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N Angry Men

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Abstract

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We develop the basic results of Bayesian Networks and propose these Networks as a setting for the Classical Condorcet Jury Theorem (CCJT) and related results. Bayesian Networks will allow us to address the plausibility of one of the central assumptions of the CCJT, the independence of individual votes, as well as provide a setting for attempts to generalize the CCJT to situations in which individual votes are not independent.

In the second chapter we define *CJT Networks*, a family of Bayesian Networks in which we interpret the CCJT. We begin with the classical result for juries with homogeneous competence and independent votes and then turn to comparing simple majority rule and random dictatorship for juries with non-homogeneous competence (and independent votes). The main contribution is an elegant combinatorial proof that simple majority rule is preferred to random dictatorship for juries with member competences all in the interval $[\frac{1}{2}, 1)$ with at least one competence greater than $\frac{1}{2}$.

In the third chapter we address the source and consequences of *dependence* between juror votes. Our primary concern is with Dietrich and Spiekermann's observation (e.g. [9]) that even in the simplified case where deliberation between jurors is not permitted, it is likely that the individual votes are not mutually independent due to common causes between individual votes. Once again we use the framework of Bayesian Networks to make the nature of the dependence explicit. We examine Dietrich and Spiekermann's generalization of the homogeneous CJT to situations where the individual votes are not independent. We argue that their theorem depends on implausible assumptions and show how there does not appear to a reasonable substitute in sight. We close by looking at an entirely different approach to dependence, which models the group deliberation process as a linear dynamical system, and we introduce the Cesaro voting method to extend on the results of DeGroot.

To my family,

Contents

List of Figures

Acknowledgments

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Chapter 1

The Formal Theory of Voting

1.1 Introduction

Consider a set J of agents faced with the task of making some binary choice. The group is required to settle on exactly one of two alternatives, say A_0 or A_1. Assume further that there is some property C such that exactly one of A_0 or A_1 has property C, and that each group member $j \in J$ has some probability p_j (which we will call the *competence* of member j) of voting for the alternative with property C.

There are many judgement aggregation methods such a group might use to produce their group decision. Two traditional methods are:

1. Simple Majority Rule: Each member writes their vote (A_0 or A_1) on a ballot and puts the ballot into a hat. The votes are then tallied, and the alternative with the most votes becomes the group decision (with some procedure to determine the group decision in the case of ties, perhaps flipping a coin or revoting).

2. The Random Ballot: Each member writes their individual vote on a ballot and puts the ballot into a hat. A single ballot is selected at random from the hat, and the alternative written on that ballot becomes the group decision.

Let us say that that a judgement aggregation method M_1 is *preferable* to another method M_2 (with respect to C) just in case the probability that M_1 will result in the alternative with property C is greater than than the probability that M_2 will result in the alternative with property C. We then ask

> For a fixed property C, under what circumstances is Simple Majority Rule preferable to Random Ballot with respect to C?

Credit is given to the Marquis de Condorcet[1] for initiating a mathematical discussion of this question in his 1785 work [7]. His result is now known as the Condorcet Jury Theorem.

[1]Marie Jean Antoine Nicolas de Caritas, Marquis de Condorcet

We give a formal statement and proof of this theorem in the next chapter after we have introduced the framework of Bayesian Networks, but we can state the result informally as follows: In addition to the setup above with a set J of size N, suppose that $N = 2n + 1$ is odd and that group competence is *homogeneous*, that is, there is some $p \in [0, 1]$ such that $p_i = p$ for all $1 \leq i \leq N$. Suppose further that the events that the individual jurors vote correctly/incorrectly are mutually independent. Then

1. If $\frac{1}{2} < p < 1$ then the probability that simple majority rule results in the alternative with property C is strictly increasing in odd N, and tends to 1 as N tends to infinity.

2. If $0 < p < \frac{1}{2}$ then the probability that simple majority rule results in the alternative with property C is strictly decreasing in odd N, and tends to 0 as N tends to infinity.

In particular, under these assumptions, simple majority rule is preferable to random ballot when $\frac{1}{2} < p < 1$, since, under this framework, random ballot is equivalent to a one member jury. Likewise, random ballot is preferable to simple majority rule when $0 < p < \frac{1}{2}$.

Traditionally, the property C is taken to be something like “correctness,” and so the alternative with property C is often referred to as the “the correct alternative,” “the better alternative,” or “the right alternative.” We will follow this tradition. The reader should keep in mind, however, that the results we will prove hold with respect to any *any* property C so long as both of the following are satisfied:

1. Exactly one of the two alternatives in question has property C.

2. For all $j \in J$, the competence p_j is just the probability that group member j will vote for the alternative with property C.

We will mention the above theorem frequently throughout this dissertation and will refer to it as the (homogeneous) *Classical* Condorcet Jury Theorem (CCJT) to distinguish it from any of the many “Condorcet Jury Theorems” (CJTs) that followed the original when it was reintroduced into the contemporary decision theory and social choice literature by Black in his 1958 paper [4]. Since that time, the research arising from Condorcet’s original result has branched in many directions including

1. Groups with *heterogeneous* competences, where the p_i depend on the member $i \in J$, as opposed remaining constant across all group members.

2. Situations in which the events of each of the jurors voting correctly are not mutually independent.

3. Finding an optimal sequence of of non-equal weights for different jurors votes given that we know their competences ahead of time (so-called *non-simple majority rule*, or *weighted voting*),[2]

[2] See Shapley and Grofman [19].

4. Choosing an optimal sub-collection of jurors to decide the issue by simple majority rule, given that we know the competences of the agents ahead of time,

5. Taking "strategic voting" into account, that is, considering situations in which an agent might have a reason to vote for what they believe to be the incorrect option,

6. Generalizing to non-binary decisions: for example, if the group is asked to *rank* three or more alternatives.

For a concise survey of the central results springing from the Classical Condorcet Jury Theorem during the past century see [11].

This dissertation will focus on the first and second items on the above list. This means

1. We will focus solely on binary choice scenarios.

2. We are motivated by the desire to know how to utilize a group of agents when we have little knowledge of their competencies beyond a simple lower or upper bound.

3. We are only interested in comparing probabilities with which the application of various methods result in a group decision for an alternative with property C given the probabilities p_i that the individuals will vote for the alternative with property C. In particular we do not concern ourselves with "strategic voting."

One of the primary goals of this paper is to situate the CCJT and related results inside the framework of Bayesian Networks, introduced in the following section. The use of causal networks in the voting literature to examine dependence between votes is introduced and used in a semi-formal manner by Dietrich and Spiekermann in [9]. In this dissertation we explicitly develop the basic results of Bayesian Networks and propose them as a setting for the CCJT and related results. This will allow us to address the plausibility of one of the central assumptions of the CCJT, the independence of individual votes, as well as provide a setting for attempts to generalize the CCJT to situations in which individual votes are not independent.

In the second chapter we define *CJT Networks*, a family of Bayesian Networks in which we interpret the CCJT. We begin with the classical result for juries with homogeneous competence and independent votes and then turn to comparing simple majority rule and random ballot for juries with non-homogeneous competence (and independent votes). The main contribution is an elegant combinatorial proof that simple majority rule is preferred to random ballot for juries with member competences all in the interval $[\frac{1}{2}, 1)$ with at least one competence greater than $\frac{1}{2}$.

In the third chapter we address the source and consequences of *dependence* between juror votes. In the first section our primary concern is with Dietrich and Spiekermann's observation (e.g. [9]) that even in the simplified case where deliberation between jurors is not permitted, it is likely that the individual votes are not mutually independent due to common causes between individual votes. Once again we use the framework of Bayesian Networks to make the nature of the dependence explicit. We examine Dietrich and Spiekermann's attempt to generalize homogeneous CCJT to situations where individual votes have common causes. We argue that their theorem depends on implausible assumptions and argue that there does not appear to be a reasonable substitute condition in sight.

In the second section of Chapter 3 we extend on a different approach to modeling dependence, introduced by DeGroot, which models the group deliberation process as a linear dynamical system. This approach does not consider common causes of individual votes, but rather the potential for jurors to influence each other's degree of confidence during deliberation. We introduce the Cesaro Judgement Aggregation Method to extend on the results of DeGroot.

NOTE: Throughout this paper we often use the word *jury* to refer to the body of agents trying to arrive at a decision and *juror* to refer to a given member of such a group. The purpose of this is to save space, not to restrict the discussion to court juries. The reader is welcome to substitute "collection of individual agents faced with arriving at a group decision on one of two alternatives" for "*jury*" and "individual agent within such a collection" for "*juror*".

1.2 Bayesian Networks

Bayesian Networks impose a graph structure on a set Υ of random variables in order to make explicit the dependencies between various subsets of Υ as well as to facilitate the computation of the joint distribution of those variables. We will be primarily concerned with the former feature.

To define Bayesian Networks we will need some basic concepts from probability theory and the theory of graphs.

Concepts from Probability Theory

We assume the reader is familiar with the definition of a probability space, but we recall some important additional concepts.

Definition 1.1. *Let $S = (\Omega, \Sigma, P)$ be a probability space with underlying set Ω, σ-algebra Σ, and probability measure P. Let $\Upsilon = \{X_i\}_{1\leq i\leq n}$ be a set of random variables on S with each X_i taking values in some finite set $D_{X_i} =: D_i$.*

1. *We will use defining formulas to denote sets. In particular, given formulae ϕ and ψ defining events $E, F \in \Sigma$, let $\phi \wedge \psi$ and $\phi \vee \psi$ denote the intersection $E \cap F$ and union $E \cup F$, respectively. For instance if $x \in D_X, y \in D_Y$ then*
$$(X = x \wedge Y = y) := \{\omega \in \Omega : X(\omega) = x \wedge Y(\omega) = y\}$$
2. *We will frequently consider tuples $A = (X_{j_i})_{1\leq i\leq r}$ of variables from of Υ. Formally we can define such an tuple A as the function $f : \{1, \ldots, r\} \to \Upsilon$ such that $f(i) = X_{j_i}$.*

3. *Given a tuple* $A = (X_{j_i})_{1\leq i\leq r}$ *of variables from* Υ*, define* D_A *to be the cartesian product* $D_A := \prod_{i=1}^{r} D_{j_i}$*. For* $\bar{a} \in D_A$ *we will write* $A = \bar{a}$ *for* $\bigwedge_{i=1}^{r}(X_{j_i} = a_i)$*.*

4. *Given tuples* $A = (X_{j_i})_{1\leq i\leq r}$*,* $B = (X_{k_i})_{1\leq i\leq s}$ *of variables from* Υ *we write* $A \subseteq B$ *to mean that each variable in* A *is in* B *(formally,* $range(A) \subseteq range(B)$*). Also, for* $A = (X_{j_i})_{1\leq i\leq r}$*,* $B = (X_{k_i})_{1\leq i\leq s}$ *tuples of variables from* Υ *define* $A \cup B$ *to be the concatenation of* A *and then* B *(Again, formally we can define this as the function* f *from* $\{1, \ldots, r+s\}$ *into* Υ *whose restriction to* $\{1, \ldots, r\}$ *is* A *and such that* $f(r+i) = X_{k_i}$ *for all* $1 \leq i \leq s$*.)*

5. *Given* $X \in \Upsilon$ *and* A *some tuple of variables from* Υ*, we define the* transition kernel $p_{X,A} : D_X \times D_A \to [0,1]$ *to be the conditional probability distribution* $P(X|A)$ *of* X *given the variables in* A *defined as follows: For* $x \in D_X$ *and* $\bar{a} \in D_A$

$$p_{X,A}(x, \bar{a}) := P(X = x|A = \bar{a}).$$

6. *Given tuples* A, B *and* C *of* Υ*, we say that* A *is* independent *of* B given C*, denoted* $(A \amalg B|C)$*, if for all* $\bar{a} \in D_A$*,* $\bar{b} \in D_B$*,* $\bar{c} \in D_C$ *such that* $P(B = \bar{b} \wedge C = \bar{c}) > 0$ *we have*

$$P(A = \bar{a}|B = \bar{b} \wedge C = \bar{c}) = P(A = \bar{a}|C = \bar{c}).$$

Otherwise we say that A *and* B *are* dependent *given* C*.*

The restriction that $P(B = \bar{b} \wedge C = \bar{c}) > 0$ in the definition of independence has the following consequence.

Lemma 1.2. 1. *If* A, B, C *are tuples from* Υ *and* $B \subseteq C$ *then* $(A \amalg B|C)$*.*

2. *If* A, B, C *are tuples from of* Υ *with* B *and* C *disjoint, then*

$$(A \amalg B|C) \Rightarrow (A \amalg (B \cup C)|C).$$

3. *If* A, B, C *are tuples from* Υ *such that* $(A \amalg B|C)$*, then for all* $\bar{a} \in D_A$*,* $\bar{b} \in D_B$*,* $\bar{c} \in D_C$ *such that* $P(C = \bar{c}) > 0$*,*

$$P(A = \bar{a}|C = \bar{c})P(B = \bar{b}|C = \bar{c}) = P(A = \bar{a} \wedge B = \bar{b}|C = \bar{c}).$$

Proof. The first part follows immediately from the definitions since if $\bar{a} \in D_A$, $\bar{b} \in D_B$, $\bar{c} \in D_C$ with $P(B = \bar{b} \wedge C = \bar{c}) > 0$ then, since $B \subseteq C$ we see that $B = \bar{b} \wedge C = \bar{c}$ and $C = \bar{c}$ are the same event. In particular

$$P(A = \bar{a}|B = \bar{b} \wedge C = \bar{c}) = P(A = \bar{a}|C = \bar{c}).$$

For the second part, let $\bar{a} \in D_A, \bar{d} \in D_{B\cup C}, \bar{c} \in D_C$ with

$$P(B \cup C = \bar{d} \wedge C = \bar{c}) > 0.$$

Then $\bar{d} = \bar{b}\bar{c}$ for some $\bar{b} \in D_B$, and

$$P(A = \bar{a}|(B \cup C) = \bar{d} \wedge C = \bar{c}) = P(A = \bar{a}|B = \bar{b} \wedge C = \bar{c}) = P(A = \bar{a}|C = \bar{c}).$$

For the third part, let $\bar{a} \in D_A$, $\bar{b} \in D_B$, $\bar{c} \in D_C$ such that $P(C = \bar{c}) > 0$, and note that if either of $P(A = \bar{a}|C = \bar{c})$ or $P(B = \bar{b}|C = \bar{c})$ is equal to zero then the equality to be shown follows immediately. And otherwise we have

$$P(A = \bar{a} \wedge B = \bar{b}|C = \bar{c}) = P(A = \bar{a}|B = \bar{b} \wedge C = \bar{c})P(B = \bar{b}|C = \bar{c}) =$$

$$P(A = \bar{a}|B = \bar{b})P(B = \bar{b}|C = \bar{c})$$

□

Concepts from Graph Theory

We assume the reader is familiar with the graph theoretic notions of *adjacency*, *incidence*, and *directed/undirected* graphs.

If $G = (V, E)$ is an undirected graph we will identify E with a set of unordered pairs or singletons from V, that is, $E \subseteq \{\{v, u\} : v, u \in V\}$ (note that v may be equal to u). We may also express the fact that $\{v, u\} \in E$ by vEu, and we will express the fact that e is an edge between v and u by $v -^e u$

If $G = (V, E)$ is a directed graph we will identify E with a set of ordered pairs $E \subseteq V^2$. Again we may also express the fact that $(v, u) \in E$ by vEu, and we will express the fact that e is an edge from v to u by $v \rightarrow^e u$ or $u \leftarrow^e v$.

Definition 1.3 (Skeletons and Paths)**.**

1. *Skeleton: If $G = (V, E)$ is a directed graph, define the* skeleton *of G to be the undirected graph $G' = (V, E')$ where $E' = \{\{v, u\} : (v, u) \in E\}$. Given $e = (v, u) \in E$, we will refer to the edge $\{v, u\} \in E'$ as the* image *of e in G'.*

2. *Path: If $G = (V, E)$ is an undirected graph, with $u, v \in V$, a* path *from u to v (of length r) is a tuple of edges $\{e_i\}_{1 \le i \le r}$ such that for some vertices $\{w_i\}_{0 \le i \le r}$ with $w_0 = u$, $w_r = v$ we have, for all $0 \le i < r$, the relation $w_i -^{e_{i+1}} w_{i+1}$.*

3. *Directed Path: If $G = (V, E)$ is a directed graph, with $u, v \in V$, a* directed path *from u to v (of length r) is a tuple of edges $\{e_i\}_{1 \le i \le r}$ such that for some vertices $\{w_i\}_{0 \le i \le r}$ with $w_0 = u$, $w_r = v$ we have, for all $0 \le i < r$, the relation $w_i \rightarrow^{e_{i+1}} w_{i+1}$.*

4. *Trail: If $G = (V, E)$ is a directed graph, with $u, v \in V$, an* undirected path *or* trail *from u to v (of length r) is a tuple of edges $\{e_i\}_{1 \le i \le r}$ such that the list of images of the edges e_i (in the same order) in the skeleton G' of G form a path from u to v in G' of length r.*

5. *Cycle:* *A* cycle *is a directed path from a vertex* v *to itself. We say that a directed graph* G *is* acyclic *if it does not have any cycles, and we refer to a directed acyclic graph as a DAG.*

Definition 1.4 (Descendant, Ancestor, Parent, Child)**.** *Let* $D = (V, E)$ *be a DAG, and let* $u, v \in V$.

1. *We say that* u *is a* descendant *of* v *(and* v *is an* ancestor *of* u*) if* $v = u$ *or there is a directed path from* v *to* u.

2. *We say that* v *is a* parent *of* u *(and* u *a* child *of* v*) if* vEu.

3. *For any* $v \in V$*, define* $par_D(v)$ *to be the set of parents of* v *in* D.

4. *For any* $v \in V$*, define* $des_D(v)$ *to be the set of descendants of* v *in* D.

This is enough to define Bayesian Networks.

Definition 1.5 (Bayesian Network)**.** *Let* $S = (\Omega, \Sigma, P)$ *be a probability space and let* $\Upsilon = \{X_i\}_{1 \le i \le n}$ *be a set of random variables on* Ω *such that each* $X \in \Upsilon$ *takes on finitely many values. Let* $D = (V, E)$ *be a DAG with* $V = \Upsilon$.

For any $\bar{x} \in D_\Upsilon$*, and for* $1 \le i \le n$ *let the tuple* $\bar{x}_i \in D_{par(X_i)}$ *be the sub-tuple of* $\bar{x}$ *consisting of exactly the elements of* $\bar{x}$ *corresponding to parents of* X_i *and in the same order in which they appear in* $\bar{x}$.

We say D *is a* Bayesian Network for Υ *if we can express the joint probability distribution of the variables* Υ *as*

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{X_i, par(X_i)}(x_i, \bar{x}_i) \tag{1.1}$$

We will now prove that a DAG is a Bayesian Network just in case each of its vertices are independent of their non-descendants given their parents.

Theorem 1.6. *Let* $S = (\Omega, \Sigma, P)$ *be a probability space and let* $\Upsilon = \{X_i\}_{1 \le i \le n}$ *be a set of random variables on* Ω *such that each* $X \in \Upsilon$ *takes on finitely many values. Let* $D = (V, E)$ *be a DAG with* $V = \Upsilon$.

Then D *is a Bayesian Network for* Υ *if and only if*

$$\text{For all } X \in \Upsilon, \quad X \amalg (\Upsilon \setminus des_D(X)) | par_D(X). \tag{1.2}$$

Proof. First we assume D is a Bayesian Network and prove the independence statement (1.2).

Fix $1 \le i \le n$ and consider the variable $X_i \in \Upsilon$. We can assume without loss that $\Upsilon = \{X_i\}_{1 \le i \le n}$ is indexed so that $dec_D(X_i) = \{X_j \in \Upsilon : j \ge i\}$.

Let $\bar{x} \in D_\Upsilon$ such that the first $i - 1$ coordinates satisfy $P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) > 0$.

Since D is a Bayesian Network, we have

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{j=1}^{n} p_{X_j, par(X_j)}(x_j, \bar{x}_j)$$

A straightforward induction on size $|dec_D(X_i)| = n - i$, summing over all possible values in D_Y for each $Y \in dec_D(X_i)$, shows that the joint probability distribution of the variables $\{\Upsilon \setminus dec_D(X_i)\} \cup \{X_i\}$ is given by

$$P(X_1 = x_1, \ldots, X_i = x_i) = \prod_{j=1}^{i} p_{X_j, par(X_j)}(x_j, \bar{x}_j)$$

Similarly, summing over all possible values in D_{X_i} shows that the joint probability distribution of the variables $\{\Upsilon \setminus dec_D(X_i)\}$ is given by

$$P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) = \prod_{j=1}^{i-1} p_{X_j, par(X_j)}(x_j, \bar{x}_j)$$

Thus

$$P(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) =$$
$$\frac{P(X_1 = x_1, \ldots, X_i = x_i)}{P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1})} = p_{X_i, par(X_i)}(x_i, \bar{x}_i)$$

This establishes the left to right direction of the theorem.
For the other direction we will need a nice ordering of D.

