Algorithmic Mechanism Design in Dynamic Environments

by

Christos Alexandros Psomas

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Computer Science

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Christos H. Papadimitriou, Chair
Professor Shachar Kariv
Professor Satish Rao
Professor Jean Walrand

Spring 2017

Algorithmic Mechanism Design in Dynamic Environments

Abstract

Algorithmic Mechanism Design in Dynamic Environments

by

Christos Alexandros Psomas

Doctor of Philosophy in Computer Science

University of California, Berkeley

Professor Christos H. Papadimitriou, Chair

Over the past few decades, a new field has emerged from the interaction between Computer Science and Game Theory: Algorithmic Mechanism Design. The field has seen tremendous growth and has been extremely successful in tackling a wide variety of problems. Despite all this progress, the vast majority of the literature to date focuses on static, one-time decisions. In many situations of interest, however, this simplification is too far from reality. For example, a search engine must choose how to allocate its advertising inventory in the face of changing search queries and advertiser budgets. In a cloud computing center resources need to be dynamically reallocated in response to the arrival of new computational tasks of varying priority. This thesis explores the interplay of incentives and the dynamic nature of decision-making in the design of efficient mechanisms.

In the first part of this thesis we study Dynamic Auction Design. We introduce a novel class of dynamic auction problems in which a monopolist is selling m items in m consecutive stages to n buyers. We study these problem from several different perspectives: *Computational Complexity*, i.e. how hard is it to compute the optimal auction, *Competition Complexity*, i.e. how much additional competition is necessary for a standard Vickrey (second-price) auction at every stage to extract more revenue than the optimal auction, *Power of Adaptivity*, i.e. what is the revenue gap between adaptive and non-adaptive auctions, *Power of Commitment*, i.e. what happens if the seller cannot commit to her future behavior.

In the second part of this thesis we study Dynamic Fair Division. We introduce a novel class of resource allocation problems in which resources are shared between agents dynamically arriving and departing over time. For a single resource, when n agents are present, there is only one truly “fair” allocation: each agent receives $1/n$ of the resource. Implementing this static solution over time is notoriously impractical. There are too many disruptions to existing allocations: for a new agent to get her fair share, all other agents must give up a small piece. What if, at every c arrivals we could only reclaim resources from a fixed number of agents d? We provide non-wasteful such algorithms that are almost optimal with respect to fairness, even for multiple, heterogeneous resources.

To my parents

Fotis and Alexandra

Contents

List of Figures

List of Tables

Acknowledgments

First and foremost, I wish to thank my advisor Christos from the bottom of my heart. Being his student in the past five years at Berkeley has been an honor and a privilege. I am indebted to him for taking me under his wing and for being there for me as a mentor and as a friend. I aspire to be more like him every day.

I want to thank Eric Friedman and Nikhil Devanur for the amazing summers I spent with them. I am grateful for their advice and support. I am also grateful to Vangelis Markakis for introducing me to theoretical computer science research during my undergraduate studies, and without whom I would have never been admitted to Berkeley in the first place. Thanks to Satish Rao for teaching me all I know about teaching.

This thesis would not be possible without my collaborators: Frank Ban, the JBC, Nikhil Devanur, Eric Friedman, Ali Ghodsi, Vasilis Gkatzelis, Nima Haghpanah, Zhiyi Huang, Kamal Jain, Thomas Kesselheim, Siqi Liu, Vangelis Markakis, Christos Papadimitriou, Sergios Petridis, George Pierrakos, Aviad Rubinstein, Scott Shenker, Shai Vardi, and Matt Weinberg.

Thanks to all the theory graduate students in Berkeley for creating an awesome environment: Nima Anari, Frank Ban, Antonio Blanca, Benjamin Caulfield (Evil Ben), the JBC, Siu Man Chan, Siu On Chan, Paul Christiano, Lynn Chua, James Cook, Anindya De, David Dinh, Raf Frongillo, Rishi Gupta, Fotis Iliopoulos, Marc Khoury, Jingcheng Liu, Urmila Mahadev, Pasin Manurangsi, Peihan Miao, George Pierrakos, Aviad Rubinstein, Manuel Sabin, Aaron Schild, Tselil Schramm, Jarett Schwartz, Jonah Sherman, Seung Woo Shin, Akshay Srinivasan, Piyush Srivastava, Isabelle Stanton, Ning Tan, Di Wang, Benjamin Weitz (Good Ben), Chris Wilkens, Sam Wong, TongKe Xue and those who I undoubtedly forgot to mention; you might find consolation in the fact that I will feel terribly guilty once I realize.

I am indebted to Stelios Kypriotis, my rowing coach in Greece, for being a force of nature in my non-academic life.

I want to thank my family and friends for their endless love and support during my PhD. Most importantly, I want to thank Kira Weinberg for putting up with the crazy person that is me. Without you none of this really matters.

Chapter 1

Introduction

In traditional algorithm design we are given an input and we must design an efficient algorithm that produces a desired output. A subtle but very important assumption is that the input is accurate. What happens when the input is controlled by selfish agents that have their own preferences over the outputs? The challenge we are faced with is designing efficient algorithms that are robust to strategic manipulations. The field that studies these questions and lies at the intersection of computer science and economics is called Algorithmic Mechanism Design. At a high level, Algorithmic Mechanism Design can be classified in two categories, based on whether monetary transfers can be used as a tool in incentivizing agents to behave honestly:

- **Mechanism Design with Money.** The centerpiece for this category is the *auction*: a seller is auctioning off m items to n interested buyers, each with a private valuation for different subsets of the items. The abstraction of the auction allows us to model and study many important problems. Some applications include Internet search auctions, wireless spectrum auctions, as well as - in the case of "reverse" auctions - goverment procurement of goods and services from the private sector.

- **Mechanism Design without Money.** In many situations of interest, even though the input comes from selfish agents, using monetary tranfers might be illegal or immoral, e.g. political decisions or organ donations. In such settings, in addition to strategyproofness, there is also an emphasis on fairness and envy-freeness. Applications include matching doctors to hospitals, kidney exchange markets, resource allocation in the cloud and elections.

Despite the explosion of work in this realm, most of the literature focuses on static, one-time decisions. Nevertheless, many environments of interest are inherently dynamic. In this thesis we explore the interplay of incentives and the dynamic nature of decision-making in the design of efficient mechanisms in two subareas: Auctions and Fair Division.

1.1 Part I: Dynamic Auction Design

One of the most celebrated results of Game Theory is Myerson's theorem [86], for which he was awarded the 2007 Nobel Prize in Economics. Myerson gave an elegant solution to the following problem: a single item is auctioned off to potential buyers. The auctioneer wants to design a mechanism to maximize her revenue, while simultaneously incentivizing the buyers to participate, but doesn't know how much the buyers might be willing to pay; she only has a prior. One of the most salient problems in the field is generalizing Myerson's auction to more general settings. Typically, the literature has focused on one-shot mechanisms. However, mechanisms of an implicit multi-stage nature are a lot more common. For example, ad auctions, like the ones used by Facebook and Google, are in truth dynamic; they are happening in distinct and correlated stages. In fact, bidders participating in these auctions change their bids multiple times per hour, suggesting that they're using learning algorithms to handle the complicated dynamics that arise.

In the first part of this thesis we introduce and study the problem of a monopolist that is auctioning off m items in m consecutive stages to n interested buyers. A buyer learns her value for the k-th item at the beginning of stage k. For the future items only a prior distribution is known. The prior of buyer i can depend on the values of that buyer for the items so far. How should this monopolist behave in order to maximize her expected revenue?

For example, consider a wireless company that wants to sell you two phones, one now and one in three years. You know precisely how much the phone is worth to you now (the company has some prior X_1 from which your value was drawn), but for the future you only have a prior X_2, that is also known to the company. To make the problem even simpler, suppose for now that X_2 does not depend on your current value. One way the seller could go about this is to sell you today's phone optimally - Myerson's theory readily applies - and make you a take-it-or-leave-it offer for the future phone: "pay $\mathbb{E}[X_2]$ now, and in three years you'll get the new phone for free.". The unsettling feature of this mechanism is that, for some realizations of the future value, the buyer ends up with negative utility. This feature gets exacerbated as the number of stages increases and the same kind of auction can be used. How could we fix this? A natural approach is to ask for stronger participation guarantees. The utility at every period should be non-negative; the new goal is to design an *ex-post individually rational* mechanism.

Subject to ex-post individual rationality, what can the auctioneer do to maximize revenue? One approach would be to completely ignore the dynamic nature of the problem, and run the optimal static, single-shot mechanism at each stage. When the values for today's and future phone are independent random variables, it is tempting to assume that this should surely be a good approximation to the optimal mechanism, or perhaps even be optimal. As we see in the next example, this is far from true.

Example 1 ([90]). *Let X_1 and X_2 be the random variables indicating the value of the buyer for the first and second stage item. X_1 takes value 2^i with probability 2^{-i} for $i = 1, \ldots, n$, and value 0 with probability 2^{-n}. X_2 takes value 2^i with probability 2^{-i} for $i = 1, \ldots, 2^n$, and* *value* 0 *with probability* 2^{-2^n}. *It can be verified that the optimal static auction for both* X_1 *and* X_2 *extracts revenue at most* 2 *(Consider setting some price* 2^k. *The expected revenue is at most* $2^k \cdot \sum_{i \geq k} 2^{-i} \leq 2$ *). Therefore, running the optimal static auction at each stage extracts revenue at most* 4. *On the other hand, there exists a dynamic mechanism that extracts revenue* n*: on the first stage the buyer pays her report* $\hat{v}$. *On the second stage the item is given for free with probability* $\frac{\hat{v}}{\mathbb{E}[X_2]}$[1]. *An easy calculation shows that truthful reporting is a weakly dominant strategy. The revenue extracted is* $\mathbb{E}[X_1] = n$.

In the first part of this thesis we explore several different facets of the problem of designing dynamic mechanisms. We are first interested in the revenue gaps introduced by the dynamic nature of the problem - what is the *Power of Adaptivity*? We present strong separations between: Non-adaptive mechanisms and the optimum deterministic adaptive mechanism (even for uncorrelated distributions); the optimum deterministic and the optimum randomized mechanism; the optimum randomized mechanism and the optimum social welfare. Another subtlety of our model is *Commitment*: What happens when contracts about future behavior cannot be written and enforced? We demonstrate an interesting facet of the complexity of dynamic mechanisms: even for two stages and one buyer, the revenue-optimal randomized auction requires the auctioneer and buyer to interact through multiple rounds of communication in the first period.

We proceed to study the *Computational Complexity* of designing dynamic auctions. Can the optimal adaptive auction be computed efficiently? We prove that the problem of finding the optimum randomized mechanism can be solved in polynomial time, and in fact for any finite number of periods of sale and for any finite number of buyers. But, is this a reasonable mechanism? In Example 1 we already saw how a randomized auction can extract a lot of revenue by offering “lotteries”. Can we efficiently compute a more reasonable auction? We focus on deterministic and ex-post IR dynamic mechanisms. The reason we insist on such mechanisms is because we believe that they draw the boundary of mechanisms in which people are likely to choose to participate - and of course because Myerson’s archetype is such a mechanism. We prove that it is strongly NP-complete to find the optimum such mechanism, even for a single buyer and two stages. Since we cannot think of a simpler dynamic mechanism design problem, this result suggests that there is no grand sweeping, Myerson-like positive result lurking somewhere in the realm of dynamic mechanism design.

All told, optimality is riddled with complexity issues, both computationally and in terms of its description. Even worse, the optimal mechanism depends on detailed knowledge of the buyers’ distributions (across time) in intricate ways. Our final topic on dynamic auctions is revenue guarantees for simple, prior-independent auctions. We study the *Competition Complexity* of dynamic auction design: *How many additional buyers are necessary and sufficient for a second price auction at each stage to extract revenue at least that of the optimal dynamic auction?* We prove that the Competition Complexity of dynamic auctions is at most $3n$ - and at least linear in n - even when the buyers’ values are correlated across stages, under

[1]Notice that $\mathbb{E}[X_2] > 2^n$, therefore $\frac{\hat{v}}{\mathbb{E}[X_2]}$ is a probability.

a monotone hazard rate assumption on the stage (marginal) distributions. This assumption can be relaxed if one settles for independent stages. We also prove results on the number of additional buyers necessary for VCG at every stage to be an α-approximation of the optimal revenue; we term this number the α-approximate Competition Complexity. For example, under the same mild assumptions on the stage distributions we prove that one extra buyer suffices for a $\frac{1}{e}$-approximation. As a corollary we get results on prior-independent dynamic auctions.

Organization of Part I

The results in this part are based on joint work with Christos Papadimitriou, George Pierrakos and Aviad Rubinstein ([90]), and Siqi Liu ([80]). In Chapter 2 we formally present the model, related work, and present our results *Power of Adaptivity*. In the same Chapter we study the *Power of Commitment*. In Chapter 4 we study the *Computational Complexity*; we present the NP-hardness proof for deterministic auctions and the algorithm for computing the optimal randomized auction. In Chapter 4 we present our results on the *Competition Complexity* of dynamic auctions.

1.2 Part II: Dynamic Fair Division

In the second part of this thesis we study settings where monetary transfers cannot be used to incentivize agents to behave honestly. Fair division has been a central topic in economics and mathematics (e.g., Alon [1987], Brams and Taylor [1995], Dubins and Spanier [1961], Pazner and Schmeidler [1978], Steinhaus [1948], Stromquist [1980], Aziz and Mackenzie [2016], Aziz and Mackenzie [2016]). More recently, it has received more attention in computer science due to its applications to resource sharing in data centers and the cloud (e.g., Ghodsi et al. [2011], Dolev et al. [2012], Popa et al. [2012], Bhattacharya et al. [2013],Wang, Li, and Liang [2014]). Traditionally, research on fair division has focused on static allocations; contemporary resource allocation protocols, however, need to be dynamic in nature. This has led to more research on fairness in the dynamic setting (e.g.,Walsh [2011], Kash, Procaccia, and Shah [2013], Aleksandrov et al. [2015], Friedman, Psomas, and Vardi [2015], Friedman, Psomas, and Vardi [2017]): There is some finite amount of resource(s), agents arrive and depart, and the goal is to constantly maintain allocations that are "fair". There are several accepted notions of fairness in the literature, for example, *envy-freeness* (no agent would like to exchange shares with any other agent) and *equitability* (every agent has the same utility). Arguably the most widely accepted notion is *proportionality*: if there are n agents in the system, each agent is allocated at least a $1/n$ fraction of what she would receive if she were allocated all of the resource. If there are multiple resources and agents have heterogeneous demands, the notions of fairness become more complex; two of the most common notions in this setting are *Competitive Equilibrium from Equal Incomes* (CEEI) and *Dominant Resource Fairness* (DRF). We elaborate more on these later on.

A major difficulty in maintaining fairness in dynamic settings comes from the price of reallocating resources. Even if there exist good solutions for k agents and for $k+1$ agents, these solutions do not typically include instructions for transitioning from one solution to the other without reclaiming and reallocating all the resources. As an example, consider the case of a single homogeneous resource. If reallocating resources is cheap and efficient, it is trivial to satisfy any and all of the above fairness notions in the dynamic setting: when there are k agents, allocate each one $1/k$ of the resource. When a new agent arrives, reallocate the resource evenly once more; note that this necessitates reducing the allocation of *all* agents, whenever a new agent arrives. If there are many successive arrivals, much of the time could be spent on reallocation, instead of resource consumption. This is an important issue in practice (e.g., Milojičić et al. [2000], Isard et al. [2009], Verma et al. [2015]).

> "The offline scheduler is not applicable in this environment because, if we simply called it each time a resource freed up, we might have to reallocate a large number of machines to obtain the configuration it returns. " Ghodsi et al. [2013].

Motivated by these issues, we study simple dynamic fair division problems where the amount of disruption is a hard constraint. We introduce a natural benchmark - the fairness ratio - the ratio of the minimal share to the ideal share ($1/k$ when there are k agents in the system). We describe an algorithm that obtains the optimal fairness ratio when $d \geq 1$ disruptions are allowed per arriving agent. However, in systems with high arrival rates even 1 disruption per arrival can be too costly. We proceed to study the scenario when fewer than one disruption per arrival is allowed. We show that we can maintain high levels of fairness even with significantly fewer than one disruption per arrival. In particular, we present an instance-optimal algorithm (the input to the algorithm is a vector of allowed disruptions) and show that the fairness ratio of this algorithm decays logarithmically with c, where c is the longest number of consecutive time steps in which we are not allowed any disruptions. We then consider dynamic fair division with multiple, heterogeneous resources. In this model, agents demand the resources in fixed proportions, known in economics as Leontief preferences. We show that the general problem in NP-hard, even if the resource demands are binary and known in advance. We study the case where the fairness criterion is Dominant Resource Fairness (DRF), and the demand vectors are binary. We design a generic algorithm for this setting using a reduction to the single-resource case. To prove an impossibility result, we take an integer program for the problem and analyze an algorithm for constructing dual solutions to a "residual" linear program. The results in this part are based on joint work with Eric Friedman and Shai Vardi ([47, 49]), and can be found in Chapter 5.

Part I

Dynamic Auction Design

Chapter 2

Dynamic Auctions: Definitions, Separations and Commitment

In this chapter we formally introduce the dynamic auctions model and present some preliminary results on dynamic auctions. We define the model in Section 2.1. In Section 2.2 we mention the related work. We present our separations between non-adaptive, deterministic and randomized auctions in Sections 2.3 for the case of a single buyer and two stages. For the same setting, we study the situation when contracts about future behavior cannot be written and enforced in Section 2.4.

2.1 The Model

A seller is auctioning off m items to n buyers in m consecutive stages. The value of buyer i for the item on stage k is $v_k^i \in V_k^i = [\underline{v}_k^i, \overline{v}_k^i]$ and is distributed according to a random variable X_k^i. These random variables are independent among buyers, but for the same buyer can be correlated across stages, i.e. X_k^i can be correlated with $X_{k'}^i$, but not with $X_k^{i'}$. X_k^i has distribution D_k^i with density f_k^i and cumulative density F_k^i. It will be often convenient to use random variables rather than distributions and thus we use X and D interchangeably. Throughout Part I of this thesis we use superscript to denote an agent and subscript to denote the stage. We write $\mathbf{X}_k = \prod_{i=1}^n X_k^i$ for the product distribution for stage k (across all buyers). Let $\mathcal{X}$ be the input to the seller's problem; $\mathcal{X}$ includes all the stage distributions, as well as their correlation. We assume that the value for each item is revealed at the beginning of each stage: at the beginning of stage k, buyer i knows her private history $v_{<k}^i = \left(v_1^i, v_2^i, \ldots, v_{k-1}^i\right)$, her value v_k^i for the item in stage k, as well as the public history $\hat{\mathbf{v}}_{<k} = \left(\hat{v}_{1:k-1}^1, \hat{v}_{1:k-1}^2, \ldots, \hat{v}_{1:k-1}^n\right)$, where $\hat{v}_{a:b}^i = \left(\hat{v}_a^i, \hat{v}_{a+1}^i, \ldots, \hat{v}_b^i\right)$ are the reported values of buyer i for stages a through b.

From the revelation principle, it is sufficient to consider direct revalation mechanisms. A mechanism M is a sequence of m allocation functions $(x_1, \ldots x_m)$ and m payment functions $(p_1, \ldots p_m)$, both taking as input all the reported valuations so far $\mathbf{v}_{\leq k}$. The allocation function for stage k has a component $x_k^i(\mathbf{v}_{\leq k})$ that represents the probability that buyer i gets the item in stage k. Similarly the payment function for stage k has a component $p_k^i(\mathbf{v}_{\leq k})$ for the payment of buyer i in stage k. A mechanism is feasible if for all stages k and all histories $\mathbf{v}_{\leq k}$, $x_k^i(\mathbf{v}_{\leq k}) \in [0,1]$ for all agents i, and $\sum_{i=1}^{n} x_k^i(\mathbf{v}_{\leq k}) \leq 1$. We assume quasi-linear utilities; the utility of buyer i in stage k is $v_k^i \cdot x_k^i(\mathbf{v}_{\leq k}) - p_k^i(\mathbf{v}_{\leq k})$.

Incentive Compatibility. At stage k we would like to have a mechanism where agent i, with real value v_k^i, maximizes her utility when reporting v_k^i, among all possible reports $\hat{v}_k^i$. This utility is in expectation over the other agents' current values, as well as her own and other agents' future values. When deciding what value $\hat{v}_k^i$ to report in stage k, the agent has to take into account that the future allocation and payments (and therefore the future utility) will be affected by this report. It could be the case that lying only in stage k or lying only in stage $k+1$ results in lower overall utility, but lying in both stages results in a higher utility! Thus, when deciding when to lie, the buyer must choose in advance a strategy that deviates from the truth now and in the future.

Let $\mathcal{S}_k$ be the set of all "future deviation strategies" at stage k. A deviating strategy $s \in \mathcal{S}_k$ is a function from possible "futures", i.e. elements of $\prod_{i=1}^{n} \prod_{t=k+1}^{m} V_t^i$ to possible reports, i.e. elements of $(\hat{v}_{k+1}, \ldots, \hat{v}_m) \in V_{k+1:m} = \prod_{i=k+1}^{m} V_i$. Note that in our definition of $\mathcal{S}_k$ the deviation in stage k is not included. A mechanism is incentive compatible if, for every buyer i, every stage k, all possible histories $\mathbf{v}_{<k}$, for all values $v_k^i \in V_k$ on stage k, and all possible current deviations $\hat{v}_k^i \in V_k$ and future deviation strategies $s \in \mathcal{S}_k$:

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{v}_{\geq k}^{-i}, v_{k+1:m}^i}\left[\sum_{j \geq k} v_j^i x_j^i\left(\mathbf{v}_{\leq j}^{-i}, v_{1:k-1}^i, v_k^i, v_{k+1:j}^i\right) - p_j^i\left(\mathbf{v}_{\leq j}^{-i}, v_{1:k-1}^i, v_k^i, v_{k+1:j}^i\right)\right] \geq \\
&\mathbb{E}_{\mathbf{v}_{\geq k}^{-i}, v_{k+1:m}^i}\left[\sum_{j \geq k} v_j^i x_j^i\left(\mathbf{v}_{\leq j}^{-i}, v_{1:k-1}^i \hat{v}_k^i, s\left(\mathbf{v}_{k:j}^{-i}, v_{k:j}^i\right)\right) - p_j^i\left(\mathbf{v}_{\leq j}^{-i}, v_{1:k-1}^i, \hat{v}_k^i, s\left(\mathbf{v}_{k:j}^{-i}, v_{k:j}^i\right)\right)\right]
\end{aligned}
\tag{2.1}
$$

Intuitively, buyer i at stage k compares her expected utility when telling the truth now and in the future, with her expected utility for reporting $\hat{v}_k^i$ now, $s\left(\mathbf{v}_{k:j}^{-i}, v_{k:j}^i\right)$ in stage j, where $\mathbf{v}_{k:j}^{-i}$ is the rest of the buyers' values in stages k through j and $v_{k:j}^i$ are the true values of buyer i in stages k through j.

If $\mathcal{S}_k$ is the set of all function from $\prod_{i=1}^{n} \prod_{t=k+1}^{m} V_t^i$ to $\prod_{i=k+1}^{m} V_i$, and Equation 2.1 is satisfied, we say that the mechanism is *incentive compatible in a perfect Bayesian equilibrium*. If $\mathcal{S}_k$ only includes the "identity function", i.e. the agent assumes truthful reporting for future stages, we say that the mechanism is *periodic incentive compatible*.

Individual Rationality. We focus on ex-post individual rationality. Every buyer's utility is non-negative *at every stage*, no matter what the other buyers' valuations are. Formally, for every buyer i, stage k, and possible history $\mathbf{v}_{\leq k}$:

$$v_k^i \cdot x_k^i(\mathbf{v}_{\leq k}) - p_k^i(\mathbf{v}_{\leq k}) \geq 0. \tag{2.2}$$

The seller's problem. The seller's goal is to find the revenue optimal mechanism that is incentive compatible and individually rational. Let $\text{OPT}[\mathcal{X}, n, m]$ denote the revenue of the optimal mechanism for n buyers and m stages, when the buyers' valuations are drawn according to $\mathcal{X}$. For the special case of $m = 1$ the solution is given by Myerson [1981]. For a general m, the seller's problem can be expressed as a linear program with variables x_k^i and p_k^i, objective

$$\max \mathbb{E}\left[\sum_{k=1}^{m}\sum_{i=1}^{n} p_k^i(\mathbf{v}_{\leq k})\right],$$

subject to constraints 2.1 and 2.2, as well as the allocation x being feasible (a linear constraint). Note that the seller's revenue is smaller for ex-post IR than ex-ante IR. Furthermore, it weakly decreases as the set $\mathcal{S}_k$ of deviations considered becomes larger. Therefore, the best upper bounds possible would be for ex-ante IR and periodic IC.

2.2 Related Work

Dynamic Auctions. Dynamic mechanisms have been studied extensively in quite general settings; see [13] for a recent survey. Many works study problems where agents arrive and depart dynamically, e.g. [92, 89, 50, 51], or problems with evolving private information, e.g. [32, 72, 93, 24, 25, 75].

The study of dynamic auctions where ex-post individual rationality is a hard constraint, was first studied in Papadimitriou et al. [2016]. Ashlagi, Daskalakis, and Haghpanah [2016] provide characterizations of the optimal ex-post IR, periodic IC dynamic mechanism, with m *independent* stages and n buyers. They show that there exists an optimal mechanism that has stage utility equal to zero for all stages, except maybe the last. In last stage the seller might have to pay the buyers[1]. Surprisingly, their mechanism can be described via updates, at every stage, to a scalar variable that guides the future allocation and payments. The authors use this characterization to give a mechanism that obtains a $\frac{1}{2}$ approximation to the optimal revenue for the single buyer problem.

Mirrokni et al. [2016] study dynamic mechanisms with an ex-interim IR constraint. They define a class of mechanisms called *bank account* mechanisms. Bank account mechanisms maintain a state variable, the balance, that is updated throughout the execution of the mechanism depending on a "spending" and "depositing" policy. The allocation and payment at

[1]To see this most clearly, consider a single agent, two stage situation where X_1 and X_2 are such that $\mathbb{E}[X_1] > \textsc{Mye}[X_1] + \mathbb{E}[X_2]$; the RHS is an upper bound to the optimal revenue by Lemma 32 in Chapter 4. The characterization of Ashlagi, Daskalakis, and Haghpanah [2016] says that there exists an optimal mechanism that extracts $\mathbb{E}[X_1]$ in the first stage; in the second stage the seller must pay back at least $\mathbb{E}[X_1] - (\textsc{Mye}[X_1] + \mathbb{E}[X_2])$

each stage depend on the report and the balance. Mirrokni et al. [2016] study revenue maximization for bank account mechanisms subject to an ex-post IR constraint. Mirrokni et al. [2016] study the design of *oblivious* dynamic mechanisms. An oblivious dynamic mechanism decides on the allocation and payment for stage k using information only about the current and past stages, i.e. it is oblivious about the buyers' value distributions $D_{k+1}, \ldots, D_m$. Their mechanism *ObliviousBalance* runs at each stage a combination of Myerson's optimal auction, a second price auction, and the money burning mechanism of Hartline and Roughgarden [2008]. Their mechanism obtains a $\frac{1}{5}$ approximation to the optimal revenue.

2.3 The Power of Adaptivity

In this section we compare deterministic and randomized auctions for a two stage setting in terms of the revenue generated, against each other and against two other benchmarks:

- the optimal non-adaptive auction — i.e. running an independent Myerson's auction on each stage; and
- the optimal social welfare SW — the expected utility of the buyer from receiving both items for free.

The following is immediate:

Fact 2. *For any distribution of valuations,*

$$Rev\left(non\text{-}adaptive\right) \leq Rev\left(deterministic\right) \leq Rev\left(randomized\right) \leq SW$$

But are these inequalities strict for some valuation distributions? And by how much?

Theorem 3. *Let v^* be the maximal buyer's valuation in any stage, and assume that all valuations are integral. Then in any two-stage auction, the maximum, over all auctions, ratio:*

- *between SW and any of $\{Rev\left(non\text{-}adaptive\right), Rev\left(deterministic\right), Rev\left(randomized\right)\}$ is exactly the harmonic number of v^*, $H_{v^*} = \sum_{i=1}^{v^*} 1/i$;*
- *between either of $\{Rev\left(deterministic\right), Rev\left(randomized\right)\}$ and $Rev\left(non\text{-}adaptive\right)$ is at least $\Omega\left(\log^{1/2} v^*\right)$ (and at most $O\left(\log v^*\right)$); and*
- *between $Rev\left(randomized\right)$ and $Rev\left(deterministic\right)$ is at least $\Omega\left(\log^{1/3} v^*\right)$ (and at most $O\left(\log v^*\right)$).*

Furthermore, even when the valuations on the different stages are independent, there exists a two-stage auction with ratio of $\Omega\left(\log\log v^*\right)$ between either of $Rev\left(deterministic\right)$, $Rev\left(randomized\right)$ and $Rev\left(non\text{-}adaptive\right)$.

Warm up: Revenue vs Social Welfare

To compare non-adaptive auctions to optimal social welfare, we can assume with no loss of generality that the auction occurs in a single stage.

Proposition 4. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. For a single stage auction, the maximum ratio between SW and Rev (non-adaptive) is at least $\frac{\log(v^*)}{2}$.*

Proof. Suppose that the buyer has valuation 2 with probability 1/2, 4 with probability 1/4, etc. until 2^z with probability 2^{-z} (and 0 also with probability 2^{-z}). Now, if the auctioneer hands out the item for free, the expected social welfare is $\mathrm{SW} = \sum_{i=1}^{z} 2^{-i} \cdot 2^i = z$.

For any choice of price 2^k chosen by the non-adaptive auction, the expected revenue is

$$\mathrm{Rev}\left(\text{non-adaptive}\right) = 2^k \cdot \sum_{i=k}^{n} 2^{-i} < 2. \qquad \square$$

The construction above is extremely useful in proving such lower bounds. In fact it is also used in our NP-hardness result. The distribution used is approximately the well known equal-revenue distribution. We will refer to it as $\mathrm{POW2}\left[1, z\right]$ to unify our notation. In general:

Definition 5. *We say that $v \sim c \cdot \mathrm{POW2}\left[a, b\right]$ if $v = c \cdot 2^{a+i}$ with probability 2^{-i-1} for all $i \in [b-a]$, and $v = 0$ with probability 2^{a-b-1}. Note in particular that the expectation is*

$$\mathbf{E}\left[\mathrm{POW2}\left[a, b\right]\right] = 2^{a-1}\left(b - a + 1\right) \ .$$

We conclude this introductory subsection by proving a tight version of the above proposition, namely

Lemma 6. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. The maximum, over all single stage auctions, ratio between SW and Rev (non-adaptive) is exactly the harmonic number of v^*.*

Proof.

$$\begin{aligned} \mathrm{SW} &= \sum_{t=1}^{v^*} t \Pr\left[v = t\right] = \sum_{t=1}^{v^*} t\left(\Pr\left[v \geq t\right] - \Pr\left[v \geq t+1\right]\right) = \sum_{t=1}^{v^*} \Pr\left[v \geq t\right] \\ &= \sum_{t=1}^{v^*} \frac{\mathrm{Rev}\left(p = t\right)}{t} \leq \sum_{t=1}^{v^*} \frac{\mathrm{Rev}\left(\text{non-adaptive}\right)}{t} = \mathrm{Rev}\left(\text{non-adaptive}\right) \cdot H_{v^*} \end{aligned}$$

where $\mathrm{Rev}\left(p = t\right)$ denotes the expected revenue from charging t. Finally, note the inequality can be made tight by setting $\Pr\left[v \leq t\right] = \frac{1}{t}$ for all $1 \leq t \leq v^*$. $\square$

Note that in the single stage setting, the optimal randomized auction does not achieve more revenue than Myerson's fixed price; therefore the same bound immediately holds for adaptive deterministic and randomized auctions.

Corollary 7. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. The maximum (over all single stage auctions) ratio between Rev(deterministic) and SW, and between Rev(randomized) and SW is exactly the harmonic number of v^*.*

Independent valuations

Surprisingly, adaptive auctions achieve a higher revenue even when the valuations on the different stages are independent. A well-known approach for extracting the entire social welfare under ex-ante individual rationality is the sale of "lottery-tickets", i.e. sell the item before the buyer sees her valuation. A rational, risk-neutral buyer would be willing to pay the expected social welfare. Here, ex-post individual rationality excludes many such auctions. In the two-stage setting this may still be possible: We could sell on the first stage a "lottery-ticket" for the second stage; we will remain ex-post IR because of the utility derived from the first-stage item.

This sounds promising, but there is one more obstacle to overcome: If the value of the first stage is higher than the cost of the lottery ticket, why can't we extract it by a fixed price auction on the first stage? We will use the same construction from Proposition 4 to ensure that the welfare on the first stage cannot be extracted using a fixed price mechanism. Informally, we are hiding the ex-post vs ex-ante IR issue in the IC constraints, which we only require to be satisfied ex-interim.

Lemma 8. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. For a two-stage auction, the ratio between the Rev(deterministic) and Rev(non-adaptive) can be as large as $\frac{(\log\log v^*)}{4}$,* even when the valuations on each stage are independent.

Proof. Let $Z = 2^z$. Let the valuation the first stage be distributed as $v_1 \sim \textsc{pow}2\left[1, z\right]$, and on the second stage $v_2 \sim \textsc{pow}2\left[1, Z\right]$. The optimal revenue for running two separate fixed-price auctions is a constant $\text{Rev}\left(\text{non-adaptive}\right) < 4$.

What about deterministic adaptive auctions? The same idea works, except that in the deterministic case, the auctioneer "punishes" the buyer for lower bids by charging higher prices on the second stage.

On the first stage, the deterministic adaptive mechanism will charge the buyer almost the full price $v_1 - (2 - 2^{-v_1})$. On the second stage, we will offer the item for price $p_2\left(v_1\right) = 2^{Z-v_1}$. The buyer's expected utility from the second stage is now exactly

$$\begin{aligned}\sum_{i:\, 2^i \geq p_2(v_1)} 2^{-i}\left(2^i - p_2\left(v_1\right)\right) &= v_1 - \sum_{i:\, 2^i \geq p_2(v_1)} 2^{-i} p_2\left(v_1\right) \\ &= v_1 - \sum_{0 \leq i \leq v_1 - 1} 2^{-i} \\ &= v_1 - \left(2 - 2^{-v_1}\right)\end{aligned}$$

Once again, the buyer's expected utility on the second stage exactly covers the price on the first stage, which guarantees that this auction satisfies IC. Finally, note the expected revenue is almost as large as the expected valuation on the first stage $\mathrm{Rev}\,(\text{deterministic}) > z-2$. □

Stronger adaptivity gaps for correlated valuations

When the valuations are correlated, we can show stronger adaptivity gaps.

Lemma 9. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. For a two-stage auction, the ratio between the Rev (deterministic) and Rev (non-adaptive) can be as large as* $\sqrt{\log v^*}/4$

Proof. Let the first-stage valuation be distributed $v_1 \sim \text{POW2}\,[1,z]$. The second-stage valuation v_2 will be conditioned on the first stage: $v_2 \mid v_1 \sim (v_1/z)\cdot \text{POW2}\,[1,z^2]$. We already saw that the non-adaptive policy's revenue on the first stage is less than 2. What is the optimal price for the second stage? To answer this question we must consider the marginal distribution of the second stage:

$$\Pr\left[v_2 = 2^l/z\right] = \sum_{k\in[z]} \Pr\left[v_1 = 2^k/z\right]\Pr\left[v_2 = 2^l \mid v_1 = 2^k\right] \le \sum_{k\in[z]} 2^{-k}2^{k-l} = z\cdot 2^{-l}.$$

Therefore, $\Pr\left[v_2 \ge 2^l/z\right] \le z\cdot 2^{1-l}$, which implies $\mathrm{Rev}\,(\text{non-adaptive}) < 4$. Now, consider the randomized mechanism that on the first stage charges the buyer $v_1 = 2^k$ (and allocates the item), and on the second stage allocates the item for free with probability k/z. When the buyer's true valuation on the first stage is 2^k, her the expected utility from reporting 2^l is given by

$$\mathrm{U}\left(2^k,2^l\right) = (l/z)\,\mathbf{E}\left[v_2 \mid v_1 = 2^k\right] - 2^l = l\cdot 2^k - 2^l\ ,$$

which is maximized by $l\in\{k,k+1\}$. The expected revenue from this randomized auction is again z. Similarly, a deterministic auction can charge $v_1 = 2^k$ on the first stage, and offer the item on the second stage for price $p_2\left(2^k\right) = 2^{n^2-nk}/z$

$$\begin{aligned}
\mathrm{U}\left(2^k,2^l\right) &= \sum_{i:\,2^{k+i}/z\ge p_2\left(2^l\right)} 2^{-i}\left(2^{k+i}/z - p_2\left(2^l\right)\right) - 2^l \\
&= \left(l-\frac{k}{z}\right)2^k - \sum_{i:\,2^{k+i}/z\ge p_2\left(2^l\right)} 2^{-i}\cdot p_2\left(2^l\right) - 2^l = \left(l-\frac{k}{z}\right)2^k - \sum_{0\le i\le zl+k-1} 2^{-i}\left(2^k/z\right) - 2^l \\
&= \left(l-\frac{k}{z}\right)2^k - \left(2-2^{-(zl+k)}\right)\left(2^k/z\right) - 2^l = \left(l+\frac{2^{-(zl+k)}}{z}\right)2^k - 2^l - \left(\frac{k+2}{z}\cdot 2^k\right)
\end{aligned}$$

The second line follows because there are $zl-k$ i's for which $i\colon 2^{k+i}/z \ge p_2\left(2^l\right)$. Notice that indeed, $\left(l+\frac{2^{-(zl+k)}}{z}\right)2^k - 2^l$ is maximized at $l=k$. □

Deterministic vs randomized auctions

Naturally, one would expect that deterministic and randomized auctions yield different revenues because we can optimize the latter in polynomial time, while optimizing over deterministic auctions is NP-hard. In this subsection we show that randomized auctions can in fact yield much more revenue.

Lemma 10. *Let v^* be the maximal buyer's valuation, and assume that all valuations are integral. For a two-stage auction, the ratio between the Rev(randomized) and Rev(deterministic) can be as large as* $\frac{(\log v^*)^{1/3}}{7}$

Our proof builds on the constructions in the proof of Lemma 9. A key observation is that by modifying the parameters for the second stage distribution, we can shift the prices without changing the expected utility. Choosing those parameters based on the valuation in the first stage, will allow us to break the deterministic auctioneer's strategy, without changing the revenue of the randomized auction.

Proof. Let $v_1 \sim \text{POW2}[1, z]$. For type i with value 2^i on the first stage, the valuation on the second stage will be 0 with probability $1 - 2^{-2n^2 i}$. The remaining $2^{-2z^2 i}$ will be distributed according to $\frac{2^{\left(2z^2+1\right)i}}{z}\text{POW2}[1, z^2]$. For any $i \in [z]$, let $V_2^i \setminus \{0\}$ be the set of nonzero feasible valuations on the second stage, conditioned on valuation 2^i on the first stage. Notice that for any $i < j$, all the values in $V_2^i \setminus \{0\}$ are much smaller than all the values in $V_2^j \setminus \{0\}$.

The randomized mechanism, again charges full price $v_1 = 2^k$ on the first stage, and gives the item for free on the second stage, with probability k/z. The buyer's utility from reporting 2^l is:

$$\text{U}\left(2^k, 2^l\right) = (l/z)\,\mathbf{E}\left[v_2 \mid 2^k\right] - 2^l = l \cdot 2^k - 2^l \ ,$$

which is maximized by $l \in \{k, k+1\}$. The expected revenue from this randomized auction is again z.

What about the deterministic auctioneer? Given any deterministic mechanism, let k^* be the minimal k for which a buyer with first-stage valuation 2^k has a nonzero probability of affording both items. In other words, after declaring valuation 2^{k^*} for the first stage, her second-stage price is at most $p_2\left(2^{k^*}\right) \leq \frac{2^{\left(2z^2+1\right)k^*+1}}{z} < 2^{2z^2(k^*+1/2)}$.

Assume that the buyer has valuation $v_1 = 2^l > 2^{k^*}$. If she deviates and declare type 2^{k^*}, she receives the first item, and she also receives the second time whenever she has nonzero valuation. On the second stage, she pays less than $2^{2z^2(k^*+1/2)}$ with probability $2^{-2z^2 l} \leq 2^{-2z^2(k^*+1)}$. Therefore her expected pay on the second stage has a negligible expected cost (less than 2^{-z^2}). On the first stage, her price cannot be greater than 2^{k^*}. The total expected payment made by the buyer with $v_1 = 2^l > 2^{k^*}$ is bounded by $2^{k^*} + 2^{-z^2}$. Summing over the probabilities of having first-stage valuation $v_1 = 2^l > 2^{k^*}$, this is still less than 1.

Consider all the types whose first-stage valuations are lower than 2^{k^*}, and yet they receive the first item. Since they can never afford the second item, on the first stage they must all be charged the same price, thus yielding a total revenue less than 2. Similarly, the types for which the first-stage item is not allocated, must all be charged the same price on the second stage. Finally, by IR constraints the expected revenue from $v_1 = 2^{k^*}$ is at most 2. Therefore, the total expected revenue is less than 7. □

2.4 The Power of Commitment

In this section we restrict the two-stage auction problem to the design of mechanism where the auctioneer cannot commit to an action in the future. There are indeed many well studied situations in economics in which contracts are impossible, legally problematic, or costly to enforce (see for example [1, 2]). But beyond this consideration, the no-contract case raises hopes of escaping the negative results in Chapter 3: Since the second stage of any no-contract mechanism is trivial (the designer will make a Myerson offer), perhaps the overall complexity can be more modest. Let us clarify the model a bit: "No contract" means that it is impossible to sign and enforce contracts that span the two periods. However, the auctioneer can commit to any (possibly randomized) behavior during the first period, including in future stages of a multi-stage communication that takes place during the first period.

We point out that the no-contract dynamic mechanism design problem faces an obstacle of a very different nature: the revelation principle no longer holds on the first stage, and in a very strong sense. More specifically, we prove that the optimal no-contract mechanism requires multiple rounds of communication on the first stage. Our lower bound does not depend on any computation or communication limitations and is based purely on the structure of the agents' information (in contrast to e.g. [31, 39]). Before we continue into the details we remark that there is a beautiful literature by economists and game theorists on lower bounds on the number of rounds in cheap talk (e.g. [46, 6, 77, 76]). The concepts there are quite similar to what happens here, but the techniques are rather different.

Model and result

In the No-contract Two-stage Auction problem, we have one buyer and two items auctioned in two stages. The communication between the auctioneer and the bidder on the first stage is used to determine the price and allocation of the first item, as well as update the auctioneer's prior about the bidder's type. On the second stage, the auctioneer offers the second item for the Myerson optimal price given the updated prior. Our goal is to design an IC and ex-post IR mechanism for the first stage that maximizes the auctioneer's expected total revenue from both stages.

Theorem 11. *The optimal mechanism for the* No-contract Two-stage Auction *requires multiple rounds of communication on the first stage.*

How can extensive communication increase revenue?

