therefore concluding the verification proof and the theorem.

Step (c): I slightly change the notation only in this part and denote $v_q(\cdot) := v(\cdot, q)$. Then, for every $(v_a, v_b, w) \in C[0,1]$, define

$$\begin{aligned} \mathbb{T}_q w(\pi) := \sup_\tau \Bigg\{ &\mathsf{E}_q\left[e^{-r\sigma} - c\int_0^\sigma e^{-rs}\mathrm{d}s + e^{-r\sigma}w(\pi_\sigma); \sigma \leq \tau\right] \\ &+ \mathsf{E}_q\left[-c\int_0^\tau e^{-rs}\mathrm{d}s + e^{-r\tau}w(\pi_\tau); \sigma > \tau\right]\Bigg\} \quad \text{for } q \in \{a, b\}, \end{aligned} \tag{B.42a}$$

$$\mathbb{T}_0[v_a, v_b, w](\pi) := r^{-1}\kappa \sum_{q \in \mathcal{M}(\pi)} \left[v_q(\pi) - w(\pi)\right]\varphi_q, \tag{B.42b}$$

where E_q is the expectation w.r.t to the Poisson process with intensity λ_q and $\mathcal{M}(\pi) = \{q : v_q(\pi) > w(\pi)\}$. Define $\mathbb{T} := (\mathbb{T}_a, \mathbb{T}_b, \mathbb{T}_0)$. The goal of this part of the proof is to show the fixed-point of $\mathbb{T}$ exists and is unique. Given the definition of $\mathcal{M}(\pi)$ one can see that $\mathbb{T}_0$ preserves the continuity. Then, I show $T_q(C[0,1]) \subset C[0,1]$. For this assume $w \in C[0,1]$ and rewrite (B.42a) as

$$\mathbb{T}_q w(\pi) = \sup_\tau \left\{\mathsf{E}_q\left[Z_1(\sigma); \sigma \leq \tau\right] + \mathsf{E}_q\left[Z_2(\tau); \sigma > \tau\right]\right\}. \tag{B.43}$$

For the first term,

$$\begin{aligned} \mathsf{E}_q\left[Z_1(\sigma); \sigma \leq \tau\right] &= \int_0^\infty Z_1(t)\mathsf{P}_q\left(t < \sigma \leq t + \mathrm{d}t, \sigma \leq \tau\right) \\ &= \int_0^\infty Z_1(t)\mathsf{P}_q\left(t < \sigma \leq t + \mathrm{d}t\right)\mathsf{P}_q\left(\tau \geq t \,|\, t < \sigma \leq t + \mathrm{d}t\right) \\ &= \int_0^\tau Z_1(t)\lambda_q\pi e^{-\lambda_q t}\mathrm{d}t. \end{aligned} \tag{B.44}$$

And for the second term,

$$\begin{aligned} \mathsf{E}_q\left[Z_2(\tau);\sigma>\tau\right] &= \mathsf{E}_q\left[\mathsf{E}_q\left[Z_2(\tau)1_{\{\sigma>\tau\}}\mid\mathcal{I}_\tau\right]\right] \\ &= \mathsf{E}_q\left[Z_2(\tau)\mathsf{P}_q\left(\sigma>\tau\mid\mathcal{I}_\tau\right)\right] \\ &= Z_2(\tau)\pi_\tau e^{-\lambda_q\tau}, \end{aligned} \tag{B.45}$$

where in last line, I used the fact that τ is inevitably $\mathcal{I}_0$-measurable because of the Poissonian underlying process, and π_τ is the posterior belief at τ given that the success will not have arrived by then. Hence,

$$\mathbb{T}_q w(\pi) = \sup_\tau\left\{\pi\int_0^\tau Z_1(t)\lambda_q e^{-\lambda_q t}\mathrm{d}t + Z_2(\tau)\pi_\tau e^{-\lambda_q\tau}\right\}. \tag{B.46}$$

Because of the Bayes law, $\frac{\pi_\tau}{1-\pi_\tau}=\frac{\pi}{1-\pi}e^{-\lambda_q\tau}$, so π_τ is continuous in the initial belief π. Therefore, the above representation together with the continuity of w amount to the continuity of $\mathbb{T}_0[w]$. So, we can now deduce that $\mathbb{T}:(C[0,1])^3\to(C[0,1])^3$.

The next step is to investigate the contraction property of $\mathbb{T}$. For this, let us equip $(C[0,1])^3$ with the following norm,

$$\|(v_a,v_b,w)\|_\varsigma := \varsigma\left(\|v_a\|_\infty+\|v_b\|_\infty\right)+\|w\|_\infty, \tag{B.47}$$

where $\varsigma>0$ is to be determined. First, I examine the contraction coefficient of $\mathbb{T}_q$. For every $w,\tilde{w}\in C[0,1]$:

$$\begin{aligned} \left|\mathbb{T}_q[w]-\mathbb{T}_q[\tilde{w}]\right|(\pi) &\le \sup_\tau\Big\{\mathsf{E}_q\left[e^{-r\sigma}\left|w(\pi_\sigma)-\tilde{w}(\pi_\sigma)\right|;\sigma\le\tau\right] \\ &\qquad +\mathsf{E}_q\left[e^{-r\tau}\left|w(\pi_\tau)-\tilde{w}(\pi_\tau)\right|;\sigma>\tau\right]\Big\} \\ &\le \|w-\tilde{w}\|_\infty\sup_\tau\mathsf{E}_q\left[e^{-r(\tau\wedge\sigma)}\right]=\|w-\tilde{w}\|_\infty. \end{aligned} \tag{B.48}$$

Let $\phi:=\varphi_a+\varphi_b$ be the total steady state mass of startups, and let $v,\tilde{v}\in(C[0,1])^2$, that respectively enforce the matchings sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$. Then:

$$\begin{aligned}
\left(\mathbb{T}_0[v,w]-\mathbb{T}_0[\tilde{v},\tilde{w}]\right)(\pi) &= r^{-1}\kappa\left(\sum_{q\in\mathcal{M}(\pi)}\left(v_q(\pi)-w(\pi)\right)\varphi_q-\sum_{q\in\widetilde{\mathcal{M}}(\pi)}\left(\tilde{v}_q(\pi)-\tilde{w}(\pi)\right)\varphi_q\right)\\
&= r^{-1}\kappa\sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)}\left(v_q(\pi)-\tilde{v}_q(\pi)-w(\pi)+\tilde{w}(\pi)\right)\varphi_q\\
&+ r^{-1}\kappa\sum_{q\in\mathcal{M}(\pi)\cap\widetilde{\mathcal{M}}(\pi)}\left(v_q(\pi)-\tilde{v}_q(\pi)-w(\pi)+\tilde{w}(\pi)\right)\varphi_q\\
&+ \underbrace{r^{-1}\kappa\sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)}\left(\tilde{v}_q(\pi)-\tilde{w}(\pi)\right)\varphi_q}_{=0}-\underbrace{r^{-1}\kappa\sum_{q\in\widetilde{\mathcal{M}}(\pi)\setminus\mathcal{M}(\pi)}\left(\tilde{v}_q(\pi)-\tilde{w}(\pi)\right)\varphi_q}_{\geq 0}\\
&\leq r^{-1}\kappa\phi\left(\sum_{q\in\{a,b\}}\|v_q-\tilde{v}_q\|_\infty+\|w-\tilde{w}\|_\infty\right).
\end{aligned} \tag{B.49}$$

Putting together the preceding bounds yields:

$$\begin{aligned}
\left\|\mathbb{T}\left[(v_a,v_b,w)\right]-\mathbb{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_\varsigma &= \varsigma\left(\|\mathbb{T}_a[w]-\mathbb{T}_a[\tilde{w}]\|_\infty+\|\mathbb{T}_b[w]-\mathbb{T}_b[\tilde{w}]\|_\infty\right)\\
&\quad+\left\|\mathbb{T}_0\left[(v_a,v_b,w)\right]-\mathbb{T}_0\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_\infty\\
&\leq 2\varsigma\|w-\tilde{w}\|_\infty+r^{-1}\kappa\phi\left(\sum_{q\in\{a,b\}}\|v_q-\tilde{v}_q\|_\infty+\|w-\tilde{w}\|_\infty\right)\\
&= r^{-1}\kappa\phi\|v_a-\tilde{v}_a\|_\infty+r^{-1}\kappa\phi\|v_b-\tilde{v}_b\|_\infty\\
&\quad+\left(2\varsigma+r^{-1}\kappa\phi\right)\|w-\tilde{w}\|_\infty
\end{aligned} \tag{B.50}$$

Assume $r^{-1}\kappa\phi < 1/3$, and find $\varepsilon > 0$ such that $r^{-1}\kappa\phi < 1/(1+\varepsilon)(3+2\varepsilon)$, and let $\varsigma = (1+\varepsilon)r^{-1}\kappa\phi$, then

$$\begin{aligned}
\left\|\mathbb{T}\left[(v_a,v_b,w)\right]-\mathbb{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_\varsigma &\leq \frac{r^{-1}\kappa\phi}{\varsigma}\times\\
&\left(\varsigma\|v_a-\tilde{v}_a\|_\infty+\varsigma\|v_b-\tilde{v}_b\|_\infty+\overbrace{\frac{\varsigma(2\varsigma+r^{-1}\kappa\phi)}{r^{-1}\kappa\phi}}^{<1}\|w-\tilde{w}\|_\infty\right)\\
&\leq \frac{1}{1+\varepsilon}\|(v_a,v_b,w)-(\tilde{v}_a,\tilde{v}_b,\tilde{w})\|_\varsigma.
\end{aligned} \tag{B.51}$$

