Title
Relational Reasoning with Rational Numbers: Developmental and Neuroimaging Approaches

Author
DeWolf, Melissa

Publication Date
2016

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Relational Reasoning with Rational Numbers:

Developmental and Neuroimaging Approaches

A dissertation submitted in partial satisfaction of the

Requirements for the degree Doctor of Philosophy

in Psychology

by

Melissa Elizabeth DeWolf

2016

ABSTRACT OF THE DISSERTATION

Relational Reasoning with Rational Numbers:

Developmental and Neuroimaging Approaches

by

Melissa Elizabeth DeWolf

Doctor of Philosophy in Psychology

University of California, Los Angeles 2016

Professor Keith J. Holyoak, Chair

The study of how children and adults learn mathematics has given rise to a rich set of psychological phenomena involving mental representation, conceptual understanding, working memory, relational reasoning and problem solving. The subfield of understanding rational number processing and reasoning focuses on mental representation and conceptual understanding of rational numbers, and in particular fractions. Fractions differ from other number types, such as whole numbers, both conceptually and in format. Previous research has highlighted the extent to which fractions and other rational numbers pose challenges for children and adults with respect to magnitude estimation and misconceptions. The goal of this dissertation is to highlight the distinct differences in reasoning with different types of rational numbers. First, a neuroimaging study provides evidence that fractions yield a distinct pattern of neural activation during magnitude estimation that differs from both decimals and integers (Chapter 2). Second, a set of behavioral studies with adults highlights the affordances of the bipartite format of fractions

for relational reasoning tasks (Chapter 3). Finally, a developmental study with pre-algebra students provides evidence for a significant relationship between relational understanding of fractions and algebra performance, and specifically algebraic modeling. This work is presented in the context of viewing mathematical notation as a type of conceptual modeling. In particular, decimals have advantages in measurement and representing magnitude. Fractions, on the other hand, have advantages in *relational* contexts, due to the fact that fractions, with their bipartite (*a/b*) format, inherently specify a relation between the cardinalities of two sets. When mathematics is viewed as a type of relational modeling, rational expressions provide a gateway to more complex mathematical notations and concepts, such as those in algebra.

The dissertation of Melissa DeWolf is approved.

Jennie Katherine Grammer

Martin M. Monti

James W. Stigler

Keith Holyoak, Committee Chair

University of California, Los Angeles

2016

DEDICATION

This dissertation is dedicated to the best teachers that I know—my parents. My interest in education and how people learn stems from their emphasis on the importance of my own education and in encouraging me to ask questions about the things I don't know, and to try to find the answers to them. This dissertation is also dedicated to the best mathematician that I know—my husband, Sam. His expert mathematical thinking in part inspired my own interest in understanding how people master math.

TABLE OF CONTENTS

LIST OF FIGURES

List of Tables

ACKNOWLEDGEMENTS

Chapter 2 of this dissertation is a version of DeWolf, Chiang, Bassok, Holyoak and Martin (under review). Martin, Holyoak and I developed the paradigm for the study and Chiang and I collected data using the fMRI scanner at the Staglin IMHRO Center for Cognitive Neuroscience with pilot funding from the center. The manuscript was written with theoretical contributions from Bassok and Holyoak and technical contributions and analyses from Chiang and Martin. Chapter 3 a version of DeWolf, Bassok and Holyoak (2015a). Bassok and Holyoak helped to develop the paradigms for each of the experiments and both contributed to the theoretical components of the manuscript, especially in the introduction and discussion. Chapter 4 of this dissertation is a version of DeWolf, Bassok and Holyoak (2015b). Bassok and Holyoak contributed to the theoretical components of the introduction and discussion of the manuscript.

I would like to thank my advisor, Keith Holyoak, for his guidance and support in developing and carrying out these studies. Miriam Bassok also provided invaluable support across various projects and throughout my graduate program. My co-authors and I thank our team of undergraduate research assistances for help collecting data for the experiments. In addition, we thank James Stigler, Belinda Thompson, Caroline Montojo, Monica Rapp, and anonymous reviewers for feedback on earlier drafts of the various manuscripts. We also thank members of the UCLA Reasoning Lab and Teaching and Learning Lab for their thoughtful feedback throughout the development of these various projects.

This dissertation research was supported by National Science Foundation Graduate Research Fellowship DGE-1144087, Grant R305C080015 from the Institute of Education Sciences, UCLA's Staglin Center pilot funding, as well as UCLA's Distinguished University Fellowship and Graduate Summer Research Mentorship.

VITA

Education

M.A. Cognitive Psychology 2012 - 2013
University of California, Los Angeles

B.S. Psychology 2008 – 2012
Minor: History
Carnegie Mellon University

Publications

Peer Reviewed Journal Articles

Lee, H. S., DeWolf, M., Bassok, M., & Holyoak, K. J. (2016). Conceptual and procedural distinctions between fractions and decimals: A cross-national comparison. *Cognition, 147,* 57-69.

Fazio, L.K. , DeWolf, M., Siegler, R. S. (2015). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory and Cognition.* Advanced online publication. http://dx.doi.org/10.1037/xlm0000153

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). From rational numbers to algebra: Separable contributions of decimal magnitude and relational understanding of fractions. *Journal of Experimental Child Psychology, 133*, 72-84.

DeWolf, M., Bassok, M., & Holyoak, K.J. (2015). Conceptual structure and the procedural affordances of rational numbers: Relational reasoning with fractions and decimals. *Journal of Experimental Psychology: General, 144*(1), 127- 150.

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction, 37*, 39 – 49.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals. *Journal of Experimental Psychology: Applied, 21*(1), 47-56.

Guthormsen, A. M., Fisher, K. J., Bassok, M., Osterhout, L., DeWolf, M. & Holyoak, K. J. (2015). Conceptual integration of arithmetic operations with real-world knowledge: Evidence from event-related potentials. *Cognitive Science*. DOI: 10.1111/cogs.12238.

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Representation and comparison of magnitudes for different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance, 40*(1), 71-82.

Griffin, M., DeWolf, M., Keinath, A., Liu, X., & Reder, L. M. (2013). Identical vs. conceptual repetition FN400 and parietal old/new ERP components occur during encoding and predict subsequent memory. *Brain Research, 1512,* 68 – 77.

Conference Proceedings

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). Modeling relational priming and multiplicative reasoning with rational numbers. In R. Dale, C. Jennings, P. Maglio, T. Matlock, D. Noelle, A. Warfaumont & J. Yoshimi (Eds.), *Proceedings of the 37th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

DeWolf, M., Son, J. Y., Bassok, M., & Holyoak, K. J. (2015). Implicit understanding of arithmetic with rational numbers: The impact of expertise. In R. Dale, C. Jennings, P.

Maglio, T. Matlock, D. Noelle, A. Warfaumont & J. Yoshimi (Eds.), *Proceedings of the 37th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

Lee, H. S., DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). Semantic alignment of fractions and decimals with discrete versus continuous entities: A cross-national comparison. In R. Dale, C. Jennings, P. Maglio, T. Matlock, D. Noelle, A. Warfaumont & J. Yoshimi (Eds.), *Proceedings of the 37th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

DeWolf, M. & Holyoak, K. J. (2014). Reciprocal and multiplicative relational reasoning with rational numbers. In B. Bello, M. Guarini, M. McShane, & B. Scassellati (Eds.), *Proceedings of the 36th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

DeWolf, M., Rapp, M., Bassok, M., & Holyoak, K.J. (2014). Semantic alignment of fractions and decimals with discrete versus continuous entities: A textbook analysis. In B. Bello, M. Guarini, M. McShane, & B. Scassellati (Eds.), *Proceedings of the 36th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

DeWolf, M., Bassok M., & Holyoak, K. J. (2013). Analogical reasoning with rational numbers: semantic alignment based on discrete versus continuous quantities. In M. Knauff, M. Pauen, N. Sebanz, & I. Wachsmuth (Eds.), *Proceedings of the 35th Annual Conference of the Cognitive Science Society* (pp. 388 – 393). Austin, TX: Cognitive Science Society.

DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons in adults. In L. Carlson, C. Hoelscher, & T. F. Shipley (Eds.), *Proceedings of the 33rd Annual Conference of the Cognitive Science Society* (pp. 1751 – 1756). Austin, TX: Cognitive Science Society.

Other Conference Presentations

DeWolf, M., Bassok, M. & Holyoak, K. J. (2015, November). Multiplicative reasoning with fractions and decimals. Poster presented at the 56th Annual Meeting of the Psychonomic Society, Chicago, Illinois.

DeWolf, M., Bassok, M. & Holyoak, K. J. (2015, October). Relational reasoning with fractions primes algebraic modeling. Poster presented at the 9th Biennial Meeting of the Cognitive Development Society, Columbus, Ohio.

DeWolf, M., Holyoak, K. J., & Bassok, M. (2015, March). Relational reasoning with fractions predicts algebra understanding. Poster presented at the 2015 Biennial Meeting of the Society for Research in Child Development, Philadelphia, Pennsylvania.

DeWolf, M., Geller, E., Thai, K.P., & Walker, J.M. (2014, May). *Relational Reasoning with Rational Numbers*. In J. Stigler (Chair), *Relational Understanding in Mathematics: Issues and Interventions.* Symposium conducted at the 26th Annual Convention of the Association of Psychological Science, San Francisco, California.

Fellowships

NSF Graduate Research Fellowship	2013 – 2016
UCLA Graduate Summer Research Mentorship	2013
UCLA Distinguished University Fellowship	2012 – 2014

CHAPTER 1: INTRODUCTION

American schools have struggled to teach many of their students a conceptual basis for understanding mathematics in a way that can support flexible transfer and generalization to novel problems in math, science and engineering. Many students in the United States lag behind students from other industrialized countries, with the US ranking 26th out of 34 industrialized nations on the Programme for International Student Assessment (PISA) math section (OECD, 2012). Recent research has focused on two central areas of understanding in mathematics that seem to be critical gateways for future success: fractions and algebra. Algebra understanding is essential for success in college math and providing further career opportunities in STEM fields (NMAP, 2008; Sadler & Tai, 2007). Similarly, fraction knowledge in elementary school is uniquely predictive of high school success in mathematics controlling for family education and income, intellectual capacity, and knowledge of whole number arithmetic (Siegler et al., 2012).

General difficulties often begin in elementary school when students first move beyond natural numbers. A great deal of evidence indicates that children have difficulty understanding fractions and decimals (Stafylidou & Vosniadou, 2004; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). Often, knowledge of natural numbers leads to initial misconceptions about fractions. One key difference between these number types is their perceptual form: fractions are not represented with a unitary symbol, but rather have a bipartite structure, with a separate numerator and denominator. This formal difference between fractions and natural numbers appears to contribute to early misconceptions, and also to later use of fallible short-cut strategies for performing computations with fractions (Bonato et al., 2007). However, the bipartite structure of fractions may prove especially useful when it is construed as a *relational structure*, rather than only as a complex expression of magnitude.

Other misconceptions about rational numbers reflect negative transfer of natural numbers properties. For example, unlike natural numbers, fractions do not have unique successors. This and related conceptual differences make it difficult for children to integrate fractions into their already well-established understanding of natural numbers (Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; see Siegler et al., 2013, for a review). For example, Vamvakoussi and Vosniadou (2010) found that Greek 12-, 14-, and 16-year-olds made several typical mistakes when completing tasks relating to the infinite number of rational numbers in any interval of a number line. Their misconceptions included the idea that fractions are discrete and have unique successors as natural numbers do; and that only fractions can lie between two fraction endpoints, whereas only natural numbers can lie between two whole-number endpoints.

Like fractions, decimals denote rational numbers that lack a unique successor. But in terms of their form decimals are not explicitly decomposed into two major parts (numerator and denominator) as are fractions, but rather are more similar to natural numbers. Decimals are typically introduced into the curriculum after fractions (and natural numbers). Vamvakoussi and Vosniadou (2010) found that children inappropriately extend properties of natural numbers to decimals. Rittle-Johnson, Siegler and Alibali (2001) found that children mistakenly view the number of digits in a decimal as an indication of its magnitude (e.g., a majority of fifth and sixth graders claimed that .274 is larger than .83). Such misconceptions suggest that the conceptual distinctions that separate fractions and decimals from natural numbers may be especially significant in making the rational numbers harder to grasp.

However, there is also evidence suggesting that children can integrate decimals into their mathematical knowledge more easily than they can fractions. Iuculano and Butterworth (2011) used a number line task to assess the accuracy of adults and children in locating natural numbers, decimals, and fractions on a number line. They found that performance was quite similar for decimals and natural numbers. In contrast, adults and children (aged 10 years) were the least accurate when placing a mark on the number line at a certain fraction value or generating a fraction value when shown a mark on a number line. Such findings suggest that, at least for well-educated adults, magnitudes are easier to grasp using decimals than fractions, and that processing decimals may be more similar to processing natural numbers.

Magnitude Representations of Rational Numbers

While there is general agreement that rational numbers (especially fractions) pose difficulties for students, there has been some controversy regarding whether adults with high education levels (e.g., college students) eventually acquire magnitude representations for other rational numbers that are basically similar to their representations of natural number magnitudes. Many researchers have examined the extent to which the mental representation of fractions is the same as or different from representations of natural numbers (e.g., Bonato et al., 2007; Schneider & Siegler, 2010; Meert, Gregoire, & Noel, 2010). Bonato et al. (2007) found no evidence for a distance effect related to the magnitudes of the fractions in the comparisons. These investigators argued that adults process fractions in a piece-meal manner, dividing up the parts of the fractions (the numerator and denominator), and using heuristics based on these separate parts to determine the relative sizes of the fractions. However, Schneider and Siegler (2010) found evidence that a distance effect can be obtained for fractions, but only if the stimuli require processing of the whole magnitude of the fraction, rather than just its parts. For example, Bonato et al. (2007) examined unit fractions in the form $1/n$, which do not require adults to compare the integrated magnitudes of the fractions. Instead, people could use a simple heuristic: fractions that have larger denominators are smaller overall. When Schneider and Siegler (2010) tested comparisons based on fractions for which this heuristic could not be used, they found a pattern consistent with the hypothesis that adults do in fact access the whole magnitude of the fraction to perform the comparison.

DeWolf, Grounds, Bassok and Holyoak (2014) directly compared adult performance in comparing magnitudes with matched fractions, decimals, and multi-digit natural numbers. Their results indicated that while the pattern of response times for comparing decimals is almost identical to that for comparing multi-digit natural numbers, the pattern for fraction comparisons is dramatically slower, and shows a more exaggerated distance effect, indicative of less precise magnitude representations for fractions. Thus, decimals appear much more effective than fractions in conveying information about magnitudes.

In addition to using behavioral techniques, neuroimaging offers a complementary approach to building a more complete model of how rational numbers are mentally represented. Studies using fMRI and neuropsychological methods have provided insight into how numbers are processed and, more specifically, how numbers are processed in relation to the quantities that they are meant to model. In particular, magnitude processing appears to be a function of the intraparietal sulcus (IPS), as the horizontal segment of the IPS has been implicated in the processing of quantities, and is associated with a "mental number line" (Dehaene, Piazza, Pinel, & Cohen, 2003).

Neuroimaging studies of fraction processing have, much like behavioral research, focused on magnitude processing (for a review see Jacob et al., 2012). For example, Ischebeck, Shocke and Delazer (2009) investigated whether adults process fractions holistically as single magnitudes, or separately based on their numerators and denominators. Results from this study provided evidence that IPS activity was consistent with a distance effect between the whole magnitude of the fraction, rather than comparisons between numerators and denominators. This finding is consistent with the results of analogous behavioral studies that have investigated magnitude representation for fractions in adults (Schneider & Siegler, 2010). However, Ischebeck et al. did not compare processing of fractions with other number types (as in DeWolf et al., 2014), leaving open the question of whether other regions are differentially recruited for fraction magnitude processing as compared to the IPS regions implicated in number processing for natural numbers (and perhaps decimals). Other researchers have also found evidence for holistic processing of fractions and proportions. Jacob and Nieder (2009) found activation in the anterior IPS and prefrontal cortex for processing of fractions represented with either numerals (e.g., ½) and or fraction words. The authors interpreted this as evidence for holistic processing, but the additional activation in the prefrontal cortex may be indicative of higher-level relational processing.

Research on the neural basis of exact calculation suggests that different brain areas are recruited depending on an individual's level of familiarity with the specific calculation. Butterworth and Walsh (2011) report that when students solve novel arithmetic problems, the IPS is engaged to represent the magnitude of the numbers in the problem. In addition, the prefrontal cortex appears to be involved in goal setting, working memory, and attention during exact calculation. In contrast, recall of previously-learned facts appears to be a memory-based process, primarily relying on the angular gyrus. Montojo and Courtney (2008) also observed prefrontal activity during a cue period in preparation for performing an exact calculation. Monti, Parsons and Osherson (2012) found that similar areas are recruited in the processing of more relationally complex expressions, such as algebraic operations. Jacob, Vallentin and Nieder (2012) review evidence that the same areas involved in mental calculation, such as the division operation (the inferior frontal gyrus, premotor cortex, presupplementary motor area, and the IPS), are also involved in representation of fraction magnitudes. However, a direct comparison of fractions and other types of numbers, such as decimals, has not yet been conducted. The present proposal aims to extend neuroimaging investigations of processing rational numbers beyond magnitude comparisons, by directly comparing regions involved in the processing of fraction versus decimal rational numbers during a novel mathematical judgment task.

Rational Numbers as Relational Models

If the only purpose of rational numbers was to convey information about magnitudes, then based on the research findings reviewed above, fractions would appear to be greatly inferior to decimals. However, it may be that rational numbers in general, and fractions in particular, should not be considered as solely a tool to express magnitudes, but also to *model conceptual relations*. Kieran (1976) argued that a key conceptual component of rational numbers is their ability to represent a variety of mathematical relations, such as part-whole, quotient, ratio number, operator and measure. Although both decimals and fractions can represent magnitudes of these types of relations, fractions seem to be particularly well-suited for representing relations between two distinct sets because of their bipartite structure, whereas decimals are better suited to represent one-dimensional measures of magnitude (DeWolf et al., 2014).

More generally, I hypothesize that mathematical understanding fundamentally involves grasping that mathematics is a system of quantitative relations among concepts, and that one important goal of math education should be to teach how the formal rules of mathematics can be applied to real-life situations (Martin & Bassok, 2005). When mathematics is viewed as a type of conceptual modeling, the notation serves a dual purpose: first as a bridge between quantitative concepts and the structure of the world being modeled, and second as a platform for performing mathematical procedures to solve problems defined in relation to the structure of the world. Procedures then derive meaning from their place in the conceptual structure, rather than standing in isolation as rote routines (e.g., English & Halford, 1995; Hinsley, Hayes & Simon, 1977; Kintsch & Greeno, 1985; Mochon & Sloman, 2004).

Previous research suggests that people's grasp of intuitive relations in the real world can be used to bolster their understanding of quantitative relations within arithmetic. For example, Bassok and her colleagues have shown that, in dealing with word problems, people are guided by *semantic alignment*: an analogical mapping between the relation between the objects in a problem statement and the relation between the arguments of arithmetic operations (see Bassok, 2001, for a review). Bassok, Chase and Martin (1998) demonstrated that the basic mathematical operations of addition, subtraction, multiplication, and division are typically conceptualized within a system of relations between mathematical relations and the ontological relations among objects in the real world. For example, addition is aligned with categorical object relations (e.g., people find it natural to add two daffodils plus three tulips, because both are subtypes of a common category, flowers), whereas division is aligned with functional object relations (e.g., a natural problem would be to divide ten tulips between two vases). Both children and adults are sensitive to semantic alignment (e.g., Martin & Bassok, 2005), and for many adults the process is highly automatic (Bassok, Pedigo, & Oskarsson, 2008). Semantic alignment may thus offer a potential base for building deeper understanding of mathematics (cf. Goldstone & Barsalou, 1998).

Viewing mathematical competence in terms of modeling and alignment may help to understand the psychological representation of rational numbers, particularly fractions and decimals. Research on understanding of rational numbers has largely focused on their function as notations for expressing numerical magnitudes. Indeed, fractions and decimals are typically viewed as simply alternative notations, which (other than rounding error) are equivalent in magnitude (e.g., 3/8 vs. 0.375). Differences between notational variants have generally been treated as conceptually inconsequential (but see Cohen, Ferrell & Johnson, 2002, and Goldstone, Landy & Son, 2010, for important exceptions).

What is often overlooked within this standard conceptual framework is the fact that the core definition of a rational number (i.e., one that can be expressed as the quotient of two natural numbers, a/b, where $b \neq 0$) is inherently *relational*. A rational number indeed expresses a magnitude, but one that quantifies the relation defining the number. Specifically, the bipartite structure of the a/b notation expresses a relational model for the quantities corresponding to the numerator and denominator. The form a/b expresses a division relation between natural numbers, which has the form of a fraction; whereas the output of the division, c, expresses the magnitude of that relation and can be a decimal (with magnitude less than one when $a < b$). The fraction and the decimal indeed convey the same magnitude, but their distinct roles in the relational model give rise to important conceptual distinctions. As illustrated in Figure 1.1, the bipartite structure of the fraction provides a direct mapping to countable subsets in a visual display (e.g., a

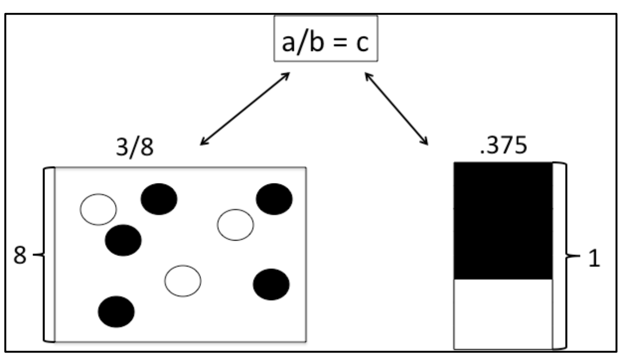


Figure 1.1. The bipartite structure of a fraction maps to countable subsets in a visual display (left), whereas the decimal equivalent represents a one-dimensional magnitude (right).

subset of 3 white objects within a set of 8). In contrast, the decimal equivalent loses the two-dimensional structure of the fraction. Instead, it provides a one-dimensional measure of a portion of a continuous unit quantity.

In considering the role of modeling in understanding rational numbers, a particularly important ontological distinction involves the nature of the two be modeled quantities, which can be viewed as either *discrete* or *continuous*. Roughly, some entities are viewed as comprising a set of individual objects (e.g., a number of apples in a basket), whereas others are viewed as a continuous mass without individuation (e.g., a bucket of water). Continuous quantities can be subdivided into equal-sized units (i.e., discretized) for the purpose of measurement, but the divisions are arbitrary in the sense that they do not isolate conceptual parts (e.g., one can distinguish between specific subsets of red and green apples in a basket by noting that 2/3 of the apples are red and 1/3 are green, but there is no psychological difference between the water contained in 2/3 of a bucket and in the complementary 1/3 of the bucket).

Importantly, the contrast between discreteness versus continuity (closely related to the count versus mass distinction in linguistics; see Bloom, 1994; Bloom & Wynn, 1997) is based fundamentally not on physics, but on psychology. For example, a pile of sand is viewed as a continuous quantity even though we know it is composed of individual grains, because those units are too small and interchangeable to be typically viewed as "important". The impact of this basic ontological distinction has been documented both in young babies (e.g., Spelke, Breilinger, Macomber, & Jacobson, 1992) and in adults (Bassok & Olseth, 1995). For example, Bassok and Olseth found that college students viewed an increase in attendance at an annual conference as discrete (since it is based on a change between magnitudes associated with two discrete events well-separated in time), but viewed an annual increase in a city's population growth as continuous (since it is based on changes stemming from the psychologically-constant process of gaining and losing undifferentiated individual residents).

Rapp, Bassok, DeWolf and Holyoak (2015) surveyed a series of textbooks as well as college students to investigate how rational numbers are used with certain types of quantities. The results indicated that both textbook writers and college students construct a majority of decimal word problems with continuous quantities and fraction word problems with discrete quantities. One possible explanation for this mapping is that some discrete entities (e.g., a person, a ball) can be counted but cannot be partitioned. Sets of such entities can appear in the numerator and the denominator of fractions (e.g. 3/7 of the children are girls), but the decimal value of such fractions (~ 0.429) may be meaningless in a decimal representation. Continuous entities can be always partitioned into arbitrary units and can therefore be represented meaningfully by either fractions (e.g., 3/5 of a pie) or decimals (e.g., .60 of a pie). Yet, decimals may be preferred over fractions for one-dimensional measurement, especially with standard metric units (centimeters, grams).

Fractions as a Stepping Stone to Algebra

Fractions thus have a dual status that poses a particular challenge for students: a fraction is at once a relationship between two quantities, expressed as a/b, and also the magnitude corresponding to the division of a by b. Similar dualities arise in algebra (Sfard & Linchevski, 1994). There is evidence that children who initially struggle with fractions also struggle with algebra. A survey of Algebra I teachers found that poor fraction knowledge is one of two major difficulties facing math students as they begin learning algebra (NORC, 2008). Further, the National Mathematics Advisory Panel (2008) also found that learning of fractions is essential for mastering algebra and more complex mathematics. If students fail to grasp the concept of a fraction as a mathematical expression, this can make understanding algebraic expressions all the more difficult.

In general, establishing a strong understanding of fractions is foundational for future math performance (Siegler et al., 2012, 2011, Booth et al., 2014). Siegler et al. (2012) found that performance on conceptual fractions questions on a standardized math exam in middle school was uniquely predictive of overall math performance and math achievement into later high school. However, the specific aspects of fraction understanding that are important for later performance have not yet been identified. Booth and colleagues (Booth et al., 2014; Booth & Newton, 2012) found that performance on a fraction number line estimation task is predictive of algebra performance. But they also found that other measures, including procedural and declarative fraction knowledge, are also predictive. It is therefore hard to tease apart, based on prior studies, what specific aspects of fraction understanding are important for later math performance.

I propose that while understanding fraction magnitudes is important, it is the understanding of fractions as a relational expression that is the gateway to future algebra performance. One basic component of algebraic thinking is the understanding that algebraic expressions represent both a process (some number a times 4) and an outcome (the number $4a$). Sfard and Linchevski (1994) have argued that this understanding is essential for the conceptual understanding of algebraic expressions. Due to their two-dimensional structure, fractions can be understood as both the process (4 divided by 5) and the outcome or magnitude (4/5), introducing students for the first time to mathematical expressions that do not need to be "solved." Thus, given a bowl containing 3 red and 5 green apples, it is easy to count and determine that the proportion of red apples is 3/8; there is no need to execute the further step of evaluating the magnitude of 3/8 (e.g., by applying division). The multidimensional structure of numbers eventually proves to be critical in understanding even the simplest algebraic expressions, such as $\frac{2}{3}x$. Thus, those students who eventually succeed in grasping the very concept of a multidimensional number, first instantiated by fractions, will have mastered a fundamental prerequisite for advanced mathematics.

This approach to characterizing psychological understanding of fractions and relational reasoning may have implications for how these types of rational numbers could be most effectively taught. The difficulty in understanding a fraction as a relational expression may explain why young children appear to understand quantitative relations such as part-whole or proportions with visual displays (Goswami, 1989; Mix, Levine & Huttenlocher; Boyer, Levine & Huttenlocher, 2008; Sophian, 2000), but not when they have to answer comparable questions with fraction notation (Ball & Wilson, 1996; Mack, 1995). For example, Ni and Zhou (2005) found that most children could answer the question, "How much is one third plus one third?" verbally as "two thirds". But when asked the question using symbolic fraction notation, "1/3 + 1/3 = ?", most children answered "2/6" (often claiming that both 2/6 and 2/3 are correct answers). Children thus seem to have some intuitive understanding of how rational quantities relate to one another, but have difficulty understanding the novel symbolic expressions (perhaps because they are only taught as magnitudes, rather than as relations).

The conceptualization of fractions as relations such as part/whole, subset/set, ratio and proportion may have implications for how children can be effectively taught to solve problems using fractions in these contexts. Typically, school instruction only emphasizes the part-to-whole relationship (Sophian, 2007; Mack, 1993), which most clearly relates to the understanding of fractions as magnitudes. Children are first introduced to fractions using pictorial representations intended to help students understand the meaning of a value smaller than one. But as we observed earlier, magnitudes are far easier to process using one-dimensional decimals than bipartite fractions, even for adults (DeWolf et al., 2014). Thus, it might well be easier for children to learn about magnitudes less than one by being introduced to decimals prior to fractions. Fractions might first be introduced with an emphasis on their status as a relationship between two natural numbers (cf. Moss and Case, 1999).

More generally, understanding how rational numbers align to specific types of quantities, and how the internal structure of mathematical expressions can affect ease of alignment to the perceptual or semantic relations being modeled, has important implications for how to best conceptualize and teach fractions and decimals. It is important to foster understanding of mathematics in a way that goes beyond teaching algorithmic procedures. A mathematical expression can represent not just a procedure, and not just a magnitude, but also a relational structure that maps onto the structure of the world.

Teaching students rational numbers in a way that allows for a conceptually complex understanding can enhance students' mathematics and science learning, as they will be better prepared to build connections between this foundational knowledge and its applications. Most importantly, most current practices do not encourage learners to build the necessary conceptual connections between different types of rational numbers and more complex relational concepts. In effect, students are left as "perpetual mathematical novices," lacking the relational connections that define expertise (Chi, Feltovich & Glaser, 1981; diSessa, 1988; Larkin et al., 1980). A goal of this research is to eventually lead to pedagogical innovations that have the potential to improve students' understanding of rational numbers.

Goals of the Dissertation

This dissertation unites three lines of research that investigate how rational numbers are mentally represented (Chapter 2), how the differing formats of rational numbers might be differentially effective for various tasks (Chapter 3), and how rational expressions might build a foundation for more complex mathematical thinking like relational modeling in algebra (Chapter 4). Together, these lines of research seek to advance our understanding of numerical cognition in general and to unite this line of research with the more general literature on mathematical and general problem solving. Specifically, a central goal of this dissertation is to point out that numerical cognition should not be thought of as isolated from other more general types of reasoning required for success in mathematics such as relational reasoning and abstract modeling. Indeed, uniting these various literatures and lines of research can help to make various educational interventions and improvements.

CHAPTER 2:

NEURAL REPRESENTATIONS OF MAGNITUDE FOR NATURAL AND RATIONAL NUMBERS

Introduction

Representations of Symbolic Number Types

Humans are unique in having developed symbolic notations for numbers. Given that a primary function of numbers is to convey magnitude values, it is important to understand the mental and neural representations of numerical magnitudes. The goal of the current study was to address the question of how different symbolic notations (natural numbers, fractions, and decimals) map onto magnitude codes. Specifically, we sought to determine whether different notations map onto a single, fully abstract, magnitude code, or whether separate representations exist for specific number types (e.g., natural versus rational) or number representations (e.g., base-10 versus fractions).

Numerous studies of numerical magnitude comparisons have yielded a *symbolic distance effect*: comparisons of numbers that are closer in magnitude (e.g., 7 vs. 8) are slower and more error prone than comparisons of numbers that are farther apart (e.g., 2 vs. 8; Moyer & Landauer, 1967; Holyoak, 1978). A similar distance effect is observed in children (Bath, La Mont, Lipton & Spelke, 2005; Brannon, 2002). Rhesus monkeys display a distance effect for numerosity comparisons; moreover, they are capable of learning shapes (Arabic numerals) corresponding to small numerosities (1-4 dots), such that the shapes acquire neural representations overlapping those of the corresponding perceptual numerosities (Diester & Nieder, 2007).

The distance effect and other phenomena have been interpreted as indications that numerical magnitudes (at least for integers) are associated with an analog magnitude representation akin to a mental number line (Dehaene & Changeux, 1993; Gallistel, 1993; Opfer

& Siegler, 2012). Neuroimaging studies with both adults and children have implicated the intraparietal sulcus (IPS) as the central area for representing and comparing symbolic integer magnitudes (and also non-symbolic magnitudes) (Dehaene, Piazza, Pinel & Cohen, 2003; Nieder & Dehaene, 2009; Piazza, Pinel, Le Bihan, & Dehaene, 2007; Pinel, Dehaene, Riviere, & Le Bihan, 2001). Further, IPS activation is inversely related to the numerical distance between two numbers being compared (Cohen Kadosh et al., 2005; Kaufmann et al., 2005), consistent with the behavioral distance effect.