We construct an example where the bidder's valuations on the two stages are independent, yet on the first stage she has a more refined prior over her second-stage valuations. Furthermore, she has a strong incentive to share her information about the second period with the auctioneer (while the auctioneer is approximately indifferent). In order to *credibly* report her information about the second period, she needs the auctioneer's help in setting up an incentive compatible mechanism. Informally, the auctioneer now has another "product" she can sell for profit: the opportunity to report information about the second period. We will refer to this new product as *OTR*, for the "opportunity to report".

The OTR has two important properties that distinguish it from the real items sold in the auction: (1) because it is not a real item, it does not contribute to the bidder's valuation when evaluating the IR constraints; and (2) the auctioneer knows its ex-interim value to the bidder (we'll set things up so that this value is independent of the partial information the bidder has in the first period; see Bullet 12 in Lemma 12). This latter property is useful when considering the IC constraints.

How does the OTR lead to multiple rounds of communication? In order to satisfy the IR constraints, the OTR must be bundled with the first (real) item. Given the results from recent years about menu complexity (e.g [65, 34]), it is not surprising that the optimal way to sell this bundle is fractional; i.e. for each price π, the bidder receives the OTR with some probability $\rho^{\text{OTR}}(\pi)$ (and the real item with probability $\rho^{(1)} = 1$ to satisfy the ex-post IR constraints). Thus we have: in round 1, the bidder places a bid; in round 2, the auctioneer allocates the OTR with some probability that depends on the bid; and in round 3, if allocated the OTR, the bidder reports her information about the second period. We next present the details of the construction.

Construction

The bidder's valuation on the second stage is drawn from one of two distributions D_1, D_2. Before the auction, the auctioneer has a prior of $(1/2, 1/2)$ over (D_1, D_2), but the bidder knows on the first stage from which of the two distributions she will draw her valuation on the second stage. We denote the mixed distribution known to the auctioneer by $\left(\frac{1}{2}D_1 + \frac{1}{2}D_2\right)$. We will introduce many constraints on those distributions, but the most important one for now is that the Myerson price[2] for each separate distribution is low (either 1 or $1 + \epsilon$ for $\epsilon \ll 1$), while the Myerson price for the mixed distribution (i.e. the auctioneer's prior) is high: k, for some sufficiently large integer $1 \ll k \ll 1/\epsilon$. In order to compensate a truthful bidder who may end up paying the slightly higher price $(1 + \epsilon)$ on the second stage, the auctioneer gives her a discount of $\epsilon/5$ on the first stage. The following lemma lists all the properties we require from D_1 and D_2, as well as some useful notation. We will assume that we have such distributions for now, and construct them explicitly later.

[2]Throughout this section, we use the term *Myerson price* of a distribution D to refer to the revenue-maximizing price for a single bidder who samples her valuations for a single item from D.

Lemma 12. *There exist distributions D_1, D_2 that satisfy all of the following conditions:*

Myerson pricing *The Myerson price given prior D_1 over the valuations is $1+\epsilon$, for prior D_2 it is 1, and for prior $\frac{1}{2}D_1 + \frac{1}{2}D_2$, it is k. Furthermore, for any convex combination of D_1 and D_2, every Myerson price is one of these three possible prices.*

Bidder's utility *Let $u_2(D' \mid D)$ denote the bidder's expected utility from the second stage auction when her true distribution is D, but the auctioneer runs a Myerson auction against a (possibly misreported) prior of D'. Then we require:*

Truthfulness $u_2(D_1 \mid D_1) + \epsilon/5 > u_2(D_2 \mid D_1)$ *and* $u_2(D_2 \mid D_2) > u_2(D_1 \mid D_2) + \epsilon/5$. *(Note that in this case we can ensure incentive compatibility by giving a discount of $\epsilon/5$ on the first stage whenever the bidder reports D_1.)*

Value of OTR *The value of the OTR to the bidder does not depend on her private information. We use θ to denote this value.*

$$u_2(D_1 \mid D_1) + \epsilon/5 - u_2\left(\tfrac{1}{2}D_1 + \tfrac{1}{2}D_2 \mid D_1\right) = \theta = u_2(D_2 \mid D_2) - u_2\left(\tfrac{1}{2}D_1 + \tfrac{1}{2}D_2 \mid D_2\right).$$

Auctioneer's revenue *By learning whether the second stage's valuation is drawn from distribution D_1 or D_2, the optimal expected revenue increases by at most $O(\epsilon)$.*

$$\texttt{Rev}\left(\frac{1}{2}D_1 + \frac{1}{2}D_2\right) \le \frac{1}{2}\texttt{Rev}(D_1) + \frac{1}{2}\texttt{Rev}(D_2) \le \texttt{Rev}\left(\frac{1}{2}D_1 + \frac{1}{2}D_2\right) + O(\epsilon).$$

Given the distributions guaranteed by Lemma 12, we travel back in time and construct a price distribution for the first stage. We want to construct a distribution where the revenue that the auctioneer can generate by using a Myerson single-item auction is significantly lower than the optimal social welfare (i.e. the bidder's expected valuation). Intuitively, the latter can only be translated into revenue by bundling with the OTR. In particular, this property can be achieved by the well-known equal-revenue distribution. Let δ be a small parameter ($\epsilon \ll \delta \ll 1/k$), and let l be a sufficiently large integer (say, $l = 100$). We set

$$\Pr[v_1 = \delta j] = \begin{cases} \frac{1}{j} - \frac{1}{j+1} & j \in \{1, \dots l-1\} \\ \frac{1}{j} & j = l \end{cases}$$

Let R denote the expected revenue from the second stage when the auctioneer knows which distribution is used, i.e. $R = \frac{1}{2}\texttt{Rev}(D_1) + \frac{1}{2}\texttt{Rev}(D_2)$. We prove that with three rounds of communication, the expected revenue is at least

$$\texttt{Rev}_3 \ge R + \delta H_l - O(\epsilon), \tag{2.3}$$

where H_l is the l-th harmonic number. With one round the expected revenue is at most

$$\texttt{Rev}_1 \le R + 3\delta. \tag{2.4}$$

Three rounds of communication

We begin by describing an approximately optimal protocol:

1. The bidder sends her true valuation v_1;
2. With probability v_1/θ, the auctioneer allocates the OTR (i.e. the auctioneer asks the bidder for her prior on the second stage's valuation);
3. If allocated the OTR, the bidder reports from which distribution (D_1 or D_2) she will draw her valuation on the second stage.

- The first item is always allocated ($x_1 = 1$), and the auctioneer charges price $p_1 = v_1 - \epsilon/5$ if the bidder reported prior D_1, and $p_1 = v_1$ otherwise. (On the second stage, the second item is offered for the Myerson price for the auctioneer's updated prior.)

It is easy to see that the IR constraints are satisfied because $p_1 \leq v_1$, and the first item is always allocated. We continue to compute the expected revenue from this auction, assuming the bidder reports truthfully. On the first stage, the revenue is at least $\mathbb{E}[v_1] - \epsilon/5 = \delta H_l - O(\epsilon)$. On the second stage, by Bullet (12) in Lemma 12, the expected revenue is at least $R - O(\epsilon)$. Overall we match our guarantee (2.3).

Finally, we prove that this mechanism satisfies the IC constraints. Given that she is allocated the OTR, it follows by Bullet (12) in Lemma 12 that she maximizes her utility by reporting truthfully on the third round. We now consider two cases, based on the bidder's prior for the second stage. We prove that in both cases, the bidder's expected utility does not depend on the reported valuation (and thus satisfies IC):

D_1 The bidder's expected utility on the second stage from the OTR is $\theta - (\epsilon/5)$. Upon reporting a first-stage valuation v', she receives the OTR with probability v'/θ, so her added utility is $v' - (\epsilon/5)\, v'/\theta$. Similarly, on the first stage her expected price is $v' - (\epsilon/5)\, v'/\theta$. Her total utility is therefore independent of the valuation she reports in the first stage.

D_2 Analogously to the previous case, the bidder's expected utility on the second stage from the OTR is θ. Upon reporting a first-stage valuation v', she receives the OTR with probability v'/θ, so her added utility is v'. On the first stage her price is always v'. Her total utility is again independent of the valuation she reports in the first stage.

One round of communication

Recall that by our construction for the second-stage distributions, for any convex combination of D_1 and D_2, the Myerson price charged in the second stage by the greedy auctioneer is always one of three possibilities: $p_2 \in \{1, 1+\epsilon, k\}$. The choice of p_2 depends on the updated prior based on the bidder's single message.

We divide the universe of legal bidder's messages into three: $M_p \subset \Sigma^*$ is the subset of messages for which the second-stage price is p, for each of $p \in \{1, 1+\epsilon, k\}$. Note that for each subset M_p, the allocation and price on the second stage are independent of the choice of message $m \in M_p$. In particular, any difference between messages must be due to different outcomes on the first stage.

Fix any $p \in \{1, 1+\epsilon, k\}$, and let $R_p^{(1)}$ be the expected revenue on the first stage when the bidder's message is in M_p. We show that $R_p^{(1)} \leq \delta$ by a reduction to selling only the first item. For message $m \in M_p$, let x_m denote the probability that the auctioneer allocates the first item to the bidder, and let π_m denote the expected price when allocated. In fact, since both the bidder and the auctioneer are risk-neutral, we can assume wlog that the auctioneer charges exactly π_m whenever the first item is allocated (note that ex-post IR constraints are preserved). Consider the single-item auction (for the first item) which requires the bidder to submit a message $m \in M_p$, and then with probability x_m offers the item for price π_m. By IC constraints, whenever the bidder submitted a message $m \in M_p$ in the original (two-stage) auction, she will continue to submit the same message in the modified (single-item) auction. Therefore, the revenue collected from this single-item auction is at least $R_p^{(1)}$. Finally, observe that due to the equal-revenue construction, the revenue from from selling the first item independently is at most δ.

Adding the expected revenues from all the feasible p's we have that the total expected revenue on the first stage at most 3δ. Since the revenue on the second stage is always at most R, (2.4) follows. This completes the proof of Theorem 11. □

Construction of D_1 and D_2: Proof of Lemma 12

Proof. We explicitly define D_1 and D_2, and then check that they satisfy all the requirements. We use $D(v)$ to denote the probability that distribution D assigns to value v. Let $O(1/\ln(k)) \leq \alpha < 1/5$ and $1/2 \leq \beta \leq 1$ be parameters to be defined soon. We define the first distrbution as follows:

$$D_1(v) = \begin{cases} 1-\alpha & v=0 \\ \left(\alpha \cdot \frac{1}{2}\right) - 2\epsilon^2 & v = 1+\epsilon \\ \alpha \cdot \left(\frac{1}{v} - \frac{1}{v+1}\right) & v \in \{2, \dots, k\} \\ \left(\alpha \cdot \frac{1}{v}\right) + 2\epsilon^2 & v = k \end{cases}$$

Notice that prices $1+\epsilon$ and k have probabilities higher than the equal-revenue curve for $v \in \{2, \dots, k\}$; one of them will always be optimal. Similarly, we let

$$D_2(v) = \begin{cases} 1-\beta & v=0 \\ \left(\beta \cdot \frac{1}{2}\right) + \left(\frac{k}{2}+1\right)\epsilon^2 & v = 1 \\ \beta \cdot \left(\frac{1}{2} - \frac{1}{k}\right) - \frac{k}{2}\epsilon^2 & v = 2 \\ \left(\beta \cdot \frac{1}{v}\right) - \epsilon^2 & v = k \end{cases}$$

Price 1 has relatively high probability, and price k comes after; thus for D_2 alone price 1 will be optimal, but together with D_1, price k maximizes the revenue.

Myerson pricing For prior D_1 the maximal revenue is achieved by $p_2 = 1 + \epsilon$:

$$\forall p' \in \{2, \ldots, k\} \quad (1+\epsilon) \cdot \Pr_{v_2 \sim D_1} [v_2 \geq 1 + \epsilon] = \alpha (1+\epsilon) > \alpha + 2p'\epsilon^2 = p' \cdot \Pr_{v_2 \sim D_1} [v_2 \geq p'] . \quad (2.5)$$

Similarly, for D_2 the revenue is maximized by $p_2 = 1$:

$$\forall p' \in \{2, k\} \;\; 1 \cdot \Pr_{v_2 \sim D_2} [v_2 \geq 1] = \beta > \beta - k\epsilon^2 = p' \cdot \Pr_{v_2 \sim D_1} [v_2 \geq p'] . \quad (2.6)$$

For the auctioneer's initial prior, $\frac{1}{2}D_1 + \frac{1}{2}D_2$, the revenue is maximized by k:

$$k \cdot \Pr_{v_2 \sim \frac{1}{2}D_1 + \frac{1}{2}D_2} [v_2 \geq k] = \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{k}{2}\epsilon^2 > \frac{1}{2}\alpha + \frac{1}{2}\beta = 1 \cdot \Pr_{v_2 \sim \frac{1}{2}D_1 + \frac{1}{2}D_2} [v_2 \geq 1] . \quad (2.7)$$

Finally, we show that for any convex combination $\lambda D_1 + (1-\lambda) D_2$ of the distributions, the revenue is maximized by some price $p_2 \in \{1, 1+\epsilon, k\}$. It is easy to see that the the optimal price belongs to the support of the mixed distribution. Yet, for any $p' \in \{3, \ldots, k-1\}$ we have:

$$\begin{aligned} k \cdot \Pr_{v_2 \sim \gamma D_1 + (1-\gamma) D_2} [v_2 \geq k] &= \gamma\alpha + (1-\gamma)\beta + (3\gamma - 1) \cdot k\epsilon^2 \\ &> \gamma\alpha + (1-\gamma)\beta \cdot \frac{p'}{k} + (3\gamma - 1) p'\epsilon^2 \\ &= p' \cdot \Pr_{v_2 \sim \gamma D_1 + (1-\gamma) D_2} [v_2 \geq p'] \end{aligned}$$

Similarly, for $p' = 2$ and $\gamma < 1$, we still have

$$k \cdot \Pr_{v_2 \sim \gamma D_1 + (1-\gamma) D_2} [v_2 \geq k] > \gamma\alpha + (1-\gamma)\beta + (6\gamma - k)\epsilon^2 = 2 \cdot \Pr_{v_2 \sim \gamma D_1 + (1-\gamma) D_2} [v_2 \geq 2] .$$

Bidder's utility Recall that $u_2(D' \mid D)$ denotes the bidder's expected utility from the second-stage auction when her true distribution is D, but the auctioneer runs a Myerson auction against a (possibly misreported) prior of D'.

Truthfulness When the bidder draws her second stage valuation from D_1 and the price is $p(D_1) = 1 + \epsilon$, her utility is

$$u_2(D_1 \mid D_1) = \alpha \cdot (H_k - 1 - \epsilon/2) + 2\epsilon^2 \cdot (k - 1 - \epsilon)$$

together with a discount of $\epsilon/5$ on the first stage, it is greater than the utility from price $p(D_2) = 1$:

$$u_2(D_2 \mid D_1) = \alpha \cdot (H_k - 1 + \epsilon/2) + 2\epsilon^2 \cdot (k - 1 - \epsilon) = u_2(D_1 \mid D_1) + \alpha\epsilon.$$

On the other hand, if the bidder draws her valuation from D_2, then we have:

$$u_2(D_1 \mid D_2) = \left(\beta\left(\frac{1}{k}\right) - \epsilon^2\right) \cdot (k-1-\epsilon) + \left(\beta\left(\frac{1}{2} - \frac{1}{k}\right) - \frac{k}{2}\epsilon^2\right) \cdot (1-\epsilon);$$

as well as

$$\begin{aligned} u_2(D_2 \mid D_2) &= \left(\beta\left(\frac{1}{k}\right) - \epsilon^2\right) \cdot (k-1) + \left(\beta\left(\frac{1}{2} - \frac{1}{k}\right) - \frac{k}{2}\epsilon^2\right) \\ &= u_2(D_1 \mid D_2) + \left(\left(\beta \cdot \frac{1}{2}\right) - \left(\frac{k}{2}+1\right)\epsilon^2\right) \cdot \epsilon. \end{aligned}$$

Therefore, the discount must satisfy $\left(\beta \cdot \frac{1}{2} - \left(\frac{k}{2}+1\right)\epsilon^2\right)\epsilon > \epsilon/5 > \alpha\epsilon$

Value of OTR The value of the OTR for a bidder with prior D_1 is given by

$$u_2(D_1 \mid D_1) + \epsilon/5 - u_2\left(\frac{1}{2}D_1 + \frac{1}{2}D_2 \mid D_1\right)$$

$$= \alpha \cdot (H_k - 1 - \epsilon/2) - 2\epsilon^2 \cdot (k-1-\epsilon) + \epsilon/5 = \alpha \cdot (H_k - 1) + O(\epsilon).$$

The value for a bidder with prior D_2 is given by

$$u_2(D_2 \mid D_2) - u_2\left(\frac{1}{2}D_1 + \frac{1}{2}D_2 \mid D_2\right)$$

$$= \left(\beta\left(\frac{1}{k}\right) - \epsilon^2\right) \cdot (k-1) + \left(\beta\left(\frac{1}{2} - \frac{1}{k}\right) - \frac{k}{2}\epsilon^2\right) = \beta\left(1.5 - O\left(\frac{1}{k}\right)\right) + O\left(\epsilon^2\right).$$

Finally, in order to achieve equal value of the OTR, choose α and β such that

$$\begin{aligned} &\alpha \cdot (H_k - 1 - \epsilon/2) - 2\epsilon^2 \cdot (k-1-\epsilon) + \epsilon/5 \\ &\qquad = \left(\beta\left(\frac{1}{k}\right) - \epsilon^2\right) \cdot (k-1) + \left(\beta\left(\frac{1}{2} - \frac{1}{k}\right) - \frac{k}{2}\epsilon^2\right) \end{aligned}$$

(In particular we can take $\beta = 1$ and $\alpha \approx 1.5/H_k$.) □

Auctioneer's revenue As we already showed in (2.5)-(2.7), the optimal expected revenue from the second item is approximately the same whether the auctioneer learns the bidder's partial information or not:

$$\frac{1}{2}\texttt{Rev}(D_1) + \frac{1}{2}\texttt{Rev}(D_2) = \frac{1}{2}\alpha(1+\epsilon) + \frac{1}{2}\beta = \frac{1}{2}\alpha + \frac{1}{2}\beta + O(\epsilon)$$

versus

$$\texttt{Rev}\left(\frac{1}{2}D_1 + \frac{1}{2}D_2\right) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{k}{2}\epsilon^2 = \frac{1}{2}\alpha + \frac{1}{2}\beta + O\left(\epsilon^2\right).$$

Chapter 3

The Computational Complexity of Dynamic Mechanism Design

In this Chapter we study the Computational Complexity of Dynamic Auctions; we prove the following two Theorems, in Sections 3.1 and 3.2 respectively.

Theorem 13. *Finding the optimal deterministic auction is strongly* NP*-hard, even for* $n = 1$ *buyers and* $m = 2$ *stages.*

Theorem 14. *For any number of stages* m*, and a constant number of independent buyers* n*, the optimal adaptive randomized auction can be found in time polynomial in the number of types and in the number of stages.*

3.1 Computing the Optimal Deterministic Auction

In this section we prove that finding the optimal deterministic dynamic auction is NP-hard even for two stages and a single buyer; we call this the Two-stage Auction problem. In order to make the reduction cleaner we slightly alter the notation. The buyer can have one of N *types*. The i-th type has probability f_i, valuation v_1^i for the first item, and probability distribution X_i over valuations for the second item. We will assume that 0 is always in the support of X_i, for all i. Our goal is to design an auction that maximizes the designer's revenue, subject to incentive compatibility in a perfect Bayesian equilibrium and ex post individual rationality.

What can we say about the structure of revenue-optimal deterministic dynamic auctions? The point of this chapter is that they are quite complex. Nonetheless we can significantly restrict our search space. Notice that by the revelation principle the most general (*adaptive*) mechanism can be described as a function that maps declared types to a price for the first item, and the combination of declared type and second stage valuation to a price for the second item. Call a mechanism *semi-adaptive* if it depends only on the buyer's declared type. In such a mechanism the buyer submits a bid for the first stage, and the seller, based on it, produces a price p_1 for the first stage and a price p_2 for the second (a price can be infinity, in which case the seller does not offer this item).

Semi-adaptive auctions are optimal

Rather surprisingly, this seemingly weak protocol is optimal.

Lemma 15. *There is a revenue-optimal deterministic mechanism which is semi-adaptive.*

Proof. Suppose that in a deterministic revenue-optimal auction satisfying incentive compatibility and ex-post individual rationality, the price on the second stage $p_2(v_1, v_2)$ depends on the buyer's valuations on both stages, v_1 and v_2. Fix any first-stage valuation $v_1 = w$, and let $u^* = \arg\min_{u \geq p_2(w,u)} p_2(w, u)$ be the second-stage valuation which minimizes that second stage price, among all second-stage valuations for which the item is allocated.

- $v_2 > p_2(w, u^*)$: the buyer could declare valuation u^* in order to buy the item for the minimum price. Therefore, since the auction is incentive compatible, it must charge $p_2(w, v_2) = p_2(w, u^*)$.
- $v_2 < p_2(w, u^*)$: we can assume wlog that the price is again $p_2(w, u^*)$, since the buyer would not buy the item anyway for the current price $p_2(w, v_2)(\geq p_2(w, u^*))$.
- $v_2 = p_2(w, u^*)$: the buyer's utility remains zero for any price $p_2(w, v_2) \geq p_2(w, u^*)$; however, the auctioneer's revenue is clearly maximized when selling the item for price $p_2(w, v_2) = q(w, u^*)$

Finally, any buyer with a different first-stage valuation $v_1 = w'$ that attempts to deviates to a bid w on the first stage, would wlog also deviate her second-stage valuation to u^*. □

Note that it is not clear whether the same is true for randomized auctions, because we do not have an order over distributions of prices: one distribution may be more attractive to one type, while another distribution is more attractive for another type.

Incentive compatibility constraints

Once we restrict ourselves to semi-adaptive auctions, the auction becomes two functions p_1, p_2 mapping the support of the prior to the reals. Let $p_1(v)$ be the price charged for the first stage item, and $p_2(v)$ the price charged for the second stage item, when the bidder reports valuation v. Let $u(v, v')$ be the expected utility of the bidder when her true value in stage one is v and she declares v'. This utility is the utility of the first stage plus the expected utility for the second stage, when offered a take-it-or-leave-it price $p_2(v')$. We want $u(v, v) \geq u(v, v')$ for all v, v'.

A nice, compact form to express our IC constraints is using the cumulative distribution of the second stage: $\bar{F}_v(x) = Pr[v_2 \geq x | v_1 = v]$. The observation here is that the buyer's second stage utility for valuation v, when charged price p_2 in stage 2, is $\int_{p_2}^{\infty} \bar{F}_v(x)dx$. So, for any two possible first-stage valuations v and v', the IC constraints are:

- If both v and v' receive the item on the first stage:

$$\int_{p_2(v')}^{p_2(v)} \bar{F}_{v'}(x)dx \geq p_1(v') - p_1(v) \geq \int_{p_2(v')}^{p_2(v)} \bar{F}_v(x)dx$$

- If neither receives the item on the first stage:

$$p_2(v) = p_2(v')$$

- If v' receives the item on the first stage, but v does not:

$$\int_{p_2(v)}^{p_2(v')} \bar{F}_v(x)dx \geq v - p_1(v')$$

$$v' - p_1(v') \geq \int_{p_2(v)}^{p_2(v')} \bar{F}_{v'}(x)dx$$

We write $Rev(t^i, p_1^i, p_2^i)$ to denote the auctioneer's revenue, when charging type t^i the first stage price p_1^i and second stage price p_2^i. We write Rev_1 or Rev_2 when we refer only to the revenue from the first or second stage, respectively.

Theorem 13. *Finding the optimal deterministic auction is strongly* NP*-hard, even for $n = 1$ buyers and $m = 2$ stages.*

Outline

Given a graph $G = (V, E)$, we construct a joint distribution of valuations such that the optimal feasible revenue (for deterministic IC and IR auctions) is a strictly increasing function of the maximum independent set in G.

More specifically, with each vertex $i \in G$ we associate a type t^i with valuation $v_1 = B_i$ for the first stage. For each type t^i, we want to have two candidate price pairs: (B_i, C_i) or (A_i, D_i). The former will give more revenue, but for every edge $(i, j) \in E$, it will be a violation of the IC constraints to charge both type t^i (B_i, C_i) and type t^j (B_j, C_j). Thus, if the difference r in expected revenue between (B_i, C_i) and (A_i, D_i) is the same for all i, charging the former for all the vertices of an independent set S and the latter for the rest of the vertices will be a valid pricing, with revenue $\sum_{i \in V} Rev(t^i, A_i, D_i) + r|S|$.

In order to impose the (B_i, C_i) vs (A_i, D_i) structure, we have an extra type t^*, with valuation $v_1 = P^*$ on the first stage. t^* appears with very high probability. This way we make most of our revenue from this type, and thus force every revenue-optimal auction to charge this type the optimal prices, (P^*, Q^*). The IC constraints for type t^* introduce strong restrictions on the prices for other types.

The restriction on each edge (i, j) is forced by the IC constraints for t^i and t^j, via a careful construction of the distributions over their second-stage valuations. The second stage distribution of t^i will be $\bar{F}_i$ and will change behavior between D_{j-1} and D_j depending on whether or not $(i, j) \in E$. See Figure 3.1.

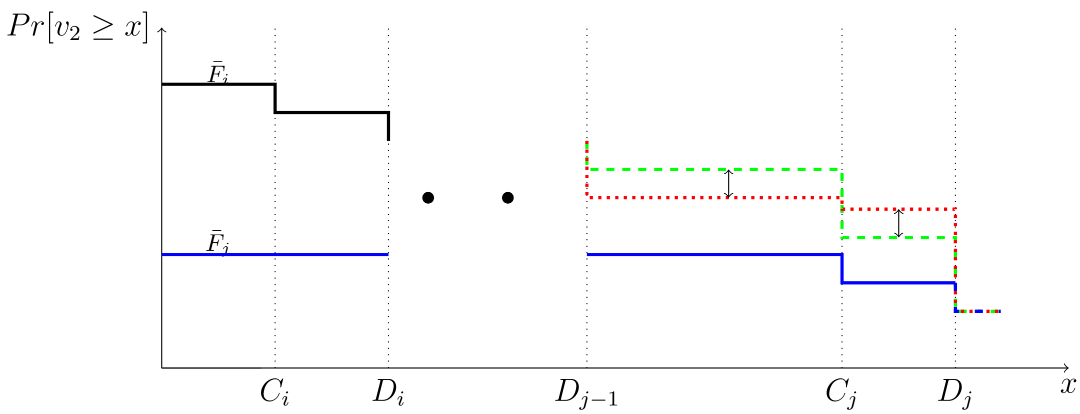

Figure 3.1: $\bar{F}_i$ when there is (dotted) an (i, j) edge for $j > i$, and when there isn't (dashed).

Construction

The distribution of valuations on the first stage is rather simple. Let $N = |V|$ denote the number of vertices in G. With probability $1 - p$, the buyer is of type t^* and has first-stage valuation $v_1 = P^* = N$; with probability $p \cdot w_i$, the buyer is of type t^i and has first-stage valuation $v_1 = B_i = N^2 + 2N + 1 - i, i \in [N]$. The parameters p and w_i will be defined soon. Notice that the first stage has support of size $N + 1$.

We will show that it is always possible to charge type i either her full value B_i on stage 1, or slightly less: $A_i = B_i - \epsilon$, for $\epsilon = 1/N^2$. For type t^*, we always want to charge the full price, P^*. Observe that

$$P^* < A_N < B_N < \cdots < A_1 < B_1.$$

Furthermore, $B_i - B_j = j - i = A_i - A_j$, $B_i - A_j = j - i + \epsilon$, and $A_i - B_j = j - i - \epsilon$.

For the second stage we are interested in pricings C_i or D_i for t^i, and Q^* for t^*. Although we only have N types, it will be convenient to think about two more special prices, which we denote C_{N+1} and D_{N+1}. We will define C_i, D_i and Q^* later; for now let us mention that

$$C_1 < D_1 < \cdots < C_N < D_N < C_{N+1} < D_{N+1} < Q^*.$$

Second stage valuations

The crux of the reduction lies in describing the distributions of the second-stage valuations for each type. It will be convenient to describe the cumulative distributions $\bar{F}_i(z) = \Pr[v_1 \geq z | t^i]$ and $\bar{F}_*(z) = [v_1 \geq z | t^*]$.

The choices of the cumulative distributions in our construction are summarized in Table 3.1. Type t^i never has nonzero second-stage valuation less than C_i, thus the cumulative distribution $\bar{F}_i(x)$ for $x \in (0, C_i)$ is $h_i = \gamma^{-4i}$, for $\gamma = 1 + 1/N$. Intuitively, this will make C_i an attractive price for the seller. Notice that $\gamma^N \approx e$ is a constant.

At each special price thereafter, $\bar{F}_i$ decreases by some multiplicative factor that is related to γ. The exact value of $\bar{F}_i(x)$ for $x \in (D_{j-1}, D_j)$ depends on whether there is an edge (i, j) in G.[1] After D_{N+1}, the distribution for all types t^i is the same. $\bar{F}_i$ halves at each $2^k D_{N+1}$, and it is 0 after $Q^* = 2^{8\gamma^{4(N+1)}} D_{N+1}$.

The distribution $\bar{F}_*$ is simpler to describe. $\bar{F}_*(x)$ is h_1 for $x \in (0, C_1)$, and decreases by a multiplicative factor of γ^2 at each special price thereafter. Type t^* never has valuations between D_{N+1} and $Q^* = 2^{8\gamma^{4(N+1)}} D_{N+1}$. $\bar{F}_*$ is constant in this domain; in particular $\bar{F}_*(x) = h_* = \frac{A_{N+1} - P^*}{Q^* - D_{N+1}}$. Intuitively, this will make Q^* an attractive price for the auctioneer. Notice also the contrast between this and the gradual decrease of $\bar{F}_i$'s. Next, we describe how to fix the last parameters.

Fixing the last parameters

One of the most important parameters in our construction is r_i: we later prove that r_i is the difference in expected revenue, conditioned on type t^i, between pricing at (B_i, C_i), and pricing at (A_i, D_i).

We set $r_{N+1} = \frac{(A_{N+1} - P^*)(\gamma - 1)}{2\gamma^{4(N+1)}} = \Theta(N)$; the rest of r_i's are defined recursively:

$$r_i = \gamma^4 r_{i+1} - (\gamma - 1)[\epsilon(\gamma^3 - \gamma) + \gamma]. \tag{3.1}$$

Notice that $\frac{r_1}{r_{N+1}} \leq \gamma^{4(N+1)} = \Theta(1)$.

Let $C_i = \frac{\frac{\gamma}{\gamma-1} r_i - \epsilon}{h_i}$ and $D_i = \frac{\gamma r_i}{(\gamma-1) h_i}$. Observe that with the recursive definition of r_i (3.1) we can get a nice expression for the following difference:

$$C_{i+1} - D_i = \gamma^2 \frac{1 - \epsilon}{h_i}.$$

The differences between pairs of special prices are summarized in Table 3.2.

Finally, we want the contribution towards the revenue from each vertex in the independent set to be the same. To that end, we set $r = \sum 1/r_i = \Theta(1)$, and weight the probability of observing each type t^i by $w_i = r/r_i$. We set the total probability of observing any of the t^i's to be $p = \frac{\epsilon}{16Nr} = \Theta(N^{-3})$.

[1]For the extra special prices, C_{N+1} and D_{N+1}, assume that all $\bar{F}_i$'s behave as in the "no edge" case.

Type	$0 \to C_i$	$C_i \to D_i$	$D_i \to C_{i+1}$	$D_{j-1} \to C_j$	$C_j \to D_j$	$2^k D_{N+1} \to 2^{k+1} D_{N+1}$	
$\bar{F}_i$	h_i	$\frac{h_i}{\gamma}$	$\frac{h_i}{\gamma^2}$	$\frac{1-\epsilon(2-\frac{1}{\gamma})}{1-\epsilon} \frac{h_{j-1}}{\gamma^2}$	$(2-\frac{1}{\gamma})h_j$	$\frac{h_{N+1}}{2^{k+1}\gamma}$	edge
			$\frac{1-\frac{\epsilon}{\gamma}}{1-\epsilon} \frac{h_i}{\gamma^2}$	$\frac{h_{j-1}}{\gamma^2}$	h_j		no edge
$\bar{F}_*$		h_i	$\frac{h_i}{\gamma^2}$	$\frac{h_{j-1}}{\gamma^2}$	h_j	h_*	

Table 3.1: Cumulative distributions

$B_i - A_i$	$A_i - B_{i+1}$	$A_{n+1} - P^*$	$D_i - C_i$	$C_{i+1} - D_i$	$Q^* - D_{N+1}$
ϵ	$1-\epsilon$	$N^2 + N - \epsilon$	$\frac{\epsilon}{h_i}$	$\gamma^2 \cdot \frac{1-\epsilon}{h_i}$	$\left(2^{8\gamma^{4(N+1)}} - 1\right) D_{N+1}$

Table 3.2: Differences between prices

Recall that the IC constraints depend on the integrals of the cumulative distribution functions. The values of the $\bar{F}_i$'s and $\bar{F}_*$ in our construction are tailored to make sure that their integrals have the values described in Table 3.3.

Type	$C_i \to D_i$	$D_i \to C_{i+1}$	$D_{j-1} \to C_j$	$C_j \to D_j$	$D_{N+1} \to Q^*$	
$\int \bar{F}_i$	$\frac{\epsilon}{\gamma}$	$1-\epsilon$	$1-(2-\frac{1}{\gamma})\epsilon$	$(2-\frac{1}{\gamma})\epsilon$	$A_{N+1} - P^*$	edge
		$1-\frac{\epsilon}{\gamma}$	$1-\epsilon$	ϵ		no edge
	$\int_{C_i}^{D_j} \bar{F}_i = B_i - A_j$					
$\int \bar{F}_*$	ϵ	$1-\epsilon$	$1-\epsilon$	ϵ	$A_{N+1} - P^*$	
	$\int_{C_i}^{Q^*} \bar{F}_* = B_i - P^*$					

Table 3.3: Integrals of cumulative distributions

Claim 16. *The integrals of the $\bar{F}_i$'s and $\bar{F}_*$ have values as stated in the Table 3.3*

Proof. Follows from multiplying the correct combination of entries of Table 3.1 and Table 3.2. □

This completes the construction of the instance of Two-stage Auction, starting from the instance of Independent Set. Incidentally, notice that the numbers used are polynomial in the size of the input graph.

Proof of the NP-hardness construction

Completeness

In this subsection we show that any independent set S in G corresponds to a feasible pricing in our auction: (B_i, C_i) for $i \in S$, (A_j, D_j) for $j \notin S$, and (P^*, Q^*) for type t^*.

Lemma 17. *Let S be an independent set of G. There exists a pricing for our auction that satisfies IC and IR and achieves revenue:*

$$(1-p)Rev(t^*, P^*, Q^*) + p\sum_{i \in V} w_i Rev(t^i, A_i, D_i) + pr|S|$$

We first show that the IC constraints are satisfied between any pair of types t^i and t^j that are not both charged (B_i, C_i) - edge or no edge in the graph (Claim 18). Then, we show that the IC constraints are satisfied between type t^* and type t^i, for any $i \in [N]$ (Claim 19). Finally we prove that charging (B_i, C_i) and (B_j, C_j) does not violate the IC constraints if there is no (i, j) edge in the graph (Claim 20).

Claim 18. *Charging types t^i and t^j, for $j > i$, any of the pairs $(B_i, C_i)/(A_j, D_j)$, $(A_i, D_i)/(B_j, C_j)$ or $(A_i, D_i)/(A_j, D_j)$, satisfies the IC constraints between t^i and t^j.*

Proof. We need to show that all the following are always true:

1.
$$\int_{C_i}^{D_j} \bar{F}_i(x)dx \geq B_i - A_j \geq \int_{C_i}^{D_j} \bar{F}_j(x)dx$$

2.
$$\int_{D_i}^{D_j} \bar{F}_i(x)dx \geq A_i - A_j \geq \int_{D_i}^{D_j} \bar{F}_j(x)dx$$

3.
$$\int_{D_i}^{C_j} \bar{F}_i(x)dx \geq A_i - B_j \geq \int_{D_i}^{C_j} \bar{F}_j(x)dx$$

It follows from Table 3.3 that the left hand sides hold. For the right hand sides, first notice that F_j is always lower than $\bar{F}_i$ in the intervals we're interested in. The first inequality is tight for $\bar{F}_i$, thus $\int_{C_i}^{D_j} \bar{F}_j(x) \leq B_i - A_j$. For $(A_i, D_i)/(B_j, C_j)$ and $(A_i, D_i)/(A_j, D_j)$ we will use induction:

- Basis $j = i+1$:

  $\int_{D_i}^{C_{i+1}} \bar{F}_{i+1}(x)dx = (C_{i+1} - D_i)h_{i+1} = (C_{i+1} - D_i)\frac{h_i}{\gamma^4} = \frac{1-\epsilon}{\gamma^2} < 1 - \epsilon = A_i - B_{i+1}$.

  And:

  $\int_{D_i}^{D_{i+1}} \bar{F}_{i+1}(x)dx = (C_{i+1} - D_i)h_{i+1} + (D_{i+1} - C_{i+1})\frac{h_{i+1}}{\gamma} = \frac{1-\epsilon}{\gamma^2} + \frac{\epsilon}{\gamma} < 1 = A_i - A_{i+1}$.

- For j we have the following:

  $\int_{D_i}^{C_j} \bar{F}_j(x)dx \leq \int_{D_i}^{D_{j-1}} \bar{F}_{j-1}(x)dx + \int_{D_{j-1}}^{C_j} \bar{F}_j(x)dx \leq (A_i - A_{j-1}) + (A_{j-1} - B_j) = A_i - B_j$

  and

  $\int_{D_i}^{D_j} \bar{F}_j(x)dx \leq \int_{D_i}^{D_{j-1}} \bar{F}_{j-1}(x)dx + \int_{D_{j-1}}^{D_j} \bar{F}_j(x)dx \leq A_i - A_{j-1} + A_{j-1} - A_j = A_i - A_j$. □

Claim 19. *When type t^* is charged (P^*, Q^*), charging t^i the pair (B_i, C_i) or the pair (A_i, D_i) doesn't violate the IC constraints between t^i and t^*.*

Proof. The IC constraints between t^i and t^* are either

$$\int_{C_i}^{Q^*} \bar{F}_i(x)dx \geq B_i - P^* \geq \int_{C_i}^{Q^*} \bar{F}_*(x)dx$$

or

$$\int_{D_i}^{Q^*} \bar{F}_i(x)dx \geq A_i - P^* \geq \int_{D_i}^{Q^*} \bar{F}_*(x)dx$$

In both cases, the inequalities can be verified easily using Table 3.3. □

Claim 20. *If $(i,j) \notin E$ the charging type t^i the pair (B_i, C_i) and type t^j the pair (B_j, C_j) doesn't violate the IC constraints between t^i and t^j.*

Proof. The IC constraint between t^i and t^j for this pricing is:

$$\int_{C_i}^{C_j} \bar{F}_i(x)dx \geq B_i - B_j \geq \int_{C_i}^{C_j} \bar{F}_j(x)dx$$

- $j = i+1$: $\int_{C_i}^{C_{i+1}} \bar{F}_i(x)dx = \int_{C_i}^{D_i} \bar{F}_i(x)dx + \int_{D_i}^{C_{i+1}} \bar{F}_i(x)dx$. The first term is equal to $\frac{\epsilon}{\gamma}$, and when there is no $(i, i+1)$ edge, the second term is equal to $1 - \frac{\epsilon}{\gamma}$, thus the left hand side is immediate. The right hand side is satisfied trivially, since $\bar{F}_{i+1}$ is always below $\bar{F}_i$ between C_i and C_{i+1} and $\bar{F}_i$ gives a tight constraint.

- $j > i+1$: Again, $\int_{C_i}^{C_j} \bar{F}_i(x)dx = \int_{C_i}^{D_{j-1}} \bar{F}_i(x)dx + \int_{D_{j-1}}^{C_j} \bar{F}_i(x)dx$. From Table 3.3 we can see that the first term is always $j - 1 - i + \epsilon$, and the second term is $1 - \epsilon$ when $(i,j) \notin E$.

  For the right hand side we have $\int_{D_i}^{C_j} \bar{F}_j(x)dx \leq A_i - B_j$ from Claim 18. Since $\bar{F}_j$ is below $\bar{F}_i$ between C_i and D_i, and $\int_{C_i}^{D_i} \bar{F}_i(x)dx = \frac{\epsilon}{\gamma} < \epsilon$ we get that:

$\int_{C_i}^{C_j} \bar{F}_j(x)dx = \int_{C_i}^{D_i} \bar{F}_j(x)dx + \int_{D_i}^{C_j} \bar{F}_j(x)dx < \int_{C_i}^{D_i} \bar{F}_i(x)dx + A_i - B_j < \epsilon + A_i - B_j = B_i - B_j$. □

Soundness

Lemma 21. *Let S be a maximum independent set in G. Then any IC and IR auction has expected revenue at most*

$$(1-p)\,Rev\left(t^{*},P^{*},Q^{*}\right)+p\sum_{i\in V}w_{i}Rev(t^{i},B_{i},D_{i})+pr\left|S\right| \tag{3.2}$$

Proof outline

We first show that charging the pair (P^{*},Q^{*}) maximizes the revenue that can be obtained from type t^{*} (Claim 22), and that (B_{i},C_{i}) yields the optimal revenue from type t^{i} (Claim 23). Observe that even if we could charge the optimal prices from every type, our expected revenue would be $(1-p)Rev(t^{*},P^{*},Q^{*})+p\sum w_{i}Rev(t^{i},B_{i},C_{i})$, which improves over (3.2) by less than $prN=\epsilon/16$. Intuitively, this means that any deviation that results in a loss of prN in terms of revenue, cannot compete with (3.2).

Next, we show (Claim 24) that if $(i,j)\in E$, then we cannot charge both t^{i} and t^{j} the optimal prices (B_{i},C_{i}) and (B_{j},C_{j}). In fact, we need a robust version of this statement: Specifically, for some small parameters $\zeta^{(1)},\zeta_{i}^{(2)}$ (to be defined later), we show that we cannot charge both t^{i} and t^{j} prices in $\left[B_{i}-\zeta^{(1)},B_{i}\right]\times\left[C_{i}-\zeta_{i}^{(2)},C_{i}\right]$ and $\left[B_{j}-\zeta^{(1)},B_{j}\right]\times\left[C_{j}-\zeta_{j}^{(2)},C_{j}\right]$, respectively.

What can we charge type t^{i} instead? In Claim 25 we show that charging less than C_{i} would require us to either not sell the item on the first stage, or charge type t^{*} less than the optimal price. On the former case, we would lose $pw_{i}\cdot B_{i}>\epsilon/16$ revenue, and would immediately imply smaller revenue than (3.2). On the latter case, we can use the robustness of Claim 24; namely, we use the fact that we cannot charge i prices that are $\left(\zeta^{(1)},\zeta_{i}^{(2)}\right)$-close to (B_{i},C_{i}). This will imply that we must change the prices for type t^{*} by some $\zeta_{*}^{(1)}$ on the first stage or $\zeta_{*}^{(2)}$ on the second stage. In either case the lost revenue is again greater than what we could potentially gain over (3.2). Therefore, we must charge t^{i} more than C_{i} on the second stage. Claim 26 shows that charging D_{i} is the best option in this case.