Definition 1.7 (Topological Ordering)**.** *Given a directed graph $G = (V, E)$, a* topological ordering *for G is a linear ordering $(V, <)$ of the vertex set V such that,*

$$\textit{For all } u, v \in V, \quad (u, v) \in E \Rightarrow u < v. \tag{1.3}$$

Lemma 1.8. *A directed graph G has a topological ordering if and only if it is acyclic.*

Proof. Clearly a directed graph with a cycle cannot have a topological ordering. For an algorithm to compute a topological ordering for a DAG see, e.g. Kahn [14]. □

Now assume that the independence assumption (1.2) holds. We must establish (1.1) from Definition 1.5
We can assume without loss that $D_X = \mathbf{m} := \{1, 2, \ldots, m\}$ for all $X \in \Upsilon$. Also, by Lemma 1.8 we can assume that $\Upsilon = \{X_i\}_{1 \leq i \leq n}$ is indexed in accordance with a topological ordering for D.
Let $\bar{x} \in \mathbf{m}^n$ be any n-tuple of elements from $\mathbf{m}$.

Then simple probability tells us that

$$P(\bigwedge_{i=1}^{n}(X_i = x_i)) = P(X_1 = x_1)P\left(\left[\bigwedge_{i=2}^{n}(X_i = x_i)\right] \mid X_1 = x_1\right) =$$

$$P(X_1 = x_1)P(X_2 = x_2|X_1 = x_1)P\left(\left[\bigwedge_{i=3}^{n}(X_i = x_i)\right] \mid X_1 = x_1) \wedge X_2 = x_2\right) =$$

$$\ldots = \prod_{i=1}^{n} P\left(X_i = x_i \mid \left[\bigwedge_{j=1}^{i-1} X_j = x_j\right]\right)$$

Now because of the way we ordered Υ, if $X_j \in par_D(X_i)$ then $j < i$, and conversely if $j < i$ then $X_j \in V \setminus des_D(X_i)$. Thus, by (1.2) we see that any factor in the above product reduces to

$$P\left(X_i = x_i \mid \left[\bigwedge_{1\leq j<i} X_j = x_j\right]\right) =$$

$$P\left(X_i = x_i \mid \left[\bigwedge_{\substack{1\leq j\leq n\\ X_j\in par(X_i)}} X_j = x_j\right]\right) = p_{X_i,par(X_i)}(x_i|\bar{x}_i),$$

where, as in the definition of Bayesian Networks, $\bar{x}_i$ denotes the sub-tuple of $\bar{x}$ consisting of coordinates k for $X_k \in par_D(X_i)$. Thus

$$P(\bigwedge_{i=1}^{n}(X_i = x_i)) = \prod_{i=1}^{n} p_{X_i,par(X_i)}(x_i|\bar{x}_i),$$

and this completes the proof. □

We are interested in another characterization of Bayesian Networks that involves the notion of *d-connectedness*. For this we will need to define

Definition 1.9 (Linear, converging, and diverging nodes)**.** *Let $D = (V, E)$ be a DAG, let $p = \{e_i\}_{1\leq i\leq r}$ be a trail in D. Let $v \in V$ be a node on p other than the first or last node on the trail, so that there are exactly two consecutive edges e_k, e_{k+1} on the trail p that are incident with v. These occur in exactly one of four possible arrangements, which we will group into three types:*

1. *Linear*

$$\longrightarrow^{e_k} v \longrightarrow^{e_{k+1}} \qquad or \qquad \longleftarrow^{e_k} v \longleftarrow^{e_{k+1}}$$

2. *Converging*

$$\longrightarrow^{e_k} v \longleftarrow^{e_{k+1}}$$

3. *Diverging*

$$\longleftarrow^{e_k} v \longrightarrow^{e_{k+1}}$$

Definition 1.10 (D-connectedness). *Let $D = (V, E)$ be a DAG, let A, B, C be disjoint subsets of V, and let $p = \{e_i\}_{1\leq i\leq r}$ be a trail from a vertex $a \in A$ to a vertex $b \in B$. We call p a* d-connecting path with respect to C *(or a* d-connecting path given C*) if, for every node v on the path that is not equal to a or b, the node v is either*

1. *linear and in $V \setminus C$,*

2. *diverging and in $V \setminus C$, or*

3. *converging and having a descendant (possibly itself) in C.*

If such a trail exists we say that A and B are d-connected with respect to C *or* d-connected given C. *Otherwise we say that A and B are* d-separated with respect to C.

Theorem 1.11. *Let $S = (\Omega, \Sigma, P)$ be a probability space and let $\Upsilon = \{X_i\}_{1\leq i\leq n}$ be a set of random variables on Ω such that each $X \in \Upsilon$ takes on finitely many values. Let $D = (V, E)$ be a DAG with $V = \Upsilon$.*
Suppose that for any $X \in \Upsilon$ and any disjoint subsets $A, B \subseteq \Upsilon \setminus \{X\}$ such that $\neg(X \amalg A|B)$ we have $\{X\}$ and A d-connected given B.
Then D is a Bayesian Network with respect to Υ.

Proof. Suppose for contradiction that for some $X \in \Upsilon$ we have

$$\neg(X \amalg (\Upsilon \setminus des_D(X))|par_D(X)).$$

Then letting $T = (\Upsilon \setminus des_D(X)) \setminus par_D(X)$ we see by the second part of Lemma 1.2 that

$$\neg(X \amalg T|par_D(X)).$$

Then by assumption there is a d-connecting path p given $par_D(X)$ from the variable X to some variable Y that is not a descendant of X nor a parent of X. We consider two cases.

First suppose that the first arrow on the path p upon leaving X points *away from* X. Since Y is not a descendant of X there must be some first converging node v on the path. But since p is d-connecting with respect to $par_D(X)$, either v or one of its descendants must be in $par_D(X)$, and this gives a contradiction since D is acyclic.

Now assume the first arrow on the path p upon leaving X points *towards* X. The node u that this arrow points away from is a parent of X. Now u cannot be equal to Y since Y is not a parent of X. But then u must be a either a diverging or a linear node, which gives a contradiction since the path p is d-connecting given $par_D(X)$.

Having reached a contradiction in both cases, we conclude that (1.2) in the statement of Theorem 1.6 is satisfied, so that D must be a Bayesian Network with respect to Υ.

□

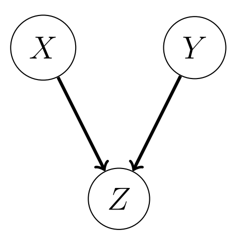
We can say more about the relation between Bayesian Networks and d-connectedness. Verma, Pearl, and Geiger have proved the following

Theorem 1.12 (Verma, Pearl, Geiger)**.** *Suppose D is a Bayesian Network for variables Υ on probability space (Ω, Σ, P) with A, B, C disjoint subsets of Υ such that $\neg(A \amalg B|C)$. Then A and B are d-connected with respect to C.*
Conversely, if D is a DAG on Υ such that A and B are d-connected with respect to C then there is at least one distribution P that such that D is a Bayesian Network with respect to Υ and such that $\neg(A \amalg B|C)$.

Proof. See [20] and [10]. □

In [18] Pearl notes that the converse above is an understatement: *Most* distributions P make $\neg(A \amalg B|C)$, since a "precise tuning of parameters" is required to ensure $(A \amalg B|C)$ when A and B are d-connected with respect to C. Here is a simple example of such a precise tuning.

Let X, Y, Z be random variables taking values in $\{0, 1\}$. Let X and Y take each value with probability .5, and let Z be distributed as the indicator function for $(X = 1 \wedge Y = 0) \vee (X = 0 \wedge Y = 1)$. Let D be the DAG with vertex set $\Upsilon = \{X, Y, Z\}$, with one edge from X to Z, one edge from Y to Z, and no other edges. It is easy to check that D is a Bayesian Network for the variables $\{X, Y, Z\}$, that $X \amalg Z|\emptyset$, and yet $\{X\}$ and $\{Z\}$ are d-connected with respect to the empty set. Also notice that removing either of the two edges results in a DAG that is not a Bayesian Network for Υ.



Chapter 2

Independent Voter Competence

The Classical Condorcet Jury Theorem for juries with homogeneous voter competence involves a positive integer N and a parameter $p \in [0,1]$, where N represents the size of a jury faced with making some verdict, and p represents the *competence* of any member of the jury, that is, the probability of any one member arriving at the correct alternative when deliberating alone. Any sort of generalization of this theorem to juries with members of heterogeneous competence will take as its initial data - instead of a single parameter p - a tuple $\{p_i\}_{1 \leq i \leq N} \in [0,1]^N$ of parameters where for each $1 \leq i \leq N$ the parameter p_i represents the competence of the ith juror.

We begin this chapter by situating these parameters inside of the framework of Bayesian Networks. This will allow us to analyze their structure in terms of a family of random variables in a *DAG* and, in so doing, give a more satisfying account of potential dependence relationships between these competencies. This will come in handy when we turn to the issue of dependent voter competence in the next chapter.

This is not the first attempt to situate the CCJT within some kind of DAG of random variables. Dietrich and Spiekermann introduce the general idea of a "causal network" in [9] enough to address the technique of conditioning on common causes (see Chapter 3 for more on this), but insofar as we can understand their putting forward any kind of network to accurately interpret the classical theorem, their network is largely a straw man to illustrate how independence between votes is highly implausible.

Consider the following Bayesian Network, which we might call the *Naive Network* for the CCJT and which is not unlike the aforementioned network proposed by Dietrich and Spiekerman. There is a single binary variable X denoting the true state of the world (think *suspect actually guilty* or *suspect actually innocent*), and for each juror i there is a binary random variable V_i which indicates the vote of the ith voter. In the DAG of the Naive Network, there is one edge from X to each of the V_i, and there are no other edges. Now certainly in such a network, since the variable X is unknown, the variables V_i are d-connected (with respect to the empty set) and so by the results at the end of the last chapter, the variables V_i will "almost never" be independent (with respect to the empty set of variables).

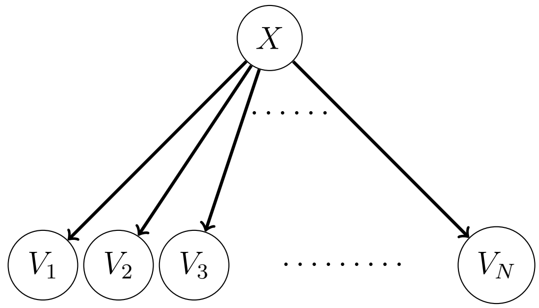


Figure 2.1: A Naive Bayesian Network for the CCJT

We agree that the CCJT requires that the jurors exist in a highly idealized setting. Nevertheless, to claim that networks like the one just given are the most accurate representation of the classical theorem inside a Bayesian Network is to disregard an implicit but necessary application condition of the CCJT . Let us elaborate: The original statement of the CCJT for juries with homogeneous competence involves nothing more than a competence p, the probability that any individual voter will vote correctly when deliberating alone, and the theorem gives a condition on p (in particular the condition that $p > \frac{1}{2}$) which, once satisfied, ensures that simple majority rule is preferred to random ballot. Furthermore, and importantly, the *attractiveness* of the theorem is supposed lie with the assumption that this condition is *reasonable*: In order for simple majority rule to be preferred to random ballot, "all that is required" is the "reasonable requirement" that jurors have individual competence p greater than $\frac{1}{2}$.

But to claim that $p > \frac{1}{2}$ is a reasonable condition is to assume that, in some way, "The truth, while perhaps not totally perspicuous, *is there to be found* in the court evidence; all the pieces of the puzzle are present and only need to be put together; and the truth will *usually* be discovered (by "reasonable" jurors)." Such an assumption takes a highly idealized notion of the quality of court evidence, and the plausibility of these kinds of assumptions is beyond the scope of this paper. Our point is simply that *any* defense of the CCJT as being at all *useful* is going to come with the aforementioned assumptions about the nature of the court evidence.

So, given that this is the case, and given that we *are* going to interpret the CCJT inside a Bayesian Network, what should that network look like?

The goal of the *CJT Network*, defined below, is to provide the simplest possible Bayesian Network where we can interpret the parameter p (in the homogeneous case) or the parameters $\{p_i\}$ (in the heterogeneous case) in terms of some family of random variables which are mutually independent, perhaps conditional on some restriction about the nature of the court evidence. For the sake of argument, we provide conditions under which the philosophical baggage of the CCJT is satisfied: where the truth, though perhaps clouded, is right there in the evidence waiting to be found, and yet the approach does not introduce dependencies between the competencies.

2.1 A Bayesian Network for the CCJT

Definition 2.1 (CJT Network). *Let $N = 2n + 1$ be an odd positive integer, and let $\Phi = \{p_i\}_{1 \le i \le N} \in [0, 1]^N$. Let $S = (\Omega, \Sigma, P)$ be a probability space and $\Upsilon = (X, E, \{V_i\}_{1 \le i \le N}, V)$ be binary random variables on Ω. We say that a directed acyclic graph Ξ is a* CJT Network (of order N with competence Φ and with respect to variables Υ) *if*

1. *Ξ is a Bayesian Network with respect to the set of variables $\{X, E, V\} \cup \{V_i\}_{1 \le i \le N}$,*
2. *The edges in Ξ are given by Figure 2.2. Specifically, the $\{V_i\}$ are all parents of V, E is a parent of each of the $\{V_i\}$, X is a parent of E, and there are no other edges,*
3. *V is the indicator function for the event*

$$\sum_{i=1}^{N} V_i > \frac{N}{2}. \tag{2.1}$$

4. *For all $i \in J$, $p_i = P(V_i = 1|E = 1) = P(V_i = 0|E = 0)$.*

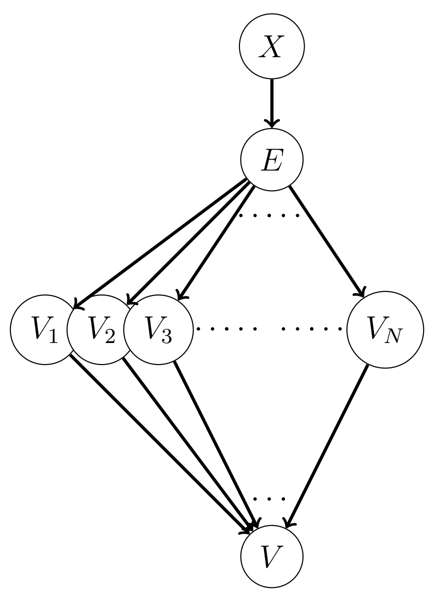


Figure 2.2: A CJT Network of order N

Each random variables in Υ takes values in the set $\{0, 1\}$ which we interpret as the two alternatives from which the group must choose. The value of X is the *correct* alternative. The value of E should be thought of as the alternative toward which the evidence points. For $1 \le i \le N$, the value of V_i represents the alternative voted for by the ith juror. The value of V represents the alternative ultimately chosen by the group of all N jurors via simple majority rule.

Notice that we do not claim to know the marginal distributions any of the variables in Υ. The only requirements we make on the various distributions are that the conditional probabilities $\Phi = \{p_i\}_{1\leq i\leq N} = P(V_i = 1|E = 1) = P(V_i = 0|E = 0)$ are known, that V is distributed as in equation 2.1 above, and that Ξ is a Bayesian Network for the variables Υ.

We now assert that the CCJT can then be viewed as an examination of the value $P(V = X|E = X)$ as a function of the jury size N and the competencies Φ.

Definition 2.2. *Let Ξ be a CJT Network of order $N = 2n + 1$ with competences $\Phi = \{p_i\}_{1\leq i\leq N} \in [0, 1]^N$ and with respect to variables $\Upsilon = (X, E, \{V_i\}_{1\leq i\leq N}, V)$. We make the following definitions*

1.
$$\Gamma := P(V = X|E = X)$$
is the probability that the jury will come to the truth given that the evidence points to the truth.

2. *Let $\mathbf{N} := \{1, \ldots, N\}$. Given $k \in \mathbb{N}$, let*
$$\mathbf{N}_k := \{\sigma \subseteq \mathbf{N} : |\sigma| = k\},$$
and let
$$M(N) := \{\sigma \subseteq \mathbf{N} : |\sigma| > \frac{N}{2}\}$$
be the set of all "majorities" from $\mathbf{N}$.

3. *Given $\sigma \subseteq \mathbf{N}$, and $\alpha \in \{0, 1\}$, we define*
$$N_{\sigma,\alpha} := \left(\bigwedge_{j\in\sigma} V_j = \alpha\right) \wedge \left(\bigwedge_{j\notin\sigma} V_j \neq \alpha\right)$$
to be the event that the set σ will consist of exactly the jurors who voted for α.

4. *Given $\sigma \subseteq \mathbf{N}$, and $\alpha \in \{0, 1\}$, we define*
$$M_{\sigma,\alpha} := P\left(N_{\sigma,\alpha}|E = \alpha\right).$$

Lemma 2.3 (Independence Lemma)**.** *Let P be the probability measure corresponding to a CJT Network Ξ of order N with competence $\Phi = \{p_i\}_{1\leq i\leq N}$. Let $\sigma \subseteq \mathbf{N}$. If we define*
$$M_\sigma := \prod_{j\in\sigma} p_j \prod_{j\in\mathbf{N}\setminus\sigma} (1 - p_j),$$

then

$$M_{\sigma,1} = M_{\sigma,0} = M_\sigma$$

Proof.

$$M_{\sigma,1} = P\left(\left(\bigwedge_{j\in\sigma} V_j = 1\right) \wedge \left(\bigwedge_{j\in\mathbf{N}\setminus\sigma} V_j = 0\right) |E = 1\right)$$

$$= \prod_{j\in\sigma} P(V_j = 1|E = 1) \prod_{j\in\mathbf{N}\setminus\sigma} P(V_j = 0|E = 1) = M_\sigma,$$

and the case for $M_{\sigma,0}$ is similar.

□

We can use this lemma to express the group competence as a neat sum of products

Corollary 2.4. *If P is the probability measure corresponding to a CJT Network Ξ of order N with competence $\Phi = \{p_i\}_{1\leq i\leq N}$ then*

$$\Gamma = \sum_{\sigma\in M(N)} \left(\prod_{i\in\sigma} p_i \prod_{i\in\mathbf{N}\setminus\sigma} (1-p_i)\right) = \sum_{k=n+1}^{N} \sum_{\sigma\in\mathbf{N}_k} \left(\prod_{i\in\sigma} p_i \prod_{i\in\mathbf{N}\setminus\sigma} (1-p_i)\right)$$

Proof. By lemma 2.3

$$\Gamma = P(V = X|E = X) = P(V = X = 1|E = X) + P(V = X = 0|E = X) =$$

$$P(V = X|E = X = 1)P(X = 1|E = X) + P(V = X|E = X = 0)P(X = 0|E = X) =$$

$$P(V = 1|E = X = 1)P(X = 1|E = X) + P(V = 0|E = X = 0)P(X = 0|E = X) =$$

$$P(V = 1|E = 1)P(X = 1|E = X) + P(V = 0|E = 0)P(X = 0|E = X)$$

where in the last equality we have used the fact that V is independent of X given E by Lemma 1.6. Now since

$$P(X = 1|E = X) + P(X = 0|E = X) = 1$$

it suffices to show that

$$P(V = 1|E = 1) = P(V = 0|E = 0) = \sum_{\sigma\in M(N)} \left(\prod_{i\in\sigma} p_i \prod_{i\in\mathbf{N}\setminus\sigma} (1-p_i)\right)$$

But this is true by Lemma 2.3 since

$$P(V = 1|E = 1) = \sum_{\sigma\in M(N)} M_{\sigma,1}$$

and similarly

$$P(V = 0|E = 0) = \sum_{\sigma \in M(N)} M_{\sigma,0}.$$

□

Now recall the method of *random ballot*, in which, after the jurors deliberate individually on the available evidence, and cast their ballots, one ballot is chosen at random (via a uniform distribution) and the alternative written on that ballot become the group decision. Note that the method of random ballot is formally equivalent to a judgement aggregation method which we might call *random dictatorship* or *random temporary executive*, in which, after the jurors deliberate individually on the available evidence, one special juror is chosen at random (via a uniform distribution) and this special juror is instructed to pronounce the verdict as an executive decision.

One could give the method of random ballot its own Bayesian Network or just combine it with a preexisting CJT Network. We have chosen the latter. Given a CJT Network Ξ with respect to variables Υ, we can add two more random variables R and D to Υ to capture the random ballot method. The variable R will model the random selection process, and the value of the variable D is the alternative chosen by the randomly selected juror.

Definition 2.5 (Augmented CJT Network)**.** *Suppose $\Xi = (\Upsilon, EDGE)$ is a CJT Network of order $N = 2n+1$ with competence $\Phi = \{p_i\}_{1\leq i\leq N} \in [0,1]^N$, and with respect to variables $\Upsilon = (X, E, \{V_i\}_{1\leq i\leq N}, V)$ on a probability space $S = (\Omega, \Sigma, P)$.*

Let R and D be two additional random variables with respect to S, and let $\Upsilon^* = (\Upsilon, R, D)$

We say that a directed acyclic graph $\Xi^* = (\Upsilon^*, EDGE^*)$ is an Augmented CJT Network *(of order N, with competence Φ, and with respect to the variables Υ^*) if*

1. *Ξ^* is a Bayesian Network with respect to the variables $\{X, E, V, R, D\} \cup \{V_i\}_{i\in J}$.*
2. *The edges in $EDGE^* \setminus EDGE$ consist of an arrow from each of the V_i to the variable D and an arrow from R to D. There are no other arrows in $EDGE^* \setminus EDGE$.*
3. *The variable R takes values in $\mathbf{N}$ and is distributed uniformly.*
4. *The variable D is the indicator function for the event*

$$\bigvee_{i=1}^{N} (R = i \wedge V_i = 1).$$

It is easy to see that Definitions 2.2, Lemma 2.3, and Corollary 2.4 continue to be true in an Augmented CJT Network. We can also define a counterpart to Γ above.

