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Patterson-Sullivan theory for Anosov subgroups

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Abstract

We extend several notions and results from the classical Patterson-Sullivan theory to the setting of Anosov subgroups of higher rank semisimple Lie groups, working primarily with invariant Finsler metrics on associated symmetric spaces. In particular, we prove the equality between the Hausdorff dimensions of flag limit sets, computed with respect to a suitable Gromov (pre-)metric on the flag manifold, and the *Finsler critical exponents* of Anosov subgroups.

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Consider a discrete group Γ of isometries of the n -dimensional hyperbolic space \mathbb{H}^n . The *critical exponent* δ is a fundamental numerical invariant associated with Γ , which measures the

asymptotic growth rates of Γ -orbits in \mathbb{H}^n . The relation between the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ and its critical exponent is now a classical result. In an influential paper [Sul79], Sullivan proved the following theorem extending pioneering work by Patterson ([Pat76]) on Fuchsian groups:

Theorem ([Sul79, Thm. 8]). *Let Γ be a convex-cocompact subgroup of the isometry group of \mathbb{H}^n . Then the critical exponent δ of Γ equals to the Hausdorff dimension of $\Lambda(\Gamma)$.*

Later Sullivan generalized this theorem for geometrically finite Kleinian groups ([Sul84]). An important ingredient of Sullivan’s proof of this theorem is the existence of a finite, non-null Borel measure on $\Lambda(\Gamma)$ that changes *conformally* under the Γ -action. The construction of such measure goes back to Patterson’s original idea in [Pat76]. Measures of this type (resp. a class of “well-behaved” measures) are commonly referred as *Patterson-Sullivan measures* (resp. *densities*). We refer to Nicholls’ book ([Nic89]) for a self-contained exposition on these results.

Since its introduction, the theory of Patterson and Sullivan has attracted a lot of attention. Further developments have been made by various people who analyzed more general classes of discrete groups and their limit sets. We list some of these developments here. Corlette ([Cor90]) and Corlette-Iozzi ([CI99]) proved the above theorem for geometrically finite groups of isometries of rank-one symmetric spaces, and Bishop-Jones ([BJ97]) extended these results to arbitrary discrete isometry groups of rank-one symmetric spaces. Yue ([Yue96]) studied the case of Hadamard spaces of negative curvature. Burger ([Bur93]), Albuquerque ([Alb99]) and Quint ([Qui02b], [Qui02a]) considered the case of Zariski-dense discrete subgroups in the isometry groups of higher-rank symmetric spaces, while Link ([Lin10]) studied the case of products of rank one symmetric spaces.

In the more abstract setting of Gromov hyperbolic spaces, Coornaert ([Coo93]) generalized much of Sullivan’s work in [Sul79] to the class of quasi-convex-cocompact groups. See also work of Paulin ([Pau97]) on actions of subgroups of Gromov hyperbolic groups. More recent developments by Das-Simmons-Urbański ([DSU17]) (see also Fishman-Simmons-Urbański, [FSU18]) achieved far reaching generalizations the Patterson-Sullivan theory (e.g., a generalization of Bishop-Jones’ theorem) in the case of “infinite-dimensional” Gromov hyperbolic spaces.

The goal of this paper is to study the Patterson-Sullivan theory for *Anosov subgroups*. Anosov subgroups, first introduced by Labourie ([Lab06]) and then further developed by Guichard-Wienhard ([GW12]) and Kapovich-Leeb-Porti ([KLP14, KLP17, KLP18]), extend the class of convex-cocompact subgroups of rank-one semisimple Lie groups to higher rank. In this paper, we mainly work with Kapovich-Leeb-Porti’s characterizations of Anosov subgroups. We briefly review some of these characterizations (and related background) in Section 1.

Let G be a noncompact real semisimple Lie group, $X = G/K$ be the associated symmetric space and Γ be a τ_{mod} -Anosov subgroup of G . We will be assuming several conditions on G and X ; they are labeled as “assumption” in Section 1. We consider two types of G -invariant (pseudo-)metrics on X , namely, one is the Riemannian metric of the symmetric space X and the other one is Finslerian. The critical exponents of Γ with respect to these two metrics, denoted by $\delta_{\mathbb{R}}$ and $\delta_{\mathbb{F}}$, respectively, are defined in the usual fashion, i.e. as the exponents of convergence of associated Poincaré series (see Section 2). Using the classical construction of Patterson, we define a Γ -invariant conformal density on the flag limit set of Γ (see Section 3).

Throughout this paper, the Finsler metric is given more emphasis than its Riemannian counterpart. For example, the construction of the above mentioned Patterson-Sullivan density is carried

out in terms of the Finsler metric. The main reason for this choice is that Finsler metrics reflect the asymptotic geometry of Γ better than the Riemannian metric.

We should note that many of the results in this paper are often proven for more general classes of discrete subgroups of G with the hope that the results may be useful, for instance, in the study of *relatively Anosov subgroups*.¹ Regarding Anosov subgroups, the main results of this paper are summarized below.

Let σ_{mod} be a maximal simplex in the Tits building of X , $\iota : \sigma_{\text{mod}} \rightarrow \sigma_{\text{mod}}$ be the opposition involution, τ_{mod} be an ι -invariant face of σ_{mod} , P be the maximal parabolic subgroup of G that stabilizes τ_{mod} , and $\text{Flag}(\tau_{\text{mod}}) = G/P$ be the partial flag manifold associated to the face τ_{mod} (see Subsection 1.2).

Main theorem. *Let Γ be a nonelementary τ_{mod} -Anosov subgroup of G and δ_{F} be the Finsler critical exponent for the action of Γ on the symmetric space $X = G/K$. Then the Patterson-Sullivan density μ (constructed with respect to the Finsler metric on X) on the flag limit set $\Lambda_{\tau_{\text{mod}}}(\Gamma) \subset \text{Flag}(\tau_{\text{mod}})$ is the unique (up to a constant) Γ -invariant conformal density. Moreover,*

- (i) *The density μ is non-atomic and its dimension equals to δ_{F} .*
- (ii) *The support of μ is $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ and the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is ergodic with respect to μ .*
- (iii) *The critical exponent δ_{F} (as well as the Riemannian critical exponent δ_{R}) is positive and finite.*
- (iv) *The Poincaré series of Γ diverges at the critical exponent δ_{F} . In other words, Γ has (Finsler) divergence type.*
- (v) *The δ_{F} -dimensional Hausdorff measure on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ with respect to a Gromov (pre-)metric² is a member of a Γ -invariant conformal density (called the Hausdorff density). In particular, the Hausdorff dimension of $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is δ_{F} .*

The uniqueness of conformal density is proven in Corollary 8.4. The main ingredients in the proof are (1) a generalization of Sullivan’s *shadow lemma* proven in Theorem 6.1, and (2) an ergodicity argument (see Theorem 8.1) also due to Sullivan. The proof of part (i) of the theorem is given in Corollaries 6.2 and 7.4. The second half of part (ii) follows from Theorem 8.3 while the first half follows from [Coo93, Cor. 5.2]. The part (iii) is proven in Propositions 2.3 and 3.1. See also the remarks following these propositions where δ_{R} is analyzed. The part (iv) follows from Corollary 6.5. The Hausdorff density in part (v) is studied in Section 9 (cf. Theorem 9.3). The background Gromov (pre-)metric is introduced in Section 5 where we also prove that the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ with respect to this metric is *conformal* (see Corollary 5.6).

While working on this article, we came to know about two recent developments by Pozzetti-Sambarino-Wienhard ([PSW19]) and Glorieux-Monclair-Tholozan ([GMT19]) which are related to our work. In these articles, the authors proved that the Hausdorff dimension of the limit set of a *projective Anosov subgroup* Γ in the real projective space is bounded above by a certain critical exponent, called the “simple root critical exponent” in the second article. In the second article, the

¹Relatively Anosov subgroups are recent extension by Kapovich-Leeb of the class of geometrically finite groups into the higher rank. See [KL18b].

²See Section 5.

authors also obtain upper and lower bounds for the Hausdorff dimension of the flag limit set³ of Γ while mentioning that they also “hoped to get” a lower bound for the limit set in the projective space. We obtain a lower bound for this limit set which turns out to be same as in the case of the flag limit set (see Theorem 10.1).

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§0. Notations

Here we list some commonly used notations.

- $\mathfrak{B}(Y)$: Class of Borel subsets of a topological space Y
- $B(x, r)$: (Open) ball of radius r centered at x
- d_F, d_R : Finsler and Riemannian metrics, respectively, on X (see Sec. 2)
- $\widehat{xy}, \overline{xy}$: Finsler⁴ and Riemannian geodesic segments, respectively, connecting $x, y \in X$ (see Sec. 2)
- δ_F, δ_R : Finsler and Riemannian critical exponents, respectively, of Γ (see Sec. 2)
- $d_G^{\epsilon, x}$: Gromov premetric (see Def. 5.2)
- d_τ^{hor} : Horospherical distance (see (3.1))

§1. Geometric preliminaries

In this section, we briefly present some background material needed for the paper.

1.1. Symmetric spaces: A *symmetric space* X is a Riemannian manifold that has an *inversion symmetry* or *point-reflection* with respect to each point $x \in X$: This is an isometric involution $s_x : X \rightarrow X$ fixing x and sending each tangent vector at x to its negative. In this paper we only consider symmetric spaces which are simply-connected and have *noncompact type*. The later means that X has no flat deRham factor and the sectional curvature of X is non-positive. In particular, X is a Hadamard manifold and, hence, is diffeomorphic to a euclidean space. We refer to Eberlein’s book [Ebe96] for a detailed discussion of symmetric spaces.

Assumption 1. The symmetric spaces X is simply-connected and of noncompact type.

A symmetric space X can be written as G/K where G is a semisimple Lie group whose Lie algebra does not have compact and abelian factors, and K is a maximal compact subgroup of G . Moreover, this group G can be chosen to have finite center and be commensurable with the isometry group $\text{Isom}(X)$ of X . For example, one can choose G to be the identity component of $\text{Isom}(X)$.

³In [GMT19] the flag limit set is called the “symmetric” limit set. See [GMT19, Thm. 1.1] for details.

⁴Note that Finsler geodesic segments connecting two points in X are usually non-unique.

Assumption 2. The semisimple Lie group G has finite center and is commensurable with the isometry group $\text{Isom}(X)$ of the symmetric space X .

Each point $x \in X$ determines a canonical decomposition of the Lie algebra \mathfrak{g} of G called the *Cartan decomposition*,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where \mathfrak{k} is tangent to the stabilizer of x which is a conjugate of K . The dimension of a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is called the *rank* of X . The exponential map $\exp_x : \mathfrak{g} \rightarrow X$ identifies \mathfrak{a} with a maximal flat $F \subset X$ through x and, hence, the rank of X can also be defined as the dimension of a maximal totally geodesic flat subspace in X . A chosen maximal flat $F_{\text{mod}} \subset X$ is called the *model flat* which we isometrically identify with \mathbb{R}^k where $k = \text{rank}(X)$. The image in $\text{Isom}(F)$ of the G -stabilizer of F_{mod} is isomorphic to $\mathbb{R}^k \rtimes W$, where the first factor acts on $F_{\text{mod}} \cong \mathbb{R}^k$ by translations while the second factor W , called the *Weyl group*, is finite, fixes the origin, and is generated by hyperplane reflections. The closures of the connected components of the complement of the reflecting hyperplanes (for hyperplane reflections in W) in F_{mod} are called *chambers*. A chosen chamber is called the *model Weyl chamber*; we denote it by Δ .

1.2. Boundary at infinity: For a symmetric space X , there are multiple notions of (partial) boundary at infinity. The space of equivalence classes of asymptotic rays is called the *visual boundary* of X and denoted $\partial_\infty X$. The visual boundary is naturally identified with the unit tangent sphere $T_x^1 X$ at any point $x \in X$. The topology it gets from this identification is called the *visual topology*. Attaching the visual boundary to X provides a compactification of X .

Another (strictly finer) topology on $\partial_\infty X$ is given by the G -invariant *Tits angle metric*:

$$\angle_{\text{Tits}}(\zeta, \eta) = \sup_{x \in X} \angle_x(\zeta, \eta)$$

where $\angle_x(\zeta, \eta)$ denotes the angle between the rays emanating from x and asymptotic to ζ and η . The boundary $\partial_\infty X$ with this topology is denoted by $\partial_{\text{Tits}} X$ and called the *Tits boundary*.

The Tits boundary $\partial_{\text{Tits}} X$ carries a canonical G -invariant structure of a spherical simplicial complex called the *Tits building* of X . This can be understood as follows: Consider the ideal boundary $\partial_\infty F_{\text{mod}}$ of F_{mod} where $k = \text{rank}(X)$. This is identified with the unit sphere \mathfrak{a}^1 of \mathfrak{a} and thus, we have an action of the Weyl group $W \curvearrowright \partial_{\text{Tits}} F_{\text{mod}}$. The pair $(\partial_{\text{Tits}} F_{\text{mod}}, W)$ is a spherical Coxeter complex which generates a spherical simplicial complex structure on entire $\partial_{\text{Tits}} X$ by the G -action.

Assumption 3. We assume that the Tits building is *thick*, i.e., every simplex of codimension one is a face of three maximal simplices.

We denote the intersection of Δ with the unit sphere in F_{mod} centered at the origin by σ_{mod} . This is a fundamental domain for the action $W \curvearrowright \partial_{\text{Tits}} F_{\text{mod}}$ where $\partial_{\text{Tits}} F_{\text{mod}}$ is identified with the unit sphere in F_{mod} centered at the origin. We call σ_{mod} the *model chamber*. Any other chamber (i.e. a top-dimensional simplex) in the Tits building is naturally identified with σ_{mod} via a G -equivariant map, called the *type map*,

$$\theta : \partial_{\text{Tits}} X \rightarrow \sigma_{\text{mod}}.$$

We reserve the notation τ_{mod} for the faces of σ_{mod} . An ideal point $\zeta \in \partial_{\text{Tits}}X$ (resp. a simplex $\tau \subset \partial_{\text{Tits}}X$) is called of *type* $\bar{\theta} \in \sigma_{\text{mod}}$ (resp. of *type* $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$) if $\theta(\zeta) = \bar{\theta}$ (resp. $\theta(\tau) = \tau_{\text{mod}}$). For $\bar{\theta} \in \sigma_{\text{mod}}$ and a simplex τ of type τ_{mod} , we use the notation $\bar{\theta}(\tau)$ to denote the unique point in τ of type $\bar{\theta}$. The *opposition involution* ι is an automorphism of σ_{mod} which is defined as the negative of the longest element in the Weyl group.

Two simplices τ_1, τ_2 in the Tits building are called *antipodal* if there exists a point-reflection s_x swapping these two. Their types are related by $\theta(\tau_1) = \iota\theta(\tau_2)$. In particular, when τ_1 has an ι -invariant type τ_{mod} , then any antipodal simplex τ_2 also has type τ_{mod} . In this paper, we only consider types that are ι -invariant.

We now describe an important class of partial boundaries of X which are central to our study. Consider the action of G on the Tits building. The stabilizer of a face τ_{mod} of σ_{mod} is a parabolic subgroup $P_{\tau_{\text{mod}}}$ of G and we identify the quotient $G/P_{\tau_{\text{mod}}}$ with the set of all simplices in the Tits building of type τ_{mod} . This quotient $G/P_{\tau_{\text{mod}}}$ is a smooth compact manifold, called the *partial flag manifold* of type τ_{mod} and is denoted $\text{Flag}(\tau_{\text{mod}})$. The partial compactification of X by attaching $\text{Flag}(\tau_{\text{mod}})$ is denoted

$$\bar{X}^{\tau_{\text{mod}}} = X \cup \text{Flag}(\tau_{\text{mod}})$$

which is topologized via the topology of *flag convergence* (see Subsection 1.6). In the special case when $\tau_{\text{mod}} = \sigma_{\text{mod}}$, the associated parabolic subgroup $P_{\sigma_{\text{mod}}}$ is minimal and $\text{Flag}(\sigma_{\text{mod}}) = G/P_{\sigma_{\text{mod}}}$ is the *full flag manifold*, also called the *Furstenberg boundary* of X .

A subset $A \subset \text{Flag}(\tau_{\text{mod}})$ is called *antipodal* if any two distinct simplices in A are antipodal.

1.3. Δ -valued distances and generalized triangle inequality: There is a canonical map $d_{\Delta} : X \times X \rightarrow \Delta$ which is defined as follows: For a pair of points (x, y) in X , there is an element $g \in G$ which maps x to the origin in Δ and y to a point $v \in \Delta$. We define $d_{\Delta}(x, y) = v$. Note that the norm $\|d_{\Delta}(x, y)\|$ (induced by the euclidean inner product on $F_{\text{mod}} \cong \mathbb{R}^k$) equals $d_{\mathbb{R}}(x, y)$ where $d_{\mathbb{R}}$ denotes the distance function induced by the Riemannian metric on X .

