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ABSTRACT

A dispersion relation, including strong turbulence, is derived for short-wavelength longitudinal self-perturbations of bunched electron beams. The effects of radiation damping and quantum excitation are included. Assuming that turbulence stabilizes coherent oscillations allows the derivation of a formula for bunch length which gives excellent fits to the data from SPEAR. As a special case of the formalism, an integral equation which describes the longitudinal self-perturbations of proton beams is derived.

I. INTRODUCTION

Anomalous lengthening of high intensity bunches of stored electrons and positrons has been observed at Orsay, Frascati[1], and Stanford, both with SPEAR I[2] and with SPEAR II[3]. In addition, current-dependent bunch widening has been measured at SPEAR II[3].

These observations have stimulated considerable theoretical effort, and various theories have been proposed to explain them. One group of theories, the equilibrium theories, assumes that the bunch lengthening is due to the modification of the radiofrequency potential well by the self-fields of the bunch. A general formulation of these theories in linear approximation has been given by Pellegrini and Sessler[4]. More recently, Keil[5] and Germain and Hereward[6] have considered the effect on the bunch length of the nonlinear distortion of the potential well due to self-fields. Although these theories provided reasonable fits to the data from Orsay, Frascati, and SPEAR I, they could not fit the data from SPEAR II. In particular, a threshold in current for the effect has been observed, for which the equilibrium theories could provide no explanation. In addition, the equilibrium theories could not explain the bunch widening effect.

Lebedev[7] proposed a theory in which it was assumed that the bunch lengthening effect was due to the excitation of coherent synchrotron oscillations. In response, the energy spread of the bunch increases until the coherent oscillations are stabilized. Because the energy spread increases, this theory predicts bunch widening. Since the coherent oscillations are stable below some critical current, the theory easily accounts for the threshold effect. The dependences on beam current and rf voltage which this theory predicted were in reasonable agreement with the data. However, in this theory, the effect has too strong a dependence on beam energy to fit the data.

Sessler[8] proposed that the bunch lengthening phenomenon is due to the onset of strong turbulence when coherent modes are unstable. As a result, the diffusion coefficient, and thus the energy width, increases until coherent modes are stable. The stability criterion was calculated using the strong turbulence theory of Dupree[9] and others[10-12]. However, the calculation was done for a uniform, i.e. unbunched, beam and the results were applied in an ad hoc manner to bunched beams.

In this paper, we calculate the bunch lengthening effect using one dimensional strong turbulence theory. We include the effect of external rf fields, i.e. bunching. For electrons and positrons, we include damping and quantum excitation due to particle radiation. We also include the equilibrium effect, i.e. the distortion of the rf potential well due to self-fields. As a special case of our formalism, we derive the integral equation for small longitudinal perturbations of proton beams.

In Section II we present the basic equation of strong turbulence theory and discuss its derivation. In Section III, we derive an integral equation for small longitudinal perturbations. In Section IV, we derive the integral equation for proton beams. In Section V, we present the integral equation for electron beams, derive the short-wavelength dispersion relation, and obtain the formula for the bunch length. In Section VI, we compare theory to the data from SPEAR I and from SPEAR II. In Section VII, we discuss further elaborations of the theory.

II. BASIC EQUATION

Let θ denote the angular distance around the machine of a particle, measured from the position of the "synchronous" particle[13]. Let ΔE denote the energy deviation of a particle from the average energy, E. Let f denote the revolution frequency of the bunch. Define

$$w = \Delta E/f \ . \tag{2.1}$$

For a single particle, the equation of motion for θ is[13]

$$\frac{d\theta}{dt} = k_0 w \ , \tag{2.2}$$

where

$$k_0 = \frac{2\pi f^2}{E}\alpha \ , \tag{2.3}$$

and α is the usual momentum compaction factor. The equation of motion for w is[13]

$$\frac{dw}{dt} = 2\pi R_m e\, \mathcal{E}(\theta,t) - \frac{w}{\tau_r} + g(t) - \frac{\Omega^2\theta}{k_0} \ . \tag{2.4}$$

R_m is the radius of the machine, τ_r is the radiation damping time, and Ω is the synchrotron oscillation frequency. The first term in equation (2.4) includes the effect of electric fields with the exception of the radiofrequency fields. The second term represents radiation damping. In the third term, $g(t)$ is a random function with zero mean which represents the effect of quantum excitation. The last term in equation (2.4) gives harmonic synchrotron oscillations at a frequency Ω, where we neglect the anharmonic part. We include in this last term not only the effect of external rf fields, which would give a frequency denoted ω_0, but also the shift in synchrotron frequency due to the self-fields of the bunch.

We let $\psi(\theta,w,t)$ be the distribution function of the particles. We write

$$\mathcal{E}(\theta,t) = \mathcal{E}_C(\theta,t) + \mathcal{E}_T(\theta,t) , \tag{2.5}$$

where $\mathcal{E}_C(\theta,t)$ is the coherent oscillation that we are interested in, and $\mathcal{E}_T(\theta,t)$ is a background field, assumed to be turbulent. If the modes which make up $\mathcal{E}_T(\theta,t)$ are assumed to have random phases, a Fokker-Planck type derivation can be used[14] to show that

$$\frac{\partial\psi}{\partial t} + k_0 w \frac{\partial\psi}{\partial\theta} + 2\pi\, R_m e\, \mathcal{E}_C(\theta,t) \frac{\partial\psi}{\partial w} - \frac{w}{\tau_r}\frac{\partial\psi}{\partial w} - \frac{\Omega^2\theta}{k_0}\frac{\partial\psi}{\partial w} - \frac{\psi}{\tau_r}$$

$$= \frac{D}{f^2}\frac{\partial^2\psi}{\partial w^2} , \tag{2.6}$$

where

$$D = D_Q + D_T . \tag{2.7}$$

In equation (2.7), D_Q is the usual quantum diffusion coefficient[15], and D_T is the diffusion coefficient due to turbulence. We have assumed that D_T is constant, i.e. that we have a broad-band turbulence spectrum. The diffusion coefficient is given[9] as an integral over the spectrum, but, as we will see, we will not have to calculate it.

If $D_T \to 0$ and $\mathcal{E}_C(\theta,t) \to \mathcal{E}(\theta,t)$, equation (2.6) becomes the equation which is usually used to describe electron and positron beams[13]. If $D_Q \to 0$, $\tau_r \to \infty$, it can be shown[14] that equation (2.6) reproduces the first order results of Dupree[9] and others[10-12] in this special case.

The accuracy of equation (2.6) depends on the validity of the random phase approximation for $\mathcal{E}_T(\theta,t)$. In Reference (14) it was shown, using a criterion due to Chirikov[16], that particles undergoing transverse betatron oscillations, moving in a nonlinear potential well, and interacting with very small amplitude longitudinal waves have stochastic trajectories. Because the particles interact with the waves, the particle stochasticity causes a random shifting of wave phases. If we assume, as seems likely, that mode-mode coupling occurs, then the randomness in phase will be communicated to all parts of the spectrum and the random phase approximation seems well justified.

Note that equation (2.6) includes both the effect of turbulence and the effect of distortion of the rf potential well.

III. INTEGRAL EQUATION

In this section we derive an integral equation which describes small perturbations about an equilibrium state.