Definition 2.6. *Given an augmented CJT Network Ξ^*, define*

$$\Delta := P(D = X|E = X)$$

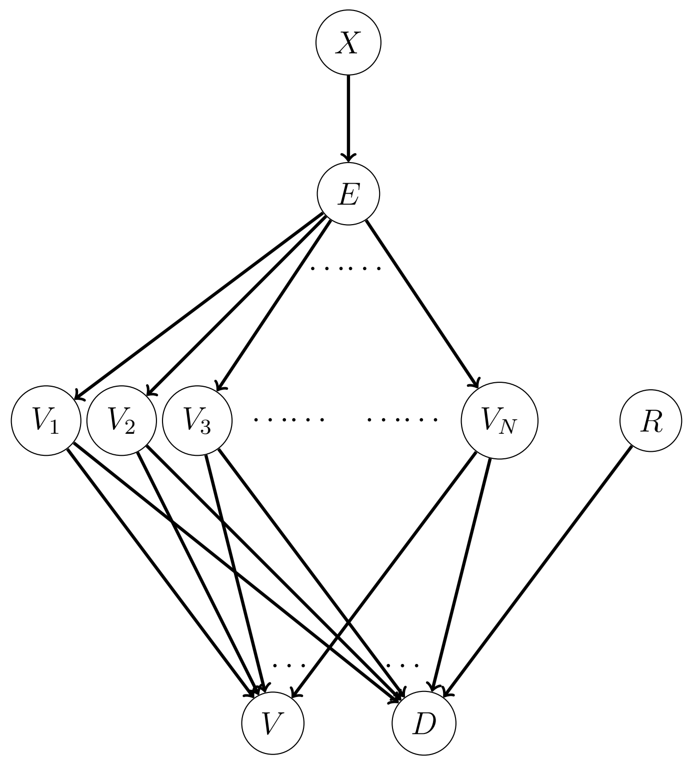


Figure 2.3: Including the process of random ballot

Lemma 2.7. *If Ξ^* is an augmented CJT Network of order N with competencies $\Phi = \{p_i\}_{1\leq i\leq N}$ then*

$$\Delta = \frac{1}{N}\sum_{i=1}^{N} p_i.$$

Proof.

$$\Delta = P(D = X|E = X) = \sum_{i=1}^{N} P(R = i \wedge D = X|E = X) =$$

$$\sum_{i=1}^{N} P(D = X|R = i \wedge E = X)P(R = i|E = X) = \frac{1}{N}\sum_{i=1}^{N} P(D = X|R = i \wedge E = X) =$$

$$\frac{1}{N}\sum_{i=1}^{N} P(V_i = X|R = i \wedge E = X).$$

Now any one of these last summands can be written

$$P(V_i = X|R = i \wedge E = X) =$$

$$P(V_i = X = 1|R = i \wedge E = X) + P(V_i = X = 0|R = i \wedge E = X) =$$
$$P(V_i = X|R = i \wedge E = X = 1)P(X = 1|E = X \wedge R = i)+$$
$$P(V_i = X|R = i \wedge E = X = 0)P(X = 0|E = X \wedge R = i) =$$
$$P(V_i = 1|R = i \wedge E = X = 1)P(X = 1|E = X \wedge R = i)+$$
$$P(V_i = 0|R = i \wedge E = X = 0)P(X = 0|E = X \wedge R = i) =$$
$$P(V_i = 1|E = 1)P(X = 1|E = X \wedge R = i) + P(V_i = 0|E = 0)P(X = 0|E = X \wedge R = i) =$$
$$p_i \left[P(X = 1|E = X \wedge R = i) + P(X = 0|E = X \wedge R = i)\right] = p_i$$

So the lemma is proved.

□

We will often want to think of Γ and Δ as functions of a positive integer N and Φ, independently of any CJT Network Ξ. While we will primarily be concerned with the functions Γ_N, Δ_N with N odd, we will occasionally make use of the functions Γ_{2n} and Δ_{2n}.

Definition 2.8. *Let N be a positive integer, and $\mathbf{N} = \{1, 2, \ldots, N\}$.*

1. *Define $\Gamma_N : [0, 1]^N \to \mathbb{R}$ by*

$$\Gamma_N(\bar{x}) := \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \sum_{\sigma \in \mathbf{N}_k} \left(\prod_{i \in \sigma} p_i \prod_{i \in \mathbf{N} \setminus \sigma} (1 - p_i) \right)$$

2. *Define $\Delta_N : [0, 1]^N \to \mathbb{R}$.*

$$\Delta_N(\bar{x}) := \frac{1}{N} \sum_{i=1}^{N} x_i.$$

2.2 Juries with homogeneous competence

The Classical Condorcet Jury Theorem considers the special case where all competencies in Φ are equal:

Definition 2.9. *We say that a CJT Network Ξ of order N with competence $\Phi = \{p_i\}_{1 \le i \le N}$ has* homogeneous competence *if there exists some $p \in [0, 1]$ such that $p_i = p$ for all $1 \le i \le N$. Otherwise we say that Ξ has* heterogeneous competence.

When homogeneity is satisfied, the values M_σ, Γ, and Δ, simplify quite a bit.

Lemma 2.10. *Let $n \in \mathbb{N}$ and let Ξ be a CJT Network (possibly Augmented) of order $N = 2n + 1$, homogeneous competence p, and with respect to variables Υ.*

1. *For all* $\sigma \subseteq \mathbf{N}$
$$M_\sigma = p^{|\sigma|}(1-p)^{N-|\sigma|}$$

2.
$$\Gamma = \sum_{i=n+1}^{N} \binom{N}{i} p^i (1-p)^{N-i}$$

3.
$$\Delta = p$$

Once again it will be nice to consider Γ and Δ (for Ξ of homogeneous competence) as functions in their own right, although in this case the latter is just the inclusion map.

Definition 2.11. *Let* N *be a positive integer.*

1. *Define* $\Gamma_N : [0,1] \to \mathbb{R}$ *by*
$$\Gamma_N(x) := \sum_{i=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{i} x^i (1-x)^{N-i}.$$

2. *Define* $\Delta : [0,1] \to \mathbb{R}$ *by*
$$\Delta(x) := x.$$

An Example / Contrasting with Random Ballot

Let us look at a simple example. We take a jury of size $N = 3$ with homogeneous juror competence p. If we name the jurors, $1, 2$, and 3, then the possible majorities are the sets $\{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$. We see that
$$\Gamma_3(p) = 3p^2(1-p) + p^3.$$

If the jurors have a competence level of, say, $p = \frac{2}{3}$, then $\Gamma_3(p) = \Gamma_3(\frac{2}{3}) = \frac{20}{27}$.

Notice that $\Gamma_3(\frac{2}{3}) > \frac{2}{3} = \Delta(\frac{2}{3})$. Thus, given that $E = X$, the probability of of arriving at the correct result by method of random ballot is strictly less reliable than the method of majority rule. We will prove this for general $p \in (\frac{1}{2}, 1)$ in Corollary 2.17 below.

The Classical Condorcet Jury Theorem

Theorem 2.12 (Classical Condorcet Jury Theorem)**.** *Let* $n \in \mathbb{N}$, *and let* $N = 2n + 1$.

1. *The sequence* $\{\Gamma_{2n+1}(x)\}_{n\in\mathbb{N}}$ *is strictly increasing in the index* n *for all* $x \in (\frac{1}{2}, 1)$.

2. *The sequence* $\{\Gamma_{2n+1}(x)\}_{n\in\mathbb{N}}$ *is strictly decreasing in the index* n *for all* $x \in (0, \frac{1}{2})$.

3. *If* $x \in (\frac{1}{2}, 1)$, *then* $\Gamma_{2n+1}(x) \to 1$ *as* $n \to \infty$.

4. *If* $x \in (0, \frac{1}{2})$, *then* $\Gamma_{2n+1}(x) \to 0$ *as* $n \to \infty$.

5. *If* $x \in \{0, \frac{1}{2}, 1\}$, *then* $\Gamma_{2n+1}(x) = x$.

The standard proof of the CCJT uses the weak law of large numbers.[1] Our proof uses basic analytical techniques (and Stirling's approximation) to analyze the behavior of the function $\Gamma_N(p)$ more closely. Our proof also makes explicit a difference formula for the function $\Gamma_{2n+1}(p)$ (in the variable n) revealing the extent of the monotonicity of $\Gamma_{2n+1}(p)$ in the variable n. This difference formula will come in handy in Chapter 3.

First a lemma.

Lemma 2.13. *For* $n \in \mathbb{N}$ *and* $N = 2n+1$,

1.

$$\frac{d}{dx}\Gamma_N = \Gamma_N' = \binom{2n+1}{n+1}(n+1)(x(1-x))^n.$$

2.

$$\Gamma_N\left(\frac{1}{2}\right) = \frac{1}{2}.$$

3.

$$\Gamma_N(1) = 1.$$

(i). Taking the first derivative of Γ_N gives us

$$\Gamma_N'(x) = \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \left[-x^i(2n+1-i)(1-x)^{2n-i} + ix^{i-1}(1-x)^{2n+1-i}\right]$$

$$= \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \left[-x^i(2n+1-i)(1-x)^{2n-i}\right] + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \left[ix^{i-1}(1-x)^{2n+1-i}\right]$$

$$= \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \left[-x^i(2n+1-i)(1-x)^{2n-i}\right] + \sum_{i=n}^{2n} \binom{2n+1}{i+1} \left[(i+1)x^i(1-x)^{2n-i}\right].$$

Naming the summands on the left T_i and those the right S_i, this says that

$$\Gamma_N'(x) = \sum_{i=n+1}^{2n+1} T_i + \sum_{i=n}^{2n} S_i = T_{2n+1} + S_n + \sum_{i=n+1}^{2n} (T_i + S_i).$$

[1] One can use the WLLN or CLT to show that the limit of the average of the voter competencies must be equal to p and thus greater than one half. The theorem follows quickly from this.
Note that the laws of large numbers and the central limit theorem will be of no use once we get to heterogeneous jury competence in the next section.

We find that for $n+1 \leq i \leq 2n$ we have

$$T_i + S_i = x^i(1-x)^{2n-i}\binom{2n+1}{i}\left[-(2n+1-i)+K(i+1)\right],$$

where

$$K = \frac{\binom{2n+1}{i+1}}{\binom{2n+1}{i}} = \frac{(2n+1)!}{(2n-i)!(i+1)!}\frac{(2n+1-i)!i!}{(2n+1)!}$$
$$= \frac{(2n+1-i)!i!}{(2n-i)!(i+1)!} = \frac{2n+1-i}{i+1}.$$

Thus $\sum_{i=n+1}^{2n}(T_i+S_i) = 0$, and so $\Gamma'_N = T_{2n+1} + S_n$.

It is also immediate that $T_{2n+1} = 0$. So

$$\Gamma'_N = S_n = \binom{2n+1}{n+1}\left[(n+1)x^n(1-x)^n\right].$$

□

(ii). By the binomial theorem,

$$\Gamma_N\left(\frac{1}{2}\right) = \sum_{i=n+1}^{2n+1}\binom{2n+1}{i}\frac{1}{2^{2n+1}}$$

$$= \frac{1}{2^{2n+1}}\sum_{i=n+1}^{2n+1}\binom{2n+1}{i}(1)^i(1)^{2n+1-i} = \frac{1}{2^{2n+1}}\frac{1}{2}(1+1)^{2n+1} = \frac{1}{2}.$$

□

(iii). Obvious. □

Definition 2.14. *For $n \in \mathbb{N}$, $N = 2n+1$, and $x \in [0,1]$, define*

$$D_N(x) := \Gamma_{N+2}(x) - \Gamma_N(x).$$

Lemma 2.15. *For $n \in \mathbb{N}$, $N = 2n+1$, and $x \in [0,1]$,*

$$D_N(x) = \binom{2n+1}{n+1}(2x-1)((1-x)x)^{n+1}.$$

Proof. It follows from Lemma 2.13 that

$$\Gamma_N(x) = \Gamma_N\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{x}\Gamma'_N(t)dt = \frac{1}{2} + \int_{\frac{1}{2}}^{x}\binom{2n+1}{n+1}\left[(n+1)t^n(1-t)^n\right]dt.$$

Thus

$$\Gamma_{N+2}(x) - \Gamma_N(x) =$$

$$\int_{\frac{1}{2}}^{x} \binom{2n+3}{n+2} \left[(n+2)t^{n+1}(1-t)^{n+1}\right] dt - \int_{\frac{1}{2}}^{x} \binom{2n+1}{n+1} \left[(n+1)t^{n}(1-t)^{n}\right] dt =$$

$$\binom{2n+1}{n+1} \int_{\frac{1}{2}}^{x} \frac{(2n+3)(2n+2)}{(n+2)(n+1)} \left[(n+2)t^{n+1}(1-t)^{n+1}\right] - \left[(n+1)t^{n}(1-t)^{n}\right] dt =$$

$$\binom{2n+1}{n+1} \int_{\frac{1}{2}}^{x} \left[(4n+6)t^{n+1}(1-t)^{n+1} - (n+1)t^{n}(1-t)^{n}\right] dt =$$

$$\binom{2n+1}{n+1} \int_{\frac{1}{2}}^{x} (t(1-t))^{n} \left[(4n+6)t(1-t) - (n+1)\right] dt.$$

It can be checked that an antiderivative of this last integrand is $(2t-1)((1-t)t)^{n+1}$, and the lemma is shown. □

Corollary 2.16. *Let $n \in \mathbb{N}$ and $N = 2n+1$,*

1. $D_N(x) > 0$ *for* $x \in (\frac{1}{2}, 1)$,
2. $D_N(x) < 0$ *for* $x \in (0, \frac{1}{2})$,
3. $D_N(x) = 0$ *for* $x \in \{0, \frac{1}{2}, 1\}$.

Proof. Straightforward. □

This establishes parts (1), (2), and (5) of Theorem 2.12.
We take this moment to present a corollary saying when majority rule is preferred to random ballot.

Corollary 2.17. *For $n \in \mathbb{N}$ and $N = 2n+1$,*

1. $\Gamma_N(x) > \Delta(x) = x$ *for* $x \in (\frac{1}{2}, 1)$,
2. $\Gamma_N(x) < \Delta(x) = x$ *for* $x \in (0, \frac{1}{2})$,
3. $\Gamma_N(x) = \Delta(x) = x$ *for* $x \in \{0, \frac{1}{2}, 1\}$.

Proof. By 2.16, since $\Gamma_1(x) = \Delta(x)$. □

Lastly we prove part (3) of the theorem. Part (4) will follow by a symmetrical argument.

Part (3). The general strategy is as follows. Again let $n \in \mathbb{N}$ and $N = 2n+1$. We know that on $(\frac{1}{2}, 1)$ the functions $\{\Gamma_{2n+1}\}$ are pointwise increasing in n (by 2.16(1)), have positive x derivative (by 2.13(1)) , and thus are bounded above by 1 (by 2.13(3)). So for each $x \in (\frac{1}{2}, 1)$ a pointwise limit $L(x)$ exists.

For a fixed $x \in (\frac{1}{2}, 1)$, let $I_x := [x, 1]$, and let $G_{N,x} : I_x \to \mathbb{R}$ be the restriction of Γ'_N to I_x. That is

$$G_{N,x}(z) = \Gamma'_N(z)|_I = \binom{2n+1}{n+1}(n+1)(z(1-z))^n$$

for all $z \in I_x$.

Then if we can show that the sequence of functions $\{G_{N,x}\}_{n\in\omega}$ converges *uniformly* on I_x to the zero function, it follows that the restrictions $\Gamma_N|_{I_x}$ converge to $L|_{I_x}$, which must be a constant function and indeed must equal $1|_{I_x}$ by 2.13(3). Doing this for an arbitrary $x \in (\frac{1}{2}, 1)$ gives us the the conclusion that we want.

So fix $x \in (\frac{1}{2}, 1)$. The last factor in $G_{N,x}(x)$ above appears frequently enough for us to give its base a name. Define $y(z) := z(1-z)$. As z increases through the domain I_x, y decreases from $y(x) < \frac{1}{4}$ to zero. In particular, we have $y(x) \in (0, \frac{1}{4})$ and so for all $z \in I_x$ we have $y(z) \in (0, y(x))$.

We wish to find, for each $\epsilon > 0$, a positive integer M such that, for all $n > M$ we have $G_{N,x}(z) < \epsilon$ for all $z \in I_x$. Equivalently, we wish to show that, as $n \to \infty$, we have $\ln G_{N,x}(z) \to -\infty$ *independent of* $z \in I_x$.

First notice that for $z \in I_x$ we have $\ln(y(z)) \in (-\infty, \alpha]$ where $\alpha = \ln y(x) < -4$.

We see that

$$\begin{aligned}\ln G_{N,x}(z) &= \ln\left(\binom{2n+1}{n+1}(n+1)(y(z))^n\right)\\ &= \ln((2n+1)!) - 2\ln(n!) + n\ln(y(z)).\end{aligned}$$

Using Stirling's approximation on the first two terms (and using big-O notation) we write

$$\ln G_{N,x}(z) = (2n+1)\ln(2n+1) - (2n+1) + O(\ln(2n+1)) - 2n\ln(n) + 2n - 2O(\ln(n)) + n\ln(y(z))$$

$$\begin{aligned}&= 2n\ln(2n+1) - 2n\ln(n) + n\ln(y(z)) \pm O(\ln(n))\\ &= n\left[\ln\left(\frac{2n+1}{n}\right)^2 + \ln(y(z)) \pm \frac{O(\ln(n))}{n}\right] \leq\\ &= n\left[\ln\left(\frac{2n+1}{n}\right)^2 + \ln y(x) \pm \frac{O(\ln(n))}{n}\right]\end{aligned}$$

But as $n \to \infty$ the expression in square brackets converges to $\ln 4 - \ln y(x) < 0$ and this expression is independent of $z \in I_x$. Thus since the factor of n outside the square brackets goes to ∞, we conclude that $\ln G_{N,x}(z)$ goes to $-\infty$ independent of $z \in I_x$.

Thus we have shown that $\{G_{2n+1,x}\}_{n\in\omega}$ converges uniformly to the zero function, thus $\Gamma_N(z)|_{I_x} \to 1$. Since x was arbitrary this gives $\Gamma_N(x) \to 1$ for $x \in (\frac{1}{2}, 1)$. This shows part

(3) of the theorem, and part (4) follows in a similar manner. This completes the proof of the CCJT. □

Figure 1.1 gives a graph of the functions $y = \Gamma_N(x)$ for juries of $1, 3, 7, 31$, and 141 jurors with homogeneous competence x .

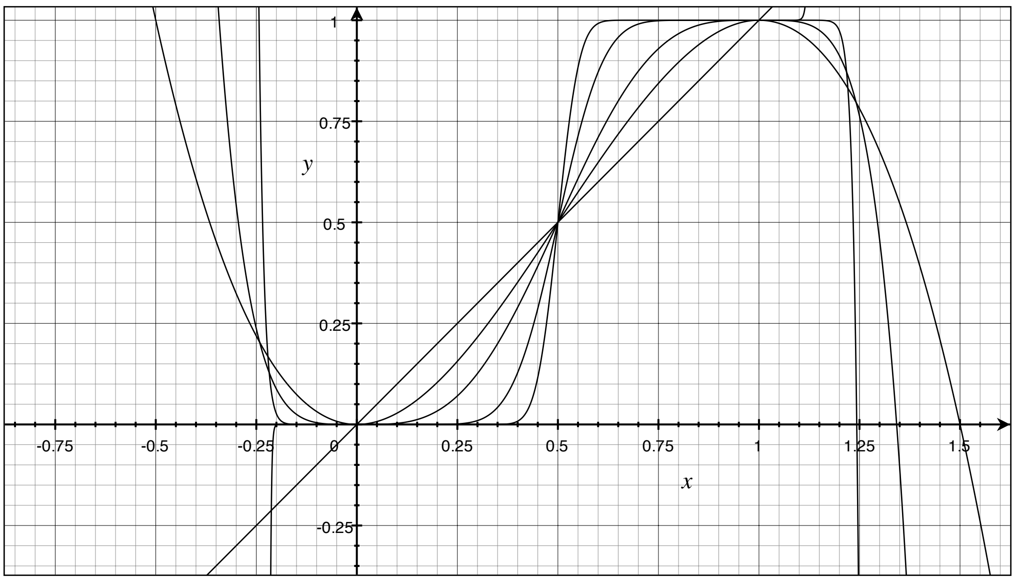


Figure 2.4: $y = \Gamma_N(x)$ for $N \in \{1, 3, 7, 31, 141\}$ jurors with homogeneous competence x

2.3 Juries with heterogeneous competence

In this section we consider how to generalize some of the previous results to situations in which the jury has heterogeneous competence.

Theorem 2.12 itself does not have an obvious translation to the heterogeneous case without some additional information about the individual competences of the incoming jurors, e.g. competences following some sequence or governed by some probability distribution.