So the contraction is resulted, and the Banach fixed-point theorem implies that there exists a unique fixed-point in the space of bounded continuous functions, so long as $r > 3\kappa\phi$. □

Proof of proposition 5

Let us emphasize that the planner's problem is still subject to search frictions and incomplete information about the VCs' types. That is the hypothetical planner only knows that a fraction $\mathfrak{p}$ of VCs have high types. In this regard, the only choice variable would be the matching sets $\mathcal{M}_a$ and $\mathcal{M}_b$. Specifically, the venture capitalists can not decide whether to match or not upon being contacted by the entrepreneurs. Further, they have no control right on when to terminate the funding. In the planner's problem all these rights are conferred to the benevolent planner. So, equation (2.8) carries through but with the indicator functions chosen by the planner. The planner's problem thus reduces to maximizing

$$S(\mathcal{M}) = n(1)w(1) + m_a(1)v(1,a) + m_b(1)v(1,b), \tag{B.52}$$

in that $n(1) := G(\{1\})$, $m_a(1) := F(\{1\},\{a\})$, $m_b(1) := F(\{1\},\{b\})$ and $n(\alpha) := G(\{\alpha\})$ subject to (2.15). The solution to this system when $\chi_a(1) = \chi_b(1) = 1$ is

$$\begin{aligned}&(n(\alpha), n(1), m_a(1), m_b(1)) = \\ &\left(\frac{1-\mathfrak{p}}{1-\alpha}, \frac{(\mathfrak{p}-\alpha)\,/\,(1-\alpha)}{1+\frac{\kappa\varphi_a}{\lambda_a}+\frac{\kappa\varphi_b}{\lambda_b}}, \frac{\kappa\varphi_a}{\lambda_a}\frac{(\mathfrak{p}-\alpha)\,/\,(1-\alpha)}{1+\frac{\kappa\varphi_a}{\lambda_a}+\frac{\kappa\varphi_b}{\lambda_b}}, \frac{\kappa\varphi_b}{\lambda_b}\frac{(\mathfrak{p}-\alpha)\,/\,(1-\alpha)}{1+\frac{\kappa\varphi_a}{\lambda_a}+\frac{\kappa\varphi_b}{\lambda_b}}\right),\end{aligned} \tag{B.53}$$

when $\chi_a(1) = 1$ and $\chi_b(1) = 0$ is

$$(n(\alpha), n(1), m_a(1)) = \left(\frac{1-\mathfrak{p}}{1-\alpha}, \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\kappa\varphi_a/\lambda_a}, \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\lambda_a/\kappa\varphi_a}\right), \tag{B.54}$$

and finally when $\chi_a(1) = 0$ and $\chi_b(1) = 1$:

$$(n(\alpha), n(1), m_b(1)) = \left(\frac{1-\mathfrak{p}}{1-\alpha}, \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\kappa\varphi_b/\lambda_b}, \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\lambda_b/\kappa\varphi_b}\right). \tag{B.55}$$

Denote the social welfare function in the first case by $S_{a,b}$, in the second case by S_a and lastly in the third case by S_b, then

$$S_{a,b} = \frac{(\mathfrak{p}-\alpha)\,/\,(1-\alpha)}{1+\frac{\kappa\varphi_a}{\lambda_a}+\frac{\kappa\varphi_b}{\lambda_b}}\left(\frac{\kappa\varphi_a v(1,a)+\kappa\varphi_b v(1,b)}{r+\kappa\varphi_a+\kappa\varphi_b}+\frac{\kappa\varphi_a}{\lambda_a}v(1,a)+\frac{\kappa\varphi_b}{\lambda_b}v(1,b)\right), \tag{B.56a}$$

$$S_a = \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\kappa\varphi_a/\lambda_a}\left(\frac{\kappa\varphi_a v(1,a)}{r+\kappa\varphi_a}+\frac{\kappa\varphi_a}{\lambda_a}v(1,a)\right), \tag{B.56b}$$

$$S_b = \frac{(\mathfrak{p}-\alpha)/(1-\alpha)}{1+\kappa\varphi_b/\lambda_b}\left(\frac{\kappa\varphi_b v(1,b)}{r+\kappa\varphi_b}+\frac{\kappa\varphi_b}{\lambda_b}v(1,b)\right). \tag{B.56c}$$

Conditioned on $\chi_q(1) = 1$, we have $(r + \lambda_q)v(1, q) = -c + \lambda_q (1 + w(1))$. Using this and the planner's version of (2.8), one can apply some algebraic simplifications on the above expressions and obtain:

$$rS_a = \frac{\mathfrak{p} - \alpha}{1 - \alpha} \frac{\kappa\varphi_a/\lambda_a}{1 + \kappa\varphi_a/\lambda_a} (\lambda_a - c) \tag{B.57a}$$

$$rS_b = \frac{\mathfrak{p} - \alpha}{1 - \alpha} \frac{\kappa\varphi_b/\lambda_b}{1 + \kappa\varphi_b/\lambda_b} (\lambda_b - c) \tag{B.57b}$$

$$rS_{a,b} = \frac{\mathfrak{p} - \alpha}{1 - \alpha} \frac{1}{1 + \kappa\varphi_a/\lambda_a + \kappa\varphi_b/\lambda_b} \left(\frac{\kappa\varphi_a}{\lambda_a} (\lambda_a - c) + \frac{\kappa\varphi_b}{\lambda_b} (\lambda_b - c) \right) \tag{B.57c}$$

Suppose α is the lowest boundary point of $\mathcal{M}_q$, then continuity requires $v(\alpha, q) = 0$, which is also reinforced by the principle of optimality. For otherwise, if $v(\alpha, q) > 0$ one can reduce α and increase the welfare functions while leaving all value functions still positive. In a neighborhood above α the value function $v(\cdot, q)$ takes the form of

$$\begin{aligned} v(\pi, q; \alpha) = &-\frac{c}{r} + \frac{\lambda_q}{r + \lambda_q} \left(1 + w(1) + \frac{c}{r}\right) \pi \\ &+ \left\{ \frac{c}{r} - \frac{\lambda_q}{r + \lambda_q} \left(1 + w(1) + \frac{c}{r}\right) \alpha_q \right\} \left(\frac{1 - \pi}{1 - \alpha_q}\right)^{1 + r/\lambda_q} \left(\frac{\pi}{\alpha_q}\right)^{-r/\lambda_q}. \end{aligned} \tag{B.58}$$

At $\pi = \alpha$,

$$\left. \frac{\partial v}{\partial \pi} \right|_{\pi = \alpha} = \frac{(1 + w(1))\, \alpha\lambda_q - c}{\lambda_q(1 - \alpha)\alpha}, \tag{B.59}$$

so a necessary condition for v to be increasing is $\alpha \geq \frac{c}{\lambda_q(1+w(1))}$. The welfare expressions in (B.57) are *decreasing* in α, compatible with the learning effect. So choosing α exactly equal to $\frac{c}{\lambda_q(1+w(1))}$ leads to an upper-bound on the welfare function, that could be attained if one finds its corresponding matching sets. This argument provides another support for the smooth-fit principle at the lowest boundary point. In addition, on $\pi \geq \alpha$,

$$\frac{\partial^2 v}{\partial \pi \partial \alpha} = \frac{(r + \lambda_q \pi) \left[(1 + w(1))\, \alpha\lambda_q - c\right]}{\lambda_q^2 \alpha^2 (1 - \alpha)^2} \left(\frac{1 - \pi}{1 - \alpha}\right)^{r/\lambda_q} \left(\frac{\pi}{\lambda_q}\right)^{-(1 + r/\lambda_q)} \geq 0, \tag{B.60}$$

which implies that $\partial v/\partial \pi$ remains positive for all $\pi \geq \alpha$ and in $\mathcal{M}_q$. As claimed before the lowest boundary point is in the set $\{c/\lambda_q (1 + w(1)) : q = a, b\}$. So long as $\lambda_b > \lambda_a$ the efficient choice is to set α as the lowest boundary point of $\mathcal{M}_b$, and hence $\alpha = \frac{c}{\lambda_b(1+w(1))}$.

One can now verify that in the *high-cost* regime S_b is the largest of all in (B.57), and in the *low-cost* regime $S_{a,b}$ is the largest. Therefore, the equilibrium matching sets found in proposition 4 are in fact constrained efficient. □

Social optimum in section 2.5

The planner maximizes the present value of social surplus S that is presented in equation (2.23), subject to the population dynamics in (2.24) and (2.25). The instruments that the planner has at her disposal is the choice of matching sets: $\{\chi_q(\pi) : q \in \{a, b\} \text{ and } \pi \in [0, 1]\}$. The current value Hamiltonian for this problem is

$$\begin{aligned}\mathcal{H} = &\sum_q \left[(\lambda_q - c)m_q(1) + \int (\lambda_q \pi - c)m_q(\pi)\mathrm{d}\pi\right] \\ &+ \sum_q v_*(1, q)\left[-\lambda_q m_q(1) + \kappa\varphi_q n(1)\chi_q(1)\right] \\ &+ w_*(1)\left[\sum_q \lambda_q m_q(1) - \sum_q \kappa\varphi_q n(1)\chi_q(1) + \sum_q \int \lambda_q \pi m_q(\pi)\mathrm{d}\pi\right] \\ &+ \sum_q \int v_*(\pi, q)\left[-\lambda_q \pi m_q(\pi) + \kappa\varphi_q n(\pi)\chi_q(\pi) + \lambda_q \partial_\pi \left(\pi(1-\pi)m_q(\pi)\right)\right]\mathrm{d}\pi \\ &+ \int w_*(\pi)\left[-\sum_q \kappa\varphi_q n(\pi)\chi_q(\pi)\right]\mathrm{d}\pi \\ &+ \rho\left[\zeta\lambda_a\left(m_a(1) + \int \pi m_a(\pi)\mathrm{d}\pi\right) - \kappa\varphi_b\left(n(1)\chi_a(1) + \int n(\pi)\chi_q(\pi)\mathrm{d}\pi\right)\right].\end{aligned} \tag{B.61}$$