Equation (2.6) has a steady state solution given by

$$\psi_0(\theta,w) = M \exp\left\{ -\frac{w^2}{2\, w_{rms}^2} - \frac{\theta^2}{2\,\theta_{rms}^2} \right\} , \tag{3.1}$$

where M is a normalization constant and

$$w_{rms}^2 = D\,\tau_r/f^2 , \tag{3.2}$$

$$\theta_{rms}^2 = D\,\tau_r k_0^{\,2}/f^2\Omega^2 . \tag{3.3}$$

Let $\psi = \psi_0 + \psi_1$. Keeping only terms linear in ψ_1 , from equation (2.6) we get

$$\frac{\partial\psi_1}{\partial t} + k_0 w \frac{\partial\psi_1}{\partial\theta} - \left(\frac{w}{\tau_r} + \frac{\Omega^2\theta}{k_0}\right)\frac{\partial\psi_1}{\partial w} - \frac{\psi_1}{\tau_r}$$

$$= \frac{D}{f^2}\frac{\partial^2\psi_1}{\partial w^2} - 2\pi R_m e\, \mathcal{E}_C(\theta,t)\frac{\partial\psi_0}{\partial w} . \qquad (3.4)$$

In order to solve equation (3.4) for ψ_1, we consider the equation

$$\frac{\partial h}{\partial t} + k_0 w \frac{\partial h}{\partial\theta} - \left(\frac{w}{\tau_r} + \frac{\Omega^2\theta}{k_0}\right)\frac{\partial h}{\partial w} - \frac{h}{\tau_r} = \frac{D}{f^2}\frac{\partial^2 h}{\partial w^2} . \qquad (3.5)$$

In terms of the solution to equation (3.5) with initial condition

$$h(\theta,w,t_0;\ \theta_0,w_0,t_0) = \delta(\theta-\theta_0)\,\delta(w-w_0) , \qquad (3.6)$$

the solution of equation (3.4) satisfies

$$\psi_1(\theta,w,t) = \int h(\theta,w,t;\ \theta_0,w_0,0)\ \psi_1(\theta_0,w_0,0)d\theta_0\, dw_0$$

$$- 2\pi R_m e\int_0^t d\tau\int d\theta_0\, dw_0 h(\theta,w,t;\ \theta_0,w_0,\tau)\left(\mathcal{E}_C(\psi_1)\frac{\partial\psi_0}{\partial w}\right)\Bigg|_{\theta_0,w_0,\tau} . \qquad (3.7)$$

We have written $\mathcal{E}_C = \mathcal{E}_C(\psi_1)$ to indicate its dependence on ψ_1. The method of solution that we are using is a generalization of the method of integration over characteristics of the unperturbed equation. Note that the solution of equation (3.5) that satisfies initial condition (3.6) is known[17].

Since equation (3.5) is homogeneous in time, the solution satisfies

$$h(\theta,w,t;\ \theta_0,w_0,\tau) = h(\theta,w,t-\tau;\ \theta_0,w_0,0) . \qquad (3.8)$$

Thus, we can Laplace transform equation (3.7) in time and use the convolution theorem for Laplace transforms to obtain

$$\tilde\psi_1(\theta,w,s) = \int\tilde h(\theta,w,s;\ \theta_0,w_0,0)\ \psi_1(\theta_0,w_0,0)d\theta_0\, dw_0$$

$$- 2\pi R_m e\int d\theta_0\, dw_0\, \tilde h(\theta,w,s;\ \theta_0 w_0,0)\left(\mathcal{E}_C(\tilde\psi_1)\frac{\partial\psi_0}{\partial w}\right)\Bigg|_{\theta_0,w_0,s} , \qquad (3.9)$$

where $(\sim)$ denotes the Laplace transformed function of s. We have assumed that $\mathcal{E}_C$ is a linear functional of ψ_1. Let

$$F(\theta,w,s) = \int \tilde h(\theta,w,s;\ \theta_0,w_0,0)\ \psi_1(\theta_0,w_0,0)d\theta_0\, dw_0 \qquad (3.10)$$

be the initial value term. Let

$$L(\theta,w,\theta_0,s) = \int dw_0\ \tilde h(\theta,w,s;\ \theta_0,w_0,0)\frac{\partial\psi_0}{\partial w}(\theta_0,w_0) . \qquad (3.11)$$

Equation (3.9) can now be written as

$$\tilde\psi_1(\theta,w,s) = F(\theta,w,s) - 2\pi R_m e\int d\theta_0\ L(\theta,w,\theta_0,s)\,\mathcal{E}_C(\tilde\psi_1)\Big|_{\theta_0,s} . \qquad (3.12)$$

We now use

$$\mathcal{E}_C(\theta_0,t) = \frac{fR_m}{\beta c}\int d\theta'\ \lambda(\theta',t)\,G\big(R_m(\theta'-\theta_0)\big) , \qquad (3.13)$$

where βc is the velocity of the bunch and where $\lambda(\theta',t)$, the charge density per unit length, is given by

$$\lambda(\theta',t) = \frac{e}{R_m}\int \psi_1(\theta',w,t)dw \ . \qquad (3.14)$$

Equation (3.13) is a consequence of the linearity of Maxwell's equations. In particular, it is exactly the same form used by Pellegrini and Sessler[4]. G is a Green's function which includes the effect of beam surroundings.

Using equations (3.13) and (3.14), equation (3.12) becomes

$$\tilde{\psi}_1(\theta,w,s) = F(\theta,w,s) - \frac{2\pi R_m f e^2}{\beta c}\int d\theta_0 \, L(\theta,w,\theta_0,s)$$
$$\otimes \int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\int \tilde{\psi}_1(\theta',w,s)dw \ . \qquad (3.15)$$

It can be shown[14] that in the uniform beam case, i.e. in the limit $\Omega \to 0$, equation (3.15) gives a dispersion relation which is identical with that derived by more straightforward means[8].

Since we are concerned with bunches short compared to the machine circumference, we extend the angular integrations in equation (3.15) to the interval $[-\infty, \infty]$. The rapid falloff in angle of $\psi_0(\theta,w)$ makes this an excellent approximation.

Let us define

$$\xi(\theta,s) = \int dw \, \tilde{\psi}_1(\theta,w,s) \ , \qquad (3.16)$$

$$\sigma(\theta,s) = \int dw \, F(\theta,w,s) \ , \qquad (3.17)$$

$$K(\theta,\theta_0,s) = \int dw \, L(\theta,w,\theta_0,s) \ . \qquad (3.18)$$

With these definitions, equation (3.15) implies

$$\xi(\theta,s) = \sigma(\theta,s) - \frac{2\pi R_m f e^2}{\beta c}\int d\theta_0 \, K(\theta,\theta_0,s)\int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\xi(\theta',s) . \qquad (3.19)$$

Let us Fourier transform from θ to p and denote Fourier transforms by a bar;

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - \frac{2\pi R_m f e^2}{\beta c}\int d\theta_0 \, \bar{K}(p,\theta_0,s)\int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\xi(\theta',s) \ . \qquad (3.20)$$

Recall that for any function, f,

$$\int d\theta_0 \, f(\theta_0) = 2\pi \, \bar{f}(0) \ . \qquad (3.21)$$

The last term in equation (3.20) is precisely of this form. Since the integrand of the last term in equation (3.20) is a product, its Fourier transform is a convolution; i.e.

$$\overline{\bar{K}(p,\theta_0,s)\int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\,\xi(\theta',s)}\,\Bigg|_{p,r_1,s}$$
$$= \int dr \, \bar{\bar{K}}(p,r_1-r,s)\,\overline{\int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\,\xi(\theta',s)}\,\Bigg|_{r,s} \qquad (3.22)$$

The convolution theorem is

$$\overline{\int d\theta' G\Big(R_m(\theta' - \theta_0)\Big)\,\xi(\theta',s)}\,\Bigg|_{r,s} = 2\pi \, \overline{G(R_m\theta)}\,\Bigg|_{r} \, \bar{\xi}(r,s) \ . \qquad (3.23)$$

Let us note that

$$\overline{G(R_m\theta)}\Big|_r = \frac{\beta c}{2\pi R_m} Z(-r\omega_0) , \tag{3.24}$$

where

$$\omega_0 = 2\pi f , \tag{3.25}$$

and where $Z(-r\omega_0)$ is the usual impedance function[4].

Using equations (3.21) - (3.24), equation (3.20) becomes

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - (2\pi)^2 fe^2 \int dr\, \bar{\bar{K}}(p,-r,s) Z(-r\omega_0) \bar{\xi}(r,s) . \tag{3.26}$$

From equations (3.11) and (3.18) we find that the equation for $\bar{\bar{K}}(p,-r,s)$ is

$$\bar{\bar{K}}(p,-r,s) = \frac{-M}{(2\pi)^2} \int_0^{\infty} dt\, e^{-st} \int d\theta_0\, d\theta\, dw\, dw_0 \frac{w_0}{w_{rms}^2}$$

$$\otimes \exp\left\{ ir\theta_0 - ip\theta - \frac{w_0^2}{2\, w_{rms}^2} - \frac{\theta_0^2}{2\, \theta_{rms}^2} \right\} h(\theta, w, t;\ \theta_0, w_0, 0) , \tag{3.27}$$

where, for bunched beams,

$$M = \frac{N}{2\pi\, w_{rms}\, \theta_{rms}} , \tag{3.28}$$

where N is the number of particles in the bunch. Equation (3.26) is an integral equation for $\bar{\xi}(p,s)$ with a term, $\bar{\sigma}(p,s)$, from initial values. In order to apply equation (3.26), the impedance, $Z(-r\omega_0)$ and the function, $\bar{\bar{K}}(p, -r, s)$, must be known. Fortunately, in cases of interest, $\bar{\bar{K}}(p,-r,s)$ can be calculated.

