Reliability-Based Design Optimization of Series Structural Systems

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Abstract

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Algorithms for solving three classes of reliability-based optimal structural design problems are developed. The first class of problems is to minimize the cost of the design, subject to failure probability and structural constraints. The second class is to minimize the failure probability of the design, subject to cost and structural constraints. The third class of problems is to minimize the initial cost plus the expected cost of failure, subject to failure probability and structural constraints. The failure probabilities can describe component failures or series structural system failures. The third class of problems is further extended to the optimal design of a portfolio of structures.

Based on a first-order approximation to the failure probability, we construct approximating problems that can be solved repeatedly to obtain approximations to the solutions of the original design problems. The approximating problems are made dependent on a set of parameters that can be adjusted to improve the accuracy of the first-order approximations. The adjustment of the parameters is based on separate evaluations of the failure probability. In special cases, we show that the approximating problems are identical to the original ones. We develop a set of parameter-adjustment rules that, together with subroutines for solving the approximating problems, composes the collection of new algorithms for solving reliability-based optimal structural design problems. Effectively, the new algorithms solve a sequence of first-order approximating problems that are constructed as the computations progress. It is observed in a set of numerical design examples that the parameter-adjustment rules are

efficient tools for improving the accuracy of the first-order approximations.

The reliability-based optimal structural design algorithms developed in this study present significant departures from the state-of-the-art. In particular, careful attention is given to the underlying assumptions and approximations to ensure a rigorous mathematical foundation for the algorithms. This, together with the fact that first-, or second-order reliability methods, Monte Carlo Simulation, or any other computational reliability method can be employed, makes the algorithms efficient, robust and versatile tools for solving reliability-based optimal structural design problems.

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Conventions and Symbols

$\triangleq$ denotes "equal by definition."

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

$\mathbb{N} \triangleq \{1, 2, 3, ...\}$ denotes the natural numbers.

$\varphi(\cdot)$ denotes the standard normal probability density function.

$\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

Superscript ^{T} denotes the transpose of a matrix.

Vectors and matrices denoted by roman letters are in bold.

x_j denotes the j-th component of a vector $\mathbf{x}$.

$(x_j)_i$ denotes the j-th component of a vector $\mathbf{x}_i$.

$\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$.

$\|\mathbf{x}\|$ denotes the norm in $\mathbb{R}^n$ and is defined by $\|\mathbf{x}\| \triangleq \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$.

$a_+ \triangleq \max\{0, a\}$ for any $a \in \mathbb{R}$.

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Chapter 1

Introduction

In the design of structures, a systematic exploration of a broad range of alternatives is necessary to identify the "best" design. The large number of design alternatives dictate the use of an automatic, computer-based procedure for examining these alternatives. Optimization theory is the mathematical framework for deriving such procedures. A similar situation arises in the design of inspection and maintenance plans for infrastructure systems, such as highway bridges.

The process of finding the "best" design is complicated by the presence of uncertainties in the structure and its environment. Uncertainties can be caused by random variations in properties of materials, insufficient information about the aging and degradation rates of a structure, the random occurrence of events such as earthquakes or explosions, and the inaccurate representation of the real-world structure and its surrounding environment by idealized mathematical models. These uncertainties can be significant factors in determining the present and future performance of a structure, and must be accounted for to ensure safe and reliable structures, see, e.g., Ang and Tang (1984), Madsen *et al.* (1986), and Ditlevsen and Madsen (1996). By using the theory of structural reliability (see, e.g., Ditlevsen and Madsen (1996) and Chapter 2), the uncertainty can be quantified, and the failure probability or its compliment, the reliability, of the structure can be determined.

Reliability-based optimal design of a structure is the process of finding the "best"

design, while including the effect of uncertainty as expressed by the failure probability or reliability of the structure. This design philosophy can potentially lead to significant social benefits in the form of more economical, efficient and reliable structures. However, the application of reliability-based optimal design in practical structural design is severely hampered due the difficulty in solving most reliability-based optimal design problems. For example, typical problems are not compatible with standard nonlinear optimization algorithms such as NLPQL (Schittkowski 1985), DOT (Vanderplaats 1992), LANCELOT (Conn *et al.* 1992), CFSQP (Lawrence *et al.* 1997), NPSOL (Gill *et al.* 1998), SNOPT (Gill *et al.* 1998), and MINOS (Murtagh and Saunders 1998). The reason for this is two-fold: (i) Standard nonlinear optimization algorithms require that all the functions involved in the problem are continuously differentiable, which is not known to be the case for the failure probability as a function of design variables (see Polak *et al.* (2000)). (ii) Standard nonlinear optimization algorithms also require that all the functions involved can be evaluated exactly in finite computational time. This is not the case for the failure probability of a structure, which is usually defined in terms of a high-dimensional integral (see Chapter 2).