For a pair $(x, y) \in X \times X$, the value $d_{\Delta}(x, y)$ is called the Δ -valued distance between x and y . This is a complete G -congruence invariant for oriented line segments in X . The Δ -valued distances satisfy generalized triangle inequalities (see [KLM09]). In the paper we will need the following triangle inequality. For $x, y, z \in X$,

$$\|d_{\Delta}(x, y) - d_{\Delta}(x, z)\| \leq d_{\mathbb{R}}(y, z). \quad (1.1)$$

1.4. Parallel sets, cones, and diamonds: For a detailed discussion on this subsection, we refer to [KLP14, Subsec. 2.4], [KLP17, Subsec. 2.5].

Let τ_{\pm} be a pair of antipodal simplices in the Tits building of X . The *parallel set* $P(\tau_+, \tau_-)$ is the union of all maximal flats in X whose ideal boundary contains $\tau_+ \cup \tau_-$ as a subset. This is a totally geodesic submanifold of X .

For a simplex τ , the *star* $\text{st}(\tau)$ of τ is the union of all chambers in the Tits building containing τ . The *open star* $\text{ost}(\tau)$ of τ is the union of all the open simplices whose closures contains τ . For a face τ_{mod} of σ_{mod} (viewed as a complex), define the open star $\text{ost}(\tau_{\text{mod}})$ similarly. The *boundary* $\partial\text{st}(\tau_{\text{mod}})$ is the complement of $\text{ost}(\tau_{\text{mod}})$ in σ_{mod} .

Let τ_{mod} be an ι -invariant face of σ_{mod} . An ideal point $\xi \in \partial_{\infty}X$ is called τ_{mod} -*regular* if its type is contained in $\text{ost}(\tau_{\text{mod}})$. Moreover, given an ι -invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$, an

ideal point $\xi \in \partial_\infty X$ is called Θ -regular if its type is contained in Θ . A nondegenerate geodesic segment (or line or ray) in X is called τ_{mod} -regular (resp. Θ -regular) if the ideal endpoints of its line extension are τ_{mod} -regular (resp. Θ -regular).

For a simplex τ in the Tits building and a point $x \in X$, the τ_{mod} -cone $V(x, \text{st}(\tau))$ with apex x is the union of all rays emanating from x asymptotic to a point $\xi \in \text{st}(\tau)$. For a τ_{mod} -regular geodesic segment $\overline{xy} \subset X$, the τ_{mod} -diamond $\diamond_{\tau_{\text{mod}}}(x, y)$ is the intersection of the opposite cones $V(x, \text{st}(\tau_+))$ and $V(y, \text{st}(\tau_-))$ containing it. The points x and y are called the *endpoints* of $\diamond_{\tau_{\text{mod}}}(x, y)$. The cones and parallel sets can be interpreted as limits of diamonds where, respectively, one or both endpoints diverges to infinity. All of these are convex subsets of X (see [KLP14, Prop. 2.14], [KLP17, Prop. 2.10]). In particular, the cones are *nested*: For every $y \in V(x, \text{st}(\tau))$, $V(y, \text{st}(\tau)) \subset V(x, \text{st}(\tau))$.

Let Θ be an ι -invariant compact subset $\text{ost}(\tau_{\text{mod}})$. In a similar way as above, the Θ -cone $V(x, \text{ost}_\Theta(\tau))$ with apex x is the union of all rays emanating from x asymptotic to a point $\xi \in \text{st}(\tau)$ of type Θ . Note that $V(x, \text{ost}_\Theta(\tau))$ is strictly contained inside $V(x, \text{st}(\tau))$.

1.5. Morse embeddings: The *Morse property* in higher rank was introduced by Kapovich-Leeb-Porti in [KLP14].

Recall that a *quasigeodesic* in X is a quasiisometric embedding $\phi : I \rightarrow X$ of an interval $I \subset \mathbb{R}$. We say that ϕ is τ_{mod} -regular *quasigeodesic* if for all sufficiently separated points $t_1, t_2 \in I$, the segment $\overline{\phi(t_1)\phi(t_2)}$ is τ_{mod} -regular. We say that ϕ is a τ_{mod} -Morse *quasigeodesic* if it is τ_{mod} -regular and for all sufficiently separated points $t_1, t_2 \in I$, the image $\phi([t_1, t_2])$ is uniformly close to $\diamond_{\tau_{\text{mod}}}(\phi(t_1), \phi(t_2))$.

Let Z be a geodesic Gromov-hyperbolic metric space (cf. Definition 4.2). A quasiisometric map $\phi : Z \rightarrow X$ is called a τ_{mod} -Morse *embedding* if the image of every geodesic is a τ_{mod} -Morse quasigeodesic with uniformly controlled coarse-geometric quantifiers: There exists a constant $D > 0$ and an ι -invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$ such that if $z_1 z_2$ is a geodesic segment in Z of length $\geq D$, then $\overline{\phi(z_1)\phi(z_2)}$ is a Θ -regular geodesic in X and the image $\phi([z_1, z_2])$ is D -close to $\diamond_{\tau_{\text{mod}}}(\phi(z_1), \phi(z_2))$.

A discrete finitely generated subgroup (equipped with a word metric) $\Gamma < G$ is called τ_{mod} -Morse if it is hyperbolic and, for an(y) $x \in X$, the orbit map $\Gamma \rightarrow \Gamma x$ is a τ_{mod} -Morse embedding.

1.6. Discrete subgroups of G and their limit sets: We consider discrete subgroups with various levels of regularity and their flag limit sets. Most of these notions first appear in the work of Benoist ([Ben97]); our discussion follows [KLP14] and [KLP17].

We first recall the notion of *regular* sequences in X . Let τ_{mod} be an ι -invariant face of σ_{mod} . Let $V(0, \partial \text{st}(\tau_{\text{mod}}))$ denote the union of all rays in Δ emanating from 0 asymptotic to points $\xi \in \partial \text{st}(\tau_{\text{mod}})$. A sequence (x_n) on X diverging to infinity is τ_{mod} -regular if for all $x \in X$, the sequence $(d_\Delta(x, x_n))_{n \in \mathbb{N}}$ in Δ diverges away from $V(0, \partial \text{st}(\tau_{\text{mod}}))$. Furthermore, a τ_{mod} -regular sequence (x_n) is called *uniformly* τ_{mod} -regular if the sequence $(d_\Delta(x, x_n))_{n \in \mathbb{N}}$ in Δ diverges away from $V(0, \partial \text{st}(\tau_{\text{mod}}))$ at a linear rate,

$$\liminf_{n \rightarrow \infty} \frac{d(d_\Delta(x, x_n), V(0, \partial \text{st}(\tau_{\text{mod}})))}{d(0, d_\Delta(x, x_n))} > 0.$$

where d denotes the euclidean distance on Δ . Accordingly, a sequence (g_n) in G is τ_{mod} -regular (resp. uniformly τ_{mod} -regular) if for some (equivalently, every) $x \in X$, the sequence $(g_n(x))$ is τ_{mod} -regular (resp. uniformly τ_{mod} -regular).

For $x \in X$ and $A \subset X$, define the *shadow of A on $\text{Flag}(\tau_{\text{mod}})$ from x* as

$$S(x : A) = \{\tau \in \text{Flag}(\tau_{\text{mod}}) \mid A \cap V(x, \text{st}(\tau)) \neq \emptyset\}. \quad (1.2)$$

Let (g_n) be a τ_{mod} -sequence on G . A sequence (τ_n) on $\text{Flag}(\tau_{\text{mod}})$ is called a *shadow sequence* of (g_n) if there exists $x \in X$ such that, for every $n \in \mathbb{N}$, $\tau_n = S(x : \{g_n x\})$. A τ_{mod} -regular sequence (g_n) is said to be *τ_{mod} -flag-convergent* to $\tau \in \text{Flag}(\tau_{\text{mod}})$ if a(ny) shadow sequence (τ_n) of (g_n) converges to τ .

The notion of flag-convergence leads to the definition of *flag limit sets* of discrete subgroups $\Gamma < G$. The (τ_{mod}) -*flag limit set* of Γ denoted by $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is the subset of $\text{Flag}(\tau_{\text{mod}})$ which consists of all limit simplices of τ_{mod} -flag-convergent sequences on Γ . The flag limit set $\Lambda_{\tau_{\text{mod}}}$ is Γ -invariant.

More generally, one defines τ_{mod} -flag-limit sets of a subset $Z \subset X$ as the accumulation subset of Z in $\text{Flag}(\tau_{\text{mod}})$ with respect to the topology of flag-convergence.

Next, we review definitions of several classes of discrete subgroups of G .

(R) A discrete subgroup $\Gamma < G$ is *τ_{mod} -regular* if for all $x \in X$ and all sequences of distinct elements (γ_n) in Γ , the sequence $(\gamma_n x)$ is τ_{mod} -regular. For τ_{mod} -regular subgroups Γ , the flag limit set $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ provides a compactification of the orbit $\Gamma x \subset X$, i.e. $\Gamma x \sqcup \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is compact.

(RA) A τ_{mod} -regular subgroup Γ is *τ_{mod} -RA (regular antipodal)* if its limit set $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is antipodal, i.e. every two distinct elements of $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ are antipodal to each other. For τ_{mod} -RA subgroups Γ , the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a convergence action (see [KLP14, Prop. 5.38]). A τ_{mod} -RA subgroup Γ is called *nonelementary* if $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ consists of at least three (hence infinitely many) points; otherwise Γ is called *elementary*. If Γ is nonelementary then the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is *minimal*, i.e. every orbit of Γ is dense, and $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is perfect.⁵

(RC) For a τ_{mod} -regular subgroup Γ , a limit simplex $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a *conical limit point* if there exists $x \in X$, $c > 0$ and a sequence (γ_n) of pairwise distinct isometries on Γ such that

$$d_{\mathbb{R}}(\gamma_n x, V(x, \text{st}(\tau))) \leq c$$

where $d_{\mathbb{R}}$ denotes the Riemannian distance on X . The set of all conical limit simplices is denoted by $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$. A subgroup $\Gamma < G$ is called *τ_{mod} -RC* if $\Lambda_{\tau_{\text{mod}}}(\Gamma) = \Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$.

(RCA) A subgroup Γ is *τ_{mod} -RCA* if it is both τ_{mod} -RA and τ_{mod} -RC.

(U) A finitely generated subgroup $\Gamma < G$ (equipped with the word metric) is said to be *undistorted* if one (equivalently, every) orbit map $\Gamma \rightarrow \Gamma x \subset X$ is a quasiisometric embedding.

(UR) A discrete subgroup $\Gamma < G$ is *uniformly τ_{mod} -regular* if for all $x \in X$ and all sequences of distinct elements (γ_n) in Γ , the sequence $(\gamma_n x)$ is uniformly τ_{mod} -regular.

(URU) A subgroup $\Gamma < G$ is said to be *τ_{mod} -URU* if it is both τ_{mod} -uniformly regular and undistorted.

⁵This follows from a general result for convergence actions by Gehring-Martin ([GM87]) and Tukia ([Tuk94]). See also [KLP14, Subsec. 3.2] or [KLP17, Subsec. 3.3].

In [KLP17, Equiv. Thm. 1.1] and [KLP18], the properties Morse, RCA and URU are proven to be equivalent to the Anosov property defined by Labourie [Lab06] and Guichard-Wienhard [GW12].

Theorem 1.1 ([KLP17, Equiv. Thm. 1.1]). *The following classes of nonelementary discrete subgroups of G are equal:*

- (i) $\tau_{\text{mod-RCA}}$,
- (ii) $\tau_{\text{mod-Morse}}$,
- (iii) $P_{\tau_{\text{mod}}}$ -Anosov,
- (iv) $\tau_{\text{mod-URU}}$.

Illustrating examples: In this paper, we consider the following two classes of examples.

Example 1.2 (Product of rank-one symmetric spaces). Let X be a product of k rank-one symmetric spaces,

$$X = X_1 \times \cdots \times X_k.$$

The rank of X is k . Let G be a semisimple Lie group commensurable with the isometry group of X . (For example, we may take $G = \text{Isom}(X_1) \times \cdots \times \text{Isom}(X_k)$.) The Assumption 3 amounts to the requirement that G preserves the factors of the direct product decomposition of X .

The model maximal flat F_{mod} can be viewed as the product of some chosen geodesic lines (coordinate axes), one for each deRham factor. The Weyl group W is generated by reflections along the coordinate hyperplanes and the longest element in it is the reflection about the origin. The model Weyl chamber Δ can be realized as the nonnegative orthant. The opposition involution ι acts on it trivially.

Recall that the Tits boundary of a product of two symmetric spaces is the simplicial join of their individual Tits buildings and, for rank-one symmetric spaces, the Tits boundary is discrete. These two facts imply that the $(p-1)$ -simplices in the Tits building of X for $1 \leq p \leq k$ can be parametrized by p -tuples $(\xi_{r_1}, \dots, \xi_{r_p}) \in \partial_{\infty} X_{r_1} \times \cdots \times \partial_{\infty} X_{r_p}$, $1 \leq r_1 < \cdots < r_p \leq k$,

$$(\xi_{r_1}, \dots, \xi_{r_p}) \leftrightarrow \tau = \text{span}\{\xi_{r_1}, \dots, \xi_{r_k}\}.$$

We say that such a simplex τ has type $\tau_{\text{mod}} = (r_1, \dots, r_p)$. The incidence structure can be understood as follows: Two simplices have a common q -face if and only if they have q equal coordinates.

The star $\text{st}(\tau)$ of $\tau = (\xi_{r_1}, \dots, \xi_{r_p})$ is the minimal subcomplex of the Tits building containing all chambers $(\zeta_1, \dots, \zeta_p)$ satisfying $\zeta_{r_i} = \xi_{r_i}$, for all $i \in \{1, \dots, p\}$.

Since the opposition involution ι fixes each chamber point-wise, every face τ_{mod} of σ_{mod} and every type is ι -invariant. Every two chambers (resp. faces of the same type) in $\partial_{\text{Tits}} X$ are antipodal to each other unless they have a common face (resp. sub-face).

Example 1.3 ($X = \text{SL}(k+1, \mathbb{R})/\text{SO}(k+1, \mathbb{R})$). We take $G = \text{SL}(k+1, \mathbb{R})$, $K = \text{SO}(k+1, \mathbb{R})$; the symmetric space $X = G/K$ is identified with the set of all positive definite, symmetric matrices in $\text{SL}(k+1, \mathbb{R})$. In this case $\text{rank}(X) = k$ and X is irreducible. The standard choice of a model flat F_{mod} is the subset of all diagonal matrices $a = \text{diag}(a_1, \dots, a_{k+1}) \in \text{SL}(k+1, \mathbb{R})$ with positive diagonal entries. We identify the model flat with a via the logarithm map

$$\log : a = \text{diag}(a_1, \dots, a_{k+1}) \mapsto (\log a_1, \dots, \log a_{k+1})$$

where \mathfrak{a} is viewed as the hyperplane in \mathbb{R}^{k+1} consisting of all points with zero sum of coordinates.

The Weyl group $W = \text{Sym}_{k+1}$ acts on \mathfrak{a} by permuting the coordinates. The standard choice for the model Weyl chamber $\Delta = \mathfrak{a}_+$ consists of all the points in \mathfrak{a} with decreasing coordinate entries. The Cartan projection⁶ $\rho : \text{SL}(k+1, \mathbb{R}) \rightarrow \mathfrak{a}_+$ can be written as $g \mapsto \log a$ where a is associated to g via the *singular value decomposition* $g = uav$, $u, v \in \text{SO}(k+1, \mathbb{R})$. The logarithm of i -th singular value of g will be denoted by $\sigma_i(g)$. The opposition involution ι sends $(\sigma_1, \dots, \sigma_{k+1}) \in \mathfrak{a}_+$ to $(-\sigma_{k+1}, \dots, -\sigma_1)$.

The Tits building of X can be identified with the incidence geometry of flags in \mathbb{R}^{k+1} . The Fursternberg boundary consists of full flags

$$V_1 \subset \dots \subset V_{k+1} = \mathbb{R}^{k+1}, \quad \dim(V_i) = i.$$

The partial flags are

$$V : V_{r_1} \subset \dots \subset V_{r_p} \subset V_{r_{p+1}} = \mathbb{R}^{k+1}, \quad \dim(V_{r_i}) = r_i,$$

$1 \leq r_1 < \dots < r_p < r_{p+1} = k+1$, which are elements of $\text{Flag}(\tau_{\text{mod}})$ where $\tau_{\text{mod}} = (r_1, \dots, r_p)$. The opposition involution sends τ_{mod} to $\iota\tau_{\text{mod}} = (k+1-r_p, \dots, k+1-r_1)$. It follows that τ_{mod} is ι -invariant if and only if $r_i + r_{p+1-i} = k+1$, for each $i = 1, \dots, p$. The partial flag manifold $\text{Flag}(\tau_{\text{mod}})$ consisting of all partial flags V of type $\tau_{\text{mod}} = (r_1, \dots, r_p)$ naturally embeds into the product of Grassmanians $\text{Gr}_{r_1}(\mathbb{R}^{k+1}) \times \dots \times \text{Gr}_{r_p}(\mathbb{R}^{k+1})$.