IV. BUNCHED PROTON BEAMS WITHOUT TURBULENCE

In this section we consider bunched proton beams in the absence of turbulence. The dispersion relation for proton beams has been considered previously[18,19]. Sacherer[18], in particular, has given a description of the modes in special cases. Previous treatments have not been able to include Landau damping in the general case. In addition, previous treatments have considered only simple impedance functions. In our equations the impedance function enters in a relatively simple way. Thus, more general impedance functions can be assumed. We show how to include Landau damping exactly in the integral equation and present a rough approximation which simplifies the dispersion relation considerably and which is adequate for our purposes.

Since protons don't radiate significantly and since turbulence is assumed absent, we take the limit $D \to 0$, $\tau_r \to \infty$, $D\tau_r$ fixed. In this limit the solution of equation (3.5) which satisfies the initial condition (3.6) is

$$h(\theta, w, t;\ \theta_0, w_0, 0) = \delta\big(\theta - A(\theta_0, w_0, t)\big)\delta\big(w - B(\theta_0, w_0, t)\big) , \tag{4.1}$$

where

$$A(\theta_0, w_0, t) = \theta_0 \cos \Omega t + (k_0 w_0/\Omega) \sin \Omega t , \tag{4.2}$$

$$B(\theta_0, w_0, t) = w_0 \cos \Omega t + (\Omega\theta_0/k_0) \sin \Omega t . \tag{4.3}$$

Using equations (4.1) and (4.2) in equation (3.27), we find

$$\bar{\bar{K}}(p,-r,s) = \frac{i\,N\,k_0\,p}{4\pi\,\Omega} \exp\left\{-\frac{\theta_{rms}^2(p^2+r^2)}{2}\right\}$$

$$\otimes \int_0^{\infty} dt\, e^{-st} \sin \Omega t \exp\left\{\theta_{rms}^2\, rp \cos \Omega t\right\} . \tag{4.4}$$

Now we use the fact that

$$\exp\left\{\theta_{rms}^2\, rp \cos \Omega t\right\} = I_0(\theta_{rms}^2\, rp) + 2 \sum_{k=1}^{\infty} I_k(\theta_{rms}^2\, rp) \cos k\,\Omega t , \tag{4.5}$$

where I_k is the modified Bessel function of the first kind. Equation (4.4) then becomes

$$\bar{\bar{K}}(p,-r,s) = \frac{i\,N\,k_0\,p}{4\pi} \exp\left\{-\frac{\theta_{rms}^2(p^2+r^2)}{2}\right\}$$

$$\otimes \left\{\frac{I_0(\theta_{rms}^2\, rp)}{s^2+\Omega^2} + \sum_{k=1}^{\infty} I_k(\theta_{rms}^2 rp)\left[\frac{k+1}{s^2+\Omega^2(k+1)^2} - \frac{k-1}{s^2+\Omega^2(k-1)^2}\right]\right\} . \tag{4.6}$$

We note that $\bar{\bar{K}}(p,-r,s)$ is large when $s \simeq in\Omega$, $n = \pm 1, \pm 2, \cdots$. Thus, coherent oscillations occur at multiples of the synchrotron frequency, as was to be expected. When $s \simeq in\Omega$, we shall keep only the largest terms. Recalling that

$$I_{n-1}(Z) - I_{n+1}(Z) = \frac{2n}{Z} I_n(Z) , \tag{4.7}$$

we get

$$\bar{\bar{K}}(p,-r,s) = \frac{i\,N\,k_0\,n^2\, I_n(\theta_{rms}^2\, rp)}{2\pi\, r\, \theta_{rms}^2(s^2+n^2\,\Omega^2)} \exp\left\{-\frac{\theta_{rms}^2\,(r^2+p^2)}{2}\right\} . \tag{4.8}$$

We have not, up to this point, included Landau damping since we have assumed linear binding forces. Landau damping results from the spread in synchrotron frequencies due to nonlinear binding forces. In order to include it exactly, we need only make the substitutions in equations (3.27) and (4.1)

$$\frac{-M\,w_0}{w_{rms}^2} \exp\left\{\frac{-w_0^2}{2\,w_{rms}^2} - \frac{\theta_0^2}{2\,\theta_{rms}^2}\right\} \rightarrow \frac{\partial\psi_0(\theta_0,w_0)}{\partial w} , \tag{4.9}$$

$$A(\theta_0,w_0,t) \rightarrow A'(\theta_0,w_0,t) , \tag{4.10}$$

where $A'(\theta_0,w_0,t)$ is the solution for $\theta(t)$ with the force under consideration. If we make these substitutions, it is no longer possible, in general, to do all of the integrals in equation (3.27). It can be shown[14], however, that a fair approximation results if we make the substitution

$$s \rightarrow s + \delta_L , \tag{4.11}$$

where

$$\delta_L = \frac{p\,\theta_{rms}\,(\Delta\,\Omega)_{rms}}{2} . \tag{4.12}$$

In equation (4.12), $(\Delta\,\Omega)_{rms}$ is the root mean square synchrotron frequency spread in the bunch. The substitution (4.11) is probably not adequate for detailed applications to proton beams, and, if not, the substitutions (4.9) and (4.10) must be used in conjunction with equations (3.26) and (3.27) to obtain the dispersion relation. For electron beams, diffusion damping dominates and the approximation is adequate.

If we do use the approximation (4.11), equation (3.26) becomes, using equation (4.8),

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - \frac{2\pi\, f\, e^2\, i\, N\, k_0\, n^2}{\theta_{rms}^2\left[(s+\delta_L)^2 + n^2\,\Omega^2\right]}$$

$$\otimes \int dr\; I_n(\theta_{rms}^2 rp)\, \exp\left\{-\frac{\theta_{rms}^2(r^2+p^2)}{2}\right\} \frac{Z(-r\omega_0)}{r}\; \bar{\xi}(r,s)\;. \tag{4.13}$$

The singularity at $r = 0$ is only apparent, since $Z(0) = 0$. Since modes are confined to the bunch, it is probably a good approximation to take

$$pr\theta_{rms}^2 \gtrsim (2\pi)^2\;. \tag{4.14}$$

We can then use the large argument approximation for the Bessel function

$$I_n(Z) \simeq \frac{e^Z}{\sqrt{2\pi Z}}\;. \tag{4.15}$$

Equation (4.13) then becomes

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - \frac{\sqrt{2\pi}\;\; f\, e^2\, i\, N\, k_0\, n^2}{\theta_{rms}^3\left[(s+\delta_L)^2 + n^2\,\Omega^2\right]}$$

$$\otimes \int dr\; \exp\left\{\frac{-\theta_{rms}^2(p-r)^2}{2}\right\} \frac{Z(-r\omega_0)}{p^{1/2}\, r^{3/2}}\; \bar{\xi}(r,s)\;. \tag{4.16}$$

For special assumptions about $Z(-r\omega_0)$ equation (4.16) is easy to solve. We will not attempt to solve equation (4.16) because of the necessity of a more accurate treatment of Landau damping in the proton beam case.

V. ELECTRON AND POSITRON BEAMS

In this section we derive the short-wavelength dispersion relation for longitudinal perturbations of electron and positron beams. We then assume that turbulence has sufficiently lengthened the bunch so that coherent oscillations are stable. Thus, we obtain a formula for bunch length.