Applying the integration by part implies that

$$\int v_*(\pi, q)\lambda_q \partial_\pi \left(\pi(1-\pi)m_q(\pi)\right)\mathrm{d}\pi = -\int \lambda_q \pi(1-\pi)v_*'(\pi, q)m_q(\pi)\mathrm{d}\pi. \tag{B.62}$$

Substituting this in the Hamiltonian and regrouping with respect to the population measures amount to

$$\begin{aligned}\mathcal{H} = &\sum_q m_q(1)\left[\lambda_q - c + \lambda_q\left(w_*(1) - v_*(1, q)\right) + \rho\zeta\lambda_a 1_{\{q=a\}}\right] \\ &+ n(1)\left[\sum_q \kappa\varphi_q\left(v_*(1, q) - w_*(1)\right)\chi_q(1) - \rho\kappa\varphi_b\chi_b(1)\right] \\ &+ \sum_q \int m_q(\pi)\left[\lambda_q\pi - c + \lambda_q\pi\left(w_*(1) - v_*(\pi, q)\right) - \lambda_q\pi(1-\pi)v_*'(\pi, q) + \rho\zeta\lambda_a\pi 1_{\{q=a\}}\right]\mathrm{d}\pi \\ &+ \int n(\pi)\left[\sum_q \kappa\varphi_q\left(v_*(\pi, q) - w_*(\pi)\right)\chi_q(\pi) - \rho\kappa\varphi_b\chi_b(\pi)\right]\mathrm{d}\pi.\end{aligned} \tag{B.63}$$

The planner's optimization problem, as expressed above, features a *continuum* of control and state processes. Therefore, I appeal to the heuristic method of [73] (chapter 6) to interpret the integrals as the summation of discrete variables over intervals of length $\mathrm{d}\pi$. The first implication of the above representation is that from the planner's viewpoint the optimal matching indicator χ^* satisfies:

$$\chi_q^*(\pi) = 1 \Leftrightarrow v_*(\pi, q) > w_*(\pi), \tag{B.64}$$

that is a match is socially optimal if the social marginal value of the partnership (i.e v_*) dominates the social marginal value of reputation while being unmatched (i.e w_*).

Next, I express the co-state equations for each of the social marginal values. In that, I will use the Gâteaux derivative (see chapter 7 in [2]) of the Hamiltonian w.r.t the associated probability measure. For instance, to find out the derivative of $\mathcal{H}$ w.r.t $n(x)$ (for $x < 1$), define δ_x as the Dirac mass concentrated at x, then:

$$\begin{aligned}\mathbb{D}_{n(x)}\mathcal{H} &:= \lim_{\varepsilon\to 0} \frac{\mathcal{H}[n(x) + \varepsilon\delta_x] - \mathcal{H}[n(x)]}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int \varepsilon\delta_x(\pi) \left[\sum_q \kappa\varphi_q \left(v_*(\pi, q) - w_*(\pi)\right) \chi_q^*(\pi) - \rho\kappa\varphi_b\chi_b^*(\pi)\right] \mathrm{d}\pi \\ &= \sum_q \kappa\varphi_q \left[v_*(x, q) - w_*(x)\right] \chi_q^*(x) - \rho\kappa\varphi_b\chi_b^*(x)\end{aligned} \tag{B.65}$$

Hence the co-state equations are ordered as follows:

$$\begin{aligned} rv_*(1, q) - \dot{v}_*(\pi, q) &= \mathbb{D}_{m_q(1)}\mathcal{H} = \lambda_q - c + \lambda_q \left(w_*(1) - v_*(1, q)\right) + \rho\zeta\lambda_a 1_{\{q=a\}} \\ rw_*(1) - \dot{w}_*(1) &= \mathbb{D}_{n(1)}\mathcal{H} = \sum_q \kappa\varphi_q \left[v_*(1, q) - w_*(1)\right] \chi_q^*(1) - \rho\kappa\varphi_b\chi_b(1) \\ rv_*(\pi, q) - \dot{v}_*(\pi, q) &= \mathbb{D}_{m_q(\pi)}\mathcal{H} \\ &= \lambda_q\pi - c + \lambda_q\pi \left(w_*(1) - v_*(\pi, q)\right) - \lambda_q\pi(1 - \pi)v_*'(\pi, q) + \rho\zeta\lambda_a\pi 1_{\{q=a\}} \\ rw_*(\pi) - \dot{w}_*(\pi) &= \mathbb{D}_{n(\pi)}\mathcal{H} = \sum_q \kappa\varphi_q \left[v_*(\pi, q) - w_*(\pi)\right] \chi_q^*(\pi) - \rho\kappa\varphi_b\chi_b^*(\pi) \end{aligned} \tag{B.66}$$

The social shadow value of the mass of late stage projects, i.e ρ, satisfies the following first-order condition:

$$\begin{aligned} r\rho - \dot{\rho} &= \frac{\partial\mathcal{H}}{\partial\varphi_b} \\ &= \kappa n(1) \left(v_*(1, b) - w_*(1) - \rho\right) \chi_b^*(1) + \int \kappa n(\pi) \left(v_*(\pi, b) - w_*(\pi) - \rho\right) \chi_b^*(\pi) \mathrm{d}\pi \end{aligned} \tag{B.67}$$

In the steady state the above representation leads to (2.31).

Proof of proposition 7

I need the following lemma to prove the proposition.

Lemma 11. *In any normal equilibrium $w(\cdot)$ is increasing iff $\{v(\cdot,a),v(\cdot,b)\}$ are increasing.*

Proof. In the normal equilibria $w(\pi)$ follows

$$w(\pi)=\max_{\chi}\left\{\frac{r^{-1}\kappa\psi(\pi)\left[v(\pi,a)\varphi_a\chi_a(\pi)+v(\pi,b)\varphi_b\chi_b(\pi)\right]}{\mu+r^{-1}\kappa\psi(\pi)\left[\varphi_a\chi_a(\pi)+\varphi_b\chi_b(\pi)\right]}\right\}. \tag{B.68}$$

Assume first, that $\{v(\cdot,a),v(\cdot,b)\}$ are increasing. It is known that the maximum of increasing functions remains increasing, therefore I have to show for any combination of χ's the *rhs* of the above expression is increasing in π. For example, let $\chi_a=\chi_b=1$, then its derivative is positively proportional to

$$r^{-1}\kappa\psi(\pi)\left(v'(\pi,a)\varphi_a+v'(\pi,b)\varphi_b\right)+r^{-1}\kappa\mu\psi'(\pi)\left(v(\pi,a)\varphi_a+v(\pi,b)\varphi_b\right)\geq 0. \tag{B.69}$$

The other permutations of χ_a,χ_b can also be checked, and one can similarly verify that for each combination, the *rhs* is increasing in π, therefore $w(\cdot)$ becomes increasing.

Conversely, now assume $w(\cdot)$ is increasing. Then the same analysis presented in lemma 5 implies that $\{v(\cdot,a),v(\cdot,b)\}$ become increasing.‖

Proof of part (i): For proving the existence of a normal equilibrium, I first establish the existence of a fixed-point α_e to the system (2.43) and (2.44). To fix ideas, let us define the following mappings $\mathsf{M}:[0,1]\to[0,1]$, $\mathsf{W}:[0,1]\to\mathbb{R}_+$ and $\mathsf{A}:\mathbb{R}_+\to[0,1]$:

$$\begin{aligned}
\mathsf{M}(x) &:= \frac{1-\mathfrak{p}}{1-x}\psi(x)+\frac{\mathfrak{p}-x}{1-x}\\
\mathsf{W}(\mu) &:= \max_{\chi}\left\{\frac{r^{-1}\kappa\left[\varphi_b\left(\lambda_b-c\right)\left(r+\lambda_a\right)\chi_b(1)+\varphi_a\left(\lambda_a-c\right)\left(r+\lambda_b\right)\chi_a(1)\right]}{\left(r+\lambda_a\right)\left(r+\lambda_b\right)\mu+\kappa\varphi_b\left(r+\lambda_a\right)\chi_b(1)+\kappa\varphi_a\left(r+\lambda_b\right)\chi_a(1)}\right\}\\
\mathsf{A}(w) &:= \frac{c}{\lambda_b(1+w)}
\end{aligned} \tag{B.70}$$

Then, α_e is the fixed point of $\mathfrak{F}:[0,1]\to[0,1]$, where $\mathfrak{F}:=\mathsf{A}\circ\mathsf{W}\circ\mathsf{M}$. Since this map is continuous on $[0,1]$, the existence of fixed-point is obvious. However, the normal equilibrium requires $\alpha_e<\mathfrak{p}$. For this note that $\mathfrak{F}(0)>0$ and

$$\mathfrak{F}(\mathfrak{p})=\frac{c}{\lambda_b\left(1+\mathsf{W}(\psi(\mathfrak{p}))\right)}<\frac{c}{\lambda_b}<\mathfrak{p}. \tag{B.71}$$

The mean-value theorem therefore implies that there always exists a normal equilibrium with $0 < \alpha_e < \mathfrak{p}$.