Therefore an upper bound to the revenue we can make is the following: charge (B_{i},C_{i}) for all i belonging to some independent set S', and (B_{j},D_{j}) for all other $j\notin S'$. (It is easy to see than in our construction even these prices won't satisfy the IC constraints). Now, the revenue given by these prices is:

$$(1-p)\,Rev\left(t^{*},P^{*},Q^{*}\right)+p\sum_{i\in S'}w_{i}Rev\left(t^{i},B_{i},C_{i}\right)+p\sum_{j\notin S'}w_{j}Rev\left(t^{j},B_{j},D_{j}\right)$$

Notice that

$$\begin{aligned}\sum_{i\in S'}w_{i}Rev\left(t^{i},B_{i},C_{i}\right) &\leq \sum_{i\in S'}w_{i}\left(Rev\left(t^{i},B_{i},C_{i}\right)-Rev\left(t^{i},A_{i},D_{i}\right)+Rev\left(t^{i},B_{i},D_{i}\right)\right)\\ &= \sum_{i\in S'}w_{i}\left(r_{i}+Rev\left(t^{i},B_{i},D_{i}\right)\right)\end{aligned}$$

Therefore, the total expected revenue

$$(1-p)\,Rev\left(t^*,P^*,Q^*\right)+pr|S'|+p\sum_{i\in S'}w_iRev\left(t^i,B_i,D_i\right)+p\sum_{j\notin S'}w_jRev\left(t^j,B_j,D_j\right)$$

which is at most the expression in (3.2)

Preliminaries

We begin by setting our padding parameters: let $\zeta^{(1)}=\frac{\epsilon}{4}$, and for each i let $\zeta_i^{(2)}=\frac{\epsilon}{4\gamma^2h_i}$. In particular, this implies that for every i, $\zeta_i^{(2)}h_i+\zeta^{(1)}<\frac{\epsilon}{2}<\epsilon-\epsilon'$. Next, let $\zeta_*^{(1)}=\frac{\epsilon}{8}$, and $\zeta_*^{(2)}=\frac{\epsilon}{8h_*}$. We now have that $\zeta_i^{(2)}h_i\gamma^2=\zeta^{(1)}=\zeta_*^{(2)}h_*+\zeta_*^{(1)}$, which we will use later in the proof. Most importantly, recall that losing $\frac{\epsilon}{8}$ from the revenue from type t^*, is equivalent to a loss of $(1-p)\frac{\epsilon}{8}>\frac{\epsilon}{16}$ from the total expected revenue, which immediately implies that the expected revenue is less than (3.2).

Optimality of (P^*,Q^*)

We will now prove that prices (P^*,Q^*) maximize the revenue from type t^*, in a robust sense:

Claim 22. *Charging type t^* prices (P^*,Q^*) maximizes the revenue from that type. Furthermore, if $p_1^*<P^*-\zeta_*^{(1)}$ or $p_2^*<Q^*-\zeta_*^{(2)}$, then the revenue from type t^* is lower than the maximal revenue by at least $\zeta_*^{(1)}$ or $\zeta_*^{(2)}h_*$, respectively.*

Proof. Clearly, P^* is the most that we can charge type t^* on the first stage. It is left to show that Q^* maximizes the revenue on the second stage.

On the second stage, we have: $Rev_2\left(t^*,Q^*\right)=Q^*h_*>A_{N+1}-P^*$. Recall that $\bar{F}_*$ changes on C_i's and D_i's, so those are the only candidates we should compare with Q^*. For any C_i, we have

$$Rev_2\left(t^*,C_i\right)=C_ih_i\gamma^2<\frac{\gamma^3r_i}{\gamma-1}\le\frac{\gamma^3r_1}{\gamma-1}\le\frac{\gamma^{4(N+1)}r_{N+1}}{\gamma-1}=\frac{A_{N+1}-P^*}{2}.$$

Similarly, for D_i,

$$Rev_2\left(t^*,D_i\right)=D_ih_i<\frac{\gamma r_i}{\gamma-1}<\frac{A_{N+1}-P^*}{2}.$$ □

Optimality of (B_i,C_i)

Similarly, we show that (B_i,C_i) maximize the revenue from type t^i.

Claim 23. $\forall x\neq C_i\;\; Rev_2\left(t^i,C_i\right)>Rev_2\left(t^i,x\right)$.

Proof. Since $\bar{F}_i$ is constant for all $x \leq C_i$, the claim for this domain follows trivially. We will prove that $Rev_2(t^i, C_i) > Rev_2(t^i, D_i)$ and deduce from Claim 26 that the claim continues to holds for any other x.

$$Rev_2\left(t^i, C_i\right) = C_i \cdot \bar{F}_i(C_i) = \frac{\gamma}{\gamma-1} r_i - \epsilon = \frac{r_i}{\gamma-1} + r_i - \epsilon > \frac{r_i}{\gamma-1} = Rev_2(t_i, D_i). \quad \square$$

Condition on edges

Below we show that if there is an edge (i,j), then we cannot charge both t^i and t^j close to their optimal prices:

Claim 24. *If $(i,j) \in E$ then it cannot be that $(p_1^i, p_2^i) \in \left[B_i - \zeta^{(1)}, B_i\right] \times \left[C_i - \zeta_i^{(2)}, C_i\right]$ and $\left(p_1^j, p_2^j\right) \in \left[B_j - \zeta^{(1)}, B_j\right] \times \left[C_j - \zeta_j^{(2)}, C_j\right]$*

Proof. Wlog, let $i < j$. Assume by contradiction that the conclusion is false.

Then we get $\int_{p_2^i}^{p_2^j} \bar{F}_i < p_1^i - p_1^j$, which is a contradiction to IC constraints for type i:

$$\begin{aligned}
\int_{p_2^i}^{p_2^j} \bar{F}_i &= \int_{p_2^i}^{C_i} \bar{F}_i + \int_{C_i}^{C_j} \bar{F}_i + \int_{C_j}^{p_2^j} \bar{F}_i \\
&\leq \int_{C_i}^{C_j} \bar{F}_i + \zeta_i^{(2)} h_i = j - i - \epsilon + \epsilon' + \zeta_i^{(2)} h_i \\
&< j - i - \zeta^{(1)} \\
&= B_i - B_j - \zeta^{(1)} \\
&\leq p_1^i - p_1^j ,
\end{aligned}$$

where the third line follows by $\zeta_i^{(2)} h_i + \zeta^{(1)} < \epsilon - \epsilon'$. $\square$

Restriction imposed by charging (P^*, Q^*) for type $*$

The claim below essentially shows that we cannot go around the restriction on prices for neighbors by reducing the prices:

Claim 25. *If $p_1^* > P^* - \zeta_*^{(1)}$ and $p_2^* > Q^* - \zeta_*^{(2)}$, then in any IC solution either:*

- *$p_1^i > B_i$ - note that this means that type i cannot purchase the item on the first stage; or*
- *$p_2^i > C_i$ - note that this substantially decreases our revenue for type i on the second stage; or*
- *$p_1^i \geq B_i - \zeta^{(1)}$ and $p_2^i \geq C_i - \zeta_i^{(2)}$*

Proof. The negation of the claim gives us two configurations: having $p_1^i \leq B_i$ and $p_2^i < C_i - \zeta_i^{(2)}$, and having $p_1^i < B_i - \zeta^{(1)}$ and $p_2^i \leq C_i$. We show the Claim is true by contradiction, i.e. both these configurations are violating.

Assume first that $p_1^i \leq B_i$ and $p_2^i < C_i - \zeta_i^{(2)}$. Consider the IC constraint comparing t^*'s utility when telling the truth and when claiming that she is type t_i:

$$\begin{aligned}\int_{p_2^i}^{p_2^*} \bar{F}_* &= \int_{p_2^i}^{C_i} \bar{F}_* + \int_{C_i}^{Q^*} \bar{F}_* + \int_{Q_*}^{p_2^*} \bar{F}_* \\ &> \int_{C_i-\zeta_i^{(2)}}^{C_i} \bar{F}_* + \int_{C_i}^{Q^*} \bar{F}_* + \int_{Q_*}^{Q^*-\zeta_*^{(2)}} \bar{F}_* = \int_{C_i}^{Q^*} \bar{F}_* + \zeta_i^{(2)} \frac{h_{i-1}}{\gamma^2} - \zeta_*^{(2)} h_* \\ &= \int_{C_i}^{Q^*} \bar{F}_* + \zeta_*^{(1)} = B_i - P^* + \zeta_*^{(1)} \\ &\geq p_1^i - p_1^*\end{aligned}$$

where the third line follows from $\zeta_i^{(2)} \frac{h_{i-1}}{\gamma^2} = \zeta_*^{(2)} h_* + \zeta_*^{(1)}$.

We now return to the other violating configuration, namely $p_1^i < B_i - \zeta^{(1)}$ and $p_2^i \leq C_i$. We now have

$$\begin{aligned}\int_{p_2^i}^{p_2^*} \bar{F}_* &= \int_{p_2^i}^{C_i} \bar{F}_* + \int_{C_i}^{Q^*} \bar{F}_* + \int_{Q_*}^{p_2^*} \bar{F}_* \\ &> \int_{C_i}^{C_i} \bar{F}_* + \int_{C_i}^{Q^*} \bar{F}_* + \int_{Q_*}^{Q^*-\zeta_*^{(2)}} \bar{F}_* = \int_{C_i}^{Q^*} \bar{F}_* - \zeta_*^{(2)} h_* \\ &= \int_{C_i}^{Q^*} \bar{F}_* - \zeta^{(1)} + \zeta_*^{(1)} = B_i - \zeta^{(1)} - P^* + \zeta_*^{(1)} \\ &\geq p_1^i - p_1^*\end{aligned}$$

where the third line follows from $\zeta_i^{(2)} \frac{h_{i-1}}{\gamma^2} = \zeta_*^{(2)} h_* + \zeta_*^{(1)}$. □

Optimality of (B_i, D_i)

We now show that D_i is the optimal price on the second stage for type t^i, conditioned on charging more than C_i.

Claim 26. $\forall y > C_i \;\; Rev_2\left(t^i, D_i\right) \geq Rev_2\left(t^i, y\right)$

Proof. It is easy to see that the second stage revenue is maximal for one of the "special points" where $\bar{F}_i$ changes. At D_i we have:

$$Rev_2\left(t^i, D_i\right) = D_i \cdot \bar{F}_i\left(D_i\right) = \frac{\gamma r_i}{\left(\gamma - 1\right) h_i} \cdot \frac{h_i}{\gamma_i} = \frac{r_i}{\gamma - 1}.$$

We now compare with each of type of special point:

- What happens if we set $p_2^i = C_{i+1}$?

$$\begin{aligned} Rev_2\left(t^i, C_{i+1}\right) &= C_{i+1} \cdot \bar{F}_i\left(C_{i+1}\right) \leq \frac{\frac{\gamma}{\gamma-1} r_{i+1} - \epsilon}{h_i \gamma^{-4}} \cdot \frac{h_i}{\gamma^2}\left(\frac{1-\epsilon/\gamma}{1-\epsilon}\right) \\ &\leq \frac{\gamma^5 r_{i+1}}{\gamma^2\left(\gamma-1\right)}\left(1+2\epsilon\right) = \frac{\gamma r_i + \left(\gamma-1\right)\left[\epsilon\left(\gamma^4-\gamma^2\right)+\gamma^2\right]}{\gamma^2\left(\gamma-1\right)}\left(1+2\epsilon\right) \\ &\leq \frac{1+2\epsilon}{\gamma\left(\gamma-1\right)} r_i + \left[\epsilon\left(\gamma^2-1\right)+1\right]\left(1+2\epsilon\right) \\ &\leq \frac{\gamma}{\gamma\left(\gamma-1\right)} r_i - \left(\frac{\gamma-\left(1+2\epsilon\right)}{\gamma\left(\gamma-1\right)}\right) r_i + \left[\epsilon\left(\gamma^2-1\right)+1\right]\left(1+2\epsilon\right) \\ &\leq \frac{r_i}{\gamma-1} - \frac{r_i}{2\gamma} + \left[\epsilon\left(\gamma^2-1\right)+1\right]\left(1+2\epsilon\right). \end{aligned}$$

  The equation in the second line follows from the recursive definition of r_i; the last inequality follows from $\gamma > 1+4\epsilon$. Now, using that $r_i > 2\gamma\left[\epsilon\left(\gamma^2-1\right)+1\right]\left(1+2\epsilon\right)$ for all i, we have that $Rev_2\left(t^i, C_{i+1}\right) < Rev_2\left(t^i, D_i\right)$.

- What happens if we set $p_2^i = D_{i+1}$?

$$\begin{aligned} Rev_2\left(t^i, D_{i+1}\right) &= D_{i+1} \cdot \bar{F}_i\left(D_{i+1}\right) \leq \frac{\gamma r_{i+1}}{\left(\gamma-1\right) h_{i+1}} h_{i+1}\left(2-1/\gamma\right) \\ &\leq \frac{r_{i+1}}{\left(\gamma-1\right)}\left(2\gamma-1\right) = \frac{2\gamma-1}{\gamma^3} \cdot \frac{\gamma r_i + \left(\gamma-1\right)\left[\epsilon\left(\gamma^4-\gamma^2\right)+\gamma^2\right]}{\gamma^2\left(\gamma-1\right)} \\ &\leq \frac{\gamma r_i + \left(\gamma-1\right)\left[\epsilon\left(\gamma^4-\gamma^2\right)+\gamma^2\right]}{\gamma^2\left(\gamma-1\right)} \\ &\leq Rev_2\left(t^i, D_i\right) \end{aligned}$$

  where the last inequality follows from the analysis for $Rev_2\left(t^i, C_{i+1}\right)$.

- What about the revenue when we charge C_{i+2}, D_{i+2} ? We reduce this case to what we already know about the revenue from type $i+1$:

  Observe that $\bar{F}_i\left(C_{i+1}\right) > \bar{F}_{i+1}\left(C_{i+1}\right)$, but $\bar{F}_i\left(C_{i+2}\right) < \bar{F}_{i+1}\left(C_{i+2}\right)$. Therefore,

$$Rev_2\left(t^i, C_{i+2}\right) < Rev_2\left(t^{i+1}, C_{i+2}\right) \leq Rev_2\left(t^{i+1}, C_{i+1}\right) < Rev_2\left(t^i, C_{i+1}\right) \ .$$

  A similar argument works for D_{i+2}, and the claim follows by induction for all C_j, D_j.

- Finally, for points $x > D_{N+1}$, we will show that $Rev_2\left(t^N, D_{N+1}\right)$ is greater than $Rev_2\left(t^N, x\right)$, and the claim will follow for all $i \leq N$ by the previous argument. (Recall that in the domain $x > D_{N+1}$, $\bar{F}_i$ is the same for all i.)

  $\bar{F}_n$ changes its values at points $2^k D_{N+1}$. We have:

$$Rev_2\left(t^N, 2^k D_{N+1}\right) = 2^k D_{N+1} \cdot \bar{F}_N\left(2^k D_{N+1}\right) = \tfrac{D_{N+1} h_{N+1}}{2\gamma} < \tfrac{D_N h_N}{2\gamma} = \tfrac{Rev_2\left(t^N, D_N\right)}{2}. \quad \square$$

Putting it all together

In Lemma 17 we saw that if there exists an independent set of size $|S|$ there exists an IC and IR satisfying pricing which yields revenue $(1-p)Rev(t^*, P^*, Q^*) + p\sum_{i\in V} w_i Rev(t^i, A_i, D_i) + pr|S|$. In Lemma 21 we saw that no auction can revenue more than $(1-p)\,Rev\,(t^*, P^*, Q^*) + p\sum_{i\in V} w_i Rev(t^i, B_i, D_i) + pr\,|S|$, where $|S|$ is the size of the maximum independent set in G.

All that's left is to show that a graph with maximum independent set of size $|S|-1$ cannot yield revenue $(1-p)Rev(t^*, P^*, Q^*) + p\sum_{i\in V} w_i Rev(t^i, A_i, D_i) + pr|S|$. To this end we need to show that,

$$(1-p)Rev(t^*, P^*, Q^*) + p\sum_{i\in V} w_i Rev(t^i, A_i, D_i) + pr|S| > \\ (1-p)Rev(t^*, P^*, Q^*) + p\sum_{i\in V} w_i Rev(t^i, B_i, D_i) + pr(|S|-1).$$

or equivalently,

$$\begin{aligned} & pr > p\sum_{i\in V} w_i(Rev(t^i, B_i, D_i) - Rev(t^i, A_i, D_i)) = p\sum_{i\in V} w_i\epsilon \\ \iff & r > \sum_{i\in V} \frac{r\epsilon}{r_i} \\ \iff & 1 > \sum_{i\in V} \frac{\epsilon}{r_i}, \end{aligned}$$

which is true since $\epsilon = \frac{1}{N^2}$, and each $r_i = O(N)$. With this the reduction is complete.

3.2 Computing the Optimal Randomized Auction

Semi-adaptive Auctions

Can we do better by using randomization? We first construct an LP that gives a randomized mechanism that performs at least as well — and sometimes much better — than the optimal deterministic mechanism. A randomized semi-adaptive auction takes as input the buyer's declared type on stage one, and outputs a distribution over pairs of prices. It is not clear that the optimum randomized auction is semi-adaptive. But the optimum semi-adaptive randomized auction has at least as good a revenue as any two-stage deterministic auction.

A key observation that significantly reduces the search space is the following: when considering randomized auctions, we can assume without loss of generality that every price in the support is exactly equal to a feasible valuation in the support of buyer's types, zero, or infinite.

Lemma 27. *Let V_k be the set of possible valuations on stage k. Then given any semi-adaptive auction $\mathcal{A}$ (randomized or deterministic), there exists a randomized semi-adaptive auction $\mathcal{A}'$, with at least as good revenue, which on stage k only offers prices in $V_k \cup \{0, \infty\}$.*

Proof. Let $V_k = \{v_{k,1} \leq v_{k,2} \leq \cdots \leq v_{k,n}\}$, and let $v_{k,0} = 0$ and $v_{k,n+1} = \infty$. Let $p_k \in (v_{k,i}, v_{k,i+1})$ be a possible price $\mathcal{A}$ charges on stage k. We can construct an auction $\mathcal{A}'$ identical to $\mathcal{A}$, except that $\mathcal{A}'$ asks for $v_{k,i}$ and $v_{k,i+1}$ on stage k with different probabilities, and never asks for p_k. Applying this argument recursively proves the claim.

Let π, π_i, π_{i+1} be the probabilities that $\mathcal{A}$ charges p_k,v_i and v_{i+1} respectively, on stage k, and $p_k = \alpha v_{k,i} + (1-\alpha)v_{k,i+1}$. Then $\mathcal{A}'$ simply charges $v_{k,i}$ with probability $\pi_i + \alpha\pi$ and $v_{k,i+1}$ with probability $\pi_{i+1} + (1-\alpha)\pi$. Observe that the expected price is the same.

Notice that the probability of allocation of the item on stage k can only increase: if the buyer bought the item on stage k for price p_k, her valuation could only be $v_{k,j} \geq v_{k,i+1}$, so the probability she buys is unchanged. If she didn't buy with price p_k and her valuation was $v_{k,i}$, she will buy with $(\alpha\pi)$ higher probability in $\mathcal{A}'$. When $v_{k,j} < v_{k,i}$ nothing changes.

Since the probability of allocation increases, with the same expected prices, the revenue also increases. On the other hand the buyer's utility remains unchanged between $\mathcal{A}$ and $\mathcal{A}'$: the expected price is the same, and the extra probability of allocation comes from cases when the buyer pays her valuation. Thus, the IC constraints of $\mathcal{A}$ continue to hold in $\mathcal{A}'$. □

It is easy to see that this proof did not use in any crucial way the semi-adaptive property, so this normalization is possible for all randomized auctions. We are now ready to describe our LP for two-stage semi-adaptive randomized auctions:

Theorem 28. *The optimal two-stage semi-adaptive randomized auction can be found in time polynomial in the number of types.*

Proof. We construct an LP of size $O(|V_1| \cdot (|V_1| + |V_2|))$ that optimizes over all two-stage semi-adaptive randomized auctions:

- **Variables:** The variables in our LP will specify the distribution of prices given the valuation on the first stage. Notice that because of the restriction on the class of auctions, we can assume wlog that given the valuation on the first stage, the prices on the two stages are independent. Let the variable $x(k, p, v_1)$ denote the probability that we offer the item on stage k for price p given first-stage valuation v_1. By Lemma 27, we only need to consider $V_k + 2$ different prices on each stage.

- **Objective:** Our expected revenue on the first stage is given by:

$$\text{Rev}_1 = \sum_{v_1} \sum_{p \leq v_1} x(1, p, v_1) \Pr[v_1] \cdot p$$

  On the second stage, we must also sum over the new valuations v_2

$$\text{Rev}_2 = \sum_{v_1} \sum_{v_2} \sum_{q \leq v_2} x(2, q, v_1) \Pr[(v_1, v_2)] \cdot q$$

  Our objective is to maximize the total revenue: $\max \text{Rev}_1 + \text{Rev}_2$.

- **Feasibility constraints:** In order for $x(k,p,v_1)$ to be feasible probabilities, their sum, for each k and v_1 must be one: $\forall k \in \{1,2\} \forall v_1 \ \sum_p x(k,p,v_1) = 1$. Similarly, they should all be non-negative: $\forall k \in \{1,2\} \forall v_1 \ \forall p \ \ x(k,p,v_1) \geq 0$.

- **IC constraints:** Given that the buyer's true valuation on the first stage is v_1, her utility from declaring u is given by:

$$\mathrm{U}(v_1,u) = \sum_{p \leq u} x(1,p,u) \cdot (v_1 - p) + \sum_{v_2} \sum_{q \leq v_2} x(2,q,u) \cdot (v_2 - q) \cdot Pr[v_2|v_1]$$

  The IC constraints require that

$$\forall (v,u) \quad \mathrm{U}(v,v) \geq \mathrm{U}(v,u)$$

Notice that the IR constraints are implied by the fact that we only add up prices smaller than valuations. The variables $x(k,p,u)$ for $p > u$ don't show up anywhere in the LP. □

Adaptive Auctions

In this subsection we write an LP for the revenue-optimal and adaptive randomized auction, that is incentive compatible in a perfect Bayesian equilibrium and ex-post individually rational, for n independent bidders and m stages. The main challenge with optimizing over dynamic (and adaptive) auctions is the structure of the IC constraints. As was the case for semi-adaptive auctions, we do not lose generality by restricting to prices that are exactly feasible valuations:

Claim 29. *Let V_k^i be the set of possible valuations of buyer i on stage k. Then, given any feasible adaptive auction $\mathcal{A}$, there exists a feasible randomized adaptive auction $\mathcal{A}'$ where i is always charged on stage k prices in $V_k^i \cup \{0,\infty\}$.*

Proof. Essentially the same as the proof of Lemma 27. □

Finally, we are ready to describe our LP a randomized adaptive auction for $n > 1$ bidders and $m > 2$ stages.

Theorem 14. *For any number of stages m, and a constant number of independent buyers n, the optimal adaptive randomized auction can be found in time polynomial in the number of types and in the number of stages.*

Note that typically, the number of types grows exponentially with m.

Proof. We define an LP that optimizes over all feasible joint distributions of allocations and prices. The most interesting part is the dynamic programming definition of the IC constraints.

- **Probability Variables:** We define variables that specify the joint distribution of prices and allocations. Let $\mathbf{v}_{\leq k}$ denote the history of valuations up to stage k, and let $\mathbf{p}_{<k}$ and $\mathbf{x}_{<k}$ denote the prices and allocations determined prior to stage k. Let the variable $\pi(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k})$ denote the probability that we offer allocate all the items up to time k to bidders $\mathbf{x}_{\leq k}$ and charge prices $\mathbf{p}_{\leq k}$ given valuations $\mathbf{v}_{\leq k}$. (Of course, we only maintain variables that corresponds to feasible and IR allocations and prices.)

- **Objective:** Our expected revenue from allocating all the items up to time m to bidders $\mathbf{x}_{\leq m}$ and charge prices $\mathbf{p}_{\leq m}$ given valuations $\mathbf{v}_{\leq m}$ is $f(\mathbf{v}_{\leq m}) \sum_{k \leq m} \mathbf{p}_{\leq k}$. Our overall expected revenue is therefore:

$$\text{Rev} = \sum_{\mathbf{v}_{\leq m}} \sum_{\mathbf{x}_{\leq m}, \mathbf{p}_{\leq m}} \pi(m, \mathbf{x}_{\leq m}, \mathbf{p}_{\leq m}, \mathbf{v}_{\leq m}) \cdot f(\mathbf{v}_{\leq m}) \sum_{k \leq m} \mathbf{p}_{\leq k}$$

- **"Ignorance about the future" constraints:** Although our distribution is fully described by the variables $\pi(m, \mathbf{x}_{\leq m}, \mathbf{p}_{\leq m}, \mathbf{v}_{\leq m})$ corresponding to the last stage m, we must make sure that the marginals on stages $k < m$ do not depend on future valuations $v^i_{k'}$ for $k' > k$. For every feasible choice of $k, \mathbf{v}_{\leq k}, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}$:

$$\pi(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k}) =$$
$$\sum_{\mathbf{u}_{k+1}} \sum_{\mathbf{y}_{k+1}, \mathbf{q}_{k+1}} \pi(k, [\mathbf{x}_{\leq k}; \mathbf{y}_{k+1}] [\mathbf{p}_{\leq k}; \mathbf{q}_{k+1}], [\mathbf{v}_{\leq k}; \mathbf{u}_{k+1}]) \cdot f(\mathbf{u}_{k+1} \mid \mathbf{v}_{\leq k})$$

  where $[\mathbf{x}_{\leq k}; \mathbf{y}_{k+1}]$ denotes the concatenation of $\mathbf{x}_{\leq k}$ and $\mathbf{y}_{k+1}$, and similarly $[\mathbf{p}_{\leq k}; \mathbf{q}_{k+1}]$ and $[\mathbf{v}_{\leq k}; \mathbf{u}_{k+1}]$.

- **Feasibility constraints:** In order for $\pi(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k})$ to be feasible probabilities, their sum, for each k and $\mathbf{v}_{\leq k}$ must be one:

$$\forall k \, \forall \mathbf{v}_{\leq k} \quad \sum \pi(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k}) = 1$$

  Similarly, they should all be non-negative:

$$\forall k \, \forall \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k}, \quad \pi(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \mathbf{v}_{\leq k}) \geq 0$$

- **IC constraints:** As we mentioned earlier, at any stage k, each bidder must choose among exponentially many strategies to deviate from the truth: today, and in the future.

  Our LP will use dynamic programming to recursively define the optimal deviation, starting from the last stage m and moving back in time. The utility for any action today is the value of today's allocation minus today's price, plus the expected utility of the maximum among all future deviations. Since the maximum is not a linear function, we use $\mathrm{U}(\cdot)$ to denote an upper bound on the utility from the optimal deviation. Then we use $\mathrm{V}(\cdot)$ to denote the expected utility from always reporting truthfully (now and in the future); here there is no maximum so the LP can compute this quantity exactly. Finally the IC constraint will require that $\mathrm{V}(\cdot) \geq \mathrm{U}(\cdot)$.

Begin from the last stage m. Suppose that bidder's i current valuation is v_m^i, that her previous valuations were $v_{<m}^i$, and she has reported $u_{<m}^i$; suppose further that the history of other bidders' valuation is $v_{\leq m}^{-i}$, and that items were allocated according to $\mathbf{x}_{<m}$ for prices $\mathbf{p}_{<m}$. Then bidder's i utility from reporting u_m^i on stage m is given by:

$$\mathrm{U}\left(m, v_m^i, u_m^i \mid v_{<m}^{-i}, u_{<m}^i, \mathbf{x}_{<m}, \mathbf{p}_{<m}\right) =$$
$$\sum_{v_m^{-i}} f\left(v_m^{-i} \mid v_{<m}^{-i}\right) \sum_{x_m \,:\, x_m^i > 0} \pi\left(m, \mathbf{x}_{\leq m}, \mathbf{p}_{\leq m}, \left[u_{\leq m}^i; v_{\leq m}^{-i}\right]\right)\left(v_m^i - p_m\right)$$

Similarly, the utility from reporting truthfully is simply given by:

$$\mathrm{V}\left(m, v_m^i \mid \mathbf{v}_{<m}, \mathbf{x}_{<m}, \mathbf{p}_{<m}\right) =$$
$$\sum_{v_m^{-i}} f\left(v_m^{-i} \mid v_{<m}^{-i}\right) \sum_{x_m \,:\, x_m^i > 0} \pi\left(m, \mathbf{x}_{\leq m}, \mathbf{p}_{\leq m}, \left[u_{\leq m}^i; v_{\leq m}^{-i}\right]\right)\left(v_m^i - p_m\right)$$

Next, for any $k \leq m$, given history $\mathbf{v}_{<k}, u_{<k}^i, \mathbf{x}_{<k}, \mathbf{p}_{<k}$, we let U^* denote an upper bound on the buyer's maximal utility from any current and future deviation:

$$\mathrm{U}^*\left(k, v_k^i \mid \mathbf{v}_{<k}, u_{<k}^i, \mathbf{x}_{<k}, \mathbf{p}_{<k}\right) \geq \max_{u_k} \mathrm{U}\left(k, v_k^i, u_k^i \mid \mathbf{v}_{<k}, u_{<k}^i, \mathbf{x}_{<k}, \mathbf{p}_{<k}\right)$$

Now, given the values of U^* for stage $k+1$, we can compute the utility of the buyer from deviating on stage k:

$$\begin{aligned}\mathrm{U}\left(k, v_k^i, u_k^i \mid \mathbf{v}_{<k}, u_{<k}^i, \mathbf{x}_{<k}, \mathbf{p}_{<k}\right) = &\sum_{\mathbf{x}_k \,:\, x_k^i > 0} \pi\left(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \left[u_{\leq k}^i; v_{\leq k}^{-i}\right]\right)\left(v_k^i - p_k\right) \\ &+ \sum_{v_{k+1}^i} f\left(v_{k+1}^i \mid v_{\leq k}^i\right) \mathrm{U}^*\left(k+1, v_{k+1}^i \mid \mathbf{v}_{\leq k}, u_{\leq k}^i, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}\right)\end{aligned}$$

For the truthful bidding, we simply have

$$\begin{aligned}\mathrm{V}\left(k, v_k^i \mid \mathbf{v}_{<k}, u_{<k}^i, \mathbf{x}_{<k}, \mathbf{p}_{<k}\right) = &\sum_{\mathbf{x}_k \,:\, x_k^i > 0} \pi\left(k, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}, \left[v_{\leq k}^i; v_{\leq k}^{-i}\right]\right)\left(v_k^i - p_k\right) \\ &+ \sum_{v_{k+1}^i} f\left(v_{k+1}^i \mid v_{\leq k}^i\right) \mathrm{V}\left(k+1, v_{k+1}^i \mid \mathbf{v}_{\leq k}, u_{\leq k}^i, \mathbf{x}_{\leq k}, \mathbf{p}_{\leq k}\right)\end{aligned}$$

Finally, our IC constraints require that whenever the buyer reported truthfully so far, she must maximize her value by continuing to report truthfully. For every feasible choice of $u_k^i, v_k^i, \mathbf{v}_{<k}, \mathbf{x}_{<k}, \mathbf{p}_{<k}$,

$$\mathrm{V}\left(k, v_k^i \mid \ldots\right) \geq \mathrm{U}\left(k, v_k^i, u_k^i \mid \ldots\right).$$

□

Chapter 4

The Competition Complexity of Dynamic Mechanism Design

4.1 Introduction

As we've already seen, the optimal dynamic mechanism can be extremely complex, both computationally and in terms of its description. Even worse, the optimal mechanism depends on detailed knowledge of the buyers' distributions (across time) in intricate ways. This Chapter aims to answer the following question: *Can we design simple dynamic mechanisms that do not depend on details of the underlying distributions?*

We are not the first to face such problems. Optimal mechanisms can be extremely complex even for a static auction with a single additive buyer (e.g. m uniform i.i.d. items on $[c, c+1]$ as observed by Daskalakis, Deckelbaum, and Tzamos [2015]). But even for the case of n i.i.d. buyers and a single item for sale, where Myerson's theory Myerson [1981] readily applies, the necessity of good prior information makes the solution somewhat less appealing in practice. Can this be avoided? An elegant result from auction theory, by Jeremy Bulow and Peter Klemperer Bulow and Klemperer [1996], states that the revenue of a second price auction with $n+1$ buyers with valuations drawn i.i.d. from a regular distribution[1] is at least that of the optimal auction, *tailored* for the exact distributions, for n buyers. In other words, it is better to invest in recruiting one more agent and run VCG, than to invest in *exactly* learning the underlying value distribution and then run the revenue-maximizing auction *tailored* to this distribution. One of the reasons this theorem is so appealing is because VCG is *prior-independent*, meaning its description is independent of the underlying distribution. The Bulow-Klemperer theorem can also be seen as a "resource augmentation" argument. The optimal auction extracts more revenue than a second price auction by definition; their theorem gives an intuitive characterization of how much. A generalization to non-identical distributions was given by Hartline and Roughgarden [2009], and more recently, Eden et

[1]A distribution D with density f and cumulative density F is regular if Myerson's virtual function $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is monotone non-decreasing.

al. [2016] proved the first *full* Bulow-Klemperer result for multi-dimensional static settings[2].

Here, we prove the first Bulow-Klemperer type results for dynamic auctions. We are interested in the number of additional buyers necessary (these extra buyers will be present in all stages) for a second price auction *at every stage* to have expected revenue at least that of the optimal dynamic auction. We adopt the terminology of Eden et al. [2016] and call this number the *Competition Complexity*. We also define the α-*approximate Competition Complexity* to be the extra number of buyers necessary for a second price auction at each stage to be an α approximation (in terms of revenue) of the optimal dynamic auction. Thus, in this terminology, Bulow and Klemperer's theorem says that the Competition Complexity of a single item static auction with n buyers is 1.

The original result of Bulow-Klemperer stems from our deep understanding of revenue optimal auctions in the single item case, thanks to Myerson's work. In contrast, maximizing revenue in dynamic or other multi-dimensional environments is poorly understood. Moreover, the gaps between the revenue of adaptive and non-adaptive dynamic auctions we presented in Chapter 2 imply that assumptions stronger than regularity of the stage distributions will be necessary. Despite these obstacles, for *correlated stages* we can show the following bounds:

Informal Theorem 1. *In the case of n buyers and m correlated stages (the value of buyer i on stage k can be correlated with past and future values, but not with other buyers' values), if all the stage (marginal) distributions have monotone hazard rate (MHR)*[3]*:*

- *The Competition Complexity is at most $3n$ and at least $(e-1)n$.*
- *The $\frac{1}{e}$-approximate Competition Complexity is 1.*
- *The $\frac{1}{3}$-approximate Competition Complexity is 0.*

In other words, if the stage distributions have monotone hazard rate, recruiting $3n$ additional buyers is *strictly* better than learning all m stage distributions *exactly*, including the possible correlation between stages. If an approximation of $\frac{1}{e}$ suffices, only one additional buyer is necessary. Even more suprisingly, simply running a second price auction at each stage extracts a $\frac{1}{3}$-approximation of the optimal revenue. These bounds hold even against ex-ante IR dynamic auctions. In other words, running a second price auction at each stage, which requires no distributional knowledge whatsoever, is a $\frac{1}{3}$ approximation to the impossible benchmark of an *optimal*, *ex-ante IR* auction that uses full knowledge of the buyers' *arbitrarily correlated distributions*!

Many common families of distributions such as the Uniform, Exponential and Normal have MHR. To get the theorem we prove new bounds on the expected order statistics of

[2] By "full" we mean that the revenue of VCG with $n + c$ buyers is at least that of the optimal auction with n buyers. No approximations.

[3] A distribution has *monotone hazard rate* (MHR) if its hazard rate $h(v) = \frac{f(v)}{1-F(v)}$ is monotone non-decreasing.

MHR distributions and combine them with an upper bound (for the optimal dynamic revenue) equal to the social welfare, i.e. the sum (over stages) of the expected first order statistic $\mathbb{E}[(X_k)_{1:n}]$ of n i.i.d. samples from the stage distribution X_k. The expected second order statistic $\mathbb{E}[X_{2:n}]$, i.e. the expected second highest value, from n i.i.d. samples from distribution X is equal to the expected revenue of a second price auction with n buyers from X. For our Competition Complexity result we need a strict inequality between the expected second order statistic $\mathbb{E}[X_{2:n+c}]$ of $n+c$ samples and the expected first order statistic $\mathbb{E}[X_{1:n}]$ (the expected highest value, the social welfare) of n samples. More specifically, we need to find a c large enough for $\mathbb{E}[X_{2:n+c}]$ to be at least as large as $\mathbb{E}[X_{1:n}]$. For MHR distributions strong tail bounds are known (see for example [19]), that can be used for approximations of the expected order statistics, by combining for example with Markov-type inequalities[4]. If we insist on strict bounds though, we cannot afford to use such lossy arguments, and new ideas are necessary. We postpone further discussion until the next Section. We proceed to explore the extend to which the assumption on the stage distributions can be relaxed.

Informal Theorem 2. *In the case of n buyers and m independent stages, ex-post IR dynamic auctions, if $m-1$ stage distributions have monotone hazard rate and the remaining stage distribution is regular, the bounds on the Competition Complexity and α-approximate Competition Complexity are the same.*

This theorem requires much better upper bounds on the optimal dynamic revenue, which we obtain via an extension of the duality framework recently proposed by Cai, Devanur, and Weinberg [2016]. The upper bound in Cai, Devanur, and Weinberg [2016] improves on the trivial upper bound of the social welfare by substituting the value of each buyer's favorite item with the corresponding Myerson's virtual value. Our bound resembles this format: the largest expected first order statistic is substituted by the corresponding expected virtual value, i.e. optimal (static) revenue. Moreover, our dual solution also induces a virtual value function $\Phi_k(.)$ at each stage k. The benchmarks we get from duality do not depend on "tail assumptions" like regularity and monotone hazard rate, but they do need stage distributions to be independent; the tail assumptions are necessary for the revenue of VCG to surpass the benchmark.

Can the tail assumptions on the stage distributions be relaxed even further? Surprisingly, the answer is no! If two stage distributions are regular, even if the stages are independent, there is only a single buyer and we ask for non-negative utility at each stage (ex-post IR), the α-Competition Complexity is unbounded, for all constants $\alpha > 0$. This fact is a corollary of Example 1 from Chapter 2.

[4]Incidentally, the approximations between the expected first and second order statistics we present here don't use this approach directly. We instead take an "auction flavored" approach and go through a comparison with the revenue of Myerson's auction.

4.2 Preliminaries

Competition Complexity.

Let $\textsc{Rev}[M, \mathcal{X}, n, m]$ denote the expected revenue of running auction M at every stage, for m stages, with n buyers whose values are drawn according to $\mathcal{X}$. We are interested in the number c of extra buyers necessary such that $\textsc{Rev}[VCG, \mathcal{X}, n+c, m]$ to be at least $\text{OPT}[\mathcal{X}, n, m]$, where VCG is simply a second price auction. This number is called the *Competition Complexity* with respect to VCG, defined by Eden et al. [2016]. We also study *approximations*:

Definition 30 (α-approximate Competition Complexity). *The α-approximate Competition Complexity with respect to VCG is the minimum number c such that $\textsc{Rev}[VCG, \mathcal{X}, n+c, m]$ is at least $\alpha \cdot \text{OPT}[\mathcal{X}, n, m]$.*

Note that we can define the α-approximate Competition Complexity and Competition Complexity to be with respect to any prior-independent auction M. In this chapter we focus on VCG. At a high level our approach is the following: (1) Find an upper bound B to $\text{OPT}[\mathcal{X}, n, m]$. (2) Prove that the revenue of running a second price auction at every stage with c additional buyers (present in every stage) yields revenue at least B. We present each step seperately. For different distributions and different constraints the bounds in steps (1) and (2) are different. Our final bounds on the Competition Complexity come from mix and matching these different bounds.

Upper bounds on OPT.

Buyers' valuations X_k^i on stage k are independent draws from a distribution X_k. We can arrange the values in a descending order: $(X_k)_{1:n} \geq (X_k)_{2:n} \geq \cdots \geq (X_k)_{n:n}$. We call $(X_k)_{r:n}$ the r-th order statistic[5]. Given a product distribution $\mathbf{X}_k = \prod_{i=1}^n X_k$, let $\textsc{Mye}[\mathbf{X}_k]$ be the revenue of Myerson's optimal auction, i.e. the revenue optimal mechanism for a one-shot auction where buyers' valuations are drawn i.i.d. from X_k. Our upper bounds on $\text{OPT}[\mathcal{X}, n, m]$ do not require any tail assumptions.

Social welfare. Our first upper bound on $\text{OPT}[\mathcal{X}, n, m]$ is the trivial one: the social welfare. At every stage k we can extract revenue at most the expected maximum valuation at that stage.

Claim 31. $\text{OPT}[\mathcal{X}, n, m] \leq \sum_{k=1}^m \mathbb{E}\left[(X_k)_{1:n}\right]$, *where $(X_k)_{1:n}$ is the highest-order statistic of n i.i.d. samples drawn from X_k.*

[5]We write $X_{r:n}$ for the r-th order statistic of n samples from a distribution X. In the auctions context we write $v_{t:t'} = (v_t, v_{t+1}, \ldots, v_{t'})$ for the reported values in stages t through t'. It will be clear from context which of the two notions we refer to.

Note that this bound holds even for ex-ante IR and periodic IC mechanisms, and even if the stages are correlated[6]. Surprisingly, as we see later, if we restrict the marginal distributions at each stage to have monotone hazard rate we can still provide bounds on the Competition Complexity even with this trivial upper bound.

Duality based bounds. To improve on the trivial bound we ask for (1) independent stages, (2) ex-post IR mechanisms. Our improved bound is as follows. Choose any stage j; the contribution of stage j is $\textsc{Mye}[\mathbf{X}_j]$, the optimal revenue we would extract from stage j in a non-dynamic setting. The contribution of every other stage $k \neq j$ is the expected maximum $\mathbb{E}[(X_k)_{1:n}]$ of n samples drawn from X_k.

Lemma 32. *For independent stages, ex-post IR and periodic IC dynamic mechanisms*

$$\mathrm{OPT}[\mathcal{X}, n, m] \leq \min_{j=1,\ldots,m} \left\{ \textsc{Mye}[\mathbf{X}_j] + \sum_{k=1,k\neq j}^{m} \mathbb{E}[(X_k)_{1:n}] \right\},$$

where $(X_k)_{1:n}$ is the highest-order statistic of n samples drawn from X_k, and $\mathbf{X}_j$ is the product distribution for stage j.

For the special case of $j = m$ the Lemma could perhaps be proven via a combinatorial argument of the form: "Because of the IR constraints, in the first $m-1$ stages one could make at most the expected maximum of n samples. The last stage is a static problem, and therefore by Myerson's theorem the revenue is at most $\textsc{Mye}[\mathbf{X}_m]$". But, even for example for two stages and a single buyer, it is not clear how to prove that the optimal revenue is at most $\textsc{Mye}[X_1] + \mathbb{E}[X_2]$.