In order to derive the dispersion relation, we need to know the function $\bar{\bar{K}}(p,-r,s)$. In Reference (14), a formula for $\bar{\bar{K}}(p,-r,s)$ was derived. Let

$$\omega = \Omega\sqrt{1 - \frac{1}{4\,\tau_r^2\,\Omega^2}}\;, \tag{5.1}$$

$$x = \sin^{-1}\frac{1}{2\,\Omega\,\tau_r}\;. \tag{5.2}$$

Then, a long, but straightforward, derivation shows that

$$\bar{\bar{K}}(p,-r,s) = \frac{i\, N\, p\, k_0}{4\pi\,\omega} \exp\left\{\frac{-\,\theta_{rms}^2(p^2+r^2)}{2}\right\}$$

$$\otimes \int_0^\infty dt\; \exp\left\{-st - \frac{t}{2\tau_r}\right\} \sin\omega t\; \exp\left\{\frac{\theta_{rms}^2 rp\Omega\; e^{-t/2\tau_r}\cos(\omega t - x)}{\omega}\right\}. \tag{5.3}$$

Note that in the limit $D \to 0$, $\tau_r \to \infty$, $D\tau_r$ fixed, equation (5.3) goes over into equation (4.4).

Since $\tau_r \Omega >> 1$ in all cases that we are interested in, we expand to first order in $1/2\tau_r\Omega$. Then, using the Bessel function expansion of the exponential, we find

$$\bar{\bar{K}}(p,-r,s) = \frac{i\,N\,p\,k_0}{4\pi\,\Omega} \exp\left\{\frac{-\theta^2_{rms}(p^2+r^2)}{2}\right\}$$

$$\otimes \int_0^\infty dt \exp\left\{-st - \frac{t}{2\tau_r}\right\} \sin \Omega t \left\{ I_0(\theta^2_{rms}\, rp\, e^{-t/2\tau_r}) \right.$$

$$\left. + 2 \sum_{\ell=1}^{\infty} I_\ell(\theta^2_{rms}\, rp\, e^{-t/2\tau_r}) \left[\cos \ell\Omega t + \ell x \sin \ell\Omega t\right]\right\}. \tag{5.4}$$

Since we expect resonances near $s \simeq in\Omega$, we get, keeping only the dominant terms

$$\bar{\bar{K}}(p,-r,s) = \frac{i\,N\,p\,k_0}{4\pi\,\Omega} \exp\left\{\frac{-\theta^2_{rms}(p^2+r^2)}{2}\right\}$$

$$\otimes \int_0^\infty dt \exp\left\{-st - \frac{t}{2\tau_r} + in\Omega t\right\}\left\{\frac{1}{2i}\left[I_{n-1}(\theta^2_{rms} rp\, e^{-t/2\tau_r})\right.\right.$$

$$\left. - I_{n+1}(\theta^2_{rms} rp\, e^{-t/2\tau_r})\right] + \frac{x}{2}\left[(n+1) I_{n+1}(\theta^2_{rms} rp\, e^{-t/2\tau_r})\right.$$

$$\left.\left. - (n-1) I_{n-1}(\theta^2_{rms}\, rp\, e^{-t/2\tau_r})\right]\right\}. \tag{5.5}$$

We expect, once again, that

$$\theta^2_{rms}\, rp >> 1 .$$

Thus, for times $t \lesssim 6\,\tau_r$, a good approximation is

$$I_n(\theta^2_{rms}\, rp\, e^{-t/2\tau_r}) \simeq \frac{e^{t/4\tau_r}}{\sqrt{2\pi\,\theta^2_{rms}\, rp}} \exp\left\{\theta^2_{rms}\, rp\, e^{-t/2\tau_r}\right\}. \tag{5.6}$$

As $t \to \infty$, the left-hand side of equation (5.6) goes to zero, while the right-hand side goes to infinity. A better approximation results if we expand the exponentional inside the exponential. Thus,

$$I_n(\theta^2_{rms}\, rp\, e^{-t/2\tau_r}) \simeq \frac{e^{t/4\tau_r}}{\sqrt{2\pi\,\theta^2_{rms}\, rp}} \exp\left\{\theta^2_{rms}\, rp\left(1 - \frac{t}{2\tau_r}\right)\right\}. \tag{5.7}$$

For times, $t \lesssim 6\tau_r$, equation (5.7) is a good approximation. Also, as $t \to \infty$, both sides of equation (5.7) go to zero. Since $\theta^2_{rms} rp/\tau_r >> 1/\tau_r$, we drop all terms in t/τ_r. Thus, equation (5.5) becomes

$$\bar{\bar{K}}(p,-r,s) \simeq \frac{i\,N\,p\,k_0}{4\pi\,\Omega} \exp\left\{-\frac{\theta^2_{rms}}{2}(p-r)^2\right\}$$

$$\otimes \int_0^\infty dt \exp\left\{-st + in\,\Omega\, t - \frac{D\,k_0^{\,2}\, rp}{2f^2\,\Omega^2}\, t\right\}$$

$$\otimes \left\{\frac{-i\,n}{\sqrt{2\pi}\;\theta^3_{rms}\,(rp)^{3/2}} + \frac{x}{\sqrt{2\pi\, rp\,\theta^2_{rms}}}\right\}. \tag{5.8}$$

Note that the diffusion damping term in equation (5.8) is the same as that derived by Dupree[20] for a magnetized plasma. Thus, our theory reproduces the first order results of previous strong turbulence theories.

Performing the integration in equation (5.8), and substituting the result in equation (3.26), we obtain

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - \sqrt{\frac{\pi}{2}}\ \frac{e^2 N f i p k_0}{\Omega\ \theta_{rms}} \int dr \frac{\exp\left\{-\frac{\theta_{rms}^2}{2}(r-p)^2\right\}}{\sqrt{rp}}$$

$$\otimes \frac{Z(-r\omega_0)\ \bar{\xi}(r,s)}{s - in\Omega + \frac{D k_0^2 rp}{2f^2 \Omega^2}} \left[\frac{1}{2\tau_r \Omega} - \frac{in}{rp\ \theta_{rms}^2}\right] . \tag{5.9}$$

For short-wavelength modes, a good approximation is

$$\exp\left\{-\frac{\theta_{rms}^2}{2}(p-r)^2\right\} \simeq \sqrt{\frac{2\pi}{\theta_{rms}^2}}\ \delta(p-r) . \tag{5.10}$$

Using this, equation (5.9) is approximately

$$\bar{\xi}(p,s) = \bar{\sigma}(p,s) - \frac{\pi e^2 N f i k_0}{\Omega\ \theta_{rms}^2} \frac{Z(-p\omega_0)\ \bar{\xi}(p,s)}{s - in\Omega + \frac{D k_0^2 p^2}{2f^2 \Omega^2}}$$

$$\otimes \left[\frac{1}{2\tau_r \Omega} - \frac{in}{p^2\ \theta_{rms}^2}\right] . \tag{5.11}$$

Thus, the dispersion relation is

$$s = in\Omega - \frac{D k_0^2 p^2}{2f^2 \Omega^2} - \frac{\pi e^2 N f i k_0 Z(-p\omega_0)}{\Omega\ \theta_{rms}^2}\left[\frac{1}{2\tau_r \Omega} - \frac{in}{p^2 \theta_{rms}^2}\right] . \tag{5.12}$$

Stability occurs when $\mathrm{Re}(s) \leqslant 0$.

Let us note several things about equation (5.12). First, both resistance, $\mathrm{Re}(Z) \neq 0$, and inductance, $\mathrm{Im}(Z) \neq 0$, can cause instability. Previously[18),19)], only the resistive instability has been found, because only proton beams have been considered. For electron and positron beams an additional dissipative process, namely radiation damping, is present, hence instability can arise from the inductive part of Z . In fact, the inductive term decreases less rapidly with beam energy than the resistive term, because of the increase in radiation damping. Thus, while the resistive term, considered by Lebedev[7)], has a strong dependence on energy, the inductive term has a dependence on energy which is much weaker. Let us also observe that not only $\mathrm{Im}(Z)$ but also $\mathrm{Re}(Z)$ contributes to a shift in frequency and should be included in the formula for Ω . At high energy, the resistive frequency shift dominates. This suggests that calculations of the equilibrium lengthening effect should include a contribution from resistance. Previous theories[4)] have only included the inductive bunch lengthening.

It is clear that it would be difficult to include Landau damping in the calculation. However, we will assume that Landau damping is adequately represented by equations (4.11) and (4.12). It then turns out that in the cases that we are interested in Landau damping is small compared to diffusion damping and can be neglected.