While the representation of whole-number magnitude has received considerable attention, far less is known about the representation of other symbolic number types, such as the rational numbers (fractions and decimals). Some have argued that the representation of magnitude in general is entirely abstract, and that all symbolic and non-symbolic magnitudes can be represented using a single mental (and neural) number line (Eger, Sterzer, Russ, Giraud & Kleinschmidt, 2003; Naacache & Dehaene, 2001; Siegler, Thompson & Schneider, 2011). However, studies investigating this topic have as yet failed to reach a consensus. Previous behavioral research has mainly focused on the extent to which fractions are represented holistically. This work has focused on the issue of whether the overall (holistic) magnitude of a fraction is accessed automatically, like an integer (Kallai & Tzelgov, 2009; Meert, Gregoire & Noel, 2010a, 2010b; Schneider & Siegler, 2010; Sprute & Temple, 2011). Evidence for holistic magnitude representation come from studies examining the distance effect during fraction comparisons. Many studies (e.g. Schneider & Siegler, 2010) have found that adults show a distance effect when representing fractions during comparisons. However, other studies have shown that depending on the stimuli and availability of various shortcut strategies, adults may represent only the whole-number components of the fraction and not its holistic magnitude (e.g. Bonato, Fabbri, Umilta & Zorzi, 2007; Fazio, DeWolf & Siegler, 2015).

Moreover, other work has shown that even when a distance effect is found for fraction comparisons, the size and scale of the effect is entirely different for fractions relative to either integers or decimals. DeWolf, Grounds, Bassok and Holyoak (2014) had adults compare fractions, matched decimals (rounded to three digits) and integers (created by multiplying the equivalent decimal by 1000 to obtain a three-digit integer). Comparisons for all three number types yielded reliable distance effects, based on the holistic magnitudes of the numbers being compared. Importantly, however, response times and error rates for the fraction comparisons were much higher than for comparisons of either decimals or integers, with the latter number types showing no differences in response times or errors. Moreover, the distance effect was much more pronounced for fractions, with response times averaging between 2 and 8 seconds for far versus near number pairs. In contrast, response times for integers and decimals overlapped with one another, and generally were no longer than 2 seconds. This dramatic difference in the scale of the distance effect across number types suggests that the magnitude information associated with fractions may be less precise than that associated with integers or decimals, and that the process of accessing magnitudes is more effortful and less automatic for fractions than for either integer or decimal formats.

Using fMRI to Investigate Magnitude Representation

Behavioral research investigating rational number magnitudes suggests there are important differences between magnitude processing for fractions relative to other number types. Although neuroimaging methods, and functional magnetic resonance imaging (fMRI) in particular, have been employed to assess the neural substrates of numerical magnitude representation (e.g., Darmla & Just, 2013), numerical symbols representations (see Ansari, 2016) and algebra (e.g., Monti, Parsons & Osherson, 2012), there is no consensus regarding the interpretation of the behavioral differences observed between fractions and other number types. The present study applied neuroimaging methods to assess the relationships among the neural representations of magnitude for different symbolic formats. If the representation of magnitude is entirely abstract, then the neural representations of a fraction and its magnitude-equivalent decimal (e.g., 2/5 vs. .40) in the IPS might be expected to be identical. In contrast, if fractions and decimals are processed very differently (as some behavioral studies suggest), then the neural codes for the different notations may differ. To date, these alternative predictions remain untested. In fact, only two studies have ever probed the neural representations underlying the processing of fractional numbers (Ischebeck, Schoke & Delazer, 2009; Jacob & Nieder, 2009a), and neither of these assessed the neural representations underlying decimal numbers, or the relationship between neural representations of magnitude across different formats for rational numbers.

A few other studies have examined how neural representations of magnitude differ as a function of notation by comparing neural responses to whole numbers versus their verbal equivalents (e.g., “12” versus “twelve”). Some studies have found that IPS activation was notation-independent (Eger et al., 2003; Naccache & Dehaene, 2001), whereas other studies suggest there may be both notation-specific and notation-independent areas (Bluthe, De Smedt & Beeck, 2015; Cohen Kadosh et al., 2007; Damarla & Just, 2013). However, these studies all compared a single mathematical notation (whole numbers) versus natural language (number names). No work has been done to investigate the question of whether alternative mathematical formats, such as fractions versus decimals, evoke similar or distinct neural representations of magnitude.

As noted above, only two studies have investigated the representation of symbolic fraction magnitudes using fMRI. Jacob and Nieder (2009a) used an adaptation paradigm to test symbolic fraction magnitudes (single and multi-digit fractions). Recovery in the BOLD signal after habituation was observed in the frontoparietal cortex, and specifically the IPS. The pattern of signal recovery was the same after presentation of either a new symbolic fraction (e.g., "1/2") or a new fraction written as a word (e.g., "half"), suggesting that fractions and their verbal equivalents recruit the same or overlapping neural areas.

The second study that investigated symbolic fraction notation with fMRI used a magnitude comparison paradigm, rather than an adaptation paradigm. Ischebeck et al. (2009) had adult participants perform a simple magnitude comparison task with fractions, in which participants saw two fractions simultaneously on the screen and pressed a button to indicate which was larger in numerical magnitude. The stimuli included different types of fraction pairs, some with common components, in order to enable a variety of potential strategies during the comparison process. The results showed that activity in the right IPS was inversely correlated with the distance between the two fractions based on their holistic magnitude difference, and not with the distances between any component parts. Ischebeck et al. interpreted their fMRI results as supporting the hypothesis that (despite an opportunity to use componential strategies) fraction comparisons were performed using holistic magnitudes.

However, neither Ischebeck et al. (2009) nor Jacob and Nieder (2009a) directly compared processing of fractions with that of other symbolic formats. Although previous work indicates that magnitude representations for fractions involve roughly the same general neural area (the

IPS) as do magnitude representations for symbolic integers (and non-symbolic numerosities; see Jacob & Nieder, 2009b; Jacob, Vallentin & Nieder, 2012), the extent to which processing and representation of magnitude is the same or different for fractions relative to other number types has not been examined. Furthermore, the more general question of whether different symbolic formats for numbers evoke the same or different abstract magnitude representations remains unanswered.

The Present Study

In the present experiment, we employ univariate and multivariate analysis of fMRI data to compare, in a within-subject design, the neural representations of magnitude across different symbolic notations (integers, decimals, and fractions). We hypothesized that, consistent with previous research, all of the number types would activate the IPS. The main questions concerned possible differences between the number types. If all number types activate the same abstract neural representation (based on relative rather than absolute magnitude, to take account of the scale difference between integers and rational numbers), then no differences among the number types would be expected. A second possibility is that neural activation of integers will differ from that of rational numbers (either fractions or decimals), both because the latter are more complex and because the overall magnitude scale differs. A third possibility, based on the behavioral findings of DeWolf et al. (2014), is that fractions will evoke a neural signature distinct from that of either magnitude-equivalent decimals or integers, whereas the latter two number types will evoke similar activation patterns.

Methods

Participants

Sixteen participants (12 female, mean age 21 years) with no documented history of neurological disorders were recruited at the University of California, Los Angeles (UCLA) through a flyer distributed in the Psychology department. Participants signed informed consent prior to the experimental session, and were paid $30 for their participation in the 1-hour study, in compliance with the procedures accepted by the local institutional review board (IRB).

Stimuli

Stimuli consisted of pairs of numbers in one of three possible symbolic types: fractions (e.g., 1/2, 3/4), decimals (e.g., .50, .75) or integers (e.g., 50, 75). Table 2.1 lists the complete set of pairs for each number type. Within each pair, numbers were always of the same type. In order to control for the number of digits on the screen across symbolic types, only single-digit fractions, double-digit decimals, and double-digit whole numbers were presented. Thus, instances of the three symbolic number types were always constructed from exactly two digits. All of the fraction comparison pairs were comprised of fractions that did not have any common components. This constraint served to minimize the use of shortcut strategies, thereby encouraging participants to access the holistic magnitude of each individual fraction. Magnitude-equivalent decimals were created by dividing out the corresponding fraction and rounding the result to two decimal places.[1] Integers were created by multiplying the matched decimal by 100 to create a two-digit number.

A total of 40 unique comparison pairs were generated for each number type. Because the numbers in the comparisons were shown sequentially, rather than simultaneously, each pair was shown twice, once in each order. Accordingly, there were a total of 80 trials for each of the three number types.

Table 2.1 List of the stimuli used for each of the fraction, decimal, and integer comparisons. Each pair was shown twice in each order.

Fraction Pairs		Decimal Pairs		Integer Pairs	
1/9	3/7	.11	.43	11	43
1/8	3/7	.13	.43	13	43
1/8	4/9	.13	.44	13	44
1/7	3/8	.14	.38	14	38
1/6	2/7	.17	.29	17	29
1/5	3/8	.20	.38	20	38
1/5	7/8	.20	.88	20	88
2/9	4/7	.22	.57	22	57
1/4	2/7	.25	.29	25	29
1/4	5/7	.25	.71	25	71
2/7	5/8	.29	.63	29	63
1/3	2/7	.33	.29	33	29
1/3	3/4	.33	.75	33	75
3/8	2/7	.38	.29	38	29
2/5	5/9	.40	.56	40	56
3/7	2/9	.43	.22	43	22
3/7	5/9	.43	.56	43	56
4/9	1/7	.44	.14	44	14
4/9	2/3	.44	.67	44	67
1/2	2/5	.50	.40	50	40
1/2	2/3	.50	.67	50	67
5/9	6/7	.56	.86	56	86
4/7	1/4	.57	.25	57	25
4/7	3/4	.57	.75	57	75
3/5	2/9	.60	.22	60	22
3/5	5/8	.60	.63	60	63
5/8	1/9	.63	.11	63	11

2/3	1/8	.67	.13	67	13
2/3	4/7	.67	.57	67	57
5/7	4/9	.71	.44	71	44
5/7	7/8	.71	.88	71	88
7/9	1/3	.78	.33	78	33
4/5	1/9	.80	.11	80	11
4/5	7/8	.80	.88	80	88
5/6	1/5	.83	.17	83	17
5/6	7/9	.83	.78	83	78
6/7	2/5	.86	.40	86	40
6/7	3/5	.86	.60	86	60
8/9	2/5	.89	.40	89	40
8/9	3/4	.89	.75	89	75

Behavioral Task

Participants were given instructions before entering the scanning room, after performing a routine safety check. Participants were told that they would see a series of numbers presented sequentially in pairs. Each trial started with a fixation cross, at the middle of the screen, for .5 s followed by a brief blank screen jittered for .1-2 s. The first number was then presented for 1.5 s followed by a brief blank screen, which was jittered for 2-7 s, and then a second number (see Figure 2.1). Participants controlled the length of presentation of the second number by pressing a button to indicate whether the second number was larger or smaller than the first number. They were instructed to try to go as fast as possible without sacrificing accuracy.

The 240 total trials (80 per symbolic type) were evenly distributed across four runs. Allocation of pairs across the four runs and order of presentation within each run was determined randomly for each participant.

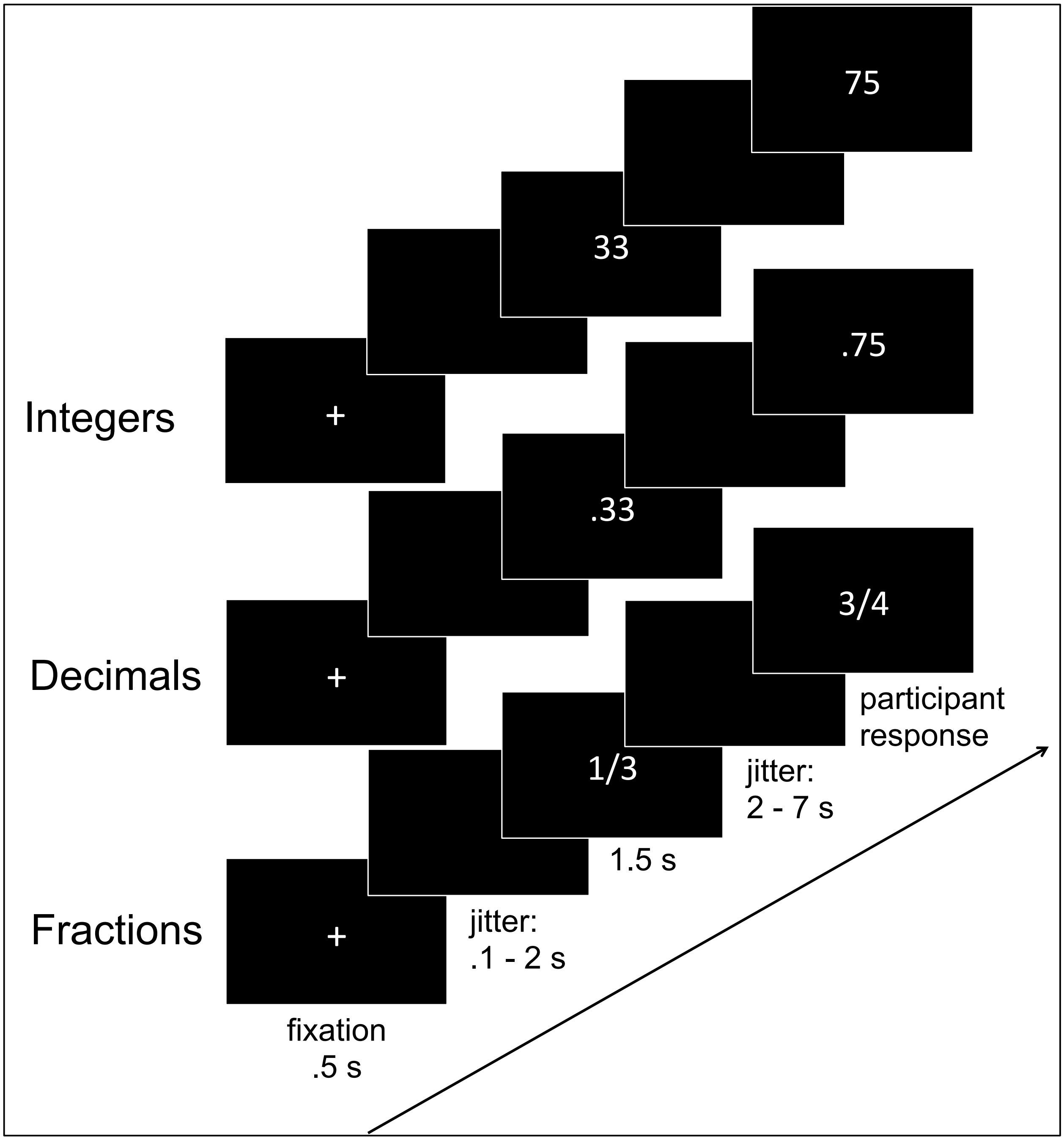


Figure 2.1 Trial procedure for each of the number type conditions (integers, decimals, fractions).

fMRI Data Acquisition

Data were acquired on a 3 Tesla Siemens Tim Trio Magnetic Resonance Imaging (MRI) scanner at the Staglin IMHRO Center for Cognitive Neuroscience at UCLA. Structural data were acquired using a T1-weighted sequence (MP RAGE, TR = 1,900 ms, TE = 2.26 ms, voxel size 1 mm3 isovoxel). Blood oxygenation level dependent (BOLD) functional data were acquired with a T2-weighted Gradient Recall Echo sequence (TR = 2,000 ms, TE = 30 ms, 32 interleaved slices, voxel size 3 × 3 × 4 mm, Flip Angle = 78 degrees). Overall, individual runs lasted an average of 566 s (min=492 s, max = 756 s).

fMRI Data-Analysis Procedures

Data preprocessing. Data analysis was carried out using FSL (Smith et al., 2004). Prior to analysis, data underwent a series of conventional preprocessing steps including motion correction (Jenkinson et al., 2002), slice-timing correction (using Fourier-space time-series phase- shifting), spatial smoothing using a Gaussian kernel of 5 mm full-width half-max, and high-pass temporal filtering (Gaussian-weighted least-squares straight line fitting, with sigma = 50 s). Data from each individual run were analyzed employing a univariate general linear model approach (Monti, 2011) with pre-whitening correction for autocorrelation (Woolrich et al., 2001).

Univariate analysis. For each run of each participant, a univariate GLM analysis was conducted with three regressors of interest marking the onset time and duration of the presentation of the *first number* of each pair, separately for each notation type (fractions, decimals, and integers). These analyses completely avoid any confounding with problem difficulty: because the second number is yet to be presented, the comparison process cannot yet be initiated. A number of additional regressors modeled the second number presentation, cue periods, and motion (first and second derivatives, and their difference). Data from the presentation of the second number was not analyzed further because it was confounded with movement (from pressing the response button) and cognitive processes relating to the comparison task (cf., Todd, Nystrom, & Cohen, 2013). For each run we computed seven contrasts. These were based on the data collected during the presentation of the first number in a comparison pair. These included the simple effects of each notation type (fraction vs. baseline, decimal vs. baseline and integer vs. baseline), as well as the pairwise differences between them (fractions > decimals; fractions > integers; integers > decimals; and decimals > integers). Prior to group analysis, individual statistical maps were transformed into MNI template space via a 2-step procedure concatenating a boundary-based co-registration to align functional data to single-subject anatomical data and a 12 degrees of freedom linear co-registration to align single-subject anatomical data to the MNI template. Data from individual runs were aggregated using a mixed-effects model (i.e., employing both the within- and between-subject variance), using automatic outlier detection. Z (Gaussianised t) statistic images were thresholded using a cluster correction of $Z > 2.3$ and a (corrected) cluster significance threshold of $p = .05$.

In order to avoid reverse subtractions (Morcom & Fletcher, 2007), for each A>B contrast (e.g., fractions > decimals), we employed a "greater-than-zero-sum" masking procedure. In this approach, analysis is restricted to voxels for which the sum of the Z statistic associated with task A (compared to fixation; Z_A) and the Z statistic associated with task B (compared to fixation; Z_B) resulted in a number greater than zero (i.e., $(Z_A+Z_B)>0$). In other words, for a voxel to be included in the contrast analysis, either Z_A and Z_B have to both be positive values (in which case it is not possible to have reverse subtractions), or Z_A had to be more positive than Z_B was negative, thereby preventing the possibility of a brain activation resulting from a weakly positive

Z_A coupled with a strongly negative Z_B. This latter point is particularly important since it highlights the advantage of our approach over the more conventional Z_A>0 masking which, indeed, would return an activation in the case where a weakly positive Z_A (e.g., Z_A=0.01) were associated with a sufficiently negative Z_B. Furthermore, our approach also has the advantage of not excluding voxels in which Z_A might be strongly positive while Z_B is (weakly) negative, unlike a Z_B>0 masking procedure.

MVPA analysis. The input to the multivariate pattern analysis (MVPA) was a set of volumes of regression coefficients (i.e., "beta" values) marking the magnitude of activation, for each voxel, in each trial (per participant). These trial-wise "patterns of activations" were obtained by employing the iterative Least Squares – Separate approach (LS-S; Mumford, Turner, Ashby & Poldrack, 2012) in which a separate GLM is run (here, using FILM with local autocorrelation; Woolrich et al., 2001) for each trial. The patterns of activation were then concatenated across time to construct a subject-wise "beta-series" of activation magnitude per trial per voxel (Rissman et al, 2004).

Representational Similarity Analysis (RSA) was run on the beta-series of activation magnitudes, in MATLAB using the RSA toolbox (Nili et al., 2014). RSA characterizes the representation in a brain region by a representational dissimilarity matrix (RDM), and compares the empirical matrix with a model. An RDM is a square symmetric matrix, with each entry referring to the dissimilarity between the activity patterns associated with two trials (e.g., entry (1,2) would represent the dissimilarity between activity patterns of trial 1 and trial 2 for a given participant). Each element of the RDM is calculated as 1 minus the Spearman correlation between the beta-series for each pair of trials. Models were manually generated to reflect idealized RDMs expected if the group of voxels was indeed modulating its activity with respect to the manipulation (see models in Figure 2.2). The Number Type Model (Figure 2.2a) was designed to test the overall ability to distinguish between each of the three number types. We then compared each of the pairwise number-type combinations to attempt to distinguish between each number type. The assumption behind the model RDMs was that a group of voxels sensitive to an experimental condition would display lower dissimilarity for same-condition trials as opposed to different-condition trials.

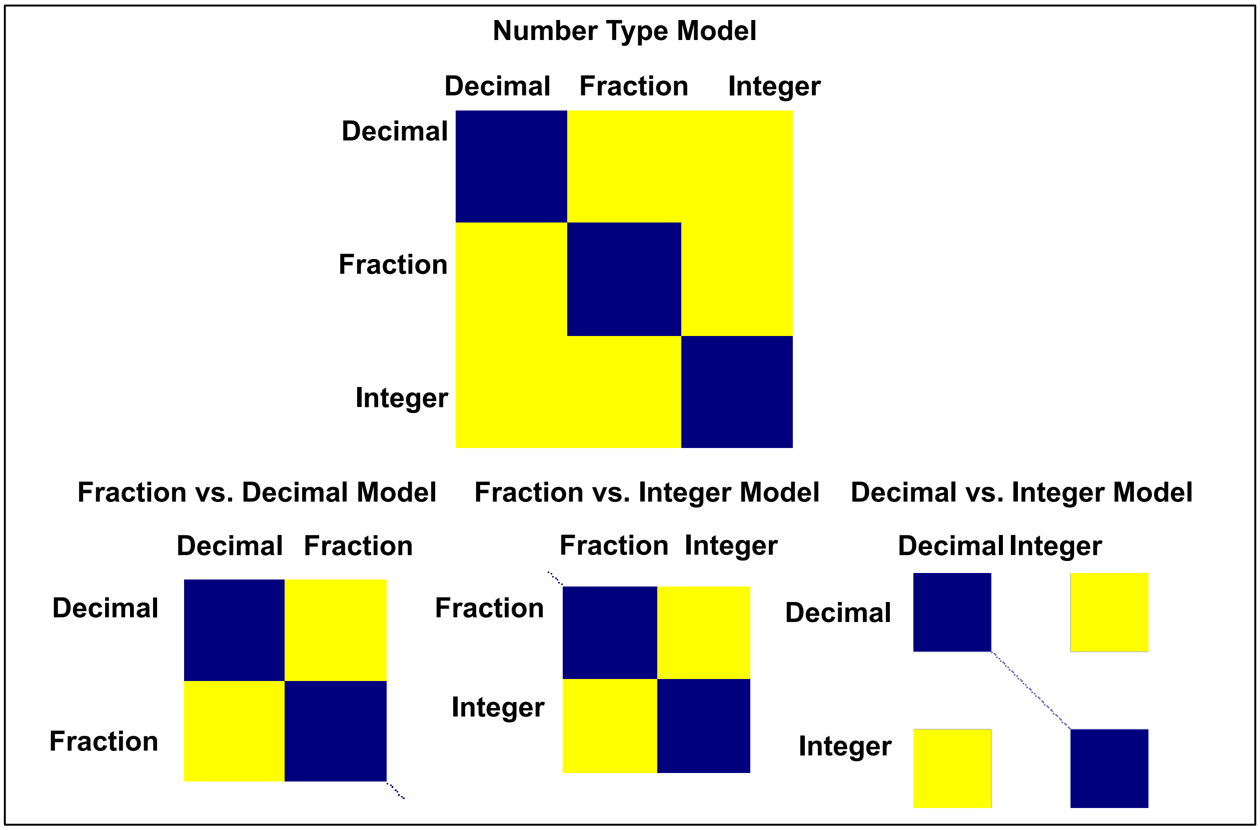


Figure 2.2 Ideal models generated for the RSA searchlight MVPA. Each matrix represents a dissimilarity matrix where yellow (1) denotes completely dissimilar items and blue (0) denotes maximally similar items.

The RSA was performed with a searchlight approach (searchlight radius: 6mm or 2 voxels; cf. Kriegeskorte, Goebel & Bandettini, 2006) within an anatomical mask of the IPS as defined by the Jülich Histological Atlas (available in FSL; Choi et al., 2006; Scheperjan et al., 2008). Within each searchlight sphere, a Spearman coefficient was computed between the empirical and model RDMs, yielding a single second-order similarity value per voxel, which reflected the resemblance of searchlight sphere activity with the hypothesized model. These coefficients were registered to the standard template, with the same 2-step procedure employed for univariate single-subject statistical parametric maps, and assessed for significance ($\rho > 0$) using FSL's randomize with threshold-free cluster enhancement (corrected $p < .05$) (Smith & Nichols, 2009; Winkler et al., 2014).

Results

Behavioral Results

Mean accuracy on the magnitude comparison task for each number type was obtained by averaging over all participants. A one-way repeated measures ANOVA revealed a significant effect of number type ($F(2, 30) = 23.23$, $MSE = .002$, $p < .001$), with fractions having lower accuracy than decimals (fractions: 84% vs. decimals: 92%, $t(15) = 6.72$, $p < .001$) and integers (91%, $t(15) = 4.82$, $p < .001$). There was no difference in accuracy between decimals and integers ($t(15) = .69$, $p = .50$).

Mean response times (RTs) for correct trials were averaged for each number type across participants. A one-way repeated measures ANOVA revealed a significant effect of number type ($F(2, 30) = 24.34$, $MSE = .09$, $p < .001$), with fractions being compared more slowly than either decimals (fractions: 1.91 s vs. decimals: 1.30 s, $t(15) = 5.22$, $p < .001$) or integers (1.24 s, $t(15) =$

5.19, $p < .001$). There was no significant difference in response time between decimals and integers ($t(15) = 1.15$, $p = .27$).

In order to assess the distance effect, average accuracies and response times were calculated for each trial across participants for fraction, decimal, and integer conditions. Regression analyses across individual items were conducted for accuracies and response times based on a logarithmic distance measure, log(|first number – second number|), which we will abbreviate as "log Dist" (see DeWolf et al., 2014; Hinrichs et al., 1981). Log Dist significantly predicted accuracy outcomes for fractions ($\beta = -.65$, $t(37) = 26.91$, $p < .001$) but not for decimals ($\beta = .02$, $t(37) = .12$, $p = .90$) or integers ($\beta = .23$, $t(37) = 1.44$, $p = .16$). The lack of a distance effect in accuracy for decimals and integers likely reflects the fact that accuracy for these number types was near ceiling. Accordingly, we focused more closely on distance effects based on RTs for correct responses. Figure 2.3 shows the average RT for each trial across participants for fractions, decimals, and integers. Log Dist significantly predicted RT outcomes for each of the number types (fractions: $\beta =$ [illegible]64, $t(37) = 5.01$, $p < .001$; decimals: $\beta =$ [illegible]33, $t(37) = 2.16$, $p = .04$; integers: $\beta =$ [illegible]33, $t(37) = 2.12$, $p = .04$). These results replicate the pattern of distance effects observed by DeWolf et al. (2014), including (as evidenced by the much larger beta coefficient for fractions) a more pronounced distance effect for fractions than for either of the other two number types.

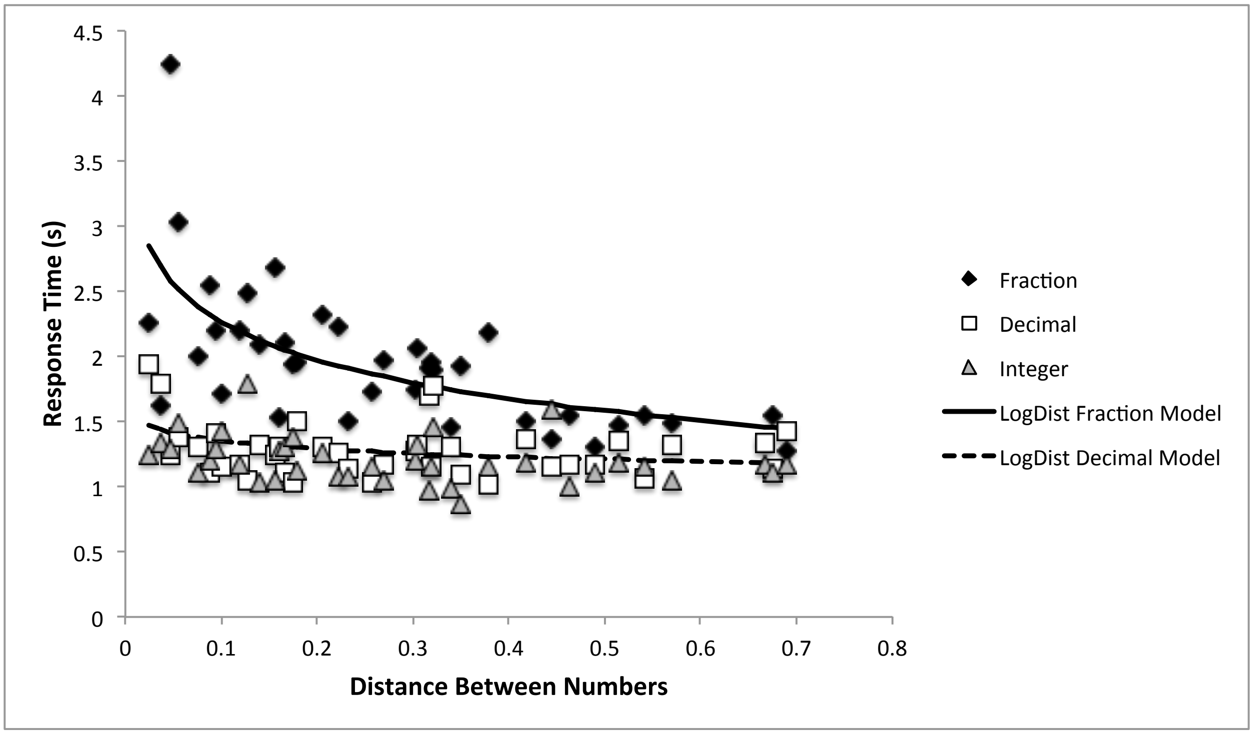


Figure 2.3 Average correct response times for each trial across participants for fractions, decimals, and integers. Fitted lines represent predictions derived from LogDist models for Fractions and Decimals. (Because predictions of the Integer model were nearly identical to those of the Decimal model, the Integer model is excluded here for simplicity.)

fMRI Results

Univariate analyses. The contrast of fractions versus decimals resulted in extensive activations within and around the left horizontal segment of the intraparietal sulcus, spanning inferior (Brodmann Area [BA] 40) and superior (BA 7) parietal lobuli, as well as the junction of the intraparietal and intraoccipital sulci. Additional left hemispheric activations were detected in frontal cortex, centered around the precentral gyrus (BA 6) together with smaller foci within the superior (BA 6) and middle (BA 9) frontal gyri, and in temporal cortex, spanning the most caudal segments of the inferior and middle temporal gyri (BA 37). Finally, right lateralized activations were observed in the cerebellum, with foci in Crus I and Lobules VI and VIIB (see Figure 2.4a and Table 2.2 for complete list of local maxima).

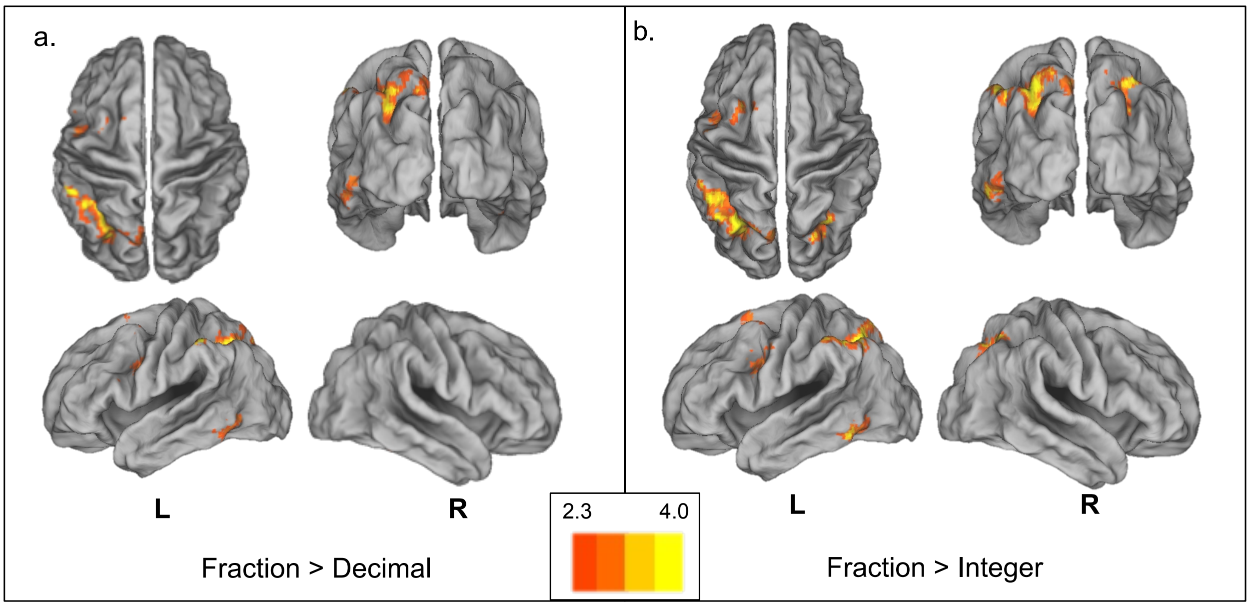


Figure 2.4 Results of the univariate analysis for (a) comparison of fraction and decimal activation and (b) comparison of fraction and integer activation, from dorsal, posterior, and lateral views. Red areas represent significant differences in activations. The color scale represents *z*-values for significant activations.

