Labeling Common and Uncommon Fractions Across Notation and Education

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Abstract

A surge of recent research on fraction representation has provided substantial insight into how people think about proportional information in written, symbolic form and in visual, non-symbolic form. However, how fractions and decimals are verbally labeled is an often-overlooked aspect of proportion representation. In the current study, we investigated how adults label fractions and decimals (Study 1) and how children in a range of grades label fractions (Study 2), using a novel web-based platform for accessing student data from real classrooms (ASSISTments). In both studies, children and adults showed remarkable consistency in the kinds of labels they used. However, there were some differences in label preferences across notation and grade-level. Although the relations between fraction labeling and fraction ability remain unclear, these studies provide a first look at the kinds of labels that people typically use and provide some initial hypotheses for future research into symbolic representations of proportion.

Keywords: fractions; decimals; labels; ASSISTments

Introduction

Fractions have been an important topic in educational research for decades. Recently, however, there has been a resurgence of interest in the topic by both psychology and educational researchers, motivated by recent findings that fraction ability and knowledge may be an important gatekeeper to later math skills (Booth & Newton, 2012).

This research has provided substantial insight into the mental representation of fraction magnitudes (e.g., Schneider & Siegler, 2010) and how people think about symbolic fractions relative to symbolic decimals (e.g., DeWolf, Grounds, Bassok, & Holyoak, 2014; Hurst & Cordes, 2016, 2018) and non-symbolic representations of proportion (e.g., Matthews & Chesney, 2015). However, little research has investigated *fraction language.* The current study is a first step to investigating the variety of ways in which symbolic fractions and decimals are labeled, and how the use of these labels may relate to children's and adults' experience and ability with rational numbers.

Fraction and decimal symbolic notations are complex, involving both Arabic numerals and non-numerical symbols (e.g., vinculum, decimal point), and are quite different from each other. Importantly, fractions and decimals can also convey distinct concepts. For example, adults and children are better able to use magnitude information from symbolic decimals than fractions (Hurst & Cordes, 2016, 2018). On the other hand, fractions may be better suited for discrete contexts in which part-whole information is necessary (DeWolf, Bassok, & Holyoak, 2015). Although substantial recent work has shed light on how symbolic, written notations are processed, substantially less work has investigated how fraction and decimals may be *verbally labeled*.

From research with whole numbers, we know that the language used to label quantities can have dramatic impacts on the way people think about the magnitudes represented by them. For example, children learning languages that use a more explicit base-10 number system for their verbal number labels tend to have better counting skills than children learning English (Fuson & Kwon, 1992; Laski & Yu, 2014). Some cross-linguistic work with fraction labels suggests that transparency in fraction labels may also be critically important. For example, using the labels "of five parts, three" or "three of five parts" may help young children learn symbolic fractions better than traditional English labels like "three fifths" (Mix & Paik, 2008).

Importantly, however, even within the English language there are a variety of ways to label symbolic fractions. For example, if presented with the symbol 3/5, we may refer to it as "three-fifths", "three over five", or "three out of five". Importantly, all three of these labels are correct and commonly used in everyday contexts. Yet, these labels are not the same; each has a different structure. In particular, "three-fifths" is the abstract and formal label that is taught in school, but it does not describe the symbol or the relation in any concrete way. On the other hand, "three over five" directly describes the formal fraction symbol (i.e., that the number three is written over the number five) and "three out of five" describes the fractional relation. Relatedly, decimals are often taught using formal fraction and place-value labels (e.g., "seventy five hundredths"), but colloquially the symbol is simply described (e.g., "zero point seven five" or "point seven five").

Thus, these distinct ways of talking about fractions and decimals may differentially rely on our formal knowledge of rational numbers, the symbolic notation, and the magnitude being represented. Despite the high flexibility in the way we label symbolic fractions and decimals, little is known about how these types of labels are used across distinct notations (fractions, decimals) or may depend on the familiarity or simplicity of the magnitude (unit fractions, non-unit fractions) and the individuals' experience and/or ability with symbolic rational numbers.