For proton beams, on the other hand, Landau damping is the only damping mechanism and must be carefully treated.

The stability threshold, which is the formula for bunch length, is, from equation (5.12),

$$\frac{p^2 \theta^4_{rms}}{2\,\tau_r} = \frac{\pi\, e^2\, N\, f\,|k_0|}{\Omega} \left[\frac{|\,\mathrm{Im}(Z)\,|}{2\,\tau_r\,\Omega} + \frac{n\,|\,\mathrm{Re}\,Z\,|}{p^2\,\theta^2_{rms}} \right] . \tag{5.13}$$

Equation (5.13) should be applied only when the bunch length predicted is greater than that calculated with turbulence absent, since in that case coherent oscillations are stable. Note that the equilibrium effect, i.e. distortion of the rf potential well, is included in equation (5.13) since Ω , and not Ω_0 , appears.

VI. COMPARISON WITH EXPERIMENTS

In order to apply equation (5.13), we must specify p and $Z(-p\omega_0)$. In principle, $Z(-p\omega_0)$ is determined by the details of the machine. However, the calculation or measurement of $Z(-p\omega_0)$ in real situations is very difficult. Thus, we will assume that

$$\frac{Z(-p\omega_0)}{p} = \text{constant} . \tag{6.1}$$

This is physically reasonable for high frequency resonant structures. We will also assume that

$$p\,\theta_{rms} = \text{constant} . \tag{6.2}$$

Since disturbances are confined to the bunch, the wavelength of a mode changes in proportion to bunch length. Thus, we obtain equation (6.2).

We also need an equation for Ω . We will use the expression of Pellegrini and Sessler[4])

$$\left(\frac{\Omega_0}{\Omega}\right)^2 = 1 + \frac{kI}{E^3 \Delta} , \tag{6.3}$$

where I is the current in the bunch, k is a constant, and Δ is the bunch length. As noted earlier, equation (6.3) does not include resistance-induced equilibrium bunch lengthening. However, in the absence of a theory which does, we must use equation (6.3). For SPEAR II, as we will see, this makes no difference. The impedance in equation (6.1) and that included in the constant k may be quite different since they correspond to different frequency ranges.

Let us define

$$\theta^2_{rmso} = \frac{D_Q\,\tau_r\,k_0^2}{f^2\,\Omega_0^2} . \tag{6.4}$$

Then our formula for bunch length implies

$$R^3 = \left(\frac{\theta_{rms}}{\theta_{rmso}}\right)^3 = \frac{2\pi\, e^2\, N\, f\,|k_0|\,\tau_r}{\theta^3_{rmso}\,\Omega\, c} \left(\frac{A}{2\tau_r\,\Omega} + B \right) , \tag{6.5}$$

where A and B are dimensionless constants to be determined. R is the ratio of predicted bunch length to natural bunch length. In equation (6.5) there are three constants to be determined, A, B, and k which enters through equation (6.3).

Equation (6.5) can be fit to the data from SPEAR I[2]). The results are shown in figures (1 - 3). Solid lines represent equation (6.5), dashed lines represent the bunch lengthening due to potential distortion. The three circled points were used to determine the three constants. The values are

$$k = 0.35 \text{ GeV-ns/mA}$$

$$A = 2.39 \times 10^{-4}$$

$$B = 8.6 \times 10^{-8} . \qquad (6.6)$$

At the higher energies, in fig. (1), turbulence is absent. At 1.5 GeV and at the higher rf voltages, in fig. (2), turbulence is responsible for a significant part of the effect. At the lower rf voltages, in fig. (3), turbulence is less significant. As can be seen, the fit is quite good. Only in fig. (3) is there significant deviation from the fit. The experimentalists reported that when half the kicker magnets were removed, the bunch lengthening effect was reduced by about 50% at lower voltages, but only by about 20% at higher voltages. If we assume that the ferrite magnets are responsible for the equilibrium lengthening but not for the turbulence, then these observations are perfectly consistent with figs. (2) and (3). All of the ferrite magnets were removed in SPEAR II.

The magnitude of the impedances that equation (6.6) implies depends on a further assumption about p . However, assuming that the wavelength is about one tenth of the bunch length,

$$\frac{|\mathrm{Re}(Z)|}{p} \sim \frac{|\mathrm{Im}(Z)|}{p} \sim 1 \text{ ohm} . \qquad (6.7)$$

This is a rather reasonable value.

On SPEAR II the bunch width, and thus the energy width, has been accurately measured. A comparison of the energy width increase and the bunch length increase is shown[3] in fig. (4). The results are consistent with the effect being entirely due to turbulence.

The fit to the data[3] from SPEAR II is shown in figs. (5) and (6). The values of the constants are

$$k = 0$$

$$A = 4.2 \times 10^{-4}$$

$$B = 4 \times 10^{-7} . \qquad (6.8)$$

The circled points were used to determine A and B . We took $k = 0$ since equilibrium effects seemed absent. From equation (6.8) we see that the equilibrium effect was eliminated but the turbulence was slightly increased in the transition from SPEAR I to SPEAR II.

The points at 3 GeV, in fig. (6), are high at higher currents, but this is probably due to inadequate control of the rf system, since the synchrotron frequency was observed to vary. Note, by the way, the threshold effect at 3 GeV; a phenomenon that seems to confirm the choice $k = 0$.

The points at 1.5 GeV and 0.6 MV, in fig. (5), reveal a definite saturation phenomenon. There are at least two possible theoretical reasons for this phenomenon. The first is that it is due to Landau damping. The approximate treatment of Landau damping that we presented cannot explain the saturation. However, since it is probable that the nonlinearity of the potential well increases with current[5),6)] due to self-fields, the Landau damping probably increases also and may lead to saturation. The second theoretical reason is that the saturation phenomenon is due to mode-mode coupling. We have discarded mode-mode coupling terms, but it is possible that they contribute to saturation. Prior to calculation we do not know if Landau damping or mode-mode coupling, or some combination of them contributes to saturation.

For PEP, if we assume impedances per unit length which are similar to those found at SPEAR I and SPEAR II, we find

$$R^3 = 1.1 \times 10^{-2} I' \tag{6.9}$$

where I' is the current in milliamps. Thus $R \geq 1$ implies

$$I \geqslant 100 \text{ mA} . \tag{6.10}$$

In equation (6.10), we have included only turbulent bunch lengthening. Since PEP plans to operate at currents per bunch lower than that given by equation (6.10) bunch lengthening due to turbulence should be no problem. This result emphasizes the importance of developing an equilibrium theory which includes resistive effects along with nonlinear effects.

VII. DISCUSSION

The theory that we have presented provides quite good fits to almost all the data. We have seen that, depending on the impedances of the machine, either potential distortion or turbulence or both may cause bunch lengthening.

The probable absence of turbulent bunch lengthening in PEP makes it important to develop completely the equilibrium theory. In

order to understand the saturation phenomenon observed in SPEAR II, it is necessary to develop the theory of the effect of nonlinear potential distortion on Landau damping. An understanding of Landau damping is also required for applications of the theory to proton beams.

FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Fig. 1. Ratio of rms measured bunch length to zero current rms bunch length, R, as a function of beam current, I, for two energies and RF voltages on SPEAR I. Squares are experimental points, solid lines are the theoretical fit. The circled points in Figs. (1A), (2A), and (3A) were used to fit three constants in the theory [see Eq.(6.5)]. At these energies the effect is due solely to potential well distortion with no contribution from turbulence.

Fig. 2. Same as Fig. 1 but with $E = 1.5$ GeV and two different RF voltages. The solid lines are the theoretical fit including turbulence, the dashed lines are the fit without turbulence. As can be seen, the contribution of turbulence is significant.

Fig. 3. Same as Fig. 2 but with two different RF voltages. Turbulence is less significant at these lower RF voltages.

Fig. 4. The ratio of rms bunch length to zero current rms bunch length, R, and the ratio of rms energy spread to zero current rms energy spread, T, as a function of current for SPEAR II. The solid line is a fit by hand to the data. These data indicate that the contribution of potential well distortion on SPEAR II is negligible; the effect is entirely due to turbulence.