Now I analyze the best-response correspondence for a generic VC. Suppose all VCs except one follow the investment strategy induced by $\mathcal{M}_b = (\alpha_e, 1]$ and $\mathcal{M}_a \subset \mathcal{M}_b$. Then, $\mu = \mathsf{M}(\alpha_e)$. Using the machinery developed in proposition 4 and the previous lemma, one can easily confirm the unique best-response of the potential deviant VC is the above matching sets $(\mathcal{M}_a, \mathcal{M}_b)$. $\|$

Proof of part (ii): Assuming $\psi'' \leq 0$ implies that

$$\mathsf{M}'(x) = \frac{1-\mathfrak{p}}{1-x}\left(\psi'(x) - \frac{1-\psi(x)}{1-x}\right) \geq 0. \tag{B.72}$$

Hence, the composition map becomes increasing from $[0,1]$ to itself, because W and A are both decreasing. Further, define

$$\Psi := \{\psi : [0,1] \to [0,1] |\, \psi(0) = 0, \psi(1) = 1, \psi' \geq 0, \psi'' \leq 0\}\,, \tag{B.73}$$

and endow Ψ with the pointwise order $\succsim$, i.e $\psi_2 \succsim \psi_1$ iff $\psi_2(x) \geq \psi_1(x)$, $\forall x \in [0,1]$. So, $(\Psi, \succsim)$ becomes a partially ordered set that is used as the underlying parameter space for the fixed-point map $\mathfrak{F}$. With slight abuse of notation, I extend the domain of $\mathfrak{F}$ as $\mathfrak{F} : [0,1] \times \Psi \to [0,1]$. Holding x constant, $\mathfrak{F}(x, \psi)$ is increasing in ψ w.r.t $\succsim$ order. Therefore, the mapping $\mathfrak{F}$ is an increasing function from $[0,1] \times \Psi$ to $[0,1]$. Now one can apply corollaries 2.5.1 and 2.5.2 of [71] to conclude that the set of fixed-points is a complete lattice and its greatest (least) element is increasing in $\psi \in \Psi$. Finally, the lattice of fixed-points, i.e α_e's, completely Pareto rank the equilibria. Because smaller values of α_e lead to smaller μ and hence larger $w(1)$ and $\{v(1,b), v(1,a)\}$. In addition, it is associated to larger masses of $\{n(1), m(1,b), m(1,a)\}$. Therefore, the welfare ranking of equilibria coincides inversely with the ranking of fixed-points of $\mathfrak{F}$. $\square$

Proof of lemma 6

In the steady state the time derivatives in (2.47) must be equal to zero, therefore (2.47a) and (2.47b) amount to:

$$n(1) = \frac{\delta + \lambda}{\kappa\varphi/\mu} m(1) \tag{B.74a}$$

$$(\delta + \lambda + \kappa\varphi/\mu)\delta m(1) = \kappa\varphi/\mu \int_{\alpha}^{\mathfrak{p}} \lambda \pi m(\pi) \mathrm{d}\pi. \tag{B.74b}$$

Also at $\pi = \mathfrak{p}$, equation (2.47c) implies that $n(\mathfrak{p}) = \delta/(\delta + \kappa\varphi\psi(\mathfrak{p})/\mu)$. Next, the expression found in (2.49) translates to

$$\int_{\alpha}^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi = \frac{\lambda m(\alpha)}{\delta + \lambda}\Upsilon_2(\alpha). \tag{B.75}$$

The *rhs* to (2.50) can be simplified using the steady state ODE resulted from $\dot{m}(\pi) = 0$:

$$\begin{aligned}
\kappa\varphi\frac{\psi(\mathfrak{p})}{\mu}n(\mathfrak{p}) = \frac{\delta\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta + \kappa\varphi\psi(\mathfrak{p})/\mu} &= \lambda\int_{\alpha}^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi + \delta\int_{\alpha}^{\mathfrak{p}} m(\pi)\mathrm{d}\pi + \delta n(\alpha) \\
&= \lambda\int_{\alpha}^{\mathfrak{p}} \partial_{\pi}\left(\pi(1-\pi)m(\pi)\right)\mathrm{d}\pi + \delta n(\alpha) \\
&= \lambda\left[\mathfrak{p}(1-\mathfrak{p})m(\mathfrak{p}) - \alpha(1-\alpha)m(\alpha)\right] + \delta n(\alpha) \\
&= \lambda m(\alpha)\Upsilon_1(\alpha) + \delta n(\alpha)
\end{aligned} \tag{B.76}$$

Recall that because of Bayesian learning over matches the steady state average reputation must be equal to $\mathfrak{p}$:

$$m(1) + n(1) + \mathfrak{p}n(\mathfrak{p}) + \int_{\alpha}^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi + \alpha n(\alpha) = \mathfrak{p} \tag{B.77}$$

Simplifying this relation using (B.74b) and (B.75) implies

$$\lambda m(\alpha)\Upsilon_2(\alpha) + \alpha\delta n(\alpha) = \frac{\delta\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta + \kappa\varphi\psi(\mathfrak{p})/\mu}\mathfrak{p}. \tag{B.78}$$

It is now straightforward to solve for $n(\alpha), m(\alpha)$ using (B.76) and (B.78), thereby obtaining (2.51d) and

$$m(\alpha) = \frac{\delta\kappa\varphi\psi(\mathfrak{p})/\mu}{\lambda\left(\delta + \kappa\varphi\psi(\mathfrak{p})/\mu\right)}\frac{\mathfrak{p} - \alpha}{\Upsilon_2(\alpha) - \alpha\Upsilon_1(\alpha)}. \tag{B.79}$$

Substituting $m(\alpha)$ from above into (B.75) yields the lemma's claim for $\int_{\alpha}^{\mathfrak{p}} \pi m(\pi)\mathrm{d}\pi$, i.e equation (2.51b). Subsequently, $m(1)$ can be found from (B.74b) thus verifying (2.51c). Finally, from the second line in (B.76) one obtains the following expression

$$\int_{\alpha}^{\mathfrak{p}} m(\pi)\mathrm{d}\pi = \frac{\lambda m(\alpha)}{\delta}\left(\Upsilon_1(\alpha) - \frac{\lambda}{\delta + \lambda}\Upsilon_2(\alpha)\right), \tag{B.80}$$

that amounts to (2.51a) by substituting $m(\alpha)$ in the above expression. □

Stochastic ordering results in subsection 2.6

For a better understanding of the stochastic ordering on the steady distribution of $\boldsymbol{\pi}_\infty$, I would first express the CDF of the density $m(\cdot)$:

$$\int_\alpha^\pi m(x)\mathrm{d}x = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\mathfrak{p}-\alpha}{\Upsilon_{2,1}(\alpha,\mathfrak{p})-\alpha\Upsilon_{1,1}(\alpha,\mathfrak{p})}\left[\frac{\delta}{\delta+\lambda}\Upsilon_{2,1}(\alpha,\pi)+\Upsilon_{1,2}(\alpha,\pi)\right],$$
$$\Upsilon_{i,j}(x,y) := \left(\frac{y}{x}\right)^{(\delta/\lambda-1)}\left(\frac{1-y}{1-x}\right)^{-(\delta/\lambda+2)} y^i(1-y)^j - x^i(1-x)^j \tag{B.81}$$

In addition, using the solution found for $m(\pi)$ and the expression (B.79) for $m(\alpha)$ it is easy to verify that for $\pi\in[\alpha,\mathfrak{p}]$

$$m(\pi) = \frac{\delta\kappa\varphi\psi(\mathfrak{p})/\mu}{\lambda\left(\delta+\kappa\varphi\psi(\mathfrak{p})/\mu\right)}\left(\frac{\pi}{\mathfrak{p}}\right)^{(\delta/\lambda-1)}\left(\frac{1-\pi}{1-\mathfrak{p}}\right)^{-(\delta/\lambda+2)}\frac{1}{\mathfrak{p}(1-\mathfrak{p})}, \tag{B.82}$$

therefore for a fixed μ the above density is independent of α.

My next goal is to show that $\mathsf{M}(\mu,\alpha)$ is increasing in each argument holding the other one constant. For this, I appeal to the theory of stochastic orders, and in particular I employ the second-order stochastic dominance. For two real-valued random variables X and Y, it is said that $X \succsim_{\mathsf{SSD}} Y$ if $\mathsf{E}u(X)\geq\mathsf{E}u(Y)$ for every increasing and concave function u. An equivalent definition is that $X\succsim_{\mathsf{SSD}} Y$ if $\mathsf{E}\left[(X-t)_-\right]\geq\mathsf{E}\left[(Y-t)_-\right]$ for every $t\in\mathbb{R}$ provided that the expectations exist.[2] The next lemma offers a sufficient condition for second-order stochastic dominance that originates from the work of [34].

Lemma 12 (Sufficient condition for SSD)**.** *Suppose the following two conditions hold:*

(i) $\mathsf{E}[X]\geq\mathsf{E}[Y]$.

(ii) There exists $t_0\in\mathbb{R}$ *such that for all* $t\leq t_0$, $\mathsf{P}(X\geq t)\geq\mathsf{P}(Y\geq t)$ *and for all* $t>t_0$, $\mathsf{P}(X\geq t)\leq\mathsf{P}(Y\geq t)$.

Then $X\succsim_{\mathsf{SSD}} Y$.

Proof. For every $t\leq t_0$,

$$\begin{aligned}\mathsf{E}\left[(X-t)_-\right] &= -\int_0^\infty \mathsf{P}\left(-(X-t)_->u\right)\mathrm{d}u = -\int_0^\infty \mathsf{P}\left(X<t-u\right)\mathrm{d}u\\ &= -\int_{-\infty}^t \mathsf{P}\left(X<z\right)\mathrm{d}z \geq -\int_{-\infty}^t \mathsf{P}\left(Y<z\right)\mathrm{d}z = \mathsf{E}\left[(Y-t)_-\right].\end{aligned} \tag{B.83}$$

[2]For every $r\in\mathbb{R}$, $(r)_- := \min\{r,0\}$. The reader can refer to chapter 4 of [65] for the proof of the equivalence.