Suppose that $\tau_{\text{mod}} = (r_1, \dots, r_p)$ is ι -invariant. Then a pair $V^\pm \in \text{Flag}(\tau_{\text{mod}})$ is antipodal if and only if $V_{r_i}^+ + V_{r_{p+1-i}}^- = \mathbb{R}^{k+1}$ for each $i = 1, \dots, p$.

§2. Critical exponent

On a symmetric space $X = G/K$, we consider two natural (pseudo-)metrics. Let $d_{\mathbb{R}}(\cdot, \cdot)$ denote the distance function on X of the (fixed) G -invariant Riemannian metric on X . Furthermore, for a fixed ι -invariant face τ_{mod} of σ_{mod} and a fixed ι -invariant type $\bar{\theta}$ in the interior of τ_{mod} , we let $d_{\mathbb{F}}$ denote the polyhedral Finsler (pseudo-)metric on X :

$$d_{\mathbb{F}}(x, y) = \langle d_{\Delta}(x, y) | \bar{\theta} \rangle \quad (2.1)$$

(cf. [KL18a, Subsec. 5.1]). The inner product above is the euclidean inner product on F_{mod} coming from the Riemannian metric on X . These two metrics are related by the inequality

$$d_{\mathbb{F}}(x, y) \leq d_{\mathbb{R}}(x, y). \quad (2.2)$$

Since the Finsler metric $d_{\mathbb{F}}$ inherently depends on the choice of τ_{mod} and $\bar{\theta}$, from now on we fix $\bar{\theta}$ and use the notation $d_{\mathbb{F}}$ to denote the corresponding Finsler metric.

The metric space $(X, d_{\mathbb{R}})$ is a complete Riemannian manifold and, in particular, it is *geodesic*, i.e. any two points in X can be connected by a geodesic segment. The (pseudo-)metric space $(X, d_{\mathbb{F}})$ is also a geodesic space. The geodesics in $(X, d_{\mathbb{F}})$ are called *Finsler geodesics*. All the Riemannian geodesics are also Finsler, however, there are other Finsler geodesics when $\text{rank}(X) \geq 2$. The precise description of all Finsler geodesics is given in [KL18a, Subsec. 5.1.3]. We merely use this description as a definition of Finsler geodesics.

⁶Or the Δ -valued distance in the sense that $d_{\Delta}(x, gx) = \rho(g)$.

Definition 2.1 (Finsler geodesics). Let $I \subset \mathbb{R}$. A path $\ell : I \rightarrow X$ is called a *Finsler geodesic* if there exists a pair of antipodal flags $\tau_{\pm} \in \text{Flag}(\tau_{\text{mod}})$ such that $\ell(I) \subset P(\tau_+, \tau_-)$ and

$$\ell(t_2) \in V(\ell(t_1), \text{st}(\tau_+)), \quad \forall t_1 \leq t_2.$$

Moreover, given an ι -invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$, a Finsler geodesic $\ell : I \rightarrow X$ is called a Θ -*Finsler geodesic* if, in addition to the above, it satisfies the following stronger condition:

$$\ell(t_2) \in V(\ell(t_1), \text{ost}_{\Theta}(\tau_+)), \quad \forall t_1 \leq t_2.$$

Remark. Finsler geodesics give alternative description of diamonds, namely, the τ_{mod} -diamond $\diamond_{\tau_{\text{mod}}}(x, y)$ is the union of all Finsler geodesics connecting the endpoints x and y . See [KL18a, Subsec. 5.1.3].

Notation. In this paper, we use the notation \overline{xy} to denote the Riemannian geodesic segment connecting a pair of points $x, y \in X$. To denote a Finsler geodesic segment connecting x and y , we use the notation \widehat{xy} .

Below we let $*$ be either \mathbb{R} or \mathbb{F} . Let $\Gamma < G$ be a subgroup, and $x, x_0 \in X$. Define the *orbital counting function* $N_*(r, x, x_0) : [0, \infty) \rightarrow [0, \infty]$,

$$N_*(r) = N_*(r, x, x_0) = \text{card}\{\gamma \in \Gamma \mid d_*(x, \gamma x_0) < r\}.$$

Using $N_*(r)$, following [Alb99] and [Qui02b], we define the *critical exponent* δ_* of Γ by

$$\delta_* = \limsup_{r \rightarrow \infty} \frac{\log N_*(r)}{r} \in [0, \infty]. \quad (2.3)$$

The critical exponents $\delta_{\mathbb{F}}$ and $\delta_{\mathbb{R}}$ will be called the *Finsler critical exponent* and *Riemannian critical exponent*, respectively.

Remark. The discussion in [Alb99] and [Qui02b] is mostly limited to the case when $\bar{\theta}$ is regular, i.e. belongs to the interior of σ_{mod} .

We note that the critical exponent is independent of the chosen points x and x_0 . This can be proved as follows: Consider the Poincaré series

$$g_s^*(x, x_0) = \sum_{\gamma \in \Gamma} \exp(-s d_*(x, \gamma x_0)). \quad (2.4)$$

It is a standard fact that $g_s^*(x, x_0)$ converges if $s > \delta_*(x, x_0)$ and diverges if $s < \delta_*(x, x_0)$ where $\delta_*(x, x_0)$ denotes the right side of (2.3). Using the triangle inequality, we obtain

$$\exp(-s d_*(x, x_0)) g_s^*(x_0, x_0) \leq g_s^*(x, x_0) \leq \exp(s d_*(x, x_0)) g_s^*(x_0, x_0).$$

Hence, convergence or divergence of $g_s^*(x, x_0)$ is independent of the choice of x and so is $\delta_*(x, x_0)$. For a similar reason, it is also independent of the choice of x_0 .

Definition 2.2. A discrete subgroup Γ of G is of (*Finsler*) *convergence type* if the Poincaré series $g_s^{\mathbb{F}}(x, x_0)$ converges at the critical exponent $\delta_{\mathbb{F}}$. Otherwise, we say that Γ has (*Finsler*) *divergence type*.

Since the action $\Gamma \curvearrowright X$ is properly discontinuous, δ_R is bounded above by the *volume entropy* of X which is finite. For the Finsler critical exponent, (2.2) implies the following lower bound,

$$\delta_R \leq \delta_F. \quad (2.5)$$

Finiteness of δ_F is more subtle because, in general, d_F is only a pseudo-metric and therefore, the orbital counting function N_F may take infinity as a value. However, if the angular radius of the model Weyl chamber σ_{mod} with respect to $\bar{\theta}$ is $< \pi/2$, then d_F is a metric equivalent to d_R and, consequently, δ_F is finite in this case. In particular, when G is simple, then diameter of σ_{mod} is $< \pi/2$ and therefore, δ_F is finite.

The following finiteness result holds in the general pseudo-metric case.

Proposition 2.3. *For a uniformly τ_{mod} -regular subgroup $\Gamma < G$, the Finsler critical exponent δ_F is finite.*

Proof. When Γ is uniformly τ_{mod} -regular, the Riemannian and Finsler (pseudo-)metrics restricted to an orbit Γx are coarsely equivalent: There exist $L \geq 1, A \geq 0$ such that, for all $x_1, x_2 \in \Gamma x$,

$$L^{-1} d_R(x_1, x_2) - A \leq d_F(x_1, x_2) \leq d_R(x_1, x_2). \quad (2.6)$$

The right side of this inequality comes from (2.2). From this we get $\delta_R \leq \delta_F \leq L\delta_R$. Since δ_R is finite, δ_F is also finite. \square

Remark. 1. It is clear from the proof of Proposition 2.3 that when Γ is uniformly τ_{mod} -regular, then δ_F is positive if and only if δ_R is positive.

2. As Anosov subgroups are uniformly regular (see Theorem 1.1), the above proposition applies to the class of Anosov subgroups.

Before closing this section, we compute Finsler metrics in two examples.

Example 2.4 (Product of rank-one symmetric spaces). We continue with the discussion from Example 1.2. The Finsler metric can be described as follows. Let $\tau_{\text{mod}} = (r_1, \dots, r_p)$ be a face of the model chamber, let $\bar{\theta} = (1/\sqrt{p}, \dots, 1/\sqrt{p})$ be its barycenter, and let d_F be the corresponding metric on X . Given $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in X$, the Δ -valued distance is

$$d_\Delta(x, y) = (d_{X_1}(x_1, y_1), \dots, d_{X_k}(x_k, y_k))$$

where d_{X_i} denotes the Riemannian distance function on X_i . Then

$$d_F(x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^p d_{X_{r_j}}(x_{r_j}, y_{r_j}). \quad (2.7)$$

Example 2.5 ($X = \text{SL}(k+1, \mathbb{R})/\text{SO}(k+1, \mathbb{R})$). We continue with the discussion from Example 1.3. The Riemannian metric on X is given by the restriction of the Killing form B of $\mathfrak{g} = \mathfrak{sl}(k+1, \mathbb{R})$ to \mathfrak{p} ,

$$B(P, Q) = 2(k+1) \text{tr}(PQ^T), \quad P, Q \in \mathfrak{g}. \quad (2.8)$$

Note that the inner product B on \mathfrak{a} (which we identify with F_{mod}) can be written as

$$\langle (\sigma_1, \dots, \sigma_{k+1}) | (\sigma'_1, \dots, \sigma'_{k+1}) \rangle = 2(k+1) \sum_{i=1}^{k+1} \sigma_i \sigma'_i. \quad (2.9)$$

Let $\tau_{\text{mod}} = (r_1, \dots, r_p)$ be an ι -invariant face of the model chamber σ_{mod} and let $\Delta_{\tau_{\text{mod}}}$ be the corresponding face of the model euclidean Weyl chamber Δ ,

$$\Delta_{\tau_{\text{mod}}} = \left\{ \boldsymbol{\sigma} = \underbrace{(\sigma_1, \dots, \sigma_1)}_{r_1\text{-times}}, \dots, \underbrace{(\sigma_i, \dots, \sigma_i)}_{(r_i - r_{i-1})\text{-times}}, \dots, \underbrace{(\sigma_{p+1}, \dots, \sigma_{p+1})}_{(k+1 - r_p)\text{-times}} \in \mathfrak{a}_+ \mid i = 2, \dots, p \right\}.$$

For notational convenience we denote $\boldsymbol{\sigma}$ in the above expression simply by the $(p+1)$ -vector $(\sigma_1, \dots, \sigma_{p+1})$ (by identifying the repeated entries). With this convention, the opposition involution acts by

$$\iota(\sigma_1, \dots, \sigma_{p+1}) = (-\sigma_{p+1}, \dots, -\sigma_1).$$

We identify τ_{mod} with the unit sphere (w.r.t. the metric in (2.9)) in $\Delta_{\tau_{\text{mod}}}$ centered at the origin, i.e., τ_{mod} consists of all elements $(\sigma_1, \dots, \sigma_{p+1}) \in \Delta_{\tau_{\text{mod}}}$ satisfying $2(k+1) \sum_{i=1}^{p+1} (r_i - r_{i-1}) \sigma_i^2 = 1$. An element $\bar{\theta} = (\sigma_1, \dots, \sigma_{p+1}) \in \tau_{\text{mod}}$ lies in the interior of τ_{mod} if and only if $\sigma_1 > \dots > \sigma_{p+1}$. Moreover, $\bar{\theta}$ is ι -invariant if and only if $\sigma_i + \sigma_{p+2-i} = 0$ for all $i = 1, \dots, p+1$.

The Finsler metric corresponding to $\bar{\theta}$ can be calculated explicitly in terms of the above formulas. In the special case when $\tau_{\text{mod}} = (1, k)$ and $\bar{\theta} = (1/2\sqrt{k+1}, 0, 1/2\sqrt{k+1})$, for all $g \in \text{SL}(k+1, \mathbb{R})$ and all $x \in X$, we have

$$d_F(x, gx) = \sqrt{k+1} (\sigma_1(g) - \sigma_{k+1}(g)). \quad (2.10)$$

§3. Conformal densities

Recall that Busemann functions define the notion of “distance from infinity”. For $\tau \in \text{Flag}(\tau_{\text{mod}})$, let $b_\tau : X \rightarrow \mathbb{R}$ denote the Busemann function based at the ideal point $\bar{\theta}(\tau) \in \partial_\infty X$ normalized at x_0 (i.e., $b_\tau(x_0)$ is set to be zero). Using Busemann functions, one defines the *horospherical signed distance functions* as

$$d_\tau^{\text{hor}}(x, y) = b_\tau(x) - b_\tau(y). \quad (3.1)$$

(Note that these functions can take negative values. However, their absolute values $|d_\tau^{\text{hor}}(x, y)|$ satisfy the triangle inequality and, hence, are pseudo-metrics on X .) These functions are related by Finsler distance functions by

$$d_\tau^{\text{hor}}(x, y) = \lim_{n \rightarrow \infty} (d_F(x, z_n) - d_F(y, z_n)) \quad (3.2)$$

whenever (z_n) is a sequence in X flag-converging to τ , cf. [KL18a, Prop. 5.43].

We define conformal densities on $\text{Flag}(\tau_{\text{mod}})$ using these horospherical distance functions.

For a topological space S , we let $M_+(S)$ denote the set of positive, totally finite, regular Borel measures on S . Recall that a group H of self-homeomorphisms of S acts on $M_+(S)$ by *pull-back*: For every $B \in \mathfrak{B}(S)$, $h \in H$,

$$\mu \mapsto h^* \mu, \quad h^* \mu(B) = \mu(h^{-1}(B)).$$

Let $\Gamma < G$ be a discrete subgroup and let $A \subset X$ be a nonempty Γ -invariant subset. By a Γ -invariant conformal A -density μ of dimension $\beta \geq 0$ (or ‘‘conformal A -density’’ in short) on $\text{Flag}(\tau_{\text{mod}})$, we mean a continuous Γ -equivariant map

$$\mu : A \rightarrow M^+(\text{Flag}(\tau_{\text{mod}})), \quad a \mapsto \mu_a,$$

satisfying the following properties:

1. For each $a \in A$, $\text{supp}(\mu_a) \subset \Lambda_{\tau_{\text{mod}}}(\Gamma)$.
2. (Invariance) μ is Γ -invariant, i.e. $\gamma^* \mu_a = \mu_{\gamma a}$ for each $\gamma \in \Gamma$ and each $a \in A$.
3. (Conformality) For every pair $a, b \in A$, $\mu_a \ll \mu_b$, i.e., μ_a is absolutely continuous with respect to μ_b , and the Radon Nikodym derivative $d\mu_a/d\mu_b$ can be expressed as

$$\frac{d\mu_a}{d\mu_b}(\tau) = \exp\left(-\beta d_\tau^{\text{hor}}(a, b)\right), \quad \forall \tau \in \text{Flag}(\tau_{\text{mod}}). \quad (3.3)$$

Remark. Though we define conformal densities for any discrete subgroup of G , for the purpose we restrict our discussion only to τ_{mod} -regular subgroups.

A conformal X -density μ is simply called a *conformal density*. Note that conformal X -densities and conformal A -densities are in a one-to-one correspondence:

$$\{\text{conformal } X\text{-densities}\} \longleftrightarrow \{\text{conformal } A\text{-densities}\}. \quad (3.4)$$

From an X -density, define an A -density by restricting the family. On the other hand, given an A -density μ , extend it to an X -density $\{\mu_x\}_{x \in X}$ by

$$d\mu_x(B) = \int_B \exp\left(-\beta d_\tau^{\text{hor}}(x, a)\right) d\mu_a(\tau), \quad B \in \mathfrak{B}(\text{Flag}(\tau_{\text{mod}}))$$

where μ_a is a density in the family μ . Note that this extension is unique because μ_x and μ_a are absolutely continuous with respect to each other. To check Γ -invariance, note that

$$\begin{aligned} \gamma^* \mu_x(B) &= \int_{\gamma^{-1}B} \frac{d\mu_x}{d\mu_a}(\tau) d\mu_a(\tau) = \int_B \exp\left(-\beta d_{\gamma^{-1}\tau}^{\text{hor}}(x, a)\right) d\mu_a(\gamma^{-1}\tau) \\ &= \int_B \exp\left(-\beta d_\tau^{\text{hor}}(\gamma x, \gamma a)\right) d\mu_{\gamma a}(\tau) = \int_B \frac{d\mu_{\gamma x}}{d\mu_{\gamma a}}(\tau) d\mu_{\gamma a}(\tau) = \mu_{\gamma x}(B), \end{aligned}$$

for every $B \in \mathfrak{B}(\text{Flag}(\tau_{\text{mod}}))$. The other two defining properties are also satisfied.