Our proof is via an extension of the Cai-Devanur-Weinberg duality framework to the dynamic setting. The work of Cai, Devanur, and Weinberg [2016] unified many different recent advances in Bayesian mechanism design by providing an approximately tight upper bound for the optimal revenue using a single dual solution. In their work, they start from a certain linear program for revenue maximization and Lagrangify the Bayesian IC and IR constraints. Their partial Lagrangian function has the following interesting property: dual solutions λ that yield finite upper bounds form a flow for every buyer i. The nodes in buyer i's flow correspond to possible valuations of this buyer. The flow $\lambda^i(v, \hat{v})$ from node v to node $\hat{v}$ captures the IC constraints between v and $\hat{v}$. All the nodes have default incoming flow equal to the probability $f(v)$ that value v is realized. Furthermore, these flows induce a "virtual valuation function" similar to Myerson's virtual value $\phi(v) = v - \frac{1-F(v)}{f(v)}$[7].

By Lagrangifying the IC and IR constraints of our linear program for maximizing revenue in the dynamic setting we can get similar characterizations. Interestingly, dual solutions

[6] Recall that we allow the value v_k^i of agent i in stage k to be correlated with her value in stage k'. We don't allow this value to be correlated with some other value v_k^j of a different agent.

[7] In fact, for the case of a single item, their virtual value function is identical to Myerson's virtual value function.

with finite upper bounds correspond to flows at every stage with the following twist: a node v_k^i, of some player i in stage k, can push flow $\lambda_s^i(v_k^i, \hat{v}_k^i)$ to a node $\hat{v}_k^i$ in the same stage, corresponding to the IC constraints between the two types and a deviation strategy s. But, it can also push flow $\kappa_k^i(v_k^i)$ to its children nodes in the next stage $k+1$, corresponding to the IR constraint of type v_k^i. The amount of flow pushed from v_k^i to its children in the next stage controls their "default" incoming flow, which depending on the choice of κ_k^i, can vary from zero to $f(v_{k+1}^i)$. The bound of Cai, Devanur, and Weinberg [2016] improves on the social welfare upper bound by substituting the value of each buyer's favorite item with the corresponding Myerson's virtual value. Our bound substitutes the largest expected order statistic with the corresponding expected virtual value. Even more interestingly, the same Myerson-like virtual valuation function is induced for every stage! We get the Lemma by carefully constructing a dual solution. In order to develop some intuition we first prove the single agent, two stage case in Section 4.3. The general proof can be found in Section 4.4.

Discrete vs Continuous. For simplicity of presentation, we prove Lemma 32 for distributions with discrete support. Our proof can be easily modified to hold for continuous distributions.

Lower bounds on the revenue of Vickrey.

After finding suitable bounds for $\text{OPT}[\mathcal{X}, n, m]$, we need to show that the revenue of VCG (a second price auction at each stage) with additional buyers surpasses these bounds. A first observation is that the expected revenue of VCG is the expected second order statistic:

Observation 33. *The expected revenue of a second price auction with n agents whose values are drawn i.i.d. from X is $\mathbb{E}[X_{2:n}]$.*

Our upper bounds on $\text{OPT}[\mathcal{X}, n, m]$ (Claim 31 and Lemma 32) are sums over m terms. The term that corresponds to stage k is either the expected revenue of the optimal static auction $\text{MYE}[\mathbf{X}_k]$ for that stage, or the expected highest order statistic $\mathbb{E}[X_{1:n}]$ of n samples. We upper bound each term separately. This gives us a sufficient number of extra buyers, i.e. an upper bound on the Competition Complexity. For the terms involving the optimal static auction, the original Theorem of Bulow and Klemperer provides a good bound for $\mathbb{E}[X_{2:n+c}]$ and regular distributions:

Theorem 34 (Bulow-Klemperer)**.** *Let X be a random variable with a regular distribution D. Let $\mathbf{X}$ be the product distribution of n samples drawn i.i.d from D. Then $\mathbb{E}[X_{2:n+1}] \geq \text{MYE}[\mathbf{X}]$.*

The following corollary can be shown:

Corollary 35. *Let X be a random variable with a regular distribution D. Let $\mathbf{X}$ be the product distribution of n samples drawn i.i.d from D. Then $\mathbb{E}[X_{2:n}] \geq \frac{n-1}{n}\text{MYE}[\mathbf{X}]$. In* *other words, the revenue of a second price auction is an* $\frac{n-1}{n}$ *approximation to the revenue of the optimal auction.*

We first note that for terms involving the expected highest order statistic similar bounds are impossible for regular distributions, as we've already seen in Example 1:

Example 36. *Let* X *be a random variable following the equal revenue distribution with* $F(x) = 1 - \frac{1}{x}$*, for* $x \in [1, \infty)$*.* $\mathbb{E}[X] = \int_0^\infty \frac{1}{x} dx$ *is unbounded, while* $\mathbb{E}[X_{2:n}] = n - 1$*, i.e. bounded for all* n*.*[8]*.The example can be modified to hold for truncated distributions: for all* n*, there exists a truncation value* V*, such that for the truncated distribution* $\mathbb{E}[X] > \mathbb{E}[X_{2:n}]$*.*

Therefore, in order to get a bound on the Competition Complexity we need to impose a restriction stronger than regularity on some stage distributions. A natural candidate is distributions with *Monotone Hazard Rate*; a distribution has *Monotone Hazard Rate* (MHR) if its hazard rate $h(v) = \frac{f(v)}{1-F(v)}$ is monotone non-decreasing. MHR distributions are a subset of regular distributions and have various nice properties, like bounded expected order statistics, small tails, etc. In this Chapter we show the following new bounds:

Theorem 37. *Let* $X_{r:n}$ *the* r*-th order statistic of* n *i.i.d samples from a continuous distribution* X *with monotone hazard rate. Then:*

1. *If* X *has bounded support, then* $\mathbb{E}[X_{2:4n}] \geq \mathbb{E}[X_{1:n}]$*.*
2. $\mathbb{E}[X_{2:n+1}] \geq \frac{1}{e}\mathbb{E}[X_{1:n}]$*.*
3. $\mathbb{E}[X_{2:n}] \geq \frac{1}{3}\mathbb{E}[X_{1:n}]$*.*

MHR distributions have been studied extensively in the Statistics literature under the (perhaps better) name of IHR, Increasing Hazard Rate, and IFR, Increasing Failure Rate (e.g. Barlow and Proschan [1966], Barlow and Proschan [1996])[9]. A common trick when working with MHR distributions is to write the cdf as $F(x) = 1 - e^{-\int_0^x h(z)dz}$. Then, since $h(x)$ is non-decreasing, $H(x) = \int_0^x h(z)dz$ is a convex function. Using one sided bounds for $H(x)$ (for example a linear approximation) one can provide lower bounds and upper bounds for quantities like the expected minimum of two samples (e.g. [15, 38, 68]). When working with order statistics of many samples though, one sided bounds such as these do not work, since the closed form for the expected second order statistic has both positive and negative terms that involve $H(x)$. Moreover, taking more samples will not compensate for a lossy argument; the proofs need to work for distributions that are essentially point masses, where all the order statistics are equal for any number of samples.

For each of the bounds in Theorem 37 we take a different approach. For the first one, we start by showing a one sample bound: $\mathbb{E}[X_{2:4}] \geq \mathbb{E}[X]$. This bound is tight for the exponential distribution (in a sense that $\mathbb{E}[X_{2:3}] < \mathbb{E}[X]$). We first prove that the inequality

[8]See Claim 93 in Appendix A.2 for a calculation.

[9]To keep consistency with the auctions community we refer to them as MHR distributions in this Chapter.

is strict when $H(x)$ is piece-wise linear and convex, and then show that every convex function with bounded domain can be approximated by a piece-wise linear function. We combine the one sample result, the fact that order statistics of MHR distributions have MHR distributions themselves, and a coupling argument to generalize to n samples.

For the second bound, $\mathbb{E}[X_{2:n+1}] \geq \frac{1}{e}\mathbb{E}[X_{1:n}]$, we take an "auction flavored" approach. First, the LHS is at least $\textsc{Mye}[\mathbf{X}]$ using Bulow and Klemperer's result. Second, we compare $\textsc{Mye}[\mathbf{X}]$ with $\textsc{Mye}[X_{1:n}]$, the expected revenue of the optimal auction in a (one-shot) single agent auction with distribution $X_{1:n}$, using a coupling argument. Third, since order statistics of MHR distributions have MHR distributions, we can use known bounds to compare $\textsc{Mye}[X_{1:n}]$ and $\mathbb{E}[X_{1:n}]$.

For the last bound, $\mathbb{E}[X_{2:n}] \geq \frac{1}{3}\mathbb{E}[X_{1:n}]$, we combine a (known) bound on the expected minimum of two samples from an MHR distribution with the (also known) fact that spacings of order statistics of MHR distributions, i.e. $\mathbb{E}[X_{1:n}] - \mathbb{E}[X_{2:n}]$, are non-increasing functions of the number of samples. We prove Theorem 37 as three separate Lemmas in Section 4.5.

Discrete vs Continuous. For a distribution over a discrete domain $\{1, 2, \ldots, N\}$, the definition of hazard rate is $h(i) = \frac{p(i)}{\sum_{j \geq i} p(j)}$ (see Barlow and Proschan [1996]). Some known inequalities for MHR distributions fail for the discrete case. For example, for continuous MHR distributions one can show that $\Pr[X \geq \mathbb{E}[X]] \geq \frac{1}{e}$; the inequality fails for a geometric distribution. Our proofs hold only for continuous distributions. It remains open whether the statements are true for discrete MHR distributions.

Putting everything together.

By combining the different upper bounds on $\text{OPT}[\mathcal{X}, n, m]$ with the corresponding lower bounds for VCG we can get upper bounds for the Competition Complexity and approximate Competition Complexity of dynamic auctions. We prove our lower bounds for the Competition Complexity in Section 4.6. Our lower bounds work for (1) independent stages, m MHR distributions, for ex-ante IR auctions (applied in Theorem 38), and (2) independent stages, $m-1$ MHR and 1 regular distribution, for ex-post IR auctions (applied in Theorem 39). The proofs are similar. The auction in the second bound is a generalization of Example 1 from Chapter 2. The auction in the first bound exploits an unsettling feature of ex-ante IR mechanisms: the buyers are willing to give up their expected future (net) utility just to be able to participate in the future auction. An ex-ante IR auction that extracts all of the social welfare is the following: at every stage the seller runs a second price auction, but before that, all buyers pay an entree fee equal to their expected utility for participating (expected value subject to being the winner, minus expected second highest value, multiplied by probability of winning). A common difficulty in both proofs is the algebraic manipulations of the expected first and second order statistics[10]. Combining with the upper bounds we get the following Theorems:

[10] The Lambert W-function makes an appearance.

Theorem 38. *For a dynamic environment where every stage distribution X_k is continuous and bounded, and has monotone hazard rate then, even for periodic IC and ex-ante IR dynamic auctions (by combining Claim 31 + Theorem 37):*

- *The Competition Complexity is at least $(e-1)n$ and at most $3n$.*
- *The $\frac{1}{e}$-approximate Competition Complexity is 1.*
- *For $n \geq 2$, the $\frac{1}{3}$-approximate Competition Complexity is 0.*

Theorem 39. *For a dynamic environment where every stage distribution X_k is continuous and bounded, the stage distributions are independent, $m-1$ stage distributions have monotone hazard rate and the remaining stage distribution is regular, then even for periodic IC and ex-post IR dynamic auctions:*

- *The Competition Complexity is at least $(e-1)n$ and at most $3n$. (For the upper bound: Lemma 32 + Thms 34, 37)*
- *The $\frac{1}{e}$-approximate Competition Complexity is 1. (Lemma 32 + Thms 34, 37)*
- *For $n \geq 2$, the $\frac{1}{3}$-approximate Competition Complexity is 0. (Lemma 32 + Corollary 35, Thm 37)*

Our lower bound for the Competition Complexity in Theorem 39 uses $m-1$ MHR distributions and one regular distribution. It remains open whether the Competition Complexity is sublinear for the case of m independent and MHR stages.

Related Work

Bulow-Klemperer Type Results. Prior-independent mechanisms have been developed in both single and multi-dimensional *static* settings, e.g. [7, 37, 57, 97]. Sivan and Syrgkanis [2013] give a version of the Bulow-Klemperer theorem for non-i.i.d. irregular distributions. Eden et al. [2016] provide the first full Bulow-Klemperer result for multidimensional static auctions, i.e. without any loss or approximation. They introduce the term *Competition Complexity*, that we also adopt here. Their main result is that the Competition Complexity of n buyers with additive valuations over m independent, regular items is at most $n+2m-2$ and at least $\log(m)$. Their upper bounds on the optimal static revenue is also via an extension of the duality framework of Cai, Devanur, and Weinberg [2016]. More recently, Feldman, Friedler, and Rubinstein [2017] also study a relaxed notion of Competition Complexity; they show that when auctioning m items separately the 99%-Competition Complexity is $O(\log m)$, and (for regular distributions) this further goes down to constant when auctioning the items as one bundle. A closely related line of work considers mechanism design with limited information in the form of samples, e.g. [38, 30, 35, 85].

The Cai-Devanur-Weinberg Duality Framework, Extensions and Related Techniques. Multiple strong duality frameworks have been developed recently, e.g. [33, 34, 56, 55], that can be seen as an optimal transport/bipartite matching problem. Haghpanah and Hartline [2015] provide an alternative strong duality framework. Closer to the framework we extend, Carroll [2017] takes a partial Lagrangian over IC and IR constraints; the application is a screening problem. In Chapter 4 we present an extension of the duality framework of Cai, Devanur, and Weinberg [2016] for dynamic settings. This framework was used to unify and improve the results of several recent works on Bayesian mechanism design (e.g. [64, 79, 10, 108, 26, 28, 27]). It was recently extended by Cai and Zhao [2016] to prove approximation results for simple mechanisms in settings with multiple subadditive bidders. It was also extended in a different way by Eden et al. [2016] for a single buyer with values that exhibit a "limited complementarity" property.

4.3 Warm up: One Buyer, Two Independent Stages

In this Section we prove the special case of Lemma 32 for one buyer and two stages.

Lemma 40. *For single agent, two independent stages, ex-post IR and periodic IC dynamic mechanisms*

$$\mathrm{OPT}\left[\mathcal{X},1,2\right] \leq \min\left\{\textsc{Mye}\left[X_1\right] + \mathbb{E}\left[X_2\right], \mathbb{E}\left[X_1\right] + \textsc{Mye}\left[X_2\right]\right\}.$$

The Partial Lagrangian.

The optimal dynamic auction needs to satisfy the following two types of constraints:

- Periodic incentive compatibility (PIC). At any stage k, assuming truthfulness in the future stages, truthfully revealing v_k maximizes the buyer's expected utility, among all possible reports $\hat{v}_k$. For the first stage, this constraint can be expressed as: for all $v_1, \hat{v}_1$ in V_1

$$v_1 x_1(v_1) - p_1(v_1) + \mathbb{E}_{v_2 \in V_2}\left[v_2 x_2\left(v_1, v_2\right) - p_2\left(v_1, v_2\right)\right] \geq \\ v_1 x_1(\hat{v}_1) - p_1(\hat{v}_1) + \mathbb{E}_{v_2 \in V_2}\left[v_2 x_2\left(\hat{v}_1, v_2\right) - p_2\left(\hat{v}_1, v_2\right)\right].$$

  For the secong stage: for all v_1 in V_1, and all $v_2, \hat{v}_2$ in V_2

$$v_2 x_2(v_1, v_2) - p_2(v_1, v_2) \geq v_2 x_2(v_1, \hat{v}_2) - p_2(v_1, \hat{v}_2).$$

- Ex-post individual rationality. The buyer's stage utility is non-negative at every stage k, no matter what the reports were in the previous stage (in the case of stage 2).

$$v_k x_k(v_{\leq k}) - p_k(v_{\leq k}) \geq 0$$

The revenue objective can be written as:

$$\mathbb{E}_{v_1,v_2}\left[p_1(v_1) + p_2(v_1, v_2)\right]$$

Thus, we have the following primal program[11]:

$$\max \sum_{v_1 \in V_1} f(v_1)p_1(v_1) + \sum_{v_1 \in V_1} f(v_1) \sum_{v_2 \in V_2} f(v_2)p_2(v_1, v_2)$$

subject to:

$$\begin{aligned}
\forall v_1, \hat{v}_1 \in V_1 : \quad & v_1 x_1(v_1) - p_1(v_1) + \sum_{v_2 \in V_2} f(v_2)\left(v_2 x_2(v_1, v_2) - p_2(v_1, v_2)\right) \geq \\
& \qquad v_1 x_1(\hat{v}_1) - p_1(\hat{v}_1) + \sum_{v_2 \in V_2} f(v_2)\left(v_2 x_2(\hat{v}_1, v_2) - p_2(\hat{v}_1, v_2)\right) \\
\forall v_1 \in V_1, \forall v_2, \hat{v}_2 \in V_2 : \quad & v_2 x_2(v_1, v_2) - p_2(v_1, v_2) \geq v_2 x_2(v_1, \hat{v}_2) - p_2(v_1, \hat{v}_2) \\
\forall v_1 \in V_1 : \quad & v_1 x_1(v_1) - p_1(v_1) \geq 0 \\
\forall v_1 \in V_1, v_2 \in V_2 : \quad & v_2 x_2(v_1, v_2) - p_2(v_1, v_2) \geq 0 \\
\forall v_1 \in V_1 : \quad & x_1(v_1) \in [0, 1] \\
\forall v_1 \in V_1, \forall v_2 \in V_2 : \quad & x_2(v_1, v_2) \in [0, 1]
\end{aligned}$$

We introduce a variable $\lambda_k(v_{\leq k}, \hat{v}_k)$ for the periodic IC constraints for stage k and a variable $\kappa_k(v_{\leq k})$ for the ex-post IR constraints for stage k. In other words, the dual variables are $\lambda_1(v_1, \hat{v}_1)$, $\lambda_2(v_1, v_2, \hat{v}_2)$, $\kappa_1(v_1)$ and $\kappa_2(v_1, v_2)$. Cai, Devanur, and Weinberg [2016] include the IR constraints with the IC constraints, by introducing a null type $\perp$, with zero allocation and zero payment; in our case, this is possible only for the ex-post IR constraint in the last stage. Similarly to them, we do not take Lagrangian multipliers for the feasibility constraints. The partial Lagrangian $\mathcal{L}(\lambda, \kappa, x, p)$ of the primal program is as follows:

$$\begin{aligned}
\mathcal{L}(\lambda, \kappa, x, p) = & \sum_{v_1 \in V_1} f(v_1)p_1(v_1) + \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} f(v_1) f(v_2) p_2(v_1, v_2) \\
& + \sum_{v_1 \in V_1} \sum_{\hat{v}_1 \in T_1} \lambda_1(v_1, \hat{v}_1)\left(v_1 x_1(v_1) - p_1(v_1) - v_1 x_1(\hat{v}_1) + p_1(\hat{v}_1)\right) \\
& + \sum_{v_1 \in V_1} \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) \sum_{v_2 \in V_2} f_2(v_2)\left(v_2 x_2\left(v_1, v_2\right) - p_2\left(v_1, v_2\right) - v_2 x_2\left(\hat{v}_1, v_2\right) + p_2\left(\hat{v}_1, v_2\right)\right) \\
& + \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, v_2, \hat{v}_2)\left(v_2 x_2(v_1, v_2) - p_2(v_1, v_2) - v_2 x_2(v_1, \hat{v}_2) + p_2(v_1, \hat{v}_2)\right) \\
& + \sum_{v_1 \in V_1} \kappa_1(v_1)\left(v_1 x_1(v_1) - p_1(v_1)\right) + \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} \kappa_2(v_1, v_2)\left(v_2 x_2(v_1, v_2) - p_2(v_1, v_2)\right)
\end{aligned}$$

Re-grouping gives the following form:

[11] The support is discrete for simplicity of presentation.

$$\begin{aligned}\mathcal{L}(\lambda, \kappa, x, p) = &\sum_{v_1 \in V_1} p_1(v_1)\left(f(v_1) - \kappa_1(v_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) + \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1)\right)\\ &+ \sum_{v_1 \in V_1} x_1(v_1)\left(v_1\kappa_1(v_1) + \sum_{\hat{v}_1 \in V_1} v_1\lambda_1(v_1, \hat{v}_1) - \sum_{\hat{v}_1 \in V_1} \hat{v}_1\lambda_1(\hat{v}_1, v_1)\right)\\ &+ \sum_{v_1 \in V_1, v_2 \in V_2} p_2(v_1, v_2)\Bigg(f(v_1)f(v_2) - \kappa_2(v_1, v_2) - \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, v_2, \hat{v}_2) +\\ &\qquad \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, \hat{v}_2, v_2) + f(v_2)\left(\sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1)\right)\Bigg)\\ &+ \sum_{v_1 \in V_1, v_2 \in V_2} x_2(v_1, v_2)\Bigg(v_2\kappa_2(v_1, v_2) + \sum_{\hat{v}_2 \in V_2} v_2\lambda_2(v_1, v_2, \hat{v}_2) - \sum_{\hat{v}_2 \in V_2} \hat{v}_2\lambda_2(v_1, \hat{v}_2, v_2)\\ &\qquad + v_2 f(v_2)\left(\sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1)\right)\Bigg).\end{aligned}$$

Duality theory tells us that for any choice of $\lambda, \kappa \geq 0$, the primal objective $\mathrm{OPT}[\mathcal{X}, 1, 2]$ is upper bounded by $\max_{x \in \mathcal{F}, p} \mathcal{L}(\lambda, \kappa, x, p)$, where $\mathcal{F}$ is the set of feasible allocations:

$$\mathrm{OPT}[\mathcal{X}, 1, 2] \leq \max_{x \in \mathcal{F}, p} \mathcal{L}(\lambda, \kappa, x, p) \tag{4.1}$$

In order to get non-trivial upper bounds, we need $\max_{x \in \mathcal{F}, p} \mathcal{L}(\lambda, \kappa, x, p)$ to be bounded. Next, we give constraints on λ and κ for this to be true. Since $p_1(v_1)$ is an unconstrained variable, if its multiplier is non-zero setting $p(v_1)$ to ∞ or $-\infty$ will make $\mathcal{L}(\lambda, \kappa, x, p)$ unbounded. Therefore:

$$f(v_1) - \kappa_1(v_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) + \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) = 0 \tag{4.2}$$

Similarly for the multiplier of $p_2(v_1, v_2)$:

$$\begin{aligned}&f(v_1)f(v_2) - \kappa_2(v_1, v_2) - \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, v_2, \hat{v}_2) + \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, \hat{v}_2, v_2)\\ &\qquad\qquad + f(v_2)\left(\sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1)\right)\\ &=^{\text{Eq. 4.2}} f(v_1)f(v_2) - \kappa_2(v_1, v_2) - \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, v_2, \hat{v}_2) + \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, \hat{v}_2, v_2)\\ &\qquad\qquad + f(v_2)\left(\kappa_1(v_1) - f(v_1)\right)\\ &= f(v_2)\kappa_1(v_1) - \kappa_2(v_1, v_2) - \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, v_2, \hat{v}_2) + \sum_{\hat{v}_2 \in V_2} \lambda_2(v_1, \hat{v}_2, v_2) = 0\end{aligned} \tag{4.3}$$

Similarly to Cai, Devanur, and Weinberg [2016] we call dual solutions that satisfy Constraints 4.2 and 4.3 **useful**. Useful solutions can be seen as flows in a certain tree. At the top of a tree we have a source. The nodes in the first level correspond to values in the support of the first stage; a node v_1 receives flow $f(v_1)$ from the source. v_1 can push flow $\lambda_1(v_1, v_1')$ to some other node v_1' on the same level, or push flow $\kappa_1(v_1)$ to its children. A child-node (v_1, v_2), or simply v_2 (we explicitly say the parent when necessary), receives incoming flow $\kappa_1(v_1) \cdot f(v_2)$ from its parent. See Figure 4.1. A similar structure is satisfied for more stages and multiple agents. We note that for correlated stages this structure fails; the incoming flow of a child-node v_2 depends on the flow pushed to and from its parent v_1.

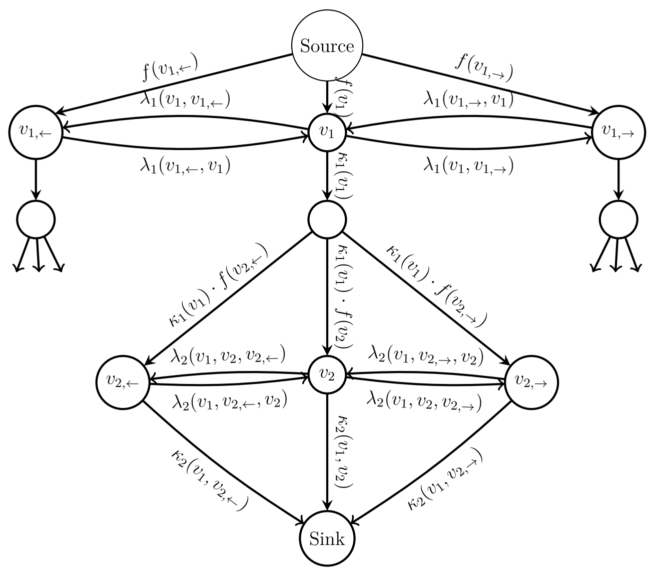

Figure 4.1: Constraints of useful dual solutions form a flow.

It is possible to derive familiar expressions for the multipliers of x_1 and x_2. Gathering all the terms that x_1 appears in $\mathcal{L}(\lambda, \kappa, x, p)$ we have:

$$
\begin{aligned}
&\sum_{v_1 \in V_1} x_1(v_1) \left(v_1 \kappa_1(v_1) + v_1 \sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) - \sum_{\hat{v}_1 \in V_1} \hat{v}_1 \lambda_1(\hat{v}_1, v_1) \right) \\
&=^{\text{Eq. 4.2}} \sum_{v_1 \in V_1} x_1(v_1) \left(v_1 \kappa_1(v_1) + v_1 \left(f(v_1) - \kappa_1(v_1) + \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) \right) - \sum_{\hat{v}_1 \in V_1} \hat{v}_1 \lambda_1(\hat{v}_1, v_1) \right) \\
&= \sum_{v_1 \in V_1} x_1(v_1) f(v_1) \left(v_1 - \frac{1}{f(v_1)} \sum_{\hat{v}_1 \in V_1} (\hat{v}_1 - v_1) \lambda_1(\hat{v}_1, v_1) \right) \\
&= \sum_{v_1 \in V_1} x_1(v_1) f(v_1) \Phi_1(v_1),
\end{aligned}
$$

where $\Phi_1(v_1) = v_1 - \frac{1}{f(v_1)} \sum_{\hat{v}_1 \in V_1} (\hat{v}_1 - v_1)\lambda_1(\hat{v}_1, v_1)$. Therefore, every useful dual solution induces a "virtual value" function $\Phi_1(.)$, such that the contribution of the first stage to the $\mathcal{L}(\lambda, \kappa, x, p)$ is the expected virtual value. A similar structure is derived for the terms involving of x_2:

$$\begin{aligned}
&\sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) \Bigg(v_2 \kappa_2(v_1, v_2) + \sum_{\hat{v}_2 \in V_2} v_2 \lambda_2(v_1, v_2, \hat{v}_2) - \sum_{\hat{v}_2 \in T_2} \hat{v}_2 \lambda_2(v_1, \hat{v}_2, v_2) \\
&\qquad + v_2 f(v_2) \Bigg(\sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) \Bigg) \Bigg) \\
&=^{\text{Eq. 4.3}} \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) \Bigg(v_2 f(v_2) \kappa_1(v_1) + \sum_{\hat{v}_2 \in V_2} (v_2 - \hat{v}_2) \lambda_2(v_1, \hat{v}_2, v_2) \\
&\qquad + v_2 f(v_2) \Bigg(\sum_{\hat{v}_1 \in V_1} \lambda_1(v_1, \hat{v}_1) - \sum_{\hat{v}_1 \in V_1} \lambda_1(\hat{v}_1, v_1) \Bigg) \Bigg) \\
&=^{\text{Eq. 4.2}} \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) \Bigg(v_2 f(v_2) \kappa_1(v_1) + \sum_{\hat{v}_2 \in V_2} (v_2 - \hat{v}_2) \lambda_2(v_1, \hat{v}_2, v_2) \\
&\qquad + v_2 f(v_2) \left(f(v_1) - \kappa_1(v_1) \right) \Bigg) \\
&= \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) \Bigg(v_2 f(v_2) f(v_1) + \sum_{\hat{v}_2 \in V_2} (v_2 - \hat{v}_2) \lambda_2(v_1, \hat{v}_2, v_2) \Bigg) \\
&= \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) f(v_1) f(v_2) \Bigg(v_2 - \frac{1}{f(v_1) f(v_2)} \sum_{\hat{v}_2 \in V_2} (\hat{v}_2 - v_2) \lambda_2(v_1, \hat{v}_2, v_2) \Bigg) \\
&= \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) f(v_1) f(v_2) \Phi_2(v_1, v_2),
\end{aligned}$$

where $\Phi_2(v_1, v_2) = v_2 - \frac{1}{f(v_1)f(v_2)} \sum_{\hat{v}_2 \in T_2} (\hat{v}_2 - v_2)\lambda_2(v_1, \hat{v}_2, v_2)$. Combining all the observations so far, we have that given a **useful** dual solution λ, κ:

$$\mathcal{L}(\lambda, \kappa, x, p) = \sum_{v_1 \in V_1} x_1(v_1) f(v_1) \Phi_1(v_1) + \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} x_2(v_1, v_2) f(v_1) f(v_2) \Phi_2(v_1, v_2)$$

Therefore, given a useful dual solution λ, κ, the revenue of any dynamic mechanism $M = (x, p)$ that is ex-post IR and periodic IC, is at most the virtual welfare of x with respect to the virtual value functions Φ_1 and Φ_2 corresponding to λ and κ. In other words,

$$\begin{aligned}
&\sum_{v_1 \in V_1} f(v_1) \Bigg(p_1(v_1) + \sum_{v_2 \in V_2} f(v_2) p_2(v_1, v_2) \Bigg) \\
&\qquad \leq \sum_{v_1 \in V_1} f(v_1) \Bigg(x_1(v_1) \Phi_1(v_1) + \sum_{v_2 \in V_2} f(v_2) x_2(v_1, v_2) \Phi_2(v_1, v_2) \Bigg).
\end{aligned}$$

Canonical Flows.

The Lemma 40 is proved as two separate Claims. We need the following definitions.

Definition 41. *For all $v_k^i \in V_k^i$ define $v_{k,\rightarrow}^i$ and $v_{k,\leftarrow}^i$ to be the values in V_k^i immediately larger and immediately smaller than v_k^i (respectively) :*

$$v_{k,\rightarrow}^i = \inf_{\hat{v}_k^i \in V_k^i : v_k^i < \hat{v}_k^i} \hat{v}_k^i \qquad\qquad v_{k,\leftarrow}^i = \sup_{\hat{v}_k^i \in V_k^i : v_k^i > \hat{v}_k^i} \hat{v}_k^i.$$

Definition 42. *Myerson's virtual value for distribution X_k is*

$$\phi(v_k) = v_k - \frac{(v_{k,\rightarrow} - v_k) \cdot \Pr_{v \sim X_k}[v > v_k]}{f(v_k)} = v_k - \frac{(v_{k,\rightarrow} - v_k) \cdot (1 - F(v_k))}{f(v_k)}.$$

Claim 43. *For a single agent, two independent stages, ex-post IR and PIC dynamic mechanisms:*

$$\mathrm{OPT}[\mathcal{X}, 1, 2] \leq \mathbb{E}[X_1] + \textsc{Mye}[X_2].$$

Proof. Consider the following dual solution:

$$\kappa_1(v_1) = f(v_1) \qquad\qquad \lambda_1(v_1, \hat{v}_1) = 0$$

$$\kappa_2(v_1, v_2) = \begin{cases} f(v_1) & \text{if } v_2 = \underline{v}_2 \\ 0 & \text{o.w.} \end{cases} \qquad \lambda_2(v_1, v_2, \hat{v}_2) = \begin{cases} f(v_1)(1 - F(\hat{v}_2)) & \text{if } \hat{v}_2 = v_{2,\leftarrow} \\ 0 & \text{o.w.} \end{cases}$$

It's easy to verify that constraints 4.2 and 4.3 are satisfied; the solution is useful. See Figure 4.2. These flows induce virtual values $\Phi_1(v_1) = v_1$ for the first stage nodes. For the second stage nodes, $\Phi_2(v_1, v_2)$ becomes equal to $\phi(v_2)$, Myerson's virtual value for X_2. For

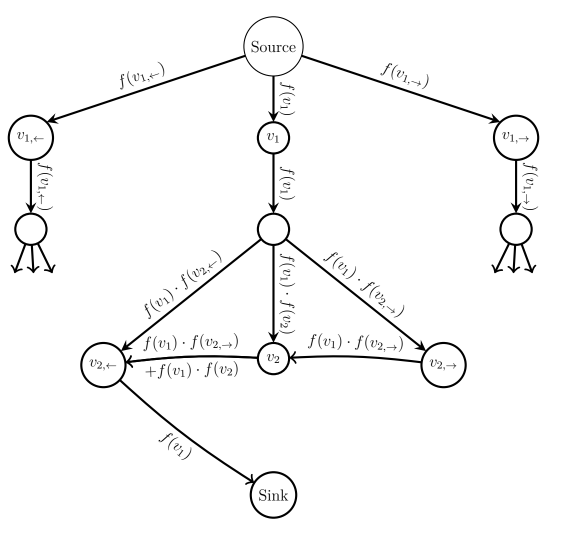

Figure 4.2: An example with support 3 of the flow with Lagrangian $\mathbb{E}[X_1] + \textsc{Mye}[X_2]$.

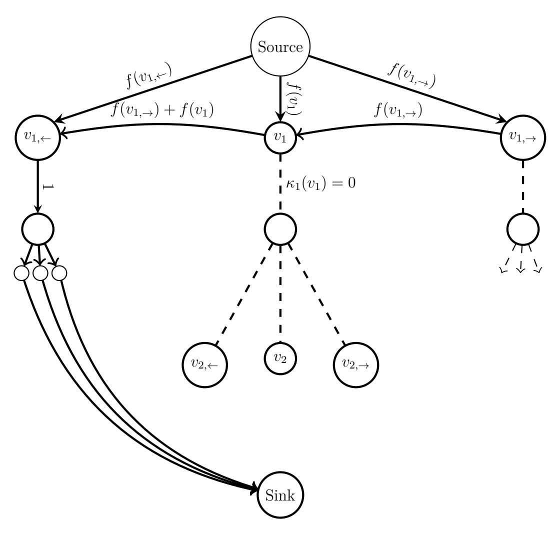

Figure 4.3: An example with support 3 of the flow with Lagrangian $\textsc{Mye}[X_1] + \mathbb{E}[X_2]$.

simplicity we assume that X_2 is regular, i.e. the virtual value Φ_2 induced by our flow is monotone non-decreasing; if this is not the case we can "iron" our flow by adding loops (See the ironing procedure in [20]). By Equation 4.1,

$$\begin{aligned}
\mathrm{OPT}\left[\mathcal{X},1,2\right] &\leq \max_{x,p} \mathcal{L}(\lambda,\kappa,x,p) \\
&= \max_{x,p} \sum_{v_1\in V_1} x_1(v_1)f(v_1)\Phi_1(v_1) + \sum_{v_1\in V_1, v_2\in V_2} x_2(v_1,v_2)f(v_2)f(v_1)\Phi_2(v_1,v_2) \\
&= \max_{x,p} \sum_{v_1\in V_1} x_1(v_1)f(v_1)v_1 \\
&\qquad + \sum_{v_1\in V_1, v_2\in V_2} x_2(v_1,v_2)f(v_2)f(v_1)\left(v_2 - \frac{1}{f(v_1)f(v_2)}\sum_{\hat{v}_2\in V_2}(\hat{v}_2-v_2)\lambda_2(v_1,\hat{v}_2,v_2)\right) \\
&= \mathbb{E}\left[X_1\right] \\
&+ \max_{x,p} \sum_{v_1\in V_1, v_2\in V_2} x_2(v_1,v_2)f(v_2)f(v_1)\left(v_2 - \frac{1}{f(v_1)f(v_2)}(v_{2,\rightarrow}-v_2)f(v_1)\left(1-F(v_2)\right)\right) \\
&= \mathbb{E}\left[X_1\right] + \sum_{v_1\in V_1} f(v_1)\max_{x,p}\sum_{v_2\in V_2} x_2(v_1,v_2)f(v_2)(v_2 - \frac{1-F(v_2)}{f(v_2)}(v_{2,\rightarrow}-v_2)) \\
&= \mathbb{E}\left[X_1\right] + \sum_{v_1\in V_1} f(v_1)\max_{x,p}\sum_{v_2\in V_2} x_2(v_1,v_2)f(v_2)\phi(v_2) \\
&= \mathbb{E}\left[X_1\right] + \sum_{v_1\in V_1} f(v_1)\textsc{Mye}\left[X_2\right] \\
&= \mathbb{E}\left[X_1\right] + \textsc{Mye}\left[X_2\right] \qquad \square
\end{aligned}$$

Claim 44. *For single agent, two independent stages, ex-post IR and PIC dynamic mechanisms:*

$$\mathrm{OPT}\left[\mathcal{X},1,2\right] \leq \textsc{Mye}\left[X_1\right] + \mathbb{E}\left[X_2\right]$$

Proof. Similar to the proof of Claim 43. Start with an assignment of λ,κ:

$$\kappa_1(v_1) = \begin{cases} 1 & \text{if } v_1 = \underline{v}_1 \\ 0 & \text{o.w.} \end{cases} \qquad \lambda_1(v_1,\hat{v}_1) = \begin{cases} 1-F(\hat{v}_1) & \text{if } \hat{v}_1 = v_{1,\leftarrow} \\ 0 & \text{o.w.} \end{cases}$$

$$\kappa_2(v_1,v_2) = \begin{cases} f(v_1)f(v_2) & \text{if } v_1 = \underline{v}_1 \\ 0 & \text{o.w.} \end{cases} \qquad \lambda_2(v_1,v_2,\hat{v}_2) = 0$$

It's easy to verify that Constraints 4.2 and 4.3 are satisfied. This time, $\Phi_1(v_1)$ is Myerson's virtual value and $\Phi_2(v_1,v_2) = v_2$, for every node v_2, expect the children of $\underline{v}_1$. Note that, given

$\kappa_1(v_1) = 0$, the every child-node under v_1 has no incoming flow; therefore, it is unavoidable for their virtual value to be equal to their value. See Figure 4.3.

$$\begin{aligned}
\mathrm{OPT}\left[\mathcal{X},1,2\right] &\leq \max_{x,p} \mathcal{L}(\lambda,\kappa,x,p) \\
&= \max_{x,p} \sum_{v_1\in V_1} x_1(v_1)f(v_1)\Phi_1(v_1) + \sum_{v_1\in V_1, v_2\in V_2} x_2(v_1,v_2)f(v_2)f(v_1)\Phi_2(v_1,v_2) \\
&= \max_{x,p} \sum_{v_1\in V_1} x_1(v_1)f(v_1)\left(v_1 - \frac{1-F(v_1)}{f(v_1)}(v_{1,\rightarrow} - v_1)\right) \\
&\quad + \sum_{v_1\in V_1}\sum_{v_2\in V_2} x_2(v_1,v_2)f(v_1)f(v_2)v_2 \\
&= \max_{x,p} \sum_{v_1\in V_1} x_1(v_1)f(v_1)\phi(v_1) + \mathbb{E}\left[X_2\right] = \textsc{Mye}\left[X_1\right] + \mathbb{E}\left[X_2\right]. \qquad \square
\end{aligned}$$

4.4 Revenue Upper Bounds for n Buyers and m Independent Stages

In this Section we prove Lemma 32, using an approach similar to the previous Section.

The Partial Lagrangian.

The optimal dynamic mechanism needs to satisfy the following two types of constraints:

- Periodic incentive compatibility (PIC). At any stage k for every buyer i, assuming truthfulness in all future stages, revealing the true value v_k^i maximizes the buyer's expected utility among all possible values $\hat{v}_k^i$. For stage k and buyer i, this constraint can be expressed as: for all $\mathbf{v}_{\leq k-1}$ in $\mathbf{V_{k-1}}$:

$$\begin{aligned}
&\mathbb{E}_{\mathbf{v}_k^{-i}}\left[v_k^i x_k^i(\mathbf{v}_{\leq k}^{-i}, v_{\leq k}^i) - p_k^i(\mathbf{v}_{\leq k}^{-i}, v_{\leq k}^i) + \mathbb{E}_{\mathbf{v}_{k+1:m}^{-i}, v_{k+1:m}^i}\left[\sum_{j>k} v_j^i x_j^i(\mathbf{v}_{\leq j}^{-i}, v_{\leq j}^i) - p_j^i(\mathbf{v}_{\leq j}^{-i}, v_{\leq j}^i)\right]\right] \\
&\quad \geq \mathbb{E}_{\mathbf{v}_k^{-i}}\left[v_k^i x_k^i(\mathbf{v}_{\leq k}^{-i}, v_{<k}^i, \hat{v}_k^i) - p_k^i(\mathbf{v}_{\leq k}^{-i}, v_{<k}^i, \hat{v}_k^i) + \right. \\
&\qquad \left. \mathbb{E}_{\mathbf{v}_{k+1:m}^{-i}, v_{k+1:m}^i}\left[\sum_{j>k} v_j^i x_j^i(\mathbf{v}_{\leq j}^{-i}, v_{<k}^i, \hat{v}_k^i, v_{k+1:j}^i) - p_j^i(\mathbf{v}_{\leq j}^{-i}, v_{<k}^i, \hat{v}_k^i, v_{k+1:j}^i)\right]\right]
\end{aligned}$$

- Ex-post individual rationality. At any stage k, the stage utility of every buyer i is non-negative regardless of the reports from previous stages. For all $\mathbf{v}_{\leq k}$ in $\mathbf{V}_{\leq k}$:

$$v_k^i x_k^i(\mathbf{v}_{\leq k}) - p_k^i(\mathbf{v}_{\leq k}) \geq 0 \tag{4.4}$$

As noted in Ashlagi, Daskalakis, and Haghpanah [2016], the following constraints are equivalent to Constrains 4.4: for all $\mathbf{v}_{\leq k-1}$ in $\mathbf{V}_{k-1}$, and v_k^i in V_k

$$\mathbb{E}_{\mathbf{v}_k^{-i}}[v_k^i x_k^i(\mathbf{v}_{\leq k}) - p_k^i(\mathbf{v}_{\leq k})] \geq 0 \tag{4.5}$$

Any x_k^i, p_k^i satisfying Constraint 4.4 naturally satisfy Constraint 4.5; furthermore, for any auction M with x_k^i, p_k^i satisfying 4.5, there exists another auction $\hat{M}$ defined as:

$$\hat{p}_k^i(\mathbf{v}_{\leq k}) = \frac{\mathbb{E}_{\mathbf{v}_k^{-i}}[p_k^i(\mathbf{v}_{\leq k})]}{\mathbb{E}_{\mathbf{v}_k^{-i}}[x_k^i(\mathbf{v}_{\leq k})]}$$

$$\hat{x}_k^i(\mathbf{v}_{\leq k}) = \begin{cases} 1 & \text{if item } k \text{ is allocated to buyer } i \text{ in } M \\ 0 & \text{o.w.} \end{cases}$$

It's easy to verify that $\hat{x}_k^i, \hat{p}_k^i$ satisfy 4.4 and $\textsc{Rev}\left[\hat{M}, \mathcal{X}, n, m\right] = \textsc{Rev}\left[M, \mathcal{X}, n, m\right]$. We use Constraints 4.5 for the primal program.