Also, an equivalent representation for $\mathsf{E}\left[(X-t)_-\right]$ is

$$\begin{aligned}\mathsf{E}\left[(X-t)_-\right] &= \mathsf{E}\left[(X-t); X<t\right] \\ &= \mathsf{E}[X]-t-\mathsf{E}\left[X-t; X\geq t\right] = \mathsf{E}[X]-t-\int_t^{\infty}\mathsf{P}\left(X\geq z\right)\mathrm{d}z.\end{aligned} \tag{B.84}$$

Therefore,

$$\mathsf{E}\left[(X-t)_-\right]-\mathsf{E}\left[(Y-t)_-\right] = \mathsf{E}[X]-\mathsf{E}[Y]+\int_t^{\infty}\left[\mathsf{P}\left(Y\geq z\right)-\mathsf{P}\left(X\geq z\right)\right]\mathrm{d}z. \tag{B.85}$$

The first term is positive and the integral term is also positive for all $t>t_0$, so $\mathsf{E}\left[(X-t)_-\right]\geq \mathsf{E}\left[(Y-t)_-\right]$ for $t>t_0$ as well. □

I will use the technique offered in this lemma to prove that an increase in α or μ *positively* shifts the steady state distribution of $\boldsymbol{\pi}_{\infty}$. This distribution is completely described by the measures found in lemma 6. For every Borel subset $B\subset[0,1]$:

$$\mathsf{P}\left(\boldsymbol{\pi}_{\infty}\in B\right) = \left(m(1)+n(1)\right)\delta_1(B)+n(\mathfrak{p})\delta_{\mathfrak{p}}(B)+\int_B m(\pi)\mathrm{d}\pi+n(\alpha)\delta_{\alpha}(B) \tag{B.86}$$

Lemma 13. *Let $\alpha_1\leq\alpha_2<\mathfrak{p}$ and $\mu_1\leq\mu_2$, then*

(i) Holding α constant, $\boldsymbol{\pi}_{\infty}(\mu_2)\succsim_{\mathsf{SSD}}\boldsymbol{\pi}_{\infty}(\mu_1)$.

(ii) Holding μ constant, $\boldsymbol{\pi}_{\infty}(\alpha_2)\succsim_{\mathsf{SSD}}\boldsymbol{\pi}_{\infty}(\alpha_1)$.

Proof. Part (i): I show that

$$\mathsf{P}\left(\boldsymbol{\pi}_{\infty}(\mu_2)\geq t\right)\begin{cases}\geq \mathsf{P}\left(\boldsymbol{\pi}_{\infty}(\mu_1)\geq t\right) & \forall t\leq\mathfrak{p}\\ \leq \mathsf{P}\left(\boldsymbol{\pi}_{\infty}(\mu_1)\geq t\right) & \forall t>\mathfrak{p}.\end{cases} \tag{B.87}$$

Note that for every $t>\mathfrak{p}$

$$\mathsf{P}\left(\boldsymbol{\pi}_{\infty}\geq t\right) = m(1)+n(1) = \frac{\kappa\varphi\psi(\mathfrak{p})/\mu}{\delta+\kappa\varphi\psi(\mathfrak{p})/\mu}\frac{\lambda}{\delta+\lambda}\frac{(\mathfrak{p}-\alpha)\Upsilon_{2,1}(\alpha,\mathfrak{p})}{\Upsilon_{2,1}(\alpha,\mathfrak{p})-\alpha\Upsilon_{1,1}(\alpha,\mathfrak{p})}, \tag{B.88}$$

that is obviously decreasing in μ, hence proving the second assertion in (B.87). For every $t\leq\mathfrak{p}$,

$$\mathsf{P}\left(\boldsymbol{\pi}_{\infty}\geq t\right) = 1-\mathsf{P}\left(\boldsymbol{\pi}_{\infty}<t\right) = 1-\left(n(\alpha)+\int_{\alpha}^{t}m(\pi)\mathrm{d}\pi\right). \tag{B.89}$$

According to (2.51d), the mass $n(\alpha)$ is decreasing in μ, so is $\int_{\alpha}^{t}m(\pi)\mathrm{d}\pi$ according to (B.81). Hence, $\mathsf{P}\left(\boldsymbol{\pi}_{\infty}\geq t\right)$ must be increasing in μ for every $t\leq\mathfrak{p}$, thus establishing the first line of

(B.87). Given that $\mathsf{E}\left[\boldsymbol{\pi}_\infty(\mu_2)\right] = \mathsf{E}\left[\boldsymbol{\pi}_\infty(\mu_1)\right] = \mathfrak{p}$, then both parts of lemma 12 are satisfied to conclude part (i).

Part (ii): Holding μ constant, for every $t \leq \alpha_2$

$$\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_2) \geq t\right) = 1 \geq \mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_1) \geq t\right). \tag{B.90}$$

Alternatively, for every $t > \alpha_2$

$$\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha) \geq t\right) = 1_{\{t \leq \mathfrak{p}\}} \left(\int_t^{\mathfrak{p}} m(\pi) \mathrm{d}\pi + n(\mathfrak{p}) \right) + n(1) + m(1). \tag{B.91}$$

Because of (B.82) the integral term is independent of α (for a fixed μ). This is the case for $n(\mathfrak{p})$ as well. Therefore, it is sufficient to show holding μ constant, $n(1) + m(1)$ is decreasing in α. This is equivalent to verifying the following expression is decreasing in α:

$$\begin{aligned} \frac{(\mathfrak{p}-\alpha)\Upsilon_{2,1}(\alpha,\mathfrak{p})}{\Upsilon_{2,1}(\alpha,\mathfrak{p}) - \alpha\Upsilon_{1,1}(\alpha,\mathfrak{p})} &= \frac{(\mathfrak{p}-\alpha)\Upsilon_{2,1}(\alpha,\mathfrak{p})}{(\mathfrak{p}-\alpha)\mathfrak{p}(1-\mathfrak{p})\left(\frac{\mathfrak{p}}{\alpha}\right)^{\delta/\lambda-1}\left(\frac{1-\mathfrak{p}}{1-\alpha}\right)^{-(\delta/\lambda+2)}} \\ &= \mathfrak{p}\left[1 - \frac{\alpha^2(1-\alpha)}{\mathfrak{p}^2(1-\mathfrak{p})}\left(\frac{\alpha}{\mathfrak{p}}\right)^{\delta/\lambda-1}\left(\frac{1-\alpha}{1-\mathfrak{p}}\right)^{-(\delta/\lambda+2)}\right] \\ &= \mathfrak{p}\left[1 - \left(\frac{\alpha}{1-\alpha}\right)^{\delta/\lambda+1}\left(\frac{\mathfrak{p}}{1-\mathfrak{p}}\right)^{-(\delta/\lambda+1)}\right] \end{aligned} \tag{B.92}$$

Since $\alpha/(1-\alpha)$ is increasing in α, then the above expression is decreasing in α, so as a result of this, for every $\alpha_1 < \alpha_2 < \mathfrak{p}$ and $t > \alpha_2$:

$$\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_2) \geq t\right) \leq \mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_1) \geq t\right) \tag{B.93}$$

Hence, lemma 12 can be applied to conclude part (ii). □

B.1 General type space

The goal of this appendix is to extend the results of section 2.2 to the general type space for projects. Specifically, I show there always exists an *increasing* reputation function w that satisfies the investors fixed-point problem. Suppose the startups' types are drawn from an arbitrary distribution with CDF $\phi(\cdot)$ and a bounded support $[a, b]$. The success arrival intensity takes the general form of $\lambda_q(\theta)$, for which I denote $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, and assume $\underline{\lambda}_q \leq \bar{\lambda}_q \leq \boldsymbol{\lambda}$ for all $q \in \mathsf{Supp}(\phi)$.

The reputation value function satisfies

$$w(\pi) = \frac{\kappa}{r} \int [v(\pi, q) - w(\pi)]^+ \phi(\mathrm{d}q), \tag{B.94}$$

therefore for every subset $B \subset [a, b]$, one can see the equilibrium value functions (w, v) satisfy

$$w(\pi) \geq \frac{\kappa}{r} \int_B [v(\pi, q) - w(\pi)] \phi(\mathrm{d}q) \Rightarrow w(\pi) \geq \frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)}. \tag{B.95}$$

Setting $B^* = \{q : v(\pi, q) > w(\pi)\}$ to bind the above inequality, one can propose the following equivalent representation for the reputation value function:

$$w(\pi) = \sup \left\{ \frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : B \subset [a, b] \right\} \tag{B.96}$$

On the other hand, given the reputation function w, each investor solves the stopping time problem when matched with a project of type q:

$$v(\pi, q) = \sup_{\tau} \mathsf{E} \left[1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau}) \right] \tag{B.97}$$

For a given q, let $\mathbb{T}_q w$ be the matching value function resulted from the above stopping time problem, hence from (B.96) it follows that w is the fixed-point to the following operator:

$$\mathcal{A}w := \sup \left\{ \frac{\int_B \mathbb{T}_q w \ \phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : B \subset [a, b] \right\} \tag{B.98}$$

In what follows I propose the appropriate function space on which $\mathcal{A}$ will be defined, and advance the study of its fixed-point with its properties.