Next we construct a conformal density using the Patterson-Sullivan construction. This definition is standard and already appeared in the work of Albuquerque and Quint, although only in the setting of Zariski dense subgroups $\Gamma < G$ and regular vectors $\bar{\theta}$; we present it here for the sake of completeness. We let $\Gamma < G$ be a τ_{mod} -regular subgroup and let Z denote the Γ -orbit of a point $x_0 \in X$. The union

$$\bar{Z} = Z \cup \Lambda_{\tau_{\text{mod}}}(\Gamma) \subset \bar{X}^{\tau_{\text{mod}}},$$

equipped with the topology of flag-convergence, is a compactification of Z .

For $s > \delta_F$ we define a family of positive measures $\mu_s = \{\mu_{x,s}\}_{x \in X}$ on \bar{Z} by

$$\mu_{x,s} = \frac{1}{g_s^F(x_0, x_0)} \sum_{\gamma \in \Gamma} \exp(-s d_F(x, \gamma x_0)) D(\gamma x_0), \quad (3.5)$$

where $D(\gamma x_0)$ denotes the Dirac point mass of weight one at γx_0 . Note that $\mu_{x,s}$ is a probability measure when $x \in Z$. Also, note that $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a null set for these measures. For $\gamma \in \Gamma$ a straightforward computation shows that

$$\gamma^* \mu_{x,s} = \mu_{\gamma x, s}. \quad (3.6)$$

Moreover, it is easy to see that the measures in the family μ_s are absolutely continuous with respect to each other. Using (3.5) we compute the Radon-Nikodym derivatives $d\mu_{x,s}/d\mu_{x_0,s}$,

$$\psi_s(z) = \frac{d\mu_{x,s}}{d\mu_{x_0,s}}(z), \quad (3.7)$$

where for $s \geq 0$,

$$\psi_s(z) := \exp(-s(d_F(z, x) - d_F(z, x_0))).$$

The formula for ψ_s above only makes sense when $z \in Z$. Since $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a null set, we extend ψ_s continuously to $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ by setting

$$\psi_s(\tau) = \exp(-s d_\tau^{\text{hor}}(x, x_0)).$$

The continuity of this function can be verified using properties of Finsler distances (e.g., see [KL18a, Sec. 5.1.2] and (3.2)).

Next we prove that $\psi_s \rightarrow \psi_{\delta_F}$ uniformly as $s \rightarrow \delta_F$. For $S \geq s, s' > \delta_F$ and $z \in Z$,

$$\begin{aligned} |\psi_{s'}(z) - \psi_s(z)| &= |\exp(-s'(d_F(z, x) - d_F(z, x_0))) - \exp(-s(d_F(z, x) - d_F(z, x_0)))| \\ &= \exp(-s(d_F(z, x) - d_F(z, x_0))) |\exp((s - s')(d_F(z, x) - d_F(z, x_0))) - 1| \\ &\leq \exp(Sd_R(x, x_0)) |\exp((s - s')(d_F(z, x) - d_F(z, x_0))) - 1|. \end{aligned}$$

Switching s and s' in the above, we also get

$$|\psi_{s'}(z) - \psi_s(z)| \leq \exp(Sd_R(x, x_0)) |\exp((s' - s)(d_F(z, x) - d_F(z, x_0))) - 1|.$$

Combining the above two inequalities, we get

$$\begin{aligned} |\psi_{s'}(z) - \psi_s(z)| &\leq \exp(Sd_R(x, x_0)) (\exp(|s' - s| \cdot |d_F(z, x) - d_F(z, x_0)|) - 1) \\ &\leq \exp(Sd_R(x, x_0)) (\exp(|s' - s|d_R(x, x_0)) - 1). \end{aligned}$$

Since Z is dense in \bar{Z} , the above yields

$$\|\psi_s - \psi_{s'}\|_\infty \leq \exp(Sd_R(x, x_0)) (\exp(|s' - s|d_R(x, x_0)) - 1)$$

Therefore, $\psi_s \rightarrow \psi_{\delta_F}$ uniformly as $s \rightarrow \delta_F$.

Now we construct a conformal density as a limit of the family of densities $\{\mu_s\}_{s>\delta_F}$. We first assume that Γ has divergence type.⁷ Then, as s decreases to δ_F , the family $\mu_s = \{\mu_{x,s}\}_{x \in X}$ weakly accumulates to a density μ supported on some subset of $\Lambda_{\tau_{\text{mod}}}(\Gamma)$.

By (3.6) we have the Γ -invariance of μ , namely, for $\gamma \in \Gamma$,

$$\gamma^* \mu_x = \mu_{\gamma x}. \quad (3.8)$$

Any such limit density is called a *Patterson-Sullivan density*.

Since μ_x is obtained as a weak limit of the measures $\mu_{x,s}$ and the derivatives $\psi_s = d\mu_{x,s}/d\mu_{x_0,s}$ converge uniformly to ψ_{δ_F} , it follows that the Radon-Nikodym derivative $d\mu_x/d\mu_{x_0}$ exists and equals to the limit

$$\lim_{s \rightarrow \delta_F} \frac{d\mu_{x,s}}{d\mu_{x_0,s}} = \psi_{\delta_F},$$

or more explicitly,

$$\frac{d\mu_x}{d\mu_{x_0}}(\tau) = \exp\left(-\delta_F d_\tau^{\text{hor}}(x, x_0)\right). \quad (3.9)$$

Note that in general weak limits are not unique. In Corollary 8.4 we will prove that for Anosov subgroups Γ we get a unique density in this limiting process.

When Γ has convergence type, we change weights of the Dirac masses by a small amount (as in [Nic89, Sec. 3.1]) in the definition (3.5) to force the Poincaré series to diverge. Define

$$\mu_{x,s} = \frac{1}{\bar{g}_s^F(x_0, x_0)} \sum_{\gamma \in \Gamma} \exp(-s d_F(x, \gamma x_0)) h(d_F(x, \gamma x_0)) D(\gamma x_0)$$

where $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a subexponential function such that the following modified Poincaré series

$$\bar{g}_s^F(x, x_0) = \sum_{\gamma \in \Gamma} \exp(-s d_F(x, \gamma x_0)) h(d_F(\gamma x, x_0))$$

diverges at the critical exponent $s = \delta_F$. In this case also, limit density μ has the properties (3.8) and (3.9).

The existence of a conformal density implies that the Finsler critical exponent of Γ is positive.

Proposition 3.1. *Suppose that Γ is a nonelementary τ_{mod} -regular antipodal subgroup. Then, the critical exponent δ_F is positive.*

Proof. Suppose to the contrary that $\delta_F = 0$. Let μ be a Patterson-Sullivan density constructed above. It follows from the Γ -invariance and conformality that for all $\gamma \in \Gamma$,

$$\mu_x(\gamma A) = \mu_{\gamma^{-1}x}(A) = \mu_x(A), \quad \forall A \in \mathfrak{B}(\Lambda_{\tau_{\text{mod}}}(\Gamma)). \quad (3.10)$$

Note that this implies that μ is atom-free. For if $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ were an atom, then, by the minimality of the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and (3.10), $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ would have infinite μ_x -mass,

Let (γ_n) be a sequence on Γ such that $\gamma_n^{\pm 1} \rightarrow \tau_{\pm} \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$. Let (U_n) , $U_n \subset \text{Flag}(\tau_{\text{mod}})$, be a contraction sequence⁸ for (γ_n) . By the definition,

⁷This will be the case for Anosov subgroups. See Corollary 6.5.

⁸See [KLP14, Def. 5.9, Prop. 5.14] or [KLP17, Defn. 4.1, Prop. 4.13].

- (1) (U_n) exhausts $\text{Flag}(\tau_{\text{mod}})$ in the sense that every compact set antipodal to τ_- is contained in U_n for all sufficiently large n .
- (2) The sequence $\gamma_n U_n$ Hausdorff-converges to τ_+ .

Let $A \subset \Lambda_{\tau_{\text{mod}}}(\Gamma) - \{\tau_+\}$ be a compact set of positive mass (this exists because τ_+ has zero mass). Therefore, by property (1), there exists $n_0 \in \mathbb{N}$ such that $\mu_x(U_n) \geq \mu_x(A) > 0$, for all $n \geq n_0$, and together with property (2) above, we get

$$\mu_x(\tau_+) \geq \lim_{n \rightarrow \infty} \mu_x(\gamma_n U_n) \geq \mu_x(A) > 0$$

Hence τ_+ is an atom which gives a contradiction. □

Remark. As a corollary to the above proposition, the Riemannian critical exponent $\delta_{\mathbb{R}}$ of a nonelementary uniformly τ_{mod} -regular antipodal subgroup is also positive. See the remark after Proposition 2.3.

§4. Hyperbolicity of Morse image

In this section we prove that the image of a Morse map is Gromov-hyperbolic with respect to the Finsler pseudo-metric $d_{\mathbb{F}}$. As a corollary, we prove that each orbit of an Anosov subgroup is also Gromov-hyperbolic with respect to the Finsler metric.

We first recall two notions of hyperbolicity.

Definition 4.1 (Rips hyperbolic). Let (Z, d) be a geodesic metric space. Then, (Z, d) is called $\delta(\geq 0)$ -hyperbolic in the sense of Rips (or Rips hyperbolic) if every geodesic triangle Δ is δ -thin, i.e. each side of Δ lies in the δ -neighborhood of the union of the other two sides.

Definition 4.2 (Gromov hyperbolic). Let (Z, d) be a geodesic metric space. For any three points $z, z_1, z_2 \in Z$, the Gromov product is defined as

$$\langle z_1 | z_2 \rangle_z = \frac{1}{2} [d(z, z_1) + d(z, z_2) - d(z_1, z_2)].$$

Then (Z, d) is called $\delta(\geq 0)$ -hyperbolic in the sense of Gromov (or Gromov hyperbolic) if the Gromov product satisfies the following ultrametric inequality: For all $z, z_1, z_2, z_3 \in Z$,

$$\langle z_1 | z_2 \rangle_z \geq \min\{\langle z_1 | z_3 \rangle_z, \langle z_2 | z_3 \rangle_z\} - \delta.$$

It should be noted that Gromov's definition applies to all metric spaces whereas Rips' definition works only for geodesic metric spaces. Moreover, Gromov hyperbolicity is not quasiisometric invariant whereas Rips hyperbolicity is (as a consequence of Morse lemma, cf. [DK18, Cor. 11.43])). For geodesic metric spaces, these two notions of hyperbolicity are equivalent (e.g., see [DK18, Lemma 11.27]).

Let (Z', d') be Rips hyperbolic and $f : (Z', d') \rightarrow (X, d_{\mathbb{R}})$ be a τ_{mod} -Morse map. We denote the image $f(Z')$ by Z . Recall that the Finsler metric is coarsely equivalent to the Riemannian metric on

Z .⁹ Therefore, since f is a quasiisometric embedding with respect to d_R , it is also a quasiisometric embedding with respect to d_F . Moreover, the image of a geodesic (of length bounded below by a constant) in Z' stays within a uniformly bounded Riemannian distance, say $\lambda_0 \geq 0$, from a τ_{mod} -regular Finsler geodesic connecting the images of the endpoints. This is a consequence of the Morse property ([KLP18, Thm. 1.1]), see also [KL18a, Prop. 12.2]. A consequence of this is that Z is λ_0 -quasiconvex in X with respect to the Finsler metric (or *Finsler quasiconvex*).

For $\lambda \geq \lambda_0$, let $Y = Y_\lambda$ be the Riemannian λ -neighborhood of Z in X . From the discussion above, it is clear that any two points $z_1, z_2 \in Z$ (with $d_R(z_1, z_2)$ sufficiently large) can be connected by a Finsler geodesic $\widehat{z_1 z_2}$ in Y .

Proposition 4.3. *Let c and c' be two Finsler geodesics in Y connecting two points z_1, z_2 . Then they are uniformly Hausdorff close. Here the Hausdorff distance is induced by either of Riemannian or Finsler metric.*

Proof. Since Riemannian and Finsler metrics are comparable on Y , it is enough to prove the proposition for the Riemannian metric.

Let \bar{c} and \bar{c}' be the respective nearest point projections of c and c' to Z . Applying the coarse inverse of f , \bar{c} and \bar{c}' map to uniform quasigeodesics \tilde{c} and \tilde{c}' , respectively, in Z' . Since Z' is Rips hyperbolic, \tilde{c} and \tilde{c}' are uniformly close. Applying f to \tilde{c} and \tilde{c}' , we see that \bar{c} and \bar{c}' are uniformly close. Hence c and c' are also uniformly close. \square

Next we observe that geodesic triangles in (Y, d_F) with vertices on Z are uniformly thin.

Proposition 4.4. *There exists $\delta \geq 0$ such that every Finsler geodesic triangle $\Delta = \Delta(z_1, z_2, z_3)$ in Y is δ -thin both in Riemannian and Finsler sense.*

Proof. Since Z' is Rips hyperbolic, geodesic triangles in Z' are δ' -thin, for some $\delta' \geq 0$. We map Δ to a uniformly quasigeodesic triangle $\Delta' \subset Z'$ via the coarse inverse map $Y \rightarrow Z'$ of the map f . Since Z' is Rips-hyperbolic, the Morse quasigeodesic triangle Δ' is uniformly thin. Therefore, Δ is also uniformly thin as well. \square

Imitating the proof of [DK18, Lem. 11.27], we prove that (Z, d_F) is Gromov-hyperbolic

Theorem 4.5 (Hyperbolicity of Morse maps). *Let $Z \subset X$ be the image of a τ_{mod} -Morse map $f : (Z', d') \rightarrow (X, d_R)$. Then (Z, d_F) is Gromov-hyperbolic.*

Proof. Let δ be as in Proposition 4.4. Then the following holds.

Lemma 4.6. *Let $z, z_1, z_2 \in Z$, and let $\widehat{z_1 z_2}$ be any Finsler geodesic in Y connecting z_1 and z_2 . Then,*

$$\langle z_1 | z_2 \rangle_z \leq d_F(z, \widehat{z_1 z_2}) \leq \langle z_1 | z_2 \rangle_z + 2\delta.$$

Proof. The proof is exactly same as [DK18, Lem. 11.22]. Note that the proof uses δ -thinness of a triangle with vertices z, z_1, z_2 . \square

⁹This is also true for any finite Riemannian tubular neighborhood of Z .

Let z, z_1, z_2, z_3 be any four points in Z , and let Δ be a Finsler geodesic triangle in Y with the vertices z_1, z_2, z_3 . Let m be a point on the side $\widehat{z_1 z_2}$ nearest to z . By Proposition 4.4, since Δ is δ -thin, $d_F(m, \widehat{z_2 z_3} \cup \widehat{z_1 z_3}) \leq \delta$. Without loss of generality, assume that there is a point n on z_2, z_3 which is δ -close to m . Then, using the above lemma, we get

$$\langle z_2 | z_3 \rangle_z \leq d_F(z, \widehat{z_2 z_3}) \leq d_F(z, \widehat{z_1 z_2}) + \delta,$$

and

$$d_F(z, \widehat{z_1 z_2}) \leq \langle z_1 | z_2 \rangle_z + 2\delta.$$

The theorem follows from this. \square

Quasiisometry of hyperbolic metric spaces extends to a homeomorphism of their Gromov boundaries. At the same time, it is proven in [KLP18] that each τ_{mod} -Morse map

$$f : Z' \rightarrow Z = f(Z') \subset X$$

extends continuously (with respect to the topology of flag-convergence) to a homeomorphism

$$\partial_\infty f : \partial_\infty Z' \rightarrow \Lambda \subset \text{Flag}(\tau_{\text{mod}}).$$

Thus, we obtain

Corollary 4.7. *The Gromov boundary $\partial_\infty Z$ of (Z, d_F) is naturally identified with the flag-limit set $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$ of Z : A sequence (z_n) in Z converges to a point in $\partial_\infty Z$ if and only if (z_n) flag-converges to some $\tau \in \Lambda$.*

For a τ_{mod} -Anosov subgroup Γ we know that the orbit map $\Gamma \rightarrow \Gamma x_0$ is a τ_{mod} -Morse embedding (see Subsection 1.6). Then, using Theorem 4.5 we obtain:

Corollary 4.8 (Hyperbolicity of Anosov orbits). *For $x_0 \in X$, let $Z = \Gamma x_0$ where Γ is a τ_{mod} -Anosov subgroup. Then (Z, d_F) is Gromov-hyperbolic. The Gromov boundary of (Z, d_F) is naturally identified with the τ_{mod} -limit set $\Lambda_{\tau_{\text{mod}}}(\Gamma)$.*

§5. Gromov distance at infinity

The definition of horospherical signed distances given in (3.1) is free of choice of any particular normalization for the Busemann functions. Note that

$$-d_F(x_1, x_2) \leq d_\tau^{\text{hor}}(x_1, x_2) \leq d_F(x_1, x_2).$$

Furthermore, d_τ^{hor} satisfies the *cocycle condition*: For each triple $x_1, x_2, x_3 \in X$,

$$d_\tau^{\text{hor}}(x_1, x_2) + d_\tau^{\text{hor}}(x_2, x_3) = d_\tau^{\text{hor}}(x_1, x_3). \quad (5.1)$$

For a pair of antipodal simplices $\tau_\pm \in \text{Flag}(\tau_{\text{mod}})$, the *Gromov product* with respect to a base point $x \in X$ is defined as

$$\langle \tau_+ | \tau_- \rangle_x = \frac{1}{2} \left(d_{\tau_+}^{\text{hor}}(x, z) + d_{\tau_-}^{\text{hor}}(x, z) \right), \quad (5.2)$$

where z is some point on the parallel set $P(\tau_+, \tau_-)$ spanned by τ_\pm .