Thereby, we obtain the following primal program:

$$\max \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{\mathbf{v}_{\leq k}} f(\mathbf{v}_{\leq k}) p_k^i(\mathbf{v}_{\leq k})$$

subject to:

$\forall i = 1, \ldots, n, \forall k = 1, \ldots m, \forall \mathbf{v}_{\leq k-1} \in \mathbf{V}_{k-1}, v_k^i, \hat{v}_k^i \in V_k$:

$$\sum_{\mathbf{v}_k^{-i}} f(\mathbf{v}_k^{-i}) \Bigg(v_k^i x_k^i(\mathbf{v}_{\leq k}^{-i}, v_{\leq k}^i) - p_k^i(\mathbf{v}_{\leq k}^{-i}, v_{\leq k}^i) + \sum_{\mathbf{v}_{k+1:m}} f(\mathbf{v}_{k+1:m}) \Bigg(\sum_{j>k} v_j^i x_j^i(\mathbf{v}_{\leq j}^{-i}, v_{\leq j}^i) - p_j^i(\mathbf{v}_{\leq j}^{-i}, v_{\leq j}^i) \Bigg) \Bigg) \geq$$

$$\sum_{\mathbf{v}_k^{-i}} f(\mathbf{v}_k^{-i}) \Bigg(v_k^i x_k^i(\mathbf{v}_{\leq k}^{-i}, v_{<k}^i, \hat{v}_k^i) - p_k^i(\mathbf{v}_{\leq k}^{-i}, v_{<k}^i, \hat{v}_k^i) + \sum_{\mathbf{v}_{k+1:m}} f(\mathbf{v}_{k+1:m}) \Bigg(\sum_{j>k} v_j^i x_j^i(\mathbf{v}_{\leq j}^{-i}, v_{<k}^i, \hat{v}_k^i, v_{k+1:j}^i) - p_j^i(\mathbf{v}_{\leq j}^{-i}, v_{<k}^i, \hat{v}_k^i, v_{k+1:j}^i) \Bigg) \Bigg)$$

$\forall k = 1, \ldots, m, \forall \mathbf{v}_{\leq k-1} \in \mathbf{V}_{k-1}, v_k^i \in V_k$:

$$\sum_{\mathbf{v}_k^{-i}} f(\mathbf{v}_k^{-i}) \left(v_k^i x_k^i(\mathbf{v}_{\leq k}) - p_k^i(\mathbf{v}_{\leq k}) \right) \geq 0$$

$\forall i = 1, \ldots, n \forall k = 1, \ldots, m, \forall \mathbf{v}_{\leq k} \in \mathbf{V}_k$: $\qquad x_k^i(\mathbf{v}_{\leq k}) \in [0, 1]$

$\forall k = 1, \ldots, m, \forall \mathbf{v}_{\leq k} \in \mathbf{V}_k$: $\qquad \displaystyle\sum_{i=1}^{n} x_k^i(\mathbf{v}_{\leq k}) \leq 1$

We introduce a Lagrangian multiplier $\lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)$ for each periodic IC constraint and $\kappa_k(\mathbf{v}_{<k}, v_k^i)$ for each ex-post IR constraint. The partial Lagrangian, after re-grouping the terms, is the following:

$$\begin{aligned}\mathcal{L}(\lambda, \kappa, x, p) = \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{\mathbf{v}_{\leq k}} p_k^i(\mathbf{v}_{\leq t}) \Bigg(& f(\mathbf{v}_{\leq k}) - f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i}(\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) \\ & -\lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)) + \sum_{j=1}^{k-1}\sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k})(\lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i) - \lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i))\Bigg) + \\ \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{\mathbf{v}_{\leq k}} x_k^i(\mathbf{v}_{\leq k}) \Bigg(& v_k^i f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i}(v_k^i \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i) \\ & -\hat{v}_k^i\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i)) + \sum_{j=1}^{k-1}\sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k})(v_k^i\lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i) - v_k^i\lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i))\Bigg)\end{aligned}$$

Duality theory tells us for any $\lambda, \kappa \geq 0$, the primal objective is upper bounded by $\max_{x\in\mathcal{F},p} \mathcal{L}(\lambda, \kappa, x, p)$, where $\mathcal{F}$ is the set of possible allocations:

$$\text{OPT}\,[\mathcal{X}, n, m] \leq \max_{x\in\mathcal{F},p} \mathcal{L}(\lambda, \kappa, x, p) \tag{4.6}$$

upper bounds the primal objective. If we want to find non-trivial upper bounds, we need to ensure that $\max_{x\in\mathcal{F},p} \mathcal{L}(\lambda, \kappa, x, p)$ is bounded. This requires that the free variables $p_k^i(\mathbf{v}_{\leq k})$ have multipliers equal to zero. Therefore: for stage k, buyer i and reports $\mathbf{v}_{\leq k}$ in $\mathbf{V}_k$

$$\begin{aligned}f(\mathbf{v}_{\leq k}) - f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i}(\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) - \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)) \\ + \sum_{j=1}^{k-1}\sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k})(\lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i) - \lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i)) = 0.\end{aligned} \tag{4.7}$$

Noticing the recursive structure of Equation 4.7, we take the constraint for stage $k-1$, buyer i and reports $\mathbf{v}_{\leq k-1}$ in $\mathbf{V}_{k-1}$:

$$\begin{aligned}f(\mathbf{v}_{\leq k-1}) - f(\mathbf{v}_{k-1}^{-i})\kappa_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i) + f(\mathbf{v}_{k-1}^{-i}) \sum_{\hat{v}_{k-1}^i}(\lambda_{k-1}(\mathbf{v}_{<k-1}, \hat{v}_{k-1}^i, v_{k-1}^i) \\ - \lambda_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i, \hat{v}_{k-1}^i)) + \sum_{j=1}^{k-2}\sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k-1})(\lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i) - \lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i)) = 0.\end{aligned} \tag{4.8}$$

Next, we simplify the LHS of Equation 4.7 with Equation 4.8:

$$\begin{aligned}
0 &= f(\mathbf{v}_{\le k}) - f(\mathbf{v}_k^{-i})\kappa(\mathbf{v}_{<k}, v_k^i) + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) - \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)) \\
&\quad + \sum_{j=1}^{k-1} \sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k})(\lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i) - \lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i)) \\
&=^{\text{Eq. 4.8}} f(\mathbf{v}_{\le k}) - f(\mathbf{v}_k^{-i})\kappa(\mathbf{v}_{<k}, v_k^i) + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) - \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)) \\
&\quad + f(\mathbf{v}_k)(f(\mathbf{v}_{k-1}^{-i})\kappa_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i) - f(\mathbf{v}_{\le k-1})) \\
&= f(\mathbf{v}_{k-1}^{-i}, \mathbf{v}_k)\kappa_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i) - f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) \\
&\quad + f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) - \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i)) = 0.
\end{aligned}$$

Since the last line equals zero, we obtain:

$$\begin{aligned}
f(\mathbf{v}_{k-1}^{-i}, \mathbf{v}_k)\kappa_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i) - f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) &= \\
f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (\lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i) - \lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i)). &\quad (4.9)
\end{aligned}$$

Recall that we call solutions of $\lambda, \kappa \ge 0$ that satisfy Equation 4.7 **useful** solutions. Then, given a set of useful λ, κ we can simplify $\mathcal{L}(\lambda, \kappa, p, x)$ using Equations 4.7, 4.8 and 4.9. Gather all the terms in $\mathcal{L}(\lambda, \kappa, x, p)$ that involve x_k^i for $k > 1$ and simplify the terms using Equation 4.9:

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{k=2}^{m} \sum_{\mathbf{v}_{\le k}} x_k^i(\mathbf{v}_{\le k}) \Bigg(f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (v_k^i \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i) + v_k^i f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) - \hat{v}_k^i \lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i)) \\
&\qquad + v_k^i \sum_{j=1}^{k-1} \sum_{\hat{v}_j^i} f(\mathbf{v}_j^{-i}, \mathbf{v}_{j+1:k})(\lambda_j(\mathbf{v}_{<j}, v_j^i, \hat{v}_j^i) - \lambda_j(\mathbf{v}_{<j}, \hat{v}_j^i, v_j^i)) \Bigg) \\
&=^{\text{Eq. 4.9}} \sum_{i=1}^{n} \sum_{k=2}^{m} \sum_{\mathbf{v}_{\le k}} x_k^i(\mathbf{v}_{\le k}) \Bigg(f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (v_k^i \lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i) - \hat{v}_k^i \lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i)) \\
&\qquad v_k^i f(\mathbf{v}_k^{-i})\kappa_k(\mathbf{v}_{<k}, v_k^i) - v_k^i f(\mathbf{v}_k)(f(\mathbf{v}_{k-1}^{-i})\kappa_{k-1}(\mathbf{v}_{<k-1}, v_{k-1}^i) - f(\mathbf{v}_{\le k-1})) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{k=2}^{m}\sum_{\mathbf{v}_{\leq k}} x_k^i(\mathbf{v}_{\leq k}) \left(v_k^i f(\mathbf{v}_{\leq k-1}) f(\mathbf{v}_k) - f(\mathbf{v}_k^{-i}) \sum_{\hat{v}_k^i} (\hat{v}_k^i - v_k^i)\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) \right) \\
&= \sum_{i=1}^{n}\sum_{k=2}^{m}\sum_{\mathbf{v}_{\leq k}} x_k^i f(\mathbf{v}_{\leq k})(\mathbf{v}_{\leq k}) \left(v_k^i - \frac{1}{f(\mathbf{v}_{\leq k-1}, v_k^i)} \sum_{\hat{v}_k^i} (\hat{v}_k^i - v_k^i)\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i) \right) \\
&= \sum_{i=1}^{n}\sum_{k=2}^{m}\sum_{\mathbf{v}_{\leq k}} x_k^i f(\mathbf{v}_{\leq k}) \Phi_k(\mathbf{v}_{\leq k})
\end{aligned}
$$

where $\Phi_k^i(\mathbf{v}_{\leq k}) = v_k^i - \frac{1}{f(\mathbf{v}_{\leq k-1}, v_k^i)} \sum_{\hat{v}_k^i} (\hat{v}_k^i - v_k^i)\lambda_k(\mathbf{v}_{<k}, \hat{v}_k^i, v_k^i)$. Finally, simplify the terms involving $x_1^i(\mathbf{v}_1)$ using Equation 4.7 evaluated at $k = 1$:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{\mathbf{v}_1} x_1^i(\mathbf{v}_1) \left(v_1^i f(\mathbf{v}_1^{-i})\kappa_1(v_1^i) + f(\mathbf{v}_1^{-i}) \sum_{\hat{v}_1^i} (v_1^i \lambda_1(v_1^i, \hat{v}_1^i) - \hat{v}_1^i \lambda_1(\hat{v}_1^i, v_1^i)) \right) \\
&=^{\text{Eq. 4.7}} \sum_{i=1}^{n}\sum_{\mathbf{v}_1} x_1^i(\mathbf{v}_1) \left(v_1^i f(\mathbf{v}_1) - f(\mathbf{v}_1^{-i}) \sum_{\hat{v}_1^i} (\hat{v}_1^i - v_1^i)\lambda_1(\hat{v}_1^i, v_1^i) \right) \\
&= \sum_{i=1}^{n}\sum_{\mathbf{v}_1} x_1^i(\mathbf{v}_1) f(\mathbf{v}_1) \left(v_1^i - \frac{1}{f(v_1^i)} \sum_{\hat{v}_1^i} (\hat{v}_1^i - v_1^i)\lambda_1(\hat{v}_1^i, v_1^i) \right) \\
&= \sum_{i=1}^{n}\sum_{\mathbf{v}_1} x_1^i(\mathbf{v}_1) f(\mathbf{v}_1) \Phi_1(\mathbf{v}_1),
\end{aligned}
$$

where $\Phi_1^i(\mathbf{v}_1) = v_1^i - \frac{1}{f(v_1^i)} \sum_{\hat{v}_1^i} (\hat{v}_1^i - v_1^i)\lambda_1(\hat{v}_1^i, v_1^i)$.

Combining all the observations so far, we have that given any useful solution λ, κ:

$$
\mathcal{L}(\lambda, \kappa, x, p) = \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{\mathbf{v}_{\leq k}} x_k^i(\mathbf{v}_{\leq k}) f(\mathbf{v}_{\leq k}) \Phi_k^i(\mathbf{v}_{\leq k}).
$$

Main Claim

Claim 45. *For n agents, m independent stages, ex-post IR and PIC dynamic mechanisms*

$$
\text{OPT}\left[\mathcal{X}, n, m\right] \leq \text{M}\textsc{ye}\left[\mathbf{X}_j\right] + \sum_{k=1, k\neq j}^{m} \mathbb{E}\left[(X_k)_{1:n}\right]
$$

for any $j = 1, \ldots, m$.

Proof. Consider the following dual solution:

$$\kappa_k(\mathbf{v}_{<k}, v_k^i) = \begin{cases} f(\mathbf{v}_{<k}, v_k^i) & k < j \\ \frac{f(\mathbf{v}_{<k}, v_k^i)}{f(v_j^i)} & k \geq j \wedge v_j^i = \underline{v}_j^i \\ 0 & \text{o.w.} \end{cases}$$

$$\lambda_k(\mathbf{v}_{<k}, v_k^i, \hat{v}_k^i) = \begin{cases} f(\mathbf{v}_{<k})(1 - F(\hat{v}_k^i)) & k = j \wedge \hat{v}_k^i = v_{k,\leftarrow}^{i} {}^{\text{Dfn 41}} \\ 0 & \text{o.w.} \end{cases}$$

It's easy to verify that Constraint 4.7 is satisfied. These flows induce virtual value $\Phi_k^i(\mathbf{v}_k) = v_k^i$ for all $k \neq j$. For stage j, $\Phi_j^i(\mathbf{v}_{\leq j})$ becomes $\phi(v_j^i)$, Myerson's virtual value for X_j. For simplicity we assume that X_j is regular, so that the virtual values induced are non-decreasing; if this is not the case, we use an "ironing" procedure to the flow λ_j in stage j, similar to [20].

Then by Inequality 4.6:

$$\begin{aligned}
\text{OPT}\left[\mathcal{X}, n, m\right] &\leq \max_{x,p} \mathcal{L}(\lambda, \kappa, x, p) \\
&= \max_{x,p} \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{\mathbf{v}_{\leq k}} f(\mathbf{v}_{\leq k}) x_k^i(\mathbf{v}_{\leq k}) \Phi_k^i(\mathbf{v}_{\leq k}) \\
&=^{\text{Dnf 41}} + \sum_{i=1}^{n} \sum_{k=1, k\neq j}^{m} \sum_{\mathbf{v}_{\leq k}} f(\mathbf{v}_{\leq k}) v_k^i \\
&+ \max_{x,p} \sum_{i=1}^{n} \sum_{\mathbf{v}_{\leq j}} f(\mathbf{v}_{\leq j}) x_j^i(\mathbf{v}_{\leq j}) \left(v_j^i - \frac{1}{f(v_j^i, \mathbf{v}_{<j})} (v_{j,\rightarrow}^i - v_j^i) f(\mathbf{v}_{<j})(1 - F(v_j^i)) \right) \\
&=^{\text{Dnf 42}} \sum_{k=1, k\neq j}^{m} \mathbb{E}\left[(X_k)_{1:n}\right] + \max_{x,p} \sum_{\mathbf{v}_{<j}} f(\mathbf{v}_{<j}) \sum_{\mathbf{v}_j} f(\mathbf{v}_j) \sum_{i=1}^{n} x_j^i(\mathbf{v}_{\leq j}) \phi(v_j^i) \\
&= \sum_{k=1, k\neq j}^{m} \mathbb{E}\left[(X_k)_{1:n}\right] + \sum_{\mathbf{v}_{<j}} f(\mathbf{v}_{<j}) \max_{x,p} \sum_{\mathbf{v}_j} f(\mathbf{v}_j) x_j(\mathbf{v}_{\leq j}) \phi(\mathbf{v}_j) \\
&= \sum_{k=1, k\neq j}^{m} \mathbb{E}\left[(X_k)_{1:n}\right] + \text{MYE}\left[\mathbf{X}_j\right].
\end{aligned}$$

□

4.5 Lower Bounding the Revenue of VCG

In this section we prove Theorem 37. The proof is broken into three Lemmas. Recall that the hazard rate of a distribution F is $h(x) = \frac{f(x)}{1-F(x)}$. F has monotone hazard rate (MHR) if $h(x)$ is a non-decreasing function. We restrict ourselves to continuous distributions. For Lemma 46 we also need the distribution to be supported on $[0, \bar{V}]$.

Lemma 46. *Let $X_{r:n}$ be the r-th order statistic of n i.i.d. samples from a continuous and bounded distribution with monotone hazard rate. Then 4n samples are necessary and sufficient for $\mathbb{E}[X_{2:4n}]$ to be at least as large as $\mathbb{E}[X_{1:n}]$.*

Lemma 47. *Let $X_{r:n}$ be the r-th (highest) order statistic of n i.i.d. samples from a continuous (possibly unbounded) distribution with monotone hazard rate. Then $\mathbb{E}[X_{2:n+1}] \geq \frac{1}{e}\mathbb{E}[X_{1:n}]$.*

Lemma 48. *Let $X_{r:n}$ be the r-th (highest) order statistic of n i.i.d. samples from a continuous (possibly unbounded) distribution with monotone hazard rate. Then $\mathbb{E}[X_{2:n}] \geq \frac{1}{3}\mathbb{E}[X_{1:n}]$.*

A useful fact about order statistics of MHR distributions that we use throughout this section is that order statistics of MHR distributions have themselves an MHR distribution:

Lemma 49 (Barlow and Proschan [1996])**.** *Assume X is a random variable with distribution F and density f which is MHR. If $X_1, X_2, \ldots, X_n$, are n independent observations on X, the order statistics formed from the X_i's are also MHR.*

We break the proof of Lemma 46 into two parts. We first prove the result for $n = 1$ in Subsection 4.5. We complete the proof of general n by combining the $n = 1$ case, Lemma 49 and a coupling argument in Subsection 4.5. We prove Lemmas 47 and 48 (necessary for our $\frac{1}{e}$ and $\frac{1}{3}$-approximate Competition Complexity bounds) in Subsections 4.5 and 4.5 respectively.

Single sample bound

Lemma 50. *Let $X_{r:n}$ the r-th order statistic of n i.i.d. samples from a continuous and bounded distribution with monotone hazard rate. Then 4 samples are necessary and sufficient for $\mathbb{E}[X_{2:4}]$ to be at least as large as $\mathbb{E}[X] = \mathbb{E}[X_{1:1}]$.*

Let $H(x) = \int_0^x h(z)dz$. If F is MHR, then $H(x)$ is a convex function as it is the integral of a non-decreasing function. The proofs of the next Claims can be found in Appendix A.1.

Claim 51. *$F(x) = 1 - e^{-H(x)}$ and $\mathbb{E}[X] = \int_0^{\bar{V}} e^{-H(x)}dx$.*

Claim 52. *$\mathbb{E}[X_{2:4}] = \int_0^{\bar{V}} 3e^{-4H(x)} - 8e^{-3H(x)} + 6e^{-2H(x)}dx$.*

For the upper bound on the number of samples, it suffices to show that $\int_0^{\bar{V}} 3e^{-4H(x)} - 8e^{-3H(x)} + 6e^{-2H(x)} - e^{-H(x)}dx \geq 0$ for all non-negative, convex and continuous functions $H(x)$. We first prove this statement for all non-negative, piecewise linear and convex functions $\hat{H}(x)$ in Lemma 53. We then show how to approximate any convex function by a piecewise linear convex function in Lemma 54. We combine Lemmas 53 and 54 to prove the upper bound in Lemma 50. The lower bound comes from considering a (truncated) exponential distribution.

Lemma 53. *Let $\hat{H}(x)$ be a non-negative, piecewise linear and convex function in $[0, \bar{V}]$, with $\hat{H}(0) = 0$. Then $\int_0^{\bar{V}} 3e^{-4\hat{H}(x)} - 8e^{-3\hat{H}(x)} + 6e^{-2\hat{H}(x)} - e^{-\hat{H}(x)}dx > 0$.*

Proof. For a general piecewise linear and convex function $\hat{H}(x)$ with c linear pieces, we have

$$\hat{H}(x) = \begin{cases} a_0x + b_0 & \text{if } x_1 \geq x \geq x_0 \\ a_1x + b_1 & \text{if } x_2 \geq x \geq x_1 \\ \dots \\ a_cx + b_c & \text{if } x_{c+1} \geq x \geq x_c \end{cases}$$

where $b_0 = 0$ and $a_ix_i + b_i = a_{i-1}x_i + b_{i-1}$, $\forall i \geq 1$. $x_0 = 0$, $x_{c+1} = \bar{V}$ and $x_{i+1} > x_i$ for all $i \geq 0$. Since $\hat{H}(x)$ is convex, $a_{i+1} \geq a_i > 0$, for all i. Let $I = \int_0^{\bar{V}} 3e^{-4\hat{H}(x)} - 8e^{-3\hat{H}(x)} + 6e^{-2\hat{H}(x)} - e^{-\hat{H}(x)}dx$.

$$\begin{aligned}
I &= \sum_{i=0}^{c} \int_{x_i}^{x_{i+1}} 3e^{-4(a_ix+b_i)} - 8e^{-3(a_ix+b_i)} + 6e^{-2(a_ix+b_i)} - e^{-(a_ix+b_i)}dx \\
&= \sum_{i=0}^{c} - \left[\frac{3e^{-4(a_ix+b_i)}}{4a_i}\right]_{x_i}^{x_{i+1}} + \left[\frac{8e^{-3(a_ix+b_i)}}{3a_i}\right]_{x_i}^{x_{i+1}} - \left[\frac{3e^{-2(a_ix+b_i)}}{a_i}\right]_{x_i}^{x_{i+1}} + \left[\frac{e^{-(a_ix+b_i)}}{a_i}\right]_{x_i}^{x_{i+1}} \\
&= \sum_{i=0}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right) \\
&\qquad - \sum_{i=0}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_i+b_i)}}{4} + \frac{8e^{-3(a_ix_i+b_i)}}{3} - 3e^{-2(a_ix_i+b_i)} + e^{-(a_ix_i+b_i)}\right) \\
&= \sum_{i=0}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right) \\
&\qquad + \frac{1}{12a_0} - \sum_{i=1}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_i+b_i)}}{4} + \frac{8e^{-3(a_ix_i+b_i)}}{3} - 3e^{-2(a_ix_i+b_i)} + e^{-(a_ix_i+b_i)}\right) \\
&= \sum_{i=0}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right) \\
&+ \frac{1}{12a_0} - \sum_{i=1}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_{i-1}x_i+b_{i-1})}}{4} + \frac{8e^{-3(a_{i-1}x_i+b_{i-1})}}{3} - 3e^{-2(a_{i-1}x_i+b_{i-1})} + e^{-(a_{i-1}x_i+b_{i-1})}\right) \\
&= \sum_{i=0}^{c} \frac{1}{a_i}\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right) \\
&\frac{1}{12a_0} - \sum_{i=0}^{c-1} \frac{1}{a_{i+1}}\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right) \\
&= \frac{1}{12a_0} + \frac{1}{a_c}\left(-\frac{3e^{-4(a_cx_{c+1}+b_c)}}{4} + \frac{8e^{-3(a_cx_{c+1}+b_c)}}{3} - 3e^{-2(a_cx_{c+1}+b_c)} + e^{-(a_cx_{c+1}+b_c)}\right) \\
&+ \sum_{i=0}^{c-1}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)\left(-\frac{3e^{-4(a_ix_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_ix_{i+1}+b_i)}}{3} - 3e^{-2(a_ix_{i+1}+b_i)} + e^{-(a_ix_{i+1}+b_i)}\right).
\end{aligned}$$

We add and subtract $\sum_{i=1}^{c} \frac{1}{12a_i}$:

$$\begin{aligned}
I &= \frac{1}{a_c}\left(-\frac{3e^{-4(a_c x_{c+1}+b_c)}}{4} + \frac{8e^{-3(a_c x_{c+1}+b_c)}}{3} - 3e^{-2(a_c x_{c+1}+b_c)} + e^{-(a_c x_{c+1}+b_c)}\right) \\
&\quad + \frac{1}{12a_0} + \frac{1}{12a_1} + \frac{1}{12a_2} + \cdots + \frac{1}{12a_c} - \frac{1}{12a_1} - \frac{1}{12a_2} - \cdots - \frac{1}{12a_c} \\
&\quad + \sum_{i=0}^{c-1}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)\left(-\frac{3e^{-4(a_i x_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_i x_{i+1}+b_i)}}{3} - 3e^{-2(a_i x_{i+1}+b_i)} + e^{-(a_i x_{i+1}+b_i)}\right) \\
&= \frac{1}{a_c}\left(-\frac{3e^{-4(a_c x_{c+1}+b_c)}}{4} + \frac{8e^{-3(a_c x_{c+1}+b_c)}}{3} - 3e^{-2(a_c x_{c+1}+b_c)} + e^{-(a_c x_{c+1}+b_c)}\right) \\
&\quad + \left(\frac{1}{12a_0} - \frac{1}{12a_1}\right) + \left(\frac{1}{12a_1} - \frac{1}{12a_2}\right) + \cdots + \left(\frac{1}{12a_{c-1}} - \frac{1}{12a_c}\right) + \frac{1}{12a_c} \\
&\quad + \sum_{i=0}^{c-1}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)\left(-\frac{3e^{-4(a_i x_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_i x_{i+1}+b_i)}}{3} - 3e^{-2(a_i x_{i+1}+b_i)} + e^{-(a_i x_{i+1}+b_i)}\right) \\
&= \frac{1}{a_c}\left(-\frac{3e^{-4(a_c x_{c+1}+b_c)}}{4} + \frac{8e^{-3(a_c x_{c+1}+b_c)}}{3} - 3e^{-2(a_c x_{c+1}+b_c)} + e^{-(a_c x_{c+1}+b_c)} + \frac{1}{12}\right) \\
&+ \sum_{i=0}^{c-1}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)\left(-\frac{3e^{-4(a_i x_{i+1}+b_i)}}{4} + \frac{8e^{-3(a_i x_{i+1}+b_i)}}{3} - 3e^{-2(a_i x_{i+1}+b_i)} + e^{-(a_i x_{i+1}+b_i)} + \frac{1}{12}\right).
\end{aligned}$$

Let

$$g(y) = -\frac{3e^{-4y}}{4} + \frac{8e^{-3y}}{3} - 3e^{-2y} + e^{-y} + \frac{1}{12}.$$

Taking the derivative gives $g'(y) = 3e^{-4y} - 8e^{-3y} + 6e^{-2y} - e^{-y}$. Solving for $g'(y) = 0$ for $y \geq 0$ when $y = 0, \ln\left(\frac{5+\sqrt{13}}{2}\right)$ and as y goes to infinity. For $y = 0$, $g(0) = 0$. $lim_{y\to\infty} g(y) = \frac{1}{12}$. For $y = \ln\left(\frac{5+\sqrt{13}}{2}\right)$,

$$g\left(\ln\left(\frac{5+\sqrt{13}}{2}\right)\right) = \frac{1}{12} - \frac{12}{(5+\sqrt{13})^4} + \frac{64}{3(5+\sqrt{13})^3} - \frac{12}{(5+\sqrt{13})^2} + \frac{2}{5+\sqrt{13}} > 0.$$

Since $\frac{1}{a_i} - \frac{1}{a_{i+1}}$, $\frac{1}{a_c}$, $a_c x_{c+1} + b_c$ and $a_i x_{i+1} + b_i$ are all strictly positive, $I > 0$. □

Our next step is to show that any convex function $H(x)$ can be approximated by a piecewise linear and convex function $\hat{H}(x)$. An elementary theorem from Real Analysis tells us that for every continuous function $f(x)$ in a closed interval $[a, b]$ and every $\epsilon > 0$, there exists a piecewise linear function $g_\epsilon(x)$ such that $\forall x \in [a, b], |f(x) - g_\epsilon(x)| < \epsilon$. We show that the same is true for every continuous and convex function $f(x)$, $\epsilon > 0$, where this time the approximation is by a piecewise linear and convex function $g(x)$[12]. The proof can be found in Appendix A.1.

[12] We don't believe this Lemma to be new.

Lemma 54. *For every function $f(x)$ that is continuous and convex in a closed interval $[a, b]$, $\epsilon > 0$, there exists a convex piecewise linear function $g_\epsilon(x)$ such that for all $x \in [a, b]$: $|f(x) - g_\epsilon(x)| < \epsilon$.*

We are now ready to prove Lemma 50:

Proof of Lemma 50. Assume there is a non-negative convex and continuous function $H(x)$ in $[0, \bar{V}]$ such that

$$I[H] = \int_0^{\bar{V}} 3e^{-4H(x)} - 8e^{-3H(x)} + 6e^{-2H(x)} - e^{-H(x)} dx = -\delta,$$

for some $\delta > 0$. Let $\hat{H}_\epsilon(x)$ be a piecewise linear and convex function that ϵ-approximates $H(x)$, as in Lemma 54. Since $I[H]$ is a bounded integral, we can choose ϵ small enough such that

$$\left| I[H] - I\left[\hat{H}_\epsilon\right] \right| < \delta.$$

This would imply that $\int_0^{\bar{V}} 3e^{-4\hat{H}_\epsilon(x)} - 8e^{-3\hat{H}_\epsilon(x)} + 6e^{-2\hat{H}_\epsilon(x)} - e^{-\hat{H}_\epsilon(x)} dx < 0$, a contradiction to Lemma 53. Therefore, 4 samples are sufficient for $\mathbb{E}[X_{2:4}] \geq \mathbb{E}[X_{1:1}]$.

To see that 4 samples are also necessary, consider the second order statistic from 3 samples with cdf $F_{2:3} = 3F(x)^2 - 2F(x)^3$ and expectation $\mathbb{E}[X_{2:3}] = \int_0^{\bar{V}} 3e^{-2H(x)} - 2e^{-3H(x)} dx$. An exponential distribution with parameter $\lambda = 1$ gives $\mathbb{E}[X_{2:3}] = \frac{5}{6} < 1 = \lambda = \mathbb{E}[X] = \mathbb{E}[X_{1:1}]$, but is not bounded. We can truncate at some large $\bar{V}$ in a way that neither $\mathbb{E}[X_{2:4}]$ nor $\mathbb{E}[X]$ change by more than a negligible amount (truncated exponential distributions still have non-decreasing hazard rate). Therefore, 4 samples are also necessary. □

Proof of Lemma 46

Proof. We have already shown (for $n = 1$) that $4n$ samples are necessary in Lemma 50. Therefore, it remains to show that $4n$ samples are sufficient. Let $Y = X_{1:n}$ be the maximum of n i.i.d. samples from F. Let the F_y be the cdf of Y. If F is MHR, then so is F_y (Lemma 49).

Since Y is MHR, by Lemma 50 we have that $\mathbb{E}[Y_{2:4}] \geq \mathbb{E}[Y] = \mathbb{E}[X_{1:n}]$, where $Y_{2:4}$ is the second order statistic of 4 samples drawn from F_y. Therefore, it suffices to show that $\mathbb{E}[X_{2:4n}] \geq \mathbb{E}[Y_{2:4}]$: Draw $4n$ samples $X_1, \ldots, X_{4n}$ from F. Let $Z_1 = \max_{i=1:n} X_i, Z_2 = \max_{i=n+1:2n} X_i, Z_3 = \max_{i=2n+1:3n} X_i$ and $Z_4 = \max_{i=3n+1:4n} X_i$. $Y_{2:4}$ is the second largest of the Z_i's. On the other hand, $X_{2:4n}$ is the second largest of the X_i's, and therefore at least as large as $Y_{2:4}$, for every single outcome. □

Bounding $\mathbb{E}[X_{2:n+1}]$: Proof of Lemma 47

We are going to use the following technical lemma:

Lemma 55. *Let X be a random variable from an MHR distribution D. Let $X_{1:n}$ be the largest order statistic of n samples from D, and let $\mathbf{X} = \Pi_{i=1}^{n} X$ denote the product distribution of n agents. Then $\mathrm{OPT}[\mathbf{X}] \geq \mathrm{OPT}[X_{1:n}]$, i.e. the optimal revenue of n i.i.d. agent from X is larger than the optimal revenue of one agent from $X_{1:n}$.*

Proof. The optimal auction M on distribution $\mathbf{X}$ is a second price auction with some reserve p. The optimal auction M' on $X_{1:n}$ is a posted price auction, with some posted price p'. We can calculate the revenue of M' on $X_{1:n}$ as follows: first draw $v_1, v_2, \ldots, v_n$ be n i.i.d. samples from X, i.e. a sample $\mathbf{v}$ from $\mathbf{X}$. If the maximum of the v_i's is larger than p', then the revenue of M' on this outcome is p': $\mathrm{OPT}[X_{1:n}] = \sum_{\mathbf{v} \sim \mathbf{D}} \Pr[\mathbf{v}] \cdot p' \cdot \mathbb{1}[\max_i v_i \geq p']$.

Let $\hat{M}$ be a second price auction with reserve p'. If the maximum of the v_i's is larger than p', then the revenue of $\hat{M}$ on this outcome is the larger of p' and the second largest v_i. Therefore $\textsc{Rev}\left[\hat{M}, \mathbf{X}\right] \geq \mathrm{OPT}[X_{1:n}]$. But, M is the second price auction with the *optimal* posted price, and therefore $\mathrm{OPT}[\mathbf{X}] = \textsc{Rev}[M, \mathbf{X}] \geq \textsc{Rev}\left[\hat{M}, \mathbf{X}\right]$. The Lemma follows. □

Proof of Lemma 47. First, since X is a random variable from an MHR distribution D, we can lower bound $\mathbb{E}[X_{2:n+1}]$ (the revenue of a second price auction) using the original Theorem of Bulow and Klemperer([18]):

$$\mathbb{E}[X_{2:n+1}] = \textsc{Rev}[\text{Vickrey}, n+1 \text{ i.i.d. agents from D}] \geq \mathrm{OPT}[n \text{ i.i.d. agents from D}]. \tag{4.10}$$

Second, by Lemma 55:

$$\mathrm{OPT}[n \text{ i.i.d. agents from D}] \geq \mathrm{OPT}[\text{one agent from } X_{1:n}]. \tag{4.11}$$

Third, by Lemma 49 order statistics of MHR distributions have MHR distributions themselves, i.e. $X_{1:n}$ has monotone hazard rate. Fourth, from a known result from auction theory (e.g. [66], Lemma 5.14) we have that the optimal expected revenue from an MHR distribution is an e approximation to the optimal expected surplus. Applying to $X_{1:n}$ gives:

$$\mathrm{OPT}[X_{1:n}] \geq \frac{1}{e}\mathbb{E}[X_{1:n}]. \tag{4.12}$$

Combining Equations 4.10, 4.11 and 4.12 gives the Lemma. □

Bounding $\mathbb{E}[X_{2:n}]$: Proof of Lemma 48

Proof. We need the following two Lemmas. The first Lemma is proved in [15]. The proof uses that $F_{2:2}(x) = 1 - (1 - F(x))^2 = 1 - e^{-2H(x)}$ and the fact that $H(x)$ is convex. The second Lemma was proved by Barlow and Proschan [1966][13]. We include the proofs in Appendix A.1.

[13] Also see Szech [2011]

Lemma 56. $\mathbb{E}[X_{1:2}] - \mathbb{E}[X_{2:2}] \leq \frac{2}{3}\mathbb{E}[X_{1:2}]$.

Lemma 57. $\mathbb{E}[X_{1:n}] - \mathbb{E}[X_{2:n}] = \mathbb{E}\left[\frac{1}{h(X_{1:n})}\right]$, *where* $h(x) = \frac{f(x)}{1-F(x)}$, *and thus is a non-increasing function of* n *for MHR distributions.*

Given the two Lemmas above, we get the desired bound on $\mathbb{E}[X_{2:n}]$ as follows:

$$\mathbb{E}[X_{1:n}] - \mathbb{E}[X_{2:n}] \leq \mathbb{E}[X_{1:2}] - \mathbb{E}[X_{2:2}] \leq \frac{2}{3}\mathbb{E}[X_{1:2}] \leq \frac{2}{3}\mathbb{E}[X_{1:n}]$$ □

4.6 Lower Bounds on The Competition Complexity

Lemma 58. *For independent stages,* $m-1$ *MHR and* 1 *regular stage, and ex-post IR auctions, the Competition Complexity is at least* $(e-1)n$*, even for auctions that are incentive compatible in a perfect Bayesian equilibrium.*

Proof. For the first $m-1$ stages, let the value distributions be $X = Exp(1, V)$ is an exponential distribution with parameter $\lambda = 1$, truncated at some large value V[14]. The value distribution Y for the last stage is an equal revenue distribution truncated at some large value $\hat{V}$[15], with $F_y(x) = 1 - \frac{1}{x}$.The following auction is IC in a perfect Bayesian equilibrium and ex-post IR:

- In the first $m-1$ stages run a first price auction: the winner is the buyer with the highest value with a payment equal to that value.
- In the last stage, the item is given for free to buyer i, with probability equal to $\frac{\sum_{k=1}^{m-1} p_k^i}{\mathbb{E}[Y]}$, where p_k^i is the price payed by agent i in stage k. In other words, every buyer i wins by bidding v, the probability that she gets the last stage item is increased by $\frac{v}{\mathbb{E}[Y]}$.

The revenue of this auction is

$$(m-1) \cdot \mathbb{E}[X_{1:n}]. \tag{4.13}$$

The revenue of VCG at every stage with c additional buyers is at most

$$(m-1) \cdot \mathbb{E}[X_{2:n+c}] + \mathbb{E}[Y_{2:n+c}] \leq (m-1) \cdot \mathbb{E}[X_{2:n+c}] + n + c. \tag{4.14}$$

We used that the expected second order statistic of k samples from an equal revenue distribution is at most k. For a calculation see Claim 93 in Appendix A.2. We want to find the smallest c such that 4.14 is at least 4.13. For simplicity we compute the expected order statistics of X as if it is an untruncated exponential distribution. We can pick V large enough such that the conclusion is the same.

[14]Truncated exponential distributions have monotone hazard rate.

[15] Y has revenue 1 for a single agent, and expectation $\ln\left(\hat{V}\right)$.

For an exponential distribution $Z \sim Exp(1)$ we have that $\mathbb{E}[Z_{1:n}] = \sum_{i=1}^{n} \frac{1}{i} = H_n$ and $\mathbb{E}[Z_{2:n}] = \sum_{i=2}^{n} \frac{1}{i} = H_n - 1$, where H_n is the n-th harmonic number. For large n, H_n can be approximated by $\ln n$. Therefore, expression 4.14 being at least expression 4.13 is equivalent to :

$$
\begin{aligned}
(m-1) \cdot \mathbb{E}[X_{2:n+c}] + n + c &> (m-1) \cdot \mathbb{E}[X_{1:n}] \\
(m-1) \cdot (H_{n+c} - 1) + n + c &> (m-1) \cdot H_n \\
H_{n+c} - H_n + \frac{n+c}{m-1} &> 1 \\
\ln\left(\frac{n+c}{n}\right) + \frac{n+c}{m-1} &> 1 \\
(n+c)e^{\frac{n+c}{m-1}} &> ne \\
\frac{n+c}{m-1} e^{\frac{n+c}{m-1}} &> \frac{ne}{m-1} \\
\frac{n+c}{m-1} &> W\left(\frac{ne}{m-1}\right) \\
c &> (m-1) \cdot W\left(\frac{ne}{m-1}\right) - n
\end{aligned}
$$

where W is the Lambert function. $\lim_{k\to\infty} kW(\frac{en}{k}) = en$, therefore $c > (e-1)n$. □

Lemma 59. *For m MHR stages, and ex-ante IR auctions, the Competition Complexity is at least $(e-1)n$, even for independent stages and for auctions that are incentive compatible in a perfect Bayesian equilibrium.*

Proof. For the first stage, X_1 is uniform $[0, \epsilon]$ for some small $\epsilon > 0$. For $k = 2, \ldots, m$, $X_k = Exp(1, V)$ is an exponential distribution with parameter $\lambda = 1$, truncated at some large V. We describe an ex-ante IR auction that is incentive compatible in a perfect Bayesian equilibrium, with revenue $\sum_{k=2}^{m} \mathbb{E}[(X_k)_{1:n}]$, i.e. essentially the social welfare: on stages 2 through m the auctioneer will run a second price auction, extracting revenue $\sum_{k=2}^{m} \mathbb{E}[(X_k)_{2:n}]$. The extra $\sum_{k=2}^{m} (\mathbb{E}[(X_k)_{1:n}] - \mathbb{E}[(X_k)_{2:n}])$ will be charged upfront; in stage 1 every buyer is offered the option to pay $\frac{1}{n}\sum_{k=2}^{m} (\mathbb{E}[(X_k)_{1:n}] - \mathbb{E}[(X_k)_{2:n}])$ in order to participate in stages 2 through k. This is equal to the expected utility of each buyer, and therefore, since they are expectation maximizers, they will accept the offer.

Given this auction, the calculation for lower bounding the Competition Complexity is almost identical (in fact much simpler) to Lemma 58. □

Part II

Dynamic Fair Division

Chapter 5

Controlled Dynamic Fair Division

In this Chapter we introduce a simple dynamic fair division problem, where agents arrive and depart over time, and the amount of disruption is a hard constraint. We formally introduce the model and related work in Section 5.1. We present our algorithm for a single, homogeneous resource in Section 5.2. We then study multiple, heterogeneous resources. For this case, there are many popular notions of fairness. We show NP-hardness for a generic one in Section 5.3, and provide algorithms (and impossibility results) when the notion of fairness is Dominant Resource Fairness (DRF) in Sections 5.4 and 5.5.

5.1 The Dynamic Fair Division Model

We first consider the case of a single homogeneous resource.

Dynamic fair division of a single resource. One unit of a homogeneous resource is shared among agents that arrive and depart over time. An allocation for t agents is denoted by a vector $\textsc{alloc}^t \in [0,1]^t$. The utility of agent j at step t is proportional to $\textsc{alloc}^t(j)$. An allocation is *feasible* if $\textsc{talloc}^t = \sum_{j=1}^{t} \textsc{alloc}^t(j) \leq 1$. ALLOC and TALLOC are usually defined with respect to a resource allocation algorithm; this is omitted from the notation when the algorithm is obvious from context. An allocation algorithm is feasible if it always outputs a feasible allocation. An allocation is *Pareto optimal* if $\textsc{talloc}^t = 1$. An allocation algorithm is Pareto optimal if it always outputs Pareto optimal allocations. An allocation for t agents is *σ_t-fair* if $\min\limits_{j=1,\ldots,t} \{\textsc{alloc}^t(j)\} \geq \frac{\sigma}{t}$. An allocation algorithm is σ-fair if it outputs a *σ_t-fair* allocation in the presence of t agents, and $\sigma \leq \min_t\{\sigma_t\}$.