Let $L^1[0, 1]$ be the Banach space of Lebesgue integrable functions on the unit interval, and $L^1_+[0, 1]$ be the subset of nonnegative functions which is readily seen to be a cone[3]. Let $\succsim$ be the partial order induced by the cone $L^1_+[0, 1]$ on the Banach space $L^1[0, 1]$, that is $w_2 \succsim w_1$ iff $w_2(\pi) \geq w_1(\pi), \forall \pi \in [0, 1]$. Then, it readily follows from (B.97) that $\mathbb{T}_q$ is a *positive* and *monotone* operator, that is letting $\mathbf{0}$ to be the zero element of $L^1[0, 1]$, then $\mathbb{T}_q\mathbf{0} \succsim \mathbf{0}$, and $\mathbb{T}_q w_2 \succsim \mathbb{T}_q w_1$ for $w_2 \succsim w_1$ in $L^1_+[0, 1]$. Further, it can easily be verified that $\mathcal{A}$ inherits *positivity* and *monotonicity* from the collection $\{\mathbb{T}_q : q \in [a, b]\}$. Next, I show without loss of generality, we can restrict the search for the fixed-point to the bounded region of all $w \in L^1_+[0, 1]$ where $\|w\|_\infty \leq \boldsymbol{\lambda}/r$.[4]

[3]A cone is a subset $\mathcal{K}$ of a Banach space which is (i) closed, (ii) for every $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$: $\alpha x + \beta y \in \mathcal{K}$, and (iii) $\mathcal{K} \cap (-\mathcal{K}) = \mathbf{0}$.

[4]Henceforth, if not stated explicitly all norms are the sup-norm.

Lemma 14. *For every $w \in L^1_+[0,1]$,*

$$\|\mathbb{T}_q w(\cdot)\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\}, \quad \phi - \textit{almost surely.} \tag{B.99}$$

Proof. For every $q \in \mathsf{Supp}(\phi)$,

$$\begin{aligned}
\mathbb{T}_q w(\pi) &= \sup_\tau \mathsf{E}\left[1_{\{\sigma\leq\tau\}}e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\left[1_{\{\sigma\leq\tau\}}e^{-r\sigma} + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\left[\max\left\{e^{-r\tau}w(\pi_\tau), e^{-r\sigma}(1+w(\pi_\sigma))\right\}\right] \\
&\leq \max\left\{\|w\|, \mathsf{E}\left[e^{-r\sigma}\right](1+\|w\|)\right\}.
\end{aligned} \tag{B.100}$$

For a fixed $\pi \in [0,1]$ and $q \in [a,b]$

$$\begin{aligned}
\mathsf{E}\left[e^{-r\sigma}\right] &= \pi\int_0^\infty e^{-rt}\bar{\lambda}_q e^{-\bar{\lambda}_q t}\mathrm{d}t + (1-\pi)\int_0^\infty e^{-rt}\underline{\lambda}_q e^{-\underline{\lambda}_q t}\mathrm{d}t \\
&= \pi\frac{\bar{\lambda}_q}{r+\bar{\lambda}_q} + (1-\pi)\frac{\underline{\lambda}_q}{r+\underline{\lambda}_q} \leq \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}
\end{aligned} \tag{B.101}$$

Substituting this into the upper bound found above for $\mathbb{T}_q w(\pi)$ concludes the proof. □

I use the previous lemma to limit the search for the space of fixed-points.

Lemma 15. *Any fixed-point of $\mathcal{A}$ (if exists) is order bounded above by the constant function $\boldsymbol{\lambda}/r$.*

Proof. First, note that the supremum in (B.98) is achieved by $B_w = \{q : \mathbb{T}_q w(\pi, q) > w(\pi)\}$ for any candidate fixed-point w. Then, for any such candidate

$$\left(1+\frac{\kappa}{r}\phi(B_w)\right)w(\pi) = \int_{B_w}\mathbb{T}_q w(\pi)\phi(\mathrm{d}q), \tag{B.102}$$

therefore, using the result of the previous lemma

$$\left(1+\frac{\kappa}{r}\phi(B_w)\right)\|w\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\}\phi(B_w). \tag{B.103}$$

Assume to the contrary that $\|w\| > \boldsymbol{\lambda}/r$, then $\max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\} = \|w\|$, and (B.103) amounts to

$$\left(1+\frac{\kappa}{r}\phi(B_w)\right)\|w\| \leq \|w\|\phi(B_w). \tag{B.104}$$

Cancelling $\|w\|$ from both sides implies $1+\frac{\kappa}{r}\phi(B_w) \leq \phi(B_w)$. Since it was assumed $\|w\| > \boldsymbol{\lambda}/r$, then $\phi(B_w) > 0$. On the other hand $\phi(B_w) \leq 1$. These two together with (B.104) yield the contradiction and hence the proof of the lemma. □

Definition 9 (Regular and strongly-minihedral cones: [39] sections 1.5 and 1.7)**.** *A Banach space partially ordered by means of a cone is called regularly partially ordered, if any monotone-increasing sequence, order-bounded from above, converges in norm to a limit point. A cone which generates a regular partial ordering is called a regular cone. A cone is said to be strongly minihedral if every order bounded subset has a least upper bound (order supremum).*

Now consider the Banach space of integrable functions $L^1[0,1]$, and the positive cone of $L^1_+[0,1] = \{f \in L^1[0,1] : f(x) \geq 0 \; \forall x \in [0,1]\}$. This cone is regular, and for any monotone increasing sequence $\{f_n\} \subset L^1[0,1]$ such that $f_1 \precsim f_2 \precsim \ldots$ and order bounded from above, $\|f_n - f\|_{L^1} \to 0$ where $f(x) = \sup_n f_n(x)$ *for every* $x \in [0,1]$ (Dominated convergence theorem). In addition $L^1_+[0,1]$ is strongly minihedral (page 52 [39]).

Let $\langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle := \left\{ f \in L^1_+[0,1] : \mathbf{0} \precsim f \precsim \boldsymbol{\lambda}/r \right\}$ be the order interval of nonnegative L^1 functions, order bounded above by the constant function $\boldsymbol{\lambda}/r$. In light of the lemma 14, we have $\mathbb{T}_q : \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$ for every $q \in [a,b]$ and hence $\mathcal{A} : \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$. At this stage, I can apply part (a) of theorem 4.1 in [39] to conclude the existence of a fixed-point of $\mathcal{A}$ in $\langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$, because the mapping $\mathcal{A}$ is monotonic in a strongly minihedral cone space. However, the mere existence of the fixed-point is far from enough. In particular, we want to know whether there exists a continuous and/or increasing fixed-point for $\mathcal{A}$. To answer such questions, I will need to dig deeper into the mapping $\mathcal{A}$, beyond its monotonicty. In doing so, I shall construct a monotone sequence of functions, and show it converges in the L^1 sense to a fixed-point of $\mathcal{A}$.

Fix $w_0 := \mathbf{0}$ and recursively define $w_n = \mathcal{A} w_{n-1}$, therefore $\{w_n\} \subset \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$ is an increasing sequence order bounded from above, hence converges in L^1 to $w_\infty \in \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$ where $w_\infty(\pi) = \sup_n w_n(\pi)$ for each $\pi \in [0,1]$ (because of the regularity of the $L^1_+[0,1]$ cone). The conceptual merit of this recursive construction is summarized in the following two points:

(i) Say a property $\star$ is owned by w_0, and is preserved by the mapping $\mathcal{A}$. Then, it holds along the sequence $\{w_n\}$.

(ii) If $\star$ is stable under the L^1 limit, then w_∞ holds this property.

Therefore, if $\mathcal{A}$ is L^1 continuous along the sequence $\{w_n\}$, then w_∞ becomes the fixed-point and the presumptive property $\star$ will be inherited to the fixed-point.

Proposition 9. *For the sequence* $\{w_n\}$ *defined above, it holds that* $\|\mathcal{A} w_n - \mathcal{A} w_\infty\|_{L^1} \to 0$*, and as a result* $w_\infty = \mathcal{A} w_\infty$*.*

Proof. First note that for every $\pi \in [0,1]$,

$$\begin{aligned}\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi) &= \sup\left\{\frac{\int_B \mathbb{T}_q w_\infty(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\ &\quad - \sup\left\{\frac{\int_B \mathbb{T}_q w_n(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\ &\leq \sup\left\{\frac{\int_B \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\ &\leq \int_0^1 \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q),\end{aligned} \tag{B.105}$$

where in the last line I used the fact that $w_\infty \succsim w_n$ and the monotonicity of the operator $\mathbb{T}_q$. Therefore, the L^1-norm can be bounded above as:

$$\begin{aligned}\|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &= \int_0^1 \left(\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi)\right)\mathrm{d}\pi \\ &\leq \int_0^1\int_0^1 \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)\mathrm{d}\pi = \int_0^1 \|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q)\end{aligned} \tag{B.106}$$

For the last equality relation, I used the fact that the integrand is positive and uniformly bounded above by $\boldsymbol{\lambda}/r$ to apply the Fubini's theorem and exchange the order of integrations. Since the integrand of the last integral above is uniformly bounded (over all $q \in [a,b]$), then one can use the Lebesgue-dominated-convergence theorem to get:

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &\leq \lim_{n\to\infty}\int_0^1 \|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q) \\ &= \int_0^1 \lim_{n\to\infty}\|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q)\end{aligned} \tag{B.107}$$

Next, I propose a method to upper-bound $\left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)$, and hence its L^1-norm. For this let G represent the random variable inside the expectation operator in the definition of $\mathbb{T}_q w$:

$$\begin{aligned}\left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi) &= \sup_\tau \mathsf{E}_\pi\left[G(\sigma, w_\infty; \tau)\right] - \sup_\tau \mathsf{E}\left[G(\sigma, w_n; \tau)\right] \\ &\leq \sup_\tau \mathsf{E}_\pi\left[e^{-r(\sigma\wedge\tau)}\,(w_\infty - w_n)(\pi_{\sigma\wedge\tau})\right] \\ &\leq \mathsf{E}_\pi\left[e^{-r\sigma}\left(w_\infty - w_n\right)(\pi_\sigma)\right] + \sup_\tau \mathsf{E}_\pi\left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right]\end{aligned} \tag{B.108}$$