The following lemma proves that the Gromov products do not depend on the chosen $z \in P(\tau_+, \tau_-)$.

Lemma 5.1. For $z_1, z_2 \in P(\tau_+, \tau_-)$, one has $b_{\tau_+}(z_1) + b_{\tau_-}(z_1) = b_{\tau_+}(z_2) + b_{\tau_-}(z_2)$.

Proof. Let z be the midpoint of $\overline{z_1 z_2}$ and let $s_z : X \rightarrow X$ be the point reflection about z . Assuming that Busemann functions are normalized at z , s_z transforms $b_{\tau_+}(z_1) + b_{\tau_-}(z_1)$ into $b_{\tau_-}(z_2) + b_{\tau_+}(z_2)$. Hence the quantities are equal. \square

Using horospherical distances, we define a *premetric*¹⁰ on $\text{Flag}(\tau_{\text{mod}})$.

Definition 5.2 (Gromov premetric). Given fixed $x \in X$, $\epsilon > 0$, define the *Gromov premetric* $d_G^{\epsilon, x}$ on $\text{Flag}(\tau_{\text{mod}})$ as

$$d_G^{\epsilon, x}(\tau_1, \tau_2) = \begin{cases} \exp(-\epsilon \langle \tau_1 | \tau_2 \rangle_x), & \text{if } \tau_1, \tau_2 \text{ are opposite} \\ 0, & \text{otherwise.} \end{cases}$$

Note that a pair of points $\tau_{\pm} \in \text{Flag}(\tau_{\text{mod}})$ is antipodal if and only if $d_G^{\epsilon, x}(\tau_+, \tau_-) \neq 0$.

Lemma 5.3. $d_G^{\epsilon, x}$ is a continuous function.

Proof. The claim follows from [Bey17, Lem. 3.8]. \square

Lemma 5.4. Let $\gamma \in G$ and $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$ be a γ -invariant antipodal subset. Then the map $\gamma : \Lambda \rightarrow \Lambda$ is conformal with respect to the premetric $d_G^{x, \epsilon}$.

Proof. Given distinct points $\tau_{\pm} \in \Lambda$,

$$\begin{aligned} d_G^{x, \epsilon}(\gamma\tau_+, \gamma\tau_-) &= \exp(-\epsilon \langle \gamma\tau_+ | \gamma\tau_- \rangle_x) \\ &= \exp\left(-\frac{\epsilon}{2} \left(d_{\gamma\tau_+}^{\text{hor}}(x, z) + d_{\gamma\tau_-}^{\text{hor}}(x, z)\right)\right) \\ &= \exp\left(-\frac{\epsilon}{2} \left(d_{\tau_+}^{\text{hor}}(\gamma^{-1}x, \gamma^{-1}z) + d_{\tau_-}^{\text{hor}}(\gamma^{-1}x, \gamma^{-1}z)\right)\right) \\ &= \exp\left(-\frac{\epsilon}{2} \left(d_{\tau_+}^{\text{hor}}(\gamma^{-1}x, x) + d_{\tau_-}^{\text{hor}}(\gamma^{-1}x, x)\right)\right) d_G^{x, \epsilon}(\tau_+, \tau_-), \end{aligned}$$

where the last equality follows from the cocycle condition (5.1). Moreover, the continuity of Busemann functions b_τ as a function of τ implies that $d_{\tau_-}^{\text{hor}}(\gamma^{-1}x, x) \rightarrow d_{\tau_+}^{\text{hor}}(\gamma^{-1}x, x)$ as $\tau_- \rightarrow \tau_+$. Therefore,

$$\lim_{\tau_- \rightarrow \tau_+} \frac{d_G^{x, \epsilon}(\gamma\tau_+, \gamma\tau_-)}{d_G^{x, \epsilon}(\tau_+, \tau_-)} = E(\gamma, \tau_+) := \exp\left(-\epsilon d_{\tau_+}^{\text{hor}}(\gamma^{-1}x, x)\right). \quad (5.3)$$

The lemma follows from this. \square

The premetric $d_G^{\epsilon, x}$ is not a metric in general since:

1. Pairs of distinct non-antipodal points have zero distance.
2. The triangle inequality may fail.

However, as we shall see below, $d_G^{\epsilon, x}$ defines a metric when restricted to “nice” antipodal subsets $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$ for sufficiently small $\epsilon > 0$.

¹⁰A *premetric* on X is a symmetric, continuous function $d : X \times X \rightarrow [0, \infty)$ such that $d(x, x) = 0$ for all $x \in X$.

Theorem 5.5. *Let $Z \subset X$ be the image of a τ_{mod} -Morse map $f : (Z', d') \rightarrow (X, d)$, and $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$ be the flag limit set of Z . There exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$ and all $x \in Z$, the premetric $d_G^{\epsilon, x}$ restricts to a metric on Λ . Moreover, the topology induced by $d_G^{\epsilon, x}$ on Λ coincides with the subspace topology of $\Lambda \subset \text{Flag}(\tau_{\text{mod}})$.*

Proof. For the first part of the theorem we only need to check that $d_G^{\epsilon, x}$ satisfies the triangle inequality for sufficiently small $\epsilon > 0$. The idea of the proof is due to Gromov [Gro87]: We show that the Gromov product defined in (5.2) restricted to Λ satisfies an ultrametric inequality (see (5.7)).

Let $Y \subset X$ be a Riemannian λ -neighborhood of Z . We assume that λ here is so large such that $x \in Y$ and the image of any complete geodesic l in Z' lies within distance λ from the parallel set spanned by the images of the ideal endpoints of l under $\bar{f} : \partial_\infty Z' \rightarrow \text{Flag}(\tau_{\text{mod}})$. Note that λ satisfying the last condition exists as a consequence of the Morse property.

Observe that (Y, d_F) is a Gromov δ -hyperbolic metric space for some $\delta \geq 0$. This follows from the Gromov hyperbolicity of (Z, d_F) (cf. Theorem 4.5) and the fact that Z and Y are (Hausdorff) λ -close to each other.

We recall from Väisälä [V05, Sec. 5] that there are multiple ways to define Gromov products on Λ viewed as the Gromov boundary of (Z, d_F) and, hence, of (Y, d_F) . For a distinct pair $\tau_\pm \in \Lambda$, define using the Gromov product $\langle \cdot | \cdot \rangle_x$ on (Y, d_F) the following two products:

$$\langle \tau_+ | \tau_- \rangle_x^{\text{inf}} = \inf \left\{ \liminf_{i, j \rightarrow \infty} \langle y_i^+ | y_j^- \rangle_x \mid (y_n^\pm) \subset Y, y_n^\pm \rightarrow \tau_\pm \right\}$$

and

$$\langle \tau_+ | \tau_- \rangle_x^{\text{sup}} = \sup \left\{ \limsup_{i, j \rightarrow \infty} \langle y_i^+ | y_j^- \rangle_x \mid (y_n^\pm) \subset Y, y_n^\pm \rightarrow \tau_\pm \right\}.$$

Then the difference of the above two quantities is uniformly bounded (see [V05, 5.7]), namely, for all distinct pairs $\tau_\pm \in \Lambda$,

$$0 \leq \langle \tau_+ | \tau_- \rangle_x^{\text{sup}} - \langle \tau_+ | \tau_- \rangle_x^{\text{inf}} \leq 2\delta. \quad (5.4)$$

Finally, $\langle \cdot | \cdot \rangle_x^{\text{inf}}$ satisfies the ultrametric inequality (see [V05, 5.12]), i.e., for distinct triples $\tau_1, \tau_2, \tau_3 \in \Lambda$,

$$\langle \tau_1 | \tau_2 \rangle_x^{\text{inf}} \geq \min \left\{ \langle \tau_1 | \tau_3 \rangle_x^{\text{inf}}, \langle \tau_2 | \tau_3 \rangle_x^{\text{inf}} \right\} - \delta. \quad (5.5)$$

By (5.4), $\langle \cdot | \cdot \rangle_x^{\text{sup}}$ also satisfies the ultrametric inequality but with a different constant, 5δ .

Next we compare Väisälä's Gromov products with ours (see (5.2)). Let $\tau_\pm \in \Lambda$ be a pair of opposite points and let $P = P(\tau_+, \tau_-)$. Note that our assumption on largeness of λ implies that there exist uniformly τ_{mod} -regular sequences (y_n^+) and (y_n^-) on $Y \cap P$ such that $y_n^\pm \rightarrow \tau_\pm$ as $n \rightarrow \infty$. Let $p \in P(\tau_+, \tau_-)$. Then, the additivity of Finsler distances on τ_{mod} -cones (cf. [KL18a, Lem. 5.10]) yields, for large n , $\langle y_n^+ | y_n^- \rangle_p = 0$. By definition,

$$\langle y_n^+ | y_n^- \rangle_x = \langle y_n^+, y_n^- \rangle_z + \frac{1}{2} \left[(d_F(y_n^+, x) - d_F(y_n^+, p)) + (d_F(y_n^-, x) - d_F(y_n^-, p)) \right],$$

and for large n ,

$$\langle y_n^+ | y_n^- \rangle_x = \frac{1}{2} \left[(d_F(y_n^+, x) - d_F(y_n^+, p)) + (d_F(y_n^-, x) - d_F(y_n^-, p)) \right].$$

The limit, as $n \rightarrow \infty$, of the right side of this equation equals $\langle \tau_+ | \tau_- \rangle_x$ (cf. (3.2)). Therefore,

$$\langle \tau_+ | \tau_- \rangle_x^{\text{inf}} \leq \langle \tau_+, \tau_- \rangle_x \leq \langle \tau_+ | \tau_- \rangle_x^{\text{sup}}. \quad (5.6)$$

Hence, by (5.4) and (5.5), $\langle \cdot | \cdot \rangle_x$ satisfies the ultrametric inequality with constant 5δ , i.e., for distinct points $\tau_1, \tau_2, \tau_3 \in \Lambda$,

$$\langle \tau_1 | \tau_2 \rangle_x \geq \min \{ \langle \tau_1 | \tau_3 \rangle_x, \langle \tau_2 | \tau_3 \rangle_x \} - 5\delta. \quad (5.7)$$

This completes the proof of the first part of the theorem.

For the second part, note that the inequality (5.6) implies that $d_G^{\epsilon, x}$ induces the standard topology on Λ as the Gromov boundary of (Y, d_F) (see [V05, 5.29]). Since, as we noted earlier, this topology is the same as the subspace topology of the flag-manifold $\text{Flag}(\tau_{\text{mod}})$, the second claim of the theorem follows as well. \square

Corollary 5.6 (Conformal metric on Anosov limit set). *Let Γ be a τ_{mod} -Anosov subgroup, $x \in X$. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ and all $z \in \Gamma x$, $d_G^{\epsilon, z}$ is a metric on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$. Moreover, the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is conformal with respect to $d_G^{\epsilon, z}$.*

Proof. Since Anosov subgroups satisfy the Morse property, corollary follows from Theorem 5.5 combined with Lemma 5.4. \square

Example 5.7 (Product of rank-one symmetric spaces). We continue with Example 2.4. Let $\tau = (\xi_{r_1}, \dots, \xi_{r_p})$ be a simplex in the Tits building of type $\tau_{\text{mod}} = (r_1, \dots, r_p)$ and $\bar{\theta} = (1/\sqrt{p}, \dots, 1/\sqrt{p}) \in \tau_{\text{mod}}$. We compute the horospherical distance, Gromov distance associated with τ_{mod} and type $\bar{\theta}$.

Let $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k) \in X$. Then $d_\tau^{\text{hor}}(x, y) = \lim_{t \rightarrow \infty} d_X(\ell_k(t), x) - t$, where $\ell(t)$ is a geodesic ray emanating from y and asymptotic to $\bar{\theta}(\tau)$. A direct computation yields

$$d_\tau^{\text{hor}}(x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^p \left(b_{\xi_{r_j}}(x_{r_j}) - b_{\xi_{r_j}}(y_{r_j}) \right) = \frac{1}{\sqrt{p}} \sum_{j=1}^p d_{\xi_{r_j}}^{\text{hor}}(x_{r_j}, y_{r_j}).$$

Hence the Gromov product can be written as

$$\langle \tau_+ | \tau_- \rangle_x = \frac{1}{\sqrt{p}} \sum_{j=1}^p \langle \xi_{r_j}^+ | \xi_{r_j}^- \rangle_{x_{r_j}}, \quad \forall \tau_\pm = (\xi_{r_1}^\pm, \dots, \xi_{r_p}^\pm) \in \text{Flag}(\tau_{\text{mod}})$$

and, finally the Gromov predistance is

$$d_G^{1/\sqrt{p}, x}(\tau_+, \tau_-) = \prod_{j=1}^p d_G^{x_{r_j}}(\xi_{r_j}^+, \xi_{r_j}^-)^{\frac{1}{p}}. \quad (5.8)$$

Example 5.8 ($X = \text{SL}(k+1, \mathbb{R})/\text{SO}(k+1, \mathbb{R})$). In this case the computations of Busemann functions (see [Hat95]) and Gromov products (see [Bey17]) are explicitly known, and therefore, the Gromov distance can also be computed explicitly. We only give a formula for the Gromov distance in the special case when $\tau_{\text{mod}} = (1, k)$ that corresponds to the partial flags $\{\text{line} \subset \text{hyperplane}\}$ of \mathbb{R}^{k+1} .

We continue with the notations from Example 2.5. The unique ι -invariant type is $\bar{\theta} = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1})$. After equipping \mathbb{R}^{k+1} with the inner product induced by the choice of $x \in X$, the Gromov product (with respect to $x = I_{k+1}$, the identity matrix) can be written as

$$\langle (l_1, h_1) | (l_2, h_2) \rangle_x = -\frac{\sqrt{k+1}}{2} \log(\sin \angle(l_1, h_2) \cdot \sin \angle(l_2, h_1))$$

where $\angle(l, h)$ denotes the angle between the line l and the hyperplane h . Thus, the Gromov distance can be written as

$$d_G^{1/\sqrt{k+1}, x}((l_1, h_1), (l_2, h_2)) = \sin \angle(l_1, h_2)^{\frac{1}{2}} \cdot \sin \angle(l_2, h_1)^{\frac{1}{2}}. \quad (5.9)$$

§6. Shadow lemma

In this section we prove a generalization Sullivan's shadow lemma in higher rank. The proof we present here is inspired by that of Albuquerque's ([Alb99, Thm. 3.3]) who treated the case of full flag manifold and Quint ([Qui02b]) who treated general flag-manifolds but only in the case of regular vectors $\bar{\theta}$.

Recall the notion of *shadow* from (1.2). We mainly consider shadows of closed balls (with respect to the Riemannian metric) of non-zero radii in X from a fixed base point $x \in X$. The topology generated by these shadows is the topology of flag convergence.

The main result in this section is the following.

Theorem 6.1 (Shadow lemma). *Let Γ be a nonelementary τ_{mod} -RA subgroup, $x \in X$, and μ a Γ -invariant conformal density of dimension β . There exists $r_0 > 0$ such that for all $r \geq r_0$ and all $\gamma \in \Gamma$ satisfying $d_R(x, \gamma x_0) > r$,*

$$C^{-1} \exp(-\beta d_F(x, \gamma x_0)) \leq \mu_x(S(x : B(\gamma x_0, r))) \leq C \exp(-\beta d_F(x, \gamma x_0)),$$

for some constant $C \geq 1$.

Before presenting the proof, we note two consequences of this theorem.

Corollary 6.2. *Let Γ be a nonelementary uniformly τ_{mod} -RA subgroup. Then any conformal density μ does not have conical limit points as atoms.*

Proof. Any conical limit point $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ lies in infinitely many shadows $S(x, B(\gamma x_0, r))$ for sufficiently large $r > 0$ (depending on τ). If τ is an atom, then (by Theorem 6.1) the Poincaré series

$$g_\beta^F(x, x_0) = \sum_{\gamma \in \Gamma} \exp(-\beta d_F(x, \gamma x_0)) \quad (6.1)$$

diverges for every $\beta \geq 0$. Hence δ_F must be infinite. But this contradicts Proposition 2.3. \square

The second application of Shadow Lemma will be given for the following class of subgroups.

Definition 6.3 (Uniform conicality). A τ_{mod} -RA subgroup is called *uniformly conical* if for a given pair of points $x, x_0 \in X$, there is a constant $r > 0$ such that for each conical limit point $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$, there exists a sequence (γ_k) on Γ flag-converging to τ satisfying $d_R(\gamma_k x_0, V(x, \text{st}(\tau))) < r, \forall k \in \mathbb{N}$.