We restrict our algorithms to disrupt a small number of agents when a new agent arrives: An allocation algorithm is d-disruptive if it is allowed to decrease the allocation of at most d agents at every arrival. Upon departure, the algorithm is not allowed to reduce the allocation of any agent except the departing agent (this is known as *population monotonicity*). Augmenting the resource of any agent is always allowed (for both arrivals and departures).

Example 60. *Assume we would like to design a Pareto optimal* 1*-disruptive algorithm for a system with a capacity for* 3 *agents. The naïve algorithm divides the largest available share equally at each arrival: When the first agent arrives, she is allocated the entire resource;* $\sigma_1 = 1$*. When the second agent arrives, she is given half of the resource, and the first agent's resource is halved:* $\sigma_2 = \frac{\min_i\{\text{ALLOC}^2(i)\}}{1/2} = 1$*. When the third agent arrives, one agent is allocated half of the resources and the two other agents are allocated one quarter of the resource each:* $\sigma_3 = \frac{1/4}{1/3} = \frac{3}{4}$*. The fairness ratio of this algorithm is* $\sigma = \min_i\{\sigma_i\} = \frac{3}{4}$*. It is easy to verify that there is no algorithm that guarantees perfect fairness (*$\sigma = 1$*), even in this simple scenario.*

The optimal algorithm is the following (Friedman, Psomas, and Vardi [2015]): when the second agent arrives, she is given $3/7$ *of the resource;* $\sigma_2 = \frac{3/7}{1/2} = \frac{6}{7}$*. Then when the third agent arrives, the first agent's share is split evenly between the first and third agents, giving the allocation* $(2/7, 3/7, 2/7)$*;* $\sigma_3 = \frac{2/7}{1/3} = \frac{6}{7}$*. This fairness ratio of this algorithm is* $\frac{6}{7}$*.*

d **disruptions per arrival.** Define $\sigma^*(d)$ to be the optimal fairness ratio of any d-disruptive mechanism (with an unbounded number of agents). We prove tight bounds on $\sigma^*(d)$, for all (integer) $d \geq 1$. At first glance, one might think that a single disruption per arrival is the best we can hope for; this is true if, for example, we require Pareto optimality (Lemma 61). However, if n agents arrive, this still mandates n disruptions! Next, we consider what can be done with even fewer disruptions. In particular, we study the problem of maximizing the fairness ratio, when fewer than one disruption is allowed per arrival.

Fewer than one disruption per arrival. Consider following scenario. An algorithm designer is given a list of constraints: for each arrival, the number of disruptions allowed is denoted. The algorithm designer's goal is to design an algorithm that maximizes the fairness ratio subject to these constraints. Ideally, the algorithm designer should design an *instance-optimal* algorithm: one that takes the list of constraints as an input and outputs a set of allocations that maximizes the fairness ratio for that list. We consider the case that fewer than one disruption is allowed per arrival, and model it as follows: the algorithm takes as input a vector ψ (which we call a *control vector*), and is allowed to use d_i donors at step i if $\psi[i] = d_i$. We call this problem *Controlled Dynamic Fair Division (CDFD).* For simplicity, we only consider *binary* control vectors (i.e., at each time step, the algorithm is either not allowed to use a donor or allowed to use one). It is straightforward to extend the results to non-binary vectors. If ψ has at most c consecutive zeros, we call it a c-control vector. We say that a c-control vector implies a $\frac{1}{c+1}$-disruptive algorithm. Notice that the case of d donors per arrival is a special case; ψ has "d" at every coordinate.

Let $\sigma^*(\psi)$ be the optimal fairness ratio for a given control vector ψ. We overload the notation and define $\sigma^*((c+1)^{-1})$ to be the optimal fairness ratio for all c-control vectors ψ, i.e., $\sigma^*((c+1)^{-1}) = min_\psi\{\sigma^*(\psi)\}$. Note that, with fewer than one disruption per arrival, Pareto optimality is impossible.

Lemma 61. *For any $c > 0$, there is no Pareto optimal algorithm for $CDFD$ that guarantees a fairness ratio of $\sigma((c+1)^{-1}) > 0$.*

Proof. Assume that such an algorithm exists. Let i be the first coordinate for which $\psi[i] = 0$. As the algorithm is Pareto optimal, there is no available resource, and agent i receives nothing. □

Multi-resource fairness. Consider the case of multiple heterogeneous resources. Let r be the number of resources in the system and, without loss of generality, assume there is 1 unit of each resource available. As in the single resource case, define $\textsc{alloc}^t(j)$ to be the allocation of the j-th agent at step t. Notice that the allocation of agent j this time is a vector, not a number.

Every agent i has a demand vector $\mathcal{D}_j \in [0,1]^r$ over the resources. Let $\mathcal{D}^t$ (or simply $\mathcal{D}$) be the t by r matrix of demands for t agents. For example, if 3 agents with demands $\mathcal{D}_1 = [1, 1/2]$, $\mathcal{D}_2 = [0, 1]$, and $\mathcal{D}_3 = [1, 1]$ are present

$$\mathcal{D} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Assume that the agents demand the resources in fixed proportions, known in economics as Leontief preferences, and have binary demand vectors. Let $u_j(\textsc{alloc}^t(j))$ be the utility of agent j for an allocation $\textsc{alloc}^t(j)$. It is convenient to think of $\mathcal{D}_{j,l}$ as the amount of resource l agent j needs to execute a task. For example, if $r = 3$, agent j with demand vector $\mathcal{D}_j = [1, 0, 1]$, needs 1 unit of resource 1 and 1 unit of resource 3 to execute a task. In this example, Leontief preferences imply that given the resource vector $\textsc{alloc}^t(j) = (\frac{1}{2}, 0, \frac{1}{2})$, the agent would have utility $u_j(\textsc{alloc}^t(j)) = \frac{1}{2}$. The agent would have the same utility for resource vector $(1, 0, \frac{1}{2})$.

In the case of multiple resources, the definition of fairness is not as straightforward. One could define fairness in a very general way: at every step t the minimum utility of an agent should be $L(t)$, for some non increasing function $L(t)$, and the maximum utility should be $U(t)$. $L(t)$ dictates the lower bound on the satisfaction of each agent, and $U(t)$ maintains fairness in the sense of envy-freeness. (An allocation of $(\frac{1}{3}, \frac{1}{4}, \frac{1}{4})$ is intuitively more fair than an allocation of $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, and this notion of fairness is enforced by an upper bound.) Define *General Dynamic Fair Division* ($GDFD$) to be this problem; the inputs are agents' demands $\mathcal{D}$, the number of resources r, and vectors L and U, and the algorithm is asked to decide whether there exist feasible allocations (i.e., ones that satisfy all the constraints).

Dominant Resource Fairness. We show in Section 5.3 that $GDFD$ is computationally intractable, even in simple settings and for clairvoyant allocation algorithms. We focus on a less general, well-studied notion of fairness: *Dominant Resource Fairness* (DRF), introduced by Ghodsi et al. [2011].

The *dominant resource* of an agent is the resource for which the agent's task requires the largest fraction of total availability. The *dominant share* of an agent is the fraction of her dominant resource she receives. The DRF algorithm seeks to maximize the minimum dominant share. DRF can also be interpreted as the leximin mechanism, i.e., it maximizes the minimum utility, and subject to that, maximizes the second minimum utility, and so on, when applied to Leontief utilities (Kurokawa, Procaccia, and Shah [2015]). DRF has multiple advantages: it is Pareto optimal, strategy-proof, envy-free, and proportional: it guarantees to each agent a utility of

$$DRF_{\mathcal{D}^t} = \left(\max_{l=1,\ldots,r} \sum_{j=1}^{t} \mathcal{D}^t_{j,l} \right)^{-1}.$$

The notion of DRF is naturally extendable to the dynamic case:

Dynamic fairness for multiple resources. An allocation is σ_t-*DRF* fair at step t, if

$$\min_{j=1,\ldots,t} \{u_j(\text{ALLOC}^t(j))\} \geq \sigma_t DRF_{\mathcal{D}^t}.$$

An allocation algorithm is σ-*DRF* fair if outputs a σ_t-*DRF* fair allocation when t agents are present, and $\sigma \geq \min_t\{\sigma_t\}$. Let $\sigma^*(d, r)$ be the optimal fairness ratio in the case of $r > 1$ resources, when d donors are allowed at every arrival, over all possible agent demands $\mathcal{D}$. We allow d to be less than 1; we discuss this in more detail later on. We define *Dynamic Dominant Resource Fairness* (DDRF) to be the problem of maximizing $\sigma^*(d, r)$.

Results and Techniques

Controlled dynamic fair division. Given a binary control vector ψ as an input, the goal is to output a set of allocations that gives the best possible fairness ratio. We describe an instance-optimal algorithm for this case, which we call SKIP. SKIP has two stages; in the first (which we sometimes refer to as the *preprocessing* stage), it computes the optimal fairness ratio $\sigma^*(\psi)$; in the second, it allocates the resource frugally, giving each agent the least amount of resource possible in order to maintain $\sigma^*(\psi)$.

Theorem 62. SKIP *is an instance-optimal algorithm for controlled dynamic fair division.*

To prove Theorem 62, we show that for any allocation algorithm $\mathcal{A}$ and any control vector ψ, if the allocations produced by $\mathcal{A}$ for ψ give a fairness ratio of σ, then the allocations produced by SKIP do too. Although, by Lemma 61, SKIP cannot be Pareto optimal, it can accommodate departures, and is population-monotone - it does not disrupt any agent when there is a departure (except the departing agent); it is only allowed to re-allocate the departing agent's share.

We would like to provide a lower bound[1] σ on the fairness ratio of SKIP. This is particularly important for unbounded systems; while SKIP is optimal for any finite control vector ψ,[2] it is not defined on infinite vectors, which are necessary for systems that can accommodate an arbitrary number of agents. In such a case, we skip the preprocessing stage of SKIP, and simply give SKIP σ as an auxiliary input. As long as σ is upper bounded by the worst fairness ratio possible for any number of agents, the allocations produced by SKIP will be feasible.

We first bound $\sigma^*(d)$ for $d \geq 1$, the optimal fairness ratio possible for all control vectors ψ such that $\psi[i] = d$; at every arrival d disruptions are allowed. We show that SKIP can guarantee performance $\sigma^*(d) = \left((d+1)\ln\left(\frac{d+1}{d}\right)\right)^{-1}$, for all (integer) $d \geq 1$. Furthermore, SKIP can be modified to be Pareto Optimal at every step. In [47] we provide a different algorithm with the same fairness bounds, and also a bound of $\frac{3(d+1)^2}{2d^2}$ on the *envy ratio* of this algorithm, the ratio of the maximum and minimum allocations.

Theorem 63. *$\sigma^*(d) = \left((d+1)\ln\left(\frac{d+1}{d}\right)\right)^{-1}$, for all (integer) $d \geq 1$.*

Then, we bound $\sigma^*\left((c+1)^{-1}\right)$, the optimal fairness ratio possible for all control vectors ψ with at most c consecutive zeros (in particular, including infinite vectors such as $(1,0,0,1,0,0,\ldots)$). Showing bounds for $\sigma^*\left((c+1)^{-1}\right)$ is much more complicated than for $\sigma^*(d)$, $d \geq 1$: there are infinitely many possible control vectors, and the allocations SKIP produced for each of these are different; furthermore, they are not as "well behaved" as the allocations for $d \geq 1$. Compare Figures 5.1 and 5.2.[3]. Figure 5.1 shows the allocations created by the optimal algorithm when 1 donor is allowed at every step (for an unbounded number of agents, truncated at 100). Figure 5.2 shows the allocations created for 4 different infinite control vectors (truncated at 30 agents). The difficulty in the second setting comes from several facts: the allocations are not monotone, they are not pointwise comparable, and they are not simple transformations one of the other. Furthermore, the allocations do not necessarily take their maxima at the limit. Still, in both cases, the total allocation converges to a limit as the number of agents grows (in the second case, *all* infinite allocations that obey certain natural requirements converge to the *same* limit (Theorem 66). The horizontal line in both figures denotes this limit. It is easy to see (this is formally proved for the two cases in [47] and Section 5.2 respectively), that the first set of allocations takes its maximum at the limit, while the second case does not.[4]

[1]Note that upper bounds on the fairness ratio are negative results, while lower bounds are positive results.

[2]We do not explicitly compute the running times of the algorithms, but all are easily implemented in time linear in the size of the input.

[3]The graphs are normalized by artificially setting $\sigma = 1$, and then running the respective optimal algorithms without any constraints on the amount of available resource. While this obviously leads to the total allocation being greater than 1, it is still instructive, as the optimal fairness ratio of these algorithms can be easily be deduced from these plots - it is simply the inverse of the maximal total allocation created

[4]It might be tempting to think that, as in Figure 5.2, there always exists *some* basic c-control vector

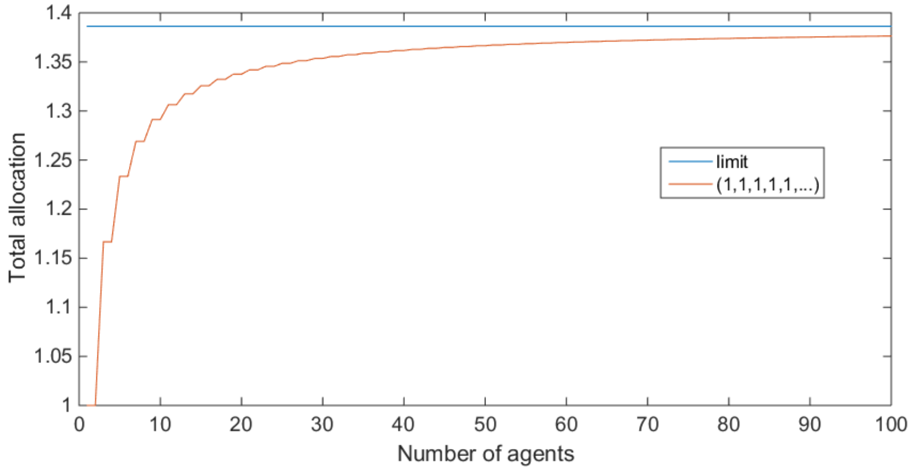

Figure 5.1: Optimal values of TALLOC for $d = 1$.

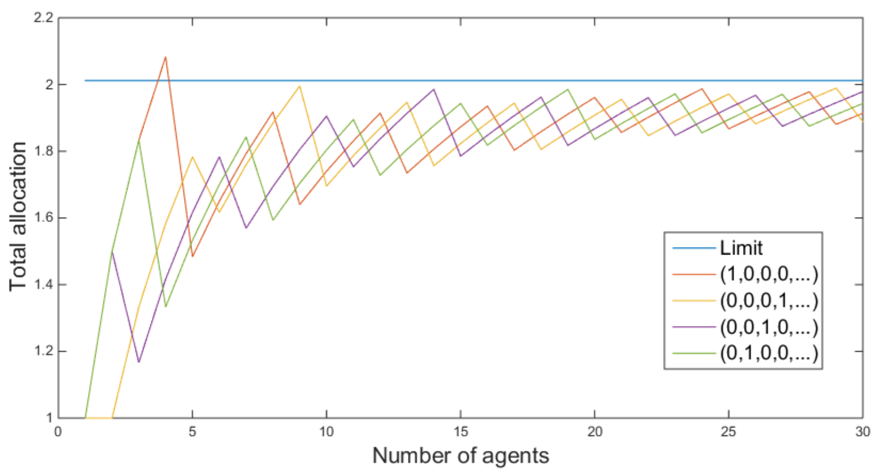

Figure 5.2: Optimal values of TALLOC for some 2-control vectors.

We show almost matching upper and lower bounds for the optimal fairness ratio attainable for any (possibly infinite) c-control vector. Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th harmonic number.

Theorem 64. *The optimal fairness ratio of CDFD for any c-control vector is bounded by*

$$(H_{c+1})^{-1} \geq \sigma^* \left((c+1)^{-1}\right) \geq \left(H_{2c+3} - \frac{1}{2}\right)^{-1}.$$

(see Section 5.2 for a definition of basic c-control vectors) that takes its maximum at the limit, and this is indeed the case for $c < 8$ (we do not prove this, but it is easy to verify). However, for $c \geq 8$, *every* basic c-control vector has its maximum at a finite number of agents (see Figure B.1 in Appendix B.5 for a pictoral example.)

In order to prove Theorem 64, we need to be able to argue about allocations of optimal algorithms for *any* ψ. For this, we define *basic* control vectors - vectors that always have exactly 1 donor every $c+1$ agents, and show that for any c-control vector, there exists some basic c-control vector whose allocations are pointwise greater. For any c, there are exactly $c+1$ basic control vectors, therefore it remains to reason about them. For the cases $c=1$ and $c=2$, it turns out that the worst allocations are always at the limit. In order to show this, we consider allocations (of the basic control vectors) that are a certain distance apart (specifically $(c+1)(c+2)$), and show that these special allocations are monotone increasing, and each is greater than the $(c+1)(c+2)$ allocations that came before it. This gives exact bounds:

Theorem 65. *(Appendix B.3) The optimal fairness ratio of $CDFD$ for any c control vector for $c=1$ and $c=2$ are*

$$\sigma^*\left((2)^{-1}\right) = 2(3\ln 3)^{-1} \textit{ and } \sigma^*\left((3)^{-1}\right) = 3(4\ln 4)^{-1}$$

respectively.

For $c>2$, we consider a set of allocations that are pointwise greater than the allocations for all basic vectors simultaneously, and bound them, to obtain Theorem 64. In addition, we show that as the number of agents grows, the optimal allocations for all basic c-control vectors converge to the same value:

Theorem 66. *There exists a number n_0 such that, for all $c>2$, all c-control vectors, and agents $t>n_0$, the allocation of* Skip *is σ_t-fair, where $\sigma_t = (c+1)\left((c+2)\ln(c+2)\right)^{-1}$.*

We show that Skip can accommodate departures, and therefore all the bounds on the fairness ratio hold even when we allow agents to depart.

We summarize our main results for $\sigma^*\left((c+1)^{-1}\right)$ in Table 5.1.

Disruptions	**Bound on the fairness ratio**
$d \geq 1$	$\left((d+1)\ln\left(\frac{d+1}{d}\right)\right)^{-1}$ (tight)
2^{-1}	$2\left(3\ln 3\right)^{-1}$ (tight)
3^{-1}	$3\left(4\ln 4\right)^{-1}$ (tight)
$(c+1)^{-1}, c>2$	$\left(H_{c+1}\right)^{-1} \geq \sigma^*\left((c+1)^{-1}\right) \geq \left(H_{2c+3} - \frac{1}{2}\right)^{-1}$
	$(c+1)\left((c+2)\ln(c+2)\right)^{-1}$ (asymptotic bound, tight)

Table 5.1: Results for 1 resource

Multiple resource dynamic fair division. In the multiple resource case, we consider the *uncontrolled* setting. The input to the algorithm is a demand matrix $\mathcal{D}$ and the number of disruptions allowed per arrival, d. We consider two settings: in the first, the *clairvoyant* setting, $\mathcal{D}$ is known in advance. In the *non-clairvoyant* (or online) setting, row i of $\mathcal{D}$ is revealed upon arrival of agent i. We focus on two notions of fairness, $GDFD$ and $DDRF$.

Our first result is that $GDFD$ is NP-hard (Section 5.3):

Theorem 67. *$GDFD$ is* NP*-hard, even for* 2 *resources,* 1 *donor, binary demand vectors, and clairvoyant algorithms.*

In Section 5.4 we consider $DDRF$, for binary demand vectors. To see why the techniques of the single resource case do not extend to the multi-resource case, consider the following simple example, for $d = 1$: Three agents arrive; the first has a demand vector $[1, 0]$, the second $[0, 1]$, and the third $[1, 1]$. The following algorithm gives the optimal allocation of $2/3$ (see Section 5.5 for the upper bound): the first two agents receive $2/3$ of their respective resource (we do not use a donor at the second arrival), and at the third arrival, one of the two agents has her allocation reduced by half, and the arriving agent is given $1/3$ of that resource. $1/3$ of the other resource is taken from the "bank" - the resource that was left unallocated. It is easy to see that a non-clairvoyant algorithm can never do as well as a clairvoyant one (in contrast to the single resource case): in the non-clairvoyant case, if an agent with demand $[0, 1]$ arrives, how much do we allocate her? Even if we know three agents will arrive in total, and the second one has demand $[1, 0]$, if the third agent has demand vector $[1, 0]$, we "should have" given the first agent all of the first resource, to get a fairness ratio of 1, but we have to allow for the possibility of the third agent's demand being $[1, 1]$. To complicate matters further, it is possible to find examples where solutions that seem intuitively correct are wrong; consider the following set of demands:

$$[0, 1], [1, 0], [0, 1], [1, 0], \mathbf{[1, 0]}, [1, 1], [1, 1].$$

Surprisingly, the only optimal algorithm uses an agent with demand $[0, 1]$ as a donor when the fifth agent (who has demand $[1, 0]$) arrives!

We prove upper and lower bounds on $\sigma^* (d, r)$ for $DDRF$ for the non-clairvoyant setting for three cases: (1) $d = kr$ for some integer k, (2) $d = 1$ and (3) $d = (c + 1)^{-1}$ for $c > 0$. For the lower bounds (positive results), we use the single resource algorithms as subroutines. The challenge is choosing which donor to use when only one donor is allowed. Our negative results make use of the single resource upper bounds bounds.

Theorem 68. *The optimal fairness ratio of $DDRF$ for $d = k \cdot r$ donors and r resources is bounded by*

$$\left((kr + 1) \ln \left(\frac{kr+1}{kr}\right)\right)^{-1} \geq \sigma^* (kr, r) \geq \left((k + 1) \ln \left(\frac{k+1}{k}\right)\right)^{-1}.$$

Theorem 69. *The optimal fairness ratio of $DDRF$ for $d = 1$ donors and $r \geq 2$ resources is bounded by*

- $\sigma^* (1, 2) \geq 2 (3 \ln 3)^{-1} \approx 0.6068$
- $\sigma^* (1, 3) \geq 3 (4 \ln 4)^{-1}$
- $\sigma^* (1, r) \geq \left(H_{2r+1} - \frac{1}{2} \right)^{-1}, r > 3.$

Theorem 70. *The optimal fairness ratio of DDRF for r resources and $(c+1)^{-1}$ donors (1 donor allowed per $c+1$ arrivals) is bounded by*

$$(H_{c+1})^{-1} \geq \sigma^* \left((c+1)^{-1}, r \right) \geq \left(H_{2r(c+1)+1} - \tfrac{1}{2} \right)^{-1}.$$

The gap between the upper and lower bounds for *DDRF* is far from tight. Even for $d = 1$, $r = 2$, a seemingly simple case, the best lower bound we have for $\sigma^* (1, 2)$ is $2 (3 \ln 3)^{-1} \approx 0.6068$, by using the single resource algorithm with 1-control vectors ($c = 1$). An immediate upper bound is $(\ln 4)^{-1} \approx 0.7213$; the fairness ratio $\sigma^* (d, r)$ is non-increasing in the number of resources r, and we have tight bounds for $\sigma^* (d, 1)$ (a subtle but necessary condition for this argument to go through is that *DRF* reduces to proportionality when there is only one resource). In Section 5.5 we improve this upper bound.

Theorem 71. *The optimal fairness ratio of DDRF for $d = 1$ and $r \geq 2$ resources is bounded by*

$$\sigma^* (1, r) \leq 0.6318 \ldots$$

Our proof technique for Theorem 71 may be of independent interest. We first identify a set of bad inputs (demand matrices $\mathcal{D}$): n $[0, 1]$'s followed by n $[1, 0]$'s, followed by a series of $2n$ $[1, 1]$'s. We describe the optimal mixed integer program for maximizing the fairness ratio. Let Z be the set of all integer variables. Given an algorithm for *DDRF* we can simulate its execution on the bad input and fix these integer variables. Fixing Z allows us to use LP duality! We take the dual with respect to the remaining (non-integer) variables, and give a procedure for constructing feasible dual solutions, for any choice of Z, and therefore any algorithm for *DDRF*. Notice that the bound is independent of r. For $r > 2$, the same technique does not seem to provide better results, at least not for choices of $\mathcal{D}$ that are the most natural; for example, for $r = 3$, one would expect that the worst inputs are n $[0, 0, 1]$'s, followed by n $[0, 1, 0]$'s, followed by n $[1, 0, 0]$'s, followed by a series of $3n$ $[1, 1, 1]$'s. Surprisingly, the fairness ratio for these inputs is an increasing function of n, at least for n small enough for us to verify computationally. Our main results for the optimal fairness ratio of *DDRF* are summarized in the following table.

r	Disruptions	Bound on the fairness ratio
2	1	$0.6318\ldots \geq \sigma^*(1,2) \geq 2(3\ln 3)^{-1} \approx 0.6068$
3	1	$0.6318\ldots \geq \sigma^*(1,3) \geq 3(4\ln 4)^{-1}$
$r > 3$	1	$0.6318\ldots \geq \sigma^*(1,r) \geq \left(H_{2r+1} - \frac{1}{2}\right)^{-1}$
r	kr	$\left((kr+1)\ln\left(\frac{kr+1}{kr}\right)\right)^{-1} \geq \sigma^*(kr,r) \geq \left((k+1)\ln\left(\frac{k+1}{k}\right)\right)^{-1}$
r	$(c+1)^{-1}$	$(H_{c+1})^{-1} \geq \sigma^*\left((c+1)^{-1}, r\right) \geq \left(H_{2r(c+1)+1} - \frac{1}{2}\right)^{-1}$

Table 5.2: Results for many resources and DRF

Related Work

Competitive Equilibrium from Equal Incomes (CEEI) was introduced by Nash Jr [1950] and has been studied extensively in the economics and theoretical computer science literature (for some recent results see Devanur et al. [2008], Budish [2011], Othman, Papadimitriou, and Rubinstein [2014], Cole and Gkatzelis [2015], Caragiannis et al. [2016]). Dominant Resource Fairness was proposed by Ghodsi et al. [2011]. DRF attracted significant attention from the computer systems community([69, 71, 100, 54]), as well as the theoretical computer science community ([91, 62, 74, 48]). It is interesting to note that DRF can also be interpreted as the Kalai-Smorodinsky bargaining solution (Kalai and Smorodinsky [1975]).

Walsh [2011] was the first to study the problem of online fair cake cutting when agents arrive, receive a piece and depart. He showed how several well-known fair division solutions (cut-and-choose, Dubins-Spanier, etc) can be adapted to satisfy desirable properties in an online setting with a single (heterogeneous) divisible cake. More recently, and closer to our problem, Kash, Procaccia, and Shah [2013] introduced a model of dynamic allocations. However, their model only considers arrivals and their main algorithm reserves resources for future arrivals; it does not allow the reallocation of resources, or agents to depart. This leads to allocations that satisfy neither fairness nor Pareto efficiency, as resources are left idle.

Guo, Conitzer, and Reeves [2009] study the problem of repeatedly allocating a single item between competing agents, and give allocation algorithms that don't allow monetary transfers with good competitive ratios with respect to optimal allocation algorithms with payments. Segal-Halevi [2016] studied the problem of re-dividing a two-dimensional resource, subject to fairness and ”geometric” constraints on the allocations.

Finally, the idea of making a small number of alterations to maintain a good solution in online settings had been studied for other problems. In online scheduling, one of the earliest works is by Phillips and Westbrook [1993], who show that linear preemption in the number of tasks yields a competitive ratio of $O(\log n)$ on makespan, where n is the number of tasks. Preemption in their context means reassigning jobs to a different machine. Sanders, Sivadasan, and Skutella [2009] study online scheduling when the migration is bounded by a constant, and Epstein and Levin [2011] who study a similar problem with bounded migrations, where the capacity of change is based on the size of the new input. Gupta, Kumar, and Stein [2014] consider online matching, online flow and online scheduling, and the number of changes they allow is an amortized constant per step. For scheduling they show a $O(\log\log(nm))$ approximation to the makespan, where n is the number of tasks and m is the number of machines. Other examples include Gu, Gupta, and Kumar [2016] and Gupta and Kumar [2014] who show how to maintain an online Steiner tree, vertices arriving online, where the algorithm can change an edge at every arrival.

5.2 Singe Resource Fair Division

In this section we study $CDFD$. The algorithm is given as input a control vector ψ (Definition 72). A solution consists of a fairness ratio σ and a set of allocations $A_1, \ldots, A_{|\psi|}$, one for each number of agents present t, such that: (1) for every $1 \le t \le |\psi|$, the fairness ratio σ_t is at least σ, and (2) some agent's resource is reduced from step t to step $t+1$ if and only if $\psi[t+1] = 1$. Note that only one agent can have their resource reduced at any time. In some cases (for example, when the control vectors are unbounded), we allow the algorithm to receive σ as an auxiliary input, in which case a possible output is "infeasible", if no such set of allocations exists. For now, we assume that agents only arrive, and do not depart; we can use t both for the time and the number of agents present. At the end of the Section we show how to augment Skip to accommodate departures.

Definition 72. *[Control vector] Let ψ^N be a binary vector of length N. $\psi^N[i] = 1$ means that we use a donor when agent i arrives, and a $\psi^N[i] = 0$ means we do not. If the maximal number of consecutive* 0*s is c, we call this a* c-control vector*.*

We define a *basic c-control vector* to be one in which there is exactly one donor every $c+1$ arrivals; otherwise the control vector is non-basic. There are $c+1$ possible basic control vectors.

Example 73. *For c = 2, the three possible basic control vectors are:*

1. *(0,1,0,0,1,0,0,1,0,...), denoted* $(0, 1, 0)^\infty$*,*
2. *(0,0,1,0,0,1,0,0,1,...), denoted* $(0, 0, 1)^\infty$*, or* $(0^2, 1)^\infty$*.*
3. *(1,0,0,1,0,0,1,0,0,...), denoted* $(1, 0, 0)^\infty$*, or* $(1, 0^2)^\infty$*.*

We will also use the following definition.

Definition 74. *Let $A = (a_1, \ldots, a_t)$ and be $B = (b_1, \ldots, b_t)$ be two sorted vectors. We say that A* dominates *B, denoted $A \succeq B$, if $\forall i \in [1, t]$, $a_i \ge b_i$.*

SKIP - an Instance-Optimal Algorithm

Definition 75 (Algorithm Skip). *Algorithm* Skip *receives as an input a control vector* ψ*. Set* $n = |\psi|$*.* Skip *has a preprocessing stage (which we describe after the main algorithm description), in which it computes the optimal* σ*. When agent* $1 \le i \le n$ *arrives, allocate her* $\frac{\sigma}{i}$ *of the resource. If* $\psi[i] = 1$*, take the agent with the most resource to be the donor, and reduce her allocation to* $\frac{\sigma}{i}$ *as well.*

The preprocessing stage is the following: simulate the arrivals of the agents $1, \dots, n$*, with* $\sigma = 1$*, and compute the sum of allocations at each step,* TALLOC_i*. Set the optimal fairness ratio to be* $\sigma = (\max_{1 \le i \le n}\{\text{TALLOC}_i\})^{-1}$*.*

First, note that the allocations created by Skip are always feasible, by the choice of σ. We prove Skip is optimal among all feasible allocation algorithms, for any control vector.

Theorem 62. Skip *is an instance-optimal algorithm for controlled dynamic fair division.*

Proof. Let ψ be an input for any allocation algorithm $\mathcal{A}$ and Skip. Let σ be the maximum fairness ratio achievable by $\mathcal{A}$ and σ' the maximum fairness ratio achievable by Skip. We consider Skip when σ' is replaced by σ during its execution. We show that Skip is feasible in this case, therefore $\sigma' \ge \sigma$. We assume w.l.o.g. that $\mathcal{A}$ never increases the allocation of any agent (except the arriving agents).

Let $\mathcal{A}^t$ and Skip^t be the sorted allocations of $\mathcal{A}$ and Skip, respectively, for t agents in the system. It suffices to show that $A^t \succeq \text{Skip}^t$ for all $t \le |\psi|$. The proof is by induction on the number of agents in the system, t.

The base case: By the definition of Skip, $\text{Skip}^1(1) = \sigma$. Because $\mathcal{A}$'s fairness ratio is σ, it must hold that $\mathcal{A}^1(1) \ge \sigma$.

The inductive step: Assume the statement holds for $t-1$ agents. We show it holds for t. There are two cases: $\psi[t] = 0$ and $\psi[t] = 1$.

Case $\psi[t] = 0$ **:** Let the allocation of Skip at time $t-1$ be $(\text{Skip}_1^{t-1}, \dots, \text{Skip}_{t-1}^{t-1})$. Then $\text{Skip}^t = (\text{Skip}_1^{t-1}, \dots, \text{Skip}_{t-1}^{t-1}, \frac{\sigma}{t})$. Algorithm $\mathcal{A}$ will allocate an amount of resource $x \ge \frac{\sigma}{t}$ to the incoming agent. Let the allocation of $\mathcal{A}$ at time $t-1$ be $(\mathcal{A}_1^{t-1}, \mathcal{A}_2^{t-1}, \dots, \mathcal{A}_{t-1}^{t-1})$. Then

$$\mathcal{A}^t = (\mathcal{A}_1^{t-1}, \mathcal{A}_2^{t-1}, \dots, \mathcal{A}_j^{t-1}, x, \mathcal{A}_{j+1}^{t-1}, \dots, \mathcal{A}_{t-1}^{t-1}),$$

for some $0 \le j \le t-1$. It is easy to see that $\mathcal{A}^t \succeq \left(A_1^{t-1}, A_2^{t-1}, \dots, A_{t-1}^{t-1}, \frac{\sigma}{t}\right)$ for any such value of j. Furthermore, by the induction hypothesis,

$$\left(\mathcal{A}_1^{t-1}, \mathcal{A}_2^{t-1}, \dots, \mathcal{A}_{t-1}^{t-1}, \frac{\sigma}{t}\right) \succeq (\text{Skip}_1^{t-1}, \dots, \text{Skip}_{t-1}^{t-1}, \frac{\sigma}{t}) = \text{Skip}^t.$$

This concludes the proof for $\psi[t] = 0$.

Case $\psi[t] = 1$ **:** Starting with the allocation vector at time $t-1$, $\mathcal{A}^{t-1}$, we break the allocation changes at time t into two steps:

1. Reduce the share of the donor, to get $\hat{\mathcal{A}}^{t-1}$.

2. Allocate x to the incoming agent, to obtain $\mathcal{A}^t$.

Denote $\mathcal{A}^{\min} = (\mathcal{A}_2^{t-1}, \ldots \mathcal{A}_{t-1}^{t-1}, \frac{\sigma}{t})$. We show that at the end of the first step, $\hat{\mathcal{A}}^{t-1} \succeq \mathcal{A}^{\min}$. The second step is identical to the $\psi[t] = 0$ case, hence we can combine the two results to conclude that $A^t \succeq \textsc{Skip}^t$. Assume algorithm $\mathcal{A}$ reduced the allocation of an agent from $\mathcal{A}_j^{t-1}$ to y, where $1 \leq j \leq t-1$ (possibly $\mathcal{A}_j^{t-1} = y$). There is some k, $j \leq k \leq t-1$ such that $\mathcal{A}_k^{t-1} \geq y \geq \mathcal{A}_{k+1}^{t-1}$. Then,

$$\hat{\mathcal{A}}^{t-1} = (\mathcal{A}_1^{t-1}, \mathcal{A}_2^{t-1}, \ldots, \mathcal{A}_{j-1}^{t-1}, \mathcal{A}_{j+1}^{t-1}, \mathcal{A}_{j+2}^{t-1}, \ldots, \mathcal{A}_k^{t-1}, y, \mathcal{A}_{k+1}^{t-1}, \ldots \mathcal{A}_{t-1}^{t-1}).$$

For $i \in [1, j-1]$, $\hat{\mathcal{A}}_i^{t-1} \geq \mathcal{A}_i^{\min}$. For $i \in [j, k]$, $\hat{\mathcal{A}}_i^{t-1} = \mathcal{A}_i^{\min}$. Then, by definition of y, $\hat{\mathcal{A}}_{k+1}^{t-1} = y \geq \mathcal{A}_{k+1}^{t-1} = A_{k+1}^{\min}$. Similarly, $\hat{A}_i^{t-1} \geq \mathcal{A}_i^{\min}$, for all $i \in [k+1, t-2]$. For the last term, we note that since $\mathcal{A}$ is σ-fair, $\mathcal{A}_{t-1}^{t-1} \geq \frac{\sigma}{t-1} \geq \frac{\sigma}{t}$ (possibly y is the last share, but then $y \geq \frac{\sigma}{t}$ as $\mathcal{A}$ is σ-fair). □

At Least One Disruption Per Arrival

We show a proof sketch of Theorem 63 for Skip and the special case of $d = 1$; the proof can be easily extended to general d. In fact, in [47] we show that a different algorithm can achieve the same bound, with additional bounds on the *envy ratio*, the ratio of the maximum and minimum allocations.

step	allocation for $(1, 1, \ldots)$	sum
1	σ	σ
2	$\frac{\sigma}{2}, \frac{\sigma}{2}$	σ
3	$\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{3}$	$\frac{7\sigma}{6}$
4	$\frac{\sigma}{3}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{4}$	$\frac{7\sigma}{6}$
5	$\frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{5}$	$\frac{7\sigma}{6}$
6	$2 \times \frac{\sigma}{4}, 2 \times \frac{\sigma}{5}, 2 \times \frac{\sigma}{6}$	$\frac{37\sigma}{30}$

Table 5.3: Allocations for the first 6 steps of Skip with basic control vector $(1, 1, \ldots)$.

Proof sketch of Theorem 63 for $d = 1$. The input to Skip is a control vector ψ, such that $\psi[i] = 1$, for all i. The allocations output by Skip have a very specific structure (see Table 5.3): On all even steps t the allocation is $\left(2 \times \frac{\sigma}{t/2+1}, 2 \times \frac{\sigma}{t/2+2}, \ldots, 2 \times \frac{\sigma}{t}\right)$. It is easy to see that the total allocation is increasing, hence focusing on the even steps is without loss of generality. Therefore, the total amount of resource allocated is $2 \cdot \sum_{i=\frac{t}{2}+1}^{t} \frac{1}{i} \approx 2 \cdot \ln\left(\frac{t}{\frac{t}{2}-1+1}\right) = 2\ln(2)$[5]. The Theorem follows. □

[5] See Appendix B.1 for the harmonic sum approximation

At Most One Disruption Per Arrival: Reduction to Basic Control Vectors

Here, we wish to compute a lower bound (positive result) on the fairness ratio of SKIP for *all* c-control vectors, but the optimal fairness ratio for any control vector depends on the vector itself. We show that basic control vectors have the worst fairness ratio; hence in order to provide a lower bound on the fairness ratio, it suffices to analyze basic control vectors. We allow SKIP to receive a fairness ratio, σ as an auxiliary input. Denote the (possibly infinite) set of allocations of SKIP when ψ^N is the control vector and σ is the fairness ratio by $\text{SKIP}(\psi^N, \sigma)$. It will be useful to think of the unallocated resource as a "bank": Let $\text{BANK}(\psi^N, t) = 1 - \text{TALLOC}(\psi^N, t)$.

We want to show that for any legal c-control vector ψ^N and real number σ, it holds that if $\text{SKIP}(\hat{\psi}^N, \sigma)$ is feasible, then $\text{SKIP}(\psi^N, \sigma)$ is feasible, where $\hat{\psi}^N$ is a basic c-control vector. To this end, we define a series of control vectors $\hat{\psi}^N = \psi_1^N, \psi_2^N, \ldots, \psi_k^N = \psi^N$ such that if $\text{SKIP}(\psi_i^N, \sigma)$ is feasible, then $\text{SKIP}(\psi_{i+1}^N, \sigma)$ is feasible, for all $1 \leq i < k$. We define these vectors inductively: Let t_i^* be the leftmost coordinate on which ψ_i^N and ψ^N differ. The first $t_i^* - 1$ entries of ψ_{i+1}^N are the same as ψ_i^N, the t_i^*-th entry becomes the same as $\psi^N[t_i^*]$, and the remainder continues as $(0^c 1)^\infty$.

Example 76. *Let $\psi^7 = (0, 1, 1, 1, 0, 1, 0, 0, 1)$. We derive the ψ_i^7s.*

$$\begin{aligned}
\hat{\psi}^N = \psi_1^7 &= (0, 1), (0, 0, 1)^\infty, \\
\psi_2^7 &= (0, 1, 1), (0, 0, 1)^\infty, \\
\psi_3^7 &= (0, 1, 1, 1), (0, 0, 1)^\infty \\
\psi_4^7 &= (0, 1, 1, 1, 0, 1), (0, 0, 1)^\infty = \psi^7 (0, 0, 1)^\infty
\end{aligned}$$

Lemma 77. *For every c-control vector ψ, there exists some basic c-control vector ψ' such that $\sigma^*(\psi) \geq \sigma^*(\psi')$.*

Proof. We prove that, if $\text{SKIP}(\psi_i^N, \sigma)$ is feasible, then $\text{SKIP}(\psi_{i+1}^N, \sigma)$ is feasible, for all steps t such that $t = t_i^* \pmod{c+1}$ in Lemma 78. We then show the same holds for steps $t \neq t_i^*$ $\pmod{c+1}$ in Lemma 79. □

Notice that $\text{ALLOC}(\psi_i^N)$ and $\text{ALLOC}(\psi_{i+1}^N)$ are identical up to step $t_i^* - 1$. Then, on step t_i^*, necessarily $\psi_i^N[t_i^*] = 0$, $\psi_{i+1}^N[t_i^*] = 1$.

Lemma 78. $\text{ALLOC}(\psi_i^N, t) \succeq \text{ALLOC}(\psi_{i+1}^N, t)$ *for all* $t = t_i^* \pmod{c+1}$.

The proof of Lemma 78 is by simple induction and is deferred to Appendix B.2. Showing that if $\text{SKIP}(\psi_i^N, \sigma)$ is feasible, then $\text{SKIP}(\psi_{i+1}^N, \sigma)$ is feasible, for steps $t \neq t_i^*$ $\pmod{c+1}$ is more involved. First observe that there exists some t_{max} such that, for all steps $t > t_{max}$, $\text{ALLOC}(\psi_{i+1}^N, t)$ is identical to the allocation of some basic control vector at step t (as the allocations produced by SKIP are memoryless). Hence, it suffices to show the result for all

$t \leq t_{max}$. The value of t_{max} is computed as follows: Let p be number of 0s between t_i^* and the previous 1, and $p' = c+1-p$. In Example 76, for ψ_4^7, $p=1$, $p'=2$. Set $k = t_i^* - (c+2) + p'$. Once the largest share is $\frac{\sigma}{k}$ (and the second largest is strictly smaller), the allocation is the same as it would have been at time k for some basic control vector (in Example 76, this allocation is $(\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{6}, \ldots))$. A straightforward calculation gives $t_{max} = t_i^* + (c+1)(k-1) = (c+2)t_i^* + (c+1)(p' - c - 3)$. Note that this is *exactly* the time at which the allocations converge.

Lemma 79. *For all* $t \neq t_i^* \pmod{c{+}1}$, $t_i^* \leq t \leq t_{max}$, *it holds that* $\textsc{bank}(\psi_{i+1}^N, t) \geq \sum_{j=1}^{c} \frac{\sigma}{t+j}$.