For example, substantial work suggests that "half" may be a privileged magnitude, even for young children (Spinillo & Bryant, 1991). Even beyond half, some magnitudes may be simpler or more common than others. For example, unit fractions (fractions with a one in the numerator) are the first types of fractions to be taught (National Governors Association Center for Best Practices, 2010), and are often considered easier than non-unit fractions. Therefore, when investigating how people think about numbers beyond the count list (i.e., natural numbers), it may be crucial to consider how the specific notation system (decimals and fractions) as well as the simplicity of the particular magnitude may impact the verbal labels used.

In the current study we report two studies addressing three specific research questions: (1) What kinds of labels do adults use for fractions and decimals, and do they differ (Study 1)? (2) Do adults' preferred labels differ across unit and non-unit magnitude values (Study 1)? And (3) What labels do children use for non-unit fractions and are labeling preferences associated with educational experience (i.e., grade) and/or ability (Study 2)?

Study 1

In Study 1, we asked adults to label either fractions or decimals, in a between-subject design. The values were taken from both unit fractions with small denominators and non-unit fractions with larger denominators, as well as the approximately equivalent decimals (up to rounding error). Our central goal was to describe the kinds of labels that adults used for symbolic rational numbers and whether they differed across notation and the type of magnitude. That is, are different notations and different magnitudes likely to be labeled in different ways? In addition, we were interested in exploring how labeling preferences may be related to fraction performance, using a symbolic magnitude comparison task. However, as revealed below, extremely high consistency in symbol labeling made it impossible to address this question and thus we will not discuss the magnitude comparison task any further.

Method

Participants Adults were recruited via Amazon Mechanical Turk (www.mturk.com). Participants were paid $0.15 for participation, which took approximately five minutes. Participants were randomly assigned to complete one of two versions of the task (fraction or decimal).

In the fraction version, our sample consisted of 109 adults (age range: 19 – 72 years, M = 35.5 years, SD = 12.7, 46% male; 12.8% completed high school only, 64.2% attended at least some college, and 22.9% attended at least some graduate school). In the decimal version, our sample consisted of 99 adults (age range: 19 – 71 years, M = 33.9 years, SD = 11.4, 47% male; 10.2% reported only completing high school, 66.3% attended at least some college, and 23.5% attended at least some graduate school).

Symbolic Labels Task Each participant was presented with three unit fractions or the approximately equivalent decimal value (1/2, 1/4, 1/3 or 0.5, 0.25, 0.3; always presented in that order) and two non-unit fractions or the approximately equivalent decimal value (7/9, 5/8 or 0.78, 0.625; presented in a random order). The order of the two types of magnitudes (unit fractions (including half) v. non-unit fractions) was counterbalanced across participants. On each trial, adults were presented with a single symbol and an empty text box and asked to type out how they would read the number out loud before advancing to the next question.

Data Analysis Responses were coded based on the type of label used. For fractions, labels were coded as: (1) Formal Labels (e.g., one quarter, seven ninths), (2) Symbol Descriptive Labels (e.g., one over four), (3) Division Labels (e.g., one divided by four), or (4) Other. For decimals, labels were coded as: (1) Formal Labels (i.e., the fraction or place value labels taught in school; e.g., for 0.25: "one quarter" or "twenty five hundredths"), (2) Symbol Descriptive Labels (e.g., "zero point two five" or "point two five"), or (3) Other. In both cases, other labels included incorrect labels, responses that were not labels, responses that included multiple labels, and non-responses.

Results and Discussion

First, we looked at the kinds of labels that adults used for both notations, separated by the type of magnitude, with "half" being considered separately (Table 1).

Notably, for adults in the fraction condition, the most common label by far was the formal fraction label: 83.5% to 91.7% of adults consistently used this label, depending slightly on the specific fraction. Regardless of whether the fractions were relatively simple unit fractions (half, third, quarter) or if they were less common and more difficult (7/9 and 5/8), adults were most likely to use a formal fraction label and relatively unlikely to use any other label. This high consistency may be due, in part, to the fact that our participants were adults who were knowingly responding to a survey and were likely extremely aware of the "correct" (i.e., formal) response. Thus, this pattern does not rule out the possibility that adults would use other labels in other instances.