We observe that Anosov subgroups satisfy the uniform conicality condition:

Proposition 6.4. *Anosov subgroups are uniformly conical.*

Proof. This follows from the fact that the orbit map $\Gamma \rightarrow \Gamma x_0 \subset X$ is a Morse embedding. Let $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ be any point and $\xi \in \partial_\infty \Gamma$ be the preimage of τ under the boundary map. Let (γ_k) , $\gamma_1 = 1_\Gamma$ be a geodesic sequence in Γ asymptotic to ξ . Then the sequence $(\gamma_k x_0)$ is a Morse quasigeodesic in X that is uniformly close to $V(x, \text{st}(\tau))$ (by definition of a Morse embedding). \square

Corollary 6.5. *Let Γ be a nonelementary uniformly conical τ_{mod} -RA subgroup and μ be a conformal density of dimension β . If the conical limit set $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$ is non-null, then the Poincaré series $g_\beta^{\text{F}}(x, x_0)$ (see (6.1)) diverges.*

Proof. Writing the elements of Γ in a sequence (γ_n) , define $S_N = \sum_{n \geq N} \exp(-\beta d_{\text{F}}(\gamma_n x_0, x))$. Convergence of the series (6.1) asserts that $\lim_{N \rightarrow \infty} S_N = 0$. Since Γ is uniformly conical, there exists $r > 0$ such that for all $N \in \mathbb{N}$,

$$\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma) \subset \bigcup_{n \geq N} S(x : B(\gamma_n x_0, r)).$$

Applying Theorem 6.1, we get

$$\mu_x(\Lambda_{\tau_{\text{mod}}}(\Gamma)) \leq \sum_{n \geq N} \mu_x(S(x : B(\gamma_n x_0, r))) \leq \text{const} \cdot S_N$$

and, the bound above approaches to zero as $N \rightarrow \infty$. Hence we must have $\mu_x(\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)) = 0$. \square

The proof of Shadow Lemma occupies the rest of the section.

Proof of Theorem 6.1. In this proof, we equip $\text{Flag}(\tau_{\text{mod}})$ with a G_x -invariant Riemannian metric. We use the notation $L(\tau)$ to denote the set of all $\tau' \in \text{Flag}(\tau_{\text{mod}})$ which are not antipodal to τ . The complement of $L(\tau)$ in $\text{Flag}(\tau_{\text{mod}})$ is denoted by $C(\tau)$. Note that $L(\tau)$ is closed and hence, compact. Moreover, if $\tau_n \rightarrow \tau_0$, then the sequence of sets $(L(\tau_n))$ Hausdorff-converges to $L(\tau_0)$.

Lemma 6.6. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\tau_0 \in \text{Flag}(\tau_{\text{mod}})$ and every $\tau \in B(\tau_0, \delta)$,*

$$N_{\varepsilon/2}(L(\tau)) \subset N_\varepsilon(L(\tau_0)).$$

Proof. We equip the set

$$Y = \{L(\tau) : \tau \in \text{Flag}(\tau_{\text{mod}})\}$$

with the Hausdorff distance d_{H} . Then, as we noted above, the function $f : \text{Flag}(\tau_{\text{mod}}) \rightarrow Y$, $\tau \mapsto L(\tau)$, is continuous and, hence, uniformly continuous. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\tau, \tau_0) < \delta$ implies $d_{\text{H}}(L(\tau), L(\tau_0)) < \varepsilon/2$, which then implies $L(\tau) \subset N_{\varepsilon/2}(L(\tau_0))$. The lemma follows from this. \square

Let $m = \mu_x(\Lambda_{\tau_{\text{mod}}}(\Gamma))$ denote the total mass of μ_x , and $l = \sup\{\mu_x(\tau) \mid \tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)\}$. Since μ_x is a regular measure and $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is compact, l is realized, i.e. if μ_x has an atomic part, then it has a largest atom. Moreover, since Γ is nonelementary, $\text{supp}(\mu_x)$ is not singleton. In fact, if τ is an atom, then the every point in the orbit $\Gamma\tau$ (which has infinite number of points) is an atom. In particular, $l < m$.

Lemma 6.7. *Given $l < q < m$, there exists an $\varepsilon_0 > 0$ such that for all $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and all $B \in \mathfrak{B}(\text{Flag}(\tau_{\text{mod}}))$ contained in $N_{\varepsilon_0}(L(\tau))$, $\mu_x(B) \leq q$.*

Proof. If this were false, then we would get a sequence (B_n) of Borel sets, a sequence (ε_n) positive numbers converging to zero, and a sequence (τ_n) on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ converging to a point τ_0 such that for every $n \in \mathbb{N}$,

$$B_n \subset N_{\varepsilon_n}(L(\tau_n)), \quad \mu_x(B_n) > q.$$

To get a contradiction, we will show that $\mu_x(\tau_0) \geq q$. Let U be an open neighborhood of $L(\tau_0)$. As $L(\tau_0)$ is compact, there exists $\varepsilon > 0$ such that $N_\varepsilon(L(\tau_0)) \subset U$. Let $\delta > 0$ be a number that corresponds to this ε as in Lemma 6.6. Choose n so large such that $\tau_n \in B(\tau_0, \delta)$ and $\varepsilon_n \leq \varepsilon/2$. By Lemma 6.6, we get $N_{\varepsilon_n}(L(\tau_n)) \subset N_\varepsilon(L(\tau_0))$ and, consequently, $B_n \subset U$. This shows that every open set U containing $L(\tau_0)$ has mass $\mu_x(U) > q$. Therefore, $\mu_x(L(\tau_0)) = \mu_x(\tau_0) \geq q$. \square

Lemma 6.8. *Given $\varepsilon > 0$ there exists $r_1 > 0$ such that for all $r \geq r_1$, the complement of $S(x : B(x_0, r))$ in $\text{Flag}(\tau_{\text{mod}})$ is contained in $N_\varepsilon(L(\tau))$, for some $\tau \in S(x_0 : \{x\})$.*

Proof. For $r > 0$ and $\tau_0 \in \text{Flag}(\tau_{\text{mod}})$, $\tau' \in C(\tau_0)$, consider

$$U(\tau_0, x_0, r) = \{\tau' \in \text{Flag}(\tau_{\text{mod}}) \mid P(\tau_0, \tau') \cap B(x_0, r) \neq \emptyset\}.$$

This is an analogue of shadows (1.2) as viewed from infinity. It is easy to verify that

$$\bigcup_{r \geq 0} U(\tau_0, x_0, r) = C(\tau_0).$$

Moreover, for $g \in G$, these shadows from infinity transform as $gU(\tau_0, x_0, r) = U(g\tau_0, gx_0, r)$.

If $k \in K = G_{x_0}$, the stabilizer of x_0 , then $kU(\tau_0, x_0, r) = U(k\tau_0, x_0, r)$. Since K is compact, there exists $M \geq 1$ such that the action $k \curvearrowright \text{Flag}(\tau_{\text{mod}})$ is M -Lipschitz for all $k \in K$. Let $r_1 > 0$ be such that $U(\tau_0, x_0, r_1/2)^c \subset N_{\varepsilon/M}(L(\tau_0))$. Here and below, for $A \subset \text{Flag}(\tau_{\text{mod}})$, $A^c = \text{Flag}(\tau_{\text{mod}}) - A$. Then, for any $\tau \in \text{Flag}(\tau_{\text{mod}})$,

$$U(\tau, x_0, r/2)^c \subset N_\varepsilon(L(\tau)), \quad \forall r \geq r_1. \quad (6.2)$$

For $x \in X$, let $\tau \in \text{Flag}(\tau_{\text{mod}})$ be a simplex such that $x \in V(x_0, \text{st}(\tau))$. Then there exists a parameterized geodesic ray x_t starting from x_0 , passing through x and asymptotic to some $\xi \in \text{st}(\tau)$.

Claim. *For all $r > 0$, $S(x : B_{2r}(x_0)) \supset U(\tau, x_0, r)$.*

Proof of claim. Pick $\tau' \in U(\tau, x_0, r)$ and let $\bar{x}_0 \in P(\tau, \tau')$ denote the nearest point projection of x_0 . In addition to the ray x_t , we define another parameterized geodesic ray \bar{x}_t , starting at \bar{x}_0 and asymptotic to ξ . Due to the convexity of the Riemannian distance function on X , the distance $d_{\mathbb{R}}(x_t, \bar{x}_t)$ monotonically decreases with t . Moreover, the cones $V(\bar{x}_t, \text{st}(\tau'))$ are nested as t decreases. Then,

$$\begin{aligned} d_{\mathbb{R}}(x_0, V(x_t, \text{st}(\tau'))) &\leq d_{\mathbb{R}}(x_0, V(\bar{x}_t, \text{st}(\tau'))) + d_{\mathbb{R}}(x_t, \bar{x}_t) \\ &\leq d_{\mathbb{R}}(x_0, V(\bar{x}_0, \text{st}(\tau'))) + r \leq d_{\mathbb{R}}(x_0, \bar{x}_0) + r \leq 2r. \end{aligned}$$

Therefore, $\tau' \in S(x : B_{2r}(x_0))$. \square

Using (6.2) it follows from the above claim that whenever $r \geq r_1$, the complement of the shadow $S(x : B(x_0, r))$ is contained in $N_\varepsilon(L(\tau))$ for some τ satisfying $x \in V(x_0, \text{st}(\tau))$. \square

Lemma 6.9. For all $r > 0$ and all $\tau \in S(x : B(x_0, r))$, $|d_F(x, x_0) - d_\tau^{\text{hor}}(x, x_0)| \leq 2r$.

Proof. We recall that the Finsler distance can alternatively be defined as

$$d_F(y, z) = \max_{\tau \in \text{Flag}(\tau_{\text{mod}})} d_\tau^{\text{hor}}(y, z),$$

where the maximum above occurs at any point in $S(y : \{z\})$ (see [KL18a, Sec. 5.1.2]). Fix some $\tau_0 \in S(x, \{x_0\})$. Then for any $\tau_1 \in S(x : B(x_0, r))$,

$$\begin{aligned} |d_F(x, x_0) - d_{\tau_1}^{\text{hor}}(x, x_0)| &= |b_{\tau_0}(x_0) - b_{\tau_1}(x_0)| \\ &= |b_{\tau_0}(x_0) - b_{\tau_0}(k^{-1}x_0)| \leq d_R(x_0, k^{-1}x_0) = d_R(kx_0, x_0), \end{aligned}$$

where $k \in K$, stabilizer of x , is some isometry satisfying $\tau_1 = k\tau_0$. In the above we chose the normalizations of the Busemann functions at x .

Let $y \in V(x, \text{st}(\tau)) \cap B(x_0, r)$. Then $y \in V(x, \sigma)$ for some chamber σ in $\text{st}(\tau)$. We identify $V(x, \sigma)$ with the model Weyl chamber Δ . Let $k_1 \in K$ such that $k_1x \in V(x, \sigma)$. Then $k_1x_0 = d_\Delta(x, x_0)$ via the identification above. Moreover, since the map

$$X \rightarrow \Delta, \quad z \mapsto d_\Delta(x, z)$$

is 1-Lipschitz (by the triangle inequality for Δ -distances (1.1)) and $d_\Delta(x, y) = y$, we obtain,

$$d_R(y, k_1x_0) \leq d(y, x_0) < r$$

and, in particular, $d(x_0, k_1x_0) < 2r$. □

Using the above lemmata, we now complete the proof of Theorem 6.1. We first fix some auxiliary quantities. Let $q \in (l, m)$ and ε_0 be corresponding constant as given in Lemma 6.7. Let δ be a constant given by Lemma 6.6 which corresponds to $\varepsilon = \varepsilon_0$. By Λ we denote the δ -neighborhood of $\Lambda_{\tau_{\text{mod}}}$ and let

$$V = \bigcup_{\tau \in \Lambda} V(x, \text{st}(\tau)) \subset X.$$

Since Γ is discrete, the elements of Γ which send x_0 outside V form a finite set Φ . Let

$$r_0 = \max\{r_1, d_R(x, \gamma x_0) \mid \gamma \in \Phi\}$$

where r_1 is a constant that corresponds to $\varepsilon_0/2$ as in Lemma 6.8.

For every $\gamma \in \Gamma$ satisfying $d_R(x, \gamma x_0) > r \geq r_0$, we assign an element $\tau_\gamma \in S(x : \{\gamma x_0\}) \cap \Lambda$ (the intersection is nonempty by above). Using Lemma 6.6, for every such τ_γ there exists $\tau_0 \in \Lambda_{\tau_{\text{mod}}}$ so that

$$N_{\varepsilon_0/2}(L(\tau_\gamma)) \subset N_{\varepsilon_0}(L(\tau_0)).$$

By Lemmata 6.7 and 6.8, $\mu_x(S(\gamma^{-1}x : B(x_0, r))) \geq m - q$ and by properties of conformal measures,

$$\begin{aligned} \mu_x(S(x : B(\gamma x_0, r))) &= \mu_{\gamma^{-1}x}(S(\gamma^{-1}x : B(x_0, r))) \\ &= \int_{S(\gamma^{-1}x : B(x_0, r))} \exp\left(-\beta d_\tau^{\text{hor}}(\gamma^{-1}x, x)\right) d\mu_x \asymp \exp(-\beta d_F(x, \gamma x_0)), \end{aligned}$$

where in the last step we have additionally used Lemma 6.9. This completes the proof. □

§7. Dimension of a conformal density

In this section, we establish a lower bound for the dimension of a conformal density. For Anosov subgroups, we prove that the dimension equals the Finsler critical exponent (see Corollary 7.4).

Theorem 7.1. *Suppose that Γ is a nonelementary τ_{mod} -RA subgroup. Let μ be a Γ -invariant conformal density of dimension β . Then β has the following lower bound:*

$$\beta \geq \delta_{\text{F}} - \delta_{\text{F}}^{\text{c}}. \quad (7.1)$$

The proof of this theorem is given at the end of this section. The number $\delta_{\text{F}}^{\text{c}}$ above quantifies the maximal exponential growth rate of the orbit Γx_0 in a conical direction. The precise definition is given below.

Definition 7.2 (Critical exponent in conical directions). Suppose that Γ is a τ_{mod} -regular subgroup. For $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$, define

$$N_{\text{F}}^{\text{c}}(r, c, x, x_0, \tau) = \text{card}\{\gamma \in \Gamma \mid d_{\text{F}}(x, \gamma x_0) < r, d_{\text{R}}(\gamma x_0, V(x, \text{st}(\tau))) < c\}$$

and

$$\delta_{\text{F}}^{\text{c}}(\Gamma) = \sup_{\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)} \left(\lim_{c \rightarrow \infty} \left(\limsup_{r \rightarrow \infty} \frac{\log N_{\text{F}}^{\text{c}}(r, c, x, x_0, \tau)}{r} \right) \right).$$

Note that it is sufficient to take the supremum in the definition of $\delta_{\text{F}}^{\text{c}}(\Gamma)$ over the conical limit set $\Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$. For rank-one symmetric spaces, and, more generally, for σ_{mod} -regular subgroups, this number is zero. Below we see that for τ_{mod} -Anosov subgroups also, $\delta_{\text{F}}^{\text{c}}(\Gamma) = 0$. It should be noted that, however, for general discrete subgroups, $\delta_{\text{F}}^{\text{c}}$ could be ∞ .

Proposition 7.3. *Suppose that Γ is a nonelementary τ_{mod} -Anosov subgroup. Then the function $N(r) = N_{\text{F}}^{\text{c}}(r, c, x, x_0, \tau)$ grows linearly with r . In particular, $\delta_{\text{F}}^{\text{c}}(\Gamma) = 0$.*

Proof. Without loss of generality, we can assume that $x = x_0$.¹¹ Fix $c > 0$, $\tau \in \Lambda_{\tau_{\text{mod}}}^{\text{con}}(\Gamma)$ and, denote the preimage of τ in $\partial_{\infty}\Gamma$ under the boundary homeomorphism $\partial_{\infty}\Gamma \rightarrow \Lambda_{\tau_{\text{mod}}}(\Gamma)$ by ζ .

Since Γ is discrete, we can arrange the elements of $\{\gamma \in \Gamma \mid d_{\text{R}}(\gamma x_0, V(x, \text{st}(\tau))) < c\} \subset \Gamma$ in a sequence (γ_n) . The sequence $x_n = \gamma_n x_0$ converges conically to τ . Let $\alpha : \mathbb{Z}_{\geq 0} \rightarrow X$ be the image (under the orbit map $\Gamma \rightarrow \Gamma x$) of a parametrized geodesic ray $\mathbb{Z}_{\geq 0} \rightarrow \Gamma$ starting at 1_{Γ} and asymptotic to ζ . Then α is a τ_{mod} -Morse quasiray starting at x_0 and asymptotic to τ . Hence α is uniformly close to $V(x_0, \text{st}(\tau))$.

