Title

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Publication Date
1995-05-01

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California PATH Working Paper
UCB-ITS-PWP-95-7

This work was performed as part of the California PATH Program of the University of California, in cooperation with the State of California Business, Transportation, and Housing Agency, Department of Transportation; and the United States Department Transportation, Federal Highway Administration.

May 1995

ISSN 1055-1417

Ideal Dynamic User-Optimal Route Choice: A Link-Based Variational Inequality Formulation

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October 4, 1992
Revised February 16, 1994

PATH Working Paper No.D94-14

Executive Summary

The ideal dynamic user-optimal (DUO) route choice problem is to determine vehicle flows on each link at each instant of time resulting from drivers using actual minimal-time routes. Actual route time is the travel time incurred while driving along the route. In a previous paper, we presented a route-based optimal control model for the ideal DUO route choice problem. However, this model is not appropriate for large scale transportation networks because some degree of route enumeration is necessary to solve the model. In this paper, we first present the traffic network constraints and link-based DUO route choice conditions. Then, we introduce a link-based variational inequality (VI) formulation for the ideal DUO route choice problem so that route enumeration can be avoided in both the formulation and the solution procedure. By proving the necessity and sufficiency of this VI, we demonstrate that the VI formulation is equivalent to the link-based DUO route choice conditions.

Contents

1 Introduction

With the advance of Intelligent Vehicle Highway Systems (IVHS), there is a crucial need to develop mathematical models to provide real-time traffic information to be used in the Advanced Traveler Information Systems (ATIS). Dynamic traffic network models are such methods which have good potential to be applied for large-scale urban transportation networks.

Yagar (1971), Hurdle (1974) and Merchant and Nemhauser (1978a) were among the first to consider dynamic models for congested traffic networks. But the assumptions of these models are very limiting and they are unsuitable for application to general large-scale networks. Important breakthroughs began to occur in the late 1980s when IVHS ignited the potential applicability of such models to the next generation of surface transportation systems.

The study of dynamic route choice models over a general road network was begun by Merchant and Nemhauser (1978a, 1978b) who presented a dynamic system-optimal (DSO) route choice model for a many-to-one network. Subsequently, Carey (1987) reformulated the Merchant-Nemhauser problem as a convex nonlinear program which has analytical and computational advantages over the original formulation. DSO route choice models over a multiple origin-destination (O-D) network were established by using optimal control theory (Friesz et al, 1989; Ran and Shimazaki, 1989a). Recently, many simulation-based DSO route choice models have also been proposed by various researchers] especially for freeway corridor problems (Chang et al, 1993).

Another important dynamic generalization of the static user equilibrium concept is dynamic user-optimal (DUO) route choice. Friesz et al (1989) proposed a DUO route choice model by considering the equilibration of instantaneous unit route costs.

Furthermore, a generalized DUO route choice model over a multiple origin-destination network was presented by Wie, F'riesz and Tobin (1990). By defining the exit flow as a *control variable,* Ran and Shimazaki (1989b) presented a DUO route choice model which considered the equilibration of instantaneous travel times. Subsequently, Ran, Boyce and LeBlanc (1993) formulated a set of new instantaneous DUO route choice models with flow propagation constraints. Recently, Janson (1991) presented a set of operational DUO route choice models using average link travel time/flow relationships and proposed a heuristic solution algorithm. Ghali and Smith (1993) also presented a set of dynamic network models using packets to represent traffic flows on links.

The choice of departure time has been addressed by several researchers, including Abkowitz (1981) and Hendrickson and Plank (1984), who developed work trip scheduling models. De Palma et al (1983) and Ben-Akiva et al (1984) modeled departure time choice over a simple network with one bottleneck using the general continuous logit model. Mahmassani and Herman (1984) used a traffic flow model to derive the equilibrium joint departure time and route choice pattern over a parallel route network. Mahmassani and Chang (1987) further developed the concept of equilibrium departure time choice and presented the boundedly-rational user equilibrium concept under which all drivers in the system are satisfied with their current travel choices, and thus feel no need to improve their outcome by changing to an alternate choice.

The static user-optimal route choice problem was formulated as an equivalent set of inequalities by Smith (1979). Later on, Dafermos (1982) developed an elastic demand model with disutility functions using the variational inequality (VI) approach. An elastic demand model with demand functions was introduced by Dafermos and Nagurney (1984b). Fisk and Boyce (1983) also presented a set of alternative VI formulations for network equilibrium travel choice problems. Nagurney (1993) summarized the modeling and algorithmic aspects of VI models for static traffic assignment problems. Recently, Friesz et al (1993) formulated a VI model for the simultaneous departure time/route choice problem. Smith (1993) also presented a route-based VI formulation using the packet representation of vehicle groups. Both dynamic models are route-based, which need explicit route enumeration in both formulation and solution.