On the other hand, for adults in the decimal condition, most used labels that described the symbol (e.g., "point five"): 79.8% to 89.9% of participants' used this label, depending on the specific decimal. In general, very few adults used the formal fraction or place-value labels for decimals equivalent to the unit fractions (excluding half; 3% of adults) or decimals equivalent to non-unit fractions (2% of adults), but a slightly greater proportion (11%) used the formal label "half" for the decimal 0.5.

Thus, adults tended to use distinct types of labels for fractions and decimals. Although the formal, school-taught labels were used for fractions, decimals were primarily labeled using a label that describes the symbolic form itself. One possibility for this is that adults were less aware of the formal, place-value labels (e.g., twenty five hundredths) for decimals, potentially because of their rarity outside the classroom. This is discussed further below.

Table 1: Percent of participants using each label type.

	Coding Category:	Formal	Descriptive	Division	Other	Inconsistent
Fractions N = 109	Half (1/2)	91.7	1.8	0.9	5.5	NA
	Unit (1/3, 1/4)	91.7	1.8	0.9	4.6	0.9
	Non-Unit (5/8, 7/9)	83.5	4.6	2.8	6.4	2.8
Decimals N = 99	Half (0.5)	11.1	81.8	NA	7.1	NA
	Equiv. Unit (0.3, 0.25)	3.0	79.8	NA	7.1	10
	Equiv. Non-Unit (0.625, 0.78)	2.0	89.9	NA	6.1	2.0

Study 2

Given the low variability in adults' labeling in Study 1, in Study 2 we investigated whether children who are actively learning symbolic fractions would show more meaningful variability in their fraction labeling. Thus, we investigated how 4th - 12th grade students label symbolic fractions and how these labeling preferences may be related to experience and/or ability with various fraction concepts and procedures. To do this in an educationally relevant setting, we used the ASSISTments platform (www.assistments.org; Heffernan & Heffernan, 2014), a web-based adaptive mathematics tutor that teachers incorporate into their lessons within the classroom or as homework and external researchers can gain access to the anonymous data of real students completing real assignments. In this way, we were able to capture the labeling preferences of a large number of students who are actively in the process of learning symbolic fractions, across multiple classrooms, schools, and states.

Method

Participants Our overall sample consisted of 1581 students who completed anywhere from 1-4 targeted assignments, resulting in a total of 2589 assignments, completed using the ASSISTments platform between 11/2015 and 12/2017. Teachers assigned the assignments as part of students' regular education. As such, students were not randomly assigned to the different assignments and the samples across the four assignments may vary markedly (e.g., grade, time of year, state, teacher) and are partially overlapping. In particular, although some students only participated in one assignment (n=811), many students participated in more than one assignment (two assignments n=618; three assignments n=66; four assignments n=86). The anonymous data were provided to us from the ASSISTments platform and the use of the data was approved through an exempt IRB from Boston College. All datasets used in the current study are made available through the ASSISTments Research Team (2017a, 2017b, 2017, 2017d).

The fraction addition sample consists of 840 students (six 4th graders, 69 5th graders, 139 6th graders, 195 7th graders, 21 8th graders, 337 12th graders, and 73 unknown). The fraction subtraction sample consists of 746 students (76 5th graders, 164 6th graders, 191 7th graders, 17 8th graders, 10 10th graders, eight 12th graders, and 280 unknown). The fraction equivalence sample consists of 417 students (eight 4th graders, 30 5th graders, 111 6th graders, 106 7th graders, two 8th graders, 15 10th graders, 51 11th graders, and 94 unknown). The fraction magnitude comparison sample consists of 586 students (18 4th graders, 49 5th graders, 168 6th graders, 140 7th graders, three 8th graders, 17 10th graders, 53 11th graders, and 138 unknown). Across all four assignments, our samples come from 86 teachers, 65 schools, 52 districts, and 12 states.

Design All students first completed two questions about how they verbally label different fractions. Students then participated in one of four assignments, tapping different skills: fraction addition, fraction subtraction, equivalent fractions, and comparing fraction magnitudes. These specific skills were chosen because the skill builders already existed within the ASSISTments platform, suggesting some ecological validity to the educational relevance of the skills, and because they cover a variety of concepts within fraction learning that span several grades of fraction instruction. All assignments were completed on a computer, as instructed by the students' teachers.