We claim that:

Claim. *The points x_n lie within a uniformly bounded distance from the image of α .*

Proof of claim. Since x_n and $\alpha(n)$ are uniformly close to $V(x_0, \text{st}(\tau))$, it is enough to understand the simpler case when for all n , both $\alpha(n), x_n \in V(x_0, \text{st}(\tau))$.

Assuming that the claim fails, we deduce that, after extraction, (x_n) diverges away from α . Since α is a Morse quasiray, α eventually enters each cone $V(x_n, \text{st}(\tau))$, but further and further away from the tip x_n as n grows. However, the separation between two successive points on α (being a quasigeodesic) is uniformly bounded. This means that we can find arbitrarily large m 's such that $\alpha(m)$ is uniformly close to the boundary of a cone $V(x_n, \text{st}(\tau))$ and is arbitrarily far away from its tip x_n . But this contradicts the τ_{mod} -regularity of the group Γ . \square

¹¹Note that the number $\delta_{\text{F}}^{\text{c}}(\Gamma)$ does not depend on x and x_0 as we have seen in the case of δ_{F} in Sec. 2.

Since each τ_{mod} -Anosov subgroup $\Gamma < G$ is uniformly τ_{mod} -regular, we may work with the Riemannian metric in place of the Finsler metric. Moreover, we may assume that the sequence (x_n) is sufficiently spaced. Let \bar{x}_n denote the nearest-point projection of x_n to the image of α . The above claim implies that $d(x_n, \bar{x}_n)$ is uniformly bounded. Since x_n 's are sufficiently spaced, \bar{x}_n 's are also sufficiently spaced which guarantees that $d_{\mathbb{R}}(\bar{x}_n, x_0) \geq \text{const} \cdot n$, for all large n , which in turn implies that $d_{\mathbb{R}}(x_n, x_0) \geq \text{const} \cdot n$. The proposition follows from this. \square

As a corollary of the above results, we obtain that any Γ -invariant conformal density must have dimension $\delta_{\mathbb{F}}$ when Γ is τ_{mod} -Anosov. The Patterson-Sullivan densities constructed in Section 3 also had this dimension.

Corollary 7.4. *Suppose that Γ is a nonelementary τ_{mod} -Anosov subgroup. Let μ be a Γ -invariant conformal density of dimension β . Then $\beta = \delta_{\mathbb{F}}$.*

Proof. By Corollary 6.5 we know that the Poincaré series $g_{\beta}^{\mathbb{F}}(x, x_0)$ diverges and, consequently, $\beta \leq \delta_{\mathbb{F}}$. The reverse inequality is obtained in combination of Theorem 7.1 and Proposition 7.3. \square

To close this section, we prove Theorem 7.1.

Proof of Theorem 7.1. We fix some $r \geq r_0$ where r_0 is given by Theorem 6.1. Passing to a finite index subgroup¹² we assume that the orbit points in Γx_0 are sufficiently spaced, say by $3r$. Further, we also assume that the stabilizer of x_0 in Γ is trivial in which case the function $N(R) = N_{\mathbb{F}}(R, x, x_0)$ counts the number of orbit points (in Γx_0) within the Finsler r -ball centered at x .

We place a Riemannian ball of radius r at each point in the orbit. In this proof, we reserve the word *ball* to specify this type of balls. The spacing between the orbit points ensures that the balls are pairwise disjoint. Note that the shadows in $\text{Flag}(\tau_{\text{mod}})$ (from x) of two distinct balls are disjoint unless they intersect some common τ_{mod} -cone with tip at x . Also note that, at large distances from x , the balls do not intersect the boundaries of the τ_{mod} -cones because of the τ_{mod} -regularity of the orbit.

Let n_R denote the maximal number of balls in $B_{\mathbb{F}}(x, R)$ that intersect a particular τ_{mod} -cone $V(x, \text{st}(\tau))$. It follows from the definition of $\delta_{\mathbb{F}}^c(\Gamma)$ that

$$\limsup_{R \rightarrow \infty} \frac{\log n_R}{R} \leq \delta_{\mathbb{F}}^c(\Gamma). \quad (7.2)$$

On the other hand, for each $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$, the maximal number of balls in $B_{\mathbb{F}}(x, R)$ whose shadows intersect τ is n_R . Therefore,

$$\frac{N_{\mathbb{F}}(R, x, x_0)}{n_R} s(R) \leq m = \text{total mass of } \mu_x, \quad (7.3)$$

where $s(R)$ is any lower bound for the measures of the shadows of balls in $B_{\mathbb{F}}(x, R)$. We note that the shadow lemma (Theorem 6.1) produces such a positive lower bound¹³, namely, we may take $s(R) = \text{const} \cdot e^{-\beta R}$. Then (7.3) yields

$$N_{\mathbb{F}}(R, x, x_0) \leq \frac{m \cdot n_R}{\text{const}} e^{\beta R}.$$

Together with (7.2), the above results in (7.1). \square

¹²This does not change the numbers $\delta_{\mathbb{F}}$ and $\delta_{\mathbb{F}}^c$.

¹³We may need to disregard a finite number of balls from the picture.

§8. Uniqueness of conformal density

Recall that an action of a group H on a measure space (S, σ) is said to be *ergodic* if each H -invariant measurable set $B \subset S$ is either null or co-null. In [Sul79], Sullivan proved that for a discrete group Γ of Möbius transformations of the Poincare ball \mathbb{B}^3 , a Γ -invariant conformal density μ of non-zero dimension is unique (here and henceforth, by “unique” we mean unique up-to a constant factor) in the class of all conformal densities of same dimension if and only if the action Γ on the limit set $\Lambda(\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$. See also [Nic89, Thm. 4.2.1]. Generalizing this statement in our setting, we obtain the following result. The proof is essentially same of Sullivan’s theorem, hence we omit the details.

Theorem 8.1. *Suppose that Γ is a nonelementary τ_{mod} -RA subgroup. A Γ -invariant conformal density μ of dimension $\beta > 0$ is unique in the class of all Γ -invariant conformal densities of dimension β if and only if the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$.*

It is then natural to ask

Question 8.2. *For which τ_{mod} -regular subgroups Γ , the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is ergodic with respect to a conformal measure?*

In this section we prove that the Anosov property is a sufficient condition:

Theorem 8.3 (Anosov implies ergodic). *Suppose that Γ is a nonelementary τ_{mod} -Anosov subgroup. Then the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is ergodic with respect to any Γ -invariant conformal density μ .*

As a corollary, we obtain that when Γ is τ_{mod} -Anosov, then, up to a constant factor, there is exactly one Γ -invariant conformal density, namely, the Patterson-Sullivan density.

Corollary 8.4 (Existence and uniqueness of conformal density). *Suppose that Γ is a nonelementary τ_{mod} -Anosov subgroup. Then, up to a constant factor, there exists a unique Γ -invariant conformal density μ , namely, the Patterson-Sullivan density.*

Proof. First of all, by Proposition 3.1, any such density must have a positive dimension. Secondly, by Corollary 7.4 this dimension equals to the critical exponent δ_{F} . Then the uniqueness follows from the combination of Theorems 8.1 and 8.3. \square

The proof Theorem 8.3 requires studying the behavior of μ near *density points*: Let Y be a metrizable topological space and σ be a Borel regular measure on Y . A point $a \in Y$ is called a *density point* of a (measurable) subset $B \subset Y$ if for every sequence (S_n) of measurable sets in Y converging to a ,

$$\lim_{n \rightarrow \infty} \frac{\sigma(S_n \cap B)}{\sigma(S_n)} = 1.$$

Hence a sequence of subsets (S_n) of a topological space Y converges to a point $a \in Y$ if for every neighborhood U of a , there exists N such that for all $n \geq N$, $S_n \subset U$. The set of all density points of B is denoted by $D(B)$. Note that $D(B)$ need not be a subset of B .

The following theorem is a generalization of the well-known Lebesgue density theorem; we refer to [Fed69, 2.9.11, 2.9.12] for a proof.

Theorem 8.5 (Density theorem). *For every measurable set $B \subset Y$, the set of density points $D(B)$ is measurable in σ . Moreover, $\sigma(A - D(A)) = 0$.*

Now we return to the proof of Theorem 8.3.

Proof of Theorem 8.3. Let μ be a Γ -invariant conformal density. Note that the dimension β of μ must be positive (by Proposition 3.1 and Corollary 7.4).

Let B be a Γ -invariant Borel subset of $\Lambda_{\tau_{\text{mod}}}(\Gamma)$. We need to prove that if B is not a null set, then it is co-null. From now on, we assume that B is not a null set, i.e. $\mu_x(B) > 0$.

We need the following lemma.

Lemma 8.6. *There exists $r_1 > 0$ such that for every $r \geq r_1$ and every $\gamma \in \Gamma$, the shadow $S(x, B(\gamma x_0, r))$ intersects $\Lambda_{\tau_{\text{mod}}}(\Gamma)$.*

Proof. The proof simply follows from the Morse property of the Anosov subgroup Γ . □

Let $\tau \in B$ be a density point (which exists since B has positive measure, see Theorem 8.5) and $(\gamma_n x_0)$ be a sequence conically converging to τ , $\gamma_n \in \Gamma$. Note that the sequence of shadows $(S_n)_{n \in \mathbb{N}}$ where $S_n = S(x : B(\gamma_n x_0, r))$, converges to τ . Since τ is a density point, we have

$$\lim_{n \rightarrow \infty} \frac{\mu_x(S_n \cap B)}{\mu_x(S_n)} = 1. \quad (8.1)$$

Also, when r is sufficiently large, by Lemma 8.6, the ratios in the above are finite. On the other hand, Γ -invariance of B and μ implies that

$$\begin{aligned} \frac{\mu_x(S(\gamma_n^{-1} x : B(x_0, r)) \cap B)}{\mu_x(S(\gamma_n^{-1} x : B(x_0, r)))} &= \frac{\mu_{\gamma_n x}(S_n \cap B)}{\mu_{\gamma_n x}(S_n)} = 1 - \frac{\mu_{\gamma_n x}(S_n - B)}{\mu_{\gamma_n x}(S_n)} \\ &= 1 - \frac{\int_{S_n - B} \exp(-\beta d_\tau^{\text{hor}}(\gamma_n x, x)) d\mu_x}{\int_{S_n} \exp(-\beta d_\tau^{\text{hor}}(\gamma_n x, x)) d\mu_x} \geq 1 - \text{const} \cdot \frac{\mu_x(S_n - B)}{\mu_x(S_n)}, \end{aligned}$$

where the inequality follows by Lemma 6.9. Together with (8.1), we get

$$\lim_{n \rightarrow \infty} \frac{\mu_x(S(\gamma_n^{-1} x : B(x_0, r)) \cap B)}{\mu_x(S(\gamma_n^{-1} x : B(x_0, r)))} = 1. \quad (8.2)$$

Note that by Corollary 6.2, μ is atom-free. Therefore, for every $\varepsilon > 0$ there exists $r > r_1$ such that

$$\mu_x(S(\gamma_n^{-1} x : B(x_0, r))) \geq m - \varepsilon,$$

for all large n , where m denotes the total mass of μ_x . The above follows from the combination of Lemmata 6.7 and 6.8. Therefore, by (8.2),

$$\mu_x(B) \geq \lim_{n \rightarrow \infty} \mu_x(S(\gamma_n^{-1} x : B(x_0, r)) \cap B) \geq m - \varepsilon,$$

which holds for every $\varepsilon > 0$. Hence $\mu_x(B) = m$. □

§9. Hausdorff density

In this section, we restrict our attention to Anosov subgroups. Usually, one defines Hausdorff measures and Hausdorff dimension for metric spaces. In the appendix to this paper we verify that the theory goes through for premetrics as well. The reader who prefers to work with metrics can assume that $\epsilon > 0$ is chosen so that $d_G^{x,\epsilon}$ defines a metric on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ (cf. Corollary 5.6).

For $\beta \geq 0$ we let \mathcal{H}_x^β denote the β -dimensional Hausdorff measure on $(\Lambda_{\tau_{\text{mod}}}(\Gamma), d_G^{x,\epsilon})$ (defined with respect to the premetric $d_G^{x,\epsilon}$ as in the appendix). The Hausdorff dimension of a Borel subset $B \subset \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is then defined as

$$\text{Hd}(B) = \inf\{\beta \mid \mathcal{H}_x^\beta(B) = 0\} = \sup\{\beta \mid \mathcal{H}_x^\beta(B) = \infty\}.$$

Note that if for some $\beta \geq 0$, $\mathcal{H}_x^\beta(B) \in (0, \infty)$, then $\text{Hd}(B) = \beta$.

Proposition 9.1. *Suppose that for some $\beta \geq 0$*

$$\mathcal{H}_x^\beta(\Lambda_{\tau_{\text{mod}}}(\Gamma)) \in (0, \infty). \quad (9.1)$$

Let $Z = \Gamma x$. Then $\mathcal{H}^\beta = \{\mathcal{H}_z^\beta\}_{z \in Z}$ is a $\beta\epsilon$ -dimensional Γ -invariant conformal Z -density.

Proof. Let $y, z \in Z$. Define a function $f : \Lambda_{\tau_{\text{mod}}}(\Gamma) \times \Lambda_{\tau_{\text{mod}}}(\Gamma) \rightarrow \mathbb{R}_{\geq 0}$ by

$$f(\tau_1, \tau_2) = \begin{cases} d_G^{y,\epsilon}(\tau_1, \tau_2) / d_G^{z,\epsilon}(\tau_1, \tau_2), & \tau_1 \neq \tau_2, \\ \exp(-\epsilon d_\tau^{\text{hor}}(y, z)), & \tau_1 = \tau_2 = \tau. \end{cases}$$

By a calculation similar to the proof of Proposition 5.4, we obtain

$$\lim_{\tau_1, \tau_2 \rightarrow \tau} \frac{d_G^{y,\epsilon}(\tau_1, \tau_2)}{d_G^{z,\epsilon}(\tau_1, \tau_2)} = \exp(-\epsilon d_\tau^{\text{hor}}(y, z))$$

which shows that f is continuous. For $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and small $\eta > 0$, let U_η be a neighborhood of τ in $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ such that $\forall \tau_1, \tau_2 \in U$,

$$d_G^{y,\epsilon}(\tau_1, \tau_2) \leq \left\{ \exp(-\epsilon d_\tau^{\text{hor}}(y, z)) + \eta \right\} d_G^{z,\epsilon}(\tau_1, \tau_2).$$

Hence the identity map $\text{id} : (\Lambda_{\tau_{\text{mod}}}(\Gamma), d_G^{z,\epsilon}) \rightarrow (\Lambda_{\tau_{\text{mod}}}(\Gamma), d_G^{y,\epsilon})$ on U_η is L_η -Lipschitz, where $L_\eta = \exp(-\epsilon d_\tau^{\text{hor}}(y, z)) + \eta$. In particular, the map id is locally Lipschitz. Therefore, for any $B \in \mathfrak{B}(U)$, $\mathcal{H}_y^\beta(B) \leq L_\eta^\beta \mathcal{H}_z^\beta(B)$. This also shows that $\mathcal{H}_y^\beta \ll \mathcal{H}_z^\beta$. Taking limit as $\eta \rightarrow 0$, we obtain

$$\frac{d\mathcal{H}_y^\beta}{d\mathcal{H}_z^\beta}(\tau) \leq \exp(-\beta\epsilon d_\tau^{\text{hor}}(y, z))$$

and switching the role of y and z in the above we also obtain the reverse inequality. Hence

$$\frac{d\mathcal{H}_y^\beta}{d\mathcal{H}_z^\beta}(\tau) = \exp(-\beta\epsilon d_\tau^{\text{hor}}(y, z))$$

which proves conformality. Suppose that $y = \gamma z$ for some $\gamma \in \Gamma$. Then for any $B \in \mathfrak{B}(\Lambda_{\tau_{\text{mod}}}(\Gamma))$,

$$\mathcal{H}_{\gamma z}^\beta(B) = \int_B \exp(-\beta\epsilon d_\tau^{\text{hor}}(\gamma z, z)) d\mathcal{H}_z^\beta = \int_B d(\gamma^* \mathcal{H}_z^\beta) = \gamma^* \mathcal{H}_z^\beta(B)$$

and Γ -invariance also follows. Therefore, \mathcal{H}^β is a conformal Z -density of dimension $\beta\epsilon$. \square

- Remark.** 1. Note that if such a family $\{\mathcal{H}_z^\beta \mid z \in Z\}$ exists, then it may be extended to a *full* conformal density via the correspondence in (3.4).
2. By the uniqueness of conformal density (Theorem 8.4), the number β in Proposition 9.1 equals to δ_F/ϵ .
3. In the following we shall see that, indeed, the δ_F/ϵ -dimensional Hausdorff measure $\mathcal{H}_x^{\delta_F/\epsilon}$ is finite and non-null (i.e. it satisfies (9.1)).

Next we show that if $\beta = \delta_F/\epsilon$, then the β -dimensional Hausdorff measure \mathcal{H}_x^β satisfies (9.1). Let us first discuss the simpler case, namely, when the Finsler pseudo-metric d_F is a metric. There is an abundance of examples when this occurs, e.g., in the case when $X = G/K$ is an irreducible symmetric space.