If, say for part (1), the only requirement was that we draft two new jurors with competence greater than $\frac{1}{2}$, then group competence is certainly not increasing in group size, as can be seen by introducing two jurors with competence .6 into a jury of one member with competence .9.

$$(.9)(.6)(.6) + (.9)(.6)(.4) + (.9)(.6)(.4) + (.1)(.6)(.6) = .792 < .9.$$

As similar claim holds for part (2).

In the case of part (3), it is easy to see that if there is some lower bound $m > \frac{1}{2}$ such that each new juror i has competence $p_i > m$ then it follows from Theorem 2.12 that the jury competence must go to 1 in the limit. Indeed if the competences of the jurors even converges to $m > \frac{1}{2}$ then the jury competence must go to 1 in the limit. If, however, we only require that each new juror has competence $p_i > \frac{1}{2}$, and if the competence of the incoming jurors converges to $\frac{1}{2}$ sufficiently fast, then we witness some interesting behavior. In particular Berend and Paroush prove the following theorem (see [1]).

Theorem 2.18 (Berend and Paroush). *Let $\{p_i\}_{1\leq i<\infty} \in [0,1]^\infty$, and let P_N be the probability that a jury with N members with respective competences $p_1, \ldots, p_N$ will arrive at the correct verdict. Then $\lim_{N\to\infty} P_N = 1$ if and only if at least one of the following conditions holds*

1.

$$\frac{\sum_{i=1}^{N} p_i - \frac{N}{2}}{\sqrt{\sum_{i=1}^{N} p_i(1-p_i)}} \to \infty$$

2. For every sufficiently large N

$$|\{i : 1 \leq i \leq N, p_i = 1\} > \frac{N}{2}$$

In this chapter we are less interested in asymptotic results than with comparing simple majority rule to random ballot for a fixed jury size N. In particular we are interested in generalizing Corollary 2.17 to juries with heterogeneous competence $\Phi = \{p_i\}_{i\in\mathbf{N}}$ when each $p_i \in (\frac{1}{2}, 1)$ (or each $p_i \in (0, \frac{1}{2})$).

In [5], Boland notices the that the following 1956 theorem by Hoeffding [13], gives an approximation to Corollary 2.17 for heterogeneous competence:

Theorem 2.19 (Hoeffding). *Let $\{X_i\}_{1\leq i\leq N}$ be independent Bernoulli random variables with respective probabilities $\{p_i\}_{1\leq i\leq N}$ of success, and let $S = \sum_{i=1}^{N} X_i$ be the random variable denoting the number of successes in all N trials. Let $\bar{p} := \frac{1}{N}\sum_{i=1}^{N} p_i$, and let b, c be two integers such that $0 \leq b \leq N\bar{p} \leq c \leq N$. Then*

$$P(b \leq S \leq c) \geq \sum_{k=b}^{c} \binom{N}{k} \bar{p}^k (1-\bar{p})^{N-k} \tag{2.2}$$

The lower bound is attained only if $p_i = \bar{p}$ for all $1 \leq i \leq N$ unless $b = 0$ and $c = N$.

Letting $c = N$ and $b = \frac{N+1}{2}$ in Hoeffding's result and then applying the Corollary 2.17 to the right hand side of (2.2), we see that a heterogeneous jury with $\bar{p} \geq \frac{1}{2} + \frac{1}{2N}$ will be more likely to arrive at the correct decision than the method of random ballot.

We are interested in proving a similar theorem but with the requirement that $p_i \geq \frac{1}{2}$ for all $1 \leq i \leq N$ and no further restriction on the average $\bar{p}$.

The theorem that we prove is 2.27 below. Note that Berend and Sapir (see [2]) have proved a more general form of this theorem in which they compare the probability of a correct verdict by simple majority rule with a jury of size N to that of the method of selecting a random sub jury of size $m \leq N$ and deciding the issue by means of simple majority rule amongst that subjury. While our result is less general, it makes use of some interesting combinatorics.

We will first need to derive some corollaries from Hall's Marriage Theorem.

Hall's Marriage Theorem on Graphs

Definition 2.20. *Let $G = (V, E)$ be an undirected graph.*

1. *We will say that G is* finite *if V and E are finite sets.*

2. *Let $U \subseteq V$. We define $N_G(U)$, the* neighborhood *of U, to be all vertices adjacent to some vertex in U.*

$$N_G(U) = \{w \in V : \exists u \in U(\{w, u\} \in E)\}$$

3. *Let $U \subseteq V$ and $F \subseteq E$. We define*

$$D_G(F, U) = \{\{x, y\} \in F : x \in U \vee y \in U\}$$

to be all edges in F that are incident with some vertex in U.

4. *Given $d \in \mathbb{N}$, we say that G is d-regular if $|N_G(\{v\})| = d$ for every $v \in V$.*

5. *We say that G is* bipartite *if we can partition the vertex set V into two disjoint subsets V_1 and V_2 such that every edge in G is incident with a vertex in V_1 and a vertex in V_2. We refer to the pair (V_1, V_2) as a* bipartition *for G.*
(NOTE: We will write $V = V_1 \oplus V_2$ to emphasize that a union is disjoint. Also, if we are building a vertex set for G out of two preexisting sets U_1 and U_2 which may have nonempty intersection, we will write $V = U_1 \oplus U_2$ to mean that we are setting V equal to the union of disjoint copies of U_1 and U_2, as in $V = (U_1 \times \{1\}) \cup (U_2 \times \{2\})$.)

6. *A subset $M \subseteq E$ is called a* matching *for G if the edges of M are pairwise non-adjacent, that is, no two edges in M share a vertex. We say that M* covers *some subset $U \subseteq V$ if, for every vertex $v \in U$, there is some edge in M incident with v.*

Theorem 2.21 (Hall's Marriage Theorem)**.** *Let G be a finite, bipartite graph with bipartition $V = V_1 \oplus V_2$. Then G has a matching that covers V_1 if and only if G satisfies the* Marriage Condition:

$$\text{For every subset } Z \subseteq V_1, \quad |Z| \leq |N_G(Z)|. \tag{2.3}$$

Proof. We refer the reader to Hall's original paper [12]. $\square$

Theorem 2.22. *Let d be a positive integer, and let $G = (V, E)$ be a finite, d-regular, bipartite graph with bipartition $V = V_1 \oplus V_2$. Then G has a matching that covers V_1.*

Proof. We will show that G satisfies the Marriage Condition (2.3).
Fix some subset $Z \subseteq V_1$. Let $H = (W, F)$ be the subgraph of G with vertices $W = Z \oplus N_G(Z)$ and edge set F equal to the restriction of E to W. Then

$$d|Z| = |D_G(E, Z)| = |F| = |D_G(F, N_G(Z))| \leq |D_G(E, N_G(Z))| = d|N_G(Z)|.$$

Dividing through by d gives the desired inequality, and therefore the theorem. $\square$

Given a set S, and $k \in \mathbb{N}$, let $S_k = \{\sigma \subseteq S : |\sigma| = k\}$.

Corollary 2.23. *Let S be a set with $|S| = N$, and let $k \in \mathbb{N}$ with $k \leq \frac{N}{2}$. Then there exists a permutation $\phi_{k,N}$ of S_k such that for all $\sigma \in S_k$*

$$\phi_{k,N}(\sigma) \cap \sigma = \emptyset.$$

Proof. Let $G = (V, E)$ be the bi-partite graph with bi-partition $V = V_1 \oplus V_2 = S_k \oplus S_k$ and edge set

$$E = \{\{\sigma, \tau\} : \sigma \in V_1, \tau \in V_2, \sigma \cap \tau = \emptyset\}.$$

G is in fact $\binom{N-k}{k}$-regular, thus Theorem 2.22 tells us that G has a matching that covers V_1. Since $V_1 = S_k = V_2$ this matching is actually a permutation with the desired property. $\square$

Corollary 2.24. *Let S be a set with $|S| = N$, and let $k \in \mathbb{N}$ with $k \leq \frac{N}{2}$. Then there exists a bijection $\psi_{k,N} : S_k \to S_{N-k}$ such that for all $\sigma \in S_k$*

$$\sigma \subseteq \psi_{k,N}(\sigma).$$

Proof. Let $\phi_{k,N}$ be as in Corollary 2.23. For $\sigma \in S_k$, define $\psi_{k,N}(\sigma) = S \setminus \phi_{k,N}(\sigma)$. $\square$

Lemma 2.25. *Let $\mathbf{N} = \{1, 2, \dots, N\}$, and let $k \in \mathbb{N}$ with $k \leq \frac{N}{2}$.*

1. *Given $\bar{y} = (y_1, \dots y_N) \in [1, \infty)^N$,*

$$\sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma| = k}} \prod_{j \in \sigma} y_j \leq \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma| = N-k}} \prod_{j \in \sigma} y_j.$$

2. *Given* $\bar{y} = (y_1, \dots y_N) \in (0,1]^N$,

$$\sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j \geq \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=N-k}} \prod_{j\in\sigma} y_j.$$

3. *In both cases equality holds if and only if*

   a) $\bar{y} = \mathbf{1} = (1,1,\dots,1)$, *or*

   b) N *is even and* $k = \frac{N}{2}$.

Proof. We will prove the first part and the half of part (3) corresponding to $\bar{y} \in [1,\infty)^N$. The second part and other half of part (3) will follow by a similar argument.
Let $\bar{y} = (y_1, \dots y_N) \in [1,\infty)^N$, and let $\psi := \psi_{k,N}$ be as in Corollary 2.24. First notice that

$$\text{for all } \sigma \in \mathbf{N}_k, \quad \prod_{j\in\sigma} y_j \leq \prod_{j\in\psi(\sigma)} y_j. \tag{2.4}$$

Thus

$$\sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j \leq \sum_{\substack{\psi(\sigma) \\ \sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\psi(\sigma)} y_j = \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=N-k}} \prod_{j\in\sigma} y_j. \tag{2.5}$$

This proves part (1).

Furthermore we have (2.5) just in case each equality holds in each instance of (2.4). Now clearly these are satisfied if $\bar{y} = \mathbf{1} = (1,1,\dots,1)$, and if N is even and $k = \frac{N}{2}$ then clearly $\psi(\sigma) = \sigma$ for all $\sigma \in \mathbf{N}_k$, and the inequalities in 2.4 are equalities.

Conversely, suppose that for some $\alpha \in \mathbf{N}$, $y_\alpha > 1$ and that $k < \frac{N}{2}$. Now ϕ is a bijection, there are $\binom{N-1}{k}$ sets $\sigma \in \mathbf{N}_k$ that do not contain α, and there are only $\binom{N-1}{N-k}$ sets $\tau \in \mathbf{N}_{N-k}$ that do not contain α. Since $k < \frac{N}{2}$ we see that $\binom{N-1}{N-k} < \binom{N-1}{k}$, and thus there must be some $\sigma \in \mathbf{N}_k$ such that $\alpha \notin \sigma$ and $\alpha \in \psi(\sigma)$. Thus the corresponding inequality in 2.4 is strict, and equality does not hold. This establishes the half of part (3) corresponding to $\bar{y} \in [1,\infty)^N$.

□

We define the probability of a tie vote.

Definition 2.26 (Probability of a Tie Vote)**.** *Given a positive integer* n*,* $N = 2n$*, and* $\bar{x} \in [0,1]^N$*, we define* β_{2n} *to be the probability of a tie:*

$$\beta_{2n} = \sum_{\sigma \in \mathbf{N}_n} \prod_{j\in\sigma} x_j \prod_{j \in \mathbf{N}\setminus\sigma} (1 - x_j)$$

For an odd integer N'*, we define*

$$\beta_{N'} = 0.$$

Comparing Simple Majority Rule and Random Ballot for Heterogeneous Juries

Armed with Lemma 2.25 and this definition, we are now ready to prove our theorem.

Theorem 2.27. *For N a positive integer (of either parity) and $\mathbf{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ of length N,*

1. $\Gamma_N(\bar{x}) \geq \Delta_N(\bar{x}) + \frac{1}{2}\beta_N(\bar{x})$ *for* $\bar{x} \in [\frac{1}{2}, 1)^N$.
2. $\Gamma_N(\bar{x}) \leq \Delta_N(\bar{x}) + \frac{1}{2}\beta_N(\bar{x})$ *for* $\bar{x} \in (0, \frac{1}{2}]^N$.
3. *The inequalities are strict for all* $\bar{x} \neq \mathbf{\frac{1}{2}}$.

Proof. Showing that equality holds for $\bar{x} = \mathbf{\frac{1}{2}}$ (for N of either parity) is similar to the proof of Lemma 2.13(ii) and is left to the reader. Now we let $D := [\frac{1}{2}, 1)^N$, and we begin by showing the inequality in (1) is strict for $\bar{x} \in D \setminus \{\mathbf{\frac{1}{2}}\}$.
Recall our definitions of Γ_N and Δ_N for heterogenous juries:

$$\Gamma_N(\bar{x}) := \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} x_j \prod_{j\in\mathbf{N}\setminus\sigma} (1-x_j),$$

and

$$\Delta_N(\bar{x}) = \frac{1}{N}\sum_{j\in\mathbf{N}} x_j.$$

The function $y(x) = \frac{x}{1-x}$ defines an increasing bijection from $[\frac{1}{2}, 1)$ onto $[1, \infty)$. Note that $x = \frac{y(x)}{1+y(x)}$ and $1 - x = \frac{1}{1+y(x)}$. For $j \in \mathbf{N}$, let $y_j = y(x_j)$, and $\bar{y} = (y_1, \ldots, y_N)$. Also, let $E = [1, \infty)^N$.

Then

$$\Gamma_N(\bar{x}) = \left(\prod_{j\in\mathbf{N}} (\frac{1}{1+y_j})\right) \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j,$$

and

$$\Delta_N(\bar{x}) = \frac{1}{N}\sum_{j\in\mathbf{N}} \left(\frac{y_j}{1+y_j}\right) = \frac{1}{N}\left(\prod_{j\in\mathbf{N}} (\frac{1}{1+y_j})\right) \sum_{j\in\mathbf{N}} \left(y_j \prod_{i\neq j} (1+y_i)\right).$$

So letting $P = \left(\prod_{j\in\mathbf{N}}(\frac{1}{1+y_j})\right)$, it suffices to show that

$$NP^{-1}\Gamma_N(\mathbf{x}) > NP^{-1}\Delta_N(\mathbf{x}) + \frac{1}{2}NP^{-1}\beta_N$$

that is, it suffices to show that, for all $\bar{y} \in E \setminus \{\mathbf{1}\}$,

$$N \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j > \sum_{j\in\mathbf{N}} \left(y_j \prod_{i\neq j} (1+y_i) \right) + \frac{1}{2} N P^{-1} \beta_N. \tag{2.6}$$

Now notice that for the first term of the right hand side of (2.6) we have

$$\sum_{j\in\mathbf{N}} \left(y_j \prod_{i\neq j} (1+y_i) \right) = \sum_{\sigma\subseteq\mathbf{N}} |\sigma| \prod_{j\in\sigma} y_j = \sum_{k=0}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j\in\sigma} y_j. \tag{2.7}$$

So making this substitution, we see that it suffices to show, for all $\bar{y} \in E \setminus \{\mathbf{1}\}$, that

$$N \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j > \sum_{k=0}^{N} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j\in\sigma} y_j + \frac{1}{2} N P^{-1} \beta_N. \tag{2.8}$$

For N an odd integer the second term on the right hand side of (2.8) is zero, and for $N = 2n$ the expression becomes

$$\frac{1}{2} N P^{-1} \beta_N = n \sum_{\sigma\in\mathbf{N}_n} \prod_{j\in\sigma} y_j. \tag{2.9}$$

Let us first consider the case for odd $N = 2n+1$. Inequality (2.8) becomes

$$(2n+1) \sum_{k=n+1}^{2n+1} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} \prod_{j\in\sigma} y_j > \sum_{k=0}^{2n+1} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j\in\sigma} y_j. \tag{2.10}$$

We break up the sum on the right into two parts for $k \leq n$ and $k \geq n+1$, and we subtract the latter from both sides to get

$$\sum_{k=n+1}^{2n+1} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} (2n+1-k) \prod_{j\in\sigma} y_j > \sum_{k=0}^{n} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j\in\sigma} y_j. \tag{2.11}$$

The $k = 2n+1$ term on the left and the $k = 0$ term on the right are equal to zero. We drop them and reindex the sum on the left from 1 to get

$$\sum_{k=1}^{n} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k+n}} (n+1-k) \prod_{j\in\sigma} y_j > \sum_{k=1}^{n} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j\in\sigma} y_j. \tag{2.12}$$

And for this it will suffice to show that the kth term on the left is no greater than the $(n+1-k)$th term on the right, and that at least one such inequality is strict. That is, it suffices to show that for all $\bar{y} \in E \setminus \{\mathbf{1}\}$, and $1 \leq k \leq n$ we have

$$\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k+n}}(n+1-k)\prod_{j\in\sigma}y_j > \sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=n+1-k}}(n+1-k)\prod_{j\in\sigma}y_j. \tag{2.13}$$

Or equivalently, dividing both sides by $n+1-k$, letting $r := n+1-k$, and recalling that $k+n = N-(n+1-k) = N-r$, it suffices to show that

$$\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=N-r}}\prod_{j\in\sigma}y_j > \sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=r}}\prod_{j\in\sigma}y_j \tag{2.14}$$

for all $\bar{y}\in E\setminus\{\mathbf{1}\}$, and $1\le r\le n$, and that at least one such inequality is strict.
But this claim follows immediately from the first part of Lemma 2.25.

We now consider the case of even $N=2n$.
Inequality (2.8) becomes

$$2n\sum_{k=n}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}\prod_{j\in\sigma}y_j > \sum_{k=0}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j + n\sum_{\sigma\in\mathbf{N}_n}\prod_{j\in\sigma}y_j. \tag{2.15}$$

We break the first sum on the right hand side into three parts for $k<n, k>n, k=n$ and combine this last part with the other term already on the right hand side to get

$$2n\sum_{k=n}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}\prod_{j\in\sigma}y_j > \sum_{k=0}^{n-1}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j + \sum_{k=n+1}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j + 2n\sum_{\sigma\in\mathbf{N}_n}\prod_{j\in\sigma}y_j. \tag{2.16}$$

Subtracting the last term of (2.16) from both sides gives

$$2n\sum_{k=n+1}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}\prod_{j\in\sigma}y_j > \sum_{k=0}^{n-1}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j + \sum_{k=n+1}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j. \tag{2.17}$$

Again subtracting the last term of (2.17) from both sides gives

$$\sum_{k=n+1}^{2n}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}(2n-k)\prod_{j\in\sigma}y_j > \sum_{k=0}^{n-1}\sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}}k\prod_{j\in\sigma}y_j. \tag{2.18}$$

Now the $k=2n$ term on the left and the $k=0$ term on the right are both equal to zero, so we drop them and reindex the sum on the left from 1 to get

$$\sum_{k=1}^{n-1} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k+n}} (n-k) \prod_{j \in \sigma} y_j > \sum_{k=1}^{n-1} \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k}} k \prod_{j \in \sigma} y_j. \tag{2.19}$$

And for this it will suffice to show that the kth term on the left is no greater than the $n-k$th term on the right, and that at least one such inequality is strict. That is, it suffices to show that for all $\bar{y} \in E \setminus \{\mathbf{1}\}$, and $1 \leq k < n$ we have

$$\sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=k+n}} (n-k) \prod_{j \in \sigma} y_j \geq \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=n-k}} (n-k) \prod_{j \in \sigma} y_j. \tag{2.20}$$

Equivalently, letting $r := n-k$, dividing both sides by $n-k$, and recalling that $n+k = N-(n-k) = N-r$, it suffices to show that

$$\sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=N-r}} \prod_{j \in \sigma} y_j \geq \sum_{\substack{\sigma \subseteq \mathbf{N} \\ |\sigma|=r}} \prod_{j \in \sigma} y_j \tag{2.21}$$

for all $\bar{y} \in E \setminus \{\mathbf{1}\}$ and $1 \leq r \leq n-1$ and that for at least some r this inequality is strict. But as in the odd case, the claim follows immediately from the first part of Lemma 2.25.

This concludes the case for $\bar{y} \in E \setminus \{\mathbf{1}\}$ and hence for $\bar{x} \in [\frac{1}{2}, 1)^N$.

The case for $\bar{x} \in (0, \frac{1}{2}]^N$ parallels the above case except that the direction of all the inequalities is reversed. The function y that we used to map the interval $[\frac{1}{2}, 1)$ to $[1, \infty)$ also maps the interval $(0, \frac{1}{2}]$ to $(0, 1]$. So to prove the theorem for $\bar{x} \in (0, \frac{1}{2}]^N \setminus \{\mathbf{\frac{1}{2}}\}$ we can proceed as we did above, using the second part of Lemma 2.25 at the end.

$\square$

A reasonable aggregation method for even sized juries

We noted that we are not suggesting that Γ_{2n} corresponds to any sensible judgement aggregation method. It makes no sense to define a decision procedure involving a rule "in case of a tie, the correct decision wins." This is not to say, however, that we should ignore the problem of finding a reasonable judgement aggregation method for even sized groups. The question is not foreign to the CJT literature itself. Berend and Sapir, for instance, suggest that, for an even sized jury, a tie should be followed by deciding the issue by means of a coin flip, and indeed Theorem 2.27 for even N entails that such a method is preferred to random ballot in the case of $\bar{x} \in [\frac{1}{2}, 1)^N \setminus \{\mathbf{\frac{1}{2}}\}$ (simply subtract $\frac{1}{2}\beta_N$ from both sides of the inequality).