Fig. 5. R as a function of I at $E = 1.5$ GeV and three different RF voltages. The squares are experimental points, and the solid lines are the theoretical fit assuming only turbulence. The circled points in Figs. (5A) and (6A) were used to fit the two constants in the theory [see Eq. (6.5)]. In (C), a saturation phenomenom, which is discussed in the text, occurs.

Fig. 6. Same as Fig. 5, but with two different energies and RF voltages. In (A), the data at the higher currents are probably inaccurate due to poor control of the RF system.

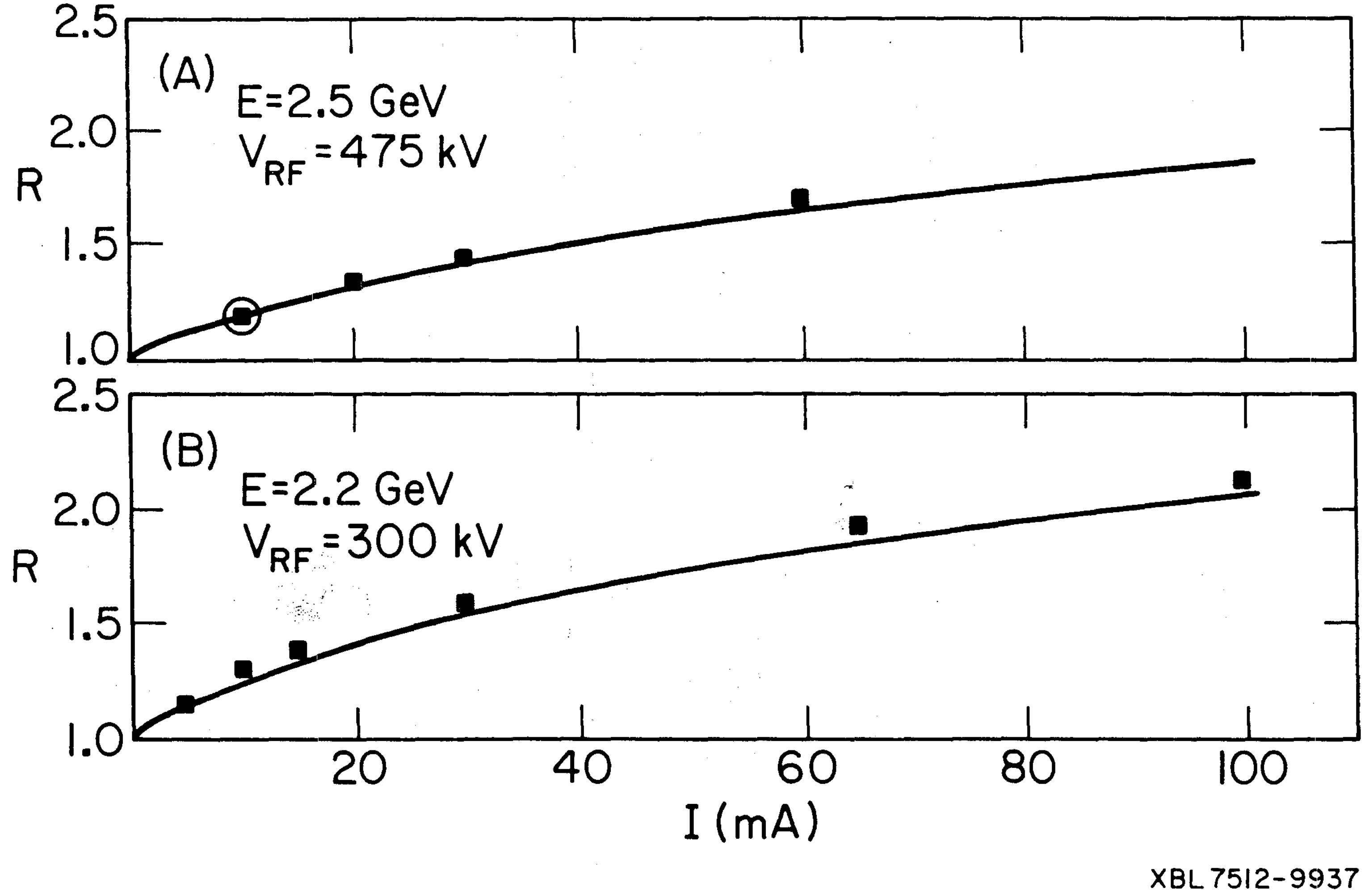
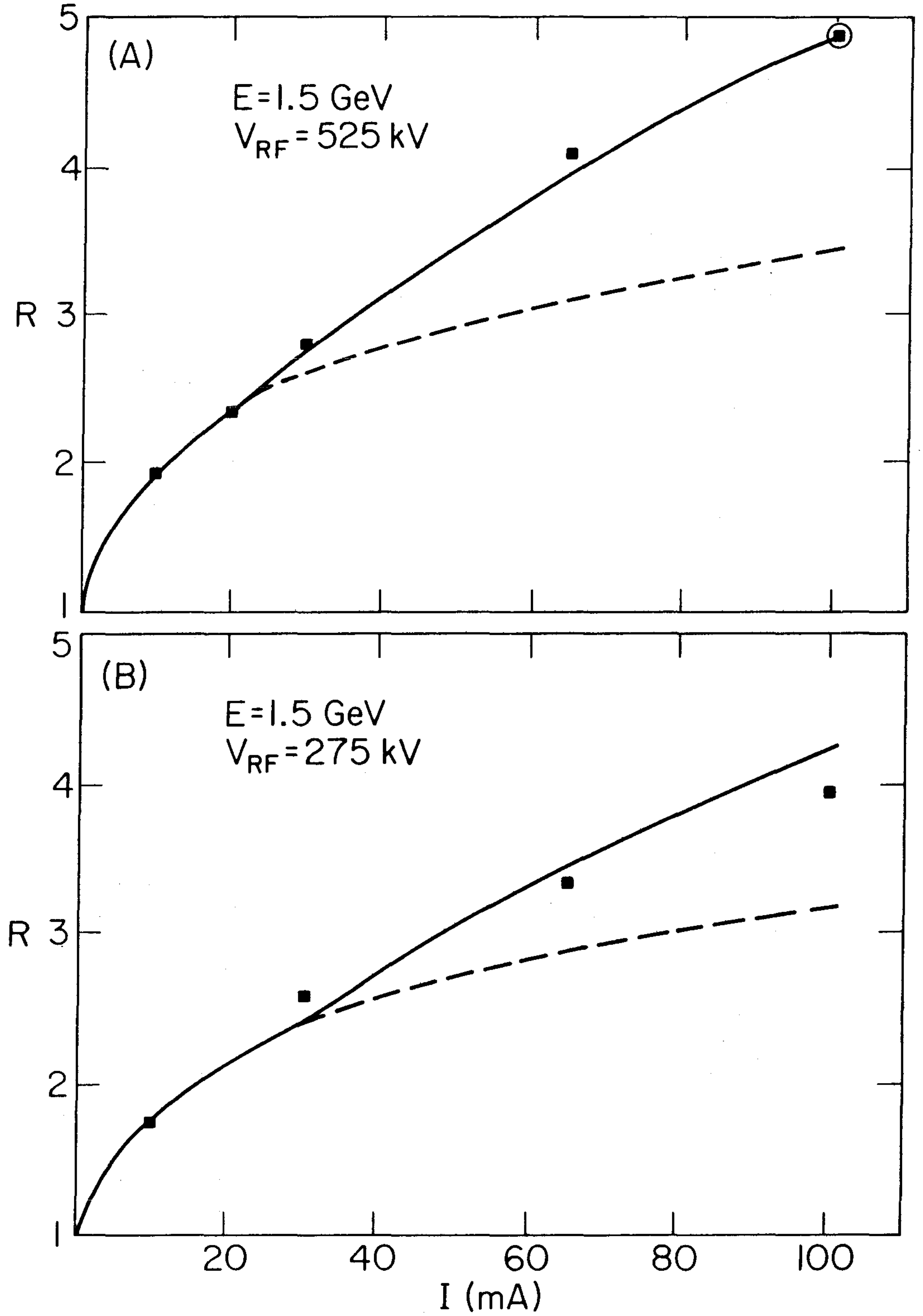
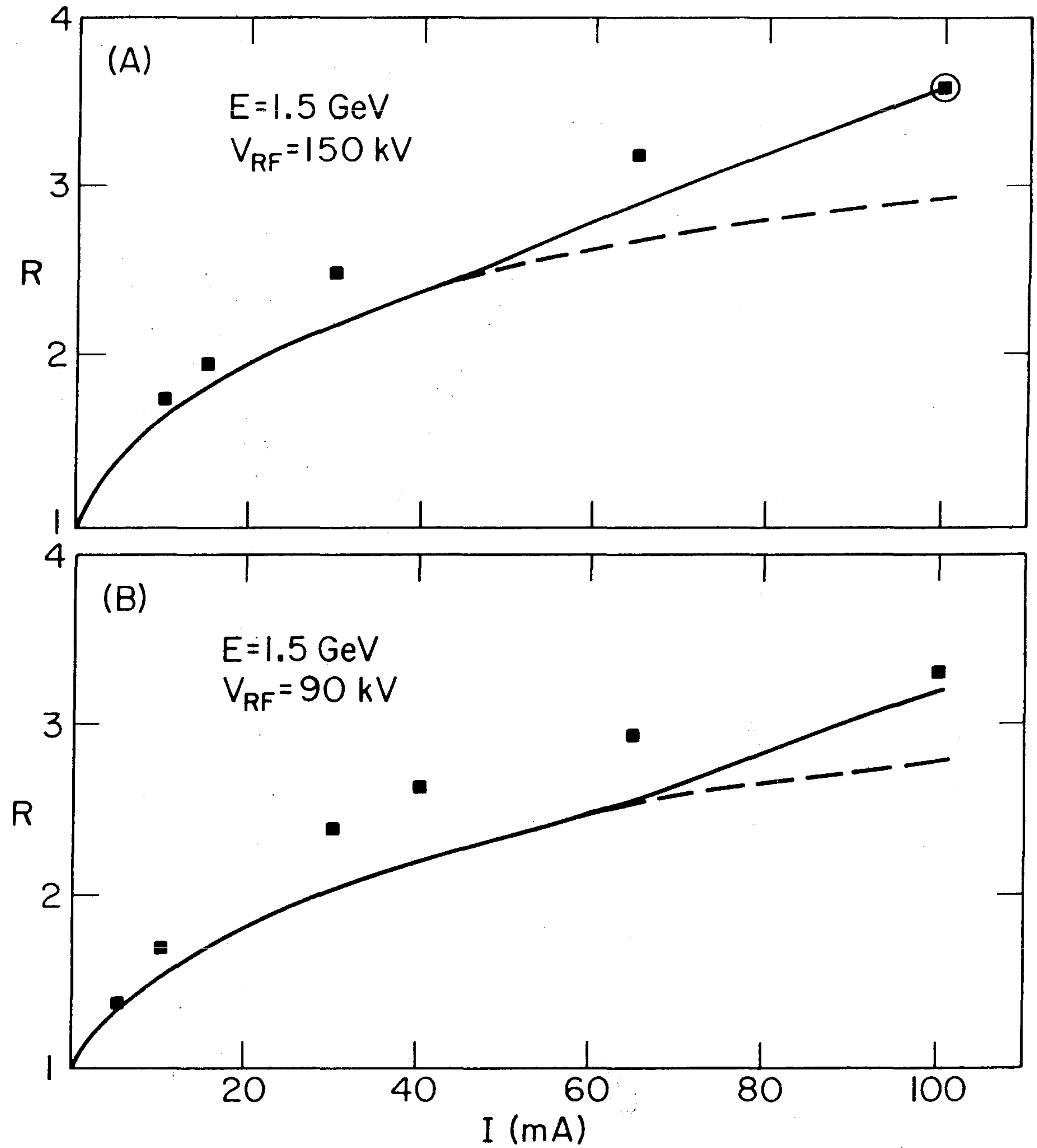