Table 2.2 Local Maxima for the Fractions > Decimals Univariate Contrast (Abrev.: hIPS: horizontal segment of the intraparietal sulcus; IOS: intraoccipital sulcus; L: left; R: Right)

MNI Coordinates					
x	y	z	Hem	Region Label (BA)	Z
Parietal					
-42	-48	48	L	Inferior Parietal Lobule (hIPS; 40)	4.30
-30	-58	44	L	Superior Parietal Lobule (hIPS; 7)	3.93
-54	-38	48	L	Inferior Parietal Lobule (40)	3.91
-28	-70	38	L	Occipito-Parietal Junction (hIPS/IOS;40/7)	3.75
-28	-76	54	L	Superior Parietal Lobule (hIPS; 7)	3.71
-34	-50	42	L	Inferior Parietal Lobule (hIPS; 40)	3.71
Frontal					
-56	14	24	L	Precentral Gyrus/Inferior Frontal Gyrus (6/44)	3.77
-50	6	32	L	Precentral Gyrus (6)	3.65
-22	6	70	L	Superior Frontal Gyrus (6)	3.27
-30	2	62	L	Precentral Gyrus/Superior Frontal Gyrus (6)	3.25
-34	2	28	L	Precentral Gyrus/Inferior Frontal Gyrus (6/44)	3.18
-36	2	62	L	Precentral Gyrus/Superior Frontal Gyrus (6)	3.16
-54	8	46	L	Precentral Gyrus (6)	2.93
-40	-4	56	L	Precentral Gyrus (6)	2.93
-28	-8	58	L	Precentral Gyrus (6)	2.91
-18	12	66	L	Superior Frontal Gyrus (6)	2.85
-50	26	28	L	Inferior Frontal Gyrus/Middle Frontal Gyrus (44/9)	2.70
-44	-4	30	L	Precentral Gyrus (6)	2.68
Temporal					
-50	-56	-22	L	Inferior Temporal Gyrus (37)	3.18
-56	-56	-14	L	Inferior Temporal Gyrus (37)	3.13
-48	-64	-4	L	Inferior Temporal Gyrus (37)	3.03
-50	-60	0	L	Middle Temporal Gyrus (37)	2.84
-48	-56	0	L	Middle Temporal Gyrus (37)	2.60
Cerebellum					
38	-56	-32	R	Crus I	3.88
38	-60	-40	R	Crus I	3.41
28	-76	-50	R	Lobule VIIB	3.19
32	-72	-26	R	Crus I	3.13
44	-66	-30	R	Crus I	2.86
26	-60	-34	R	Lobule VI	2.86

The contrast of fractions versus integers resulted in extensive activations in bilateral parietal cortex (with L>R), centered within and around the horizontal segment of the intraparietal sulci, spanning inferior (BA 40) and superior (BA 7) parietal lobuli.[2] Similarly to the contrast of fractions versus decimals, left lateralized activations were also obtained in frontal cortices, mostly within the precentral gyrus (BA 6) together with foci across superior (BA 6, 8) and middle (BA 8) frontal gyri, and in the caudal section of temporal cortex, in the inferior and middle temporal gyri (BA 37). Finally, right hemispheric activations were again observed in the cerebellum, with foci in Crus I and II, and Lobule VI (see Figure 2.4b and Table 2.3 for a complete list of local maxima).

Direct comparison of the decimal and integer conditions, in both directions (i.e., decimals > integers; integers > decimals), failed to reveal any significant activation.

Table 2.3 Local Maxima for the Fractions > Integers Univariate Contrast (Abrev.: hIPS: horizontal segment of the intraparietal sulcus; L: left; R: Right)

MNI Coordinates					
x	y	z	Hem	Region Label (BA)	Z
Parietal					
-28	-62	46	L	Superior Parietal Lobule (hIPS; 7)	4.71
-30	-74	54	L	Superior Parietal Lobule (hIPS; 7)	4.39
-30	-70	54	L	Superior Parietal Lobule (7)	4.21
-48	-50	52	L	Inferior Parietal Lobule (40)	3.99
-48	-48	48	L	Inferior Parietal Lobule (40)	3.98
-40	-46	46	L	Inferior Parietal Lobule (hIPS; 40)	3.82
32	-68	48	R	Inferior Parietal Lobule (40)	3.82
30	-60	50	R	Superior Parietal Lobule (hIPS; 7)	3.74
28	-64	50	R	Inferior Parital Lobule (40)	3.69
24	-68	48	R	Superior Parietal Lobule (7)	3.58
42	-54	58	R	Superior Parietal Lobule (7)	3.40
22	-72	50	R	Superior Parietal Lobule (7)	3.33
Frontal					

-50	10	28	L	Precentral Gyrus (6)	3.88
-52	12	24	L	Precentral Gyrus (6)	3.78
-52	10	36	L	Precentral Gyrus (6)	3.62
-24	16	54	L	Superior Frontal Gyrus (8)	3.62
-36	4	28	L	Precentral Gyrus (6)	3.51
-24	12	58	L	Middle Frontal Gyrus (8)	3.51
-46	6	34	L	Precentral Gyrus (6)	3.41
-36	2	62	L	Precentral Gyrus (6)	3.37
-52	2	42	L	Precentral Gyrus (6)	3.34
-18	12	64	L	Superior Frontal Gyrus (6)	2.97
-24	8	46	L	Middle Frontal Gyrus (8)	2.82
-24	14	66	L	Superior Frontal Gyrus (8)	2.78
Temporal					
-50	-56	-10	L	Inferior Temporal Gyrus (37)	3.76
-54	-54	-14	L	Inferior Temporal Gyrus (37)	3.65
-42	-60	-6	L	Inferior Temporal Gyrus (37)	3.08
-56	-64	-10	L	Inferior Temporal Gyrus (37)	2.98
-48	-64	-4	L	Inferior Temporal Gyrus (37)	2.96
-48	-48	-14	L	Inferior Temporal Gyrus (20/37)	2.76
Cerebellum					
38	-56	-30	R	Crus I	3.87
36	-64	-44	R	Crus II	3.25
40	-60	-38	R	Crus I	3.24
20	-66	-26	R	Lobule VI	3.21
38	-58	-42	R	Crus II	3.18
32	-72	-26	R	Crus I	3.18

Multivariate analyses. Because MVPA requires an equal number of trials across all conditions, one participant was excluded from this analysis because she did not finish one of the runs due to a computer error (missed two trials).

Figure 2.5 shows the areas within the IPS that yielded significant activations for each of the four models. The Number Type model (distinguishing between the three number types) shows a broad set of bilateral activations. Mirroring the results of the univariate analysis, the Fraction vs. Decimal model shows mostly left-lateralized IPS activation, whereas the Fraction vs. Integer model shows bilateral IPS activation. Unlike the results of the univariate analysis, the Decimal vs. Integer model yielded a small number (~3) of significant voxels that distinguished between decimals and integers. While this is a small area, it points to a possible pattern difference in the encoding of decimals and integers beyond what the univariate analysis revealed.

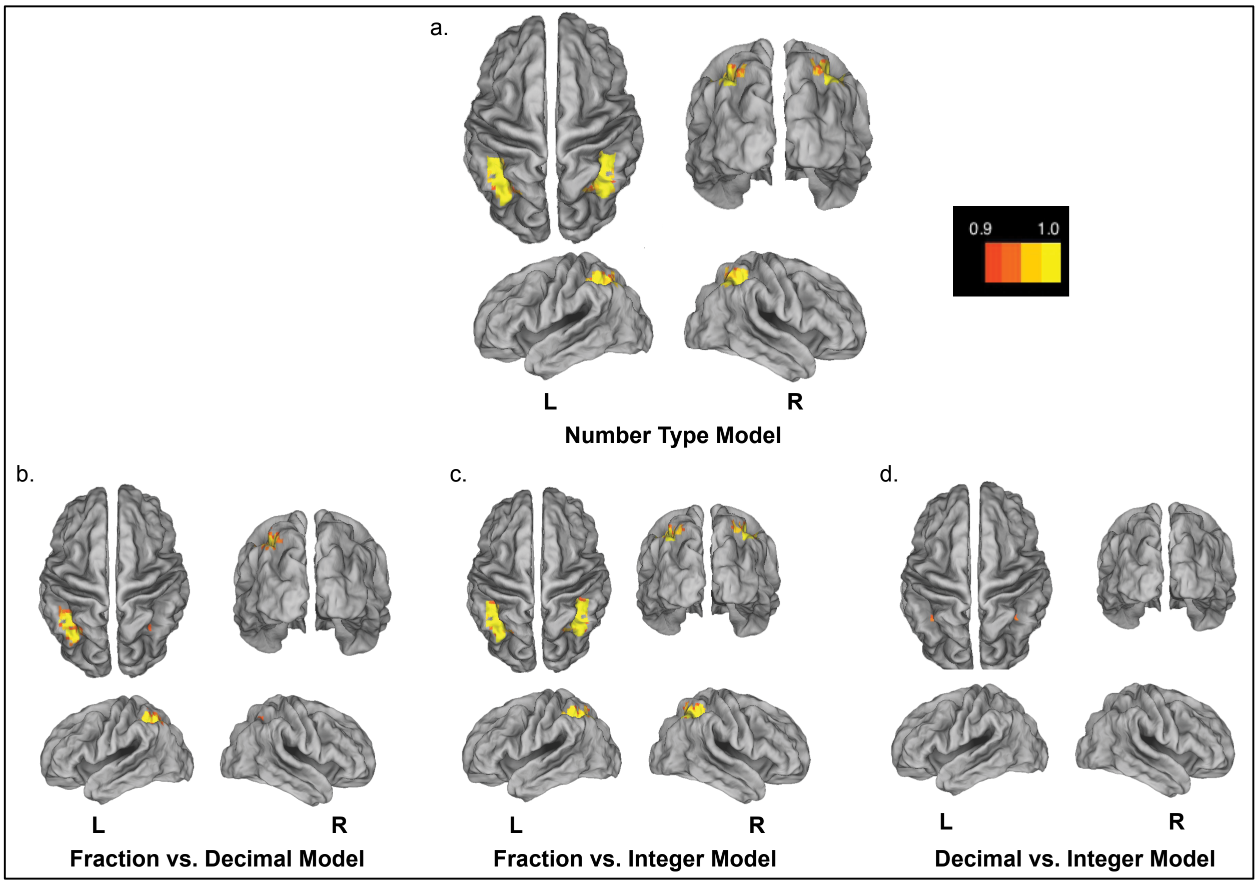


Figure 2.5 Results of the multivoxel pattern analysis (MVPA) from dorsal, posterior, and lateral views for each of the four hypothesized models (see Figure 2.2). The color scale represents 1 - *p*-values (e.g., .95 to 1 would be significant). Note: The searchlight analysis was restricted to the IPS, which was selected as a region of interest.

Discussion

Fraction Magnitudes are Neurally Distinct from Decimals and Integers

The central goal of the present study was to distinguish between possible models of neural representation for different symbolic number formats. The behavioral results showed that each of the number types elicited a reliable distance effect on correct RT. The presence of a distance effect suggests that all number types were processed holistically. However, the neuroimaging results showed that magnitudes evoked distinct neural patterns that distinguished the number types. Results of both a univariate analysis and MVPA indicate that while fractions, decimals, and integers all activate areas of the IPS, fractions yield a distinct pattern of activation associated with a unique subarea of the IPS. In contrast, decimals and integers yielded very similar and overlapping patterns, with MVPA identifying only a very small set of voxels that distinguished the latter two number types. These results suggest that while neural representations, across notations, all elicit activation within the intraparietal sulcus, neural representation appear to be sensitive to number representation (notably, base-10 numbers versus fractions), but not to number type (natural versus rational).

To our knowledge, the present neuroimaging study is the first to compare fractions with both decimals and integers. The two previous studies (Ischebeck et al., 2009; Jacob & Nieder, 2009a) that investigated the representation of fraction magnitudes using fMRI had assumed that because fractions activate the IPS (as do integers), and because fraction activation was modulated by a distance effect based on holistic magnitude, the brain represents proportional (fraction) magnitudes in the same way that it does absolute (integer) magnitudes. However, by making direct comparisons among all three number types, the present study was able to clearly dissociate magnitude activations for fractions as compared to those for either integers or decimals.

Isolating Magnitude Representations for Individual Numbers

An important methodological innovation of the present study is its use of a design based on sequential presentation of individual numbers in a magnitude comparison task. Compared to passive observation of numbers, the magnitude comparison task strongly guides participants to access holistic magnitude representations for individual numbers. Moreover, the behavioral results from the comparison task fully replicated previous work comparing performance with the three number types (DeWolf et al., 2014). Comparisons were less accurate and slower for pairs of fractions than for pairs of decimals or integers. A distance effect was obtained for all number types, but was most pronounced for fractions. Our behavioral results thus confirm that participants in our neuroimaging paradigm were performing magnitude comparisons in essentially the same way as has been observed in previous behavioral studies.

At the same time, the sequential nature of the present design allowed us to decouple the process of *accessing* a magnitude representation for an individual number from the process of magnitude *comparison*. Our fMRI analyses focused solely on the initial 1.5s period when a single number was displayed. During this period participants were motivated to access the magnitude of the presented number, but were unable to initiate a comparison because the second number in the pair had not yet appeared. Previous neuroimaging studies with fractions recorded neural signals during the comparison process itself. In contrast, our findings provide a clear picture of the neural activity underlying access to the magnitude of a single individual number, isolated from the additional activity that would be triggered by comparing two magnitudes.

What is Special About Fraction Magnitudes?

We considered three hypotheses about the relation between magnitude representations for different symbolic notations. (1) All notations might evoke some universal, fully abstract magnitude code; (2) the magnitude code might differ between natural numbers (integers) and the more complex rational numbers (fractions and decimals); or (3) the magnitude code for fractions might differ from that for the base-10 notations (decimals and integers). Our findings clearly support the third of these hypotheses. To the best of our knowledge, no previous study has shown such a strong dissociation between the neural patterns elicited by alternative notations for the same magnitude. Even though 2/5 and .40 express the same magnitude, the brain processes the two symbols very differently. In contrast, the magnitude representations for a decimal (.40) and an integer expressing a magnitude 100 times larger (40) are very similar. Importantly, the latter result implies that the neural code for numerical magnitude is on a scale that is fundamentally relative rather than absolute. Thus base-10 notations evoke similar activation patterns based on their relative magnitudes, whereas the bipartite fraction notation is processed very differently from either.

Future Directions

The present study lays the groundwork for further exploration of the differences among neural representations evoked by different symbolic number types. Behavioral evidence points to a major conceptual distinction between fractions and decimals, with the former being selectively used to code the magnitudes of *discrete* entities (which can be counted), and the latter selectively used to code the magnitudes of *continuous* quantities (which can be either estimated or measured by imposing arbitrary units; see Rapp, Bassok, DeWolf & Holyoak, 2015). In addition, it is important to examine neural processing in mathematical tasks other than those that focus on magnitudes. Whereas fractions are disadvantaged relative to decimals in magnitude comparison tasks, fractions convey reliable advantages in a variety of reasoning tasks. Because of their bipartite structure, fractions have a much more natural correspondence to relational concepts based on ratios of countable sets (DeWolf, Bassok & Holyoak, 2015a). The relational aspects of fraction representations appear to make fraction understanding a critical bridge to learning algebra (DeWolf, Bassok & Holyoak, 2015b), which depends critically on grasping the concept of a variable (understood to represent a quantity of unknown magnitude). The "isolation" technique introduced in the present paper (imaging activity evoked by an individual number as the participant prepares for a specific mathematical task performed immediately afterwards) might usefully be extended to compare the neural patterns evoked by the same symbol (e.g., a fraction) in preparation for tasks that require different types of information (e.g., magnitudes or relational concepts).

Footnotes

1. Behavioral work (DeWolf et al., 2014) has shown that decimals are processed in a similar fashion regardless of whether the stimulus set uses a fixed number of digits without a leading zero (e.g., .75), or includes numbers with leading zeros (e.g., 0.75) or additional digits after the decimal (e.g., .750). In the present study we used only 2-digit decimals without leading zeros in order to equate the number of constituent digits across the three number types.

2. The analyses comparing fractions to decimals and fractions to integers revealed hemispheric differences, with the latter comparison resulting in bilateral parietal activations and the former resulting in left lateralized activations only. However, these differences were mainly attributable to the non-linear nature of the thresholding procedure. Inspection of uncorrected statistical parametric maps resulting from the fractions-minus-decimals contrast revealed clusters of above-threshold voxels (i.e., individual $Z > 2.3$); however, these were too small to survive the cluster-extent thresholding. These subthreshold activations explain why no difference was apparent when directly comparing decimals and integers.

References

Ansari, D. (2016). Number symbols in the brain. In D. B. Berch, D. C. Geary & K. Mann Koepke (Eds.), *Mathematical cognition and learning*, Vol. 2. San Diego, CA: Elsevier. doi:10.1016/B978-0-12-801871-2.00002-2

Barth, H. L. A., Mont, K., Lipton, J., & Spelke, E. S. (2005). Abstract number and arithmetic in preschool children. *Proceedings of the National Academy of Sciences, USA, 102,* 14116–14121.

Bluthe, J., De Smedt, & Beeck, H. P. (2015). Visual number beats abstract numerical magnitude: Format-dependent representation of Arabic digits and dot patterns in human parietal cortex. *Journal of Cognitive Neuroscience, 27*(7), 1376 – 1387.

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance, 33*, 1410–1419.

Brannon, E. M. (2002). The development of ordinal numerical knowledge in infancy. *Cognition, 83*, 223–240.

Cantlon, J. F., Brannon, E. M., Carter, E. J. & Pelphrey, K. A. (2006). Functional imaging of numerical processing in adults and 4-year-old children. *PLoS Biology, 4*(5), e125.

Choi, H. J., Zilles, K., Mohlberg, H., Schleicher, A., Fink, G. R., Armstrong, E., & Amunts, K. (2006). Cytoarchitectonic identification and probabilistic mapping of two distinct areas within the anterior ventral bank of the human intraparietal sulcus. *Journal of Comparative Neurology*, *495*(1), 53-69.

Cohen Kadosh, R., Cohen Kadosh, K., Kass, A., Avishani, H. & Goebel, R. (2007). Notation-dependent and -independent representations of numbers in the parietal lobes. *Neuron, 53*(2), 307 – 314.

Cohen Kadosh, R., Henik, A., Rubinstein, O., Mohr, H., Dori, H., van de Ven, V., Zorzi, M., Hendler, T., Goebel, R., Linden, D.E. (2005). Are numbers special? The comparison systems of the human brain investigated by fMRI. *Neuropsychologia, 43*, 1238–1248.

Darmla, S. R., & Just, M. A. (2013). Decoding the representation of numerical values from brain activation patterns. *Human Brain Mapping, 34*(10), 2624 – 2634.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General*, *122*(3), 371-396.

Dehaene, S., & Changeux, J.-P. (1993). Development of elementary numerical abilities: A neuronal model. *Journal of Cognitive Neuroscience, 5*, 390–407.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*, 487–506.

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015a). Conceptual structure and the procedural affordances of rational numbers: relational reasoning with fractions and decimals. *Journal of Experimental Psychology: General, 144*(1), 127 – 150.

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015b). From rational numbers to algebra: Separable contributions of decimal magnitude and relational understanding of fractions. *Journal of Experimental Child Psychology, 133*, 72 – 84.

DeWolf, M., Grounds, M., Bassok, M. & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance, 40*(1), 71 – 82.

Diester, I., & Nieder, A. (2007). Semantic associations between signs and numerical categories in the prefrontal cortex. *PLoS Biology*, *5*, e294.

Eger, E., Sterzer, P., Russ, M. O., Giraud A-L., & Kleinschmidt, A. (2003). A supramodal number representation in human intraparietal cortex, *Neuron, 37,* 719 – 725.

Emerson, R. W., & Cantlon, J. F. (2014). Continuity and change in children's longitudinal neural responses to numbers. *Developmental Science*, *18*(2), 1-13.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2015). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Expeirmental Psychology: Learning, Memory and Cognition.* Advanced Online Publication: http://dx.doi.org/10.1037/xlm0000153

Gallistel, C. R. (1993). A conceptual framework for the study of numerical estimation and arithmetic reasoning in animals. In S. T. Boysen & E. J. Capaldi (Eds.), *Development of numerical competence: Animal and human models* (pp. 211–223). Hillsdale, NJ: Erlbaum.

Hinrichs, J. V., Yurko, D. S., & Hu, J. M. (1981). Two-digit number comparison: Use of place information. *Journal of Experimental Psychology: Human Perception and Performance, 7*, 890-901.

Holyoak, K. J. (1978). Comparative judgments with numerical reference points. *Cognitive Psychology, 10,* 203–243.

Ishebeck, A., Schoke, M., & Delazer, M. (2009). The processing and representation of fractions within the brain: An fMRI investigation. *NeuroImage, 47*(1), 403 – 413.

Jacob, S. N., & Nieder, A. (2009a). Notation-independent representation of fractions in the human parietal cortex. *Journal of Neuroscience, 29,* 4652 – 4657.

Jacob, S. N., & Nieder, A. (2009b). Tuning to non-symbolic proportions in the human frontoparietal cortex. *European Journal of Neuroscience*, *30*(7), 1432-1442.

Jacob, S. N., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: The brain's code for proportions. *Trends in Cognitive Sciences, 16*(3), 157 – 166.

Jenkinson, M., Bannister, P., Brady, M., & Smith, S. (2002). Improved optimization for the robust and accurate linear registration and motion correction of brain images. *Neuroimage*, *17*(2), 825-841.

Kallai, A. Y., & Tzelgov, J. (2009). A generalized fraction: An entity smaller than one on the mental number line. *Journal of Experimental Psychology: Human Perception and Performance, 35,* 1845–1864.

Kaufmann, L., Koppelstaetter, F., Delazer, M., Siedentopf, C., Rhomberg, P., Golaszewski, S., Felber, S., & Ischebeck, A., 2005. Neural correlates of distance and congruity effects in a numerical Stroop task: An event-related fMRI study. *NeuroImage 25*, 888–898.

Kriegeskorte, N., Goebel, R., & Bandettini, P. (2006). Information-based functional brain mapping. *Proceedings of the National Academy of Sciences of the United States of America*, *103*(10), 3863-3868.

Meert, G., Grégoire, J., & Noël, M. P. (2010a). Comparing 5/7 and 2/9: Adults can do it by accessing the magnitude of the whole fractions. *Acta Psychologica*, *135*(3), 284-292.

Meert, G., Grégoire, J., & Noël, M. P. (2010b). Comparing the magnitude of two fractions with common components: Which representations are used by 10-and 12-year-olds? *Journal of Experimental Child Psychology*, *107*(3), 244-259.

Monti, M. M. (2011). Statistical analysis of fMRI time-series: a critical review of the GLM approach. *Frontiers in Human Neuroscience*, *5*(28), 1 – 13.

Monti, M. M., Parsons, L. M., & Osherson, D. N. (2012). Thought beyond language neural dissociation of algebra and natural language. *Psychological Science*, *23,* 914 - 922.

Morcom, A. M., & Fletcher, P. C. (2007). Does the brain have a baseline? Why we should be resisting a rest. *Neuroimage*, *37*(4), 1073-1082.

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature, 215*, 1519–1520.

Mumford, J. A., Turner, B. O., Ashby, F. G., & Poldrack, R. A. (2012). Deconvolving bold activation in event-related designs for multivoxel pattern classification analyses. *Neuroimage 59*(3), 2636-2643.

Naccache, L., & Dehaene, S. (2001). The priming method: Imaging unconscious repetition priming reveals an abstract representation of number in the parietal lobes. *Cerebral Cortex, 11*, 966 – 974.

Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual Review of Neuroscience, 32*, 185 – 208.

Nili, H., Wingfield, C., Walther, A., Su, L., Marslen-Wilson, W., & Kriegeskorte, N. (2014). A toolbox for representational similarity analysis. *PLoS Computational Biology 10*(4), e1003553.

Opfer, J. E., & Siegler, R. S. (2012). Development of quantitative thinking. In K. J. Holyoak & R. G. Morrison (Eds.), *Oxford handbook of thinking and reasoning* (pp. 585-605). New York: Oxford University Press.

Piazza, M., Pinel, P., Le Bihan, D., & Dehaene, S. (2007). A magnitude code common to numerosities and number symbols in human intraparietal cortex, *Neuron, 53*, 293–305.

Pinel, P., Dehaene, S., Riviere, D., & Le Bihan, D., 2001. Modulation of parietal activation by semantic distance in a number comparison task. *NeuroImage*, *14*, 1013–1026.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals. *Journal of Experimental Psychology: Applied, 21*(1), 47 – 56.

Rissman, J., Gazzaley, A., D'Esposito, M. (2004). Measuring functional connectivity during distinct stages of a cognitive task. *Neuroimage*, 23(2):752-63.

Scheperjans, F., Hermann, K., Eickhoff, S. B., Amunts, K., Schleicher, A., & Zilles, K. (2008). Observer-independent cytoarchitectonic mapping of the human superior parietal cortex. *Cerebral Cortex*, *18*(4), 846-867.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*, 1227–1238.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive psychology*, *62*(4), 273-296.

Smith, S. M., Jenkinson, M., Woolrich, M. W., Beckmann, C. F., Behrens, T. E., Johansen-Berg, H., ... & Matthews, P. M. (2004). Advances in functional and structural MR image analysis and implementation as FSL. *Neuroimage*, *23*, S208-S219.

Smith, S. M., & Nichols, T. E. (2009). Threshold-free cluster enhancement: Addressing problems of smoothing, threshold dependence and localisation in cluster inference. *Neuroimage*, *44*(1), 83-98.

Sprute, L., & Temple, E. (2011). Representations of fractions: Evidence for accessing the whole magnitude in adults. *Mind, Brain, and Education*, *5*(1), 42-47.

Todd, M. T., Nystrom, L. E., & Cohen, J. D. (2013). Confounds in multivariate pattern analysis: Theory and rule representation case study. *Neuroimage, 77*, 157-165

Winkler, A. M., Ridgway, G. R., Webster, M. A., Smith, S. M., & Nichols, T. E. (2014). Permutation inference for the general linear model. *NeuroImage*, *92*, 381-397.

Woolrich, M. W., Ripley, B. D., Brady, M., & Smith, S. M. (2001). Temporal autocorrelation in univariate linear modeling of FMRI data. *Neuroimage*, *14*(6), 1370-1386.

CHAPTER 3:

CONCEPTUAL STRUCTURE AND THE PROCEDURAL AFFORDANCES OF RATIONAL NUMBERS: RELATIONAL REASONING WITH FRACTIONS AND DECIMALS

Introduction

Mathematical Thinking as Modeling

Mathematical understanding fundamentally involves grasping that mathematics is a system of quantitative relations among concepts. It has been argued that a core problem with math education, particularly in the United States, is that greater focus is placed on execution of mathematical procedures than on understanding of quantitative relations (Gonzales et al., 2008; Richland, Stigler & Holyoak, 2012; Stigler & Hiebert, 1999; Rittle-Johnson & Star, 2007). Nonetheless, educators do make an effort to convey the conceptual structure of mathematics, typically by presenting students with examples of analogous real-world situations. For example, the addition operation is often illustrated with examples of combining subsets of a superset category (e.g., 3 red + 5 blue marbles = 8 marbles). Adults prefer the operation of addition to be applied to sets of objects drawn from a common immediate superordinate (e.g., marbles) or from closely-related co-hyponyms (e.g., roses and daisies); in contrast, they expect division to be applied to sets that have a functional relationship (e.g., 9 cookies divided among 3 children). Such expectancies suggest that people use a process of semantic alignment (akin to analogical reasoning) to systematically connect arithmetic operations with conceptual categories (Bassok, Chase & Martin, 1998; Bassok, Pedigo, & Oskarsson, 2008).

Other work also suggests that mathematical thinking involves various forms of mapping between numbers and concepts. By far the most attention has been directed at the unidimensional concept of magnitude, which in humans and other primates enables a mapping from numerical quantities onto a mental (and neural) number line (e.g., Cantlon, Brannon, Carter & Pelphrey, 2006; Dehaene & Changeux, 1993; Moyer & Landauer, 1969; Pinel, Piazza, Bihan & Dehaene, 2004; for recent theoretical work on the acquisition of magnitudes, see Chen, Lu & Holyoak, 2014). Another type of mapping links variations in symbolic notation at a perceptual level (e.g., spacing of operator symbols) with relations among procedures (e.g., order of operations; Braithewaite & Goldstone, 2013; Landy, Brookes & Smout, 2014; Landy & Goldstone, 2007; Zahner & Corter, 2010). More generally, procedures appear to derive meaning from their place in a conceptual structure, rather than standing in isolation as rote routines (e.g., English & Halford, 1995; Hinsley, Hayes & Simon, 1977; Kintsch & Greeno, 1985; Mochon & Sloman, 2004).

Previous psychological work on mathematics as a form of modeling has focused either on the unidimensional concept of number magnitude, or on relations involving more complex mathematical expressions, such as equations. In the present paper we focus on the relational structure of numbers themselves. Whereas all numbers express a unidimensional magnitude, some notations for numbers also inherently represent multidimensional relationships. Such multidimensionality is apparent in numbers introduced in advanced mathematics (e.g., complex numbers); however, we argue it is also a property of relatively simple notations for rational numbers taught in elementary school—most notably, common fractions[1] (e.g., 2/3, 13/4).

Acquisition of Fractions and Decimals

Our focus in this paper is on how well-educated reasoners understand fractions and decimals; however, developmental work sheds important light on the difficulties associated with each notation type. After the familiar natural numbers, fractions and then decimals are typically introduced in school (in that order, at least in the curricula generally adopted in the United

States) as new types of numbers that can express magnitudes less than one. Both symbolic notations often prove problematic for students. Children, and even some adults, exhibit misconceptions about the complex conceptual structure of fractions (Siegler et al., 2011, 2013; Ni & Zhou, 2005; Stigler et al., 2010). Students often find it difficult to reconcile this perceptually and conceptually unfamiliar type of number with their well-established concept of natural numbers (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010), particularly in understanding how the whole numbers within the fraction contribute to its overall magnitude. Instead of processing fractions as one integrated quantity, both children and adults often process the natural number parts of the fraction separately (Kallai & Tzelgov, 2009; Bonato, Fabbri, Umiltà & Zorzi, 2007). Despite (or perhaps because of) their associated difficulties, an understanding of fractions appears to be very important for subsequent learning. For example, students' ability to place fractions on a number line is uniquely related to their general arithmetic ability and later performance (Siegler et al., 2011, 2012). Students also encounter problems in learning to understand decimals (Rittle-Johnson, Siegler & Alibali, 2001), but generally master the magnitudes of decimals before fractions (Iuculano & Butterworth, 2011). The relative ease of learning magnitudes of decimals presumably reflects the fact that their implicit denominator is a constant (base 10), rather than a variable, so that a decimal inherently expresses the unidimensional concept of magnitude (Halford et al., 1998, 2010).