Therefore, the L^1-norm is bounded by

$$\|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1} \leq \underbrace{\int_0^1 \mathsf{E}_\pi \left[e^{-r\sigma} \left(w_\infty - w_n\right) \left(\pi_\sigma\right)\right] \mathrm{d}\pi}_{\mathcal{I}_1 :=} + \underbrace{\int_0^1 \sup_\tau \mathsf{E}_\pi \left[e^{-r\tau} \left(w_\infty - w_n\right) \left(\pi_\tau\right); \tau < \sigma\right] \mathrm{d}\pi}_{\mathcal{I}_2 :=} . \tag{B.109}$$

The integrands of both integrals are bounded by $\boldsymbol{\lambda}/r$, hence applying the Lebesgue-dominated-convergence theorem twice for the first integral implies

$$\begin{aligned} \lim_{n\to\infty} \mathcal{I}_1 &= \int_0^1 \lim_{n\to\infty} \mathsf{E}_\pi \left[e^{-r\sigma} \left(w_\infty - w_n\right) \left(\pi_\sigma\right)\right] \mathrm{d}\pi \\ &= \int_0^1 \mathsf{E}_\pi \left[\lim_{n\to\infty} e^{-r\sigma} \left(w_\infty - w_n\right) \left(\pi_\sigma\right)\right] \mathrm{d}\pi = 0, \end{aligned} \tag{B.110}$$

because w_∞ is the pointwise supremum of the sequence $\{w_n\}$. To show the convergence for the second integral, first note that for every given $\varepsilon > 0$ one can find $T > 0$ such that

$$\sup_\tau \mathsf{E}_\pi \left[e^{-r\tau} \left(w_\infty - w_n\right) \left(\pi_\tau\right); \tau < \sigma\right] \leq \sup_{\tau \leq T} \mathsf{E}_\pi \left[e^{-r\tau} \left(w_\infty - w_n\right) \left(\pi_\tau\right); \tau < \sigma\right] + \varepsilon, \tag{B.111}$$

uniformly over all π. This is indeed due to the uniform boundedness of $(w_\infty - w_n)$ by $\boldsymbol{\lambda}/r$. Next, because of the property of supremum for every $\varepsilon > 0$, there exist $\tau_{n,\pi}$ (possibly depending on n and π) such that

$$\sup_{\tau \leq T} \mathsf{E}_\pi \left[e^{-r\tau} \left(w_\infty - w_n\right) \left(\pi_\tau\right); \tau < \sigma\right] \leq e^{-r\tau_{n,\pi}} \left(w_\infty - w_n\right) \left(\pi_{\tau_{n,\pi}}\right) \mathsf{P}_\pi \left(\tau_{n,\pi} < \sigma\right) + \varepsilon. \tag{B.112}$$

Therefore,

$$\begin{aligned} \mathcal{I}_2 &\leq \int_0^1 e^{-r\tau_{n,\pi}} \left(w_\infty - w_n\right) \left(\pi_{\tau_{n,\pi}}\right) \mathsf{P}_\pi \left(\tau_{n,\pi} < \sigma\right) \mathrm{d}\pi + 2\varepsilon \\ &= \int_0^1 e^{-r\tau_{n,\pi}} \left(w_\infty - w_n\right) \left(\pi_{\tau_{n,\pi}}\right) \left(\pi e^{-\bar{\lambda}_q \tau_{n,\pi}} + (1-\pi) e^{-\underline{\lambda}_q \tau_{n,\pi}}\right) \mathrm{d}\pi + 2\varepsilon. \end{aligned} \tag{B.113}$$

Because of the Bayes-law, $\pi_{\tau_{n,\pi}} = \frac{\pi e^{-\Delta_q \tau_{n,\pi}}}{1-\pi+\pi e^{-\Delta_q \tau_{n,\pi}}}$. Leveraging this relation and applying the change of variable to the above integral lead to

$$\begin{aligned} \mathcal{I}_2 - 2\varepsilon &\leq \int_0^1 (w_\infty - w_n)(x) \frac{e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}}}{\left(1 - x + x e^{\Delta_q \tau_{n,x}}\right)^3} \mathrm{d}x \\ &\leq \int_0^1 (w_\infty - w_n)(x) e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}} \mathrm{d}x, \end{aligned} \tag{B.114}$$

where in the last inequality I used the fact that $\left(1 - x + xe^{\Delta_q \tau_{n,x}}\right)$ is increasing in x. Since, $\tau_{n,x} \leq T$ the last integrand in (B.114) is uniformly bounded for all x and n. Hence, one can apply the Lebesgue-dominated-convergence theorem and obtain

$$\lim_{n\to\infty} \mathcal{I}_2 \leq \int_0^1 \lim_{n\to\infty} (w_\infty - w_n)(x) e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}} \mathrm{d}x + 2\varepsilon = 2\varepsilon. \tag{B.115}$$

Since this relation holds for every $\varepsilon > 0$, then $\lim_{n\to\infty} \mathcal{I}_2 = 0$. This establishes the L^1 convergence of $\mathcal{A}w_n$ to $\mathcal{A}w_\infty$ and thus proves that $w_\infty = \mathcal{A}w_\infty$. □

A very important property owned by w_0 and preserved under $\mathcal{A}$ is being increasing in π. In the next lemma, using the techniques from coupling of probability measures and stochastic dominance, I show $\mathcal{A}w$ is increasing in π when w is.

Lemma 16. *Let w be an increasing function in π, then $\mathcal{A}w$ becomes increasing in π as well.*

Proof. Fix q and suppose $\pi_2 \geq \pi_1$. Define the random variables

$$\sigma_i \overset{d}{=} \pi_i \mathsf{exp}(\bar{\lambda}_q) + (1 - \pi_i)\mathsf{exp}(\underline{\lambda}_q), \quad i \in \{1, 2\} \tag{B.116}$$

as the exponential time of success arrivals under π_1 and π_2[5]. One can easily check $\sigma_1 \succsim \sigma_2$ in the sense of first order stochastic dominance (see the supplementary material). Therefore, for every *decreasing* function f we will have $\mathsf{E}[f(\sigma_2)] \geq \mathsf{E}[f(\sigma_1)]$. Recall the definition of $\mathbb{T}_q$:

$$\begin{aligned} &\mathbb{T}_q w(\pi) = \sup_\tau \mathsf{E}_\pi \left[G(\sigma; \tau)\right] \\ &G(\sigma; \tau) := 1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau}). \end{aligned} \tag{B.117}$$

The first two terms in G are clearly decreasing in σ, so for every $q \in [a, b]$ and τ:

$$\mathsf{E}\left[1_{\{\sigma_2 \leq \tau\}} e^{-r\sigma_2} - c\int_0^{\sigma_2\wedge\tau} e^{-rs}\mathrm{d}s\right] \geq \mathsf{E}\left[1_{\{\sigma_1 \leq \tau\}} e^{-r\sigma_1} - c\int_0^{\sigma_1\wedge\tau} e^{-rs}\mathrm{d}s\right] \tag{B.118}$$

The proof for monotonicity of the last term in G is a bit more tricky, because $\pi_{\sigma\wedge\tau}$ is not just a function of σ, but it also depends on the initial π. So let us define $\boldsymbol{w}(\pi, \sigma; \tau) := e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau})$ where π is the initial belief value. To proceed, I need to define σ_1 and σ_2 on the *same* probability space, because the analysis to be presented needs more than the application of the first order stochastic dominance. For this, I use the Strassen theorem ([45]

[5]The term $\mathsf{exp}(\lambda)$ denotes an exponential random variable with the rate λ.

chapter 4) to find the coupling $(\hat{\sigma}_1, \hat{\sigma}_2)$ such that $\hat{\sigma}_i \overset{d}{=} \sigma_i$ for $i = 1, 2$, and crucially $\hat{\sigma}_1 \geq \hat{\sigma}_2$ almost surely. It is proven in the online appendix that for every τ, $\boldsymbol{w}$ is increasing in π and decreasing in σ (while holding π constant), therefore

$$\begin{aligned}
\mathsf{E}_{\pi_2}\left[e^{-r(\sigma_2\wedge\tau)}w(\pi_{\sigma_2\wedge\tau})\right] &= \mathsf{E}\left[\boldsymbol{w}(\pi_2, \hat{\sigma}_2; \tau)\right] && \text{(B.119a)}\\
&\geq \mathsf{E}\left[\boldsymbol{w}(\pi_1, \hat{\sigma}_2; \tau)\right] && \text{(B.119b)}\\
&\geq \mathsf{E}\left[\boldsymbol{w}(\pi_1, \hat{\sigma}_1; \tau)\right] && \text{(B.119c)}\\
&= \mathsf{E}_{\pi_1}\left[e^{-r(\sigma_1\wedge\tau)}w(\pi_{\sigma_1\wedge\tau})\right]. && \text{(B.119d)}
\end{aligned}$$

In (B.119a) and (B.119d), I used the fact that coupling preserves the marginal distributions. In (B.119b), I apply the increasing property of $\boldsymbol{w}$ in π, and in (B.119c) its decreasing property in σ.