2.5
2.0
1.5
1.0
R
(A)
E=2.5 GeV
V_{RF}=475 kV
2.5
2.0
1.5
1.0
R
(B)
E=2.2 GeV
V_{RF}=300 kV
20
40
60
80
100
I (mA)
XBL 7512-9937

Fig. 1



XBL 7512-9935

Fig. 2



XBL 7512-9938

Fig. 3

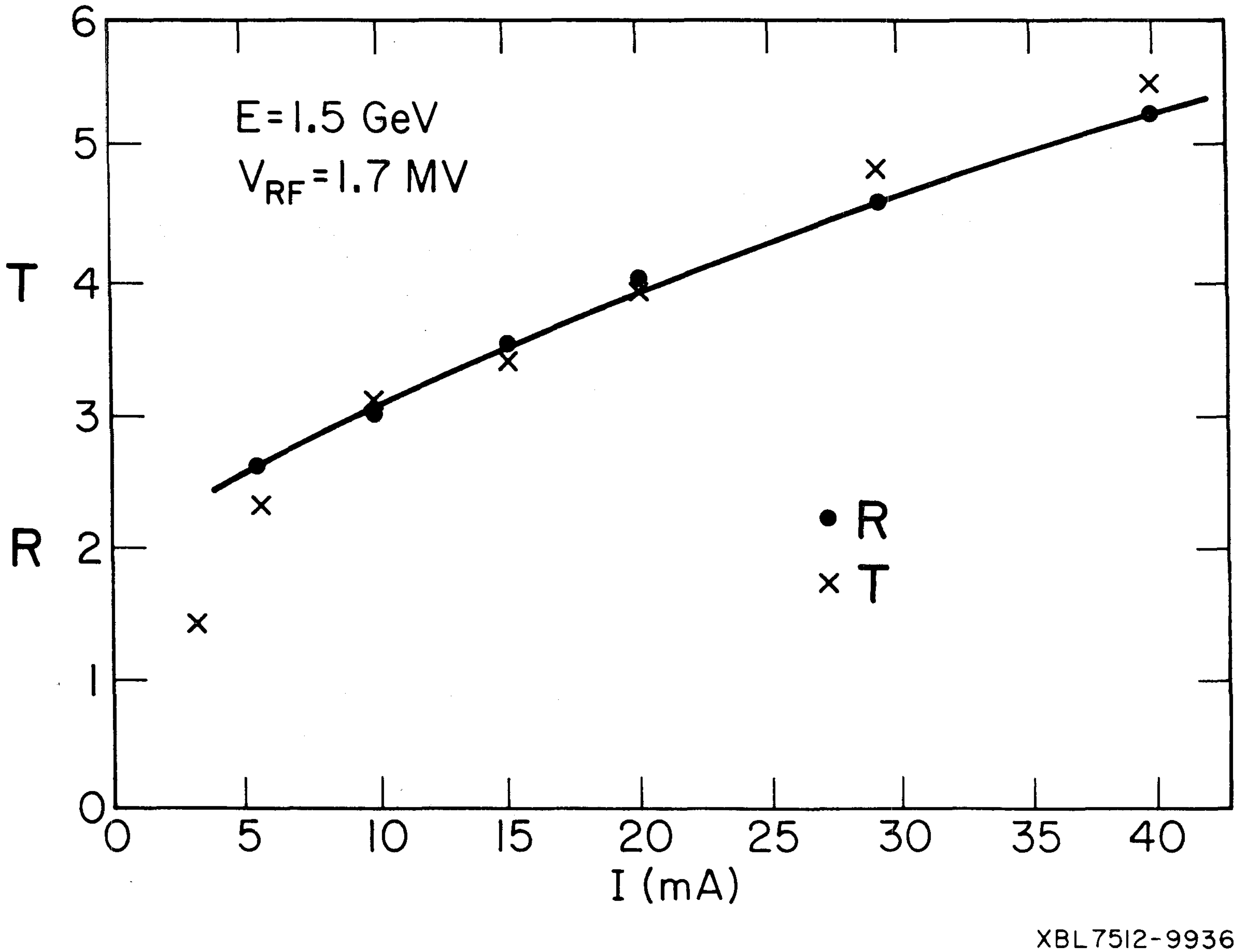

E=1.5 GeV
V_{RF}=1.7 MV
T
R
● R
× T
0 1 2 3 4 5 6
0 5 10 15 20 25 30 35 40
I (mA)
XBL7512-9936

Fig. 4