However, since the dynamic traffic flow does not have constant flow rate during propagation, the route-based VI can not be transformed into a link-based VI. Thus, it is very difficult to develop a solution algorithm for route-based VI without explicit route enumeration. It is generally understood that explicit route enumeration for large transportation networks is infeasible. Therefore, this problem is becoming the most critical constraint for route-based VI models to be applied in realistic transportation networks.

In order to overcome this problem, we develop a link-based VI route choice model in this paper. In addition to this significant contribution, we note that our formulation approach is different from others and has more of a traffic engineering background. The link-based ideal dynamic user-optimal (DUO) route choice model is presented for a network with multiple origin-destination pairs. In Section 1, the network constraints for the dynamic traffic network model are first introduced. In Section 2, we present the definition of ideal DUO and its corresponding ideal DUO route choice conditions. The dynamic traffic network constraints are summarized in Section **3.** Then, a general link-based variational inequality formulation of the ideal DUO route choice problem is proposed. The proofs of necessity and sufficiency are given to demonstrate that this VI model is equivalent to the link-based ideal DUO route choice conditions. Finally, some discussions on the VI route choice model are presented and some future studies are proposed.

2 Dynamic Network Constraints

Here, we consider a network with multiple origins and destinations. The traffic network is represented by a directed graph with nodes and directed links. A node can represent either an origin or a destination, or simply an intersection. The index r denotes an origin node and the index s denotes a destination node.

Consider a fixed time period $[0,T]$ where T is the time sufficient for all persons departing during the peak period to complete their trips. We define

$x_a(t)$ = number of vehicles traveling on link a at time t;

$x_a^{rs}(t)$ = number of vehicles traveling on link a with origin r and destination s at time t.

All variables with superscripts rs denote the variables with origin r and destination s. We have by definition that

$$\sum_{rs} x_a^{rs}(t) = x_a(t) \qquad \forall a. \tag{1}$$

Let $u_a(t)$ denote the inflow rate into link a at time t and $v_a(t)$ denote the exit flow rate from link a at time t. The inflows and exit flows, $u_a(t)$ and $v_a(t)$, are both control variables. The state variable for link a is the number of vehicles $x_a(t)$ on link a. The state equation for link a can then be written as

$$\frac{dx_a^{rs}(t)}{dt} = u_a^{rs}(t) - v_a^{rs}(t) \qquad \forall a, r, s. \tag{2}$$

We assume that the number of vehicles on link a at initial time $t = 0$ equals zero:

$$x_a^{rs}(0) = 0, \qquad \forall a, r, s. \tag{3}$$

Thus, the number of vehicles on link a at any time t is

$$x_a^{rs}(t) = \int_0^t [u_a^{rs}(\omega) - v_a^{rs}(\omega)]d\omega \qquad \forall a, r, s. \tag{4}$$

We require that all variables are nonnegative at all times:

$$x_a^{rs}(t) \geq 0, \qquad u_a^{rs}(t) \geq 0, \qquad v_a^{rs}(t) \geq 0, \qquad \forall a, r, s. \tag{5}$$

Denote the required instantaneous flows from origin node r to destination node s at time t as $f^{rs}(t)$, which is a given function of time. Also denote $A(j)$ as the set of links whose tail node is j (after j), and $B(j)$ as the set of links whose head node is j (before j), where j is any node including origin node r and destination node s. The flow conservation at an intermediate node j $(j \neq r, s)$ for each O-D pair requires that the flow exiting from links pointing into node j at time t equals the flow entering links which leave node j at time t. Thus, the flow conservation equations can be expressed as

$$\sum_{a \in B(j)} v_a^{rs}(t) = \sum_{a \in A(j)} u_a^{rs}(t) \qquad \forall j \neq r, s. \tag{6}$$

Conservation of flow at origin node r $(r \neq s)$ requires the flow originating at origin r at time t to equal the flow entering the links which leave origin r at time t. Thus, the flow conservation equations for the origin nodes can be expressed as

$$\sum_{a \in A(r)} u_a^{rs}(t) = f^{rs}(t) \qquad \forall r \neq s; s. \tag{7}$$

Denote the instantaneous flows arriving at destination node s from origin node r at time t as the control variable $e^{rs}(t)$, and let $e_p^{rs}(t)$ denote these flows over route p at time t. Conservation of flow at destination node s $(s \# r)$ requires the flow exiting at destination s at time t to equal the flow exiting the links which lead to destination s at time t. Thus, the flow conservation equations for the destination nodes can be expressed as

$$\sum_{a \in B(s)} v_a^{rs}(t) = e^{rs}(t) \qquad \forall r; s \neq r. \tag{8}$$