The proof for this Lemma is quite technical, and as it mostly involves applications of known techniques, it is deferred to Appendix B.2.

Bounding the Fairness Ratio of SKIP

The allocations created by Skip have a particular structure: they resemble a segment of the harmonic series, with some doubled entries (see Example 80 and Table B.1). Even though we showed that in order to bound the fairness ratio, it is enough to consider only basic control vectors, each basic control vector has a different fairness ratio. Nevertheless, we would like to provide some upper bound on the fairness ratio of Skip, for each c. We give two types of bounds: an upper bound that applies to all control vectors for any number of agents, and an asymptotic bound. The asymptotic bound is particularly useful for systems that wish to be able to accommodate an unbounded number of agents, and in which the amount of time the system will have fewer than n_0 agents is vanishingly small. In this case, one can set the fairness ratio to be the asymptotic fairness ratio, with an arbitrary "quick fix" heuristic for when there are fewer than n_0 agents in the system (for example, allowing slightly more disruptions to guarantee the asymptotic fairness ratio).

To better characterize these allocations, we need some notation. Elements of an allocation (an element is a real number[6]) that appear once are called *singletons*, and those that appear twice *doubles*. We use the following to make our notation more compact:

$\bowtie_c$ represents the following; the first two shares are a double, continuing the series from the last term. The allocation then alternates between c singletons and a double; the last c are singletons.

$\rightharpoonup_k$ represents k more singletons continuing the series, including the one just before the symbol, $k \geq 0$. $k = 0$ means there are no singletons (we'll use this to simplify the notation later on).

Example 80. *The following are possible allocations, written in full and abbreviated.*

1. $\left(\frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{5}, \frac{\sigma}{6}, \frac{\sigma}{7}, \frac{\sigma}{8}, \frac{\sigma}{8}, \frac{\sigma}{9}, \frac{\sigma}{10}\right)$ *or* $\left(\frac{\sigma}{3} \rightharpoonup_2, \bowtie_2, \frac{\sigma}{8}, \frac{\sigma}{8}, \frac{\sigma}{9} \rightharpoonup_2\right)$.

[6] In all of the algorithms that we describe, shares are rational numbers.

2. $\left(\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{5}, \frac{\sigma}{6}, \frac{\sigma}{7}, \frac{\sigma}{8}\right)$, *or* $\left(\frac{\sigma}{2} \rightharpoonup_3, \frac{\sigma}{5}, \frac{\sigma}{5}, \frac{\sigma}{6} \rightharpoonup_3\right)$.

3. $\left(\frac{\sigma}{4}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{6}, \frac{\sigma}{7}, \frac{\sigma}{8}, \frac{\sigma}{8}, \frac{\sigma}{9}, \frac{\sigma}{10}, \frac{\sigma}{11}, \frac{\sigma}{12}, \frac{\sigma}{12}, \frac{\sigma}{13}, \frac{\sigma}{14}, \frac{\sigma}{15}\right)$ *or* $\left(\frac{\sigma}{3} \rightharpoonup_0, \bowtie_3, \frac{\sigma}{12}, \frac{\sigma}{12}, \frac{\sigma}{13} \rightharpoonup_3\right)$.

Instead of individually analyzing each basic control vector, for any $c > 0$, we define a set of allocations:

$$S^c = S^c_{1,1}, \dots S^c_{1,c+1}, S^c_{2,0}, S^c_{2,1} \dots, S^c_{2,c+1}, S_{3,0}, S^c_{3,1} \dots, S^c_{3,c+1}, \dots$$

that simultaneously upper bounds all allocations created by all basic control vectors, for a fixed c. (This series is not a series of valid allocations; it is only used for the analysis.) At a high level, $S_{t,i}$ is the allocation created by some basic control vector ψ, such that the largest share is $\frac{1}{t}$ (all of the allocations in Example 80 are such allocations; they correspond to $S^2_{3,2}, S^3_{2,3}$ and $S^3_{3,0}$ respectively). For ψ, the allocation at this time is necessarily greater than the previous c allocations (as there was no donor for c rounds). The following observation is a characterization of the allocations just before the round when a donor is used. Note that the control vector is characterized by the variable k, which does not appear in the expression; we only claim that for every k and every round t that is just before the donor, there is some i for which the expression holds.

Definition 81. *For $t = 1$ and $j \in \{1, \dots, c+1\}$,*

$$S^c_{t,j} = \sigma \rightharpoonup_{(j)} .$$

For $t > 1$ and $j \in \{0, 1, \dots, c+1\}$, $t'_{t,j} = (t-1)(c+2) + j - c$,

$$S^c_{t,j} = \frac{\sigma}{t} \rightharpoonup_{(j)}, \bowtie_c, \frac{\sigma}{t'}, \frac{\sigma}{t'}, \frac{\sigma}{t'+1} \rightharpoonup_{(c)} .$$

It is easy to verify that these are exact descriptions of all possible allocations of Skip in the round before a donor is used; we invite the reader to consult Example 80 in which the allocations correspond to $S^2_{3,2}, S^3_{2,3}$ and $S^3_{4,0}$ respectively.

Lemma 82. *The fairness ratio of* Skip *for any $t > 1$, j is at most*

$$\left(\arg\max_{t \in \mathbb{N}^+, j \in \{0,\dots,c+1\}} \sum_{i=t}^{(t-1)(c+2)+j} \frac{1}{i} + \sum_{i=0}^{t-2} \frac{1}{t + i(c+1) + j}\right)^{-1} .$$

Proof. The maximal allocation of S is an upper bound on the maximal allocation of Skip, with a basic control vector. The allocation of $S_{t,j}$ is

$$\sum_{i=t}^{(t-1)(c+2)+j} \frac{1}{i} + \sum_{i=0}^{t-2} \frac{1}{t + i(c+1) + j},$$

by straightforward summation over the allocation vector of Definition 81, where the first is a sum of all the values appearing in the allocation vector and the second part is a sum of the duplicates. □

The following is our bound for basic control vectors, for $c > 3$. For the cases of $c = 1$ and $c = 2$, we get a stronger bounds. We note that this discrepancy between $c = 1, 2$ and $c = 3$ is unavoidable, as the asymptotic bound holds for all time periods for $c = 1, 2$, but not for $c = 3$. The plot for $c = 3$ is slightly misleading, in that it appears that for some control vectors, the asymptotic bound holds. While this is true for all $c < 8$, it ceases to be true thereafter. The plot in Appendix B.5 (taken with Lemma 77) shows that no control vector has fairness ratio at most the asymptotic bound. See Appendix B.3 for the missing proofs and the analysis for $c = 1$ and $c = 2$.

Theorem 83. *For all $c > 2$, the fairness ratio, $\sigma^*\left((c+1)^{-1}\right)$ for all c-control vectors and all steps t satisfies:*

$$\frac{1}{H_{c+1}} \geq \sigma^*\left((c+1)^{-1}\right) \geq \frac{1}{H_{2c+3} - \frac{1}{2}}.$$

Proof. We consider two cases: $t = 1$ and $t > 1$. For the former, the allocation simply is $1 + \frac{1}{2} + \cdots + \frac{1}{j}$. For control vector $(10^c)^\infty$ this is the first $c + 1$ elements of the harmonic progression. For $t > 1$, the total resource allocated is:

$$\sum_{i=t}^{(t-1)(c+2)+j} \frac{1}{i} + \sum_{i=0}^{t-2} \frac{1}{t + i(c+1) + j} \leq \sum_{i=t}^{(t-1)(c+2)+c+1} \frac{1}{i} + \sum_{i=0}^{t-2} \frac{1}{t + i(c+1)}.$$

We bound each term separately. For the term on the left, one can show that the expression is decreasing in t (Lemma 97 in Appendix B.2), and therefore is upper bounded by $\sum_{i=2}^{2c+3} \frac{1}{i} = H_{2c+3} - 1$. The term on the right is upper bounded by $\frac{1}{2}$ (Lemma 98 in Appendix B.2). Combined, we get that the total resource allocated is at most $H_{2c+3} - \frac{1}{2}$. $H_{2c+3} - \frac{1}{2} > H_{c+1}$, for all $c \geq 2$. Furthermore, the bound of the total resource for $t = 1$, is tight. □

We show a tight asymptotic bound on the fairness ratio of Skip as the number of agents tends to infinity.

Theorem 66. *There exists a number n_0 such that, for all $c > 2$, all c-control vectors, and agents $t > n_0$, the allocation of* Skip *is σ_t-fair, where $\sigma_t = (c+1)\left((c+2)\ln(c+2)\right)^{-1}$.*

Proof. We use the following fact about harmonic sums (Appendix B.1): $\sum_{x=a}^{b} \frac{1}{x} \leq \ln\left(\frac{b}{a-1}\right)$.

We bound the total allocation at step t, as it appears in Lemma 82. For the first term:

$$\sum_{i=t}^{(t-1)(c+2)+j} \frac{1}{i} \leq \ln\left(\frac{(t-1)(c+2)+j}{t-1}\right) = \ln\left(c + 2 + \frac{j}{t-1}\right),$$

which approaches $\ln(c+2)$ as $t \to \infty$. For the second term:

$$\sum_{i=0}^{t-2} \frac{1}{t+i(c+1)+j} \leq \sum_{i=0}^{t-2} \frac{1}{t+i(c+1)} = \frac{1}{c+1} \sum_{i=0}^{t-2} \frac{1}{\frac{t}{c+1}+i} = \frac{1}{c+1} \sum_{i=\frac{t}{c+1}}^{\frac{t}{c+1}+t-2} \frac{1}{i}$$
$$\leq \frac{1}{c+1} \ln\left(\frac{\frac{t}{c+1}+t-2}{\frac{t}{c+1}-1}\right) = \frac{1}{c+1} \ln\left(\frac{t+(t-2)(c+1)}{t-c-1}\right),$$

which approaches $\frac{1}{c+1}\ln(c+2)$ as $t \to \infty$. Combining the two gives a bound of $\frac{(c+2)\ln(c+2)}{c+1}$ on $\textsc{talloc}^t$ as t goes to infinity; the theorem follows. □

Accommodating Departures

So far, we have described the input to $CDFD$ as a c-control vector ψ. This was sufficient in the "arrivals-only" model, but when allowing for departures of agents the situation can get more complex; the optimal fairness ratio could depend on a number of parameters, for example who the departing agent is. In this subsection we prove that this is not the case.

The problem now is the following: we are given a c-control vector ψ. Agents arrive and depart arbitrarily. If $\psi[t] = 1$ the algorithm is allowed to use a donor when there are $t-1$ agents in the system and a new one arrives. This could happen multiple times. Call this the *arrivals-departures model*. We prove that the fairness ratio of Skip for a given control vector ψ is the same as when agents are not allowed to depart.

Theorem 84. *The optimal fairness ratio $\sigma^*(\psi)$ of* Skip *with input ψ is the same in the arrivals-departures model as in the arrivals-only model.*

Proof. Let $\textsc{alloc}^t$ and $\textsc{alloc}^{t+1}$ be the (sorted) allocations of Skip for a given vector ψ. It suffices to show that even when an arbitrary agent from $\textsc{alloc}^{t+1}$ departs, it is possible to distribute her share in a way that the sorted allocation is $\textsc{alloc}^t$. If $\psi[t+1] = 0$, this is trivial; we focus on the case that $\psi[t+1] = 1$, i.e., the agent with the highest utility at step t was a donor. The two allocations we consider are $\textsc{alloc}^t = (a_1, a_2, \ldots, a_t)$ and $\textsc{alloc}^{t+1} = \left(a_2, a_3, \ldots, a_t, \frac{\sigma}{t+1}, \frac{\sigma}{t+1}\right)$.

Assume that one of the last two agents, with a $\frac{\sigma}{t+1}$ share, departs. Since both $\textsc{alloc}^t$ and $\textsc{alloc}^{t+1}$ are feasible, $a_1 - 2\frac{\sigma}{t+1}$ is equal to the difference $\textsc{bank}^{t+1} - \textsc{bank}^t$. Therefore, there must be a way to combine the departing agent's share $\frac{\sigma}{t+1}$, with the other share equal to $\frac{\sigma}{t+1}$, and the unallocated amount $\textsc{bank}^{t+1}$, to get a_1. Assume for contradiction this is not the case; then $a_1 > 2\frac{\sigma}{t+1} + \textsc{bank}^{t+1}$, which implies that $a_1 - 2\frac{\sigma}{t+1} > \textsc{bank}^{t+1}$. The LHS is equal to $\textsc{bank}^{t+1} - \textsc{bank}^t$. Combining gives $\textsc{bank}^t < 0$, a contradiction.

If the departing agent is some agent $j \in [2, t-1]$, we can do the following: allocate the difference $a_j - \frac{\sigma}{t+1}$ (which is positive since the allocation is sorted) to one of the last two agents. The amount left to distribute is exactly $\frac{\sigma}{t+1}$; we've already shown this is sufficient to increase the share of the other of the last two agents to a_1. □

5.3 $GDFD$ is NP-hard

In this section we prove Theorem 67.

Theorem 67. *$GDFD$ is NP-hard, even for 2 resources, 1 donor, binary demand vectors, and clairvoyant algorithms.*

We show a reduction from the *Bounded Partition Problem (BPP)*: we are given a set $S = \{a_1, \ldots, a_{2n}\}$ of rational numbers, $a_i \in (\alpha, \beta)$. The goal is to partition S into two subsets of equal size, such that the sums of the numbers in each subset are equal. This problem is NP-hard for any $\beta > \alpha$. Assume w.l.o.g. that $a_1, \ldots, a_{2n}$ are non-decreasing.

In order to make the construction cleaner, and w.l.o.g., allow resources to have capacity more than 1. We reduce BPP to the following instance of $GDFD$ with 2 resources of capacity $\frac{3}{2}\sum_{i=1}^{2n} a_i$ each, 1 donor, and

- $4n$ agents: the first n have a demand vector $[0,1]$, the next n have a demand vector $[1,0]$, and the last $2n$ agents have a demand vector $[1,1]$,
- $U = (1, \ldots, 1, a_{2n})$ ($4n - 1$ "1"'s, followed by a_{2n}), and
- $L = (1, \ldots, 1, a_{2n}, a_{2n-1}, \ldots, a_1)$ ($2n$ "1"'s, followed by $a_{2n}, \ldots, a_1$).

The key observation is that up to step $2n$, the first $2n$ agents must have utility 1, but at step $4n$ they all must have utility less than or equal to a_{2n}, and thus each one of them must be a donor exactly once. Therefore, the utility of an agent i, for $i > 2n$, never changes, is at least a_{4n-i+1}, and it contributes that amount to each of the two resources. Picking a $[1,0]$ or a $[0,1]$ agent as a donor for that step is equivalent to deciding in which partition the number a_{4n-i+1} belongs.

We need to show that there is a satisfying allocation if and only if there is a partition.

Observation 85. *The following must hold:*

1. *The utilities of all agents up to time $2n$ must be exactly 1.*
2. *Since $a_{2n} < 1$, in order to meet the final upper bound $U(4n) = a_{2n}$, each of the first $2n$ agents must be a donor at least once. But since there are only $2n$ time steps, each of the first $2n$ agents must be a donor exactly once.*
3. *None of the last $2n$ agents can be donors.*

Let $x_1, \ldots, x_{2n}$ denote the final utilities of the first $2n$ agents, and let $z_1, \ldots, z_{2n}$ denote the final utilities of the last $2n$ agents (in reverse order). Consider how these utilities evolve, starting from step $2n$. Notice that, by Observation 85, the values of the last $2n$ agents do not change once they are set, and the values of the first $2n$ agents are initially set to 1 (at time $2n$), and then change once.

Therefore, if the lower bounds L are satisfied, $z_i \geq a_i$ for all $i \in [2n]$, and there is some permutation π of $X = \{x_1, \ldots, x_{2n}\}$ such that $\pi(x_i) \geq a_i$ for all $i \in 2n$. Note that each $[1, 1]$ contributes twice its utility to the total resource allocated. Hence, the total allocated resource is:

$$\sum_{i=1}^{2n} x_i + 2z_i = \sum_{i=1}^{2n} \pi(x_i) + 2z_i \geq \sum_{i=1}^{2n} 3a_i.$$

As this is exactly the total available resource, equality must hold everywhere; that is, for all $i \in [2n]$, $z_i = \pi(x_i) = a_i$. For each of the two resources, the amount allocated to the last $2n$ agents is $\sum_{i=1}^{2n} a_i$, hence $\frac{1}{2} \sum a_i$ of each resource is allocated to the first $2n$ agents; hence there is a partition of the numbers $a_i, \ldots a_{2n}$ into equal sized subsets such that the sum of elements of each subset is $\frac{1}{2} \sum a_i$, as required.

To verify that a partition implies a good allocation, we need to verify that

1. U and L are not violated.

2. The capacity of the resources is never exceeded.

The first requirement is immediate from the allocation process described above; as the final allocation satisfies the capacity bound, the following Lemma suffices to prove the second.

Lemma 86. *The allocation of each resource is monotone non-decreasing.*

Proof. There are no donors in the first $2n$ rounds, therefore it suffices to look at the final $2n$ rounds (which we now label $1, \ldots, 2n$). We have verified that in each of these rounds, an agent with demand $[1, 1]$ arrives, and either $[1, 0]$ or $[0, 1]$ is the donor. An agent with demand $[1, 1]$ arrives at round $i \in [2n]$. Assume w.l.o.g. that the donor is $[1, 0]$. The amount allocated of the second resource increases by $a_i > 0$, and of the first decreases by $1 - a_i$, and increases by a_i, for a total increase of $2a_i - 1 > 0$, as $a_i > 0.5$. □

5.4 Lower and Upper Bounds for Multiple Resources

We show lower and upper bounds for the optimal fairness ratio of *DDRF*. In Section 5.5 we show a tighter upper bound for the case $d = 1$ and r resources. We describe algorithms for the following cases:

1. the number of donors allowed is a multiple of the number of resources ($d = kr$), for some integer $k \geq 1$,

2. $d = 1$, (r is any integer),

3. one disruption is allowed for per $c + 1$ arrivals, for $c \geq 1$.

Our algorithms use solutions to the single resource algorithms as subroutines: For each resource l, we run a copy of a single resource algorithm. Let $\mathcal{SR}$ be an algorithm for the single resource case (we expand shortly about the exact nature of $\mathcal{SR}$). When the t-th agent arrives with demand $\mathcal{D}_t = (\mathcal{D}_{t,1}, \mathcal{D}_{t,2}, \ldots, \mathcal{D}_{t,r})$, the l-th copy of $\mathcal{SR}$ is given as an input the number $D_{t,l}$: a 0 or a 1. If the number is 1, $\mathcal{SR}$ behaves as if an agent arrived. If the number is 0, $\mathcal{SR}$ ignores it. The total amount of resource l allocated to agent i is dictated by the l-th copy. The following lemma enables us to combine the single resource algorithms:

Lemma 87. *Let $\mathcal{MR}$ be a multiple resource algorithm that executes a single resource algorithm $\mathcal{SR}$ with fairness ratio σ^* for each resource. Then, $\mathcal{MR}$ is σ^*-DRF fair.*

Proof. The DRF allocation at step t is: $DRF_{\mathcal{D}^t} = \left(\max_{l=1,\ldots,r} \left\{\sum_{j=1}^{t} \mathcal{D}_{j,l}^{t}\right\}\right)^{-1}$, where $\mathcal{D}_{j,l}$ is the demand for the l-th resource by agent j.

Notice that the amount of resource l agent i has at step t is at least $\sigma^* \frac{\mathcal{D}_{i,l}}{\sum_{j=1}^{t} \mathcal{D}_{j,l}} \geq \sigma^* \cdot \mathcal{D}_{i,l} \cdot DRF_t$, since the l-th copy of the single resource algorithm has received as input the numbers $\{\mathcal{D}_{1,l}, \ldots, \mathcal{D}_{t,l}\}$ by step t, and their sum is exactly the number of agents that demand resource l. The utility of agent i for a vector of resources $v = (v_1, \ldots, v_r)$ is $\min_{l=1\ldots r} \left\{\frac{v_l}{\mathcal{D}_{i,l}}\right\}$. Therefore, for the vector of resources

$$\left(\sigma_k^* \cdot \mathcal{D}_{i,1} \cdot DRF_t, \ldots, \sigma_k^* \cdot \mathcal{D}_{i,r} \cdot DRF_t\right) = \sigma^* DRF_t \left(\mathcal{D}_{i,1}, \ldots, \mathcal{D}_{i,r}\right).$$

The fairness ratio is therefore $\sigma^* \cdot DRF_t$. □

Theorem 68. *The optimal fairness ratio of DDRF for $d = kr$ donors and r resources is bounded by*

$$\left((kr+1)\ln\left(\frac{kr+1}{kr}\right)\right)^{-1} \geq \sigma^*(kr, r) \geq \left((k+1)\ln\left(\frac{k+1}{k}\right)\right)^{-1}$$

Proof. For the upper bound, notice that $\sigma^*(kr, r) \leq \sigma^*(kr, 1) \leq \left((kr+1)\ln\left(\frac{kr+1}{kr}\right)\right)^{-1}$. For the lower bound, our algorithm for $d = kr$ runs r copies of the optimal single resource algorithm, one for each resource, where every copy is allowed to use at most k donors per arrival. The total number of donors is $d = kr$, and by Lemma 87 the algorithm guarantees a σ_k^* fraction of the DRF utility for each agent. □

Theorem 69. *The optimal fairness ratio of DDRF for $d = 1$ donors and $r \geq 2$ resources is bounded by*

- $\sigma^*(1, 2) \geq 2(3\ln 3)^{-1} \approx 0.6068$
- $\sigma^*(1, 3) \geq 3(4\ln 4)^{-1}$
- $\sigma^*(1, r) \geq \left(H_{2r+1} - \frac{1}{2}\right)^{-1}$, $r > 3$.

Proof. Our algorithm for $d = 1$ will run r copies of the SKIP algorithm, one for each resource, where every copy can use a at least one donor for every r arrivals.

The algorithm maintains a priority list O_t over the copies of SKIP. Let $O_1 = (1, 2, \dots, r)$: at step $t = 1$, the first copy has the highest priority, the second copy has the second highest, and so on. At the arrival of the t-th agent with demand $D_t = (D_{t,1}, D_{t,2}, \dots, D_{t,r})$, the l-th copy of the single resource algorithm will be given as an input the number $D_{t,l}$ if that number is non-zero (otherwise it receives no input). All copies that received a non-zero input request a donor. Copies with no input stay idle. If there are conflicts, (two or more copies request *different* donors), then the copy with the highest priority in O_t, among those that requested a donor, is the only one allowed to use a donor, and is moved to have the lowest priority in O_{t+1}. The total amount of resource l allocated to agent i is dictated by the l-th copy of SKIP.

Clearly, only one donor is used per step. Therefore we only need to show that the algorithm guarantees a $\sigma^*(r^{-1})$ fraction of the *DRF* utility for each agent (recall that here $\sigma^*(r^{-1})$ is the fairness ratio of SKIP when one donor every r arrivals is allowed). Notice that for each step that a copy requests a donor but is not allowed to use one, it moves up one spot in the priority list. A copy cannot be denied a donor more than $r - 1$ times consecutively. This implies that the l-th copy, if seen independently, behaves identically to SKIP, and therefore, by Lemma 87 the algorithm is $\sigma^*(r^{-1})$-*DRF* fair. □

Using a similar approach we can show positive results for $d = (c + 1)^{-1}$. The main observation here is that each single resource subroutine cannot be denied a donor for more than $r(c+1)-1$ steps. The upper bound on the fairness ratio is implied by the upper bound on the single resource case.

Theorem 70. *The optimal fairness ratio of DDRF for r resources and $(c+1)^{-1}$ donors (1 donor allowed per $c + 1$ arrivals) is bounded by*

$$(H_{c+1})^{-1} \geq \sigma^*\left((c+1)^{-1}, r\right) \geq \left(H_{2r(c+1)+1} - \frac{1}{2}\right)^{-1}.$$

5.5 Bounds for $\sigma^*(1, r)$ via Duality

We prove the following theorem.

Theorem 71. *The optimal fairness ratio of DDRF for $d = 1$ and $r \geq 2$ resources is bounded by*

$$\sigma^*(1, r) \leq 0.6318\dots$$

The high level of our approach to the proof of Theorem 71 is the following.

1. Write the optimal mixed integer program for a general number of donors d and fix an input $\mathcal{D}$ (to be described later).

2. Notice that the only integer variables are variables $Z_{t,i} \in \{0,1\}$, encoding whether the i-th agent is a donor at step t. If we fix these variables, the remaining program is an LP. Treat these variables as constants Z and consider the dual of this LP.

3. Show an algorithm that, given Z, outputs a feasible solution for the dual with a value of at most $\sigma^* \approx 0.6318$.

The main difficulty is showing that no matter what the adversary picks Z to be, a feasible dual solution can always be constructed, such that we get the desired bound. The optimal integer program for general d is:

$$\begin{aligned}
&\max \sigma \\
\text{subject to: } \forall i \leq t, t \in [N]: \quad & u_i^t \geq \frac{\sigma}{DRF_t} \\
\forall r \in [R], t \in [N]: \quad & \sum_{i=1}^{t} u_{ir}^t D_{i,r} \leq 1, \qquad \forall t \in [N]: \quad \sum_{i=1}^{t} Z_{t,i} \leq d \\
\forall i, t \in [N]: \quad & u_i^t - u_i^{t-1} \leq 1 - Z_{t,i}, \qquad u_i^t - u_i^{t-1} \geq -z_i^t \\
& Z_{t,i} \in \{0,1\}
\end{aligned}$$

Fixing our integer variables $Z_{t,i}$ and taking the dual for the resulting linear program gives:

$$\begin{aligned}
\min \; & \sum_{t,r} f(t,r) + \sum_{t,i} x(t,i)\,(1 - Z_{t,i}) + \sum_{t,i} y(t,i) Z_{t,i} \\
\text{subject to} \; & \\
& \sum_{t,i} l(t,i) \geq 1 \\
\forall t \in [N-1], i \in [t]: \quad & \sum_{r} D_{i,r} f(t,r) + x(t,i) - x(t+1,i) + y(t+1,i) - y(t,i) \geq \frac{l(t,i)}{DRF_t} \\
\forall i \in [N]: \quad & \sum_{r} D_{i,r} f(N,r) + x(N,i) - y(N,i) \geq \frac{l(N,i)}{DRF_N}
\end{aligned}$$

A Feasible Dual Solution

Let $\hat{Z}$ be an $4n$ by $4n$ lower triangular matrix such that $\hat{Z}_{t,i} = DRF_t$ if and only if $Z_{t,i} = 1$, and $Z_{t',i} = 0$ for all $t' > t$. In other words, $\hat{Z}_{t,i} = DRF_t$ if t is the last time agent i becomes a donor. If agent i is never a donor, then we let $\hat{Z}_{i,i} = DRF_i$.

Let s_1 be the sum of all elements in the first n columns of $\hat{Z}$, s_2 the sum of all elements in the next n columns of $\hat{Z}$, and s_3 the sum of all elements in the last $2n$ columns. We say that a column i is *valid* if $i > 2n$, or $i \in [1, n]$ and $s_1 > s_2$, or $i \in [n+1, 2n]$ and $s_2 > s_1$. Finally, let $\sigma = \frac{1}{\max\{s_1,s_2\}+s_3}$.

Our dual solution for a given choice of Z is as follows:

- $x(t,i) = 0$, $\forall t, i$, and $f(t,r) = 0$, $\forall t \leq N-1, r \in \{1,2\}$.
- $f(N,1) = \sigma$ if $s_1 \geq s_2$, and zero otherwise. $f(N,2) = \sigma$ if $s_1 > s_2$, and zero otherwise.
- $y(t,i) = \sigma$, for all $t > t'$, where t' is such that $\hat{Z}_{t',i} \neq 0$, and i is *valid*. $y(t,i) = 0$ everywhere else. In other words, $y(t,i)$ is σ from the time after i was a donor for the last time, but only for i that is *valid*. Notice that this makes $y(t,i) \cdot Z_{t,i} = 0$ everywhere.
- $l(t,i) = \sigma \cdot DRF_t$ if and only if $\hat{Z}_{t,i} \neq 0$, and i is *valid*.

See Appendix B.4 for an example.

Lemma 88. *The solution described above is feasible for all Z and n, and the value of the objective is σ.*

Proof.

- **Objective:** Notice that $x(t,i) = y(t,i)Z_{t,i} = 0$, for all t, i, therefore the objective is simply $\sum_r f(N,r)$. At most one of $f(N,r)$ can be non-zero, with a value of σ.
- **First line of constraints:**

$$\sum_{t,i} l(t,i) = \sigma \left(\sum_{t,i \ valid} \hat{Z}_{t,i} \right) = \sigma \left(\max\{s_1, s_2\} + s_3 \right) = 1.$$

- **Second line of constraints:**

  These constraints reduce to $y(t+1,i) - y(t,i) \geq \frac{1}{DRF_t} l(t,i)$. $l(t,i)$ is non zero only when $\hat{Z}_{t,i}$ is non-zero and i is valid, in which case, $y(t,i) = 0$, but $y(t+1,i) = \sigma$, therefore the constraint is satisfied. When $l(t,i)$ is zero, either both $y(t,i)$, $y(t+1,i)$ are zero, or both are equal to σ, therefore the constraint is again satisfied.

- **Third line of constraints:**

  These constraints reduce to $\sum_r f(N,r) - y(N,i) = \sigma - y(N,i) \geq \frac{1}{DRF_N} l(N,i)$. $l(N,i)$ is non-zero only when $\hat{Z}_{t,i}$ is non-zero and i is valid, in which case $y(N,i) = 0$, and therefore the constraint is satisfied. If $l(N,i) = 0$, then notice that the RHS is zero, and $y(N,i) \leq \sigma$, therefore the constraint is again satisfied.

□

Bounding σ

We now show that no matter how Z is chosen, σ is at most $0.6318\ldots$ Recall that $\sigma = \frac{1}{\max\{s_1,s_2\}+s_3}$. Therefore, in order to maximize σ, an adversary that picks Z needs to minimize $\max\{s_1, s_2\} + s_3$. Every choice of Z yields $\hat{Z}$ in a unique way, with all non zero $\hat{Z}_{t,i}$s taking certain values that depend on DRF.

Define $\mathcal{D}$ as follows: n $[0, 1]$'s followed by n $[1, 0]$'s, followed by a series of $2n$ $[1, 1]$'s This input has: $DRF_t = \frac{1}{t}$ for $t = 1, \ldots, n$, $DRF_t = \frac{1}{n}$ for $t = n + 1, \ldots, 2n$, and $DRF_t = \frac{1}{t-n}$ for $t = 2n + 1, \ldots, 4n$. Instead of bounding the minimum $(\max\{s_1, s_2\} + s_3)$, we bound the minimum $\left(\frac{s_1+s_2}{2} + s_3\right)$. Notice that this value is only smaller, therefore our bound for σ is worse. Minimizing $\left(\frac{s_1+s_2}{2} + s_3\right)$ is equivalent to the following game:

We have an ordered set of $4n$ numbers A, where $A_t = DRF_t$ as described above. We also have a set of $4n - 1$ numbers $B = A \setminus \{1\}$. Replace numbers in A with numbers in B, using each number in B at most once, to obtain $\hat{A}$. s_1 is the sum of the first n elements of $\hat{A}$, s_2 the sum of the next n elements, and s_3 the sum of the last $2n$ elements. Associate each number $A_i \in A$ with a weight w_i, which is $\frac{1}{2}$ if $i \leq 2n$, and 1 otherwise. Our objective is to minimize the weighted sum $\sum_{i=1}^{4n} w_i \hat{A}_i$. We show a strategy for this game that has value 1.58258. This gives $\sigma \approx 0.6318$.

Example 89. *For $n = 2$, we have*

$$B = \{1/2, 1/2, 1/2, 1/3, 1/4, 1/5, 1/6\}.$$

A possible way to change A into $\hat{A}$ is the following (numbers that were not replaced are in bold):

w	1/2	1/2	1/2	1/2	1	1	1	1
A	1	1/2	1/2	1/2	1/3	1/4	1/5	1/6
$\hat{A}$	1/5	1/4	1/6	1/3	**1/3**	**1/4**	**1/5**	**1/6**

The sum of the first $n = 2$ elements of $\hat{A}$ is $\frac{1}{5} + \frac{1}{4} = 0.45$, the sum of the next n elements is $\frac{1}{3} + \frac{1}{6} = 0.5$, and the sum of the last $2n$ elements is $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 0.95$. The tight bound is $\max\{0.45, 0.5\} + 0.95 = 1.45$. The value of the game, i.e., the weighted sum of $\hat{A}$, $\frac{s_1+s_2}{2} + s_3$, is 1.425.

Observation 90. *In the optimal solution, $\hat{A}_i \leq \frac{1}{n}$, for all i.*

Notice that if all the weights are the same, the problem is trivial: greedily replace the largest number in A with the smallest number in B, until the largest number in A is smaller than the smallest available number in B.

In order to minimize the weighted sum of A by replacing numbers in A with numbers in B, observe the following: for each pair of numbers $A_i, A_j \in A$, with $w_i < w_j$, we can find a number $x_{i,j}$, such that replacing A_i with a number $x \geq x_{i,j}$ gives a smaller or equal weighted sum than replacing A_j with $x_{i,j}$, and $x_{i,j}$ is the smallest such number. Therefore, $x_{i,j}$ is the solution to $A_i w_i + w_j x \geq w_j A_j + w_i x$, which is $x_{i,j} = \frac{w_j A_j - w_i A_i}{w_j - w_i}$. If $w_i = w_j$ then it is better to replace the smaller of the two.

The following Lemma is immediate:

Lemma 91. *In the optimal solution: (1) all $i \geq 3n$ are never replaced, and (2) all $i \leq \frac{n}{2}$ are always replaced.*

Proof. We show that if for $i \geq 3n$, i.e., $A_i \leq \frac{1}{2n}$ and $w_i = 1$, then $x_{j,i} \leq 0$, for all $j \leq 2n$ ($w_j = 1/2$). This implies that for all positive numbers x, replacing A_j with x is better than replacing A_i with x. It suffices to consider the maximum reasonable value that A_j can take, which is $\frac{1}{n}$ (Observation 90). By replacing $w_i = 1$, $A_j = \frac{1}{n}$ and $w_j = \frac{1}{2}$ we have: $x_{j,i} = \frac{w_i A_i - w_j A_j}{w_i - w_j} = 2A_i - \frac{1}{n}$, which less than zero, exactly when $A_i \leq \frac{1}{2n}$. The proof of the second part of the lemma is similar and is omitted. □

Even though, given a number $x \in B$, we can find the optimal number to replace, it is not generally true that the optimal algorithm considers all B_i in decreasing order. Regardless, we can show the following structure for the optimal strategy:

Lemma 92. *In the optimal solution, if some $i > 2n$ is replaced by $b \in B$, then for all $b' \in B$ that replaced $j \in [0, 2n]$, we have $b \leq b'$.*

Proof. Assume this is not the case. This means that there is some $j \in [0, 2n]$ such that A_j was replaced by some $b' < b$. Let $S + \frac{b'}{2} + b$ be the value of the game in this strategy. Consider instead the following strategy: replace A_j with b and A_i with b'. The value now is $S + \frac{b}{2} + b' = S + \frac{b}{2} + \frac{b'}{2} + \frac{b'}{2} < S + b + \frac{b'}{2}$, which is the value of the optimal strategy; a contradiction. □

An immediate corollary is that the smallest numbers in B are used to replace the largest numbers A_i, for $i > 2n$. Combining with Lemma 91, we get that in the optimal solution, a fraction f^* of the n smallest numbers in B is used to replace all $i \in [2n+1, (2+f^*)n]$, while the remaining numbers in B are used to replace $i \leq 2n$. There are $2n$ numbers in B that are smaller than $\frac{1}{n+1}$, therefore for $i \leq 2n$, a $1 - \frac{f^*}{2}$ fraction of them are replaced, while the $f^* n$ numbers remaining are equal to $\frac{1}{n}$. The value of the optimal strategy is:

$$
\begin{aligned}
V^* = \sum_{i=1}^{4n} w_i \hat{A}_i &= \frac{1}{2}\sum_{i=1}^{2n} \hat{A}_i + \sum_{i=2n+1}^{4n} \hat{A}_i \\
&= \frac{1}{2}\left(\sum_{i=f^*n+1}^{n} B_i + f^* n \frac{1}{n}\right) + \sum_{i=1}^{f^*n} B_i + \sum_{i=(2+f^*)n+1}^{4n} \hat{A}_i \\
&= \frac{1}{2}\left(\sum_{i=n+1}^{(3-f^*)n} \frac{1}{i} + f^*\right) + \sum_{i=(3-f^*)n}^{3n} \frac{1}{i} + \sum_{i=(1+f^*)n+1}^{3n} \frac{1}{i}.
\end{aligned}
$$

At the limit this converges to:

$$
\frac{f^*}{2} + \frac{1}{2}\ln\left(3 - f^*\right) + \ln\left(\frac{3}{3 - f^*}\right) + \ln\left(\frac{3}{1 + f^*}\right).
$$

This function is convex for $f \in [0, 1]$, with a minimum at $\frac{5-\sqrt{17}}{2} \approx 0.4384$, with a value of 1.5825. We conclude that σ is at most $\frac{1}{1.58258} \approx 0.6318$.

Bibliography

[1] Daron Acemoglu. “Training and innovation in an imperfect labour market”. In: *The Review of Economic Studies* 64.3 (1997), pp. 445–464.

[2] Anat R Admati and Motty Perry. “Joint projects without commitment”. In: *The Review of Economic Studies* 58.2 (1991), pp. 259–276.

[3] Martin Aleksandrov et al. “Online fair division: analysing a food bank problem”. In: *Proceedings of the 24th International Conference on Artificial Intelligence*. AAAI Press. 2015, pp. 2540–2546.

[4] Noga Alon. “Splitting necklaces”. In: *Advances in Mathematics* 63.3 (1987), pp. 247–253.

[5] Itai Ashlagi, Constantinos Daskalakis, and Nima Haghpanah. “Sequential mechanisms with ex-post participation guarantees”. In: *Proceedings of the 2016 ACM Conference on Economics and Computation*. ACM. 2016, pp. 213–214.

[6] Robert J Aumann and Sergiu Hart. “Long cheap talk”. In: *Econometrica* 71.6 (2003), pp. 1619–1660.

[7] Pablo Azar et al. “Optimal and efficient parametric auctions”. In: *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2013, pp. 596–604.

[8] Haris Aziz and Simon Mackenzie. “A discrete and bounded envy-free cake cutting protocol for any number of agents”. In: *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*. IEEE. 2016, pp. 416–427.

[9] Haris Aziz and Simon Mackenzie. “A discrete and bounded envy-free cake cutting protocol for four agents”. In: *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*. ACM. 2016, pp. 454–464.

[10] Moshe Babaioff et al. “A simple and approximately optimal mechanism for an additive buyer”. In: *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*. IEEE. 2014, pp. 21–30.

[11] Richard E Barlow and Frank Proschan. “Inequalities for linear combinations of order statistics from restricted families”. In: *The Annals of Mathematical Statistics* 37.6 (1966), pp. 1574–1592.

[12] Richard E Barlow and Frank Proschan. *Mathematical theory of reliability*. SIAM, 1996.

[13] Dirk Bergemann and Maher Said. "Dynamic auctions". In: *Wiley Encyclopedia of Operations Research and Management Science* (2011).

[14] Arka A. Bhattacharya et al. "Hierarchical Scheduling for Diverse Datacenter Workloads". In: *Proceedings of the 4th Annual Symposium on Cloud Computing*. SOCC '13. Santa Clara, California: ACM, 2013, 4:1–4:15. ISBN: 978-1-4503-2428-1. DOI: 10.1145/2523616.2523637. URL: http://doi.acm.org/10.1145/2523616.2523637.

[15] *Big Red Bits*. http://www.bigredbits.com/archives/539. 2011.

[16] Steven J Brams and Alan D Taylor. "An envy-free cake division protocol". In: *American Mathematical Monthly* (1995), pp. 9–18.

[17] Eric Budish. "The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes". In: *Journal of Political Economy* 119.6 (2011), pp. 1061–1103.

[18] Jeremy Bulow and Paul Klemperer. "Auctions Versus Negotiations". In: *The American Economic Review* 86.1 (1996), pp. 180–194.

[19] Yang Cai and Constantinos Daskalakis. "Extreme-value theorems for optimal multidimensional pricing". In: *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*. IEEE. 2011, pp. 522–531.

[20] Yang Cai, Nikhil R Devanur, and S Matthew Weinberg. "A duality based unified approach to Bayesian mechanism design". In: *Proceedings of the 48th Annual ACM Symposium on Theory of Computing (STOC)*. 2016.

[21] Yang Cai and Mingfei Zhao. "Simple Mechanisms for Subadditive Buyers via Duality". In: *arXiv preprint arXiv:1611.06910* (2016).

[22] Ioannis Caragiannis et al. "The unreasonable fairness of maximum Nash welfare". In: *Proceedings of the 2016 ACM Conference on Economics and Computation*. ACM. 2016, pp. 305–322.

[23] Gabriel Carroll. "Robustness and separation in multidimensional screening". In: *Econometrica* 85.2 (2017), pp. 453–488.

[24] Ruggiero Cavallo. "Efficiency and redistribution in dynamic mechanism design". In: *Proceedings of the 9th ACM conference on Electronic commerce*. ACM. 2008, pp. 220–229.

[25] Ruggiero Cavallo, David C Parkes, and Satinder Singh. "Optimal coordinated planning amongst self-interested agents with private state". In: *arXiv preprint arXiv:1206.6820* (2012).

[26] Shuchi Chawla, Jason D Hartline, and Robert Kleinberg. "Algorithmic pricing via virtual valuations". In: *Proceedings of the 8th ACM conference on Electronic commerce*. ACM. 2007, pp. 243–251.

[27] Shuchi Chawla, David Malec, and Balasubramanian Sivan. “The power of randomness in Bayesian optimal mechanism design”. In: *Games and Economic Behavior* 91 (2015), pp. 297–317.

[28] Shuchi Chawla et al. “Multi-parameter mechanism design and sequential posted pricing”. In: *Proceedings of the forty-second ACM symposium on Theory of computing*. ACM. 2010, pp. 311–320.

[29] Richard Cole and Vasilis Gkatzelis. “Approximating the Nash social welfare with indivisible items”. In: *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*. ACM. 2015, pp. 371–380.

[30] Richard Cole and Tim Roughgarden. “The sample complexity of revenue maximization”. In: *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*. ACM. 2014, pp. 243–252.

[31] Vincent Conitzer and Tuomas Sandholm. “Computational criticisms of the revelation principle”. In: *Proceedings 5th ACM Conference on Electronic Commerce (EC-2004), New York, NY, USA, May 17-20, 2004*. 2004, pp. 262–263. DOI: `10.1145/988772.988824`. URL: `http://doi.acm.org/10.1145/988772.988824`.

[32] Pascal Courty and Li Hao. “Sequential screening”. In: *The Review of Economic Studies* 67.4 (2000), pp. 697–717.

[33] Constantinos Daskalakis, Alan Deckelbaum, and Christos Tzamos. “Mechanism design via optimal transport”. In: *Proceedings of the fourteenth ACM conference on Electronic commerce*. ACM. 2013, pp. 269–286.