Bonato et al. (2007) argue that adults interpret fractions in terms of their component integer parts, suggesting that the natural alignment of fractions may resemble that of integers (i.e., a preference for discrete quantities). But fractions may also be interpreted as holistic numbers, providing a possible basis for them to align with continuous quantities. Schneider and Siegler (2010) showed that when comparing fraction magnitudes, adults show a distance effect (response times decrease as the difference between magnitudes increases), analogous to that found with natural numbers. However, DeWolf, Grounds, Bassok and Holyoak (2014) found that while the pattern of response times for comparing decimals is almost identical to that for comparing multi-digit natural numbers, the pattern for fraction comparisons is dramatically slower, and shows a greatly exaggerated distance effect, indicative of less precise magnitude representations for fractions. Thus, decimals appear to be much more effective than fractions in conveying information about magnitudes. This line of research suggests that fractions are not always interpreted as magnitudes, or at least that magnitudes are less easily derived from fractions than from decimals (perhaps because the two-dimensional format of fractions does not match the base-10 system associated with whole numbers).

Acquisition of Discrete and Continuous Quantities

A key conceptual distinction, which we will argue is intimately related to alternative notations for rational numbers, is the distinction between countable and continuous quantities, which align with integers and real numbers, respectively. This distinction has been highlighted in developmental research (e.g., Cordes & Gelman, 2005). Counting is appropriate when the relevant entities are discrete objects (e.g., the number of girls in a group of children), whereas measurement is appropriate for continuous mass quantities (e.g., height of water in a beaker). Continuous quantities can be subdivided into equal-sized units (i.e., discretized) to render them measurable by counting (e.g., slices of pizza), but the divisions are arbitrary in the sense that they do not isolate conceptual parts. Even for adults, the distinction between continuous and discrete quantities has a strong impact on transfer of mathematical procedures (Bassok & Holyoak, 1989; Bassok & Olseth, 1995). Procedures learned using discrete concepts transfer fairly readily to continuous concepts (which can be discretized); however, procedures learned using continuous quantities are almost impossible to transfer to discrete concepts (because there is no sensible way to "fill gaps" between discrete elements to create a meaningful continuous entity).

Research suggests that appreciation of continuous magnitudes is an early developmental achievement. Prior to learning how to count, infants and young children are already able to distinguish the magnitudes of different continuous quantities (Clearfield & Mix, 2001; Feigenson, Carey & Spelke, 2002). This ability to make distinctions between continuous quantities reflects an approximate number sense, which is evolutionarily more primitive than exact calculations (Feigenson, Dehaene & Spelke, 2004; Dehaene, 1997). However, school-aged children have an advantage when performing operations (e.g., counting) with discrete quantities (Gelman, 1993) over performing measurement operations using matched continuous quantities (Nunes, Light & Mason, 1993). By this age, when students have learned to count, they can use numbers to achieve greater precision by counting than by estimation of continuous quantities.

The accuracy advantage afforded by exact computation strategies depends on acquiring competence in the necessary computation. Because exact computation is acquired later than the more elementary approximation strategies, children still acquiring competence in calculation may actually make *less* accurate judgments when quantities are discretized rather than continuous. For example, Boyer, Levine and Huttenlocher (2008) found that when children were given a continuous quantity demarcated with lines to show equal parts, they were more likely to select an incorrect proportional match than when they made proportional judgments between two continuous quantities. Despite the fact that continuous judgments could have been made in either condition (by simply ignoring the lines in the "discrete" displays), discretization apparently triggered (imperfect) use of an exact calculation strategy instead of the more primitive (but for novices, more accurate) approximation strategy. This finding provides evidence that from a young age, the ontological concept of discreteness serves as a strong cue for use of an exact calculation strategy.

Relational Structure of Rational Numbers

In the present paper we focus on adult understanding of rational numbers in relation to discrete and continuous quantities. The core definition of a rational number—one that can be expressed as the quotient of two integers, a/b, where $b \neq 0$—specifies a relation between the cardinality of two sets. As Kiernan (1976) has argued, a key conceptual function of rational numbers is to represent a variety of mathematical relations, such as part-whole, quotient, ratio number, operator and measure. As alternative notations for rational numbers, fractions and decimals have typically been viewed as simply equivalent in magnitude, other than rounding error (e.g., 3/8 vs. 0.375). For example, the Common Core State Standards Initiative (2014) for Grade 4 refers to decimals as a "notation for fractions".[2] Research has shown that decimals are more effective than fractions for representing one-dimensional values of magnitude (DeWolf et al., 2014). However, because of their bipartite structure, factions seem to be better suited (relative to decimals) for representing relations between two distinct sets. A fraction represents the ratio formed between the cardinalities of two sets, each expressed as an integer. Because both the numerator and the denominator are free to vary, a fraction is inherently a two-dimensional structure (English & Halford, 1995; Halford, Wilson & Phillips, 1998, 2010). Although fractions do express magnitudes, they first and foremost express relations between countable entities. The inherent relationality of fractions may make them an important precursor to algebra, consistent with the importance of performance with fractions as a predictor of subsequent success in mathematics (Siegler et al., 2011, 2012). Figure 3.1 illustrates a relational structure, in the form $a/b = c$, that connects rational numbers to their magnitudes. Note that fractions and decimals play distinct roles within this structure. Specifically, the *a/b* component, which expresses a ratio between integers, has the form of a fraction; in contrast, the output of the implied division, *c*,

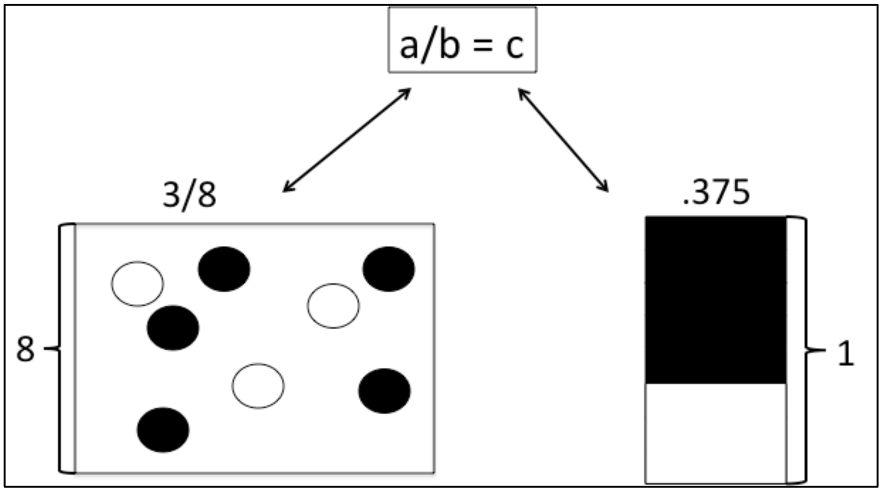


Figure 3.1. Modeling relations based on discrete or continuous quantities with fractions or decimals.

expresses the magnitude of that relation, and can be expressed using the one-dimensional decimal notation (with magnitude less than 1 when $a < b$).

Importantly, though a fraction and its corresponding decimal convey (to some arbitrarily close approximation) the same magnitude, their distinct roles in the relational model give rise to meaningful conceptual distinctions. The displays shown in Figure 3.1 involve two subsets that are either broken down into countable units (left) or shown together as one continuous mass (right). Both fractions and decimals can serve as models of such visual displays, but have different natural alignments. The bipartite structure of a fraction provides a direct mapping to countable subsets in a visual display (e.g., a subset of 3 white objects within a set of 8 white and black objects; see Figure 3.1, left).

In contrast, a decimal is inherently unidimensional because the implied denominator is fixed (base 10) rather than variable (see Halford et al., 1998, 2010). Accordingly, the decimal, relative to the fraction, yields a poorer conceptual match to the two-dimensional set structure of the discrete display. However, the decimal may provide a one-dimensional measure of a portion of a continuous unit, and hence is a better conceptual match to a visual display involving continuous masses (see Figure 3.1, right). Such continuous masses do not correspond to sets (without the additional step of dividing each mass into equal-sized discrete elements, thereby creating measurement units). Without first imposing such a measurement system on the continuous display, there is no basis for counting elements, and therefore there is no unique integer that characterizes the cardinality of each of the masses. Thus, the continuous display provides a poor conceptual match to the fraction notation. In contrast, because the decimal notation inherently represents a one-dimensional quantity conveyed by the relative magnitudes of the two masses, it yields a better conceptual match than does the fraction.

These conceptual differences between fractions and decimals are intimately linked to differences in the mathematical procedures they afford. Table 3.1 summarizes the natural correspondences between fractions and decimals, respectively, with concepts and procedures. The experiments we report are all based on the 2 x 2 design outlined in Table 3.1, formed by the factorial combination of symbolic notation (fraction vs. decimal) and display type (discrete vs. continuous). Our central hypothesis is that fractions are conceptually linked to discrete sets and decimals to continuous masses. Because small sets of discrete elements can be counted, the

Table 3.1: Correspondences Between Notations for Rational Numbers and Concepts/Procedures

Preferred Conceptual Quantity Type	Symbolic Notation	Preferred Procedure	Back-up Procedure
discrete sets	fraction	count	*continuous masses:* impose measurement scale, then count units
continuous masses	decimal	estimate relative magnitude (proportion)	*discrete sets:* count and divide *continuous masses:* impose measurement scale, then count units, then divide

preferred procedure for evaluating fractions relies on counting elements in sets. Thus, when fractions are applied to discrete displays, counting is likely to be evoked. Moreover, in many reasoning tasks, the solution can be achieved without converting the two-dimensional fraction into a one-dimensional magnitude. For example, to determine that the discrete display in Figure 3.1 (left) conveys the relation 3/8, it suffices to count the elements of each set and form the resulting fraction, which matches that given; there is no need to execute the further step of evaluating the magnitude of 3/8 (e.g., by applying division). In contrast, evaluating the fraction with respect to the continuous display (right) would require imposing (perhaps by mental "slicing") a measurement system on the masses to create equal-sized discrete units, which can then be counted (the "back-up procedure"; see Table 3.1).

In contrast to a fraction, a one-dimensional decimal does not naturally align with the two components that form a ratio, but rather with the continuous quantity corresponding to its integrated magnitude (the value of a proportion). Hence the preferred procedure for evaluating a decimal as a match to a partitioned perceptual display would be estimation of a relative magnitude. People (and perhaps other animals) appear to be able to estimate proportions based on their system for approximate magnitude (Jacob, Vallentin & Nieder, 2012; Nieder &

Dehaene, 2009; Halberda & Fiegenson, 2008). However, by their very nature, approximate magnitudes will be more error-prone than exact calculations (assuming the reasoner is competent in counting). Thus, decimals will tend to evoke estimation procedures for both discrete and continuous displays, leading to reduced accuracy relative to fractions with discrete displays that afford use of a counting procedure.

The “back-up” procedure for decimals would be to use a variant of a counting strategy (see Table 3.1). Counting is possible for a discrete display; however, it would directly create a fraction, which would then have to be converted to a decimal by division. The extra division step would create difficulty in evaluating the decimal. For a continuous display, it would be necessary to first impose a measurement scale on the continuous masses, a process likely to be cognitively demanding. Thus, the superior alignment of the conceptual structure of a decimal to the continuous display may not translate into greater accuracy or speed in evaluating a match between the value of the decimal and of the depicted relative magnitude.

We will test the hypothesized differences between reasoning with fractions versus decimals in experiments using simple visual stimuli; however, there is reason to believe that the two-dimensional structure of fractions may also be the basis for other reasoning advantages associated with “natural frequency” formats (Gigerenzer & Hoffrage, 1995; Hoffrage, Gigerenzer, Krauss & Martignon, 2002; Tversky & Kahneman, 1983). In the terminology of the present paper, a natural frequency is a specific type of fraction format. We will consider the implications of our findings for understanding natural frequencies in the General Discussion.

Overview of Experiments on Reasoning with Rational Numbers

Our analysis of fractions as relational models leads to the prediction that, even though decimals are more effective in conveying one-dimensional magnitudes than are fractions (DeWolf et al., 2014), fractions should allow more accurate reasoning about bipartite relational structures, such as ratios defined on components of discrete perceptual displays. To test this basic hypothesis, we created pictorial displays of a set comprised of two subsets, paired with either a fraction or decimal value representing a certain ratio relation within the display (see Figure 3.2). The quantities in each display were either discrete, continuous, or continuous but divided into equal units (i.e., discretized). Experiment 1 tested the hypothesis that college-educated adults will in fact exhibit a preferential alignment between fractions and sets of discrete quantities, and between decimals and portions of continuous quantities. Importantly, we employed a task involving no computation, thus ensuring that the decision would be purely based on a conceptual preference for particular symbolic notations of rational numbers, rather than on the relative ease of performing computational procedures using these notations.

The subsequent experiments involved tasks that do require computational procedures to reason about relations. We focused on two types of ratio relations, both of which can be mapped to the structure of a fraction. A part-to-part ratio (PPR) is the relation between the size of the two subsets of a whole, whereas a part-to-whole ratio (PWR) is the relation between the size of one subset and the whole.[3] These two types of ratios were chosen in part because they offered a methodological advantage for administering a forced-choice task (Experiments 2-4). However, there are also important conceptual differences between these two types of relationships. The PWR is a conventional relationship for representing continuous magnitude. Creating an equivalent decimal for a PWR fraction is conceptually straightforward, and simply indicates a one-dimensional magnitude. By contrast, the PPR more directly reflects the abstract relational nature of a fraction. The decimal equivalent of a PPR is more difficult to understand in isolation, whereas a fraction highlights the comparison of interest (e.g., 2/3 might indicate 2 boys for every 3 girls, whereas .67 would indicate .67 boy for every 1 girl). We included both types of relationships in order to evaluate the generality of fractions as notations for expressing relations.

The framework we propose (see Table 3.1) predicts that the two types of rational numbers are likely to evoke different procedures for evaluating such relations given different types of displays. When discrete units are provided, counting is a likely strategy, which (at least for well-educated students solving problems involving relatively small values) should generate accurate measures of subsets that align directly with the numerator and denominator of a fraction. For decimals, in contrast, counting of subsets/sets would require additional processing (e.g., mental division) to translate the result into decimal form. Alternatively, decimal magnitudes may be estimated directly (Jacob et al., 2012). However, estimation is likely to be less accurate than counting, resulting in more errors when decimals, rather than fractions, are paired with displays of countable items. For continuous displays, the bipartite format of fractions may still encourage counting (e.g., by mentally slicing the display into units). However, accuracy is likely to be sacrificed, reducing or eliminating the advantage of fractions over decimals.

Experiment 2 tested the hypothesis that adults will be more accurate in identifying ratio-type relations (presented in displays similar to those illustrated in Figure 3.1) when using fractions than decimals, as long as the quantities are discrete or discretized (i.e., countable). In Experiments 3-4, we extended this paradigm to examine the impact of the two types of rational numbers in a task that requires higher-order analogical reasoning. Based on the hypothesized role of conceptual alignment in mathematical modeling, we predicted that for displays of countable entities, fractions would yield greater accuracy than decimals in both relation identification and analogical reasoning.

Experiment 1

In Experiment 1 we examined whether adults in fact show consistent preferences for particular types of symbolic notations depending on the type of entities being represented. In particular, we tested the hypothesis that adults prefer to use fractions to represent countable ratio relationships and decimals to represent magnitudes of continuous quantities, even when no computation is required.

Method

Participants

Participants were 48 undergraduates at the University of California, Los Angeles (UCLA; mean age: 20.4 years; 37 females), randomly assigned in equal numbers to two between-subjects conditions (part-to-part vs. part-to-whole ratio; see below). All participants received course credit.

Materials and Design

The study was a 2 (relation type: part-to-part vs. part-to-whole ratios) X 3 (display type: continuous, discretized, discrete) design. As noted earlier, a part-to-part ratio (PPR) is the relation between the size of the two subsets of a whole, whereas a part-to-whole ratio (PWR) is the relation between the size of one subset and the whole. In our subsequent experiments, using two different relation types allowed us to create tasks involving a two-alternative forced choice. In Experiment 1 relation type was a between-subjects factor, and display type was a within-subjects factor.

Figure 3.2 depicts examples of the three display types. The discrete items were displays of circles, squares, stars, crosses, trapezoids, and cloud-like shapes. The continuous items were displays of rectangles that could differ in width, height and orientation (vertical or horizontal). The discretized items were identical to the continuous displays except that the rectangles were divided into equal-sized units by dark lines. For the stimuli used in test trials, red and green were used to demarcate the two different subsets (in practice trials, yellow and blue colors were used). The displays varied which color represented the larger subset versus the smaller subset. Displays in all experiments were sized at approximately 700 pixels (width) x 800 pixels (height), viewed from a distance of about 50 cm.

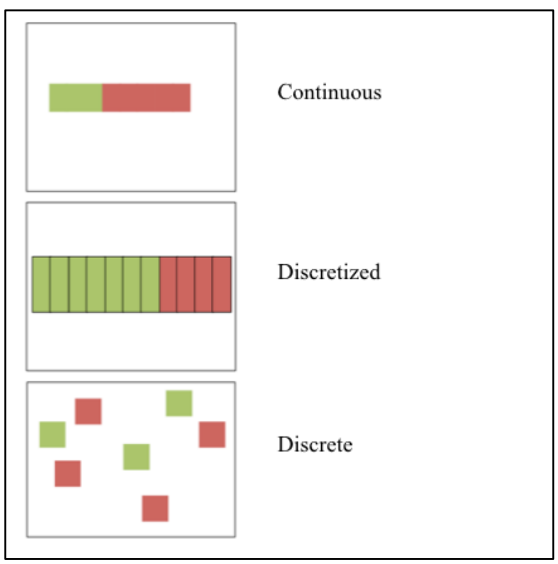


Figure 3.2. Examples of continuous, discretized and discrete displays used in experiments.

Participants were given instructions for either the part-to-part ratio (PPR) or part-to-whole ratios (PWR) condition. They were given the following instructions for the PPR condition: “In this experiment, you will see displays that show various part to part relations. In the display below [display with 1 orange circle and 2 blue crosses] this would be the number of orange circles relative to the number of blue crosses. Such relations can be represented with fractions (e.g. 3/4) or with decimals (e.g. .75). For each display your task is to choose which **notation** is a better representation of the depicted relation—a fraction or a decimal. Note that the specific values (i.e., 3/4 and .75) are just examples and do not match the values in the displays." For the PWR condition, the instructions were identical except for the description of the relations. In this condition the part-to-whole relation was defined using the example of the number of orange circles relative to the total number of blue crosses and orange circles. The relation type (PPR vs. PWR) was manipulated between subjects; thus participants in the PPR condition were only told about PPRs and participants in the PWR condition were only told about PWRs. Participants were shown examples of the continuous and discretized displays, in addition to the discrete display, and were told that displays could appear in any of those formats.

The task was simply to decide whether the relationship should be represented with a fraction or a decimal. Figure 3.3 shows an example of a discretized trial. In order to assess this preference on a conceptual level, the specific fraction and decimal shown to participants (3/4 and .75) were held constant across all trials, and never matched the number of items in the pictures. Thus (unlike the subsequent experiments we report), no mathematical task needed to be performed. There was therefore no requirement for accuracy, nor was any speed pressure imposed. Since the quantity shown in a display never matched the particular fraction and decimal values provided as response options, there was no real need to even determine the specific value represented in a display. The paradigm of Experiment 1 was thus intended to investigate participants' conceptual representations for fractions and decimals, in a situation in which mathematical procedures were not required.

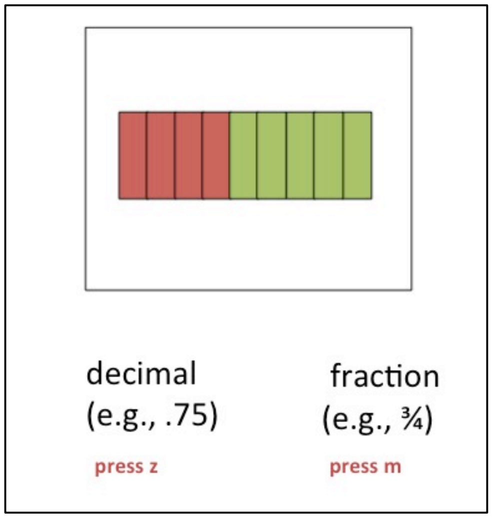


Figure 3.3. An example of a discretized trial from Experiment 1. Participants were shown the same decimal (.75) and fraction (3/4) as alternatives on every trial.

Procedure

Stimuli were displayed with Macintosh computers using Superlab 4.5 (Cedrus Corp., 2004), and participant responses were recorded. Participants were given the instructions described above for either the PPR condition or the PWR condition. Because the choice values provided did not match the values depicted in the picture (they were always .75 for decimal and 3/4 for fraction), there was no accuracy measure. Also, there was no speed pressure to respond. Participants were told to select the *z* key for decimals and the *m* key for fractions. Participants completed 60 test trials (20 for each display type). A fixation cross was displayed for 600 ms between each trial. Display types were shown in a different random order for every participant.

Results and Discussion

Because participants were forced to choose either a fraction or a decimal for each trial, the preference for each is complementary. For simplicity, we report the preference for fractions. The proportion of trials in which participants selected the fraction notation was computed for each display type for each participant. Figure 3.4 shows the proportion of trials that participants picked either fractions or decimals for each display. A 2 (relation type: PPR vs. PWR) X 3 (display type: discrete, discretized, continuous) ANOVA was used to assess differences in notation preference. There was no interaction between relation type and display type, $F(2, 45) = .53$, $p = .59$; $\eta_p^2 = .02$, and there was no significant difference between PPR and PWR conditions, .57 vs, .62, $F(1, 46) = 1.74$, $p = .19$, $\eta_p^2 = .04$. However, there was a significant main effect of display type, $F(2, 45) = 23.33$, $p < .001$, $\eta_p^2 = .31$.

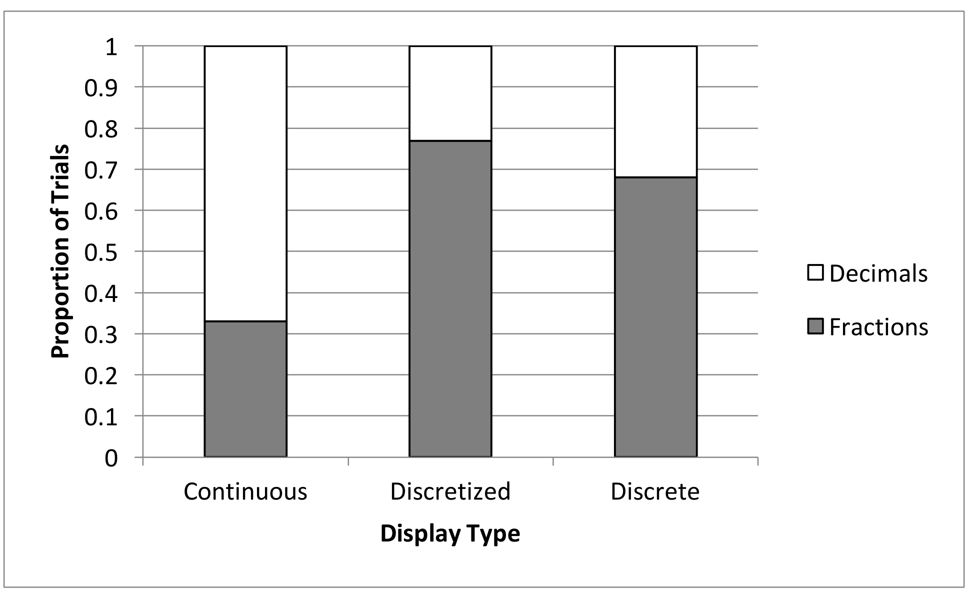


Figure 3.4. Proportion of trials for each display type in which either a fraction or decimal were chosen (Experiment 1).

Planned comparisons showed that there was no overall difference between discrete and discretized displays, .68 vs. .77; $F(1, 46) = 2.11$, $p = .15$, $\eta_p^2 = .04$. However, the preference for fractions was significantly lower for continuous displays than for either the discrete, .33 vs. .68; $F(1, 46) = 15.24$, $p < .001$, $\eta_p^2 = .25$, or discretized displays, .33 vs. .77; $F(1, 46) = 45.59$, $p < .001$, $\eta_p^2 = .50$. Thus, participants showed a strong preference for fractions with discrete and discretized displays, and a symmetrical preference for decimals with continuous displays.

The results of Experiment 1 revealed that adults prefer to represent both PPR and PWR ratio relationships with fractions when a display shows a partition of countable entities, but with decimals when the display shows a partition of continuous mass quantities. Participants picked the number format that provided the best conceptual match to either continuous or discrete displays, even though no procedural computation was required to complete the task. No mathematical task needed to be performed, and the specific quantities depicted in the displays did not match the numerical values of the fractions and decimals provided as choice options; hence our findings demonstrate that the preferential association of display types (discrete or continuous) and rational number formats (fractions or decimals) has a conceptual basis. The results of Experiment 1 closely align with evidence that college-educated adults show a preference for using continuous displays to represent decimals and countable displays to represent fractions (Rapp, Bassok, DeWolf & Holyoak, 2015).

Experiment 1 thus provides strong support for the hypothesis that the natural alignment of different symbolic notations with different quantity types has a conceptual basis. Experiments 2-4 tested whether these conceptual alignments also hold for more complex tasks that require computations and procedures.

Experiment 2

Experiment 1 established a conceptual correspondence between quantity types and symbolic notations for rational numbers. Experiments 2-4 examined whether this conceptual correspondence also makes one or the other symbolic notation more effective in a relational reasoning task. Experiment 2 tested the hypothesis that adults would be better able to identify and evaluate ratio relationships using fractions than decimals, especially for discrete (or discretized) quantities.

Method

Participants

Participants were 58 UCLA undergraduates (mean age: 20.4 years; 49 females), randomly assigned in equal numbers to the two between-subjects conditions. Course credit was provided to participants.

Materials and Design

The study was a 2 (symbolic notation: fractions vs. decimals) X 2 (relation type: part-to-part vs. part-to-whole ratios) X 3 (display type: continuous, discretized, discrete) design. Although the distinction between the two types of relations is of potential theoretical interest in its own right, our main reason for using the two types was methodological, as it enabled us to create a two-alternative forced-choice task. Symbolic notation was a between-subjects factor, and relation type and display type were within-subjects factors.

The displays were similar to those used in Experiment 1 (see Figure 3.2). The magnitudes of fractions and decimals were matched. The values of the fractions and decimals were always less than one, and decimals were shown rounded to two decimal places. The values of the rational number presented on each trial represented one of two ratio relationships within the display: part-to-whole ratio (PWR) or part-to-part ratio (PPR). These were the same relationships used in Experiment 1, but the task in Experiment 2 explicitly required participants to identify on each trial which of the two relationships matched a presented number. Thus, a number was paired with the display that specifically matched one of the relationships.

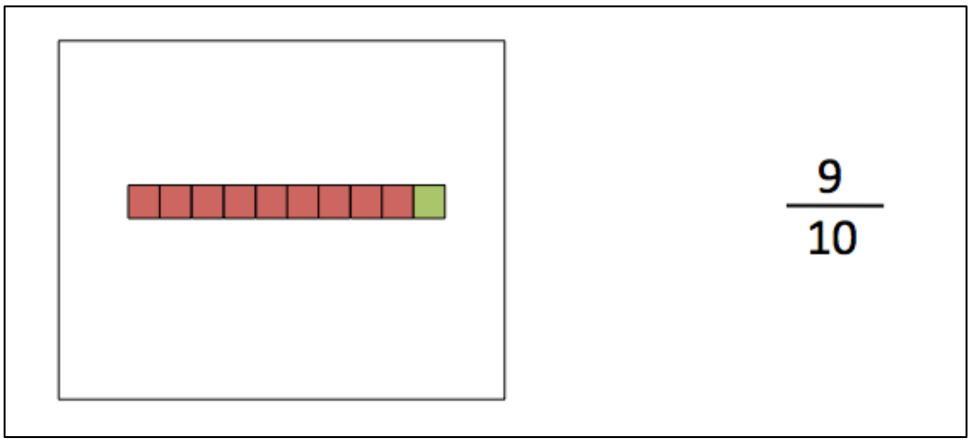


Figure 3.5. Example of a ratio identification problem (Experiment 2).

For example, Figure 3.5 shows an example of a PWR trial with a display with 9 red units out of a total of 10. The number specified is 9/10 (or .90 in a matched problem using decimals), thus corresponding to a PWR. For the corresponding PPR problem, the number would be 1/9 (or .11 in decimal notation). The smaller subset would be the numerator in this case, so that the overall magnitude was always less than one.

Procedure

Stimuli were displayed with Macintosh computers using Superlab 4.5 (Cedrus Corp., 2004), and response times and accuracy were recorded. Participants received the following instructions: "In this experiment, you will see a display paired with a value. You need to identify which of the two following relationships is shown." Below this, there were two different displays showing the PWR and PPR relations, which were simply referred to as "Relation 1" and "Relation 2". The assignment of the labels was counterbalanced for all subjects such that half were told Relation 1 was PPR and the other half were told Relation 1 was PWR. The PPR display contained1 circle and 2 crosses. For the fractions condition this was labeled as "1/2 amount of circles per amount of crosses;" for the decimals condition it was labeled as ".50 amount of circles per amount of crosses." The PWR was represented by a display of 2 circles and 3 crosses. For the fractions condition this was labeled as "2/5 of the total is the amount of circles;" for the decimal condition it was labeled as ".40 of the total is the amount of circles." The first of these explanations of the PPR and PWR relations was shown with discrete items.

The subsequent screen showed the same values paired with discretized displays. A third screen showed the same values paired with continuous displays. Half of the participants were told to select the *z* key for Relation 1 and to select the *m* key for Relation 2; the other half received the reverse key assignments.

After this introduction, participants were given an example problem and asked to identify the relation. After they made their judgment, an explanation was shown to participants about why the example showed the correct relation. The explanation also stated what the numerical value would be for the problem if it had shown the alternative relation. Participants were then given another example using the other relation, with the same explanation process. A series of practice trials was then administered. Participants had to complete at least 24 practice trials (four for each of the six within-subjects conditions). If they scored at least 17 correct (i.e., about 70%) they were able to move on to the test trials. If they did not score above this threshold, they continued with additional practice trials until they reached the threshold percentage correct. About 75% of participants passed the threshold after the first round of practice trials. The remaining participants were able to advance after a second set of practice trials. All of the practice trials were different from those used in the test trials. Feedback was given for incorrect trials, in the form of a red “X” on the screen. After the practice trials had been completed, a screen was displayed informing participants that the actual test trials were beginning. Participants were told to try to go as quickly as possible without sacrificing accuracy. They completed 72 test trials (12 for each of the 6 within subjects conditions). A fixation cross was shown for 500 ms between each trial. Feedback was continued for incorrect trials. Relation types and display types were shown in a different random order for every participant.

Results and Discussion

Accuracy and mean response time (RT) on correct trials were computed for each condition for each participant. A 3 (display: continuous, discretized, discrete) X 2 (relation type: PPR vs. PWR) X 2 (symbolic notation: fraction vs. decimal) mixed factors ANOVA was used to assess differences in RT and accuracy. As the three-way interaction was not reliable, $F(2, 112) = .97$, $p = .38$, $\eta_p^2 = .02$,, all analyses are reported after collapsing across the factor of relation type. Figure 3.6 displays the pattern of accuracy, the primary dependent measure in Experiments 2-4. Accuracy exceeded chance level (50%) for all conditions. For some conditions, accuracy was lower than the practice threshold level, likely because the practice trials comprised a different set of problems that were less challenging than the actual trials. There was a significant interaction between display type and symbolic notation, $F(2, 55) = 24.57$, $MSE = 1.7$, $p < 0.001$. $\eta_p^2 = .47$. Planned comparisons revealed that participants were more accurate when using fractions than decimals for discrete displays, 81% vs. 63%; $F(1, 56) = 23.64$, $MSE = 8.2$, $p < 0.001$ $\eta_p^2 = .30$, and discretized displays, 78% vs. 63%; $F(1, 56) = 13.92$, $MSE = 9.7$, $p < 0.001$, $\eta_p^2 = .20$. In contrast, accuracy did not differ as a function of symbolic notation for continuous displays, 62%

vs. 67%; $F(1, 56) = 1.52$, $MSE = 10.4$, $p = 0.22$, $\eta_p^2 = .03$. There was a significant main effect of relation type favoring PPR over PWR, 72% vs. 66%, $F(1, 56) = 11.00$, $MSE = .024$, $p = .002$, $\eta_p^2 = .16$; however, there was no interaction between relation type and number type, $F(1, 56) = .03$, $MSE = .024$, $p = .87$, $\eta_p^2 < .001$, nor an interaction between relation type and display type, $F(2, 112) = 1.79$, $MSE = .02$, $p = .17$, $\eta_p^2 = .03$. Thus, participants across all conditions performed better on the PPR problems than the PWR problems (perhaps because PWR problems require an extra computational step to add the two subsets to find the whole).