Denote $E_p^{rs}(t)$ as the cumulative number of vehicles arriving at destination s by time t from origin r through route p. By definition, it follows that

$$\frac{dE_p^{rs}(t)}{dt} = e_p^{rs}(t) \qquad \forall p, r, s \neq r. \tag{9}$$

At the initial time $t = 0$,

$$E_p^{rs}(0) = 0, \qquad \forall p, r, s. \tag{10}$$

By definition, the variables must be nonnegative at all times:

$$e_p^{rs}(t) \geq 0, \qquad E_p^{rs}(t) \geq 0, \qquad \forall p, r, s. \tag{11}$$

It is necessary to ensure that the entering and exiting flows, as well as the vehicles remaining on links, are consistent with the link travel times. We write these constraints using actual link travel times.

The actual travel time $\tau_a[x_a(t), u_a(t), v_a(t)]$, or simply $\tau_a(t)$, over link a is assumed to be dependent on the number of vehicles $x_a(t)$, the inflow $u_a(t)$ and the exit flow $v_a(t)$ on link a at time t. We assume the travel time $\tau_a(t)$ on link a is the sum of two components: **1)** a flow-dependent running time $g_{1a}[x_a(t), u_a(t)]$ over link a and **2)** a queuing delay $g_{2a}[x_a(t), v_a(t)]$. It follows that

$$\tau_a(t) = g_{1a}[x_a(t), u_a(t)] + g_{2a}[x_a(t), v_a(t)]. \tag{12}$$

The two components $g_{1a}[x_a(t), u_a(t)]$ and $g_{2a}[x_a(t), v_a(t)]$ of the time-dependent link travel time function $\tau_a[x_a(t), u_a(t), v_a(t)]$ are assumed to be nonnegative and differentiable with respect to $x_a(t), u_a(t)$ and $x_a(t), v_a(t)$, respectively.

We formulate the constraints relating link flows to link times as follows. Let $x_{ap}^{rs}(t)$ denote the number of vehicles on link a using route p between O-D pair rs at time t. By definition,

$$\sum_{rsp} x_{ap}^{rs}(t) = x_a(t) \qquad \forall a. \tag{13}$$

For any intermediate node $j \neq r$ on route p, denote a subroute $\tilde{p}$ as the section of route p from node j to destination s. For any link $a \in B(j)$, vehicles on link a using route p at any time t must result in either:

1. extra vehicles on downstream links on subroute $\tilde{p}$ at time $t + \tau_a(t)$, or
2. increased exiting vehicles at the destination at time $t + \tau_a(t)$.

It follows that

$$x_{ap}^{rs}(t) = \sum_{b \in \tilde{p}} \{x_{bp}^{rs}[t + \tau_a(t)] - x_{bp}^{rs}(t)\} + \{E_p^{rs}[t + \tau_a(t)] - E_p^{rs}(t)\}$$

$$\forall r, s, p, j, a; j \neq r; a \in B(j). \qquad (14)$$

We refer the reader to Ran, Boyce and LeBlanc **(1993)** for more details. There are also capacity constraints and oversaturation constraints for time-dependent traffic flows in the network. The exit flow capacity constraints are combined in the dynamic link travel time functions as suggested by Ran et al **(1992).** For other physical capacity constraints and oversaturation constraints, we leave them for the situations where the model is implemented on a realistic traffic network because those constraints will largely increase the computational burden. The detailed discussion of those constraints can be found in Ran **(1993).**

3 The Ideal DUO Route Choice Conditions

Since we are considering a continuous time problem and assuming that the link travel times are increasing with inflows and number of vehicles on links, the flow propagation constraints presented in Section 1 automatically guarantee that the first-in-first-out (FIFO) requirement for flows on a given route can be satisfied. We note that the FIFO requirement may be violated in a discrete time situation. We also note that the traditional BPR functions are no longer applicable in a dynamic traffic network problem where time-dependent queuing and spillback problems occur. A set of time-dependent link travel time functions for signalized arterial links have been proposed by Ran et al (1992). Those link travel time functions are similar to the above general link travel time functions. It is our intention that the realistic link travel time functions will be employed when our VI route choice model is implemented for realistic transportation networks.

Consider the flow which originates at node r at time t and is destined for node s. There is a set of routes $\{p\}$ between O-D pair (r,s). Define $\eta_p^{rs}(t)$ as the travel time *actually* experienced over route p by vehicles departing origin r toward destination s at time t. We use a recursive formula to compute the route travel time $\eta_p^{rs}(t)$ for all allowable routes. Assume route p consists of nodes $(r,1,2,\cdots,i,\cdots,s)$. Denote $\eta_p^{ri}(t)$ as the travel time *actually* experienced over route p from origin r to node i by vehicles departing origin r at time t. Then, a recursive formula for route travel time $\eta_p^{rs}(t)$ is:

$$\eta_p^{ri}(t) = \eta_p^{r(i-1)}(t) + \tau_a[t + \eta_p^{r(i-1)}(t)] \qquad Qp,r,i; i = 1,2,\cdots,s; \qquad (15)$$

where link $a = (i-1,i)$.