Fraction Labels Task Students were presented with two questions, one at a time. In both questions, children were shown a symbolic fraction (presented in their formal and upright format with a horizontal fraction bar) and asked: "if you were to read this fraction out loud to yourself, how would you say it?". Students were told there was no right or wrong answer because people read fractions out loud in different ways. On the first question, students were shown 5/7 and were prompted to type their answer in a textbox exactly as they would say it. On the second question, students were shown 3/8 and were given five multiple-choice options to chose from: "three-eight", "three over eight", "three", "three eighths", and "three slash eight".

Skill Builder Tasks Following the label questions, children were presented with the Skill Builder assigned by the teacher. A Skill Builder is a unique type of assignment within ASSISTments that gives students random questions from a larger question bank, allowing access to hints, until students respond accurately to three questions in a row. The basic structure of the skill builder and scoring was identical across the four skill builders, however the nature of the problems differed.

The Addition skill builder involved adding two symbolic, proper fractions with different denominators. On each question, two symbolic fractions were presented with a "+" in between and students were asked to find the sum and type it into a text box either as a reduced proper fraction or as a mixed number (e.g., 2 ¼).

The Subtraction skill builder involved subtracting one symbolic, proper fraction from another with different denominators. On each question, two symbolic fractions were presented with a "-" in between and students were asked to find the difference and type it into a text box either as a reduced proper fraction or as a mixed number.

The Equivalent Fractions skill builder involved finding a missing value in order to make two fractions equivalent or converting a single fraction into a different form (e.g., simplifying a fraction or converting into a mixed number). The problems were presented in one of three ways: (1) symbolically, with two fractions on either side of an equal sign and a variable in one of the fractions (e.g., 3/6 = x/12), (2) as a word problem with only one symbolic fraction and a written statement asking students to find the missing numerator/denominator of an equivalent fraction (e.g., 3/6: find the denominator of a fraction equivalent to the given fraction with a numerator of 30), or (3) as a single symbolic fraction the student needed to convert (e.g., convert the improper fraction 23/6 into a mixed number).

In the Magnitude Comparison skill builder, students were presented with two symbolic proper fractions and asked to choose which fraction was greatest.

Data and Data Analysis Students' responses on the open ended label question were categorized into one of the following label categories: (1) Formal (e.g., five sevenths), (2) Symbol Descriptive (e.g., "five over seven"), (3) Division (e.g., five divided by seven), (4) Out-of (e.g., "five out of seven"), (5) Other, which included incorrect labels and non-responses, and (6) Multiple or Inconsistent labels. For formal labels, the responses of "five sevenths", "five seventh", and "five sevens" were all accepted, as they *sound* highly similar to the true formal label, however "five seven" was not accepted (and would have been categorized as an incorrect label).

We will look at the data in two ways. First, to get a descriptive picture of the kinds of labels students use across grades, we will analyze all students, regardless of the skill builder they participated in. For students who participated in multiple skill builders, students who reported a consistent label will be grouped into that label category and students who were inconsistent in their labels will be categorized as such (i.e., within "multiple or inconsistent labels"). Second, to look at the relations between the kinds of labels students use and their fraction ability, we analyzed each skill builder separately.

Reported demographic information is based on self-reports within the ASSISTments platform. Since grade (i.e., year in school) was not required information many students did not report it, resulting in substantial missing data (584/1581 students did not report their grade). Thus, although we have included some analyses looking at educational level, these rely on a substantially smaller subsample of students that have reported their grade.

Results and Discussion

Descriptive Analyses First, we looked at the kinds of labels that students reported and how this differed across the students' grade (for those who self reported grade). Table 2 presents the number of students whose response was in each category for the free response (Table 2A) and multiple-choice (Table 2B) questions.