We also analyzed reaction times (RTs), primarily to assess any possible speed-accuracy trade-offs. Figure 3.7 displays the pattern of mean correct RTs across conditions. As found for the accuracy analysis, a significant interaction was obtained between display type and symbolic notation, $F(2, 55) = 3.52$, $MSE = 1369$, $p = 0.037$, $\eta_p^2 = .11$. Although RTs tended to be faster

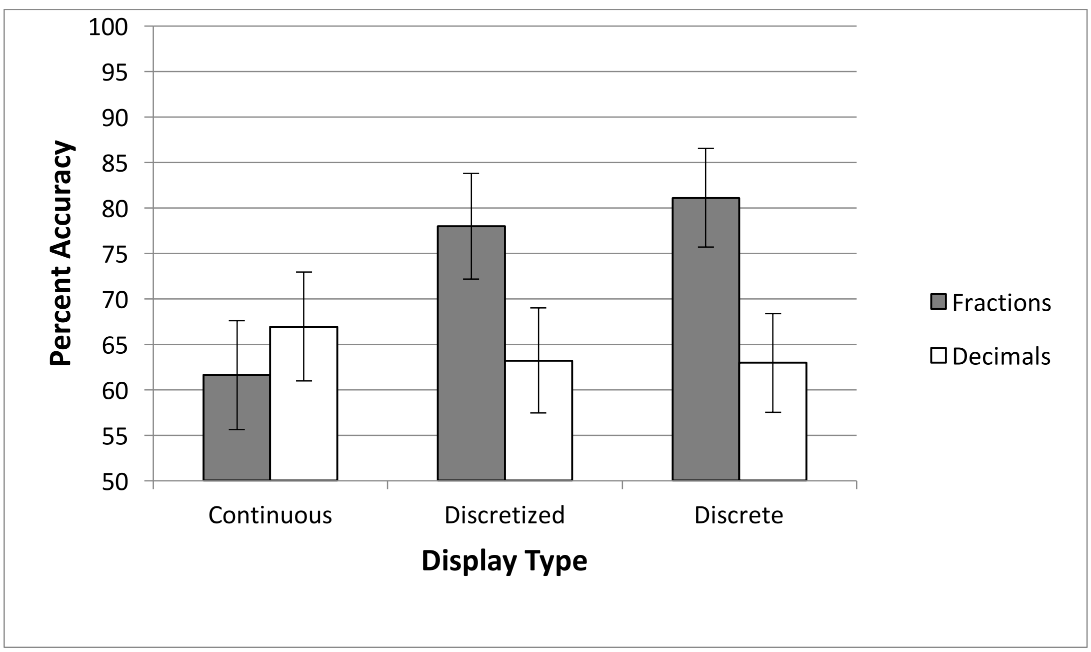


Figure 3.6. Accuracy of relation identification using fractions and decimals across different types of displays (Experiment 2). Error bars indicate standard error of the mean.

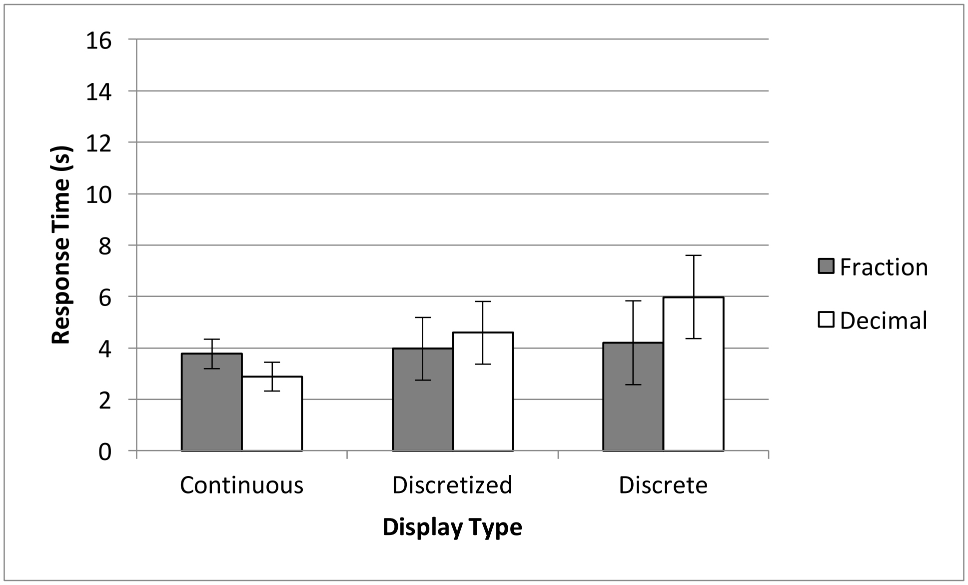


Figure 3.7. Mean response time for relation identification using fractions and decimals across different types of displays (Experiment 2). Error bars indicate standard error of the mean.

for fractions than decimals for discrete displays, the difference was not reliable, 4.20 s vs. 5.97 s; $F(1, 56) = 2.40$, $MSE = 7607$, $p = .13$, $\eta_p^2 = .04$. A non-significant trend was also obtained for discretized displays, 3.93 s vs. 4.59 s; $F(1, 56) = 0.585$, $MSE = 4357$ $p = .45$, $\eta_p^2 = .01$. These RT analyses confirm that the accuracy advantage of fractions over decimals for countable displays is not attributable to speed-accuracy trade-offs. For continuous displays, however, response times were significantly slower with fractions than with decimals, 3.77 s vs. 2.88 s; $F(1, 56) = 4.77$, $MSE = 9202$, $p = .03$, $\eta_p^2 = .08$.

As in the accuracy analyses, there was also a significant main effect of relation type such that PPR problems were solved more quickly than PWR problems, 4.07 s vs. 4.37 s; $F(1, 56) = 8.45$, $MSE = 9455$, $p = .005$, $\eta_p^2 = .13$. However there was no significant interaction between relation type and number type, $F(1, 56) = .20$, $MSE = 9455$, $p = .67$, $\eta_p^2 = .004$, nor between relation type and display type, $F(2, 112) = .65$, $MSE = 13693$, $p = .52$, $\eta_p^2 = .01$.

In order to evaluate whether participants preferentially engaged in counting strategies when evaluating fractions rather than decimals, we conducted regression analyses on the averaged data for each individual trial. The main variable of interest was the size of the denominator for the correct ratio. If a counting strategy were used, then time to identify the correct ratio would be expected to increase with the size of the larger subset (i.e., the subset corresponding to the denominator of the ratio). For each symbolic notation, analyses were performed separately for accuracy and for response times on correct trials, for each display type. To increase the sensitivity of the analyses, we included ratio type as a covariate, and also the logarithm of the numerical difference between the correct ratio and the alternative ratio. Difference between alternatives was defined as the difference between the given value and the value that would be the result of the alternative ratio. This factor was likely to predict problem difficulty, as previous research has shown that it is more difficult for adults to compare the relative sizes of two magnitudes (expressed as fractions or ratios) as the difference between the magnitudes decreases (DeWolf et al., 2014; Schneider & Siegler, 2010). Thus, we investigated whether evidence of a counting strategy would emerge after controlling for other likely sources of differences in problem difficulty. As no reliable effects of denominator size emerged for either symbolic notation in regression analyses based on accuracy, we only report analyses based on response times.

In accord with the hypothesis that fractions will be associated with use of counting, denominator size emerged as a significant predictor of RTs (after controlling for the two covariates) for each of the three display types: discrete, $b = .284$, $t(20) = 6.76$, $p < .001$, discretized, $b = .247$, $t(20) = 7.83$, $p < .001$, and continuous, $b = .16$, $t(20) = 4.56$, $p < .001$. The fact that denominator size was reliable even for the continuous displays suggests that participants who evaluated fractions used a counting-like strategy even when the display did not provide measurement units (perhaps by imagining such units using the back-up procedure specified in Table 3.1). A very different pattern emerged when evaluations were based on decimals. Denominator size was not a reliable predictor of response times for either discrete, $b = .164$, $t(20) = .958$, $p = .35$, or continuous displays, $b = -.01$, $t(20) = .267$, $p = .79$, though it was marginally significant for discretized displays. $b = .285$, $t(20) = 2.09$, $p = .05$. The regression analyses on RTs thus supported the hypothesis that counting strategies tend to be used when evaluating fractions. Decimals, by contrast, may have been evaluated using an estimation procedure, which we hypothesize is preferred (see Table 3.1), and which would be more direct (and no less accurate) than a "count-and divide" strategy.

Overall, the results of Experiment 2 revealed an advantage for identifying ratio relationships in displays when these ratios were represented by fractions rather than decimals. However, this pattern was moderated by the nature of the visual displays. When displays conveyed countable entities (sets of discrete objects, or continuous displays parsed into units of measurement), ratios were evaluated more accurately when the symbolic notation was a fraction rather than a decimal. In contrast, when the display showed continuous quantities without measurement units, accuracy in evaluating ratio relations did not differ, and decisions were made more quickly for decimals than fractions.

Although the relation-identification task used in Experiment 2 yielded an interaction between number type and display type, its quantitative form (at least for the accuracy measure) differed from the similar interaction obtained in the purely conceptual task used in Experiment 1.

Experiment 1 revealed a significant preference for decimals over fractions with continuous entities, but Experiment 2 did not find a clear accuracy advantage for decimals in any condition, even when paired with continuous displays (though the accuracy trend in the latter condition did favor decimals, and decimals yielded faster response times). The lack of a clear accuracy advantage for decimals paired with continuous displays in Experiment 2 likely reflects the important role that mathematical procedures play in relation identification (see Table 1). In particular, comparing the value of a decimal to a display requires either estimation or (for discrete or discretized displays) counting and division. As estimation is inherently more error-prone than counting strategies (at least for college students), this procedural requirement prevented decimals from showing a clear accuracy advantage over fractions even in the continuous condition. However, the estimation procedure apparently used for decimals in the continuous condition does seem to be faster to execute than the procedure used in this condition for fractions.

Overall, the findings of Experiment 2 highlight the importance of the relational structure internal to fractions. When the values within the fraction can be mapped to particular entities (i.e., when they are countable), identifying ratio relationships in visual displays is greatly facilitated. Decimals, in contrast, do not exhibit the same type of internal structure; hence it is more difficult to map the integrated decimal value to two separate subsets within a display. However, when displays are continuous, the two subsets are difficult to measure exactly. Presumably people are then forced to use more approximate estimate strategies, for which decimals yield about the same accuracy as fractions, with greater processing speed.

Experiment 3

Experiment 2 demonstrated an advantage of fractions over decimals in the identification of ratio relationships in displays with countable subsets and values. In Experiment 3 we investigated whether the fraction advantage (modulated by the nature of the visual displays) would extend to a more complex analogical reasoning task. To address this question, we extended the ratio-identification paradigm used in Experiment 2. In the analogy task introduced in Experiment 3, participants had to use the ratio relationship they identified in an initial source display to select an analogous value for the same type of ratio in a target display. Whereas the task used in Experiment 2 required identification of a first-order relation in a display (ratio between two quantities), the task used in Experiment 3 required computing a higher-order relation between the ratio relation extracted in the source display and the analogous relation in a target display. This reasoning task is more complex than that used in Experiment 2 because participants must not only identify the relationship between display and number, but also apply this relationship to a novel display and identify the correct value. If the internal relational structure of a fraction is an important aspect of its meaning, then the general pattern of performance differences observed in Experiment 2 should also be obtained in a task involving high-level analogical reasoning.

Method

Participants

Participants were 52 undergraduates at UCLA (mean age = 21; 30 females) who received course credit. They were randomly assigned in equal numbers to the two between-subjects conditions.

Materials and Design

Similar to Experiment 2, a 2 (symbolic notation: fractions vs. decimals) X 2 (relation type: part-to-part vs. part-to-whole ratios) X 3 (display type: continuous, discretized, discrete) design was employed. Symbolic notation was manipulated between subjects whereas relation type and display type were manipulated within subjects. As in Experiment 2, discrete, discretized, and continuous displays were paired with either a fraction or decimal that represented a PPR or PWR. The analogy problems used in Experiment 3 (see Figure 3.8) can be viewed as a generalization of the proportional analogy format A:B :: C:D vs. D', where A and C

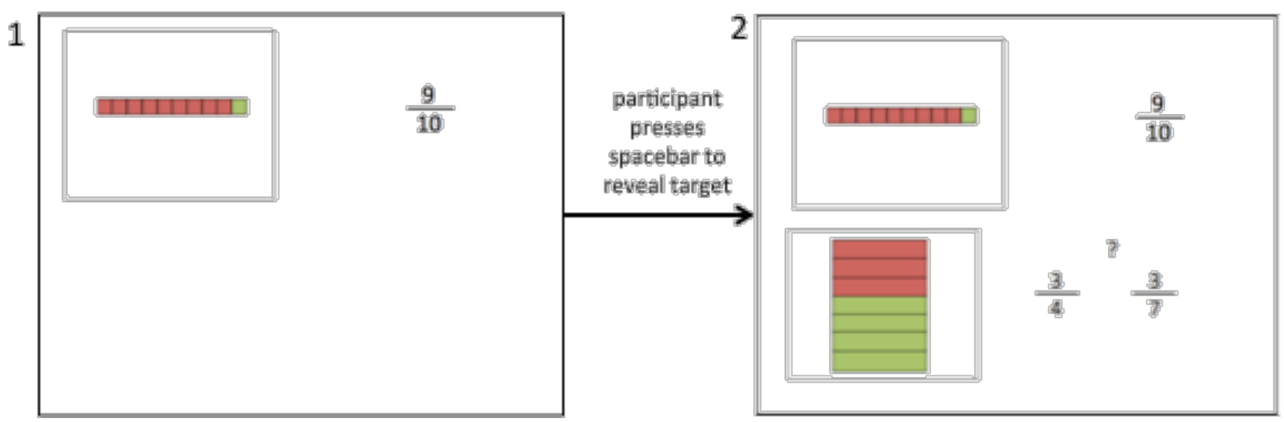


Figure 3.8. Example of an analogy trial used in Experiments 3-4.

are ratios in the source and target displays, respectively, and C, D and D' are corresponding rational numbers. (Since ratios are themselves relations, these analogy problems actually involved a third-order relation.) The source analogs (A:B) were identical to the problems used in Experiment 2. Participants needed to identify which of two numbers in the target (D vs. D') correctly matched the display with the same relationship specified in the source. The analogy task required making a choice of the correct number to complete the target analog using the same relation as in the source. The symbolic notation (fraction or decimal) was always the same across the source and target. Solving an analogy problem required first identifying the ratio relation in the A display characterized by the number given as B (as in Experiment 1). Once the higher-order relation between A and B was extracted, the solution required identifying the same relation type in target display C, and choosing the corresponding number as the D term. The D' foil was always chosen to match the alternative ratio relation in the C display.

As the analogy problems were constructed using the problems from Experiment 2 as the A:B source, the specifications of the stimuli are the same. The same two colors, green and red, were used in the A and C displays, and the color relationship was maintained, such that the same color mapped to the same part of the relation in both A and C. This constraint served to identify which part (lesser or greater) mapped to the numerator in a ratio relation. As in Experiment 2, color assignments varied across trials, so the same color might indicate the lesser subset on one trial and the greater subset on another. For each trial, the source and target were randomly assigned for each participant. Thus the only aspect that was consistent between the two analogs was the higher-order relationship (PPR or PWR) and the color mapping.

Procedure

The procedure was identical to Experiment 2, except expanded to constitute an analogy task. Participants were given the following instructions: "In this experiment, there will be two different types of relations between the picture and the numerical value shown below." Below this instruction appeared the same two displays for the two types of ratios, as in Experiment 2 with the same labeling. Participants were then told: "The first step is to identify the relation between the top picture and numerical value [shown with an example of a source display as in the top of Figure 3.8]. The second step is to select one of the two numerical values on the right that shares the same relationship with the bottom picture as the top picture" [shown with an example of a target display as in the bottom of Figure 3.8]. For the first step, instead of hitting a key that corresponded to a specific relationship (as in Experiment 2), participants simply hit the space bar when they had identified the relationship. After the space bar was pressed, the target (C:D vs. D') was shown on the screen below the source, so that both the source and target were on the screen simultaneously. Participants were asked to select which of two numbers (D or D') shared the same relationship with this display as the relationship from the source. Half of the time, D appeared on the right side of the screen. They made their selection by pressing the *z* key for the number shown on the left and the *m* key for the number shown on the right. The *z* and *m* keys were labeled with "L" and "R", respectively, so that participants could remember which key corresponded to each number. As in Experiment 2, participants were told to try to go as quickly as possible without sacrificing accuracy. After reading the instructions and completing 12 practice trials with feedback, participants proceeded to the 72 test trials.

Results and Discussion

As in Experiment 2, accuracy (the primary dependent measure) and mean RT on correct trials were computed for each condition for each participant. RTs were measured from the onset of the source problem to the selection of the correct value for the target display. A mixed factors ANOVA was used to compare differences in RT and accuracy. No reliable overall differences were obtained between the two relation types (PPR and PWR) on either measure, accuracy: $F(1, 50) = 2.58$, $p = 0.11$, $\eta_p^2 = .05$; RT: $F(1,50) = 0.73$, $p = .40$, $\eta_p^2 = .01$, so all results reported here are collapsed across this factor, as in Experiment 2.

As shown in Figure 3.9 the pattern of results for analogical accuracy largely followed the pattern observed in Experiment 2 for the simpler relation identification task. A significant interaction was obtained between display type and symbolic notation, $F(2, 49) = 20.59$, $MSE =$ 1.8 $p < .001$, $\eta_p^2 = .46$. Planned comparisons indicated that accuracy was higher for fractions than decimals in the discrete condition, 87% vs. 66%; $F(1, 50) = 28.96$, $MSE = 7.3$ $p < .001$, $\eta_p^2 = .37$, and discretized condition, 80% vs. 67%; $F(1, 50) = 10.06$, $MSE = 8.3$, $p = .003$, $\eta_p^2 = .17$, but did not differ across the two symbolic notations for the continuous condition, 61% vs. 65%; $F(1, 50) = 0.86$, $MSE = 7.7$, $p = .36$, $\eta_p^2 = .02$.

The accuracy analysis yielded a reliable 3-way interaction between relation type, display type and number type, $F(2, 100) = 6.89$, $MSE = .02$, $p = .002$, $\eta_p^2 = .12$. This interaction was driven by a significant advantage in accuracy for PWR problems over PPR problems only for decimals paired with continuous displays, 74% vs. 55%, $F(1, 50) = 23.76$, $MSE = .04$, $p < .001$, $\eta_p^2 = .32$. No such interaction was apparent in Experiment 2, and we have no explanation for this finding.

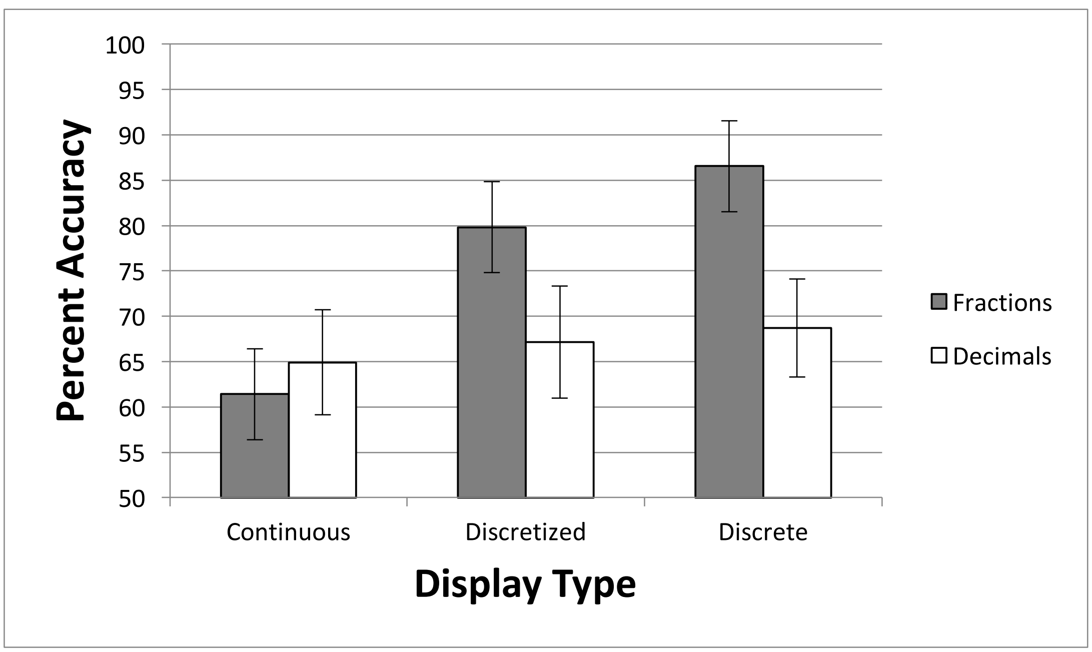


Figure 3.9. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 3). Error bars indicate standard error of the mean.

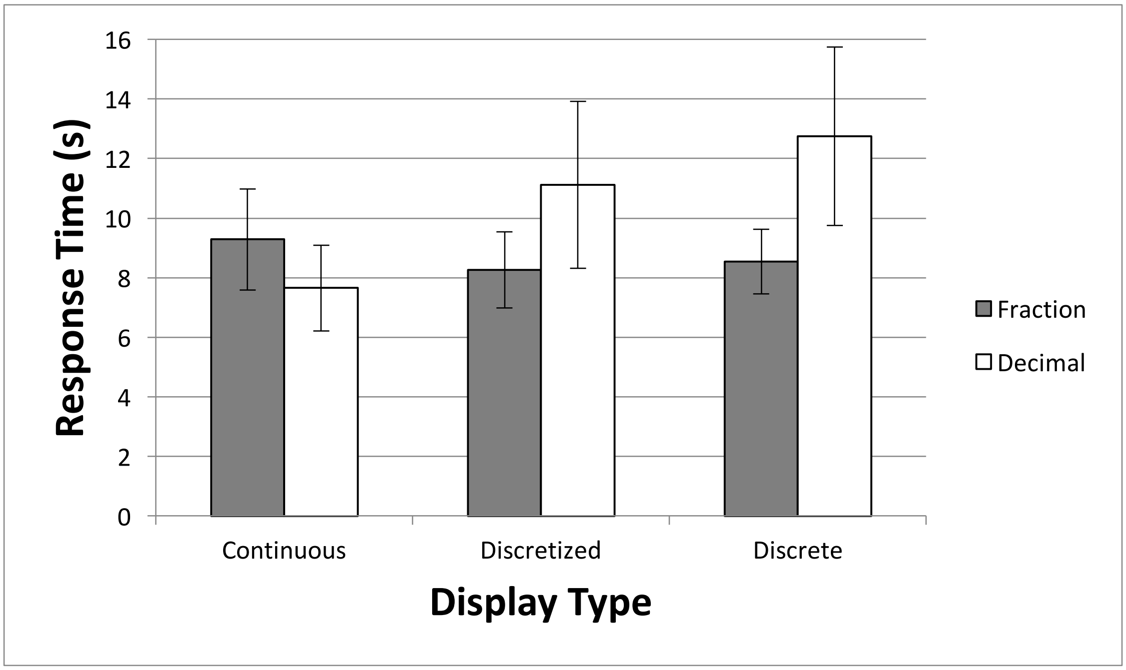


Figure 3.10. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 3). Error bars indicate standard error of the mean.

In analyzing RTs, response times for incorrect answers were excluded, as were RT outliers that were greater than three standard deviations from the mean (roughly 2% of response times). As shown in Figure 3.10, the RT pattern is consistent with the pattern of accuracy results, and largely replicates the pattern observed in Experiment 2 (though RTs in Experiment 3 were of course much longer, as they reflect the duration of the entire analogy problem, not just processing of the source). In particular, there was a reliable interaction between symbolic notation and display type, $F(2, 49) = 16.19$, $MSE = 2721$, $p < .001$, $\eta_p^2 = .40$. Planned comparisons indicated that RTs were faster with fractions than decimals for the discrete condition, 8.5 s vs. 12.8 s; $F(1, 50) = 7.10$, $p = .01$, $\eta_p^2 = .12$, with a trend for the discretized condition, 8.3 s vs. 11.2 s; $F(1, 50) = 3.51$, $p = .07$, $\eta_p^2 =.07$. For the continuous conditions, RTs for fractions versus decimals did not differ reliably, 9.3 s vs. 7.7 s; $F(1, 50) = 2.12$, $p = .15$, $\eta_p^2 = .04$. The general pattern of RT differences was thus similar to that observed in the simpler relational verification task (Experiment 2), except that there was a general shift toward an RT advantage for fractions over decimals.

No reliable interaction was observed between relation type and number type, $F(1, 50) = .27$, $MSE = 3718$, $p = .61$, $\eta_p^2 = .01$, nor between relation type and display type, $F(2, 49) = 1.85$ $MSE = 2721$, $p = .16$, $\eta_p^2 = .03$. The 3-way interaction between the three factors was also not reliable, $F(2, 100) = 3.24$, $MSE = 2728$, $p = .05$, $\eta_p^2 = .06$.

As was the case for the ratio identification task used in Experiment 2, the analogy task used in Experiment 3 showed an overall advantage for solving problems using fractions as compared to decimals. In Experiment 3, participants not only had to correctly identify a particular relationship between a numerical value and a ratio relation in a display, but also had to use this relationship to identify which of two values correctly mapped to the same type of ratio relation in a new display (where the ratio quantity differed between source and target). Unlike Experiment 2, RTs in Experiment 3 showed a significant RT advantage for fractions over decimals in solving problems using discrete and discretized displays, whereas RTs for the two symbolic notations did not differ for continuous displays (rather than showing a reversal as in Experiment 2). Thus if anything, the overall fraction advantage was yet more pronounced in the complex analogical reasoning task employed in Experiment 3. These findings imply that people are able to more quickly identify a number that correctly maps to a ratio relation when the symbolic notation of that number affords a one-to-one mapping to the conceptually relevant units within the displays, as is the case for a fraction. The fraction advantage extends beyond the identification of ratio relationships, as fractions also facilitate mapping of higher-order relations between types of ratios.

Experiment 4A

In the fraction conditions used in Experiments 2 and 3, the number of items in the display directly corresponded to the paired fraction value. For example, if the number was 2/3 and the ratio relation was a PPR, then the display would display 2 items of one type and 3 items of another type. In the corresponding decimal condition, the number would appear as .67. However, there are many item arrangements that could fit this ratio (2/3, 4/6, 8/12, etc.). Experiment 4A was designed to determine whether the fraction advantage found in the analogy task used in Experiment 3 would still be obtained if the numbers in the numerator and denominator did not directly match the specific quantities shown in the display. Experiment 4A included fractions that were equivalent in overall value, but did not map one-to-one with the quantities in the displays. For example, a display with two items of one type and three items of another might be paired with 4/6, rather than 2/3.

Method

Participants

Participants were 75 UCLA undergraduates (mean age = 20.4; 53 females) who received course credit, randomly assigned in equal numbers to the three between-subjects conditions.

Materials and Design

Experiment 4A had one within-subjects factor (relation type) with 2 levels: PPR vs. PWR, and one between-subjects factor (symbolic notation) with 3 levels: one-to-one (OTO) fractions, non-one-to-one (NOTO) fractions, and decimals. Because the findings of the previous experiments demonstrated that the largest advantage in accuracy for fractions over decimals is found for the discrete display type, only problems based on discrete displays were tested. All conditions included exactly the same set of displays. However, the OTO fractions condition used values that mapped one-to-one with the displays (e.g., if the relation was PWR with 3 items out of 7 items, the fraction would be 3/7). In contrast, NOTO fractions had numerators and denominators that were either two or three times greater or smaller than the actual number of items shown (e.g., 3 out of 7 items paired with 6/14). In the decimal condition, the same display was paired with the decimal equivalent (e.g., 3 out of 7 items paired with .43). There were 24 problems for each of the two relation conditions (PPR, PWR) for a total of 48 problems.

Procedure

The procedure was basically the same as in Experiment 3, except that only discrete displays were shown. Participants saw the same directions as in Experiment 3 (with only discrete examples and discrete practice questions). They were told to try to go as quickly as possible without sacrificing accuracy. After completing 12 practice trials with feedback, participants continued on to the test trials and were given feedback throughout the task.

Results and Discussion

As in the previous experiments, accuracy and mean RTs on correct trials were computed for each condition for each participant. RTs were again measured from the onset of the source displays to the selection of the correct value for the target display. A mixed factors ANOVA was used to compare differences in RT and accuracy. Figure 3.11 shows the pattern of accuracy across conditions. There was a significant main effect of symbolic notation, $F(2, 72) = 27.79$, $MSE = 4.2$, $p < .001$, $\eta_p^2 = .44$, and relation type, $F(1, 72) = 8.33$, $MSE = 1.1$, $p = .005$, $\eta_p^2 = .11$. Planned comparisons showed that accuracy for OTO fractions was significantly higher than accuracy for decimals, 95% vs. 65%, $F(1, 72) = 55.24$, $MSE = 8.4$, $p < .001$, $\eta_p^2 = .44$, or for

NOTO fractions, 95% vs. 78%, $F(1, 72) = 17.56$, $MSE = 8.4$, $p < .001$, $\eta_p^2 = .20$. Accuracy for NOTO Fractions was significantly higher than accuracy for decimals, 78% vs. 65%, $F(1, 72) =10.02$, $MSE = 8.4$, $p = .002$, $\eta_p^2 = .12$.. Thus, fractions maintained an accuracy advantage over decimals for analogical reasoning with discrete displays, even when the numerator and denominator of the fraction do not equal the corresponding quantities in the visual display (NOTO fraction condition).

Unlike Experiment 3, a significant main effect of relation type was found in Experiment 4A, $F(1, 72) = 8.33$, $MSE = 1.1$, $p = .005$, $\eta_p^2 = .11$. Mean accuracy was 82% for PPR problems and 77% for PWR problems. In addition, a small but significant interaction was obtained between relation type and symbolic notation, $F(2, 72) = 3.18$, $MSE = 1.1$, $p = .047$, $\eta_p^2 = .08$. As this interaction did not replicate in Experiment 4B, we do not consider it further.

Figure 3.12 shows the corresponding pattern of response times. For RTs, the interaction between relation type and symbolic notation was not reliable, $F(2, 72) = 1.63$, $MSE = 4304$, $p = .20$, $\eta_p^2 = .04$, nor was the main effect of relation type, $F(1, 72) = 3.00$, $MSE = 4304$, $p = 0.09$,

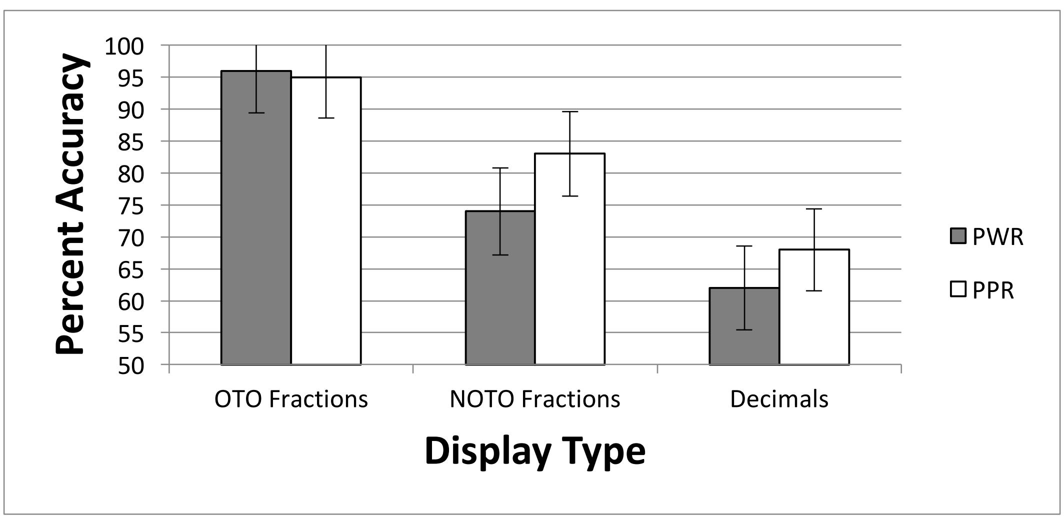


Figure 3.11. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 4A). Error bars indicate standard error of the mean.