Denote $\pi^{rs}(t)$ as the minimal travel time experienced by vehicles departing from origin r to destination s at time t. Then, $\pi^{rs}(t) = \min_p \eta_p^{rs}(t)$. $\pi^{rs}(t)$ is a functional of all link flow variables at time $\omega \geq t$, i.e., $\pi^{rs}(t) = \pi^{rs}[u(\omega), v(\omega), x(\omega); w \geq t]$. This functional is neither a state variable nor a control variable, and it is not fixed. This functional is not available in closed form. Nevertheless, it can be evaluated when $u(\omega)$, $v(\omega)$ and $x(\omega)$ are temporarily fixed (as in a Frank-Wolfe algorithm), which is all that is required for solving the model.

Then, we propose a definition of DUO that reflects the *ideal* route choice behavior of travelers as in Ran, Boyce and LeBlanc (1992). The formulation of the ideal DUO route choice problem will be based on the underlying choice criterion that each traveler uses the route that minimizes his/her actual travel time when departing from the origin or any intermediate node to his/her destination.

> **Ideal DUO:** *If, for each O-D pair at each instant of time, the actual travel times experienced by travelers departing at the same time are equal and minimal, the dynamic traffic flow over the network is in an ideal dynamic user-optimal state.*

The above definition can also be called a predictive or anticipatory DUO, since the actual route travel time is predicted using the corresponding route choice model. This model assumes each traveler will have perfect information about the future network conditions and will comply with the guidance instructions based on ideal DUO route choice conditions. We note that in an ideal DUO, a route p between r and s is being used at time t if $f_p^{rs}(t) > 0$. In the following, we write a set of route-based ideal DUO route choice conditions based on the above definition:

$$\eta_p^{rs}(t) \geq \pi^{rs}(t) \qquad \forall p, r, s; \tag{16}$$

$$f_p^{rs*}(t)\left[\eta_p^{rs}(t) - \pi^{rs}(t)\right] = 0 \qquad \forall p, r, s; \tag{17}$$

$$f_p^{rs*}(t) \geq 0 \qquad \forall p, r, s. \tag{18}$$

Most of the current ideal DUO route choice models are formulated using the above route-based DUO route choice conditions. Since explicit route enumeration can not be avoided when directly using the above conditions, we seek to derive a set of link-based ideal DUO route choice conditions as follows.

Unlike in the previous dynamic route choice models, we now write the equivalent mathematical inequalities for the ideal DUO definition using link and node variables, in contrast to the route-based formulation. In this case, for any route from origin r to destination s, link a is defined as used at time t if $u_a^{rs}(t) > 0$.

Define $\pi^{ri^*}(t)$ as the minimal travel time *actually* experienced by vehicles departing origin r to node i at time t, the asterisk denoting that the travel time is computed using ideal DUO traffic flows. For link $a = (i, j)$, the minimal travel time $\pi^{rj^*}(t)$ from origin r to j should be equal or less than the minimal travel time $\pi^{ri^*}(t)$ from origin r to i plus the actual link travel time $\tau_a[t + \pi^{ri^*}(t)]$ at time instant $[t + \pi^{ri^*}(t)]$, where this time instant is the earliest clock time when the flow departing origin r at time t can enter link a. It follows that

$$\pi^{ri^*}(t) + \tau_a[t + \pi^{ri^*}(t)] \geq \pi^{rj^*}(t) \qquad \forall a = (i, j), r. \tag{19}$$

If, for each O-D pair rs, any departure flow from origin r at time t enters link a at the earliest clock time $[t + \pi^{ri^*}(t)]$, or $u_a^{rs}[t + \pi^{ri^*}(t)] > 0$, then the ideal DUO route choice conditions require that link a is on the minimal travel time route. In other words, the minimal travel time $\pi^{rj^*}(t)$ for vehicles departing origin r toward node j at time t should equal the minimal travel time $\pi^{ri^*}(t)$ for vehicles departing from origin r to i plus the actual link travel time $\tau_a[t + \pi^{ri^*}(t)]$ at time instant $[t + \pi^{ri^*}(t)]$ It follows that

$$\pi^{rj^*}(t) = \pi^{ri^*}(t) + \tau_a[t + \pi^{ri^*}(t)], \quad \text{if } u_a^{rs^*}[t + \pi^{ri^*}(t)] > 0 \quad \forall a = (i, j), r, s. \tag{20}$$