1.1 Objective and Scope

The reliability-based design, inspection, or maintenance optimization of a structural system consists of three steps. First, the analyst needs to identify an objective (i.e., the quantity that should be optimized), the design variables, and an allowable range of the design variables and other quantities (i.e., the constraint set). If the analyst has more than one objective in mind, the procedure will also need to include the selection of a method to aggregate the different objectives. Second, the analyst needs to build models to describe the structural response to loads and to compute the failure probability of the structure. The objective and the constraint set are typically defined in terms of these quantities. Third, the analyst has to develop, or find in the literature, an appropriate (optimization) method for finding the values of the design variables that optimize the objective. This report deals with the third

step. Specifically, we construct algorithms for solving the following reliability-based optimal structural design problems:

$\mathbf{P}_1$: Minimize the cost of the design, subject to failure probability and structural constraints.

$\mathbf{P}_2$: Minimize the failure probability of the design, subject to cost and structural constraints.

$\mathbf{P}_3$: Minimize the initial cost plus the expected cost of failure, subject to failure probability and structural constraints.

The algorithms for the solution of problems in the form $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ are constructed under the assumption that the design variables are real-valued (i.e., *not* integer-valued), the failure probability is defined for a series structural system (as defined in Chapter 2), and the structural response to loads can be computed in finite time. The latter is the case when the structural response is defined in terms of the solution of a finite element model. Hence, effectively, we have assumed that the structural model is discretized in time and space.

1.2 Review of the State-of-the-Art

There is a large body of literature on reliability-based optimal design, inspection, and maintenance of structures, with contributors from the aerospace, automotive, and civil engineering community. In this section, we give an overview of the most significant contributions to the field.

Since our scope is the design of structures, i.e., the load carrying parts of buildings, bridges, offshore platforms, aerospace structures, and vehicles, we do not cover the literature on reliability-based optimal design of other engineering systems, e.g., the system of electrical components in an airplane or the cooling system of a nuclear

power plant. For such perspectives, see, e.g., Kuo *et al.* (2001), Valdez-Flores and Feldman (1989), and Chiang and Yuan (2001).

1.2.1 Overview of Studies Prior to 1990

This subsection presents the most significant contributions in the field of reliability-based optimal structural design prior to 1990. Another review of this period can be found in Frangopol and Moses (1994). For a review of the few contributions prior to 1990 on reliability-based inspection and maintenance optimization of structures, see Sommer (1993) and Sherif and Smith (1981) for steel structures, and Frangopol *et al.* (2001) for bridges. The following presentation is influenced by the review chapters in the doctoral dissertations by Leheta (1988) and Enevoldsen (1991).

The field of reliability-based optimal structural design grew out of the aerospace industry with its increasing demand for safe and economical design of novel airtransport structures. The first major effort to combine reliability and optimization considerations in the design process was made by Hilton and Feigen (1960). They assumed that applied stresses were normally distributed random variables, which resulted in a random safety margin. In a technical note, Kalaba (1962) suggested to use dynamic programming techniques to solve Hilton and Feigen's optimization problem. Moses and Kinser (1967) modeled the structural reliability in terms of series and parallel systems of statistically dependent components. Other early contributions were made by Switzky (1964) and Murthy and Subramanian (1968).

Following advances in the theory of structural reliability, attempts were made to incorporate the new reliability formulations into the reliability-based optimal design problem. Moses (1969) and Moses and Stevenson (1970) used Cornell's reliability index (see Ditlevsen and Madsen (1996)) as a measure of the structural safety. Murotsu *et al.* (1976), (1978) included model uncertainty by using a second-moment approximation of structural reliability.

Rosenblueth and Mendoza (1971) departed from the typical "minimum-weight" objective function. They considered the objective function as consisting of several

parts representing the benefit derived from the survival of the structure, the initial cost, and the expected cost of failure. In their defense related applications, Bracken and McGill (1973) considered problems with constraints given by max-functions or max-min-functions, i.e., constraints defined in terms of optimization sub-problems. They recognized that such constraint functions are not differentiable everywhere, and hence standard nonlinear optimization algorithms may jam. However, in their numerical example, they still resorted to standard nonlinear algorithms. Kwak and Haug Jr. (1976a) and Kwak and Haug Jr. (1976b) developed special algorithms for the solution of optimal design problems with max-function constraints. By taking advantage of the special structure of their design problem and by assuming that the max-function has a finite number of maximizers, they could use variational analysis to replace the max-function constraint by a linearized subproblem (Kwak and Haug Jr. 1976a), or by explicit, equivalent expressions (Kwak and Haug Jr. 1976b). The resulting reformulated problem was solved by the gradient projection method. Other contributions in the 1970s were Vanmarcke (1971), Moses (1977), and Davidson *et al.* (1977).