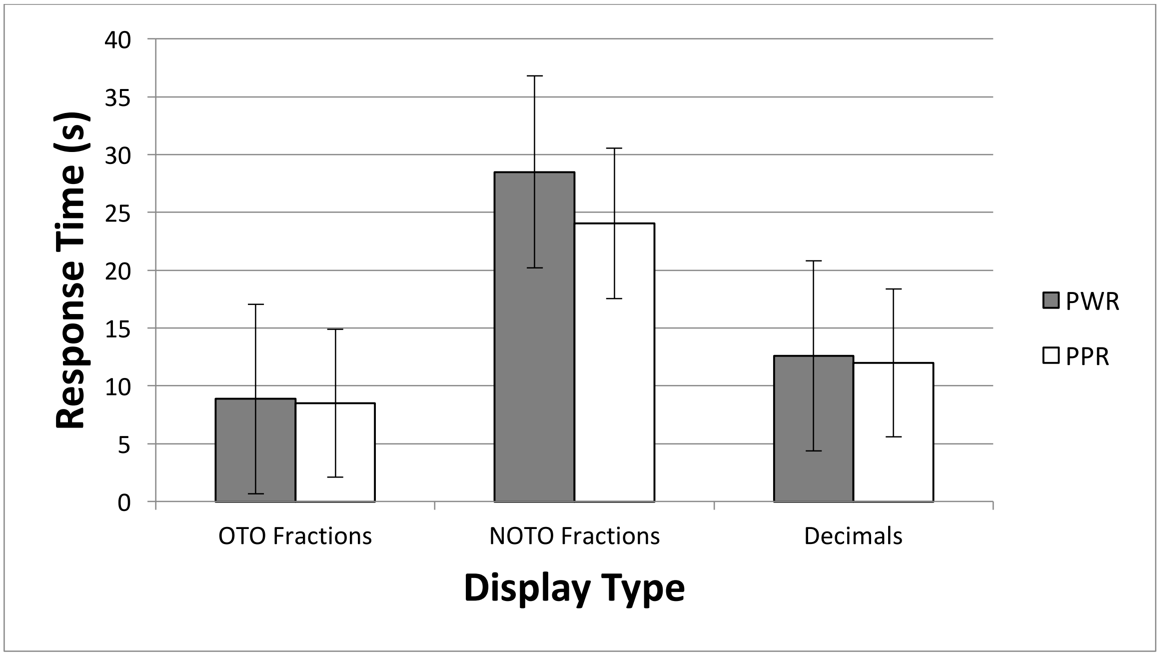


Figure 3.12. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 4A). Error bars indicate standard error of the mean.

η_p^2 = .04, nor the main effect of symbolic notation, $F(1, 72) = 1.78$, MSE =2648757, $p = .18$, η_p^2 = .05. There was no effect of relation type, PPR vs. PWR: 15.2 s vs. 17.0 s, $F(1, 71) = 3.00$, MSE = 43004, $p = .09$, $\eta_p^2 = .04$, and no interaction between relation type and number type, $F(2,71)$ = 1.63, MSE = 43004, $p = .20$, $\eta_p^2 = .04$. Clearly the considerable variance in RTs contributed to the lack of statistically reliable RT differences. Nonetheless, it is of note that the NOTO fractions condition yielded mean RTs considerably longer than those for either of the other two symbolic notations.

Experiment 4B

In Experiment 4A, response times for the NOTO fraction condition were considerably longer than those for the OTO fraction and decimal conditions. One explanation for this pattern is that the components of NOTO fractions are proportional to the relevant quantities in the displays. Even though extra time is required for a NOTO fraction because the depicted quantities do not equal the numerator and denominator, a counting strategy can yield higher accuracy than the approximate strategy associated with decimals. Nonetheless, an alternative possibility is that the accuracy advantage of the NOTO fraction condition over decimals in Experiment 4A was simply the consequence of a speed-accuracy tradeoff. To evaluate the latter possibility, Experiment 4B used an identical design and procedure as Experiment 4A, except that the instructions were altered to encourage participants to take their time and try to achieve high accuracy (whereas in Experiment 4A, participants were told to go as fast as possible without sacrificing accuracy). If decimals are able to support analogical reasoning with ratios just as well as NOTO fractions, then the two conditions should not differ in accuracy in the absence of speed pressure.

Method

Participants were 66 UCLA undergraduates (mean age: 20.6; 53 females) who received course credit, randomly assigned in equal numbers to the three between-subjects conditions. The design and procedure was identical to Experiment 4A with the exception that participants were directed to spend as much time as necessary on each problem to achieve high accuracy.

Results and Discussion

Figure 3.13 shows the pattern of accuracy across conditions. Consistent with the removal of time pressure, accuracy was generally higher in Experiment 4B than 4A. The analysis revealed a significant effect of symbolic notation, $F(2, 63) = 10.23$, $MSE = 5$, $p < .001$, $\eta_p^2 = .25$. Planned comparisons revealed that OTO fractions yielded higher accuracy than decimals, 94% vs. 73%, $F(1, 63) = 20.30$, $MSE = 9.9$, $p < .001$, $\eta_p^2 = .24$. Moreover, NOTO fractions also yielded significantly higher accuracy than decimals, 85% vs. 73%, $F(1, 63) = 6.802$, $MSE = 9.9$, $p = 0.01$, $\eta_p^2 = .10$. There was a trend toward greater accuracy for OTO fractions compared to NOTO fractions, 94% vs. 85%, $F(1, 63) = 3.60$, $MSE = 9.9$, $p = .06$, $\eta_p^2 = .05$.

In contrast to Experiment 4A, the interaction between relation type and symbolic notation was not reliable in Experiment 4B, $F(2, 63) = .68$, $MSE = 1.2$, $p = .51$, $\eta_p^2 = .02$. However, there was again a significant effect of relation type, $F(1, 63) = 4.11$, $MSE = 1.2$, $p =.047$, $\eta_p^2 = .06$. Mean accuracy was 86% for PPR problems and 82% for PWR problems.

Figure 3.14 shows the pattern of RTs across conditions. As in Experiment 4A, there was no reliable interaction, $F(2, 63) = 2.30$, $MSE = 7983$, $p = .11$, $\eta_p^2 = .07$, or main effect for relation type, $F(1, 63) = 0.95$, $MSE = 7983$, $p = 0.33$, $\eta_p^2 = .02$; however, there was a significant effect of symbolic notation, $F(2, 63) = 7.59$, $MSE = 132138$, $p = .001$, $\eta_p^2 .19$. Planned comparisons showed that there was no significant RT difference between NOTO fractions and

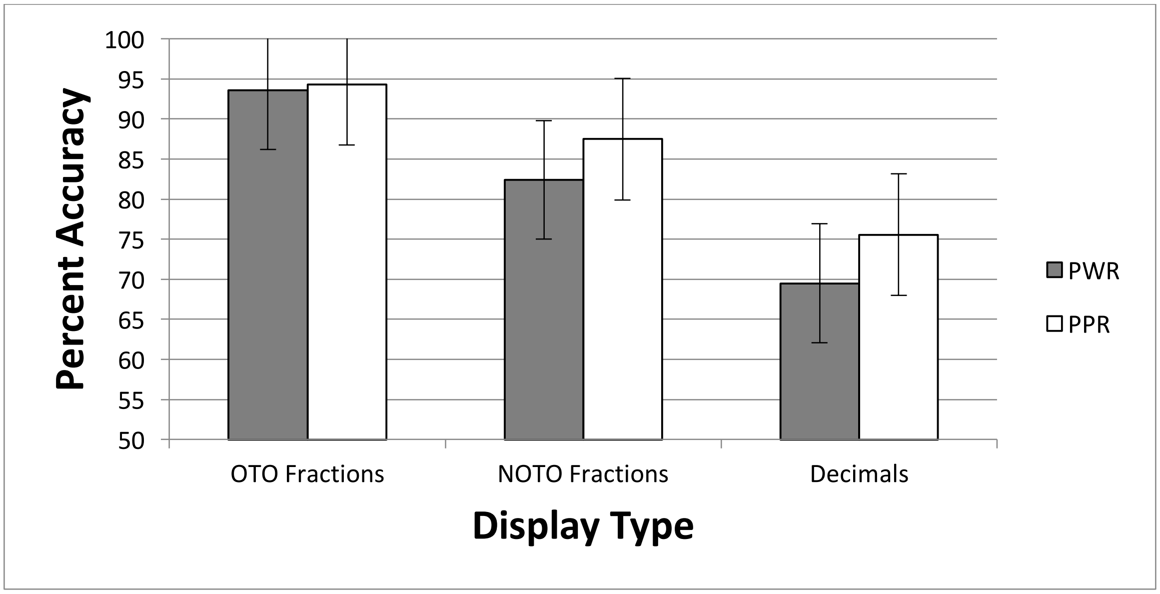


Figure 3.13. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 4B). Error bars indicate standard error of the mean.

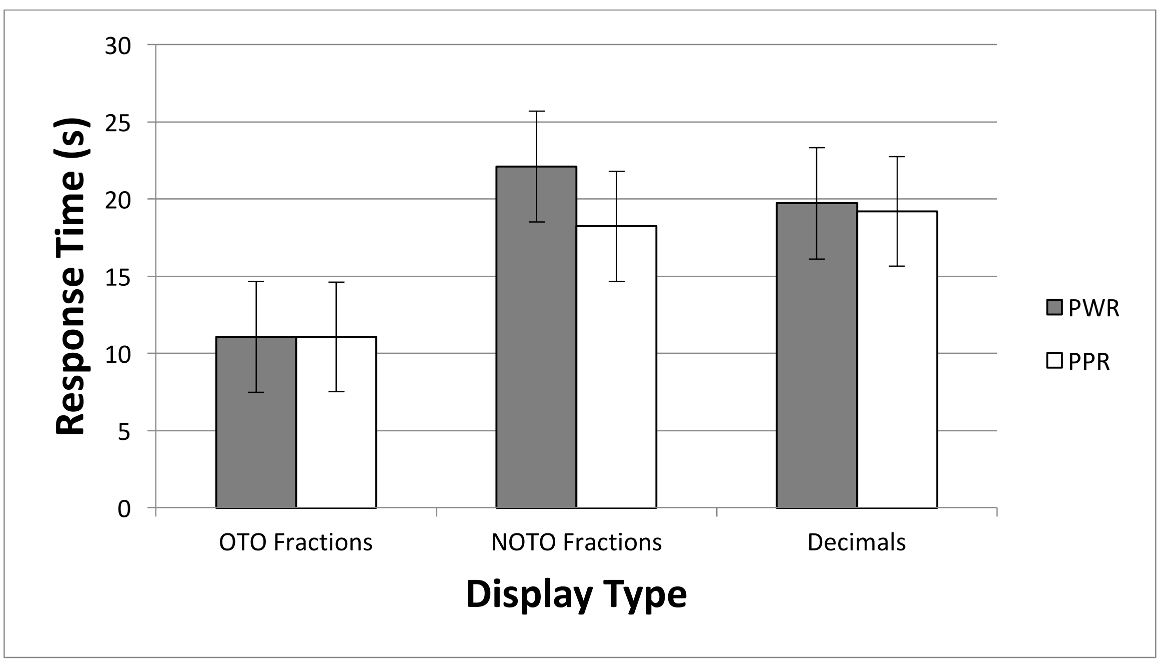


Figure 3.14. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 4B). Error bars indicate standard error of the mean.

decimals, 20.17 s vs. 19.47 s, $F(1, 63) = 0.01$, $MSE = 264737$, $p = .91$, $\eta_p^2 = 0$. However, OTO fractions yielded faster RTs than NOTO fractions, 11.06 s vs. 20.17 s ($F(1, 63) = 11.02$, $MSE = 264737$, $p = .002$, $\eta_p^2 = .15$, and also decimals, 11.06 s vs. 19.46 s, $F(1, 63) = 11.75$, $MSE = 264737$, $p = .001$, $\eta_p^2 = .16$. There was no reliable difference in RTs between PPR and PWR problems, 16.82 s vs. 16.33 s, $F(1, 63) = .95$, $MSE = 79832$, $\eta_p^2 = .02$, and no interaction between relation type and number type, $F(2, 62) = 2.30$, $p = .11$, $\eta_p^2 = .07$.

The results of Experiment 4B demonstrated that even when participants solve analogy problems without any speed pressure, and RTs for the two critical conditions are quite closely matched, accuracy is higher for NOTO fractions than for decimals. Accuracy did not significantly differ between NOTO and OTO fractions, indicating that the fraction advantage is not dependent on whether they were reduced or not. It appears that decimals simply do not align well with ratios defined over discrete visual quantities; thus allowing extra time does not eliminate their disadvantage in accuracy. Fractions with a numerator and denominator that are proportional but not equal to the depicted quantities (NOTO fractions) add extra processing time relative to fractions that have a one-to-one (OTO) mapping. But in the absence of speed pressure, even NOTO fractions yield greater accuracy in analogical reasoning with discrete displays than do decimals.

General Discussion

Summary

The present study, to the best of our knowledge, provides the first evidence that the internal structure of an individual number can provide a model of relations in the external environment, thereby altering performance in tasks that require reasoning about these relations.

Specifically, we tested the hypotheses that fractions are conceptually linked to countable discrete entities, and naturally express a two-dimensional relationship between the cardinal values of sets; whereas decimals are conceptually linked to continuous masses, and more naturally express the relative magnitude of a proportional relation. Previous work has shown that although decimals and fractions can express equivalent magnitudes (subject to rounding error on decimals), the one-dimensional nature of decimals is in fact advantageous in representing magnitudes (DeWolf et al., 2014). Here, we found that the bipartite notation of fractions is advantageous in reasoning about ratio relations (either part-to-whole or part-to-part).

Our experiments used visual displays showing either two countable subsets or two parts of a continuous area. Experiment 1 demonstrated that college-educated adults prefer to use fractions to represent ratio relations between countable sets, and prefer to use decimals to represent ratio relations of continuous masses, in a task that does not require any mathematical procedures. Hence, these findings indicate that the selective affinity of fractions with discrete entities and decimals with continuous entities has a conceptual basis.

Experiments 2-4 examined relational reasoning with fractions and decimals when procedural computations were required. The bipartite structure of fractions (*a/b*) invites counting the size of two separate sets, whereas the decimal notation invites an estimate of the one-dimensional magnitude of a ratio—a procedure that does not depend on discrete elements, but that is less accurate than counting discrete elements. The results of Experiments 2-4 demonstrated an overall advantage for fractions over decimals in relational tasks based on ratios—both relation identification (Experiment 2) and higher-order analogical reasoning (Experiments 3-4). However, this advantage was moderated by the nature of the depicted quantities. Fractions allowed more accurate relational reasoning when the depicted quantities were discrete elements, or continuous quantities that had been discretized by introducing units suitable for measurement. This fraction advantage reflects the fact that fractions align well with discrete quantities, which in turn support exact calculation procedures, such as counting. Fractions may still encourage counting (by mental "slicing" into units) even for continuous displays (as suggested by the regression analyses performed in Experiment 2), though accuracy is reduced. Performance with decimals was relatively equal, and less accurate, for all quantity types, suggesting that decimals are preferentially evaluated using estimation rather than counting. Although decimals naturally align with continuous quantities and fractions do not (as shown in Experiment 1), decimals support less accurate computational procedures in these relational tasks, and hence are no more effective than fractions when such tasks require performing computations on continuous quantities.

Alternative Interpretations

Several "deflationary" accounts of the present findings deserve consideration. It might be argued that the observed differences in reasoning with fractions versus decimals are simply another example of the general phenomenon that alternative notation systems for number provide dramatically different algorithmic affordances (e.g., computing 73 x 27 using Arabic numerals is considerably easier than LXXIII x XXVII using Roman numerals; see Zhang & Norman, 1995, for a general analysis of number systems used in different cultures). However, the differences in the affordances offered by fractions and decimals do not involve comparisons between notations drawn from different cultures and historical periods. Rather, fractions and decimals are both familiar number types defined within the Arabic system, in common use throughout the world today. The present findings show that even within the basic number system in near-universal use in the modern world, different number formats vary in their affordances for both calculation and reasoning.

It also might be argued that our findings simply show that discrete quantities elicit counting, a procedure associated with fractions, whereas continuous quantities elicit magnitude estimation, which is associated with decimals. This is indeed a reasonable summary of much of our findings, but we believe the empirical phenomena are more meaningful when placed in a theoretical framework based on the properties of symbolic notations as models. First of all, it is by no means obvious *a priori* that fractions are necessarily associated with discrete representations. Fractions might be interpreted as holistic numbers (Schneider & Siegler, 2010), providing a possible basis for them to align with continuous quantities. More generally, the primary focus of psychological research on numerical cognition in recent years has been on magnitude representation, the common property associated with all number concepts. However, the present findings do not show evidence that adult process fractions holistically. Rather, componential processing of the magnitudes associated with the numerator and denominator appeared to be sufficient to identify the relations, and thus seemed to provide the preferred strategy. Further, adults did not seem to adopt a holistic strategy even for continuous displays, as demonstrated by the poorer performance for fractions with such displays.

The framework we have presented here serves to call attention to another basic property of numerical systems—the representation of quantitative relations and procedures for reasoning with them. An analysis of the internal structure of rational numbers makes it clear why fractions are especially suited for reasoning about relations between the cardinality of sets, whereas decimals are better-suited for magnitude comparisons. We suspect that the relational structure of fractions is closely linked to the acquisition of more complex relational concepts involved in algebra, and hence has important implications for instruction (as elaborated below).

The high-level summary of our findings also misses important nuances in our data. In particular, Experiment 1 showed that adults preferentially associate discrete displays with fractions, and continuous displays with decimals, even when the task does not require any kind of computation. Thus, the conceptual linkage between fractions and decimals with discrete and continuous quantities, respectively, holds even when neither counting nor magnitude estimation is required. (Below we discuss implications of our framework for reasoning tasks involving natural frequencies, which also do not require counting.) Indeed, the pattern of results changed in an important way in the later experiments, for which computation was critical to perform the task. In these reasoning tasks people showed an advantage for fractions with discrete displays, but relatively equal (and poorer) performance for both symbol types with continuous displays. The lack of a decimal advantage for continuous displays when computation is required can best be understood by taking account not only of the preferred associations between symbol types and procedures, but also of the relative precision of those procedures. Finally, Experiments 4A and 4B showed that the advantage of fractions over decimals for discrete displays is obtained even when the numerator and denominator of the fraction do not match the cardinality of the relevant sets, as long as the proportional relation is maintained. Thus, the strategy underlying the fraction advantage is more complex than a simple “count and match”.

Natural Frequencies as Fraction-like Representations

The theoretical framework we have presented has the advantage of connecting work on numerical cognition with other research areas related to quantitative reasoning. In particular, the present evidence that fractions can function as relational models provides insight into other findings indicating that people's accuracy in relational reasoning depends on the format of rational numbers. For example, studies of tasks that require Bayesian inference (in particular, integration of base rates with likelihoods) have consistently found an advantage for natural frequency formats over probabilities, percentages, and other formats that have been "simplified" by removing or standardizing the size of the specified population (Gigerenzer & Hoffrage, 1995; Hoffrage et al., 2002; Tversky & Kahneman, 1983). For example, observing that 40 of 1000 people have a certain disease can be summarized as a natural frequency, 40/1000. As a type of fraction, the numerator and denominator in this bipartite frequency format align with the sizes of the subset and the population, respectively. In the terminology of the present paper, a natural frequency is a fraction with a one-to-one relationship to a part-to-whole ratio (a symbolic notation tested in all our reported experiments). In contrast, reduced expressions equivalent in magnitude, such as a decimal representing the proportion (0.04), or a percentage (4%), are one-dimensional, expressing the magnitude of the natural frequency but not its internal structure (Halford et al., 1998, 2010). Moreover, as Hoffrage et al. (2002) correctly emphasized, relative frequencies that use a standardized denominator, such as 100, also lose information about the size of the population. For example, although the natural frequency 40/1000 is equivalent in magnitude to the standardized relative frequency 4/100, the denominator in the latter is a constant rather than a variable, which means that the dimensionality of the structure has been reduced from two to one. Though the magnitude has been preserved, the relational structure has been obscured.

Based on a literature review, Hoffrage et al. (2002) report that Bayesian inference tends to be more accurate with natural frequencies than with any of the various one-dimensional notations (including standardized relative frequencies) that express equivalent magnitudes.

Importantly, the frequencies in these tasks were directly stated to participants as summary statistics; no counting or estimation was required. Gigerenzer and Hoffrage (1995) interpret the natural frequency advantage in terms of the evolutionary origins of frequency information in animal activities such as foraging. However, since all explicit number notations are human cultural artifacts, the connection to biological evolution would appear to be tenuous (see Barbey & Sloman, 2007, for discussion of alternative hypotheses). From our perspective, the proximal basis for the advantage of natural frequencies in supporting certain types of inferences is simply that their format affords closer alignment between the mathematical model and the structure of relevant relations in the world (as was also the case for the relational reasoning tasks investigated in the present study). This interpretation is consistent with the hypothesis that natural frequencies are particularly good cues to the structure of problems involving nested sets (e.g., “10 out of 1000 smokers in the sample developed lung cancer;” see Tversky & Kahneman, 1983; Barbey & Sloman, 2007). For different reasoning tasks (notably, magnitude comparison) where the set structure is not critical (and indeed may be a distraction), we would expect other formats (e.g., proportions) to be more effective than natural frequencies (for an example, see Over, 2007). Thus, although the present study involved reasoning with simplified visual displays devoid of meaning, our findings have implications for real-world reasoning tasks involving relations between the cardinalities of sets.

Approximate Estimation Versus Exact Calculation

The present findings are consistent with the hypothesis that decimals invite an approximate strategy for estimating magnitudes of ratios (Jacob et al., 2012). For continuous quantities, this estimation strategy is about as effective as the best strategy available for fractions. However, for discrete displays that enable exact calculations, fractions support a more accurate strategy. When the numbers forming the fraction match the quantities in the display, decisions based on fractions were more accurate than those based on decimals, with no cost in response time. Decisions for fractions were slower when their constituent numbers were not in one-to-one correspondence with the displayed quantities (NOTO fractions; Experiment 4A). However, NOTO fractions (which preserve proportionality though not equality with the displayed quantities) still yielded greater accuracy than decimals, even when time pressure was eliminated and response times equated (Experiment 4B). The fraction advantage thus extends beyond the benefit of a direct match between the numbers in the fraction and the numbers of entities in the displays. Rather, fractions have a more basic advantage attributable to the ease of aligning their bipartite structure with the natural situation model. As long as the depicted quantities are countable, fractions enable use of a more precise computation strategy, yielding greater accuracy than the estimation strategy encouraged by decimals.

It should be emphasized that the accuracy advantage afforded by exact computation strategies depends on acquiring competence in the necessary computation. For our college-student participants, exact computation of relatively small discrete quantities was presumably a well-learned skill. But because exact computation is acquired later than the more elementary approximation strategies, young children may make less accurate judgments when quantities are discretized rather than continuous (Boyer et al., 2008). The paradigms introduced in the present study may provide useful diagnostic tools for tracking developmental changes in reasoning with rational numbers

The present findings lend support to the hypothesis that fractions are not automatically processed as a holistic magnitude. In the reasoning tasks we assessed, for which holistic processing is not beneficial, participants were able to perform well using componential processing of fractions. Previous research on magnitude representations of fractions has shown that fractions can indeed be processed holistically (Schneider & Siegler, 2010), but this process in not automatic (DeWolf et al., 2014) and is not necessarily preferred over componential processing (Bonato et al., 2007). Our results support the hypothesis that the processing of fractions is task-dependent. Adults often seem to be able to adopt whatever strategy for processing fractions is most suitable for the task at hand.

In many instances of mathematical problem solving, it is not important for students to actually calculate or find a magnitude in order to solve a problem. Rather, simply understanding the relation is sufficient, and sometimes can be more useful and adaptive. For example, when deciding which is larger, $\frac{x}{5}$ vs. $\frac{x}{6}$, knowing the actual magnitude of *x* is impossible. To find the correct answer, it is sufficient to note that the same number, *x,* is divided either by a larger number, 6, or a smaller number, 5. If one knows that division by larger numbers results in smaller numbers (assuming *x* is positive), it follows that $\frac{x}{5}$ is larger than $\frac{x}{6}$. Thus the answer can be inferred by using relational knowledge about relative sizes, division, and equivalence, without any direct assessment of magnitudes. Similarly, if one knows that $m \times \frac{3}{5} = n$, one can infer that $n \times \frac{5}{3} = m$ without any information about the magnitudes of *m* and *n*, simply by understanding multiplication and reciprocal relations. For higher levels of mathematics, such as algebra and calculus, problems are often solved not by identifying specific magnitudes, but by understanding mathematical relationships.

Implications for Teaching Rational Numbers

The present findings, in conjunction with previous research on processing of fractions and decimals, may have implications for how these types of symbolic notations could be most effectively taught. The present analysis and findings shed light on why fractions are especially difficult for students to learn (Siegler et al., 2013; Ni & Zhou, 2005; Stigler et al., 2010; Vamvakoussi & Vosniadou, 2010), and also are especially important predictors of success in acquiring more advanced mathematical knowledge in high school (Siegler et al., 2011, 2012). In the United States, fractions are generally the first number type introduced in school after the familiar natural numbers. Whereas natural numbers fundamentally express a unidimensional magnitude, fractions are two-dimensional. Not only is their bipartite format unfamiliar, but their conceptual structure is inherently more complex than that of natural numbers, placing greater demands on working memory (English & Halford, 1995; Halford et al., 1998, 2010). Although fractions indeed represent magnitudes, magnitudes do not fully capture their meaning. Elementary-school students (who are likely to be unevenly developed with respect to working memory capacity) may have difficulty grasping the two-dimensional structure of fractions, particularly if instruction focuses on their magnitudes rather than on their relational meaning. But in many reasoning tasks, including those investigated in the present paper, the two-dimensional structure of fractions can in fact be exploited to advantage. Although fractions are relatively inefficient as representations of magnitudes, they can be very effective as representations of relations.

Fractions thus have a dual status that poses particular challenges for students: a fraction is at once a relationship between two quantities, expressed as a/b, and also the magnitude corresponding to the division of a by b. Gray and Tall (1994) have argued that children's understanding of arithmetic is dependent on their "proceptual" understanding: grasping that a mathematical expression containing an arithmetic operation embodies the process of obtaining a certain result (similar to the "process-product dilemma" in algebra discussed by Sfard & Linchevski, 1994). Gray and Tall (1994) found that students who become proficient in arithmetic at an earlier age show greater ability to move back and forth flexibly between an arithmetic expression and the result of that expression. The difficulty in understanding a fraction as a relational expression may explain why young children appear to understand quantitative relations such as part-whole or proportions with visual displays (Goswami, 1989; Mix, Levine & Huttenlocher; Boyer, Levine & Huttenlocher, 2008; Sophian, 2000), but not when they have to answer comparable questions with fraction notation (Ball & Wilson, 1996; Mack, 1995). For example, Ni and Zhou (2005) found that most children could answer the question, "How much is one third plus one third?" verbally as "two thirds". But when asked the question using symbolic fraction notation, "1/3 + 1/3 = ?", most children answered "2/6" (often claiming that both 2/6 and 2/3 are correct answers). Children thus seem to have some intuitive understanding of how rational quantities relate to one another, but have difficulty understanding the novel symbolic expressions.

It is therefore important to convey the dual nature of fractions to students, with a focus on their relational structure. The multidimensional structure of numbers eventually proves to be critical in understanding even the simplest algebraic expressions, such as $\frac{2}{3}x$. Thus, those students who eventually succeed in grasping the very concept of a multidimensional number, first instantiated by common fractions, will have mastered a fundamental prerequisite for advanced mathematics.

The conceptualization of fractions as relations such as part/whole, subset/set, ratio and proportions may have implications for how children are able to solve problems using fractions in these contexts. Typically, school instruction in the United States only emphasizes the part-to-whole relationship (Sophian, 2007; Mack, 1993), which most clearly relates to the understanding of fractions as magnitudes. Children are first introduced to fractions using pictorial representations intended to help students understand the meaning of a value smaller than one. As we observed earlier, magnitudes are far easier to process using one-dimensional decimals than bipartite fractions, even for adults (DeWolf et al., 2014). Thus, it might well be easier for children to learn about magnitudes less than one by being introduced to decimals prior to fractions. Fractions might be taught later than decimals, with an emphasis on their status as a *relationship* between two natural numbers and that multiple possible relationships can be of interest. For example, teaching students about the part-to-part relation in addition to the part-to-whole relation might help to expand children's understanding of fractions. This more general understanding might in turn aid students in eventually learning more abstract mathematics, such as algebra.

In fact, Moss and Case (1999) implemented a curriculum with 4th graders in Canada that reorganized the usual order of instruction in rational numbers. Children were first taught percentages (in the context of volumes, and on number lines), then decimals, and lastly fractions. Fractions were explained simply as another way to represent a decimal. By contrast, typical curricula describe teaching decimals as another way to represent a fraction. Moss and Case found that children taught the symbolic notations in this novel sequence suffered less interference from whole-number strategies when using other rational numbers, and achieved a deeper understanding of them. This approach to teaching rational numbers is supported by our framework, as this instructional method encourages students to first master the concept of rational-number magnitude with the symbolic notation that more effectively represents such magnitude: decimals. Though Moss and Case did not emphasize the role of fractions in the types of relational contexts we have discussed here, this type of curriculum could be used to allow students to master decimal magnitudes and then later learn about fractions in the context of modeling types of relations. Thus, a major issue that arises for students using current curricula in the U.S.—understanding fraction magnitudes—might be deemphasized in this context.

More generally, understanding how types of rational numbers align to specific types of quantities, and how the internal structure of mathematical expressions can affect ease of alignment to the perceptual or semantic relations being modeled, has important implications for how to best conceptualize and teach fractions and decimals. It is important to foster understanding of mathematics in a way that goes beyond teaching algorithmic procedures. A mathematical expression can represent not just a procedure, and not just a magnitude, but also a relational structure that maps onto the structure of the world.

Footnotes

1. Throughout this paper we will refer to common fractions (i.e., numbers formed by a ratio of integers) simply as “fractions”.

2. At a deeper level of analysis, the two number types are not representationally equivalent. Fractions represent rational numbers, whereas unbounded decimals represent the real numbers, of which the rational numbers are a subset. If decimals are bounded, they cannot exactly capture the magnitude of all real (or rational) numbers (e.g., 1/3); if unbounded, they also capture the irrational numbers (e.g., 🚔), which have no exact fraction equivalent. Experimental work has for obvious reasons only used bounded decimals, the magnitudes of which at least closely approximate the magnitudes of matched fractions. However, the fundamental definition of common fractions (but not decimals) in terms of the cardinality of sets demonstrates that the two formats represent conceptually distinct number types, rather than simply alternative notations or typographical conventions.

3. The term “fraction” is most commonly applied to part-whole or subset-set relations, which we term *part-to-whole* ratios. The term “ratio” is more commonly used for subset-subset relations. In the present paper we refer to the latter as *part-to-part* ratios, and use the term “fraction” to embrace both. Similarly, we use the standard fraction notation (*a/b*) to represent ratios (more commonly notated *a:b*). In short, we use the familiar term “fraction” and its notation to cover a variety of bipartite structures based on a division relation.

References

Ball, D., & Wilson, S. M. (1996). Integrity in teaching: Recognizing the fusion of the moral and intellectual. *American Educational Research Journal, 33*(1), 155–192.

Barbey, A. K., & Sloman, S. A. (2007). Base-rate respect: From ecological rationality to dual processes. *Behavioral and Brain Sciences*, *30*, 241-297.

Bassok, M. (2001). Semantic alignments in mathematical word problems. In D. Gentner, K.J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 401 – 433). Cambridge, MA: MIT Press.

Bassok, M., Chase, V. M., & Martin, S. A. (1998). Adding apples and oranges: Alignment of semantic and formal knowledge. *Cognitive Psychology, 35,* 99-134.

Bassok, M., Pedigo, S. F., & Oskarsson, A. T. (2008). Priming addition facts with semantic relations. *Journal of Experimental Psychology: Learning Memory and Cognition, 34*, 343-352.

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance*, *33*, 1410–1419.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental psychology*, *44*, 1478–90.

Braithewaite, D. W., & Goldstone, R. L. (2013). Integrating formal and grounded representations in combinatorics learning. *Journal of Educational Psychology, 105*(3), 666-682.