When looking at the correct labels for both questions, the two most common responses were the formal label (e.g., five sevenths) and the descriptive label using the term "over" (e.g., five over seven). In order to get a better sense of changes across grades in the use of these two labels in particular, we used a binary logistic regression, predicting label from grade (for the subset of students who reported grade). For both the fractions 5/7 (n=580 using formal, n=148 using "over") and 3/8 (n=688 choosing formal, n=187 choosing "over"), the model was statistically significant (5/7: $\chi^2(1) = 57.1$, $p < 0.001$; 3/8: $\chi^2(1) = 40.4$, $p < 0.001$) and accurately predicted 81.2% and 78.6% of the cases, respectively. In particular, grade was a significant predictor in both models. Odds of reporting the "over" label increase by 1.4 times with each one unit increase in grade for the fraction "5/7" (Wald $\chi^2(1) = 56.4$, $p < 0.001$) and 1.3 times with each one unit increase in grade for the fraction "3/8" (Wald $\chi^2(1) = 41.0$, $p < 0.001$). Thus, students from higher grades were more likely to use the label that describes the symbol ("over"), than students from lower grades. This might suggest that children's use of the descriptive label may be associated with more experience with fraction symbols, whereas children's early experiences with fraction symbols may be less flexible to using alternative labels, outside the formal label they are explicitly taught.

Table 2A: % of Students Using Each Label Type for 5/7

	Total N	Formal	Descriptive ("over")	Division	Out of	Other	Multiple/ Inconsistent
Full Sample	1581	58.4	17.5	1.1	1.8	6.8	14.8
Grade 4	30	73.3	6.7	0	0	16.7	3.3
Grade 5	135	72.6	3.0	0.7	1.5	10.4	11.9
Grade 6	297	52.9	12.5	2.7	2.0	5.1	24.9
Grade 7	390	61.3	13.6	0.8	2.6	6.2	15.6
Grade 8	24	58.3	12.5	0	4.2	4.2	20.8
Grade 9	0	NA	NA	NA	NA	NA	NA
Grade 10	27	7.4	51.9	3.7	3.7	11.1	22.2
Grade 11	55	65.5	21.8	0	1.8	3.6	7.3
Grade 12	39	30.8	59.0	0	0	2.6	7.7

Table 2B: % of Students Using Each Label Type for 3/8

	Total N	Formal	Descriptive ("over")	Descriptive ("slash")	Whole Numbers	Numerator Only	Inconsistent
Full Sample	1581	69.6	20.2	0.2	0.9	0	9.0
Grade 4	30	83.3	13.3	0	3.3	0	0
Grade 5	135	85.9	3.7	0	0	0	10.4
Grade 6	297	65.3	16.5	0	0.7	0	17.5
Grade 7	390	69.7	19.2	0.8	1.0	0	9.2
Grade 8	24	75.0	20.8	0	0	0	4.2
Grade 9	0	NA	NA	NA	NA	NA	NA
Grade 10	27	22.2	63.0	0	0	0	14.8
Grade 11	55	65.5	25.5	0	0	0	9.1
Grade 12	39	53.8	46.2	0	0	0	0

Relation between Labeling and Performance Next, we investigated the relations between how children labeled fractions and their fraction ability. Given the partial overlapping nature of the data from the skill builders, we opted not to use an overall analysis. Instead, we compared the average number of problems required to complete the specific skill builder (higher value = worse performance) between students who used a formal label versus an "over" label[1] (the difference in sample size across these two groups resulted in unequal variances and thus the Welch t-tests not assuming equal variances is reported for all tests). Further, we are reporting the analyses with and without controlling for grade. Given the substantial missing data for grade, these analyses rely on highly different samples and sample sizes, and may be theoretically different (we discuss this further in the General Discussion).

For students in the fraction addition and fraction subtraction skill builders, students who chose the "over" label performed slightly, though significantly, better than students who chose the formal label (Addition: $n_{formal} = 625$, $M_{formal} = 7.4$ problems; $n_{over} = 199$, $M_{over} = 6.8$ problems; $t(458.3) = 2.4$, $p = 0.02$, Cohen's d = 0.17; Subtraction: $n_{formal} = 576$, $M_{formal} = 7.4$ problems; $n_{over} = 152$, $M_{over} = 6.5$ problems; $t(405.3) = 2.4$, $p = 0.001$, Cohen's d = 0.22). This pattern remained marginal in the smaller sample when controlling for grade in the subtraction skill builder ($n_{formal} = 351$, $n_{over} = 102$; $p = 0.09$, $\eta^2_{partial} = 0.006$), but not in the addition skill builder ($n_{formal} = 351$, $n_{over} = 105$; $p = 0.3$, $\eta^2_{partial} = 0.002$).