In this section we introduce a different approach to dealing with ties. In short: In case of a tie, the jury should simply vote again (we assume that they do so with their original competences p_i). While perhaps not applicable to every real life situation, particularly when dealing with groups of stubborn voters, we believe the approach is certainly more reasonable

than that of leaving the decision up to a 50/50 lottery.
With this in mind we make the following definition.

Definition 2.28. *[Ties Result in Re-voting] Let n be a positive integer. We define $\hat{\Gamma}_{2n}$: $[0,1]^{2n} \to \mathbb{R}$ by*

$$\hat{\Gamma}_{2n}(\bar{x}) := \frac{\alpha_{2n}}{1-\beta_{2n}}$$

where

$$\alpha_{2n} := \sum_{k=n+1}^{2n} \sum_{\sigma \in \mathbf{N}_k} \prod_{j\in\sigma} x_j \prod_{j \in \mathbf{N}\setminus\sigma} (1-x_j) = \Gamma_{2n} - \beta_{2n}$$

and

$$\beta_{2n} := \sum_{\sigma \in \mathbf{N}_n} \prod_{j\in\sigma} x_j \prod_{j \in \mathbf{N}\setminus\sigma} (1-x_j).$$

So α_{2n} is the probability that a proper majority will vote correctly, and β_{2n} is the probability of a tie. The value $\hat{\Gamma}_{2n}(\bar{x})$ is both the probability of the success of the jury and the probability of success given the event of a tie. That is, it is the unique solution Z to the equation

$$Z = \alpha_{2n} + Z\beta_{2n}.$$

That is, it is the probability that a the even sized jury will either have a *proper* majority voting for the correct outcome or arrive at a tie and, when revoting, have a proper majority voting for the correct outcome or arrive at another tie and, when revoting, ...(and so on). The following theorem should be obvious, but we give a short proof nonetheless. It says that the method of resolving ties by revoting is preferred to the method of settling ties via the flip of a fair coin for $\bar{x} \in [\frac{1}{2}, 1)^{2n} \setminus \{\mathbf{\frac{1}{2}}\}$.

Theorem 2.29. *Let n be a positive integer and $\bar{x} \in [\frac{1}{2}, 1)^{2n} \setminus \{\mathbf{\frac{1}{2}}\}$. Then*

$$\hat{\Gamma}_{2n}(\bar{x}) > \alpha_{2n}(\bar{x}) + \frac{1}{2}\beta_{2n} > \Delta_{2n}(\bar{x}).$$

Proof. The second inequality is immediate from Theorem 2.27.

For the first inequality, we must show

$$\hat{\Gamma}_{2n} = \frac{\alpha_{2n}}{1-\beta_{2n}} \geq \alpha_{2n} + \frac{1}{2}\beta_{2n},$$

or equivalently

$$\alpha_{2n} \geq (1-\beta_{2n})(\alpha_{2n} + \frac{1}{2}\beta_{2n}) = \alpha_{2n} + \frac{1}{2}\beta_{2n} - \alpha_{2n}\beta_{2n} - \frac{1}{2}\beta_{2n}^2,$$

or equivalently

$$\alpha_{2n} + \frac{1}{2}\beta_{2n} > \frac{1}{2},$$

which is true since $\alpha_{2n} + \frac{1}{2}\beta_{2n} = \Gamma_{2n} - \frac{1}{2}\beta_{2n} > \Delta_{2n}$ by Theorem 2.27. □

Alternately, we could simply observe that $\alpha_{2n}(\bar{x})+\frac{1}{2}\beta_{2n}$ gives the probability of success of the Berend and Sapir's judgement aggregation method, which begins with a vote but which, in the event of a tie, decides the issue by the flip of a fair coin. Since the latter method only differs from the method corresponding to $\hat{\Gamma}$ in how it deals with ties, and since a revote is preferred to a coin flip for $\bar{x} \in [\frac{1}{2}, 1)^{2n} \setminus \{\boldsymbol{\frac{1}{2}}\}$, the theorem is shown.

Chapter 3

Dependent Voter Competence

In most real life group decision scenarios, the individual members are not isolated from one another, and indeed they spend quite a bit of time and resources attempting to influence the position of their fellow group members. An obvious example is, of course, a court jury: After the court evidence has been presented and the closing statements are made, the jurors deliberate, sometimes at great length, about the best interpretation of the evidence presented. Regardless of the direction in which the evidence points (represented by the random variable E in the previous section) it is likely that an outspoken member of the jury will influence the probability that its fellow jury members will come to the correct decision.

Another example is a democratic electoral race, during which citizens spend a great deal of energy discussing which of two or more candidates would better fill the office. Regardless of the direction that the sum total of publicly available information about the relative merits of the two candidates points, a strategic campaign by one voter or group of voters for or against a candidate can greatly influence the likelihood that another voter will cast its vote for that candidate.

The reader has probably already noticed that the model of judgment aggregation given in the previous chapter will not adequately model these types of situations, since it assumes that there is some lump of evidence E such that that the family $\{V_i\}$ of random variables giving the votes of the individual group agents are mutually independent given E. More specifically, recall that if our deliberating group consisted of N members $\mathbf{N} = \{1, \ldots, N\}$ with competencies $p_1, \ldots, p_N$, (with $p_i = P(V_i = 1|E = 1) = P(V_i = 0|E = 0)$), and if $\sigma \subseteq \mathbf{N}$, then (by 2.3) the probability $M_{\sigma,\alpha}$ that σ consists of exactly the members that voted for alternative α (given that the evidence pointed to α) was simply the product

$$M_\sigma = \prod_{i\in\sigma} p_i \prod_{i\in\mathbf{N}\setminus\sigma} (1 - p_i). \tag{3.1}$$

The literature surrounding the CJT has not been ignorant of the fact that a more complex account of dependence between voter competence is necessary if a theorem is going to have any hope of applicability to real life. Ladha [15] examined generalizations of the CJT for which voter competences are not independent but where an estimate on the pairwise correlations ρ between voter competence is known, and Boland, Proschan and Tong [6] consider situations in which each juror's vote is correlated with the vote of a *group leader*. Until recently, however, most approaches toward dependence in the voting literature have emphasized the tendency of members of the jury to *directly* influence the opinion of fellow group members and have ignored the more general problem that most dependencies between random variables representing quantities measured in nature are not a result of a direct causal interaction between these quantities but rather are due to the variables in question being effects of a common cause. In a recent paper, Dietrich and Spiekermann have examined this phenomenon in detail, advocated the use of *causal networks* to model complex dependencies, and attempted to give an analogue of the CJT that takes complex dependence into account. An evaluation of this approach is the topic of the first section of this chapter. In the subsequent section we address dependence resulting from direct causal chains between voters, namely the ability for voters to influence each other during deliberation.

3.1 Conditioning on Common Causes

Dietrich and Spiekermann refer to the independence assumption inherent in the CCJT as *Classical Independence* and give the following definition:

Definition 3.1 (Classical Independence, (p.91 [9]))**.** *If X is the random variable representing the true state of the world with regard to the binary choice (e.g.* guilty *or* innocent*, etc) and the $\{V_i\}$ are the random variables representing the votes of the group members, then the $\{V_i\}$ are independent conditional on X.*

In attempting to draw attention to how the assumption of Classical Independence may fail even when there is no causal chain leading *from* any one of the group members to any other, Dietrich and Spiekermann give the following thought experiment from historical fiction.

> Imagine a government relying on a group of economic advisers. Towards the end of 2007, when the US housing market starts dropping, the government wants to know whether a recession is imminent. It asks all advisers and adopts the majority view. To ensure that the experts do not influence each other, safeguards are in place to prevent any communication between them. If the classical CJT applied, we could conclude that the probability of a correct majority vote converges to 1 as more and more advisers are consulted. But this conclusion is unlikely to be true because Classical Independence is typically violated even though the experts do not communicate. To see this, consider a few examples. First, if all economists rely on the same publicly available evidence, then this evidence will usually cause them to vote in the same way. For instance, if all the

> evidence misleadingly suggests healthy growth (with the evidence indicating, say, that banks have much healthier balance sheets than they actually have) while a bank crash is already around the corner, then most reasonable economists will be wrong in their prediction. The votes are then dependent through consulting the same evidence. More precisely, given for instance that alternative 1 is correct, incorrect votes for 0 by some voters raise the probability of misleading evidence, which in turn raises the probability that other voters also vote incorrectly, a violation of independence. Second, if all economists rely on the same theoretical assumptions for the interpretation of the evidence (such as low correlations between market prices of certain credit default swaps), this common influence is likely to induce dependence between the votes. In the extreme, either all get it right or all get it wrong. Finally, if the experts are more likely to make wrong predictions in weather that gives headaches, then weather creates dependence between votes. Again, in the extreme either all get it right or all get it wrong (and have headaches).[p 94 [9]]

We summarize two of Dietrich and Spiekermann's main points as follows.

1. Variables that have effects on more than one of the random variables $\{V_i\}$ can induce dependence between the latter even if there are no directed causal chains between the voters.

2. In any real world situation, the network of common causes are likely to be much more complex than a network where all of the variables $\{V_i\}$ have a single ancestor, a parent X representing the true state of the world. Indeed any DAG representing a real world situation will have the $\{V_i\}$ separated from the variable X by many intermediary variables.

We concur with both of these observations. With regard to the first point, we have already noted in our discussion of Bayesian Networks in the first chapter how in a diverging node $A \leftarrow C \rightarrow B$ dependence can propagate from A to B through C if we are not conditioning on C. With respect to the second observation above, we would agree and indeed add that in most group decision making scenarios the true state of the world X is unlikely to be a parent of *any* of the $\{V_i\}$ (although it is likely to be a distant ancestor). At any rate, since there will likely be many common parents of the variables $\{V_i\}$ other than X, we will not be able to prevent dependence from propagating between the $\{V_i\}$ simply by conditioning on the distant ancestor X.

These are facts which, to a limited extent, we anticipated in Chapter 2, when we introduced CJT Networks. There we wanted to situate the CCJT inside some simple Bayesian Network including the true state of the world X with descendants $\{V_i\}$, but in the spirit of the CCJT wanted to consider the simplest possible situation in which the individual votes were independent. For this reason we were careful *not* to define the competencies p_i of the jurors as the probabilities $P(V_i = X)$ but as the probabilities $p_i = P(V_i = 1|E = 1) = P(V_i = 0|E = 0)$. Recall that in a CJT Network the range $\{0, 1\}$ of each of the variables was the set of alternatives themselves, not the *correctness/incorrectness* of the choice, but the principal remains the same. Since the variable E was the only parent of our variables $\{V_i\}$, this approach allowed us to admit the existence of the most obvious common cause between the variables $\{V_i\}$ in any voting situation - even in those where the group members are isolated from one another. Since we were conditioning on the *known* parent in letting $p_i = P(V_i = 1|E = 1) = P(V_i = 0|E = 0)$, the variables V_i were *not* d-connected (see Definition 1.10) with respect to $\{E\}$, independence of the variables V_i with respect to E was established, and we were allowed to use (3.1) above to compute the probability $P(V = X|E = X)$.

While we maintain that this basic dependence structure (X a parent of E which is in turn a parent of each of the V_i) is the best way to situate the CCJT inside the framework of Bayesian Networks, it was never meant to be less coarse of an abstraction than the CCJT itself, and there are a couple of reasons why it will never be applicable to any but the most ideal group decision making scenarios.

In the CJT Network

1. The only common cause affecting the variables $\{V_i\}$ was the unique parent E, and

2. The nature of this variable E suggested that the best reading of the CCJT was to assume that all probabilities of interest, the parameters p_i and the aggregate probability of the jury voting correctly, were actually to be thought of as being conditional on E taking a specific value.

But, as Dietrich and Spiekermann point out, there are likely to be a very complex system of variables $E_1, \ldots, E_m$ that are common causes of many different subsets of the family $\{V_i\}$. And the escape used in the CJT Network will not work in general. There will not be any "natural values" $e_1, \ldots e_m$ that we can assume the variables $\{E_i\}$ to take, that is, some natural values $e_1, \ldots e_m$ such that we can claim that the individual competences p_i are "best interpreted as" $P(V = \alpha|E_1 = e_1, \ldots, E_N = e_N)$ or that group competence Γ is "really" $P(V = X|E_1 = e_1, \ldots, E_N = e_N)$. Dietrich and Spiekermann's solution is simple: If we cannot condition on a particular sequence of values for all common causes of variables $\{V_i\}$, we can condition on *all* possible values.

Let us consider a simple example.

Let N be a positive integer, let $\mathbf{N} = \{1, \ldots, N\}$, and consider the simplified case where we have a Bayesian Network Θ with binary random variables $\{V_i\}_{i\in\mathbf{N}}$ such that there is no directed path between any two of the V_i and the $\{V_i\}_{i\in\mathbf{N}}$ have exactly one parent, a random variable F taking values in the finite set D_F. Suppose there are no other variables in Θ. (See figure 3.1.)

Thus the V_i are independent conditional on F. But suppose this time that there is no particular value in D_F that we can assume F to take. Suppose we want to compute the

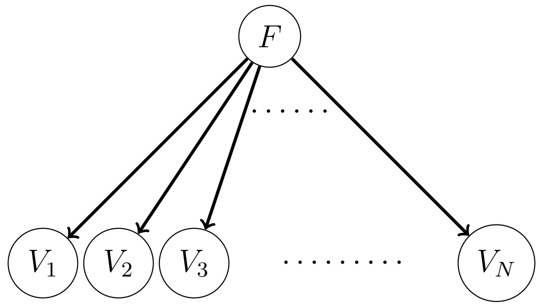


Figure 3.1: Conditioning on a single parent F taking many values.

probability M^*_σ of the event N^*_σ that a certain subset $\sigma \subseteq \mathbf{N}$ is exactly the set of indices j for which $V_j = 1$ (and all others take value 0):

If we know the marginal probability distribution on the variable F, we can compute this value as follows

$$M^*_\sigma = \sum_{v \in D_F} P(F = v) P(N^*_\sigma | F = v).$$

But since F is the unique parent of the variables $\{V_i\}$, the latter are independent conditional on $F = v$ for any $v \in D_F$, and the multiplicative property of equation 3.1 is restored

$$M^*_\sigma = \sum_{v \in D_F} P(F = v) P\left(\left(\bigwedge_{j \in \sigma} V_j = 1\right) \wedge \left(\bigwedge_{j \in \mathbf{N} \setminus \sigma} V_j = 0\right) | F = v\right)$$
$$\sum_{v \in D_F} P(F = v) \prod_{j \in \sigma} P(V_j = 1 | F = v) \prod_{j \in \mathbf{N} \setminus \sigma} P(V_j = 0 | F = v).$$

It is easy to see that we can extend this simplified example to the general case where there are any finite number $F_1 \ldots F_m$ of parents of the variables $\{V_i\}$ since if the variable F_i takes values in the set D_i, we can amalgamate the F_i into one common parent F taking values in the cartesian product of the D_i. Dietrich and Spiekermann refer to the resulting random variable simply as *the problem* and denote it by Π. Thus they advocate the following replacement for *Classical Independence* in 3.1.

Definition 3.2 (New Independence, p.91 [9])). *The random variables $\{V_i\}$ are independent conditional on the problem Π.*

The Framework

Let us once again define a Bayesian Network appropriate to the issue at hand.

Definition 3.3 (Generalized CJT Network). *Let M be a positive integer, $n \in \mathbb{N}$, and $N = 2n+1$. Let D_Π be some finite set with $|D_\Pi| = M$. Let $Y : \mathscr{P}(D_\Pi) \to [0,1]$ be a function, and let $\Phi = \{p_{i,\pi}\}_{1\leq i\leq N, \pi\in D_\Pi} \in [0,1]^{NM}$. Let $S = (\Omega, \Sigma, P)$ be a probability space and $\Upsilon = (X, \Pi, \{V_i\}_{1\leq i\leq N}, V)$ be random variables on Ω.*

We say that a directed acyclic graph Θ is a Generalized CJT Network (of order (N, M) with competence (Y, Φ) and with respect to variables Υ) *if*

1. *Θ is a Bayesian Network with respect to $\{X, \Pi, V\} \cup \{V_i\}_{1\leq i\leq N}$.*
2. *The variable Π takes values in the set D_Π.*
3. *The function Y is the push-forward distribution on Π. That is, for $\pi \in D_\Pi$,*
$$Y(\pi) = P(\Pi = \pi).$$
4. *The edges in Θ are given by Figure 3.2. Specifically, the $\{V_i\}$ are all parents of V, Π is a parent of each of the $\{V_i\}$, X is a parent of Π, and there are no other edges,*
5. *V is the indicator function for the event*
$$\sum_{i=1}^{N} V_i > \frac{N}{2}. \tag{3.2}$$
6. *For all $1 \leq i \leq N$ and $\pi \in D_\Pi$,*
$$p_{i,\pi} = P(V_i = 1|\Pi = \pi).$$

Note the difference from the original CJT Network in the interpretation of the variables $\{V_i\}_{i\in\mathbf{N}}$ and and V. In the original CJT Network the variables V_i represented the alternative for which the ith juror voted and V represented the alternative chosen by the jury as a whole. In the Generalized CJT Network the variables V_i represent the success/failure of the ith juror at arriving at the *correct* alternative X, and V represents the success/failure of the jury in arriving at X. It follows that in a Generalized CJT Network the competences $p_{i,\pi}$ must be interpreted differently as well, above and beyond their being conditional on the variable Π instead of E.

Definition 3.4. *Let M be a positive integer, $n \in \mathbb{N}$, and $N = 2n+1$. Let Θ be a CJT Network of order (N, M), with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{1\leq i\leq N, \pi\in D_\Pi} \in [0,1]^{NM}$, and with respect to variables $\Upsilon = (X, \Pi, \{V_i\}_{1\leq i\leq N}, V)$. Again, let $\mathbf{N} = \{1, \ldots, N\}$. We make the following definitions*

1. $$\Gamma^* := P(V = 1) = P\left(\sum_{i=1}^{N} V_i > \frac{N}{2}\right)$$
is the probability that the group votes correctly.

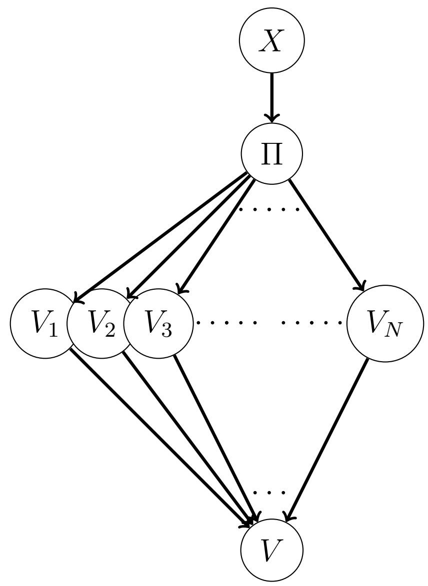


Figure 3.2: A Generalized CJT Network of order N

2. *Given $\sigma \subseteq \mathbf{N}$, define*

$$N^*_\sigma := \left(\bigwedge_{j\in\sigma} V_j = 1\right) \wedge \left(\bigwedge_{j\in\mathbf{N}\setminus\sigma} V_j = 0\right)$$

to be the event that the set σ will consist of exactly the jurors who vote correctly.

3. *Given $\sigma \subseteq \mathbf{N}$, we define*

$$M^*_\sigma := P(N^*_\sigma).$$

Then by the argument in the above example for the network with variables $\{V_i\}$ with parent F, we have the following lemma.

Lemma 3.5. *Let M be a positive integer, $n \in \mathbb{N}$, and $N = 2n+1$. Let P is the probability measure for a Generalized CJT Network Θ of order (N, M) with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{i\in J,\pi\in D_\Pi} \in [0,1]^{NM}$.*

For $\sigma \subseteq \mathbf{N}$,

$$M^*_\sigma = \sum_{\pi\in D_\Pi} Y(\pi) \prod_{j\in\sigma} p_{j,\pi} \prod_{j\in\mathbf{N}\setminus\sigma} (1 - p_{j,\pi}).$$

We can use this lemma to compute Γ^*.

Theorem 3.6. *Let M be a positive integer, $n \in \mathbb{N}$, and $N = 2n+1$. Let Γ_N be as in Definition 2.8.*

If P is the probability measure for a Generalized CJT Network Θ of order (N, M), with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{i\in\mathbf{N},\pi\in D_\Pi} \in [0,1]^{NM}$, then

$$\Gamma^* = \sum_{\pi\in D_\Pi} Y(\pi)\Gamma_N(p_{1,\pi}, p_{2,\pi}, \ldots, p_{N,\pi}).$$

Proof.

$$\Gamma^* = \sum_{k=n+1}^{N} \sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}} p_\sigma =$$

$$\sum_{k=n+1}^{N} \sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}} \sum_{\pi\in D_\Pi} P(\Pi=\pi) \prod_{j\in\sigma} p_{i,\pi} \prod_{j\in\mathbf{N}\setminus\sigma} (1-p_{i,\pi}) =$$

$$\sum_{\pi\in D_\Pi} P(\Pi=\pi) \sum_{k=n+1}^{N} \sum_{\substack{\sigma\subseteq\mathbf{N}\\|\sigma|=k}} \prod_{j\in\sigma} p_{i,\pi} \prod_{j\in\mathbf{N}\setminus\sigma} (1-p_{i,\pi}) =$$

$$\sum_{\pi\in D_\Pi} Y(\pi)\Gamma_N(p_{1,\pi}, p_{2,\pi}, \ldots, p_{N,\pi}).$$

□

Again we can augment this Generalized CJT Network to account for the method of random ballot by adding random variable R and D, where R is a uniform random variable with range $\mathbf{N}$, and D is the event that the juror pointed to by R votes correctly. See Chapter 2 for details.