Combining (B.118) and (B.119) implies that for every τ and $q \in [a, b]$: $\mathsf{E}_{\pi_2}[G(\sigma_2; \tau)] \geq \mathsf{E}_{\pi_1}[G(\sigma_1; \tau)]$, therefore, applying the supremum on both sides (w.r.t to τ) yields $\mathbb{T}_q w(\pi_2) \geq \mathbb{T}_q w(\pi_1)$. From this and expression (B.98), it is now straightforward to conclude that $\mathcal{A}w(\pi_2) \geq \mathcal{A}w(\pi_1)$. □

Now we are in a position to claim the existence of a fixed-point that is increasing, the proof of which follows from previous lemma and the fact that increasing property is closed under the L^1 limit.

Theorem 4. *The operator $\mathcal{A}$ has an increasing fixed-point function.*

For a candidate increasing fixed-point w, we can now assure that if $w(\pi') > 0$ for some π', then $w(\pi'') > 0$ for all $\pi'' > \pi'$. This means once w exceeds zero it will never fall down to zero again, therefore the union of all matching sets over $q \in [a, b]$ must be an *increasing set* in $[0, 1]$, hence there exists an equilibrium point α such that

$$\bigcup_{q\in[a,b]} \{\pi : \mathbb{T}_q w(\pi) > w(\pi)\} = (\alpha, 1]\,. \tag{B.120}$$

Next, I show how α is determined. Its location is important because it represents the point of endogenous exit from the market. In particular, the VCs with lower reputation than α would no longer invest. In the next proposition, I show under some natural assumptions, α is the boundary point of the stopping time problem that a generic investor solves when is matched to the *best* type of projects, i.e $q = b$. For this I present two notions. The profile of arrival intensity $\lambda = \{(\underline{\lambda}_q, \bar{\lambda}_q) : q \in [a, b]\}$ is called monotone if $\underline{\lambda}_q$ and $\bar{\lambda}_q$ are increasing in q. It satisfies the *increasing-differences* if $\bar{\lambda}_{q''} - \underline{\lambda}_{q''} \geq \bar{\lambda}_{q'} - \underline{\lambda}_{q'}$ for every $q'' > q'$ in $[a, b]$.

Proposition 10. *Assume the profile λ is monotone and satisfies the increasing-differences. Then, α is the lowest boundary point of $\mathcal{M}_b$, and is the unique fixed-point of*

$$\alpha = \frac{c}{\Delta_b \left(1 + w\left(\frac{\bar{\lambda}_b \alpha}{\Delta_b \alpha + \underline{\lambda}_b}\right)\right)} - \frac{\underline{\lambda}_b}{\Delta_b}. \tag{B.121}$$

Proof. Assume by contradiction that $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$, and there exists $q < b$ such that $\alpha = \inf \mathcal{M}_q$, that is a VC matched with a project of type q, terminates the funding as her reputation nears α. The principles of optimality requires smooth and continuous fit at α, namely $v'(\alpha, q) = v(\alpha, q) = 0$. From the Bellman equation for every $\pi \in \mathcal{M}_q$ it must be that

$$rv(\pi, q) = -c + \left(\bar{\lambda}_q \pi + \underline{\lambda}_q (1-\pi)\right)(1 + w \circ \jmath(\pi) - v(\pi, q)) - \pi(1-\pi)\Delta_q v'(\pi, q). \tag{B.122}$$

In that $\jmath$ returns the posterior *after* the success has taken place at time t:

$$\jmath(\pi_{t^-}) := \frac{\bar{\lambda}_q \pi_{t^-}}{\bar{\lambda}_q \pi_{t^-} + \underline{\lambda}_q (1 - \pi_{t^-})} \tag{B.123}$$

In the baseline model, the success event was *conclusive* thus $\jmath(\pi) = 1$ for every $\pi \in (0, 1]$. The optimality principles at $\pi = \alpha$ imply

$$c = \left(\alpha \Delta_q + \underline{\lambda}_q\right)(1 + w \circ \jmath(\alpha)). \tag{B.124}$$

Further, at $\pi = \alpha$, since $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$ then $v(\alpha, b) = w(\alpha) = 0$ and superhamonicity implies that

$$0 > \mathcal{L}_b v(\alpha, b) - rv(\alpha, b) - c = (\alpha \Delta_b + \underline{\lambda}_b)(1 + w \circ \jmath(\alpha)) - c. \tag{B.125}$$

Replacing (B.124) in the above inequality and canceling c from both sides amount to

$$0 > \frac{\alpha \Delta_b + \underline{\lambda}_b}{\alpha \Delta_q + \underline{\lambda}_q} - 1. \tag{B.126}$$

However the *rhs* of the above inequality is positive because of the monotonicity and increasing-differences, hence the contradiction is resulted. Therefore, it must be that $\alpha = \inf \mathcal{M}_b$.

On the uniqueness of α, note that the *lhs* of (B.121) is increasing in α, while the *rhs* is decreasing – because w is an increasing function. Therefore, upon the existence, α is uniquely determined by this equation. □

B.2 Supplementary proofs

Proof of lemma 4

Assume equation (2.7) holds, then one can check with $\chi_a(\pi) = \chi_b(\pi) = 1$ in equation (2.8) both of the conditions $v(\pi, a) > w(\pi)$ and $v(\pi, b) > w(\pi)$ are satisfied, therefore the *if* part is established. For the *only if* direction, assume $\pi \in \mathcal{M}(a) \cap \mathcal{M}(b)$, then it must be that $\chi_a(\pi) = \chi_b(\pi) = 1$. Replacing this in (2.8) and simplifying $v(\pi, b) > w(\pi)$ results in the first inequality in (2.7). Similarly, simplifying $v(\pi, a) > w(\pi)$ leads the second inequality in (2.7).‖

Proofs of section B.1

Proof for $\sigma_1 \succsim_{\mathsf{FSD}} \sigma_2$. For the two random variables defined in (B.116) we have

$$\mathsf{P}(\sigma_i > t) = \pi_i e^{-\bar{\lambda}_q t} + (1 - \pi_i) e^{-\underline{\lambda}_q t} \tag{B.127}$$

therefore,

$$\mathsf{P}(\sigma_1 > t) - \mathsf{P}(\sigma_2 > t) = (\pi_2 - \pi_1) \left(e^{-\underline{\lambda}_q t} - e^{-\bar{\lambda}_q t} \right) \geq 0, \tag{B.128}$$

because $\bar{\lambda}_q \geq \underline{\lambda}_q$ for every $q \in [a, b]$. Therefore, $\sigma_1 \succsim_{\mathsf{FSD}} \sigma_2$.‖

Properties of the transformed function $\boldsymbol{w}$. Here I prove the properties claimed about the function $\boldsymbol{w}$, namely the fact that it is increasing in π (initial belief) and decreasing in σ (success arrival time).

Decreasing in σ. Fix the initial belief π (as well as τ and q), then $\boldsymbol{w}$ is clearly continuous in σ and is constant on $[\tau, \infty)$. Further, it is decreasing on $[0, \tau]$, because $\bar{\lambda}_q \geq \underline{\lambda}_q$ so the posterior belief about $\{\theta = H\}$ falls more as the elapsed time to success gets longer. Formally, because of Bayesian learning

$$\begin{aligned} \pi_\sigma &= \pi_{\sigma^-} + \Delta \pi_\sigma \\ &= \pi_{\sigma^-} + \frac{\bar{\lambda}_q - \underline{\lambda}_q}{\pi_{\sigma^-} \left(\bar{\lambda}_q - \underline{\lambda}_q \right) + \underline{\lambda}_q} \pi_{\sigma^-} (1 - \pi_{\sigma^-}), \end{aligned} \tag{B.129}$$

where the first term π_{σ^-} is the posterior belief just before the success arrival and the second term $\Delta \pi_\sigma$ is the amount that the posterior jumps up at the time of the success. Define $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q \geq 0$, then again because of the Bayes-law:

$$\pi_{\sigma^-} = \frac{\pi e^{-\Delta_q \sigma}}{1 - \pi + \pi e^{-\Delta_q \sigma}} \Rightarrow \frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} = -\Delta_q \pi_{\sigma^-} (1 - \pi_{\sigma^-}) < 0 \tag{B.130}$$

Differentiating $\Delta\pi_\sigma$ w.r.t π_{σ^-} yields:

$$\frac{\partial \Delta\pi_\sigma}{\partial \pi_{\sigma^-}} = \frac{\Delta_q\left[(1-2\pi_{\sigma^-})(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q) - \pi_{\sigma^-}(1-\pi_{\sigma^-})\Delta_q\right]}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2} \tag{B.131}$$

I can now use the previous two relations to take the total derivative of π_σ w.r.t σ:

$$\begin{aligned}\frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} &= \left(1 + \frac{\partial \Delta\pi_\sigma}{\partial \pi_{\sigma^-}}\right)\frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \\ &= \frac{\underline{\lambda}_q(\underline{\lambda}_q + \Delta_q)}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2}\frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \leq 0\end{aligned} \tag{B.132}$$

To conclude the verification of $\boldsymbol{w}$ being decreasing in σ note that for $\sigma \in [0, \tau]$

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\sigma} = -re^{-r\sigma}w(\pi_\sigma) + e^{-r\sigma}w'(\pi_\sigma)\frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} \leq 0, \tag{B.133}$$

because of (B.132) and the fact that w is assumed increasing on $[0, 1]$ and hence is a.e differentiable with positive derivative.

Increasing in π. To show that $\boldsymbol{w}$ is increasing in π, I must hold σ fixed, thus it remains to show $w(\pi_{\sigma\wedge\tau})$ is increasing in the initial belief π. It is pretty straightforward to show that the posterior belief at any time, for Poissonian environment that we have, is increasing in the initial belief, hence the proof readily follows from the increasing property of w.‖