Conversely, for students in the fraction equivalence skill builder, students who chose the formal label performed slightly, though only marginally significantly, better than students who chose the "over" labels ($n_{formal} = 297$, $M_{formal} = 9.9$ problems; $n_{over} = 115$, $M_{over} = 11.4$ problems), $t(180.7) = 1.9$, $p = 0.06$, Cohen's d = 0.23. This pattern remained marginal when controlling for grade in the smaller sample ($n_{formal} = 221$, $n_{over} = 98$; $p = 0.09$, $\eta^2_{partial} = 0.009$). Notably, however, this was not true when students were grouped based on the open-ended problem with 5/7 ($p = 0.7$).

The magnitude comparison task did not show any significant differences between those who used the "over" versus the formal label (all ps > 0.1).

Together, this might suggest that using descriptive symbols is associated with basic arithmetic procedures whereas the formal, abstract label may be associated with the conceptual aspects of equivalent fractions. However, given that these patterns were substantially weakened when controlling for grade, it is unclear whether the relation between performance and labeling may be reflecting relations with ability per se, irrespective of experience, or more tied to experience, as measured through grade.

General Discussion

Across two studies, we explored how adults and children label symbolic rational numbers, without any overt context. Overall, the way both children and adults labeled symbolic fractions and decimals was remarkably consistent within the samples.

In Study 1, almost all of the adult participants used formal labels when labeling fractions and symbolic descriptive labels referencing the decimal point when labeling decimals. Thus, although there was not variability within our sample, there was a striking, although also unsurprising, difference across the two notation systems. That is, not only are decimals easier to use for thinking about magnitude (Hurst & Cordes, 2016), they were also labeled in a more colloquial and less formal way. This colloquial and less formal label may further highlight adults' relative comfort in working with decimal notation and/or may reflect the difficulty of the formal fraction and place-value labels typically taught in school.

In Study 2, most children labeled the fractions using formal fraction labels. However, unlike the adult sample, a sizeable group of children did use symbolic descriptive labels using the "over" phrase when labeling fractions. Furthermore, children from higher grades were more likely to use the descriptive "over" label than children from lower grades. This pattern, combined with the small evidence that children who used the "over" labeled performed better on addition and subtraction problems, may suggest that children are more able to use colloquial or less formal labels as they gain experience with the symbols, particularly for relatively basic, procedural skills. However, it is worth noting that our adult sample also overwhelmingly used the formal labels and did not rely on the "over" label. Thus, although it is possible that this is a cohort effect (i.e., differences in fraction label education across time), it may also signal a u-shaped pattern in label preferences across educational experience. Furthermore, although only marginal, it is worth noting that performance on the

[1] For brevity, we are only reporting results based on students' labeling on the multiple choice response to 3/8, however results based on labeling 5/7 show the same results with only one exception, which is highlighted in-text.

equivalence fraction skill builder showed the opposite relation: children who used "over" label performed *worse* on fraction equivalence than children who used formal labels. This may be because fraction equivalence relied on more than just procedural symbol manipulation and required a deeper understanding of the fraction concept, however, this is highly speculative and should be further investigated. Lastly, the magnitude comparison task performance was not associated with differences in label preference for the children and we were unable to investigate this question with the adults.

Taken together, these patterns provide some initial hypotheses for how labels may be implicated in the way people think about fractions and decimals. In particular, although the recent emphasis on whole number bias (e.g., Ni & Zhou, 2005) may lead to the hypothesis that labels that rely primarily on the symbolic components ("over" or "point" labels) may be associated with lower understanding and experience, this is not what we found. Across both studies, adults tended to use colloquial or symbolically descriptive labels more often for easier notation systems (decimals) and, even for fractions, children from older grades were more likely to uses this type of label. Thus, although this type of label is more informal and more related to the structure of the symbol than the underlying concept, its use may reflect increased fluency or experience with the notation. However, this symbol-focused approach to thinking about fractions in particular may not be equally beneficial across all aspects of fraction understanding (e.g., fraction subtraction v. equivalence). This may be an important avenue for future research.

In sum, the descriptive patterns of behavior by adults and children in the current samples provide some initial data for generating hypotheses about the role of labels in fraction and decimal learning, an often-overlooked aspect of rational number representation.

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