The above equation is also equivalent to the following:

$$\left[\pi^{rj^*}(t) - \pi^{ri^*}(t) - \tau_a[t + \pi^{ri^*}(t)]\right] \, u_a^{rs^*}[t + \pi^{ri^*}(t)] = 0 \qquad \forall a = (i, j), r, s. \tag{21}$$

Thus, the link-based ideal DUO route choice conditions can be summarized as follows:

$$\pi^{ri^*}(t) + \tau_a[t + \pi^{ri^*}(t)] \geq \pi^{rj^*}(t) \qquad \forall a = (i, j), r; \tag{22}$$

$$\left[\pi^{ri^*}(t) + \tau_a[t + \pi^{ri^*}(t)] - \pi^{rj^*}(t)\right] u_a^{rs^*}[t + \pi^{ri^*}(t)] = 0 \qquad \forall a = (i,j), r, s; \quad (23)$$

$$u_a^{rs}[t + \pi^{ri^*}(t)] \geq 0 \qquad \forall a = (i,j), r, s. \quad (24)$$

In the appendix, we prove that the above link-based ideal DUO route choice conditions imply the route-based ideal DUO route choice conditions (16)-(18). We note that a similar set of link-based ideal DUO route choice conditions were proposed by Kuwahara and Akamatsu **(1993).** In their formulation, they use a different representation of departure/arrival times for traffic flows.

4 Link-Based Variational Inequality Forrnulation of Ideal DUO Route Choice

The static user-optimal route choice problem has been formulated as both route-based and link-based variational inequality models by various researchers. Since the traffic flow is assumed constant with no dispersion during progression, the route-based VI and link-based VI are convertible. In a time-dependent traffic network, the dynamic flow is no longer constant and traffic dispersion has to be taken into account in flow propagation. Because of these reasons, the dynamic route-based VI route choice model can not be transformed into a dynamic link-based VI route choice model. Thus, it is very difficult to develop a solution algorithm for the dynamic route-based VI route choice model without explicit route enumeration. Therefore, a link-based dynamic VI route choice model and other link-based dynamic travel choice models are of critical importance to the success of dynamic network equilibrium modeling.

In this section, the constraint set for our dynamic route choice problem is first summarized as follows:

Relationship between state and control variables:

$$\frac{dx_a^{rs}}{dt} = u_a^{rs}(t) - v_a^{rs}(t) \qquad \forall a, r, s; \tag{25}$$

$$\frac{dE_p^{rs}(t)}{dt} = e_p^{rs}(t) \qquad \forall p, r, s \neq r; \tag{26}$$

Flow conservation constraints:

$$f^{rs}(t) = \sum_{a \in A(r)} u_a^{rs}(t) \qquad \forall r, s; \tag{27}$$

$$\sum_{a \in B(j)} v_a^{rs}(t) = \sum_{a \in A(j)} u_a^{rs}(t) \qquad \forall r, s, j; j \neq r, s; \tag{28}$$

$$\sum_{a \in B(s)} v_a^{rs}(t) = e^{rs}(t) \qquad \forall r, s; \tag{29}$$

Flow propagation constraints:

$$x_{ap}^{rs}(t) = \sum_{b \in \tilde{p}} \{x_{bp}^{rs}[t+\tau_a(t)] - x_{bp}^{rs}(t)\} + \{E_p^{rs}[t+\tau_a(t)] - E_p^{rs}(t)\} \qquad \forall a \in B(j); j \neq r; p, r, s; \tag{30}$$

Definitional constraints:

$$\sum_{rs} u_a^{rs}(t) = u_a(t), \qquad \sum_{rs} v_a^{rs}(t) = v_a(t), \qquad \forall r, s; \tag{31}$$

$$\sum_{rsp} x_{ap}^{rs}(t) = x_a(t), \qquad \sum_{rs} x_a^{rs}(t) = x_a(t), \qquad \forall r, s; \tag{32}$$

Nonnegativity conditions:

$$x_a^{rs}(t) \geq 0, \quad u_a^{rs}(t) \geq 0, \quad v_a^{rs}(t) \geq 0 \qquad \forall a, r, s; \tag{33}$$

$$e_p^{rs}(t) \geq 0, \qquad E_p^{rs}(t) \geq 0, \qquad \forall p, r, s; \tag{34}$$

Boundary conditions:

$$E_p^{rs}(0) = 0, \qquad \forall p, r, s; \tag{35}$$

$$x_a^{rs}(0) = 0, \quad \forall a, r, s. \tag{36}$$

Denote $\Omega_a^{rj^*}(t)$ as the difference between the minimal travel time from origin r to node j and the travel time from origin r to node j via the minimal travel time route from origin r to node i and link a for vehicles departing from origin r at time t. It follows that

$$\Omega_a^{rj^*}(t) = \pi^{ri^*}(t) + \tau_a[t + \pi^{ri^*}(t)] - \pi^{rj^*}(t) \qquad \forall a, r; a = (i, j). \tag{37}$$