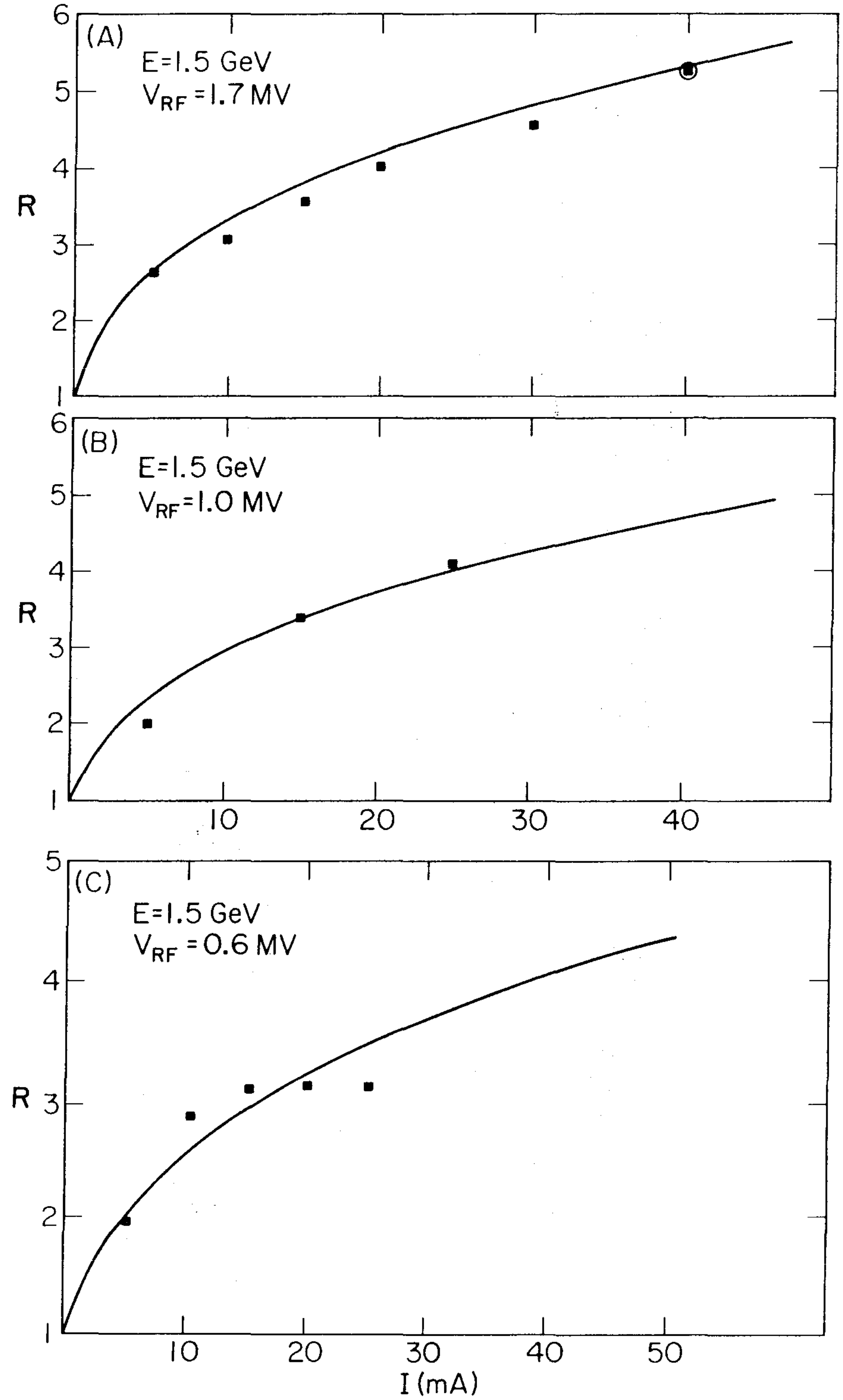
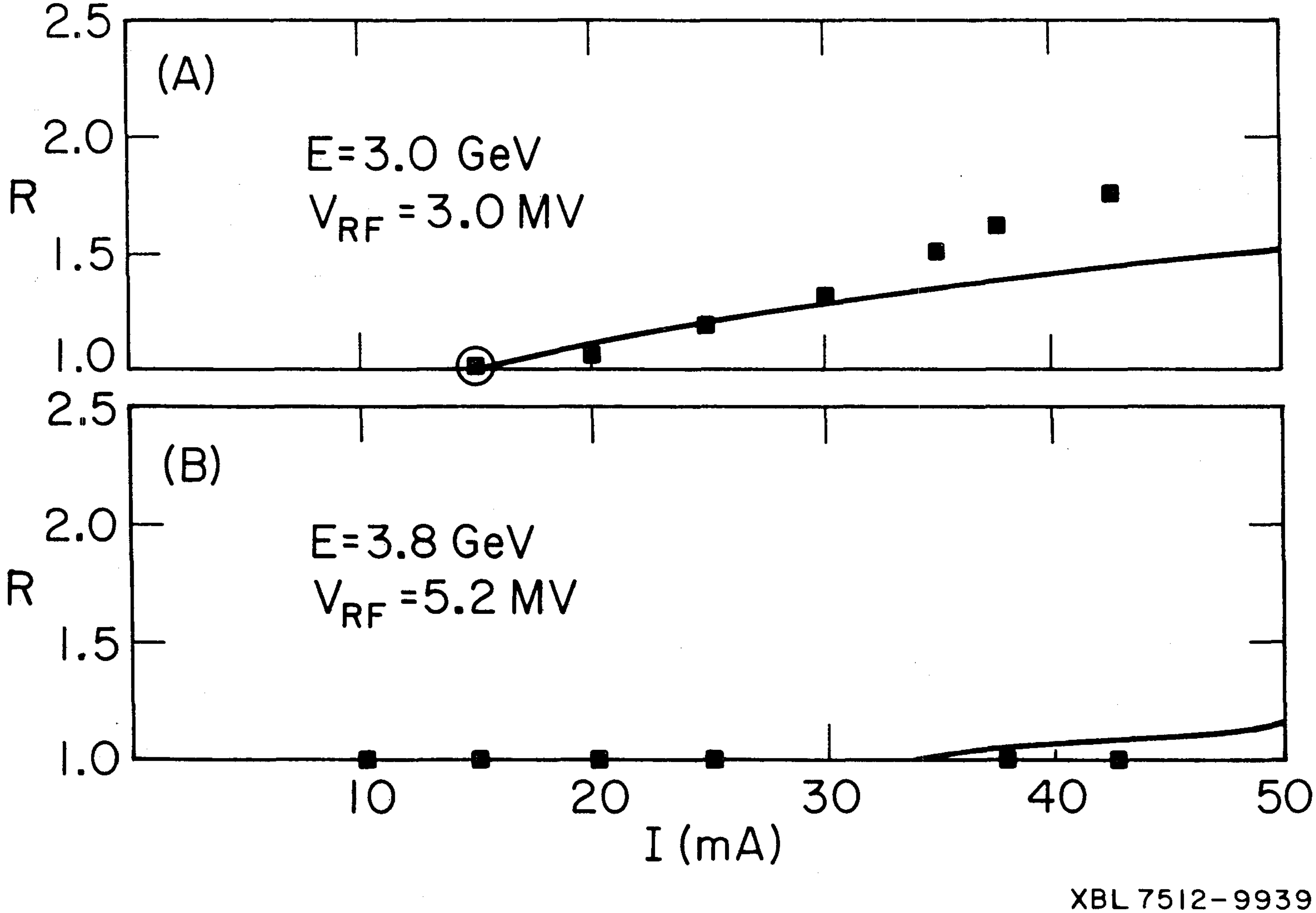

(A)
E=1.5 GeV
V_{RF}=1.7 MV
(B)
E=1.5 GeV
V_{RF}=1.0 MV
(C)
E=1.5 GeV
V_{RF}=0.6 MV
R
I (mA)
10
20
30
40
50
XBL 7512-9934

Fig. 5



2.5
2.0
1.5
1.0
R
(A)
E=3.0 GeV
V_RF = 3.0 MV
2.5
2.0
1.5
1.0
R
(B)
E=3.8 GeV
V_RF =5.2 MV
10
20
30
40
50
I (mA)
XBL 7512-9939

Fig. 6