Brankaer, C., Ghesquière, P., & De Smedt, B. (2014). Children's mapping between non-symbolic and symbolic numerical magnitudes and its association with timed and untimed tests of mathematics achievement. *PlOS ONE*, *9*(4), e93565.

Cantlon, J., Brannon, E. M., Carter, E., & Pelphrey, K. (2006). Functional imaging of numerical processing in adults and 4-yr-old children. *PLoS Biology, 4*, e125, 1-11.

Cedrus Corp. (2004). Superlab 4.0 [Computer software]. San Pedro, CA.

Chen, D., Lu, H., & Holyoak, K. J. (2014). The discovery and comparison of symbolic magnitudes. *Cognitive Psychology*, *71*, 27-54.

Clearfield, M. W., & Mix, K. S. (2001). Infants use continuous quantity—not number—to discriminate small visual sets. *Journal of Cognition and Development, 2*, 243-260.

Cohen, D. J., Ferrell, J. M., & Johnson, N. (2002). What very small numbers mean. *Journal of Experimental Psychology: General*, *131*(3), 424–442.

Common Core State Standards Initiative (2014). http://www.corestandards.org/Math/Content/4/NF/ (retrieved September 23, 2014).

Cordes, S., & Gelman, R. (2005). The young numerical mind: What does it count? In J. Campbell (Ed), *Handbook of mathematical cognition* (pp. 128-142). Hove, UK: Psychology Press.

Dehaene, S. (1997). *The number sense*. New York: Oxford University Press.

Dehaene, S., & Changeux, J.-P. (1993). Development of elementary numerical abilities: A neuronal model. *Journal of Cognitive Neuroscience, 5*, 390-407.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science*, *320*(5880), 1217-1220.

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance, 40,* 71 – 82.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes.* Mahwah, NJ: Erlbaum.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' discrimination of number vs. continuous extent. *Cognitive Psychology*, *44*, 33-66.

Feigenson, L., Dehaene, S. & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences*, *8*, 307 - 314.

Fias, W., & Verguts, T. (2008). Not all basic number representations are analog: Place coding as a precursor of the natural number system. *Behavioral and Brain Sciences*, *31*, 650-651.

Fisher, K. J., Borchert, K., & Bassok, M. (2011). Following the standard form: Effects of equation format on algebraic modeling. *Memory & Cognition, 39*(3), 502-515.

Gelman, R. (1993). A rational-constructivist account of early learning about numbers and objects. In D. L. Medin (Ed.), *Learning and motivation*, Vol. 30 (pp. 61-96). New York: Academic Press.

Gigerenzer, G., & Hoffrage, U. (1995). How to improve Bayesian reasoning without instruction: Frequency formats. *Psychological Review*, *102*, 684-704.

Goldstone, R. L., & Barsalou, L. W. (1998). Reuniting perception and conception. *Cognition*, *65*(2), 231-262.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*(2), 265-284.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). *Highlights from TIMSS 2007:Mathematics and science achievement of U.S. fourth- and eighth-grade students in an international context* (NCES 2009–001). Washington, D.C.: National Center for Education Statistics, U.S. Department of Education.

Goswami, U. (1989). Relational complexity and the development of analogical reasoning. *Cognitive Development*, *96*, 229–241.

Gray, E. M, & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115 - 141.

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology, 44,* 1457-1465.

Halford, G. S., Wilson, W. H. & Phillips, W. (1998). Processing capacity defined by relational complexity: Implications for comparative, developmental and cognitive psychology. *Behavioral Brain Sciences, 21*(6), 803-831.

Halford, G. S., Wilson, W. H., & Phillips, S. (2010). Relational knowledge: The foundation of higher cognition. *Trends in Cognitive Sciences, 14*(11), 497-505.

Hinsley, D., Hayes, J. R., & Simon, H. A. (1977). From words to equations: Meaning and representation in algebra word problems. In P. A. Carpenter & M. A. Just (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Erlbaum.

Hoffrage, U., Gigerenzer, G., Krauss, S., & Martignon, L. (2002). Representation facilitates reasoning: What natural frequencies are and what they are not. *Cognition*, *84*, 343-352.

Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology*, *103*(1), 17-29.

Iuculano, T., & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology, 64,* 2088-2098.

Izard, V., & Dehaene, S. (2008). Calibrating the mental number line. *Cognition*, *106*(3), 1221-1247.

Jacob, S. N., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: The brain's code for proportions. *Trends in Cognitive Sciences, 16*(3), 157-166.

Kallai, A. Y., & Tzelgov, J. (2009). A generalized fraction: An entity smaller than one on the mental number line. *Journal of Experimental Psychology: Human Perception and Performance, 35*, 1845-1864

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational number. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review*, *92*, 109-129.

Landy, D., Brookes, D., & Smout, R. (2014) Abstract numeric relations and the visual structure of algebra. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 40*(3), 797-810.

Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought? *Journal of Experimental Psychology: Learning, Memory, and Cognition, 33*, 720-733.

Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85-105). Mahwah, NJ: Erlbaum.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422–441.

Martin, S. A., & Bassok, M. (2005). Effects of semantic cues on mathematical modeling: Evidence from word-problem solving and equation construction tasks. *Memory & Cognition*, *33*, 471-478.

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*, 164–174.

Mochon, D., & Sloman, S. A. (2004). Causal models frame interpretation of mathematical equations. *Psychonomic Bulletin and Review*, *11*, 1099-1104.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*, 122–147.

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature*, *215*, 1519-1.520.

Mundy, E., & Gilmore, C. K. (2009). Children's mapping between symbolic and nonsymbolic representations of number. *Journal of Experimental Child Psychology*, *103*(4), 490-502.

Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual Review of Neuroscience, 32,* 185-208.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of the whole number bias. *Educational Psychologist*, *40*, 27–52.

Nunes, T., Light, P., Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction, 3,* 39-54

Over, D. E. (2007). Content-independent conditional inference. In M. J. Roberts (Ed.), *Domain general versus domain specific processes in higher cognition* (pp. 83-103). Hove, UK: Psychology Press.

Pinel, P., Piazza, M., Bihan, D. L., & Dehaene, S. (2004). Distributed and overlapping cerebral representations of number, size, and luminance during comparative judgments. *Neuron*, *41*, 1-20.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals, *Journal of Experimental Psychology: Applied, 21*(1), 47-56.

Richland, L. E., Stigler, J. W., Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist*, *47*, 189-203.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93, 346–362.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99,* 561-574.

Sasanguie, D., Göbel, S. M., Moll, K., Smets, K., & Reynvoet, B. (2013). Approximate number sense, symbolic number processing, or number–space mappings: What underlies mathematics achievement? *Journal of Experimental Child Psychology*, *114*(3), 418-431.

Schneider, M., & Siegler, R.S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, *36*, 1227-1238.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26,* 191-228.

Siegler, R. S., Duncan, G. J., Davis-Kean, P.E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*, 691- 697.

Siegler, R. S., Fazio L.K., Bailey, D.H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*, 13-19.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*, 273-296.

Sophian, C. (2007). *The origins of mathematical knowledge.* Mahwah, NJ: Erlbaum.

Stigler, J. W., Givvin, K. B., & Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *The MathAMATYC Educator, 10*, 4-16.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Sophian, C. (2000). Perceptions of proportionality in young children: Matching spatial ratios. *Cognition, 75*, 145–170.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review*, *90*, 293-315.

Vamvakoussi, X., & Vosniadou, S. (2010). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*, 453- 467.

Zahner, D., & Corter, J. E. (2010). The process of probability problem solving: Use of external visual representations. *Mathematical Thinking and Learning, 12*(1), 177-204.

Zhang, J., & Norman, D. A. (1995). A representational analysis of numeration systems. *Cognition*, *57*, 271-295.

CHAPTER 4:

FROM RATIONAL NUMBERS TO ALGEBRA: SEPARABLE CONTRIBUTIONS OF DECIMAL MAGNITUDE AND RELATIONAL UNDERSTANDING OF FRACTIONS

Introduction

Given the well-documented difficulties that American students often experience learning algebra and more advanced topics in mathematics (Gonzales et al., 2008; Richland, Stigler & Holyoak, 2012; Smith & Thompson, 2007), it is important to identify those aspects of earlier mathematics that predict success or failure on advanced topics. Decomposing the prerequisites for success at algebra can potentially guide theoretical analyses of the mental representation of mathematics, and also aid in developing more effective instructional strategies. Recent work suggests that knowledge of rational numbers, notably fractions, is closely linked to later success in mathematics. For example, in a large sample of students from the United States and United Kingdom, Siegler et al. (2012) found that fraction knowledge, measured by basic arithmetic and conceptual questions, predicted algebra knowledge and math achievement in general at age 16 (beyond what could be predicted from knowledge of whole numbers). Other researchers have also found significant connections between fraction knowledge (including basic conceptual knowledge and performance on tasks that assess grasp of fraction magnitude) and algebra performance (Booth & Newton, 2012; Booth, Newton & Twiss-Garrity, 2014; Brown & Quinn, 2007; Empson & Levi, 2011).

Though there seems to be an important link between fraction understanding and algebra performance, the nature of this link has yet to be firmly established. Wu (2009) and Siegler et al. (2011) have emphasized the fact that a fraction, like any other type of number, can be placed on a number line. This understanding requires an integration of procedural and conceptual knowledge about fractions and magnitudes. Two recent studies by Booth and colleagues (Booth & Newton, 2012; Booth et al., 2014) have provided support for this hypothesis. Among middle-school students taking elementary algebra classes, a significant correlation was observed between performance on a task requiring estimates of the positions of fractions on a number line, and a subsequent algebra test that included problems requiring solving problems, knowledge of critical features in algebraic equations, and coding of equations. Number-line estimation with fractions was a stronger predictor of algebra performance than declarative fraction knowledge, a measure of procedural knowledge of how to use fractions in equations, or number-line estimation with whole numbers. These findings raise the possibility that a key link between knowledge of rational numbers and algebra performance may involve understanding of fraction magnitudes.

Although understanding of magnitudes is without question a core aspect of mathematical knowledge, there are good reasons to believe that understanding of mathematical *relations* is also critical in grasping algebra. For example, in the algebraic expression $x = 4y$, the value of the variable x is expressed in relation to that of y without any specific magnitude being assigned to either. In recent work (DeWolf, Bassok & Holyoak, 2015; Rapp, Bassok, DeWolf, & Holyoak, 2015), we have emphasized that fractions, with their bipartite a/b structure, naturally convey the relations between the numerator and the denominator (most typically, two countable sets). Of course, a fraction also represents the magnitude that corresponds to the division of a by b. This duality in the roles of fractions as mathematical representations of relations and magnitudes is similar to the duality of algebraic expressions (Sfard & Linchevski, 1994). Students must understand that they can use algebraic expressions to represent both the relations between quantities and also the process used to find an unknown quantity. For example, the quantity of 4 boxes of equal weight can be represented as $4w$, without knowing the actual magnitude of a box's weight. The expression 4*w* represents the combined weight of the boxes and the process (multiplication) that could be used to determine the total weight given the actual weight of one box. Thus, students' conceptual understanding of fractions as representing both relations and magnitudes may be an important precursor for their subsequent understanding of algebraic expressions.

Interestingly, whereas fractions represent both relations and numerical magnitudes, magnitude-equivalent decimals lose the relational structure inherent in a fraction and more directly express one-dimensional magnitude. Studies have shown that magnitude comparisons can be made much more quickly and accurately with decimals than with fractions (Iuculano & Butterworth, 2011; DeWolf, Grounds, Bassok & Holyoak, 2014), but that fractions are more effective than decimals in tasks such as relation identification or analogical reasoning, for which relational information is paramount (DeWolf et al., 2015). Since fractions are the first numbers with an internal relational structure that students are taught, understanding of fractions as relations may be a key predictor of early algebra success. Insofar as understanding magnitudes is also important for grasping algebra, magnitude tasks involving decimals (which express magnitudes more directly than do fractions) may be more predictive than are magnitude tasks based on fractions.

At present, no study has assessed tasks involving fractions as well as tasks involving decimals as predictors of algebra performance, nor has any study attempted to tease apart relational and magnitude knowledge as predictors. Hence, our aim in the current study was to better distinguish among possible links between specific types of rational-number knowledge and early algebra performance. We extended the general design of the studies by Booth and Newton (2012) and Booth et al. (2014). In addition to assessing magnitude knowledge using number-line tasks with whole numbers and fractions, we also included a similar task with decimals. By comparing the predictive power of magnitude tasks with fractions and with decimals, we sought to determine whether fraction magnitude is a uniquely important predictor of algebra performance, or whether knowledge of rational-number magnitude in general (perhaps better assessed using decimals than fractions) is what is critical.

In addition, we added a measure designed to test students' understanding of fractions as relations. Importantly, performing well on this task did not require calculation of any particular magnitude. If knowledge of fractions predicts algebra performance because of transfer based on the status of fractions as relational expressions, then the relational test may provide a novel and unique predictor of success in algebra.

Method

Participants

All students were enrolled in introductory pre-algebra courses from two suburban Los Angeles schools. A total of 65 seventh-grade middle-school students (mean age 12.4 years; 26 male, 39 female) participated in the study near the end of the school year. Students were from five different classes consisting of students with a substantial range of skill levels.

Measures and Materials

Number-line Estimation Tasks. To measure magnitude knowledge, we adopted a pencil-and-paper number-line estimation task that has been used in many previous studies, including that of Booth et al. (2014). Our aim was to closely replicate the number-line estimation findings of Booth et al. (2014); hence we adopted their three number-line estimation tasks. In addition, we included a fourth decimal number-line task, which was created by translating the fractions used in the fraction estimation task into magnitude equivalent decimals. Thus, in total four scales were used: 0-1,000,000 Whole Numbers (12 trials), 0-62,571 Whole Numbers (12 trials), 0-1 Fractions (18 trials), and 0-1 Decimals (18 trials). The two measures of whole-number magnitude (regular scale of 0-1,000,000 and atypical scale of 0-62,571) were combined to create a composite measure of whole-number magnitude knowledge (as in Booth & Newton, 2012; Booth et al., 2014).

The two whole-number tasks (replications of those used by Booth et al., 2014) were completed with a packet of 8 1/2" x 11" paper with a 20 cm line printed across the middle. The line was marked with 0 at the left end and 1,000,000 (or 62,571) at the right end. On each page, a number was written at the top and students were instructed to put a hatch mark on the line where that number would go. The numbers used were the same as those used by Booth et al. (2014). For the 0-1,000,000 scale, these were: 3123, 7604, 12129, 20394, 85261, 132694, 298237, 358742, 453903, 595246, 724859, and 953271; for the 0-62,571 scale: 19, 44, 176, 1059, 6426, 15023, 21649, 27393, 33691, 42672, 49126, 54705.

The fraction number-line task was identical to the whole-number line tasks, except that the scale of the number line was from 0 to 1 and numbers given were fractions rather than whole numbers. The fractions used were again identical to those used by Booth et al. (2014): 1/360, 1/180, 1/45, 5/118, 1/12, 13/85, 1/5, 3/11, 2/7, 1/3, 83/215, 177/352, 3/5, 5/8, 33/47, 7/9, 5/6, and 146/149.

The decimal number-line task (a new task introduced in the present study) was identical to the fraction number-line task except that each of the fractions used in the fraction number line task were given as their decimal equivalents. The decimals varied in length from 2 to 6 digits, including some decimals with initial or terminating 0's, in order to avoid any perceptual cues that might lead students to interpret decimals using a whole-number strategy. The decimals used were: .00278, .0056, .022, .042373, .08, .1529, .200, .27, .28571, .3333, .386, .3324, .60, .62500, .702128, .78, .83333, .980.

Students were randomly assigned to complete the four scales in one of six possible orders. The trials within each of the scales were randomized. Students were given the same instructions to complete the number-line task as in Booth et al. (2014): "First we're going to work with four sets of number lines. You remember what a number line is, right? A number line is just a line with numbers across that shows us all of the numbers in order. In these number lines, only the numbers at the ends of the line will be marked, but not the ones in between. Your job is going to be to mark where you think some other numbers would go. Above each number line, there will be a number. Whenever you decide where you think the number goes, you need to place a mark on the number line where you think it goes. Make sure to pay attention to what the numbers are on the ends of the number lines because there are three different scales in your packet. Go ahead and work through your packet."

Similar to the procedure used by Booth et al. (2014), the distance between the left endpoint and the student's hatch mark on the number line was measured with millimeters (using a standard ruler). These measurements were then recorded on a computer, and used to compute what the corresponding value would be at that hatch mark on the number-line scale. These values were calculated as (distance from the left endpoint/total distance) x length of number-line scale. For example, if a student placed a hatch mark for 2500 on the 0-1,000,000 line at 40 mm, the corresponding value would be (40/2000) X 1000000 = 200000 (i.e., the student placed 2500 where 200000 should go). These estimated values were plotted against the actual values that were given, and the R^2_{lin} for each participant was calculated. In addition, the percent absolute error (PAE) was also computed for each participant as ((Estimate – Actual)/(Scale)) * 100), following the procedure used by Booth et al. (2014).

Fraction Relations Task. This measure (a novel addition in the present study) consisted of a range of questions that measured students understanding of different relational uses for fractions (see Appendix B). For example, questions measured knowledge of fraction equivalence, division, inverse, multiplying by the reciprocal, and identifying part-to-part ratios vs. part-to-whole ratios in countable sets. These questions were taken from previous published work and research projects designed to investigate conceptual rather than procedural understanding of fractions. (Sources are noted in Appendix B.) Importantly, for all of the problems, there was no need to calculate the magnitude of any fraction. These questions were therefore quite distinct from the pure measure of magnitude understanding provided by the fraction number-line estimation task. The Fraction Relations Task consisted entirely of multiple-choice questions that were scored on a binary basis (0 for incorrect, 1 for correct).

Fraction Procedures Task. An additional measure focusing on knowledge of fraction-related procedures consisted of problems taken from the "Declarative Fraction Knowledge" items used by Booth et al. (2014). These questions were mainly designed to measure understanding of how to manipulate fractions as they appear in equations (see Appendix B). For example, questions included asking students to identify what the next procedural step would be to solve a problem. This measure was designed to control for connections between fractions and algebra that are based on simply knowing how to perform particular procedures that are common between fractions and algebra (e.g., manipulation of fractions, and the division operation). To solve all of these questions, participants must be able to identify correct procedures that are used when algebra problems involve fractions.

The questions assessing knowledge of fraction relations and fraction procedures were intermixed within a single battery, with questions presented to each student in one of six random orders. All of the questions used in the battery are shown in Appendix B.

Algebra Knowledge Task. Algebra knowledge was measured with a variety of questions (see Appendix B). Some of the questions were adapted from Booth et al.'s (2014) measure of algebra knowledge, which included equation solving and "feature knowledge"—i.e., understanding of properties of algebraic equations (e.g., is 4x 🎗 3 equivalent to 3 🎗 4x?). This task also included three algebra word problems and other algebra problems taken from a bank of questions involving understanding and creating algebra expressions used in Algebra 1 courses.[1] Of the 14 total questions, six were in multiple-choice format and were scored on a binary basis (0 for incorrect, 1 for correct). The remaining eight items were equation-solving and word-problem questions, which were scored as correct if the student found the correct value for the missing variable, and wrong otherwise. No partial credit was awarded if the student wrote out the correct procedure but failed to find the correct answer.

Procedure

Students received all of these measures as pencil-and-paper tests in the following order: 1) number-line estimation, 2) fractions (relational and procedural tasks), and 3) algebra knowledge similar to the procedure followed by Booth et al. (2014) and Booth and Newton (2012). The fixed order was used mainly due to practical limitations of controlling the packet completion in a classroom setting. Students were given about 50 minutes to complete all three packets, though most took less than 30 minutes. Students were not allowed to use calculators to complete the problems, and were encouraged to write out their work on the paper.

Results

Distributions of Performance for Each Measure

Multiple regression analyses were performed to identify those tasks that reliably accounted for unique variance in algebra performance. As a prerequisite to these analyses, we examined histograms and standard descriptive statistics of performance for each measure to ensure that all measures showed a reasonable degree of variability across our participants. For the number-line tasks, the pattern of performance was very similar using either R^2_{lin} or Percent Absolute Error (PAE) to assess level of performance. These two measures have often been used interchangeably in previous number-line estimation studies (Booth et al., 2014; Siegler & Booth, 2004). For simplicity we only report results based on PAE. Average PAE scores for both the fraction number line (FNL) and decimal number line (DNL) measures were 15%. Variance for the DNL was slightly higher ($\sigma^2 = 122$) than the variance in performance for FNL ($\sigma^2 = 101$). The average PAE score for the Whole Number Line measure (WNL) was almost identical at 16%, but variance in performance was considerably smaller ($\sigma^2 = 49$). Average proportion correct on Fraction Procedures questions was .57 ($\sigma^2 = .04$). Performance on Fraction Relations questions was slightly lower (mean = .42, $\sigma^2 = .05$). Performance on Algebra questions was also in the same range (mean = .48, $\sigma^2 = .02$). All the measures showed substantial variability in performance across students, satisfying a prerequisite for serving as potential predictors.

Intercorrelations Among All Measures

Table 1 shows the raw correlations between the three number-line measures (WNL, FNL and DNL), the fraction relations and fraction procedures measures, and the algebra measure. Note that correlations of the number-line measures with the other measures are negative, because PAE is a measure of error. As observed by Booth et al. (2014), the PAE for FNL was significantly correlated with algebra performance (r = -.39, p = .001), whereas the PAE for WNL was not (r = -.14, p = .26). In addition, based on the new measures introduced in the present study, we found that the PAE for DNL (r = -.58, p < .001), performance on fraction relations (r = .47, p < .001) and on fractions procedures (r = .33, p = .008) were all significantly correlated with algebra performance. (Again, note that significant negative correlations indicate that higher performance on number-line tasks is associated with higher performance on other tasks.)

Table 4.1. Raw correlations between measures

	Fraction Relations	Fraction Procedures	Fraction Number line	Decimal Number line	Whole Number line
Fraction Procedures	**.38****				
Fraction Number line	**-.38****	-.19			
Decimal Number line	**-.45*****	**-.27***	**.50*****		
Whole Number line	.07	-.07	.08	.03	
Algebra Performance	**.47*****	**.33****	**-.39****	**-.58*****	-.14

** .01 < p < .05*
*** .001 < p < .01*
**** p < .001*

Multiple Regression Analyses

Because several predictor variables showed reliable intercorrelations, we performed a series of multiple regression analyses to distinguish among the different predictors, and to identify those that best accounted for unique components of variance in algebra performance. We first used the three number-line measures as predictors in order to distinguish among these three measures of magnitude processing. The predictors were entered first as FNL, then DNL, then WNL (as Booth et al., 2014, found FNL to be the leading predictor). The overall model accounted for a significant amount of variance in algebra performance ($F(3, 61) = 11.5$, $p < .001$); however, only DNL contributed a significant proportion of variance (*Beta* = -.51, $t(61) = 4.28$, $p < .001$; WNL: *Beta* = -.12, $t(61) = 1.15$, $p = .27$; FNL: *Beta* = -.14, $t(61) = 1.14$, $p = .26$). We repeated the regression analysis using a different order of variables (DNL, then FNL, then WNL) and obtained the same pattern of results.

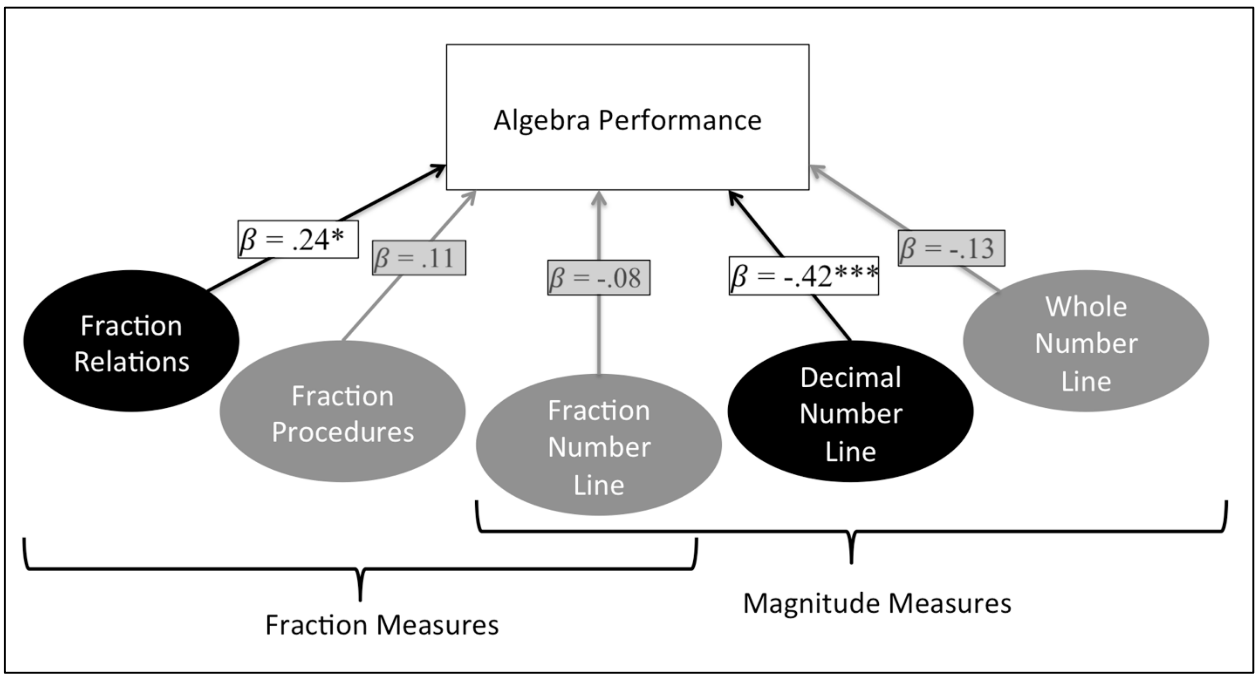


Figure 4.1. Predictive model for algebra performance derived from multiple-regression analyses.

A second set of regressions was then performed, including the Fraction Relations and Fraction Procedures measures in addition to the three number-line estimation measures. The Fraction Relations predictor was inserted first, as we hypothesized this would be a strong predictor of algebra performance. Given that DNL was shown to be the most predictive magnitude measure, we then entered DNL, then FNL, and finally WNL. The last predictor entered was Fraction Procedures, which we did not expect to be closely related to algebra performance. The model, which is depicted in Figure 4.1, accounted for a significant amount of variance in algebra performance ($F(5, 59) = 8.64$, $p < .001$). However, only scores on the DNL (*Beta* = -.42, $t(59) = 3.43$, $p = .001$) and the Fraction Relations task (*Beta* = .24, $t(59) = 2.12$, $p = .03$) accounted for a significant proportion of variance over and above the other three predictors (WNL: *Beta* = -.13, $t(59) = 1.31$, $p = .19$; FNL: *Beta* = -.08, $t(59) = .69$, $p = .50$, Fractions procedures: *Beta* = .11, $t(59) = 1.03$, $p = .31$). We also tested the model by entering the magnitude measures first, followed by Fraction Relations and Fraction Procedures, and obtained the same pattern of results. Thus, among all the predictor measures that were examined, only fractions relations and decimal number-line performance predicted unique components of variance in algebra performance while controlling for the other measures.

Discussion

The present study provides evidence that both understanding of decimal magnitudes, as assessed by a number-line task, and relational understanding of fractions, are strong predictors of algebra performance. We replicated the empirical finding of Booth et al. (2014) and Booth and Newton (2012) that accuracy in number-line estimation with fractions is related to algebra performance (whereas accuracy with whole numbers is not); however, multiple regression analyses revealed that this linkage no longer holds when performance with decimals on the same task is considered. In addition, we show for the first time that a different and distinct aspect of fraction knowledge—a measure of understanding relations involving fractions—adds a unique contribution to predicting algebra performance. Together, measures of decimal magnitude understanding and relational fraction understanding reliably predict early algebra performance.

Many previous studies have focused primarily on number-line estimation and its predictive power. Successful performance on the fraction number-line estimation task appears to involve two components: 1) performing a division operation based on the relation between the numerator and denominator, and 2) approximating the resulting magnitude on the physical line. Our results suggest that these two skills can be distinguished by separate measures. The former can be assessed by a test of relational knowledge with fractions, and the latter by the number-line task using decimals (since decimals obviate the need to perform division). These two specific measures provide more accurate predictors of algebra performance than does the fraction number-line task.

Further research is needed to understand why the number-line task with decimals predicts algebra performance more effectively than the same task with other number types. As rational numbers, both fractions and decimals are more complex than whole numbers, But as noted above, number-line placement with decimals may provide a "purer" measure of magnitude comprehension than the same task with fractions, because decimal magnitudes can be accessed without performing a division operation. Interpreting a multi-digit decimal value certainly requires a form of relational processing, as does interpretation of a multi-digit whole number. Decimal notation is distinct from that of whole numbers (particularly because a leading zero can appear to the right of the decimal point and left of any other integers). However, research on magnitude comparisons has shown that performance with decimals is similar in accuracy and speed to performance with multi-digit whole numbers, whereas comparisons with fractions are much more difficult (Iuculano & Butterworth, 2011; DeWolf et al., 2014; Huber et al., 2014). It thus seems that decimals, like whole numbers but unlike fractions, can be readily interpreted as expressing a one-dimensional magnitude.

However, there are other possible interpretations for the greater predictive power of the decimal version of the number-line task. Decimal performance was somewhat more variable than performance with other number types. This greater variability might reflect the fact that decimals are introduced to students later than fractions, and hence are the number type with which the students in our study had had the least experience. Consequently, the decimal number-line estimation task may simply be measuring general math ability (as the more precocious students may have mastered decimal magnitudes earlier than their age-matched peers). Previous research has shown that as number-line estimation tasks (with whole numbers or fractions) become progressively more difficult (by increasing the size of the scale, using unusual scales, or including fractions), performance on these tasks is correlated with the overall development of general math ability (Booth & Siegler, 2008; Siegler et al., 2011; Ramani & Siegler, 2008). Further research is needed to determine whether decimal number-line estimation is especially correlated with overall math knowledge.

An equally intriguing finding from the present study is that understanding of fraction relations is a reliable additional predictor of algebra performance, separable from measures of magnitude understanding. Relational understanding of fractions thus appears to provide a stepping-stone towards acquiring the cognitive skills needed to form and understand algebraic expressions. The bipartite format of fractions sets them apart from all other number types, enabling them to convey relations between sets more effectively than their decimal magnitude counterparts (DeWolf et al., 2015). Fractions thus provide an early opportunity for students to understand the concept of expressing relations between quantities. In fact, excessive emphasis on understanding fraction magnitudes may obscure their relational meaning.

In particular, it appears that this type of relational understanding might be most useful for understanding algebraic expressions. Understanding how to create and manipulate algebra expressions is a crucial aspect of mastering algebra, and is important for solving word problems and equations. In general, understanding how to appropriately construct and manipulate fraction expressions is necessary for successful construction and manipulation of algebraic expressions. However, this linkage goes beyond simply being able to perform the same rote procedures, as success on the common procedures measured by the Fraction Procedures questions did not uniquely predict algebra performance. Based on this finding, it seems that algebra instruction should be related more closely to fraction instruction in order to bootstrap students' understanding of algebra.

Other researchers have pointed out the connections between understanding relational concepts in arithmetic and in algebra. Empson and colleagues (Empson & Levi, 2011, Empson, Levi & Carpenter, 2011) have suggested that students' basic intuitions about arithmetic functions (across both whole numbers and fractions) can be exploited to build basic relational concepts linking arithmetic to algebra. In particular, learning about fraction relations may help students to acquire some implicit understanding of general regularities such as the associative property. For example, to add 3/4 + 1/2, a student might reason that 3/4 is equal to 1/2 + 1/4. Accordingly, one can add 1/2 + 1/2 to get 1, and have 1/4 left over (Empson, 1999). Empson argues that fraction learning can be connected to basic understanding of properties of algebra (e.g., the use of the distributive property of multiplication over addition to add $7a + 4a$ to get $11a$, which is similar to the reasoning required to understand how to add 70 + 40 or 7/5 + 4/5).