In order to simplify the presentation, we rewrite the link-based ideal DUO route choice conditions as follows:

$$\Omega_a^{rj^*}(t) \geq 0 \qquad \forall a = (i, j), r; \tag{38}$$

$$u_a^{rs^*}[t + \pi^{ri^*}(t)]\, \Omega_a^{rj^*}(t) = 0 \qquad \forall a = (i, j), r, s; \tag{39}$$

$$u_a^{rs}[t + \pi^{ri^*}(t)] \geq 0 \qquad \forall a = (i, j), r, s. \tag{40}$$

Then, the equivalent link-based variational inequality formulation of ideal DUO route choice conditions **(38)-(40)** may be stated as follows.

Theorem 1. The dynamic traffic flow pattern satisfying constraints (25)-**(36)** is in an ideal DUO route choice state if and only if it satisfies the variational inequality problem:

$$\sum_{rs} \sum_{a} \Omega_a^{rj^*}(t) \left\{ u_a^{rs}[t + \pi^{ri^*}(t)] - u_a^{rs^*}[t + \pi^{ri^*}(t)] \right\} \geq 0 \tag{41}$$

Proof of Necessity.

We need to prove that the ideal DUO route choice conditions **(38)-(40)** imply the variational inequality problem **(41).** For any link a, a feasible inflow at time $[t + \pi^{ri^*}(t)]$ is

$$u_a^{rs}[t + \pi^{ri^*}(t)] \geq 0. \tag{42}$$

Multiplying DUO route choice condition **(38)** with equation **(42)**, we have

$$u_a^{rs}[t+\pi^{ri^*}(t)]\,\Omega_a^{rj^*}(t)\geq 0 \qquad \forall a,r,s;\, a=(i,j). \tag{43}$$

We subtract the second ideal DUO route choice condition **(39)** from equation **(43)** and obtain

$$\left\{u_a^{rs}[t+\pi^{ri^*}(t)]-u_a^{rs^*}[t+\pi^{ri^*}(t)]\right\}\,\Omega_a^{rj^*}(t)\geq 0 \qquad \forall a,r,s;\, a=(i,j). \tag{44}$$

Summing equation **(44)** for all links a and all O-D pairs rs, it follows that

$$\sum_{rs}\sum_{a}\left\{u_a^{rs}[t+\pi^{ri^*}(t)]-u_a^{rs^*}[t+\pi^{ri^*}(t)]\right\}\,\Omega_a^{rj^*}(t)\geq 0 \qquad \text{where } a=(i,j). \tag{45}$$

Proof of Sufficiency.

We need to prove that any solution $u_a^{rs^*}[t+\pi^{ri^*}(t)]$ to the variational inequality problem **(41)** satisfies the ideal DUO route choice conditions **(38)-(40).** For any O-D pair rs, any link a in some route p from origin r to destination s, let $a=(i,j)$.

Case (i). If

$$u_a^{rs^*}[t+\pi^{ri^*}(t)]>0, \tag{46}$$

we define a pair of successive link inflow patters $u^+(t)$ and $u^-(t)$ for link $b=(l,m)$ and O-D pair cd at time t as follows:

$$u^+(t)=\{u_b^{+cd}[t+\pi^{cl^*}(t)],\ \forall\, b=(l,m),c,d\}, \tag{47}$$

where

$$u_b^{+cd}[t+\pi^{cl^*}(t)]=u_a^{rs^*}[t+\pi^{ri^*}(t)]+\varepsilon, \tag{48}$$

if $cd=rs$, $b=(l,m)=(i,j)=a$, otherwise,

$$u_b^{+cd}[t+\pi^{cl^*}(t)]=u_b^{cd^*}[t+\pi^{cl^*}(t)]; \tag{49}$$

and

$$u^-(t) = \{u_b^{-cd}[t + \pi^{cl^*}(t)], \ \forall\ b = (l, m), c, d\}, \tag{50}$$

where

$$u_b^{-cd}[t + \pi^{cl^*}(t)] = u_a^{rs^*}[t + \pi^{ri^*}(t)] - \epsilon, \tag{51}$$

if $cd = rs$, $b = (l, m) = (i, j) = a$, otherwise,

$$u_b^{-cd}[t + \pi^{cl^*}(t)] = u_b^{cd^*}[t + \pi^{cl^*}(t)]. \tag{52}$$

If we choose $0 < \epsilon < u_a^{rs^*}[t + \pi^{ri^*}(t)]$, then $u^+(t)$ and $u^-(t)$ induce two feasible route flow patterns through network flow constraints **(25)-(36).** Substituting now $u^+(t)$ into the variational inequality problem **(41),** we have