Definition 3.7. *If Θ^* is an augmented Generalized CJT Network of order (N, M) and with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{1\le i\le N,\pi\in D_\Pi}$ then define*

$$\Delta^* := P(D=1) = P\left(\bigvee_{i=1}^{N} (R = j_i \wedge V_i = 1)\right)$$

The following result is immediate.

Lemma 3.8. *If Θ^* is an augmented Generalized CJT Network of order (N, M) and with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{1\le i\le N,\pi\in D_\Pi}$ then*

$$\Delta^* = \frac{1}{N} \sum_{\pi\in D_\Pi} Y(\pi) \sum_{i=1}^{N} p_{i,\pi}.$$

As in Chapter 2, it will be convenient to consider Γ^* and Δ^* as functions of N and (Φ, Y) independent of any Generalized CJT Network Θ.

Definition 3.9. *Let N, M be positive integers. Let $\mathbf{N} = \{1, 2, \ldots, N\}$ and $\mathbf{M} = \{1, 2, \ldots, M\}$. Also let $Y : \mathbf{M} \to [0, 1]$ be a probability distribution on $\mathbf{M}$.*

1. *Define $\Gamma^*_{N,M,Y} : [0, 1]^{NM} \to \mathbb{R}$ by*

$$\Gamma^*_{N,M,Y}(\bar{x}) := \sum_{k\in\mathbf{M}} Y(k)\Gamma_N(x_{1,k}, x_{2,k}, \ldots, x_{N,k})$$

2. *Define $\Delta^*_{N,M,Y} : [0, 1]^{NM} \to \mathbb{R}$.*

$$\Delta^*_{N,M,Y}(\bar{x}) := \frac{1}{N} \sum_{k\in\mathbf{M}} Y(k) \sum_{i=1}^{N} x_{i,k}$$

Generalized Homogeneity

We give the following correlate of homogeneous voter competence in the case where we are conditioning on multiple states of the world.

Definition 3.10 (Generalized Homogeneity of voter competence)**.** *We say that a Generalized CJT Network Θ of order (N, M) and with competence (Y, Φ) with $\Phi = \{p_{i,\pi}\}_{1\le i\le N, \pi\in D_\Pi}$ has* generally homogeneous competence *or* competence homogeneous with respect to the problem Π *if, for all $\pi \in D_\Pi$, there is some p_π such that $p_{i,\pi} = p_\pi$ for all $i \in \mathbf{N}$.*
When competence is homogeneous with respect to the problem Π we will write $p_{i,\pi}$ as p_π and $\Phi = \{p_\pi\}_{\pi\in D_\Pi}$

Note: When competence is homogeneous with respect to the problem Π, it will be useful to think of the distribution Y as a probability distribution on a *competence random variable* $p \in [0, 1]$. This will come in handy since in many cases what is of interest is not the state of the problem Π, but the competence of the jury in that state.
Thus for $S \subseteq [0, 1]$ a Borel set, we will sometimes use the common shorthand

$$Y(S) := Y(\{\pi \in D_\Pi : p_\pi \in S\}) = P(\Pi \in \{\pi \in D_\Pi : p_\pi \in Sp\}),$$

and, when $S = \{p\}$ is a singleton, we will suppress the braces:

$$Y(p) := Y(\{\pi \in D_\Pi : p_\pi = p\}) = P(\Pi \in \{\pi \in D_\Pi : p_\pi = p\}).$$

Once again, when general homogeneity is satisfied, the values M^*_σ, Γ^*, and Δ^*, simplify quite a bit.

Lemma 3.11. *Let Θ be a CJT Network (possibly augmented) of order $N = 2n + 1$ with competence (Y, Φ), and with respect to variables Υ. Assume homogeneity with respect to the problem Π.*

1. *For all* $\sigma \subseteq \mathbf{N}$
$$M^*_\sigma = \sum_{\pi \in D_\Pi} Y(\pi)(p_\pi)^{|\sigma|}(1-p_\pi)^{N-|\sigma|}.$$

2.
$$\Gamma^* = \sum_{\pi \in D_\Pi} Y(\pi) \sum_{k=n+1}^{N} \binom{N}{k} (p_\pi)^k (1-p_\pi)^{N-k} = \sum_{\pi \in D_\Pi} Y(\pi)\Gamma_N(p_\pi).$$
And indeed we can sum over all possible values p_π *instead of the values* $\pi \in D_\pi$ *to get*
$$\Gamma^* = \sum_{p \in [0,1]} Y(p) \sum_{k=n+1}^{N} \binom{N}{k} (p)^k (1-p)^{N-k} = \sum_{p \in [0,1]} Y(p)\Gamma_N(p)$$

3.
$$\Delta^* = \sum_{\pi \in D_\Pi} Y(\pi)p_\pi.$$
or, summing over all possible values p_π,
$$\Delta^* = \sum_{p \in [0,1]} Y(p)p.$$

Once again it will be nice to consider Γ^* and Δ^* as functions in their own right.

Definition 3.12. *Given a probability distribution* P *on the interval* $[0,1]$, *define supp*(P), *the support of* P, *by*
$$supp(P) = \{r \in [0,1] : P(r) > 0\}.$$
Define $Dis[0,1]$ *to be all probability distributions on* $[0,1]$ *such that supp*(P) *is a finite set.*

Definition 3.13. *Let* N *and* M *be positive integers, and let* $\mathbf{M} = \{1, 2, \ldots, M\}$. *Also let* $P : \mathbf{M} \to [0,1]$ *be a probability distribution on* $\mathbf{M}$.

1. *Define* $\Gamma^*_{N,M,P} : [0,1]^M \to \mathbb{R}$ *by*
$$\Gamma^*_{N,M,P}(\bar{x}) := \sum_{k \in \mathbf{M}} P(k)\Gamma_N(x_k)$$

2. *Define* $\Delta^*_{N,M,P} : [0,1]^M \to \mathbb{R}$ *by*
$$\Delta^*_{N,M,P}(\bar{x}) := \sum_{k \in \mathbf{M}} P(k)x_k$$

Alternately we can define these as functions of a distribution $P \in Dis[0,1]$.

1. *Define* $\Gamma^*_N : Dis[0,1] \to \mathbb{R}$ *by*

$$\Gamma^*_N(P) := \sum_{p\in[0,1]} P(p)\Gamma_N(p)$$

2. *Define* $\Delta^*_N : Dis[0,1] \to \mathbb{R}$ *by*

$$\Delta^*(P) := \sum_{p\in[0,1]} P(p)p$$

Tendency to Exceed $\frac{1}{2}$

We turn towards giving Dietrich and Spiekermann's "new and improved" competence requirement in terms of Generalized CJT Networks.

Definition 3.14 (Tendency to Exceed $\frac{1}{2}$ and New Competence (p. 101 [9]))**.** *Let* Θ *be a Generalized CJT Network* Θ *of order* (N, M) *with* generally homogeneous *competence* (Y, Φ) *with* $\Phi = \{p_\pi\}_{\pi\in D_\Pi}$.

We say that competence Tends to Exceed $\frac{1}{2}$ *if*

$$\text{for all } \epsilon \in [0, \frac{1}{2}], \quad Y(\frac{1}{2}+\epsilon) \geq Y(\frac{1}{2}-\epsilon). \tag{3.3}$$

In this case we say that New Competence *holds.*

Note that, given that competence is generally homogeneous, Definition 3.14 only depends on the distribution Y, and so it makes sense to speak of a distribution $Y \in Dis[0,1]$ as satisfying New Competence.[1] Thus we can state Dietrich and Spiekermann's theorem as follows:

Theorem 3.15 (Dietrich and Spiekermann)**.** *Let* $N = 2n+1$*, and let* $Y \in Dis[0,1]$ *satisfy New Competence.*
Then

1. $\{\Gamma^*_{2n+1}(Y)\}_n$ *is increasing in* n.

2. $L := \lim_{n\to\infty} \Gamma^*_{2n+1}(Y)$ *exists.*

3. $L = Y\left((\frac{1}{2}, 1]\right) + \frac{1}{2}Y\left(\frac{1}{2}\right)$.

4. $L < 1$ *if* $Y((\frac{1}{2}, 1]) \neq 1$.

5. $L = 1$ *if* $Y((\frac{1}{2}, 1]) = 1$.

Proof. See pp. 116-120 of [9]. □

[1] Alternately, we could speak of a sequence of Generalized CJT Networks of orders $\{(2n+1, M)\}_{1\leq n<\infty}$ and homogeneous competence (Y, Φ) as satisfying New Competence, but we do not take this route.

An Objection

While it is unfortunate that the above theorem only applies to groups with homogeneous competence with respect to the problem Π, it is understandable that this would be the first step in a general project aiming to develop an analogue to the CCJT for a general Bayesian Network of dependence between voter competence.

There is another issue however, that is more damning.

Following Dietrich and Spiekermann's definition of *Tendency to Exceed* $\frac{1}{2}$, those authors state

> To paraphrase this condition once more, the problem is more likely to be of the sort on which voters are competent than of the sort on which they are incompetent (where voters are homogeneous in competence, as in the classical setup). [p. 101, [9]]

If this were in fact an accurate paraphrase of the requirement for Theorem 3.15 we would view that theorem as a successful generalization of the CCJT to Bayesian Networks with competence homogeneous with respect to the problem Π. Unfortunately, this is *not* an accurate paraphrase of Tendency to Exceed $\frac{1}{2}$ used in New Competence. Indeed, the "paraphrase" just given more closely corresponds to the requirement that

$$Y((\frac{1}{2}, 1]) \geq Y([0, \frac{1}{2})) \tag{3.4}$$

which is strictly weaker than the condition in Definition 3.14. For instance consider a distribution P on competence such that $P(1) = .99$ and $P(.49) = .01$. This easily satisfies (3.4) but fails (3.3).

Indeed, the fact that Theorem 3.15 fails to apply under distributions like the one just given seems to show the theorem to be of little value. In the distribution just given, the voters are very capable: Under typical circumstances they are perfect in their judgement, and the possibility of error only appears under highly unlikely circumstances, and even then they are just slightly more likely to settle upon the incorrect conclusion than the correct one. What's worse, the there is a natural kind of monotonicity which the condition *Tendency to Exceed* $\frac{1}{2}$ fails to satisfy. Consider a probability distribution similar to the one just given. Again $P(1) = .99$, but now $P(0) = .1$. This distribution does satisfy (3.3), even though it was produced from a distribution that failed to satisfy this condition by moving exactly one of the possible competencies *down* quite a bit (but maintaining its probability).

So if New Competence is not a reasonable condition to require of a group for any attempt at a CJT for general Bayesian Networks, what is?

Two approaches come readily to mind. The first is to require that the "paraphrase" above be taken literally. That is, we require the condition in inequality 3.4 above. Another approach is to require that the expected (homogeneous) competence of the group members be greater than $\frac{1}{2}$.

Note that neither of these conditions implies the other. As may be seen by examining the two distributions

$$P(.51) = .51;\; P(.01) = .49:$$

$$\text{Expected Member Competence} = .51(.51) + .01(.49) = .265 < \frac{1}{2}.$$

$$P(.99) = .49;\; P(.49) = .51:$$

$$\text{Expected Member Competence} = .49(.51) + .99(.49) = .735 > \frac{1}{2}.$$

We devote the remainder of this section to showing how both of these approaches fail to ensure the non-decreasing monotonicity of group competence Γ_N^* as a function in the size N of the group.

The Difference Function

Recall the difference function D_N from Definition 2.14. For any distribution $P \in Dis[0,1]$, we can define the increase $D_{N,P}^*$ in group competence as a result of adding two members to a size-N jury of homogeneous competence.

Definition 3.16. *Let N be an odd positive integer and $P \in Dis[0,1]$. Define*

$$D_{N,P}^* := \Gamma_{N+2}^*(P) - \Gamma_N^*(P).$$

We can express this value in terms of our original difference function from Definition 2.14.

Lemma 3.17. *Let N be an odd positive integer and $P \in Dis[0,1]$ be a probability distribution with finite support. Then*

$$D_{N,P}^* = \sum_{p\in[0,1]} P(p)(2p-1)\binom{N}{\frac{N+1}{2}}[p(1-p)]^{\frac{N+1}{2}}.$$

Proof. By Lemma 2.15

$$D_{N,P}^* = \Gamma_N^*(P) - \Gamma_{N+2}^*(P) = \sum_{p\in[0,1]} P(p)[\Gamma_{N+2}(p) - \Gamma_N(p)]$$

$$= \sum_{p\in[0,1]} P(p)D_N(p) = \sum_{p\in[0,1]} P(p)\binom{N}{\frac{N+1}{2}}(2p-1)[p(1-p)]^{\frac{N+1}{2}}.$$

$\square$

Notice that the binomial coefficient does not depend on the index of summation, allowing us to define the following *normalized difference*, which we can extend to an exponential polynomial function on $\mathbb{R}$.

Definition 3.18 (Normalized Difference Function)**.** *Let N be an odd positive integer, $x \in \mathbb{R}$, and $P \in Dis[0,1]$.*

1. *Define*
$$\overline{D}_{N,P} = \frac{D^*_{N,P}}{\binom{N}{\frac{N+1}{2}}} = \sum_{p \in [0,1]} P(p)(2p-1)[p(1-p)]^{\frac{N+1}{2}}.$$

2. *Define*
$$\overline{D}_P(x) = \sum_{p \in [0,1]} P(p)(2p-1)[p(1-p)]^{\frac{x+1}{2}}.$$

We observe that

Proposition 3.19. *$\overline{D}_{N,P}$ is positive, negative, or zero if and only if $D^*_{N,P}$ is positive, negative, or zero, respectively.*

We are interested in conditions on $P \in Dis[0,1]$ such that $\bar{D}_{N,P}$ is monotonic non-decreasing in N.

A special case

Suppose that Π admits exactly two possible competencies p, q, so that $P(p) = \alpha \neq 0$ and $P(q) = 1 - \alpha \neq 0$. Let us also assume that $\frac{1}{2} < p < 1$ and $0 < q < \frac{1}{2}$. Let $y = \frac{x+1}{2}$ so that $\bar{D}_P$ reduces to

$$\nabla(y) = \bar{D}_P(x) = \alpha(2p-1)[p(1-p)]^y + (1-\alpha)(2q-1)[q(1-q)]^y.$$

Let us consider the zeros of this function. Setting $\nabla(y) = 0$, taking logarithms, and rearranging gives

$$y \log \frac{p(1-p)}{q(1-q)} = \log \frac{(\alpha-1)(2q-1)}{\alpha(2p-1)}.$$

If p and q are symmetric about $\frac{1}{2}$, then ∇ has a zero just in case every y is a zero, which happens just in case $\alpha = \frac{1}{2}$.

On the other hand if p and q are not symmetric about $\frac{1}{2}$. Then the logarithm on the left is nonzero, and dividing both sides by this logarithm gives the unique zero

$$y = \frac{\log \frac{(\alpha-1)(2q-1)}{\alpha(2p-1)}}{\log \frac{p(1-p)}{q(1-q)}}.$$

It is straightforward to check that this root is non-singular, since the system $\nabla(y) = \nabla'(y) = 0$ entails that p and q are symmetric about $\frac{1}{2}$.

The Condition $P((\frac{1}{2}, 1]) \geq P([0, \frac{1}{2}))$

We briefly mention an example showing that the condition 3.4 does not guarantee monotonicity. The following distribution P satisfies 3.4 but makes $\bar{D}_P(3)$ negative:

$$P(.6) = P((\frac{1}{2}, 1]) = .6;\ P(.25) = P([0, \frac{1}{2})) = .4 :$$

$$\bar{D}_P(3) = .6(.2)[.6(.4)]^2 + .4(-.5)[.25(.75)]^2 < 0.$$

Expected competence greater than $\frac{1}{2}$

We define the expected competence as follows

Definition 3.20 (Expected Competence)**.** *Let Θ be a Generalized CJT Network Θ of order N with* generally homogeneous *competence* (Y, Φ)*. We define the* expected competence *as*

$$p_E(Y) := \sum_{p \in [0,1]} pY(p)$$

Notice that the value $p_E(Y)$ only depends on a distribution $Y \in Dis[0, 1]$, and so we may view p_E as a real valued function on the set $Dis[0, 1]$. The following proposition is immediate.

Proposition 3.21. *For $P \in Dis[0, 1]$,*

$$p_E(P) = \Delta^*(P).$$

A candidate constraint that is probably the most analogous with the restriction on competence in the CCJT would be to require that $p_E > \frac{1}{2}$. How much of the conclusion of Dietrich and Spiekermann's theorem can we recover with this definition of competence? It turns out that, not only can we construct distributions P with $p_E(P) > \frac{1}{2}$ where the competence fails to be *increasing* in the size of the jury, we can even find examples where picking a randomly selected juror is more reliable than simple majority rule within a jury of odd size greater than one. That is, we can find $P \in Dis[0, 1]$ such that

$$\frac{1}{2} < p_E = \Delta^*(P) > \Gamma^*_N(P)$$

This fact is already mentioned by Dietrich and Spiekermann but not elaborated on. The following theorem shows how badly the condition on the expectation fails to be a sufficient condition for a CJT.

Theorem 3.22. *Let N be any positive integer and $y \in \mathbb{R}$ with $y \geq 1$. Let $S \subseteq [0, 1]^{N+1} \times (0, 1)^N$ be defined by*

$$S = \Big\{(p_0, \ldots, p_N, Q_0, \ldots, Q_{N-1}) \in [0,1]^{N+1} \times (0,1)^N : \\ \sum_{i=0}^{N-1} Q_i < 1,\ \forall (0 \leq i < j \leq N)(p_i > p_j),\ p_0 > \frac{1}{2},\ p_N < \frac{1}{2}\Big\}$$

And let $Q_N := 1 - \sum_{i=0}^{N-1} Q_i$.
Define $\hat{D} : S \to \mathbb{R}$ *and* $E : S \to \mathbb{R}$ *by*

$$\hat{D} : (p_0, \ldots, p_N, Q_0, \ldots, Q_{N-1}) \mapsto \sum_{i=0}^{N} Q_i(2p_i - 1)(p_i(1-p_i))^y$$

$$E : (p_0, \ldots, p_N, Q_0, \ldots, Q_{N-1}) \mapsto \sum_{i=0}^{N} Q_i p_i.$$

Then there is a non-empty open subset $U \subseteq S$ *on which* $\hat{D}$ *is negative and* $E > \frac{1}{2}$.

Proof. We fix an arbitrary real $y \geq 1$, and prove the result by induction on N.

First let $N = 1$. Then

$$S = \left\{(p_0, p_1, Q_0) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}) \times (0,1)\right\}$$

$$\hat{D} : (p_0, p_1, Q_0) \mapsto Q_0(2p_0 - 1)(p_0(1-p_0))^y + (1-Q_0)(2p_1 - 1)(p_1(1-p_1))^y$$
$$E : (p_0, p_1, Q_0) \mapsto Q_0 p_0 + (1-Q_0)p_1.$$

Let F denote the $p_0 = 1$ face of S. Notice that when restricted to F the function $\hat{D}$ reduces to

$$\hat{D}(1, p_1, Q_0) = (1-Q_0)(2p_1 - 1)(p_1(1-p_1))^y,$$

which is negative on S since $p_1 < \frac{1}{2}$. Furthermore when restricted to F the function E reduces to

$$E(1, p_1, Q_0) = Q_0 + (1-Q_0)p_1,$$

which, for any $p_1 \in [0, \frac{1}{2})$ gives a path traveling from p_1 to 1 as the parameter Q_0 travels from 0 to 1. Thus for any $p_1 \in [0, \frac{1}{2})$ there is a Q_{p_1} such that the function $E(1, p_1, \cdot)$ maps the open interval $(Q_{p_1}, 1)$ above $\frac{1}{2}$. This Q_{p_1} is equal to $\frac{\frac{1}{2} - p_1}{1 - p_1}$, the solution to the equation $E(1, p_1, X) = \frac{1}{2}$, and this value changes continuously with the argument p_1.

So let S' be the subset of F that lies on the Q_0-increasing side of the curve $\langle p_1(t), Q_0(t)\rangle = \langle t, \frac{\frac{1}{2} - t}{1 - t}\rangle$ for t in the *open* interval $(0, \frac{1}{2})$. Then S' is a (two dimensional) open (in the subspace topology) subset of F; furthermore $\hat{D}$ is negative on all of S' but E is greater than $\frac{1}{2}$ on all of S'.

Now since $\hat{D}$ is continuous in C, we see that there is a (three dimensional) open subset of S on which $\hat{D}$ is negative and E is greater than $\frac{1}{2}$. This completes the $N = 1$ case.

Now let N be a positive integer greater than one, and suppose the result is true for integers less than N.