As discussed earlier, fractions are the first example of a number that is a relational expression to which students are exposed. Their format has implications for the types of procedures that are appropriate for students to perform. Taking the example from Empson and colleagues used above, one could consider 7*a* to be an expression of 7 units each of size *a*. This interpretation is similar to the interpretation of the fraction 7/5 as an expression of 7 units of 1/5. Such an understanding of 7/5 has implications for the appropriate ways to perform arithmetic operations, such as grasping the constraint that "unlike terms" cannot be combined in algebra (e.g., $7a + 8b$ does not equal $15ab$, just as 7/5 and 8/9 cannot be combined to get 15/45 or 15/14). As students become fluent with the concept of a relational expression, their understanding of fraction relations may help to bootstrap their learning about similar relations within algebra. Thus, as the present findings suggest, a conceptually rich understanding of fractions may be especially important for understanding algebra.

The current research establishes a correlation between fraction understanding and algebra performance, and shows that this factor is separable from the predictive power of measures of magnitude. However, further research is required to determine whether these linkages between rational-number knowledge and learning of algebra are causal. Future studies should test whether direct instruction in relations involving fractions improves algebra performance, over and above the potential usefulness of instruction focusing on magnitudes of rational numbers. A program of research aimed at assessing the effectiveness of alternative instructional interventions would help to determine whether fraction understanding has a causal impact on algebra performance.

Currently, recommendations to educators highlight the importance of teaching students the magnitudes of fractions, and especially emphasize use of the number-line representation to highlight relative magnitudes (NMAP, 2008; Siegler et al., 2010). The present findings suggest that training placement of decimals on a number-line representation may be at least as effective for instruction. Of equal importance, teachers may need to focus on highlighting the connections between fraction and algebraic expressions, thereby capitalizing on the relational parallels between these two important domains of mathematical knowledge.

Footnote

1. A sample of problems were taken from a pool of problems used by Belinda Thompson in work performed for her PhD dissertation (in progress) at the University of California, Los Angeles, which she kindly allowed us to use in the present study.

References

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253.

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–8.

Booth, J. L., & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development, 79,* 1016-1031.

Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *Australian Mathematics Teacher, 62*(4), 28 – 40.

Brown, G., & Quinn, R. J. (2007). Fraction proficiency and success in algebra: What does research say? *Australian Mathematics Teacher*, *63*(3), 23–31.

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). Conceptual structure and the procedural affordances of rational numbers: Relational reasoning with fractions and decimals. *Journal of Experimental Psychology: General.* Advanced online publication. dx.doi.org/10.1037/xge 0000034.

DeWolf, M., Grounds, M., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with difference types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance, 40(1),* 71-82.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

Empson, S. B., Levi, L., & Carpenter, T. P. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds.), *Early* *algebraization: A global dialogue from multiple perspectives* (pp. 409 – 428). Berlin, Heidelberg: Springer.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). *Highlights from TIMSS 2007:Mathematics and science achievement of U.S. fourth- and eighth-grade students in an international context* (NCES 2009–001). Washington, D.C.: National Center for Education Statistics, U.S. Department of Education.

Huber, S., Klein, E., Willmes, K., Nuerk, H.-C., & Moeller, K. (2014). Decimal fraction representations are not distinct from natural number representations - evidence from a combined eye-tracking and computational modeling approach. *Frontiers in Human Neuroscience*, *8*, 1–14.

Iuculano, T., & Butterworth, B. (2012). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology*, *64*(11), 2088–2098.

National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

Ramani, G. B., & Siegler, R. S. (2008). Promoting broad and stable improvements in low-income children's numerical knowledge through playing number board games. *Child Development*, *79*(2), 375-394.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals, *Journal of Experimental Psychology: Applied, 21*(1), 47-56.

Richland, L. E., Stigler, J. W., & Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist*, *47*(3), 189–203.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—the case of algebra. *Educational Studies in Mathematics, 26,* 191-228.

Siegler, R., Carpenter, T., Fennell, F., Geary, D. Lewis, J., Okamoto, Y., Thompson, L. & Wray, J. (2010). *Developing effective fractions instruction: A practice guide (NCEE No. 2010-009).* Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–7.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–96.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York: Erlbaum.

Wu, H. (2001). How to prepare students for algebra. *American Educator*, *25*(2), 10–17.

Wu, H. (2009). What's sophisticated about elementary mathematics? *American Educator, 33(3)*, 4-14.

CHAPTER 5: GENERAL DISCUSSION AND CONCLUSION

Summary

The research presented in this dissertation has investigated (1) the mental representation of rational numbers, (2) the benefits of alternative rational number formats for relational reasoning, and (3) the importance of rational expressions for later mathematics achievement, especially in algebra. Chapter 2 presented a neuroimaging study that provided evidence for a distinct mental representation of fractions compared to decimals and integers (which seem to be indistinguishable on a neural basis). Importantly, this work shows that fraction magnitude representation necessarily involves a mental calculation process that is more effortful and recruits distinct areas of the IPS compared to either decimals or integers which are fairly automatic by comparison. Chapter 3 presented behavioral evidence for the potential benefits of fractions in tasks that do not require representation of a fraction's integrated or holistic magnitude. Instead, adults show superior performance with fractions compared to decimals when the goal of the task is to identify the relation between two subsets and a numerical value. Thus, fractions are not necessarily more complex than other number types and indeed have contexts in which their unique format actually presents advantages. Finally, Chapter 4 presented developmental evidence that shows that fraction relational understanding is significantly linked and uniquely predictive of algebra performance. Together, this research suggests that mastering fractions is essential for success in later mathematics. But, importantly, this mastery requires a deep conceptual understanding of fractions that relies on understanding the relational expression and not just the magnitude.

Rational Numbers as Relational Models

When thinking of this work within the broader context of the numerical cognition and math education literature, it seems to unite two separate lines of research. The previous literature on misconceptions and on children and adults' understanding of fraction magnitudes is disconnected from an earlier line of research on rational numbers, which pointed out that fractions can be used in a variety of roles. Ohlsson (1988) summarized various definitions and uses of fractions as described by many researchers in math education and psychology (e.g., Kieren, 1975, 1980; Behr, Lesh, Post, & Silver, 1983; Vergnaud, 1983). The compiled list included representing part/whole relationships, ratio or multiplicative comparison between two quantities, probability, operator, quotient, and points on a number line. Ohlsson argued that fractions can be understood by any of three basic interpretations: comparisons (e.g., ratio), partitioning (e.g., division), and composite operations (e.g., rates). He argued that these types of interpretations are applied when students use fractions to model physical quantities in the real world. Thus, not only is it difficult for students to interpret the symbolic notation of fractions due to their competing conceptions of other number types, but in addition fractions themselves may vary in meaning and purpose depending on the context. However, the research presented in this dissertation suggests that the variety of uses and contexts may actually be advantageous when rational numbers are considered as mathematical modeling tools.

A central goal of this work is to view mathematical notation as a type of conceptual modeling. In the case of fractions, the bipartite notation can be used to represent a relation between two distinct, countable quantities. By contrast, decimals provide a one-dimensional notation and therefore are best used to represent magnitudes and used as a measurement system for continuous quantities. Indeed, empirical tests show that well-educated adults and educators prefer to represent discrete sets with fractions and continuous quantities with decimals. Furthermore, adults show advantages in using fractions in relational contexts over decimals. Conversely, decimals are advantageous in contexts that involve magnitude representation. This also has implications for how fractions and decimals form foundations for to be learned math concepts. Increased prowess and understanding of decimal magnitudes seems to be important for number sense and potentially for overall math achievement. Fractions seem to lay a foundation for contexts that require mode abstract relational modeling. This type of knowledge may also relate to build off of more general relational reasoning ability and problem solving which is also an important component to overall success in mathematics.

Future research might further explore the links between fraction knowledge and relational reasoning in general such as in tasks like the Ravens Progressive Matrices or abstract analogical reasoning tasks. More research could also be done to explore the links between decimal magnitude representation and enhanced number sense and general math achievement. At present, relatively few studies exist examining decimal magnitude estimation. Instead, many have focused on ways to improve fraction number line estimation.

This research provides evidence for a different approach for rational number education. Instead of a focus on overcoming misconceptions or issues related to fraction magnitude estimation, learners might develop a deeper understanding of rational number magnitude if they were first introduced to more straightforward magnitude representations for rational numbers (e.g. decimals; c.f. Moss & Case, 1998). Lessons on fractions could then be focused on understanding the various meanings of fractions in different contexts. Proportions and ratios could be taught within the larger umbrella of the “fraction.” Fraction notation could even be introduced as just another way to represent a division operation. More direct studies could test how to best implement these interventions to improve rational number understanding and appreciation for the affordances of alternative mathematical notations.

APPENDICES

APPENDIX A: MATERIALS USED FOR EXPERIMENTS IN CHAPTER 3

Table A1. Ratios Used in Experiment 1

Continuous	Discretized	Discrete
1/10	2/11	2/9
1/6	2/10	2/7
1/4	2/9	2/10
4/13	2/7	3/10
3/6	2/6	3/9
3/9	4/13	3/11
4/11	5/15	3/8
5/12	4/10	4/10
3/7	5/12	4/14
4/9	3/7	3/7
5/11	5/9	3/4
5/9	4/7	4/9
5/14	7/12	5/10
4/7	5/8	6/11
6/10	5/13	4/7
9/13	4/6	6/10
8/10	7/9	2/3
5/6	8/10	7/10
7/8	7/8	8/10
8/15	4/5	4/6

Note: In Experiment 1, participants were not told the actual ratio given, nor were they asked to identify this ratio. The numbers in this table indicate the number of pieces in the red and green subsets for the various displays.

Table A2. Ratios Used in Experiment 2

	PPR		PWR	
	Fraction	Decimal	Fraction	Decimal
Discrete	5/7	.71	7/10	.70
	4/6	.67	10/15	.67
	6/10	.60	5/6	.83
	6/8	.75	7/8	.88
	4/8	.50	4/7	.57
	4/5	.80	6/11	.55
	2/7	.29	2/9	.22
	3/9	.33	2/5	.40
	4/9	.44	5/15	.33
	3/8	.38	3/10	.30
	4/10	.40	4/14	.29
	3/7	.43	3/7	.43
Discretized	6/8	.75	9/10	.90
	7/8	.88	5/8	.63
	5/8	.63	7/12	.58
	5/9	.56	4/6	.67
	4/5	.80	7/9	.78
	4/7	.57	4/5	.80
	3/7	.43	4/13	.31
	2/9	.22	2/9	.22
	2/10	.20	4/10	.40
	2/7	.29	2/7	.29
	1/4	.25	5/12	.42
	3/9	.33	1/9	.11
Continuous	4/5	.80	9/16	.56
	4/7	.57	7/8	.88
	7/8	.88	3/5	.60
	6/8	.75	9/13	.69
	5/6	.83	11/14	.79
	5/9	.56	9/15	.60
	4/13	.31	1/10	.10
	5/15	.33	6/13	.46
	4/9	.44	3/9	.33
	2/8	.25	6/14	.43
	4/11	.36	4/11	.36
	5/12	.42	2/12	.17

Table A3. Ratios Used for Target Displays in Experiment 3.

	PPR				PWR			
	Fraction		Decimal		Fraction		Decimal	
	Target	Foil	Target	Foil	Target	Foil	Target	Foil
Discrete	3/5	3/8	.60	.38	3/7	3/4	.43	.75
	6/7	6/13	.86	.46	2/5	2/3	.40	.67
	5/9	5/14	.56	.36	3/8	3/5	.38	.60
	3/10	3/13	.30	.23	1/6	1/5	.17	.20
	3/7	3/10	.43	.30	4/13	4/9	.31	.44
	2/9	2/11	.22	.18	2/12	2/10	.17	.20
	4/7	4/11	.57	.36	4/9	4/5	.44	.80
	2/8	2/10	.25	.20	5/13	5/8	.38	.63
	6/8	6/14	.75	.43	6/14	6/8	.43	.75
	4/10	4/14	.40	.29	3/11	3/8	.27	.38
	2/6	2/8	.33	.25	2/8	2/6	.25	.33
	4/9	4/13	.44	.31	2/11	2/9	.18	.22
Discretized	5/7	5/12	.71	.42	5/12	5/7	.58	.71
	3/4	3/7	.75	.43	6/13	6/7	.46	.86
	3/5	3/8	.60	.38	5/11	5/6	.45	.83
	2/5	2/7	.40	.29	2/11	2/9	.18	.22
	3/10	3/13	.30	.23	3/10	3/7	.30	.43
	5/11	5/16	.45	.31	2/12	2/10	.17	.20
	7/8	7/15	.88	.47	3/7	3/4	.43	.75
	4/6	4/10	.67	.40	5/13	5/8	.38	.63
	7/8	7/15	.88	.47	4/9	4/5	.44	.80
	3/11	3/14	.27	.21	4/13	4/9	.31	.44
	2/12	2/14	.17	.14	2/9	2/7	.22	.29
	1/5	1/6	.20	.17	2/14	2/12	.14	.17
Continuous	4/6	4/10	.67	.60	7/15	7/8	.47	.88
	6/7	6/13	.86	.46	3/11	3/8	.27	.38
	6/11	6/17	.55	.35	7/16	7/9	.44	.78
	2/9	2/11	.22	.18	2/14	2/12	.14	.17
	3/8	3/11	.38	.27	4/17	4/13	.24	.31
	3/16	3/19	.19	.16	2/7	2/5	.29	.40
	7/12	7/19	.58	.37	6/13	6/7	.46	.86
	5/6	5/11	.83	.45	7/17	7/10	.41	.70
	7/9	7/16	.78	.44	8/18	8/10	.44	.80
	4/13	4/17	.31	.24	4/14	4/10	.29	.40
	2/6	2/8	.33	.25	3/14	3/11	.21	.27
	4/10	4/14	.40	.29	2/11	2/9	.18	.22

**Note:* Each of these target and foil combinations were randomly matched with a source picture of the same entity type and ratio type from Experiment 2 (as shown in Table A2).

Table A4. Ratios Used for Source Displays in Experiments 4A and 4B.

PPR			PWR		
OTO Fraction	NOTO Fraction	Decimal	OTO Fraction	NOTO Fraction	Decimal
5/7	10/14	.71	7/10	14/20	.70
4/6	2/3	.67	10/15	2/3	.67
6/10	3/5	.60	5/6	10/12	.83
6/8	3/4	.75	7/8	14/16	.88
4/8	2/4	.50	4/7	12/21	.57
4/5	8/10	.60	6/11	12/22	.55
2/7	4/14	.29	2/9	4/18	.22
3/9	1/3	.33	2/5	6/15	.40
4/9	8/18	.44	5/15	1/3	.33
3/8	6/16	.38	3/10	6/20	.30
4/10	2/5	.40	4/14	2/7	.29
3/7	6/14	.43	3/7	6/14	.43
6/11	12/22	.55	9/10	18/20	.90
7/8	14/16	.88	5/8	10/16	.63
5/8	10/16	.63	7/12	14/24	.58
5/9	10/18	.56	4/6	8/12	.67
12/14	6/7	.86	7/9	14/18	.78
4/7	8/14	.57	4/5	12/15	.80
4/11	8/22	.36	4/13	8/26	.31
2/9	4/18	.22	2/9	6/27	.22
2/10	1/5	.20	4/10	2/5	.40
3/10	6/20	.30	2/7	6/21	.29
1/4	3/12	.25	5/12	10/24	.42
1/8	3/24	.13	1/9	3/27	.11

**Note*: In Experiments 4A and 4B all displays were discretized.

Table A5. Ratios Used for Target Displays in Experiments 4A and 4B.

PPR						PWR					
OTO Fraction		NOTO Fraction		Decimal		OTO Fraction		NOTO Fraction		Decimal	
Target	Foil	Target	Foil	Target	Foil	Target	Foil	Target	Foil	Target	Foil
3/5	3/8	6/10	6/16	.60	.38	3/7	3/4	6/14	6/8	.43	.75
6/7	6/13	12/14	12/26	.86	.46	2/5	2/3	4/10	4/6	.40	.67
5/9	5/14	10/18	10/28	.56	.36	3/8	3/5	6/16	6/10	.38	.60
3/10	3/13	6/20	6/26	.30	.23	1/6	1/5	3/18	3/15	.17	.20
3/7	3/10	6/14	6/20	.43	.30	4/13	4/9	8/26	8/18	.31	.44
2/9	2/11	4/18	4/22	.22	.18	2/12	2/10	1/6	1/5	.17	.20
4/7	4/11	8/14	8/22	.57	.36	4/9	4/5	8/18	8/10	.44	.80
2/8	2/10	1/4	1/5	.25	.20	5/14	5/8	10/26	10/16	.38	.63
6/8	6/14	3/4	3/7	.75	.43	6/14	6/8	3/7	3/4	.43	.75
4/10	4/14	2/5	2/7	.40	.29	3/11	3/8	6/22	6/16	.27	.38
2/6	2/8	1/3	1/4	.33	.25	2/8	2/6	1/4	1/3	.25	.33
4/9	4/13	8/18	8/26	.44	.31	2/11	2/9	4/22	4/18	.18	.22
5/7	5/12	10/14	10/16	.71	.42	5/12	5/7	10/24	10/14	.42	.71
3/4	3/7	6/8	6/14	.75	.43	6/13	6/7	12/26	12/14	.46	.86
4/5	4/9	8/10	8/18	.80	.44	5/11	5/6	10/22	10/12	.45	.83
2/5	2/7	4/10	4/14	.40	.29	4/14	4/10	2/7	2/5	.29	.40
2/7	2/9	4/14	4/18	.29	.22	3/10	3/7	6/20	6/14	.30	.43
5/11	5/16	10/22	10/32	.45	.31	2/12	2/10	1/6	1/5	.17	.20
7/8	7/15	14/16	14/30	.88	.47	5/15	5/9	10/28	10/18	.36	.56
4/6	4/10	2/3	2/5	.67	.40	4/12	4/8	2/6	2/4	.33	.50
8/9	8/17	16/18	16/34	.89	.47	4/9	4/5	8/18	8/10	.44	.80
3/11	3/14	6/22	6/28	.27	.21	7/15	7/8	14/30	14/16	.47	.88
2/12	2/14	1/6	1/7	.17	.14	2/9	2/7	4/18	4/14	.22	.29
1/5	1/6	3/15	3/18	.20	.17	2/14	2/12	1/7	1/6	.14	.17

**Note*: In Experiments 4A and 4B all displays were discretized.

Appendix B: Materials Used For Experiments In Chapter 4

Fraction Problems

Procedural Fraction Questions (from Booth et al., 2014, "Declarative Fractions Questions")

Are either of these an effective first step toward solving for *z* in the equation $3 = \frac{1}{z}$?
Circle yes or no.

a. Multiply both sides by *z*. Yes No

b. Divide both sides by 3. Yes No

Would any of the following steps be an effective first step toward solving the equation $\frac{6}{d} = 2$? Circle yes or no.

a. Subtract 2 from both sides. Yes No

b. Multiply both sides by *d*. Yes No

If $y = 3x + 2$, which of these expresses *x* in terms of *y*? Circle the correct answer.

a. $x = \frac{y-2}{3}$ b. $x = \frac{y+2}{3}$ c. $x = \frac{y}{3} - 2$ d. $x = \frac{y}{3} + 2$

On Planet Zebula, zeds are a unit of money. On this planet, Carla paid *x* zeds for 3 cartons of juice. What is the price in zeds of 1 carton of juice? Circle all that apply.

a. $\frac{x}{3}$ b. $\frac{3}{x}$ c. $3 + x$ d. $\frac{1}{3}x$ e. $3x$

Relational Fraction Questions

Multiplicative/Division Relations

(adapted from Thompson, 2014, unpublished dissertation)

Which expression shows a way to find half a number, n? Circle yes/no for each expression:

$n \div \frac{1}{2}$ yes no

$n - \frac{1}{2}$ yes no

$n \div 2$ yes no

$n \times \frac{1}{2}$ yes no

(adapted from Brown & Quinn, 2006)

What is $\frac{7}{\frac{3}{5}}$ equal to?

a. $7 \times \frac{3}{5}$

b. $\frac{3}{5} \div 7$

c. $\frac{35}{3}$

d. $\frac{21}{5}$

Inverse Relation (adapted from Brown & Quinn, 2006)

(n is an integer greater than 0)
If *n* increases in value, then 1/*n*

a. gets very close to 1

b. gets very close to 0

c. increases in value, too

Equivalence Relation (adapted from Thompson, 2014, unpublished dissertation)
Which fraction is equal to 15/20?

a. 20/25

b. 9/12

c. 20/15

d. none of these

Which fraction is equal to 8/12?

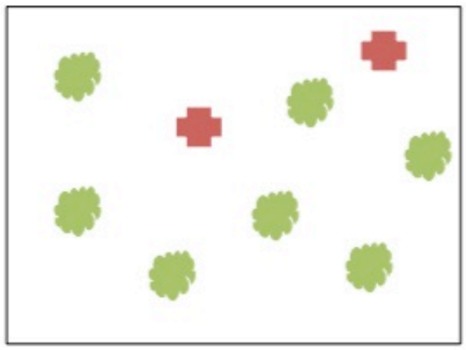
a. 24/48

b. 16/36

c. 12/18

d. none of these

Identifying Ratio Relations (adapted from DeWolf, Bassok & Holyoak, 2015)
In the pictures below, there are two types of relationships: part to part ratio (number of crosses to the number of clouds) and part to whole ratio (number of crosses to the total number of crosses and clouds OR number of clouds to the total number of crosses and clouds). Circle the relationship that each fraction represents:



$\frac{2}{7}$	a) part to part ratio	b) part to whole ratio
$\frac{4}{18}$	a) part to part ratio	b) part to whole ratio
$\frac{14}{18}$	a) part to part ratio	b) part to whole ratio
$\frac{7}{9}$	a) part to part ratio	b) part to whole ratio

Algebra Problems
Equation Solving (from Booth et al., 2014)
Solve the equations:

1. $5 = x - 7$

2. $\frac{8}{k} = 4$

3. $-4x + 5 = 8$

4. $\frac{6}{b} = 9$

5. $-3y + 6 = 8 + 5y$

Feature Knowledge (from Booth et al., 2014)

Which of the following is equal to $-4x + 3$?

a. $4x + 3$	yes	no
b. $3 - 4x$	yes	no
c. $4x - 3$	yes	no
d. $3 + (-4x)$	yes	no
e. $3 + 4x$	yes	no

If $10x - 12 = 17$ is true, which of the following must also be true?

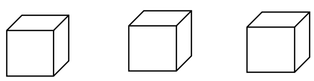
a. $10x - 12 + 12 = 17 + 12$	yes	no
b. $x - 2 = 17$	yes	no
c. $10x = 29$	yes	no
d. $10x = 17$	yes	no
e. $10x - 10 - 12 - 10 = 17$	yes	no
f. $10x - 12 + 12 = 17$	yes	no

Word Problem Solving (from Booth et al., 2014)
The Carlson family is moving today. It took 89 boxes to pack up all of their things. Dad told each of his four children to carry 16 boxes to the truck and then he would get the rest. How many boxes did Dad have to carry?

Tommy bought a pair of shoes on sale. It was 1/4 off the original price. He paid $42. What was the original price of the shoes?

(not taken from Booth et al., 2014)
Ted has $12 more than Carla, and Carla has $8 more than Devon. Together, Ted and Carla have four times as much as Devon. How much money does each person have?

Understanding Algebra Expressions (adapted from Thompson, 2014, unpublished dissertation)
Each of these three boxes weighs the same amount. If the weight of one box is x, what is the weight of the three boxes together?



a. $3x$
b. $x + 3$
c. 3
d. It's impossible to tell

What could be the value of x that makes this equation true?

$$x + x + x = 15$$

a. 5,5,5
b. 4,5,6
c. 3,3,9
d. all of these

If a+b=c and d+e=c, when is a+b=d+e true?

a. always
b. never
c. it depends on the values of *a,b,c,d,* and *e*

If n is some number, and k is 4 less than n, which expression represents k?

a. 4
b. n-4
c. 4-n
d. It's impossible to tell

References

Ball, D., & Wilson, S. M. (1996). Integrity in teaching: Recognizing the fusion of the moral and intellectual. *American Educational Research Journal, 33*(1), 155–192.

Barbey, A. K., & Sloman, S. A. (2007). Base-rate respect: From ecological rationality to dual processes. *Behavioral and Brain Sciences*, *30*, 241-297.

Bassok, M. (2001). Semantic alignments in mathematical word problems. In D. Gentner, K.J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 401 – 433). Cambridge, MA: MIT Press.

Bassok, M., Chase, V. M., & Martin, S. A. (1998). Adding apples and oranges: Alignment of semantic and formal knowledge. *Cognitive Psychology, 35,* 99-134.

Bassok, M., Pedigo, S. F., & Oskarsson, A. T. (2008). Priming addition facts with semantic relations. *Journal of Experimental Psychology: Learning Memory and Cognition, 34*, 343-352.

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance*, *33*, 1410–1419.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental psychology*, *44*, 1478–90.

Braithewaite, D. W., & Goldstone, R. L. (2013). Integrating formal and grounded representations in combinatorics learning. *Journal of Educational Psychology, 105*(3), 666-682.

Brankaer, C., Ghesquière, P., & De Smedt, B. (2014). Children's mapping between non-symbolic and symbolic numerical magnitudes and its association with timed and untimed tests of mathematics achievement. *PlOS ONE*, *9*(4), e93565.

Cantlon, J., Brannon, E. M., Carter, E., & Pelphrey, K. (2006). Functional imaging of numerical processing in adults and 4-yr-old children. *PLoS Biology, 4*, e125, 1-11.

Cedrus Corp. (2004). Superlab 4.0 [Computer software]. San Pedro, CA.

Chen, D., Lu, H., & Holyoak, K. J. (2014). The discovery and comparison of symbolic magnitudes. *Cognitive Psychology*, *71*, 27-54.

Clearfield, M. W., & Mix, K. S. (2001). Infants use continuous quantity—not number—to discriminate small visual sets. *Journal of Cognition and Development, 2*, 243-260.

Cohen, D. J., Ferrell, J. M., & Johnson, N. (2002). What very small numbers mean. *Journal of Experimental Psychology: General*, *131*(3), 424–442.

Common Core State Standards Initiative (2014). http://www.corestandards.org/Math/Content/4/NF/ (retrieved September 23, 2014).

Cordes, S., & Gelman, R. (2005). The young numerical mind: What does it count? In J. Campbell (Ed), *Handbook of mathematical cognition* (pp. 128-142). Hove, UK: Psychology Press.

Dehaene, S. (1997). *The number sense*. New York: Oxford University Press.

Dehaene, S., & Changeux, J.-P. (1993). Development of elementary numerical abilities: A neuronal model. *Journal of Cognitive Neuroscience, 5*, 390-407.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science*, *320*(5880), 1217-1220.

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance, 40,* 71 – 82.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes.* Mahwah, NJ: Erlbaum.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' discrimination of number vs. continuous extent. *Cognitive Psychology*, *44*, 33-66.

Feigenson, L., Dehaene, S. & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences*, *8*, 307 - 314.

Fias, W., & Verguts, T. (2008). Not all basic number representations are analog: Place coding as a precursor of the natural number system. *Behavioral and Brain Sciences*, *31*, 650-651.

Fisher, K. J., Borchert, K., & Bassok, M. (2011). Following the standard form: Effects of equation format on algebraic modeling. *Memory & Cognition, 39*(3), 502-515.

Gelman, R. (1993). A rational-constructivist account of early learning about numbers and objects. In D. L. Medin (Ed.), *Learning and motivation*, Vol. 30 (pp. 61-96). New York: Academic Press.

Gigerenzer, G., & Hoffrage, U. (1995). How to improve Bayesian reasoning without instruction: Frequency formats. *Psychological Review*, *102*, 684-704.

Goldstone, R. L., & Barsalou, L. W. (1998). Reuniting perception and conception. *Cognition*, *65*(2), 231-262.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*(2), 265-284.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). *Highlights from TIMSS 2007:Mathematics and science achievement of U.S. fourth- and eighth-grade students in an international context* (NCES 2009–001). Washington, D.C.: National Center for Education Statistics, U.S. Department of Education.

Goswami, U. (1989). Relational complexity and the development of analogical reasoning. *Cognitive Development*, *96*, 229–241.

Gray, E. M, & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115 - 141.

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology, 44,* 1457-1465.

Halford, G. S., Wilson, W. H. & Phillips, W. (1998). Processing capacity defined by relational complexity: Implications for comparative, developmental and cognitive psychology. *Behavioral Brain Sciences, 21*(6), 803-831.

Halford, G. S., Wilson, W. H., & Phillips, S. (2010). Relational knowledge: The foundation of higher cognition. *Trends in Cognitive Sciences, 14*(11), 497-505.

Hinsley, D., Hayes, J. R., & Simon, H. A. (1977). From words to equations: Meaning and representation in algebra word problems. In P. A. Carpenter & M. A. Just (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Erlbaum.

Hoffrage, U., Gigerenzer, G., Krauss, S., & Martignon, L. (2002). Representation facilitates reasoning: What natural frequencies are and what they are not. *Cognition*, *84*, 343-352.

Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology*, *103*(1), 17-29.

Iuculano, T., & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology, 64,* 2088-2098.

Izard, V., & Dehaene, S. (2008). Calibrating the mental number line. *Cognition*, *106*(3), 1221-1247.

Jacob, S. N., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: The brain's code for proportions. *Trends in Cognitive Sciences, 16*(3), 157-166.

Kallai, A. Y., & Tzelgov, J. (2009). A generalized fraction: An entity smaller than one on the mental number line. *Journal of Experimental Psychology: Human Perception and Performance, 35*, 1845-1864

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational number. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review*, *92*, 109-129.

Landy, D., Brookes, D., & Smout, R. (2014) Abstract numeric relations and the visual structure of algebra. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 40*(3), 797-810.

Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought? *Journal of Experimental Psychology: Learning, Memory, and Cognition, 33*, 720-733.

Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85-105). Mahwah, NJ: Erlbaum.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422–441.

Martin, S. A., & Bassok, M. (2005). Effects of semantic cues on mathematical modeling: Evidence from word-problem solving and equation construction tasks. *Memory & Cognition*, *33*, 471-478.

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*, 164–174.

Mochon, D., & Sloman, S. A. (2004). Causal models frame interpretation of mathematical equations. *Psychonomic Bulletin and Review*, *11*, 1099-1104.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*, 122–147.

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature*, *215*, 1519-1.520.

Mundy, E., & Gilmore, C. K. (2009). Children's mapping between symbolic and nonsymbolic representations of number. *Journal of Experimental Child Psychology*, *103*(4), 490-502.

Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual Review of Neuroscience, 32,* 185-208.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of the whole number bias. *Educational Psychologist*, *40*, 27–52.

Nunes, T., Light, P., Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction, 3,* 39-54

Over, D. E. (2007). Content-independent conditional inference. In M. J. Roberts (Ed.), *Domain general versus domain specific processes in higher cognition* (pp. 83-103). Hove, UK: Psychology Press.

Pinel, P., Piazza, M., Bihan, D. L., & Dehaene, S. (2004). Distributed and overlapping cerebral representations of number, size, and luminance during comparative judgments. *Neuron*, *41*, 1-20.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals, *Journal of Experimental Psychology: Applied, 21*(1), 47-56.

Richland, L. E., Stigler, J. W., Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist*, *47*, 189-203.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93, 346–362.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99,* 561-574.

Sasanguie, D., Göbel, S. M., Moll, K., Smets, K., & Reynvoet, B. (2013). Approximate number sense, symbolic number processing, or number–space mappings: What underlies mathematics achievement? *Journal of Experimental Child Psychology*, *114*(3), 418-431.

Schneider, M., & Siegler, R.S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, *36*, 1227-1238.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26,* 191-228.

Siegler, R. S., Duncan, G. J., Davis-Kean, P.E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*, 691- 697.

Siegler, R. S., Fazio L.K., Bailey, D.H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*, 13-19.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*, 273-296.

Sophian, C. (2007). *The origins of mathematical knowledge.* Mahwah, NJ: Erlbaum.

Stigler, J. W., Givvin, K. B., & Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *The MathAMATYC Educator, 10*, 4-16.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Sophian, C. (2000). Perceptions of proportionality in young children: Matching spatial ratios. *Cognition, 75*, 145–170.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review*, *90*, 293-315.

Vamvakoussi, X., & Vosniadou, S. (2010). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*, 453- 467.

Zahner, D., & Corter, J. E. (2010). The process of probability problem solving: Use of external visual representations. *Mathematical Thinking and Learning, 12*(1), 177-204.

Zhang, J., & Norman, D. A. (1995). A representational analysis of numeration systems. *Cognition*, *57*, 271-295.