$$\Omega_a^{rj^*}(t)\,\epsilon \ \geq\ 0 \tag{53}$$

which implies

$$\Omega_a^{rj^*}(t) \ \geq\ 0. \tag{54}$$

Similarly, substituting $u^-(t)$ into the variational inequality **(41)** yields

$$\Omega_a^{rj^*}(t) \ \leq\ 0. \tag{55}$$

Hence, combining equations **(54)** and **(55),** we have

$$\Omega_a^{rj^*}(t) \ =\ 0. \tag{56}$$

Case (ii). If

$$u_a^{rs^*}[t + \pi^{ri^*}(t)] = 0, \tag{57}$$

then the above defined $u^{+}(t)$ is still feasible. Therefore, equations **(54)** is valid, i.e.,

$$\Omega_a^{rj^*}(t) \geq 0. \tag{58}$$

Concluding from equations **(56)** and **(58)**, we obtain

$$\Omega_a^{rj^*}(t) \geq 0. \tag{59}$$

Clearly, from either equations **(46)** and **(56)** in Case (i) or equations **(57)** and **(58)** in Case (ii), it follows that

$$u_a^{rs^*}[t+\pi^{ri^*}(t)]\,\Omega_a^{rj^*}(t) = 0, \tag{60}$$

while, by definition, it always holds that

$$u_a^{rs^*}[t+\pi^{ri^*}(t)] \geq 0. \tag{61}$$

Since O-D pair rs and link a are assumed to be arbitrarily chosen, equations **(59)-(61)** are just the link-based ideal DUO route choice conditions **(38)-(40).** Therefore, variational inequality **(41)** is **a** sufficiency condition for ideal DUO route choice conditions **(38)-(40).** Since we proved the necessity and sufficiency of the variational inequality problem **(41)** in the above, we state that the VI problem is equivalent to the ideal DUO route choice conditions **(38)-(40).** The proof is complete.

5 Concluding Remarks

In this paper, a link-based VI model for ideal DUO route choice is proposed. The necessity and sufficiency proofs of the VI model demonstrate that this model is consistent with the link-based ideal DUO route choice conditions. Because the constraint set is compact and the travel time function is continuous, the existence of this VI can be easily proven (Nagurney, **1993).** However, the uniqueness of this VI requires strong monotonicity conditions for the travel time function. The rigorous proof of uniqueness will be given in a subsequent paper.

The main advantage of such a link-based VI formulation is that the explicit route enumeration can be avoided. This feature allows our model to be applicable for large-scale dynamic transportation networks with general link travel time functions.

The link-based VI model for ideal DUO route choice can be extended to include departure/arrival time choice and mode choice as well. Our next step is to develop some efficient solution algorithms for the ideal DUO route choice VI model. We expect that the Frank-Wolfe and diagonalization techniques proposed by Boyce et al (1991) can be applied to solve this model. Other solution algorithms, such as the projection algorithm, implemented by Nagurney (1986) for VI models for static network equilibrium problems are also extendable for our dynamic VI problem. We note that the solution algorithm for our ideal DUO route choice VI model has to be implemented on an expanded time-space network proposed in Boyce et al (1991).

Appendix: Relationship of Route-Based and Link-Based Ideal DUO Route Choice Conditions

Lemma 1: *The link-based ideal DUO route choice conditions (38)-(40) imply the route-based ideal DUO route choice conditions (16)-(18).*

Proof:

We need to prove that under the link-based ideal DUO route choice conditions (38)-(40), for each O-D pair rs, any vehicle flows departing from origin r at time t must arrive at destination s at the same time by using the minimal actual travel time routes.

For simplicity, we first consider the case having only two route departure flows $f_1^{rs}(t) > 0$, $f_2^{rs}(t) > 0$ for one O-D pair rs at time t. It follows that

$$f_1^{rs}(t) + f_2^{rs}(t) = f^{rs}(t) \tag{62}$$

Suppose $f_1^{rs}(t)$, $f_2^{rs}(t)$ take route 1 and route 2, respectively. Route 1 and route 2 are minimal-travel-time routes generated under the link-based ideal DUO route choice conditions. For simplicity, assume that route 1 comprises 4 links: $1 = (r,h), 2 = (h,i), \cdots, 4 = (j,s)$; and route 2 comprises 5 links $5 = (r,k), 6 = (k,l), \cdots, 9 = (m,s)$. Note that route 1 and route 2 may have overlapping links. Also note that this assumption can be generalized to any route with any number of links. Using the link-based ideal DUO route choice conditions (38)-(40), we have

$$u_1^{rs^*}(t) > 0, u_2^{rs}[t + \pi^{rh^*}(t)] > 0, \cdots, u_4^{rs}[t + \pi^{rj^*}(t)] > 0 \tag{63}$$

$$u_5^{rs^*}(t) > 0, u_6^{rs}[t + \pi^{rk^*}(t)] > 0, \cdots, u_9^{rs}[t + \pi^{rm^*}(t)] > 0 \tag{64}$$