Let S' be the denote the following subset of the $p_{N-1} = p_N$ face of the closure of S

$$S' = \Big\{(p_0, \ldots, p_{N-2}, p_N, p_N, Q_0, \ldots, Q_{N-1}) \in [0,1]^{N+1} \times (0,1)^N : \\ \sum_{i=0}^{N-1} Q_i < 1,\ \forall (0 \le i < j \le N)(p_i > p_j),\ p_0 > \frac{1}{2},\ p_N < \frac{1}{2}\Big\}$$

Now we can define a function $\hat{D}'$ on the set S' using the formula for $\hat{D}$

$$\hat{D}'(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-1}) = \\ \left[\sum_{i=0}^{N-2} Q_i(2p_i - 1)(p_i(1-p_i))^y\right] + (Q_{N-1} + Q_N)(2p_N - 1)(p_N(1-p_N))^y$$

And we can define a function E' on the set S' using the formula for E.

$$E'(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-1}) = \left[\sum_{i=0}^{N-2} Q_i p_i\right] + (Q_{N-1} + Q_N)p_N$$

Now $Q_N := 1 - \sum_{i=0}^{N-1} Q_i$ so $Q_{N-1} + Q_N = 1 - \sum_{i=0}^{N-2} Q_i =: \hat{Q}$. So we can write

$$\hat{D}'(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-1}) = \\ \left[\sum_{i=0}^{N-2} Q_i(2p_i - 1)(p_i(1-p_i))^y\right] + \hat{Q}(2p_N - 1)(p_N(1-p_N))^y$$

$$E'(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-1}) = \left[\sum_{i=0}^{N-2} Q_i p_i\right] + \hat{Q}p_N$$

Notice that the values of $\hat{D}'$ and E' are independent of the argument Q_{N-1}. Let's temporarily focus on the following $Q_{N-1} = 0$ subset S'' of the closure of S':

$$S'' = \Big\{(p_0, \ldots, p_{N-2}, p_N, p_N, Q_0, \ldots, Q_{N-2}, 0) \in [0,1]^{N+1} \times (0,1)^{N-1} \times \{0\} : \\ \sum_{i=0}^{N-2} Q_i < 1,\ \forall (0 \le i < j \le N)(p_i > p_j),\ p_0 > \frac{1}{2},\ p_N < \frac{1}{2}\Big\}$$

Now we can define a function $\hat{D}''$ on the set S'' using the formula for $\hat{D}'$

$$\hat{D}''(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-2}, 0) = \left[\sum_{i=0}^{N-2} Q_i(2p_i - 1)(p_i(1-p_i))^y\right] + \hat{Q}(2p_N - 1)(p_N(1-p_N))^y$$

And we can define a function E'' on the set S'' using the formula for E'.

$$E''(p_0, \ldots, p_N, p_N, Q_0, \ldots, Q_{N-2}, 0) = \left[\sum_{i=0}^{N-2} Q_i p_i\right] + \hat{Q}p_N$$

After routine relabeling of variables we see that $\hat{D}''$, E'', and S'' are (isomorphic to) the $\hat{D}$, E, and S in the $N-1$ induction hypothesis. Thus, we have an open subset of U'' of S'' on which $\hat{D}''$ is negative and $E'' > \frac{1}{2}$. By continuity we have an open subset of U' of S' on which $\hat{D}'$ is negative and $E' > \frac{1}{2}$. Finally by continuity we have an open subset of U of S on which $\hat{D}$ is negative and $E > \frac{1}{2}$. This completes the case for N and completes the proof. □

The following "existence theorem" is an understatement to say the least.

Corollary 3.23. *For and M, N positive integers, with $M \geq 2$.*
There exists $P \in Dis[0,1]$ and $\alpha, \beta \in support(P)$ such that all of the following hold:

1. $$|support(P)| = M$$

2. $$P(\alpha) < \frac{1}{2} < P(\beta),$$

3. $$p_E(P) = \Delta^*(P) > \frac{1}{2}$$

4. $$\Gamma^*_N(P) > \Gamma^*_{N+2}(P)$$

Proof. Immediate from Theorem 3.22. □

3.2 Linear Update Functions

In this section we turn to dependence induced by direct causal chains between voters. In particular we hope to model the deliberation process of groups like court juries.

Let $J = \{1, \ldots, n\}$ be our deliberating group. We can model the deliberation process by viewing our set $\{p_i\}_{1 \leq i \leq n}$ of voter competencies as *initial* competences $\bar{p}_0 = \{p_{i,0}\}_{1 \leq i \leq n}$ to be periodically updated by a set $\{U_i\}_{i\in J}$ of *update functions* with $U_i : [0,1]^n \to [0,1]$. We obtain a sequence $\{\bar{p}_j\}_{j\in\mathbb{N}}$ of *competence vectors* by letting $\bar{p}_j = \{p_{i,j}\}_{1\le i\le n}$ with the latter defined recursively by $p_{i,j+1} = U_i(\bar{p}_j)$. We interpret the value $U_i(\bar{p}_j)$ as being the posterior competence of the ith member given the prior vector $\bar{p}_j$ of publicly available competences of the jury members at a given time j.

Note that in this situation it may be at least as appropriate to refer to the values $p_{i,j}$ as *degrees of belief* or *confidences* instead of *competences* if we are also willing to make the simplifying assumption that at any time a voter will vote correctly with probability equal to its degree of belief.

We imagine that after each round of deliberation a juror i's new degree of belief is a function U_i of the degrees of belief of its and all of its fellow group members' degrees of belief just prior to the most recent deliberation. We can then measure how each member of the jury feels about an issue after j deliberations by computing $\bar{p}_j$, and we can even investigate the limiting behavior of the sequence $\{\bar{p}_j\}$ as $j \to \infty$.

For an arbitrary family $\{U_i\}_{1\le i\le n}$ of update functions, computing the vector $\bar{p}_j$ for large j, let alone computing $\lim_{j\to\infty} \bar{p}_j$, will not be computationally feasible. In what follows we examine the case where each U_i is the restriction to the n-cube $[0,1]^n$ of a linear functional on $\mathbb{R}^n$. In this case, the collection $\{U_i\}$ determines an n-by-n matrix $A = [a_{i,j}]$ where, given $\bar{x} = (x_1,\ldots,x_n) \in [0,1]^n$, we have $U_i(\bar{x}) = \sum_{k=1}^n a_{i,k}x_k$. Thus

$$\bar{p}_{j+1} = (U_1(\bar{p}_j),\ldots,U_n(\bar{p}_j)) = A\bar{p}_j,$$

and for any $m \in \mathbb{N}$ we have

$$\bar{p}_m = A^m\bar{p}_0.$$

Note that $U_i(\bar{x}) = \sum_{k=1}^n a_{i,k}x_k$ maps $[0,1]^n$ into $[0,1]$ only if the coefficients $\{a_{i,k}\}_k$ are non-negative real numbers. Furthermore, the sum of these coefficients must be between 0 and 1 inclusive, and indeed for each i the function U_i maps *onto* $[0,1]$ just in case $U_i(\bar{1}) = 1$ (where $\bar{1} = (1,\ldots,1)$), which in turn is true just in case the coefficients $\{a_{i,k}\}_k$ sum to 1.

Any non-negative, real matrix A with row sums $\|A_{i,\cdot}\|_1 = 1$ for all $1 \le i \le n$ is called a *(non-negative, row) stochastic matrix*, and we give the following definition.

Definition 3.24 (Transition Matrix)**.** *Given a set $J = \{1,\ldots,n\}$ of jurors with update functions $\{U_i : [0,1]^n \to [0,1]\}_{1\le i\le n}$ and given a non-negative row stochastic matrix $A = [a_{i,j}]$ satisfying $U_i(\bar{x}) = \sum_{k=1}^n a_{i,k}x_k$ for all $\bar{x} \in [0,1]^n$, we say that A is the* transition matrix *for the jury J and update functions $\{U_i\}_{1\le i\le n}$. We also say that the jury J has update functions* $\{U_i\}_{1\le i\le n}$ admitting *a transition matrix A.*

Given this machinery, our primary concern is the following:

> Given a jury J, with transition matrix A and initial competencies/degrees of belief given by the vector $\bar{p}_0$, does the sequence $\{\bar{p}_i\} = A^k p_0$ converge? Given

that the sequence does converge, can we compute the limit? What should we do in situations where the sequence does not converge?

The theory of Perron-Frobenius gives an partial answer to this question, and we shall begin by examining it below. This approach to modeling dependence between voter competence as a linear dynamical system was first introduced by DeGroot in [8], and certain details were clarified by Berger in [3], although both authors were primarily concerned with conditions under which the limiting vector $\bar{p}_\infty = \lim_{k\to\infty} \bar{p}_k$ had all components equal, that is, when the group reached a *consensus*. For a recent examination of the DeGroot method we refer the reader to the recent work of Mossel and Tamuz, for instance [17]. After presenting a framework similar to DeGroot's, we go on to present, for the first time, the *Cesàro Judgement Aggregation Method*, which extends on the results of DeGroot by producing a limiting vector of degrees of belief for a large number transition matrices A whose powers $\{A^k\}$ do *not* converge.

Positive Transition Matrices

Fix some positive integer n and some jury $J = \{1, \dots, n\}$. Suppose the update functions for J admit a transition matrix A. Let us assume further that A is a *positive* matrix, that is, each of the entries of A are positive. This models situations where each juror $j \in J$ is influenced to some non-zero degree by the opinion of each of its neighbors.
We can use Perron-Frobenius theory to conclude that for any such matrix A, not only does $\lim_{N\to\infty} A^N$ exist, but $\bar{p}_\infty = \lim_{N\to\infty} \bar{p}_N$ does not depend on A. We present a proof of the following theorem in Appendix A.

Theorem 3.25. *Given a jury $J = \{1, \dots, n\}$ with update functions $\{U_i\}_{1\le i\le n}$ admitting a* positive *transition matrix A, the limit $A^\infty = \lim_{N\to\infty} A^N$ exists and is equal to the orthogonal projection mapping onto the subspace spanned by the eigenvector $\langle 1, 1, \dots, 1\rangle$. Furthermore, the vector $\bar{p}_\infty = A^\infty \bar{p}_0 = \lim_{N\to\infty} \bar{p}_N = \langle p_a, \dots, p_a\rangle$ where $p_a = \frac{1}{n}\sum_{i=1}^n p_{0,i}$ is the average of the competences in the initial competence vector $\bar{p}_0$.*

Proof. See Theorem A.11 □

This result is already striking. It says that if we are using a system of deliberation in which, at each phase of the deliberation, each voter gives some non-zero weight to the current degree of belief of every member of the jury and updates their belief linearly in accordance with these weights, then the jury will converge to a homogeneous jury with uniform competence equal to the averages of the initial competences of the jurors.

One might point out that no actual jury will undergo an infinite number of deliberations nor will a real deliberation be organized neatly into a sequence of repeating sub-deliberations and tentative votes, as the framework above suggests. Yet the result above remains important since it says that the jury competence can be made arbitrarily close to the aforementioned average simply by taking a sufficiently large finite number of deliberation steps.

A more plausible objection is that, particularly in cases of large deliberating bodies like those for major elections, there are portions of the population that do not give any weight to the level of belief of members outside that group. In large bodies of naturally occurring deliberating bodies it is unlikely that every component is receiving a signal, direct or indirect, from every other component. In particular the transition matrix A for the jury J may contain zeros.

Non-negative Transition Matrices

Fortunately, there are many cases where some jurors give no weight to some of their fellow jury members, so that the update functions determine a matrix with zeros, and yet the sequence A^k still converges. To formulate the conditions under which this is the case it will be helpful to introduce the notion of the *accessibility graph* $G(A)$ of a matrix A.

Definition 3.26 (Accessibility Graph)**.** *Given an $n \times n$ matrix $A = [a_{i,j}]$, define the* accessibility graph $G(A)$ *of A to be the directed graph (V, E) on n vertices $V = \{v_1, \ldots, v_n\}$ such that there is an edge $(v_i, v_j) \in E$ between two vertices if and only if $a_{i,j} \neq 0$.*

If J is a jury with update functions $\{U_i\}$ admitting a transition matrix A as above, then there is an edge from v_i to v_j just in case voter i gives positive weight to the confidence level of juror j. In particular, if A is a positive transition matrix for a jury of n members as in the previous subsection, the graph $G(A)$ is the unique graph (V, E) on n vertices with $(i, j) \in E$ for every $i, j \in V$.

Definition 3.27 (Strongly Connected Graph)**.** *A directed graph $G = (V, E)$ is said to be* strongly connected *if for any two vertices $u, v \in V$ there is a directed path (see Definition 1.3) from u to v.*

With this definition, we can prove the following theorem. Again we present a proof of this theorem in Appendix A.

Theorem 3.28. *Let $J = \{1, \ldots, n\}$ with update functions $\{U_i\}_{1 \leq i \leq n}$ admitting a transition matrix B. Suppose that the accessibility graph $G(B)$ is strongly connected and has at least one loop (of length one).*

Then the limit $B^\infty = \lim_{N\to\infty} B^N$ exists and is equal to the orthogonal projection map onto the subspace spanned by the eigenvector $\langle 1, 1, \ldots, 1\rangle$. Furthermore, the vector $\bar{p}_\infty := B^\infty \bar{p}_0 = \lim_{N\to\infty} \bar{p}_N = \langle p_a, \ldots, p_a\rangle$ where $p_a = \frac{1}{n}\sum_{i=1}^{n} p_{0,i}$ is the average of the competences in the initial competence vector $\bar{p}_0$.

Proof. See Theorem A.16. $\square$

Thus, in order to obtain the conclusion of 3.25, it is not necessary that each component v_i of the network be receiving a direct signal from another component v_j so long as both (1) there is some chain of signals which, over a fixed finite number $N_{i,j}$ of units of time, will allow v_i to give positive weight to the state of v_j as of $N_{i,j}$ units of time in the past, and (2) at least one component is giving positive weight to its own previous value.

The Cesàro Judgement Aggregation Method

As we have noted, if the transition matrix A has eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$ then the sequence $\{A^k\}_{1 \leq k < \infty}$ will not converge. In certain circumstances, however, we can modify our strategy to produce a vector of confidences which still represent both the initial confidence vector as well as the deliberations between voters. We have the following theorem, whose proof is given in the appendix.

Theorem 3.29 (Cesaro Limit)**.** *Let $J = \{1, \ldots, N\}$ with initial confidence vector $\bar{p}_0$ and update functions $\{U_i\}_{1 \leq i \leq N}$ admitting a transition matrix B.*

Define the sequence $\{p_k\}$ recursively by

$$\bar{p}_{k+1} = B\bar{p}_k.$$

Also define the sequence $\{\bar{q}_k\}$ by

$$\bar{q}_k = \frac{1}{k}\sum_{j=0}^{k-1} \bar{p}_j.$$

Then the Cesàro Limit, defined as

$$q_\infty = \lim_{k \to \infty} \bar{q}_k$$

exists, and is equal to the orthogonal projection of $\bar{p}_0$ onto the eigenspace $N(B - I)$.

Proof. See Theorem A.18 of the appendix. □

Definition 3.30 (Cesàro Confidence Vector)**.** *Given a jury $J = \{1, \ldots, N\}$ with initial confidence vector $\bar{p}_0$ and update functions $\{U_i\}_{1 \leq i \leq N}$ admitting a transition matrix B, we define the* Cesàro Confidence Vector *for J and $\{U_i\}_{1 \leq i \leq N}$ to be the vector*

$$C(\bar{p}_0, B) := q_\infty$$

where q_∞ is as in Theorem 3.29.

Note that if the transition matrix B is reducible, (that is, if $G(B)$ is not strongly connected), the eigenspace $N(B - I)$ may have dimension greater than one. The important observation, however, is that even in the case where the transition matrix in question should happen to satisfy the hypotheses of Theorem 3.28, *the Cesàro Confidence Vector is the same final vector as would have resulted by simply taking the limit of the vectors $\bar{p}_k = B^k\bar{p}_0$.* This is important because it means that in situations where we know that a jury's update functions admit a transition matrix, but do not have further information about that matrix, we can err on the side of caution and use the Cesàro Voting Method.

We close by presenting, for the first time, the Cesàro Judgement Aggregation Method of Length T

Definition 3.31 (Cesàro Judgement Aggregation Method of Length T)**.** *Let T be a positive integer and let $J = \{1, \ldots, N\}$ be some group faced with a decision between two exhaustive and mutually exclusive alternatives A_0 and A_1. We define the* Length-T Cesàro Judgement Aggregation Method *as follows:*

1. *Each member $j \in J$ begins by writing a real number $v_{j,0}$ on a ballot and writes their name on the ballot. The number must represent that juror's confidence that alternative A_1 is correct (a number of 1 represents total confidence for alternative A_1 and 0 represents total confidence for alternative A_0.)*
2. *All ballots then are revealed to every member of the group. After each member has witnessed their fellow group members' ballots, each member revises their degree of support for the the two alternatives.*
3. *Each member j then fills out another ballot, with a new number $v_{j,1}$, and writes their name on the ballot. Again the ballots are revealed, and the group members revise their degree of support. This process continues for a total of T iterations (giving a total of $T+1$ ballots per group member).*
4. *Meanwhile, at each revelation of the ballots, a clerk records the names and numbers on the ballots. At the end of T-iterations, the clerk computes, for each $j \in J$, the average q_j of each of values $\{v_{j,i}\}_{0 \leq i \leq T}$.*
5. *The clerk then decides the group decision according a Bernoulli probability distribution on $\{A_0, A_1\}$ which selects A_1 with probability $\Gamma_N(\bar{q})$ (as defined in Definition 2.8) if N is odd, and with probability $\hat{\Gamma}_N(\bar{q})$ (as defined in Definition 2.28) if N is even.*

Our emphasis in this section has been to find a judgement aggregation method extending that of Theorem 3.28 that is guaranteed to terminate, and we have been less concerned with computing the method's probability of success. Nevertheless we include the following proposition.

Proposition 3.32. *Let $J = \{1, \ldots, N\}$ be some group faced with a decision between two exhaustive and mutually exclusive alternatives A_0 and A_1. Suppose that the initial confidences in alternative A_1 are given by the vector $\bar{p}_0$. Suppose that each group member alters their confidence in accordance with update functions $\{U_i\}_{1 \leq i \leq N}$ admitting a transition matrix B.*

If $\Gamma_C(T)$ denotes probability that the group will arrive at the correct alternative via a Length T Cesàro Judgement Aggregation Method. Then, as $T \to \infty$,

$$\Gamma_C(T) \to \Gamma_N(C(\bar{p}_0, B)) \quad \text{for } N \text{ odd, and}$$

$$\Gamma_C(T) \to \hat{\Gamma}_N(C(\bar{p}_0, B)) \quad \text{for } N \text{ even.}$$

Proof. Immediate from 3.29 □

Appendix A

Stochastic Matrices and Perron-Frobenius Theory

This appendix is meant to bridge the gap between a solid foundation in linear algebra (say up to and including the Jordan normal form of a matrix) and the theorems of the first section of Chapter 3. Many of the theorems are stated without proof if they are well known and have proofs that are widely accessible. For a more thorough treatment see any text on stochastic matrices or Perron-Frobenius Theory, for example [16].

Definition A.1 (Preliminary Definitions)**.** *Let A be a square matrix with complex entries.*

- $size(A)$ *is the dimension (or side length) of* A.
- $index(A)$ *is the smallest non-negative integer* l *such that* $null(A^l) = null(A^{l+1})$.
- *The* spectrum *of* A*, denoted* $\sigma(A)$*, is the set of eigenvalues of* A.
- *For* $\lambda \in \sigma(A)$, $index(\lambda)$ *is defined to be* $index(A - \lambda I)$.
- *For* $\lambda \in \sigma(A)$, $geomult(\lambda) = dim(null(A - \lambda I))$ *is the* geometric multiplicity *of* λ.
- *For* $\lambda \in \sigma(A)$, $algmult(\lambda) = dim(null(A - \lambda I)^{size(A)}) = dim(null(A - \lambda I)^{index(\lambda)})$ *is the* algebraic multiplicity *of* λ.
- *For* $\lambda \in \sigma(A)$*, we say that* λ *is a* simple *eigenvalue if* $algmult(\lambda) = 1$.
- *For* $\lambda \in \sigma(A)$*, we say that* λ *is a* semi-simple *eigenvalue if* $algmult(\lambda) = geomult(\lambda)$ *or equivalently if* $index(\lambda) = 1$.
- *The* spectral radius *of* A*, denoted* $\rho(A)$ *is defined as* $\max_{\lambda \in \sigma(A)} |\lambda|$.

A.1 Limits and Summability

Definition A.2. *Let A be a square matrix with complex entries.*

- *A is* convergent *if the sequence $\{A^k\}$ converges (pointwise).*
- *A is (Cesàro)* Summable *(to L) if the limit*

$$L = \lim_{k\to\infty} S_k(A)$$

exists, where

$$S_k(A) := \frac{1}{k}\sum_{j=0}^{k-1} A^j.$$

We use the following two theorems without proof.