[34] Constantinos Daskalakis, Alan Deckelbaum, and Christos Tzamos. “Strong duality for a multiple-good monopolist”. In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. ACM. 2015, pp. 449–450.

[35] Nikhil R Devanur, Zhiyi Huang, and Christos-Alexandros Psomas. “The sample complexity of auctions with side information”. In: *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*. ACM. 2016, pp. 426–439.

[36] Nikhil R Devanur et al. “Market equilibrium via a primal–dual algorithm for a convex program”. In: *Journal of the ACM (JACM)* 55.5 (2008), p. 22.

[37] Nikhil Devanur et al. “Prior-independent multi-parameter mechanism design”. In: *International Workshop on Internet and Network Economics*. Springer. 2011, pp. 122–133.

[38] Peerapong Dhangwatnotai, Tim Roughgarden, and Qiqi Yan. “Revenue maximization with a single sample”. In: *Games and Economic Behavior* 91 (2015), pp. 318–333.

[39] Shahar Dobzinski, Noam Nisan, and Sigal Oren. “Economic efficiency requires interaction”. In: *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*. ACM. 2014, pp. 233–242.

[40] Danny Dolev et al. "No Justified Complaints: On Fair Sharing of Multiple Resources". In: *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*. ITCS '12. Cambridge, Massachusetts: ACM, 2012, pp. 68–75. ISBN: 978-1-4503-1115-1. DOI: 10.1145/2090236.2090243. URL: http://doi.acm.org/10.1145/2090236.2090243.

[41] Lester E Dubins and Edwin H Spanier. "How to cut a cake fairly". In: *American mathematical monthly* (1961), pp. 1–17.

[42] Alon Eden et al. "A simple and approximately optimal mechanism for a buyer with complements". In: *arXiv preprint arXiv:1612.04746* (2016).

[43] Alon Eden et al. "The Competition Complexity of Auctions: A Bulow-Klemperer Result for Multi-Dimensional Bidders". In: *arXiv preprint arXiv:1612.08821* (2016).

[44] Leah Epstein and Asaf Levin. "Robust algorithms for preemptive scheduling". In: *European Symposium on Algorithms*. Springer. 2011, pp. 567–578.

[45] Michal Feldman, Ophir Friedler, and Aviad Rubinstein. *99% Revenue via Enhanced Competition*. Tech. rep. Working paper, 2017.

[46] Françoise Forges. "Equilibria with communication in a job market example". In: *The Quarterly Journal of Economics* 105.2 (1990), pp. 375–398.

[47] Eric J. Friedman, Christos-Alexandros Psomas, and Shai Vardi. "Dynamic Fair Division with Minimal Disruptions". In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation, EC '15, Portland, OR, USA, June 15-19, 2015*. 2015, pp. 697–713. DOI: 10.1145/2764468.2764495. URL: http://doi.acm.org/10.1145/2764468.2764495.

[48] Eric Friedman, Ali Ghodsi, and Christos-Alexandros Psomas. "Strategyproof allocation of discrete jobs on multiple machines". In: *Proceedings of the fifteenth ACM conference on Economics and computation*. ACM. 2014, pp. 529–546.

[49] Eric Friedman, Christos-Alexandros Psomas, and Shai Vardi. *Controlled Dynamic Fair Division*. Tech. rep. Working paper, 2017.

[50] Alex Gershkov and Benny Moldovanu. "Dynamic revenue maximization with heterogeneous objects: A mechanism design approach". In: *American economic Journal: microeconomics* 1.2 (2009), pp. 168–198.

[51] Alex Gershkov and Benny Moldovanu. "Efficient sequential assignment with incomplete information". In: *Games and Economic Behavior* 68.1 (2010), pp. 144–154.

[52] Ali Ghodsi et al. "Choosy: Max-min Fair Sharing for Datacenter Jobs with Constraints". In: *Proceedings of the 8th ACM European Conference on Computer Systems*. EuroSys '13. Prague, Czech Republic: ACM, 2013, pp. 365–378. ISBN: 978-1-4503-1994-2. DOI: 10.1145/2465351.2465387. URL: http://doi.acm.org/10.1145/2465351.2465387.

[53] Ali Ghodsi et al. “Dominant Resource Fairness: Fair Allocation of Multiple Resource Types”. In: *Proceedings of the 8th USENIX Conference on Networked Systems Design and Implementation*. NSDI’11. Boston, MA: USENIX Association, 2011, pp. 24–24. URL: `http://dl.acm.org/citation.cfm?id=1972457.1972490`.

[54] Ali Ghodsi et al. “Multi-resource fair queueing for packet processing”. In: *ACM SIGCOMM Computer Communication Review* 42.4 (2012), pp. 1–12.

[55] Yiannis Giannakopoulos and Elias Koutsoupias. “Duality and optimality of auctions for uniform distributions”. In: *Proceedings of the fifteenth ACM conference on Economics and computation*. ACM. 2014, pp. 259–276.

[56] Yiannis Giannakopoulos and Elias Koutsoupias. “Selling two goods optimally”. In: *International Colloquium on Automata, Languages, and Programming*. Springer. 2015, pp. 650–662.

[57] Kira Goldner and Anna R Karlin. “A Prior-Independent Revenue-Maximizing Auction for Multiple Additive Bidders”. In: *International Conference on Web and Internet Economics*. Springer. 2016, pp. 160–173.

[58] Albert Gu, Anupam Gupta, and Amit Kumar. “The power of deferral: maintaining a constant-competitive steiner tree online”. In: *SIAM Journal on Computing* 45.1 (2016), pp. 1–28.

[59] Mingyu Guo, Vincent Conitzer, and Daniel M Reeves. “Competitive repeated allocation without payments”. In: *International Workshop on Internet and Network Economics*. Springer. 2009, pp. 244–255.

[60] Anupam Gupta and Amit Kumar. “Online steiner tree with deletions”. In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2014, pp. 455–467.

[61] Anupam Gupta, Amit Kumar, and Cliff Stein. “Maintaining assignments online: Matching, scheduling, and flows”. In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2014, pp. 468–479.

[62] Avital Gutman and Noam Nisan. “Fair allocation without trade”. In: *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*. International Foundation for Autonomous Agents and Multiagent Systems. 2012, pp. 719–728.

[63] Nima Haghpanah and Jason Hartline. “Reverse mechanism design”. In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. ACM. 2015, pp. 757–758.

[64] Sergiu Hart and Noam Nisan. *Approximate Revenue Maximization with Multiple Items*. Tech. rep. The Federmann Center for the Study of Rationality, the Hebrew University, Jerusalem, 2012.

[65] Sergiu Hart and Noam Nisan. "The menu-size complexity of auctions". In: *ACM Conference on Electronic Commerce, EC '13, Philadelphia, PA, USA, June 16-20, 2013*. 2013, pp. 565–566. DOI: 10.1145/2482540.2482544. URL: http://doi.acm.org/10.1145/2482540.2482544.

[66] Jason D Hartline. *Mechanism design and approximation*.

[67] Jason D Hartline and Tim Roughgarden. "Optimal mechanism design and money burning". In: *Proceedings of the fortieth annual ACM symposium on Theory of computing*. ACM. 2008, pp. 75–84.

[68] Jason D Hartline and Tim Roughgarden. "Simple versus optimal mechanisms". In: *Proceedings of the 10th ACM conference on Electronic commerce*. ACM. 2009, pp. 225–234.

[69] Benjamin Hindman et al. "Mesos: A Platform for Fine-Grained Resource Sharing in the Data Center." In: *NSDI*. Vol. 11. 2011, pp. 22–22.

[70] Michael Isard et al. "Quincy: fair scheduling for distributed computing clusters". In: *Proceedings of the ACM SIGOPS 22nd symposium on Operating systems principles*. ACM. 2009, pp. 261–276.

[71] Carlee Joe-Wong et al. "Multiresource allocation: Fairness-efficiency tradeoffs in a unifying framework". In: *IEEE/ACM Transactions on Networking (TON)* 21.6 (2013), pp. 1785–1798.

[72] Sham M Kakade, Ilan Lobel, and Hamid Nazerzadeh. "Optimal dynamic mechanism design and the virtual-pivot mechanism". In: *Operations Research* 61.4 (2013), pp. 837–854.

[73] Ehud Kalai and Meir Smorodinsky. "Other solutions to Nash's bargaining problem". In: *Econometrica: Journal of the Econometric Society* (1975), pp. 513–518.

[74] Ian Kash, Ariel D Procaccia, and Nisarg Shah. "No agent left behind: Dynamic fair division of multiple resources". In: *Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems*. International Foundation for Autonomous Agents and Multiagent Systems. 2013, pp. 351–358.

[75] Daniel Krähmer and Roland Strausz. "Optimal sales contracts with withdrawal rights". In: *The Review of Economic Studies* 82.2 (2015), pp. 762–790.

[76] Vijay Krishna and John Morgan. "Cheap Talk". In: *The New Palgrave Dictionary of Economics*. Palgrave, 2008.

[77] Vijay Krishna and John Morgan. "The Art of Conversation: Eliciting Information from Experts through Multi-Stage Communication". In: *Journal of Economic Theory* 117.2 (2004), pp. 147–179.

[78] David Kurokawa, Ariel D Procaccia, and Nisarg Shah. "Leximin allocations in the real world". In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. ACM. 2015, pp. 345–362.

[79] Xinye Li and Andrew Chi-Chih Yao. "On revenue maximization for selling multiple independently distributed items". In: *Proceedings of the National Academy of Sciences* 110.28 (2013), pp. 11232–11237.

[80] Siqi Liu and Christos-Alexandros Psomas. *On the Competition Complexity of Dynamic Mechanism Design*. Tech. rep. Working paper, 2017.

[81] Dejan S Milojičić et al. "Process migration". In: *ACM Computing Surveys (CSUR)* 32.3 (2000), pp. 241–299.

[82] Vahab Mirrokni et al. "Dynamic auctions with bank accounts". In: *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*. 2016.

[83] Vahab Mirrokni et al. *Oblivious Dynamic Mechanism Design*. Tech. rep. Working paper, 2016.

[84] Vahab Mirrokni et al. "Optimal dynamic mechanisms with ex-post IR via bank accounts". In: *arXiv preprint arXiv:1605.08840* (2016).

[85] Jamie H Morgenstern and Tim Roughgarden. "On the pseudo-dimension of nearly optimal auctions". In: *Advances in Neural Information Processing Systems*. 2015, pp. 136–144.

[86] Roger B Myerson. "Optimal auction design". In: *Mathematics of operations research* 6.1 (1981), pp. 58–73.

[87] John F Nash Jr. "The bargaining problem". In: *Econometrica: Journal of the Econometric Society* (1950), pp. 155–162.

[88] Abraham Othman, Christos Papadimitriou, and Aviad Rubinstein. "The complexity of fairness through equilibrium". In: *Proceedings of the fifteenth ACM conference on Economics and computation*. ACM. 2014, pp. 209–226.

[89] Mallesh Pai and Rakesh V Vohra. *Optimal dynamic auctions*. Tech. rep. Discussion paper//Center for Mathematical Studies in Economics and Management Science, 2008.

[90] Christos Papadimitriou et al. "On the complexity of dynamic mechanism design". In: *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2016, pp. 1458–1475.

[91] David C. Parkes, Ariel D. Procaccia, and Nisarg Shah. "Beyond Dominant Resource Fairness: Extensions, Limitations, and Indivisibilities". In: *Proceedings of the 13th ACM Conference on Electronic Commerce*. EC '12. Valencia, Spain: ACM, 2012, pp. 808–825. ISBN: 978-1-4503-1415-2. DOI: `10.1145/2229012.2229075`. URL: `http://doi.acm.org/10.1145/2229012.2229075`.

[92] David C Parkes and Satinder P Singh. "An MDP-Based Approach to Online Mechanism Design." In: *NIPS*. 2003, pp. 791–798.

[93] Alessandro Pavan, Ilya Segal, and Juuso Toikka. "Dynamic mechanism design: A myersonian approach". In: *Econometrica* 82.2 (2014), pp. 601–653.

[94] Elisha A Pazner and David Schmeidler. "Egalitarian equivalent allocations: A new concept of economic equity". In: *The Quarterly Journal of Economics* (1978), pp. 671–687.

[95] Steven Phillips and Jeffery Westbrook. "Online load balancing and network flow". In: *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*. ACM. 1993, pp. 402–411.

[96] Lucian Popa et al. "FairCloud: sharing the network in cloud computing". In: *Proceedings of the ACM SIGCOMM 2012 conference on Applications, technologies, architectures, and protocols for computer communication*. ACM. 2012, pp. 187–198.

[97] Tim Roughgarden, Inbal Talgam-Cohen, and Qiqi Yan. "Robust Auctions for Revenue via Enhanced Competition". In: ().

[98] Peter Sanders, Naveen Sivadasan, and Martin Skutella. "Online scheduling with bounded migration". In: *Mathematics of Operations Research* 34.2 (2009), pp. 481–498.

[99] Erel Segal-Halevi. "How to Re-Divide a Cake Fairly". In: *arXiv preprint arXiv:1603.00286* (2016).

[100] David Shue, Michael J Freedman, and Anees Shaikh. "Performance isolation and fairness for multi-tenant cloud storage". In: *Presented as part of the 10th USENIX Symposium on Operating Systems Design and Implementation (OSDI 12)*. 2012, pp. 349–362.

[101] Balasubramanian Sivan and Vasilis Syrgkanis. "Vickrey auctions for irregular distributions". In: *International Conference on Web and Internet Economics*. Springer. 2013, pp. 422–435.

[102] Hugo Steinhaus. "The problem of fair division". In: *Econometrica* 16.1 (1948).

[103] Walter Stromquist. "How to cut a cake fairly". In: *American Mathematical Monthly* (1980), pp. 640–644.

[104] Nora Szech. "Optimal advertising of auctions". In: *Journal of Economic Theory* 146.6 (2011), pp. 2596–2607.

[105] Abhishek Verma et al. "Large-scale cluster management at Google with Borg". In: *Proceedings of the Tenth European Conference on Computer Systems*. ACM. 2015, p. 18.

[106] Toby Walsh. "Online cake cutting". In: *International Conference on Algorithmic DecisionTheory*. Springer. 2011, pp. 292–305.

[107] Wei Wang, Baochun Li, and Ben Liang. "Dominant resource fairness in cloud computing systems with heterogeneous servers". In: *INFOCOM, 2014 Proceedings IEEE*. IEEE. 2014, pp. 583–591.

[108] Andrew Chi-Chih Yao. "An n-to-1 bidder reduction for multi-item auctions and its applications". In: *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2015, pp. 92–109.

Appendix A

Missing Proofs From Part I

A.1 Proofs missing from Section 4.5

Proof of Claim 51. $\frac{d}{dx}\log\left(1-F(x)\right) = \frac{-f(x)}{1-F(x)} = -h(x)$. Therefore, $1-F(x) = e^{-\int_0^x h(z)dz}$. Re-arranging proves the first part of the claim. We get the second part from the definition of expectation for non-negative random variables. □

Proof of Claim 52.

$$\begin{aligned}\mathbb{E}\left[X_{2:4}\right] &= \int_0^{\bar{V}} 1 - F_{2:4}(x)dx = \int_0^{\bar{V}} 1 - 4F(x)^3 + 3F(x)^4 dx \\ &= \int_0^{\bar{V}} 1 - 4\left(1-e^{-H(x)}\right)^3 + 3\left(1-e^{-H(x)}\right)^4 = \int_0^{\bar{V}} 3e^{-4H(x)} - 8e^{-3H(x)} + 6e^{-2H(x)}dx\end{aligned}$$

□

Proof of Lemma 54. Since $f(x)$ is continuous, we can pick a $\delta(\epsilon) = \delta > 0$, such that for any $x, x' \in [a, b]$ such that $|x - x'| < \delta$, we have $|f(x) - f(x')| < \epsilon$. Let n be an integer such that $\frac{b-a}{n} < \delta$, and let $x_j = a + j \cdot \frac{b-a}{n}$, for $j = 0, 1, \ldots, n$. On each $[x_j, x_{j+1}]$ define $g_\epsilon(x)$ to be the function whose graph is the line segment connecting $(x_j, f(x_j))$ with $(x_{j+1}, f(x_{j+1}))$, i.e. $g_\epsilon(x) = f(x_j) + t \cdot (f(x_{j+1}) - f(x_j))$ for some $t \in [0, 1]$. Since $g_\epsilon(x) \in [f(x_j), f(x_{j+1})]$, the Intermediate Value Theorem implies that for all $x \in [x_j, x_{j+1}]$ there exists a $y_j \in [x_j, x_{j+1}]$, such that $g_\epsilon(x) = f(y_j)$. Since each $x \in [a, b]$ belongs in some $[x_j, x_{j+1}]$, the choice of δ implies that $|f(x) - g_\epsilon(x)| = |f(x) - f(y_j)| < \epsilon$. It remains to show that $g_\epsilon(x)$ is convex: by construction, the slope of $g_\epsilon(x)$ for $x \in [x_j, x_{j+1}]$ is $\left(f(x_{j+1}) - f(x_j)\right) / \left(x_{j+1} - x_j\right)$, which is strictly increasing with j since $f(x)$ is convex. □

Proof of Lemma 56.

$$\begin{aligned}\mathbb{E}\left[X_{2:2}\right] &= \int_0^{\infty} (1 - F(x))^2 dx = \int_0^{\infty} e^{-2H(x)}dx \geq \int_0^{\infty} e^{-H(2x)}dx \\ &= \frac{1}{2}\int_0^{\infty} 1 - F(x)dx = \frac{1}{2}\mathbb{E}\left[X\right],\end{aligned}$$

where the first line holds because $H(x)$ is a convex function. Since $\mathbb{E}[X_{1:2}] + \mathbb{E}[X_{2:2}] = 2\mathbb{E}[X]$, we have that $\mathbb{E}[X_{1:2}] = 2\mathbb{E}[X] - \mathbb{E}[X_{2:2}] \leq \frac{3}{2}\mathbb{E}[X]$. Therefore, $\mathbb{E}[X_{2:2}] \geq \frac{1}{2}\mathbb{E}[X] \geq \frac{1}{2}\frac{2}{3}\mathbb{E}[X_{1:2}] = \frac{1}{3}\mathbb{E}[X_{1:2}]$. The Lemma follows. □

Proof of Lemma 57.

$$\begin{aligned}
\mathbb{E}[X_{1:n}] - \mathbb{E}[X_{2:n}] &= \int_0^\infty F_{2:n}(x) - F_{1:n}(x)dx \\
&= \int_0^\infty nF^{n-1}(x) - (n-1)F^n(x) - F^n(x)dx \\
&= \int_0^\infty nF^{n-1}(x)\left(1 - F(x)\right)dx \\
&= \int_0^\infty nF^{n-1}(x)f(x)\frac{1}{h(x)}dx \\
&= \int_0^\infty f_{1:n}(x)\frac{1}{h(x)}dx = \mathbb{E}\left[\frac{1}{h(X_{1:n})}\right].
\end{aligned}$$

□

A.2 Proofs missing from Section 4.6

Claim 93. *The expected second order statistic of an equal revenue distribution of k samples is $k-1$.*

Proof. For simplicity we show the proof for an untruncated equal revenue distribution X. $F_{2:k}(x) = F^k(x) + kF^{k-1}(x)\left(1 - F(x)\right) = \left(1 - \frac{1}{x}\right)^k + k\left(1 - \frac{1}{x}\right)^{k-1}\frac{1}{x} = \left(\frac{x-1}{x}\right)^k\left(1 - \frac{k}{1-x}\right)$. Therefore, $\mathbb{E}[X_{2:k}] = \int_1^\infty 1 - \left(\frac{x-1}{x}\right)^k\left(1 - \frac{k}{1-x}\right)dx$. The antiderivative of $1 - \left(\frac{x-1}{x}\right)^k\left(1 - \frac{k}{1-x}\right)$ is $x\left(1 - \left(\frac{x-1}{x}\right)^k\right)$. Therefore:

$$\begin{aligned}
\mathbb{E}[X_{2:k}] &= \int_1^\infty 1 - \left(\frac{x-1}{x}\right)^k\left(1 - \frac{k}{1-x}\right)dx \\
&= \left[x\left(1 - \left(\frac{x-1}{x}\right)^k\right)\right]_1^\infty \\
&= \left(\lim_{x\to\infty}\frac{x^k - (x-1)^k}{x^{k-1}}\right) - 1 \\
&= k - 1.
\end{aligned}$$

□

Appendix B

Missing Proofs From Part II

B.1 Number Theory Facts

Fact 94.

$$H_n \geq \ln n + \gamma', \tag{B.1}$$

$$H_n \leq \ln n + \gamma' + \frac{1}{2n}, \tag{B.2}$$

It is easy to derive the following:

Lemma 95. *[47] For natural numbers $b > a > 1$,*

$$\ln(b) - \ln(a-1) - \frac{1}{2a-2} \leq \sum_{x=a}^{b} \frac{1}{x} \leq \ln(b) - \ln(a-1).$$

B.2 Missing Proofs from Section 5.2

Proof of Lemma 78

Lemma 96. $\text{ALLOC}(\psi_i^N, t) \succeq \text{ALLOC}(\psi_{i+1}^N, t)$ *for all* $t = t_i^* \pmod{c+1}$.

Proof. If $t < t_i^*$, the Lemma holds trivially. We prove the case $t \geq t_i^*$ by induction on t. For the base case, denote $\text{ALLOC}(\psi_i, t_i^* - 1) = \text{ALLOC}(\psi_{i+1}, t_i^* - 1) = (a_1, a_2, \ldots, a_{t_i^*-1})$.

Then

$$\text{ALLOC}(\psi_i^N, t_i^*) = (a_1, a_2, \ldots, a_{t_i^*-2}, a_{t_i^*-1}, \sigma/t_i^*),$$
$$\text{ALLOC}(\psi_{i+1}^N, t_i^*) = (a_2, a_3, \ldots, a_{t_i^*-1}, \sigma/t_i^*, \sigma/t_i^*).$$

Noticing that $\text{ALLOC}(\psi_i, t_i^* - 1)$ is sorted (hence $a_\ell \geq a_{\ell+1}$ for all ℓ), and $a_{t_i^*-1} = \frac{\sigma}{t_i^*-1} > \frac{\sigma}{t_i^*}$ completes the proof of the base case.

For the inductive step, let $\textsc{alloc}(\psi_i^N, t) = (a_1, a_2, \dots, a_t)$ and also let $\textsc{alloc}(\psi_{i+1}^N, t) = (b_1, b_2, \dots, b_t)$. Since $t = t_i^* \pmod{c+1}$, we know that $\psi_i^N[t+j] = 0$, for all $0 \le j \le c-1$, $\psi_i^N[t+c] = 1$ and $\psi_{i+1}^N[t+j] = 0$ for $j \ne 1 \pmod{c+1}$. p is the distance of t_i^* to the previous 1. $p' = (c+1-p)$. (In Example 76, ψ_4^7, $p = 2$.)

$$\textsc{alloc}(\psi_i^N, t) = (a_1, \dots, a_{t-1}, a_t),$$

$$\textsc{alloc}(\psi_i^N, t+p) = (a_2, \dots, a_t, \tfrac{\sigma}{t+1} \rightharpoonup_{(p-1)} \tfrac{\sigma}{t+p}, \tfrac{\sigma}{t+p}),$$

$$\textsc{alloc}(\psi_i^N, t+c+1) = (a_2, \dots, a_t, \tfrac{\sigma}{t+1} \rightharpoonup_{(p-1)} \tfrac{\sigma}{t+p}, \tfrac{\sigma}{t+p}, \tfrac{\sigma}{t+p+1} \rightharpoonup_{(c-p)})$$

and

$$\textsc{alloc}(\psi_{i+1}^N, t) = (b_1, \dots, b_{t-1}, b_t),$$

$$\textsc{alloc}(\psi_{i+1}^N, t+c+1) = (b_2, \dots, b_t, \tfrac{\sigma}{t+1} \rightharpoonup_{(c-1)} \tfrac{\sigma}{t+c}, \tfrac{\sigma}{t+c+1}, \tfrac{\sigma}{t+c+1}).$$

The inductive step holds, as $a_\ell \ge b_\ell$ for all ℓ. □

Proof of Lemma 79

Lemma 96. *For all $t \ne t_i^* \pmod{c+1}$, $t_i^* \le t \le t_{max}$, it holds that* $\textsc{bank}(\psi_{i+1}^N, t) \ge \sum_{j=1}^{c} \frac{\sigma}{t+j}$.

Proof. Recall that p' is the distance from t to the next 1. It must be that $\textsc{bank}(\psi_i^N, t+p') \ge \sum_{j=1}^{c} \frac{\sigma}{t+j+p'}$, as the c entries following the next 1 in ψ_i^N are 0. Therefore it suffices to prove that

$$\begin{aligned}\textsc{talloc}(\psi_i^N, t+p') - \textsc{talloc}(\psi_{i+1}^N, t) &\ge \sum_{j=1}^{c} \frac{\sigma}{t+j} - \sum_{j=1}^{c} \frac{\sigma}{t+j+p'} \\ &= \sum_{j=1}^{p'} \frac{\sigma}{t+j} - \sum_{j=c-p'+1}^{c} \frac{\sigma}{t+j+p'}.\end{aligned}$$

Some simple arithmetic shows that

$$\textsc{talloc}(\psi_i^N, t+p') - \textsc{talloc}(\psi_{i+1}^N, t) =$$

$$\sum_{j=1}^{p'} \frac{1}{t+j} + \left(\frac{1}{t_i^*+p'} + \frac{1}{t_i^*+c+1+p'} + \dots + \frac{1}{t+p'}\right) + \left(\frac{1}{t_i^*} + \frac{1}{t_i^*+c+1} + \dots + \frac{1}{t}\right)$$

We show that this is at least $\sum_{j=1}^{p'} \frac{\sigma}{t+j} - \sum_{j=c-p'+1}^{c} \frac{\sigma}{t+j+p'}$, or equivalently, that

$$\sum_{j=c-p'+1}^{c} \frac{1}{t+j+p'} + \left(\frac{1}{t_i^*+p'} + \frac{1}{t_i^*+c+1+p'} + \dots + \frac{1}{t+p'}\right)$$

$$-\left(\tfrac{1}{t_i^*}+\tfrac{1}{t_i^*+c+1}+\tfrac{1}{t_i^*+2(c+1)}+\cdots+\tfrac{1}{t}\right)\geq 0.$$

It is easily verifiable that the LHS is decreasing in t in this range ($t_i^* \leq t \leq t_{max}$), therefore its suffices to prove the inequality for $t = t_{max}$. Let

$$f(t_i^*)=\left(\frac{1}{t_{max}+c+1}+\frac{1}{t_{max}+c+2}+\cdots+\frac{1}{t_{max}+c+p'}\right)$$
$$+\left(\tfrac{1}{t_i^*+p'}+\tfrac{1}{t_i^*+c+1+p'}+\cdots+\tfrac{1}{t_{max}+p'}\right)+\left(\tfrac{1}{t_i^*}+\tfrac{1}{t_i^*+c+1}+\tfrac{1}{t_i^*+2(c+1)}+\cdots+\tfrac{1}{t_{max}}\right).$$

Consider $f(t_i^*+c+1)-f(t_i^*)$. We prove that $f(t_i^*+c+1)-f(t_i^*)\leq 0$, for all t_i^* and c in Lemma 96. Given this inequality holds, it remains to show that $f(t_i^*)$ is non-negative in the limit. We can re-write the function as:

$$f(t_i^*)=\sum_{j=1}^{p'}\frac{1}{t_{max}+j}+\frac{1}{c+1}\sum_{j=0}^{t_i^*+p'-c-3}\frac{1}{\frac{t_i^*+p'}{c+1}+j}+\frac{1}{c+1}\sum_{j=0}^{t_i^*+p'-c-3}\frac{1}{\frac{t_i^*}{c+1}+j}=$$
$$=\tfrac{1}{c+1}\left(\sum_{j=(t_i^*+p')/(c+1)}^{\frac{(c+2)t_i^*+(c+1)(p'-c-3)+p'}{c+1}}\tfrac{1}{j}-\sum_{j=t_i^*/(c+1)}^{\frac{(c+2)t_i^*+(c+1)(p'-c-3)}{c+1}}\tfrac{1}{j}\right)+\sum_{j=(c+2)t_i^*+(c+1)(p'-c-3)+c+1}^{(c+2)t_i^*+(c+1)(p'-c-3)+c+p'}\frac{1}{j}$$

Applying the approximations for the Harmonic number from Appendix B.1 and taking the limit as t_i^* goes to infinity completes the proof. □

Lemma 96. *$f(t_i^*+c+1)-f(t_i^*)\leq 0$, for all t_i^* and c.*

Proof. Set $t_{max}=(c+2)t_i^*+(c+1)(p'-c-3)$ (this is the t_{max} w.r.t. t_i^*) and $t'_{max}=(c+2)(t_i^*+c+1)+(c+1)(p'-c-3)=t_{max}+(c+1)(c+2)$ (this is w.r.t. t_i^*+c+1). We have that $\Delta=f(t_i^*+c+1)-f(t_i^*)$ is:

$$\Delta=\left(\frac{1}{t_{max}+(c+1)(c+2)+c+1}+\cdots+\frac{1}{t_{max}+(c+1)(c+2)+c+p'}\right)$$
$$+\left(\frac{1}{t_i^*+c+1+p'}+\cdots+\frac{1}{t_{max}+(c+1)(c+2)+p'}\right)$$
$$-\left(\frac{1}{t_i^*+c+1}+\frac{1}{t_i^*+2(c+1)}+\cdots+\frac{1}{t_{max}+(c+1)(c+2)}\right)$$
$$-\left(\frac{1}{t_{max}+c+1}+\frac{1}{t_{max}+c+2}+\cdots+\frac{1}{t_{max}+c+p'}\right)$$
$$-\left(\frac{1}{t_i^*+p'}+\frac{1}{t_i^*+c+1+p'}+\cdots+\frac{1}{t_{max}+p'}\right)$$
$$+\left(\frac{1}{t_i^*}+\frac{1}{t_i^*+c+1}+\frac{1}{t_i^*+2(c+1)}+\cdots+\frac{1}{t_{max}}\right).$$

This can be re-written as:

$$
\begin{aligned}
f(t_i^* + c + 1) - f(t_i^*) = & \sum_{j=1}^{p'} \left(\frac{1}{t_{max} + (c+1)(c+2) + c + j} - \frac{1}{t_{max} + c + j} \right) \\
& + \left(\sum_{j=1}^{c+2} \frac{1}{t_{max} + (c+1)j + p'} \right) - \frac{1}{t_i^* + p'} \\
& - \left(\sum_{j=1}^{c+2} \frac{1}{t_{max} + (c+1)j} \right) + \frac{1}{t_i^*} \\
= & \frac{p'}{t_i^*(t_i^* + p')} - \sum_{j=1}^{c+2} \frac{p'}{(t_{max} + (c+1)j + p')\,(t_{max} + (c+1)j)} \\
& - \sum_{j=1}^{p'} \frac{(c+1)(c+2)}{(t_{max} + (c+1)(c+2) + c + j)\,(t_{max} + c + j)} \\
\leq & \frac{p'}{t_i^*(t_i^* + p')} - \frac{(c+2)p'}{(t_{max} + (c+1)(c+2) + p')\,(t_{max} + (c+1)(c+2))} \\
& - \sum_{j=1}^{p'} \frac{(c+1)(c+2)}{(t_{max} + (c+1)(c+2) + c + j)\,(t_{max} + c + j)} \\
\leq & \frac{p'}{t_i^*(t_i^* + p')} - \frac{(c+2)p'}{(t_{max} + (c+1)(c+2) + p')\,(t_{max} + (c+1)(c+2))} \\
& - \frac{p'(c+1)(c+2)}{(t_{max} + (c+1)(c+2) + c + p')\,(t_{max} + c + p')}.
\end{aligned}
$$

The expression above is equal to:

$$
\begin{aligned}
f(t_i^* + c + 1) - f(t_i^*) \leq & \frac{1}{-1 + (2+c)p + (c+2)t} + \frac{1}{3 + c^2 - 2p - 2t - c(-3 + p + t)} \\
& + \frac{((c+1)(1 - 3p - 4t + c(-1 + p + t)^2 + 2(p+t)^2))}{(t(t+p)(-1 + p + 2t + c(-1 + p + t))(-1 + 2p + 2t + c(-1 + p + t)))}.
\end{aligned}
$$

This is easily verified computationally to be non-positive.

□

Lemma 97. *The function* $f(t) = \sum_{i=t}^{(t-1)(c+2)+c+1} \frac{1}{i}$ *is monotone decreasing in* t*, for integer values of* t *and integer* $c > 0$.

Proof.

$$
\begin{aligned}
f(t) - f(t+1) &= \sum_{i=t}^{(t-1)(c+2)+c+1} \frac{1}{i} - \sum_{i=t+1}^{t(c+2)+c+1} \frac{1}{i} \\
&= \frac{1}{t} - \sum_{i=t(c+2)}^{t(c+2)+c+1} \frac{1}{i} > \frac{1}{t} - (c+2)\frac{1}{t(c+2)} = 0
\end{aligned}
$$

□

Lemma 98. *For integer $c > 1$ and integer $t \geq 2$,*

$$\sum_{i=0}^{t-2} \frac{1}{t + i(c+1)} \leq 1/2.$$

Proof. It suffices to prove the Lemma for $c = 2$, as the sum decreases as c increases.

$$
\begin{aligned}
\sum_{i=0}^{t-2} \frac{1}{t+3i} &= \frac{1}{t} + \sum_{i=1}^{t-2} \frac{1}{t+3i} \\
&\leq \frac{1}{t} + \int_0^{t-2} \frac{1}{t+3x} dx = \frac{1}{t} + \frac{1}{3}\ln(4 - 6/t) \\
&\leq \frac{1}{t} + \frac{1}{3}\ln 4 \leq 1/2,
\end{aligned}
$$

for $t \geq 27$. It is easy to computationally verify the Lemma holds for smaller t. □

B.3 The Fairness ratio of CDFD for $c = 1$ and $c = 2$

We prove the following.

Theorem 65. *The optimal fairness ratio of CDFD for any c control vector for $c = 1$ and $c = 2$ are*

$$
\begin{aligned}
&\sigma^*\left((2)^{-1}\right) = 2(3\ln 3)^{-1} \text{ and} \\
&\sigma^*\left((3)^{-1}\right) = 3(4\ln 4)^{-1}
\end{aligned}
$$

respectively.

For $c = 1$, there are two basic control vectors: $(1, 0, 1, 0, \ldots)$ and $(0, 1, 0, 1, \ldots)$. For $c = 2$, there are three basic control vectors (Example 73). We prove the bound in Theorem 65 for each basic control vector separately. The theorem then follows from Lemma 77. Here we only present the proof for one of the two control vectors. We omit the proof for the latter control vector as it is virtually identical. Furthermore, we only present the proof for $c = 1$; the proof for $c = 2$ is by similar (slightly more involved) case analysis.

step	allocation for $(0, 1, \ldots)$	sum	allocation for $(1, 0, \ldots)$	sum
1	σ	σ	σ	σ
2	$\frac{\sigma}{2}, \frac{\sigma}{2}$	σ	$\sigma, \frac{\sigma}{2}$	$\frac{90\sigma}{60}$
3	$\frac{\sigma}{2}, \frac{\sigma}{2}, \frac{\sigma}{3}$	$\frac{80\sigma}{60}$	$\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{3}$	$\frac{70\sigma}{60}$
4	$\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{4}$	$\frac{80\sigma}{60}$	$\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{3}, \frac{\sigma}{4}$	$\frac{85\sigma}{60}$
5	$\frac{\sigma}{2}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{4}, \frac{\sigma}{5}$	$\frac{92\sigma}{60}$	$\frac{\sigma}{3}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{5}$	$\frac{79\sigma}{60}$
6	$\frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{6}, \frac{\sigma}{6}$	$\frac{82\sigma}{60}$	$\frac{\sigma}{3}, \frac{\sigma}{3}, \frac{\sigma}{4}, \frac{\sigma}{5}, \frac{\sigma}{5}, \frac{\sigma}{6}$	$\frac{89\sigma}{60}$

Table B.1: Allocations for the first 6 steps of Skip with basic 1-control vectors.

Lemma 99. *The allocations created by* Skip*, with basic control vector* $\psi^1 = (0, 1, \ldots)$ *(from step 6 onward) are*

1. *On steps* $t = 0 \pmod 6$*:* $\frac{\sigma}{(t/3)+1}, \bowtie_1, \frac{\sigma}{t}, \frac{\sigma}{t}$.
2. *On steps* $t = 2 \pmod 6$*:* $\left(\frac{\sigma}{((t+1)/3)+1}\right) \bowtie_1, \frac{\sigma}{t}, \frac{\sigma}{t}$.
3. *On steps* $t = 4 \pmod 6$*:* $\frac{\sigma}{((t+2)/3)}, \rightharpoonup_1, \bowtie_1, \frac{\sigma}{t}, \frac{\sigma}{t}$.

On odd steps, add $\frac{\sigma}{t}$ *to the previous step (note the denominators have "changed" relative to the new step).*

Proof. The proof is by induction on the round number. The base case (step 6) appears in Table B.1. The move from even to odd steps is immediate, as there is no donor. Because only the agent with the highest utility has her allocation decreased, it is easy to verify the transition from odd to even steps, by renaming the denominators. □

In order to compute the optimal fairness ratio achieved by Skip on ψ^1, we bound the sum of allocations at odd steps, as each odd step uses strictly more resources than the previous even step.

We bound the size of $\text{BANK}(\psi^1, t)$, noting for all $t > 0$, $\text{BANK}(\psi^1, t) < 1$. On odd steps, we take $\frac{\sigma}{i}$ from the bank. On all even steps except the second and fourth, we return some resource to the bank. (Note that to be able to reach step 5, we need $\frac{23\sigma}{15}$ in the bank; this immediately implies that $\sigma \leq \frac{15}{23}$.) First, we show that the resource allocated monotonically increases, when we look at it through a slightly wider lens:

Lemma 100. *The total resource allocated by* $\text{Skip}(\psi^1)$ *in steps* $t = 5 \pmod 6$ *is (strictly) monotone increasing.*

Proof. It is easy to verify the following, from Lemma 99:

Fix some $\tau = 0 \pmod 6$, $\tau \geq 6$.

1. On step τ, we give $\sigma\left(\frac{\sigma}{\tau/3} - \frac{2}{\tau}\right) = \frac{\sigma}{\tau}$ to the bank.
2. On step $\tau + 2$, we give $\sigma\left(\frac{1}{\tau/3+1} - \frac{2}{\tau+2}\right) = \frac{\sigma\tau}{(\tau+2)(\tau+3)}$
3. On step $\tau + 4$, we return $\sigma\left(\frac{1}{\tau/3+2} - \frac{2}{\tau+4}\right) = \frac{\sigma\tau}{(\tau+4)(\tau+6)}$.

On steps $\tau+1, \tau+3, \tau+5$, we take $\frac{\sigma}{\tau+1}, \frac{\sigma}{\tau+3}, \frac{\sigma}{\tau+5}$ respectively. It is easy to verify (by simple calculus), that for $\tau \geq 6$, the sum of what we take over these 6 steps is greater than the sum of what we give. The proof follows. □

We now complete the proof of Theorem 65 (for the control vector $(1, 0, 1, 0, \ldots)$).

Sketch of proof of Theorem 65. From Proposition 100, the amount of resource allocated increases; therefore we only need to make sure the algorithm doesn't over allocate when t goes to infinity. We analyze the case when the allocation is larger than all previous allocations, $t = 5 \pmod 6$. At time $t = 0 \pmod 6$, the total resource allocated is

$$\sigma\left(2\sum_{i=1}^{t/3}\frac{1}{t/3+2i} + \sum_{i=1}^{t/3}\frac{1}{t/3+2i-1}\right) \approx \sigma\left(\frac{3}{2}\sum_{i=t/3}^{t}\frac{1}{i}\right) \approx \sigma\frac{3\ln 3}{2}.$$

It must hold that $\sigma\frac{3\ln 3}{2} \leq 1$, and the allocations at times $t = 0 \pmod 6$ and $t = 5 \pmod 6$ are asymptotically the same. The theorem follows. □

B.4 Example of constructing a feasible dual solution

For a concrete example, let $n = 2$, with Z and $\hat{Z}$ as follows:

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{Z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ DRF_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & DRF_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & DRF_6 & 0 & DRF_6 & 0 & 0 \\ 0 & DRF_7 & 0 & 0 & 0 & 0 & DRF_7 & 0 \\ 0 & 0 & 0 & 0 & DRF_8 & 0 & 0 & DRF_8 \end{bmatrix}$$

with $s_1 = DRF_4 + DRF_7$, $s_2 = DRF_5 + DRF_6$, $s_3 = 2 * DRF_8 + DRF_6 + DRF_7$. We have that $DRF_4 = \frac{1}{2}$, $DRF_5 = \frac{1}{3}$, $DRF_6 = \frac{1}{4}$, $DRF_7 = \frac{1}{5}$ and $DRF_8 = \frac{1}{6}$. Therefore, $s_1 = 0.7$, $s_2 = \frac{7}{12}$ and $s_3 = \frac{7}{12} + \frac{1}{5}$. Therefore, we have that

$$\sigma = \frac{1}{s_1 + s_3} = 0.6741.$$

This gives us:

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & \sigma & 0 & 0 \\ \sigma & \sigma & 0 & 0 & 0 & \sigma & \sigma & 0 \end{bmatrix}$$

$$l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma DRF_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma DRF_6 & 0 & 0 \\ 0 & \sigma DRF_7 & 0 & 0 & 0 & 0 & \sigma DRF_7 & 0 \\ 0 & 0 & 0 & 0 & \sigma DRF_8 & 0 & 0 & \sigma DRF_8 \end{bmatrix}$$

and $f(N,1) = \sigma$, $f(N,2) = 0$.

B.5 Additional Figures

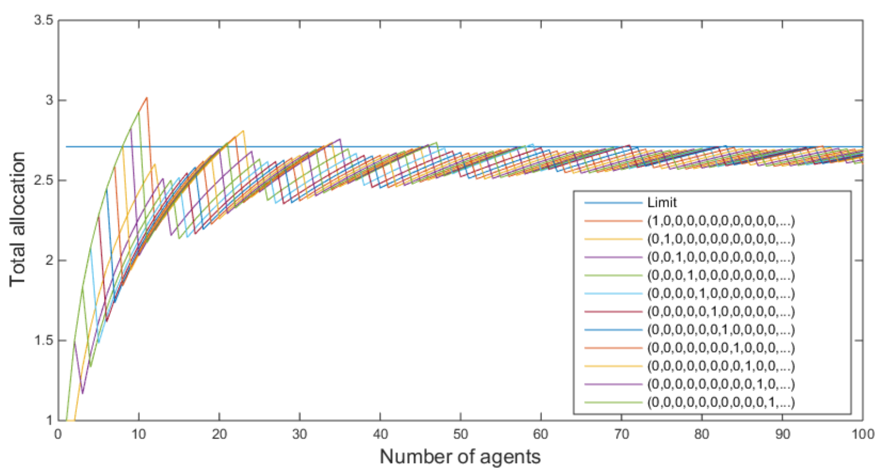

Figure B.1: Optimal values of TALLOC for different control vectors when $c = 10$.