This is because route 1 and route 2 are generated under the link-based ideal DUO route choice conditions (38)-(40) so that there are inflows over links $1, 2, \cdots, 9$ over route 1 and route 2. If route 1 and route 2 do not have overlapping links, the inflow on each link over route 1 and 2 is positive at the instant of time when departure flows arrive at the link. It follows that

$$u_{11}^{rs^*}(t) > 0, u_{21}^{rs}[t + \pi^{rh^*}(t)] > 0, \cdots, u_{41}^{rs}[t + \pi^{rj^*}(t)] > 0 \tag{65}$$

$$u_{52}^{rs^*}(t) > 0, u_{62}^{rs}[t + \pi^{rk^*}(t)] > 0, \cdots, u_{92}^{rs}[t + \pi^{rm^*}(t)] > 0 \tag{66}$$

where the second subscripts 1 and 2 represent route 1 and 2, respectively. Note that the instants of time when departure flows arrive at the links are ensured by the link-based ideal DUO route choice conditions (38)-(40). For example, $[t + \pi^{rh^*}(t)]$ is the instant of time when departure flow $f_1^{rs}(t)$ arrives at link 2. In other words, if departure flows $f_1^{rs}(t) > 0$, $f_2^{rs}(t) > 0$ satisfy the link-based ideal DUO route choice conditions (38)-(40), we can obtain equations (65)-(66).

If route 1 and route 2 have overlapping links, (65)-(66) still hold. For example, if link 2 equal link 6 (route 1 and route 2 are overlapping on link 2), $[t+\pi^{rh^*}(t)] = [t+\pi^{rk^*}(t)]$ is the instant of time when departure flows arrive at link 2. Both flows must experience the same link travel time $\tau_2[t+\pi^{rh^*}(t)]$ and exit link 2 at the same time $[t+\pi^{ri^*}(t)]$. Then the inflows on subsequent links over route 1 and route 2 still satisfy equations (65)-(66).

Denote the arrival flows over route 1 and 2 as $e_1^{rs}[t+\pi^{rs^*}(t)]$, $e_2^{rs}[t+\pi^{rs^*}(t)]$, which are associated with departure flows $f_1^{rs}(t)$, $f_2^{rs}(t)$, respectively. Note that route 1 and route 2 are minimal-travel-time routes. Using the route-based flow propagation constraints (14) for the last links $4 = (j, s)$ and $9 = (m, s)$ over route 1 and 2, we obtain that

$$e_1^{rs}[t+\pi^{rs^*}(t)] > 0, \quad e_2^{rs}[t+\pi^{rs^*}(t)] > 0 \tag{67}$$

where

$$\pi^{rs^*}(t) = \pi^{rj^*}(t) + \tau_4[t+\pi^{rj^*}(t)] = \pi^{rm^*}(t) + \tau_9[t+\pi^{rm^*}(t)]$$

Note that

$$u_{11}^{rs^*}(t), u_{21}^{rs}[t+\pi^{rh^*}(t)], \cdots, u_{41}^{rs}[t+\pi^{rj^*}(t)], e_1^{rs}[t+\pi^{rs^*}(t)]$$

and

$$u_{52}^{rs^*}(t), u_{62}^{rs}[t+\pi^{rk^*}(t)], \cdots, u_{92}^{rs}[t+\pi^{rm^*}(t)], e_2^{rs}[t+\pi^{rs^*}(t)]$$

are the two sets of inflows over route 1 and 2, respectively. Since these flows are positive, we conclude that the departure flows $f_1^{rs}(t)$, $f_2^{rs}(t)$ arrive at destination s at the same time $[t+\pi^{rs^*}(t)]$. Thus, the link-based ideal DUO route choice conditions guarantee that for O-D pair rs, flows departing at time t have the same arrival time $[t+\pi^{rs^*}(t)]$.

If we consider a general case having multiple route departure flows $f_p^{rs}(t) > 0$ for any O-D pair rs at time t, the above analysis still applies to any positive departure flow over any route p between O-D pair rs. Therefore, the link-based ideal DUO route choice conditions **(38)-(40)** imply the route-based ideal DUO route choice conditions **(16)-(18).**

Acknowledgments

The authors would like to thank Ding Zhang and Anna Nagurney at the University of Massachusetts at Amherst for their helpful suggestions. Partial support of this research was provided by the Federal Highway Administration through the ADVANCE Demonstration Project, Illinois Department of Transportation, Schaumburg, Illinois.

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