Theorem A.3. *Suppose M is an $n\times n$ matrix with $\sigma(M) = \{\lambda_1, \dots \lambda_s\}$. Let $M = N^{-1}JN$ where J is a Jordan form of M. Let k_i denote the size of the largest Jordan block for λ_i in J (so $k_i = index(\lambda_i)$). Let $f : \mathbb{C} \to \mathbb{C}$ be function such that for any $1 \le i \le s$ and $0 \le d \le k_i - 1$, the value $f^{(d)}(\lambda_i)$ (the dth derivative of f at λ_i) is defined.*

Then the following matrix $f(A)$ is well-defined (does not depend on the particular Jordan Decomposition J).

$$f(A) := N^{-1}f(J)N,$$

where $f(J)$ is defined by replacing each Jordan block

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & & \\ 0 & \lambda_i & 1 & 0 & \\ & 0 & \ddots & \ddots & 0 \\ & & 0 & \ddots & 1 \\ & & & 0 & \lambda_i \end{bmatrix}$$

of J with the matrix

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2!} & \dots & \frac{f^{(r-1)}(\lambda_i)}{(r-1)!} \\ 0 & f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2!} & \vdots \\ & 0 & \ddots & \ddots & \frac{f''(\lambda_i)}{2!} \\ & & 0 & \ddots & f'(\lambda_i) \\ & & & 0 & f(\lambda_i) \end{bmatrix},$$

where r is the size of the Jordan block.

Proof. See [16] pp. 601-603. □

Theorem A.4. *Suppose that* $\sum_{j=0}^{\infty} c_j (z-z_0)^j$ *converges to* $f(z)$ *at each point inside a circle* $|z - z_0| = r$*, and suppose that* A *is a matrix such that* $|\lambda_i - z_0| < r$ *for each eigenvalue* $\lambda_i \in \sigma(A)$.
Then $\sum_{j=0}^{\infty} c_j (A - z_0 I)^j$ *converges to* $f(A)$.

Proof. See [16] pp. 605-606. □

Corollary A.5. *Let*

$$J = \begin{bmatrix} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & 0 & \\ & 0 & \ddots & \ddots & 0 \\ & & 0 & \ddots & 1 \\ & & & 0 & \lambda \end{bmatrix}$$

with $size(J) = r$.

1. *If* $|\lambda| < 1$ *then the sequence* $\{J^k\}$ *converges to the zero matrix.*
2. *If* $|\lambda| \geq 1$ *and* $\lambda \neq 1$ *then the sequence* $\{J^k\}$ *diverges.*
3. *If* $\lambda = 1$ *then the sequence* $\{J^k\}$ *converges if and only if* $r = 1$.

Proof. Applying Theorem A.4 to $f(x) = x^k$ gives

$$J^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{r-1}\lambda^{k-r+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \vdots \\ & & & \ddots & \\ & 0 & \ddots & \ddots & \binom{k}{2}\lambda^{k-2} \\ & & 0 & \ddots & \binom{k}{1}\lambda^{k-1} \\ & & & 0 & \lambda^k \end{bmatrix}.$$

If $|\lambda| < 1$, then

$$\left| \binom{k}{i} \lambda^{k-i} \right| = \left| \frac{k!}{(k-i)!i!} \lambda^{k-i} \right| \leq \frac{k^i}{i!} |\lambda|^{k-i},$$

which goes to zero as $k \to \infty$.
If $|\lambda| \geq 1$ and $\lambda \neq 1$, it is easy to see that entries on the main diagonal (as well as any other nonzero entries) diverge as $k \to \infty$.
If $\lambda = 1$ and there are any nonzero entries above the main diagonal then these diverge as $k \to \infty$. □

Theorem A.6. *A matrix A is convergent just in case*

1. $\rho(A) < 1$ *or*
2. $\rho(A) = 1$ *with* $\lambda = 1$ *the only eigenvalue on the unit circle and* $\lambda = 1$ *semi-simple.*

When A is convergent, A has a Jordan form

$$P^{-1}AP = \begin{bmatrix} I_d & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$

where $d = algmult_A(1)$, I_d is a $d \times d$ identity matrix, and $\rho(C) < 1$. (In particular we allow $d = 0$ or $d = size(A)$.)

If A is convergent, it converges to the projection onto $null(A - I)$ along $range(A - I)$.

Proof. This follows from Corollary A.5 together with the following observations. The matrix A converges just in case each of its Jordan blocks converge.
Also, if

$$A = P \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} P^{-1}$$

converges, then, since $\rho(C) < 1$, the limit must be

$$P \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1}.$$

Since the first d columns of P are a basis for $null(A - I)$ and the remaining columns are a basis for $range(A - I)$, we conclude that this limit is the projection onto $null(A - I)$ along $range(A - I)$. □

It is easy to see that if the sequence $\{A^k\}_{1 \leq k < \infty}$ converges to some L, then A is *summable* to L as well. The converse is false, as can be seen by considering the 1×1 matrix $A = [-1]$. More generally, we have the following theorem, which the reader should compare to A.6.

Theorem A.7. *A matrix A is summable just in case $\rho(A) \leq 1$ with any eigenvalues on the unit circle being semi-simple.*

When A is summable, it has a Jordan form

$$PAP^{-1} = \begin{bmatrix} I_d & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix},$$

where $d = algmult_A(1)$, I_d is a $d \times d$ identity matrix (in particular, we allow $d = 0$ or $d = n$), $\rho(D) \leq 1$, $\lambda = 1$ is not an eigenvalue of D, and all eigenvalues λ of D on the unit circle have $index(\lambda) = 1$.
If A is summable it is summable to the projection onto $null(A - I)$ along $range(A - I)$.

Proof. If $J = P^{-1}AP$ is a Jordan form for A, then A is summable just in case J is, and J is summable just in case each of its Jordan blocks are summable. So it suffices to determine when a Jordan block

$$J^* = \begin{bmatrix} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & 0 & \\ & 0 & \ddots & \ddots & 0 \\ & & 0 & \ddots & 1 \\ & & & 0 & \lambda \end{bmatrix}$$

for an eigenvalue λ is summable.
It follows from theorem A.4 that the diagonal entries of $S_k(J^*) := \frac{1}{k}\sum_{j=0}^{k-1}(J^*)^j$ are

$$F(\lambda) := \frac{1}{k}\sum_{j=0}^{k-1}\lambda^j = \begin{cases} 1 & \text{for } \lambda = 1 \\ \frac{1}{1-\lambda}(\frac{1}{k} - \frac{\lambda^k}{k}) & \text{for } \lambda \neq 1 \end{cases}. \tag{A.1}$$

We can see that if $|\lambda| > 1$ then J^* is not summable (because the main diagonal diverges), and if $|\lambda| < 1$ then $J(\lambda)$ is summable to zero (indeed it converges to 0). It only remains to consider the case where $|\lambda| = 1$.

Suppose that $|\lambda_0| = 1$, J' is a Jordan block for λ_0, and $size(J') > 1$ (equivalently $index(\lambda) > 1$). It follows from Theorem A.4 that the entries of the superdiagonal of $S_k(J')$ consists of the derivative $\frac{dF}{d\lambda}(\lambda_0)$. When $\lambda_0 = 1$ this quantity is

$$\frac{dF}{d\lambda}(\lambda_0) = \frac{1}{k}\sum_{j=0}^{k-1} j\lambda_0^{j-1} = \frac{1}{k}(1 + 2 + \ldots + k - 1) = \frac{k-1}{2},$$

which diverges to infinity. For $\lambda_0 \neq 1$ with $|\lambda_0| = 1$ we can use A.1 to see that

$$\frac{dF}{d\lambda}(\lambda_0) = \left(\frac{1}{1-\lambda_0}\right)\left(\frac{1}{1-\lambda_0}\left(\frac{1}{k} - \frac{\lambda_0^k}{k}\right) - \lambda_0^{k-1}\right),$$

which oscillates indefinitely as $k \to \infty$.

So A is summable just in case the Jordan blocks of $J = P^{-1}AP$ are either 1×1 blocks containing a scalar λ with $|\lambda| = 1$ or blocks for eigenvalues λ satisfying $|\lambda| < 1$. The 1×1 blocks are summable to the limit of F in (A.1), which is 1 for $\lambda = 1$ and 0 for $|\lambda| = 1$ with $\lambda \neq 1$. The last claim of the theorem follows just as in the Theorem A.6 □

A.2 Perron's Theorem

Definition A.8. *Let $X = [x_{i,j}]$ be any $m \times n$ matrix with complex entries.*

- $$\|X\|_\infty := \max_{1 \le i \le m} \sum_{1 \le j \le n} |x_{i,j}|$$

  is the maximum absolute row sum of X.

- *X is* positive *(*non-negative*) if its entries are positive (resp. non-negative) real numbers.*

- *$|X|$ is the non-negative matrix whose (i, j)th entry is $|x_{i,j}|$, that is, $|X| = [|x_{i,j}|]$.*

- *If X has entries in $\mathbb{R}$ and Y is another $m \times n$ matrix with entries in $\mathbb{R}$, we write $X < Y$ ($X \leq Y$) to mean that $x_{i,j} < y_{i,j}$ (resp. $x_{i,j} \leq y_{i,j}$) for each $1 \leq i \leq m$, $1 \leq y \leq n$.*

- *A square, non-negative matrix A is* (row) stochastic *if each of its rows sums to 1.*

Theorem A.9 (Perron's Theorem)**.** *Let A be positive matrix with $\rho(A) = r$. Then*

- *$r > 0$.*

- *$r \in \sigma(A)$.*

- *$algmult_A(r) = 1$.*

- *There exists a unique positive eigenvector $\bar{p}$ for A such that $\sum_{1 \leq i \leq n} p_i = 1$, and, except for positive multiples of $\bar{p}$, there are no other non-negative eigenvectors for A of any eigenvalue.*

- *r is the only eigenvalue λ of A with $|\lambda| = r$.*

Proof. See [16]. □

Definition A.10 (Perron vector, Perron root)**.** *The value $r = \rho(A)$ in the preceding theorem is called the* Perron root *of A.*
The eigenvector $\bar{p}$ is called the Perron vector *of A.*

Theorem A.11. *Given a jury $J = \{1, \ldots, n\}$ with update functions $\{U_i\}_{1 \leq i \leq n}$ admitting a* positive *transition matrix A, the limit $A^\infty = \lim_{N \to \infty} A^N$ exists and is equal to the projection mapping onto the subspace spanned by the eigenvector $\langle 1, 1, \ldots, 1 \rangle$. Furthermore, the vector $\bar{p}^\infty = A^\infty \bar{p}_0 = \lim_{N \to \infty} \bar{p}_N = \langle p_a, \ldots, p_a \rangle$ where $p_a = \frac{1}{n} \sum_{i=1}^{n} p_{0,i}$ is the average of the competences in the initial competence vector $\bar{p}_0$.*

Proof. Perron's Theorem tells us that A satisfies the conditions for convergence in Theorem A.6. Namely, since the transition matrix A is row stochastic, the Perron root $\rho(A) = 1$ of A is the only eigenvalue on the unit circle and is semi-simple (in fact it is simple). Thus A is convergent.

In addition, the Perron vector $\bar{p}$ of A is equal to $(\frac{1}{n}, \ldots, \frac{1}{n})$. The unit vector parallel to $\bar{p}$ is $\bar{u} = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$, so the projection mapping onto $null(A - I)$ is $\bar{x} \mapsto \langle x, \bar{u} \rangle \bar{u}$, which is equal to the vector whose components are all the average of the components of $\bar{x}$. □

A.3 The Perron-Frobenius Theorem for Non-Negative Matrices

Unfortunately, the results of the Perron theorem do not hold for general non-negative matrices. For example, the 2×2 identity matrix has $\lambda = 1$ as a non-simple eigenvalue, and the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has an eigenvalue on the unit circle other than 1. In this section we will examine a subclass of non-negative matrices for which the results of the Perron theorem do hold.

Definition A.12. *Let A be a square, non-negative matrix.*

- *A* permutation matrix *is a square matrix P whose only non-zero entries are equal to 1 and such that P contains a exactly one such entry per column and exactly one per row.*
- *A is* reducible *if there exists a permutation matrix P such that*

  $$P^T B P = \begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix}$$

  with C_1 and C_3 square. Otherwise the matrix is said to be irreducible.
- *An irreducible matrix A is said to be* primitive *if it has exactly one eigenvalue on the unit circle.*
- *An irreducible matrix A is said to be* imprimitive *if it is not primitive.*

Intuitively, B is reducible if there exists a proper non-empty subset S of the basis vectors such that $span(S)$ is invariant under B. It shouldn't be surprising that this condition is equivalent to an easily stated one for the graph $G(B)$ (defined in chapter 3).

Theorem A.13. *A square matrix B is irreducible if and only if $G(B)$ is strongly connected.*

Proof. First note that if P is a permutation matrix then $G(B) \cong G(P^T BP)$ since the latter is simply a relabeling of the the vertices of $G(B)$. Thus $G(B)$ is strongly connected just in case $G(P^T BP)$ is.

Now assume that B is reducible and

$$M := P^T BP = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}$$

with M_1 and M_3 square. Then clearly there is no path in $G(M)$ from the jth vertex to the ith vertex where $i \leq (\text{size}(M_1)) < j$. Thus $G(M)$ is not strongly connected, and therefore

$G(B)$ is not strongly connected.

Next suppose that $G(B)$ is not strongly connected. Let $V = \{v_1, \ldots, v_n\}$ be the list of vertices of $G(B)$. Then there are vertices v_i and v_j such that there is no path of directed edges from v_j to v_i (it is easy to see that we can assume v_i and v_j are distinct). Let

$$C = \{v \in V : \exists \text{ a path from } v_j \text{ to } v\}.$$

Then there is a permutation τ that re-indexes V such that C is an end segment of V. Thus there is some permutation matrix P_τ such that

$$P_\tau^T B P_\tau = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}$$

with the size of M_3 being equal to the size of C. Thus B is reducible. □

Theorem A.14 (Perron-Frobenius Theorem for Non-negative Matrices)**.** *Let A be an irreducible, non-negative matrix with $r = \rho(A)$*

- $r > 0$.
- $r \in \sigma(A)$.
- $algmult_A(r) = 1$.
- *There exists a unique positive eigenvector $\bar{p}$ for the eigenvalue $\lambda = r$ such that $\sum_{1 \leq i \leq n} p_i = 1$, and, except for positive multiples of $\bar{p}$, there are no other non-negative eigenvectors of A corresponding to* any *eigenvalue.*

As in the positive case, r and $\bar{p}$ are referred to respectively as the Perron root *and* Perron vector *of A.*

Proof. See [16]. □

The reader will notice that this theorem is identical to Theorem A.9 except for the assumption of irreducibility and also that there is now a possibility of eigenvalues $\lambda \neq r$ with $|\lambda| = r$, that is, A may be *imprimitive.* Fortunately there is an easily verified condition which guarantees primitivity.

Theorem A.15. *A non-negative irreducible matrix A is primitive if at least one of its diagonal entries is non-zero.*

Proof. See [16]. □

Thus we have the desired theorem for non-negative transition matrices:

Theorem A.16. *Given a jury* $J = \{1, \dots, n\}$ *with update functions* $\{U_i\}_{1 \le i \le n}$ *admitting a transition matrix* B*, and given that the accessibility graph* $G(B)$ *is strongly connected and has at least one loop (of length one), then the limit* $B^\infty = \lim_{N\to\infty} B^N$ *exists and is equal to the projection map onto the subspace spanned by the eigenvector* $\langle 1, 1, \dots, 1\rangle$*. Furthermore, the vector* $\bar{p}^\infty = B^\infty \bar{p}_0 = \lim_{N\to\infty} \bar{p}_N = \langle p_a, \dots, p_a\rangle$ *where* $p_a = \frac{1}{n}\sum_{i=1}^{n} p_{0,i}$ *is the average of the competences in the initial competence vector* $\bar{p}_0$*.*

Proof. By Theorem A.13, the row stochastic matrix B is irreducible and thus satisfies the conditions of the Perron-Frobenius theorem for non-negative matrices. Furthermore, since $G(B)$ has at least one loop, the main diagonal of B contains some nonzero entry, and by Theorem A.15 we know that B is primitive. Thus we can once again use Theorem A.6 as in the proof of Theorem A.11 to get the desired result. □

A.4 General Stochastic Matrices

Canonical Form for Reducible Matrices

If A is reducible, there is a permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

with B, D square. If either of B, D are reducible we can apply another permutation, and, repeating this process, we eventually obtain a permutation matrix Q such that

$$Q^T A Q = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & \cdots & \cdots & A_{1,r} \\ 0 & A_{2,2} & \cdots & \cdots & \cdots & A_{2,r} \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & A_{r,r} \end{bmatrix},$$

where each matrix $A_{i,i}$ is irreducible or the 1×1 zero matrix.
Lastly, if there are rows of the above block decomposition having nonzero entries only in the corresponding diagonal block, permute these rows to the bottom.[1]

[1] More specifically, for each matrix $A_{i,i}$ residing in rows $\alpha, \dots, \alpha+\delta$, such that these rows have all their non-zero entries contained in $A_{i,i}$ itself, look for the *first* index $\beta > \alpha + \delta$ such that β is the last row in some block of rows for matrices $A_{j,\cdot}$ and such that all blocks of rows for matrices $A_{k,\cdot}$ with $k > j$ also have their non-zero entries contained in their diagonal block $A_{k,k}$. If no such β exists, do nothing to $A_{i,i}$. Otherwise let σ be the cyclic permutation of the indices α through β that sends $\alpha + \delta$ to β. Then conjugate by the permutation matrix for σ.

This gives us a matrix of the following form

$$R^T AR = \left[\begin{array}{cccc|cccc} B_{1,1} & B_{1,2} & \cdots & B_{1,s} & B_{1,s+1} & B_{1,s+2} & \cdots & B_{1,t} \\ 0 & B_{2,2} & \cdots & B_{2,s} & B_{2,s+1} & B_{2,s+2} & \cdots & B_{2,t} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & B_{s,s} & B_{s,s+1} & B_{s,s+2} & \cdots & B_{s,t} \\ \hline 0 & 0 & \cdots & 0 & D_{s+1,s+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & D_{s+2,s+2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_{t,t} \end{array}\right], \tag{A.2}$$

which is called the *canonical form* of A.
All blocks on the diagonal are either irreducible or the 1×1 zero matrix. But since rows still sum to 1, the blocks $D_{i,i}$ for $s+1 \leq i \leq t$ cannot be zero.

Theorem A.17. *If λ is an eigenvalue of a row stochastic matrix A and $|\lambda| = 1$ then λ is semi-simple.*

Proof. Let A be row stochastic with canonical form given by (A.2).
First consider the eigenvalues of a block $B_{i,i}$. Such a matrix is either the 1×1 zero matrix, in which case is has no eigenvalues on the unit circle, or $B_{i,i}$ is irreducible. If $B_{i,i}$ is irreducible, then since the rows of A sum to 1 and the matrices $B_{i,d}$ for $i < d \leq t$ are not all null, we have $B_{i,i}\bar{1} \leq \bar{1}$ and $B_{i,i}\bar{1} \neq \bar{1}$ (where $\bar{1}$ is the vector of ones with length equal to $size(B_{i,i})$.
Now recall that for all $\lambda \in \sigma(B_{i,i})$ we have $|\lambda| \leq \|B_{i,i}\|_\infty \leq 1$ (the maximum absolute row sum). So if $|\lambda| = 1$ then $\rho(B_{i,i}) = 1$. Let $\bar{q} > 0$ be the Perron vector for $B_{i,i}^T$. Then $\bar{q}^T A = \bar{q}^T$, and thus $\bar{q}^T(I - B_{i,i}) = 0$. But also $(I - B_{i,i})\bar{1} \geq 0$ and $(I - B_{i,i})\bar{1} \neq 0$, so $\bar{q}^T(I - B_{i,i})\bar{1} > 0$ which is impossible.
Thus all eigenvalues of A on the unit circle are in $\bigcup_{s+1 \leq j \leq t} \sigma(D_{j,j})$. But the matrices $D_{j,j}$ are both irreducible and stochastic, thus their eigenvalues on the unit circle have algebraic multiplicity one. Any such eigenvalue may be repeated in more than one such matrix $D_{j,j}$ but in each such matrix the algebraic multiplicity, and therefore the geometric multiplicity, is equal to one. Thus the algebraic multiplicity of λ in A is equal to its geometric multiplicity in A. □

We may now prove Theorem 3.29.

Theorem A.18. *Given a jury $J = \{1, \ldots, n\}$ with initial confidence vector $\bar{p}_0$ and update functions $\{U_i\}_{1 \leq i \leq n}$ admitting a transition matrix B, define the sequence $\{p_k\}$ recursively by*

$$\bar{p}_{k+1} = B\bar{p}_k.$$

Also define the sequence $\{\bar{q}_k\}$ by

$$\bar{q}_k = \frac{1}{k}\sum_{j=0}^{k-1} \bar{p}_j.$$

Then the limit

$$q_\infty = \lim_{k\to\infty} \bar{q}_k$$

exists, and is equal to the orthogonal projection of $\bar{p}_0$ *onto the eigenspace* $N(B-I)$.

Proof. By Theorem A.17, all eigenvalues λ of B with $|\lambda| = 1$ are semi-simple and thus B satisfies the conditions of Theorem A.7. Then the limit $L := \lim_{k\to\infty} S_j(B)$ exists and is equal to the projection onto $null(B-I)$ along $range(B-I)$. Lastly, we see that the vectors

$$\bar{q}_k = \frac{1}{k}\sum_{j=0}^{k-1} \bar{p}_j = \frac{1}{k}\sum_{j=0}^{k-1} B^j \bar{p}_0 = (S_k(B))\bar{p}_0$$

converge to

$$\bar{q}_\infty = \lim_{k\to\infty} \bar{q}_k = \lim_{k\to\infty} (S_k(B))\bar{p}_0 = L\bar{p}_0.$$

$\square$

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