Let (Y, d) be a proper, geodesic, Gromov hyperbolic metric space and Γ be a nonelementary discrete group of isometries acting properly discontinuously on Y . Let Λ be the limit set of Γ in $\partial_\infty Y$. Further, assume that Γ is *quasiconvex-cocompact*, i.e. the quasiconvex hull $\text{QCH}(\Lambda)$ is nonempty and the quotient $\text{QCH}(\Lambda)/\Gamma$ is compact. In [Coo93], Coornaert proved the following result.

Theorem 9.2 ([Coo93, Cor. 7.6]). *Suppose that the critical exponent δ of Γ is finite. Then the δ -dimensional Hausdorff measure on Λ with respect to a Gromov metric d_G is finite and non-null.*

To apply this theorem to our case, we need an appropriate setup. In Section 4, we proved that the orbit $Z = \Gamma x$ is a Gromov hyperbolic space with respect to the Finsler metric (cf. Corollary 4.8) and it is also proper. But Z fails to be geodesic. This problem can be remedied by taking a uniform neighborhood Y of Z in X such that Z is quasiconvex in Y , and then putting the intrinsic path-metric d on Y induced by d_F (this requires positivity of d_F), and finally by completing Y in this metric. Then (Y, d) is proper, geodesic and Gromov hyperbolic. Moreover, (Y, d) and the isometrically embedded (Z, d_F) are Hausdorff-close and, in particular, (Y, d) is quasiisometric to (Z, d_F) by a $(1, A)$ -quasiisometry. This implies that there is a bi-Lipschitz homeomorphism from $\partial_\infty Y$ (equipped with the metric d_G^ϵ defined by $d_G^\epsilon(\xi_1, \xi_2) = d_G(\xi_1, \xi_2)^\epsilon$ where d_G is a Gromov metric on $\partial_\infty Y$) to $(\Lambda_{\tau_{\text{mod}}}(\Gamma), d_G^{x, \epsilon})$. Note that the action $\Gamma \curvearrowright (Y, d)$ satisfies all the properties needed to apply Theorem 9.2. Therefore, by this theorem the δ_F/ϵ -dimensional Hausdorff measure on $\partial_\infty Y$ (and, consequently, also on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$) is finite and non-null.

In the general case where the positivity of d_F is unknown, the above argument still works after some modifications. Let us go back to our construction in the above paragraph. Let Y be a uniform Riemannian neighborhood of Z in which Z is Finsler quasiconvex. Define a new Γ -invariant metric \bar{d}_F on Y by

$$\bar{d}_F(y, z) = \max \{d_F(y, z), \epsilon d_R(y, z)\}, \quad \forall y, z \in Y$$

where $\epsilon > 0$ is some number that is strictly lesser than L^{-1} given in (2.6). Note that for $y, z \in Z$, if $d_F(y, z)$ is sufficiently large, then $\bar{d}_F(y, z) = d_F(y, z)$. Moreover, for a given ι -invariant compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$ and a possibly smaller ϵ (depending on the choice of Θ), any Θ -Finsler geodesic (see Definition 2.1) connecting these two points remains a geodesic in this new metric. In other words, Z remains quasiconvex in Y with respect to \bar{d}_F .

Note that the identity embedding $(Z, d_F) \rightarrow (Y, \bar{d}_F)$ is a $(1, A)$ -quasiisometric embedding for some large enough A and the image is Hausdorff-close to Y . Therefore, in this case also we get a

natural identification of the Gromov boundaries of (Z, d_F) and (Y, \bar{d}_F) . Next, considering intrinsic metrics, we complete Y as before to get a proper, geodesic, Gromov hyperbolic space (Y, d) . The rest of the argument works as before.

Using Proposition 9.1, we obtain the following result.

Theorem 9.3. *Suppose that Γ is a nonelementary τ_{mod} -Anosov subgroup. If $\beta = \delta_F/\epsilon$, then the β -dimensional Hausdorff density $\mathcal{H}^\beta = \{\mathcal{H}_z^\beta\}_{z \in \Gamma x}$ is a Γ -invariant conformal density of dimension δ_F . In particular, the Hausdorff dimension with respect to the metric $d_G^{x,\epsilon}$ satisfies*

$$\text{Hd}(\Lambda_{\tau_{\text{mod}}}(\Gamma)) = \delta_F/\epsilon.$$

Moreover, \mathcal{H}^β equals to a non-zero multiple of the Patterson-Sullivan density.

We have mostly completed the proof of this theorem. The remaining ‘‘moreover’’ part follows from the uniqueness of Γ -invariant conformal density (Theorem 8.4).

Corollary 9.4. *With respect to the Gromov premetric $d_G^x := d_G^{x,1}$ the Hausdorff dimension satisfies*

$$\text{Hd}(\Lambda_{\tau_{\text{mod}}}(\Gamma)) = \delta_F.$$

§10. Examples

10.1. Product of hyperbolic spaces: Let Γ_1, Γ_2 be isomorphic discrete cocompact subgroups of $\text{PSL}(2, \mathbb{R})$ where the isomorphism is given by $\phi : \Gamma_1 \rightarrow \Gamma_2$. We let $f : S^1 \rightarrow S^1$ be the equivariant homeomorphism of ideal boundaries of hyperbolic planes determined by ϕ .

The discrete subgroup

$$\Gamma = \{(\gamma_1, \phi\gamma_1) \mid \gamma_1 \in \Gamma_1\} < G = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$$

acts on $X = \mathbb{H}^2 \times \mathbb{H}^2$ as a σ_{mod} -Anosov subgroup. (This follows, for instance, from the fact that Γ is an URU subgroup of G .) The σ_{mod} -limit set of Γ (in the full flag-manifold $S^1 \times S^1$) equals the graph of the map f .

We denote d_1 (resp. d_2) the distance functions of the constant -1 curvature Riemannian metrics on the first (resp. second) factor of the product $\mathbb{H}^2 \times \mathbb{H}^2$. Also, we let δ_i denote the critical exponent for the action of Γ_i on the hyperbolic plane, $i = 1, 2$. Since Γ_1 and Γ_2 are cocompact, the limit sets are S^1 and, hence $\delta_1 = \delta_2 = 1$. By [Sul79, Cor. 10], the corresponding orbital counting functions (w.r.t. a fixed base point) satisfy

$$C^{-1}e^r \leq N_i(r) \leq Ce^r, \quad i = 1, 2, \tag{10.1}$$

for some constant $C > 1$.

We work with the Finsler metric on $\mathbb{H}^2 \times \mathbb{H}^2$ given by

$$d_F((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2}$$

(cf. (2.7)¹⁴). Note that

$$\min \{d_1(x_1, y_1), d_2(x_2, y_2)\} \leq d_F((x_1, x_2), (y_1, y_2)) \leq \max \{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

¹⁴We multiply the distance function (2.7), for $p = 2$, by a factor $1/\sqrt{2}$ in order to avoid unnecessary radical constants.

Together with (10.1), the above yields

$$C^{-1}e^r \leq N_{\mathbb{F}}(r) \leq Ce^r.$$

Therefore, $\delta_{\mathbb{F}} = 1$. By Theorem 9.3, the Hausdorff dimension of $\Lambda_{\sigma_{\text{mod}}}(\Gamma)$ (w.r.t. to the Gromov premetric (5.8), $p = 2$) equals to 1.

Note that the above argument also works for an arbitrary product $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$.

10.2. Projective Anosov representations: Recall that a representation $\Gamma \rightarrow \text{SL}(k+1, \mathbb{R})$, $k \geq 2$, is called *projective Anosov* if it is τ_{mod} -Anosov for $\tau_{\text{mod}} = (1, k)$ (see Examples 1.3, 2.5, and 5.8 for notations). The Finsler critical exponent associated to the ι -invariant type $\bar{\theta} = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1})$ will be denoted by $\delta_{\mathbb{F}}$.

Let $\Gamma \rightarrow \text{SL}(k+1, \mathbb{R})$ be a projective Anosov representation. In [GMT19], the authors defined the following two critical exponents of Γ , namely, the *Hilbert critical exponent*

$$\delta_{1,k+1} = \limsup_{r \rightarrow \infty} \frac{\log \text{card}\{\gamma \in \Gamma \mid \sigma_1(\gamma) - \sigma_{k+1}(\gamma) < r\}}{r}$$

and the *simple root critical exponent*

$$\delta_{1,2} = \limsup_{r \rightarrow \infty} \frac{\log \text{card}\{\gamma \in \Gamma \mid \sigma_1(\gamma) - \sigma_2(\gamma) < r\}}{r}.$$

A direct computation yields

$$\sqrt{k+1}\delta_{\mathbb{F}} = \delta_{1,k+1} \leq \delta_{1,2}/2,$$

where the left equality follows from the formula for the Finsler metric given by (2.10) and the right inequality follows from $2(\sigma_1 - \sigma_2) \leq \sigma_1 - \sigma_{k+1}$. Also note that (by (5.9)) for a pair of partial flags $(l_1, h_1), (l_2, h_2) \in \text{Flag}(\tau_{\text{mod}})$,

$$d_{\mathbb{G}}^{1/\sqrt{k+1}, x}((l_1, h_1), (l_2, h_2)) \leq \sin \angle(l_1, l_2)$$

where the right side equals the distance (with respect to the constant curvature Riemannian metric on $\mathbb{R}P^k$ determined by $x \in X$) between the lines l_1, l_2 in $\mathbb{R}P^k$. This together with Theorem 9.3 implies that

$$\delta_{1,k+1} = \delta_{\mathbb{F}}\sqrt{k+1} = \text{Hd}(\Lambda_{\tau_{\text{mod}}}(\Gamma)) \leq \text{Hd}_{\mathbb{R}}(\xi^1(\partial_{\infty}\Gamma)) \quad (10.2)$$

where $\xi^1 : \partial_{\infty}\Gamma \rightarrow \mathbb{R}P^k$ is the Γ -equivariant embedding¹⁵ of $\partial_{\infty}\Gamma$ into $\mathbb{R}P^k$ and $\text{Hd}_{\mathbb{R}}$ denotes the Hausdorff dimension with respect to the Riemannian metric. Together with a recently obtained upper-bound for $\text{Hd}_{\mathbb{R}}(\xi^1(\partial_{\infty}\Gamma))$ (see [PSW19, Prop. 4.1] or [GMT19, Thm. 4.1]), we obtain the following result.

Theorem 10.1. *Let $\Gamma \rightarrow \text{SL}(k+1, \mathbb{R})$ be a projective Anosov representation. Then*

$$\delta_{1,k+1} \leq \text{Hd}_{\mathbb{R}}(\xi^1(\partial_{\infty}\Gamma)) \leq \delta_{1,2}.$$

Also compare [GMT19, Cor. 1.2] where the authors obtain identical bounds for the Hausdorff dimension of the flag limit set equipped with a *certain* Gromov metric.

¹⁵Composition of the Γ -equivariant boundary embedding $\partial_{\infty}\Gamma \rightarrow \text{Flag}(\tau_{\text{mod}})$ and the projection map $\text{Flag}(\tau_{\text{mod}}) \rightarrow \mathbb{R}P^k = \text{Gr}_1(\mathbb{R}^{k+1})$.

Appendix: Hausdorff measures on premetric spaces

Let X be a metrizable topological space. Recall that an *outer measure* is a function $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ that satisfies

1. $\mu(\emptyset) = 0$,
2. for all $A, B \in \mathcal{P}(X)$ with $A \subset B$, $\mu(A) \leq \mu(B)$, and
3. for all countable collection $(A_k \mid k \in \mathbb{N})$ of subsets of X ,

$$\mu\left(\bigcup_{k \in \mathbb{N}} A_k\right) \leq \sum_{k \in \mathbb{N}} \mu(A_k).$$

A set $A \subset X$ is called μ -*measurable* if for every $E \in \mathcal{P}(X)$, $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$. By Carathéodory's theorem (cf. [Fol99, Thm. 1.11]), μ -measurable sets form a σ -algebra to which μ restricts as a complete measure.

Assume now that X is compact. The outer measure μ is called *good* if additionally,

4. for all $A, B \subset X$ with $\bar{A} \cap \bar{B} = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

The next lemma asserts that, for outer measures μ on compact metrizable spaces, the σ -algebra of Borel sets is a subalgebra of the σ -algebra of μ -measurable sets.

Lemma A.1. *Let X be a compact metrizable space. If μ is a good outer measure on X , then every Borel set $B \in \mathfrak{B}(X)$ is measurable.*

Proof. Let d be a metric on X . Then the condition 4 above implies that

- 4'. for all $A, B \subset X$ with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Therefore, μ is a metric outer measure on (X, d) . By [Fol99, Prop. 11.16], Borel subsets of X are measurable. \square

Definition A.2 (Premetric space). Let X be a topological space. A symmetric continuous function $d : X \times X \rightarrow [0, \infty]$ is called a *premetric* on X . A pair (X, d) consisting of a metrizable topological space X and a premetric d on X is called a *premetric space*.

In what follows, we consider only *positive* premetrics, i.e.

$$d(x, y) > 0 \iff x \neq y, \quad \forall x, y \in X$$

Let (X, d) be a compact positive premetric space. Then d satisfies the following *separation property*:

$$d(A, B) > 0 \iff \bar{A} \cap \bar{B} = \emptyset, \quad \forall A, B \subset X. \quad (\text{A.3})$$

Let $\varepsilon > 0, \beta > 0$. For every $A \subset X$, define

$$\mathcal{H}_\varepsilon^\beta(A) = \inf_{\mathcal{U}} \left\{ \sum_{k \in \mathbb{N}} \text{diam}_d(U_k)^\beta \mid \mathcal{U} = \{U_k \mid k \in \mathbb{N}\} \text{ covers } A, \text{ mesh}(\mathcal{U}) \leq \varepsilon \right\}.$$

In the above, $\text{mesh}(\mathcal{U})$ is the supremum of the d -diameters of the members of \mathcal{U} . Then

$$\mathcal{H}_\varepsilon^\beta : \mathcal{P}(X) \rightarrow [0, \infty]$$

is an outer measure on X (cf. [Fol99, Prop. 1.10]). Define the β -dimensional *Hausdorff measure* \mathcal{H}^β by

$$\mathcal{H}^\beta(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^\beta(A).$$

Theorem A.3. *The Hausdorff measure \mathcal{H}^β is a good outer measure.*

Proof. We need to check the properties 1-4 above. Since, for all $\varepsilon > 0$, $\mathcal{H}_\varepsilon^\beta$ is an outer measure, taking limit $\varepsilon \rightarrow 0$, properties 1-3 are easily verified. Therefore, we only need to check that \mathcal{H}^β satisfies property 4.

Let $A, B \subset X$ such that $\bar{A} \cap \bar{B} = \emptyset$. By (A.3), $d(A, B) = d_0 > 0$. Let $\varepsilon < d_0$ be a positive number and \mathcal{U} be a countable open cover of $A \cup B$ with $\text{mesh}(\mathcal{U}) \leq \varepsilon$. If such open cover does not exist, then $\mathcal{H}_\varepsilon^\beta(A \cup B)$ (and hence, $\mathcal{H}^\beta(A \cup B)$) is infinity. Otherwise, \mathcal{U} can be written as a disjoint union $\mathcal{U}_A \sqcup \mathcal{U}_B$ where \mathcal{U}_A consists of all open sets in \mathcal{U} that intersect A and \mathcal{U}_B consists of the rest. Clearly, \mathcal{U}_A and \mathcal{U}_B are open covers of A and B , respectively. Therefore,

$$\sum_{E \in \mathcal{U}} \text{diam}_d(E)^\beta = \sum_{E \in \mathcal{U}_A} \text{diam}_d(E)^\beta + \sum_{E \in \mathcal{U}_B} \text{diam}_d(E)^\beta \geq \mathcal{H}_\varepsilon^\beta(A) + \mathcal{H}_\varepsilon^\beta(B).$$

Since the above holds for any cover \mathcal{U} with $\text{mesh} \leq \varepsilon$, we have

$$\mathcal{H}_\varepsilon^\beta(A \cup B) \geq \mathcal{H}_\varepsilon^\beta(A) + \mathcal{H}_\varepsilon^\beta(B).$$

Taking limit $\varepsilon \rightarrow 0$, we get $\mathcal{H}^\beta(A \cup B) \geq \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. The reverse inequality follows from property 3. Therefore, $\mathcal{H}^\beta(A \cup B) = \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. This completes the proof. \square

By Lemma A.1 and the above theorem, we obtain the following result.

Corollary A.4. *Every Borel subset of X is \mathcal{H}^β -measurable.*

The *Hausdorff dimension* of a Borel subset $B \subset (X, d)$ is then defined as

$$\text{Hd}(B) = \inf\{\beta \mid \mathcal{H}^\beta(B) = 0\} = \sup\{\beta \mid \mathcal{H}^\beta(B) = \infty\}.$$

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