The 1980s brought improved standard nonlinear algorithms, such as the sequential unconstrained minimization technique (see Fiacco and McCormick (1990)), which was employed by Davidson *et al.* (1980) and Surahman and Rojiani (1981) to solve reliability-based optimal structural design problem. Around the same time, Rao (1980) solved chance constrained design problems by stochastic programming techniques.

Murotsu *et al.* (1979) and Feng and Moses (1986a) presented an approximation of the failure probability of series structural systems. The original problem, containing a constraint on the series system failure probability, was replaced by sequences of approximating problems with only component failure probability constraints. The approximating problems were solved and the component failure probability constraints were modified until the series system constraint was satisfied. This heuristic scheme was reported to work satisfactorily.

The first effort in reliability-based shape optimization, in contrast to earlier stud-

ies considering size optimization, is by Furuta (1980). Rao (1981) presented the first major effort to optimize structures subject to random vibrations. Frangopol (1984) conducted the first study focusing on the interactive aspects of the design process. Moreover, Frangopol (1985) was the first to consider multiple objectives in the reliability-based optimal structural design problem.

Application papers in structural engineering frequently considered frame structures (e.g., Carmichael (1981), Cheng and Chang (1985), and Surahman and Rojiani (1981)), and plastic design of structures (see Frangopol (1983)). Gross and Sobieszczanski-Sobieski (1980) is one of many applications in aircraft design. See also Thoft-Christensen (1991) for an overview of application papers from different fields.

The 1980s ended with attempts to use improved system reliability methods (Sorensen (1987), (1988), and Sorensen and Thoft-Christensen (1989)). Sensitivity techniques and approximating gradient calculations were developed and employed in optimal structural design problems by Sorensen (1987), (1988), Lee and Kwak (1987), and Kwak and Lee (1987). More realistic structural response models, e.g., finite element models, were introduced to the field of reliability-based optimal structural design by Feng and Moses (1986b) and Sorensen (1987), (1988).

1.2.2 Recent Studies

The majority of studies after 1990 attempt to use standard nonlinear optimization algorithms to solve the reliability-based optimal structural design problem. These efforts include Murotsu and Shao (1990) and Kim and Kwak (1996) in shape optimization; Mahadevan (1992), Liu and Moses (1992), and Lin and Frangopol (1996) in design of frames, trusses, and reinforced concrete structures, respectively; Mori and Ellingwood (1994) and Frangopol *et al.* (1997) in maintenance planning of deteriorating structures; Kim and Wen (1990) and Weiji and Li (2001) in design under multiple hazards, and Pedersen and Thoft-Christensen (1996) with special emphasis on the interactive aspect of the design process. All these studies involve the failure probability

as a constraint, or in the objective function. No attempts were made in these works to show that the resulting design problems satisfy the necessary requirements for the use of standard nonlinear optimization algorithms, which were employed. As mentioned in the introduction, these requirements are that all the functions in the problem are continuously differentiable and can be evaluated exactly in finite computational time. Additionally, it is usually required that the constraint set satisfies a constraint qualification. More theoretical studies, such as that by Enevoldsen (1991) and Enevoldsen and Sorensen (1994), also neglect these requirements, while reporting problems with convergence in certain cases. While one cannot prove that the exact failure probability is differentiable with respect to the design variables (see Polak *et al.* (2000)), one can easily show that approximations of the failure probability as obtained by first- or second-order reliability methods (FORM and SORM) or Monte Carlo simulation (see Chapter 2), which were used in most of the above studies, are not differentiable. Hence, under such conditions, standard nonlinear optimization algorithms may jam and not converge to a solution of the problem at hand.

In the literature, there are also attempts to apply gradient-free algorithms to solve optimal design problems under uncertainty. Leheta (1988) used a simple pattern search method; Itoh and Liu (1999), Nakamura *et al.* (2000), Cheng and Ang (1999), and Thampan and Krishnamoorthy (2001) used genetic algorithms, and Beck *et al.* (1999) and Tsompanakis and Papadrakakis (2000) used hybrid algorithms that combine deterministic and stochastic search methods. These algorithms are applicable to most reliability-based optimal design problems, including those that contain non-differentiable functions. However, they are known to have slow convergence, and the computational effort required to achieve a solution can be extremely high.

In the FORM, an approximation of the failure probability is itself given by an optimization problem (see Chapter 2). Madsen and Friis Hansen (1992) and later Kuschel and Rackwitz (2000b), (2000a) and Rackwitz (2000a),(2000b) replaced this "inner" optimization problem by its first-order necessary optimality conditions and solved the resulting reliability-based optimal design problem in an augmented space by a standard nonlinear algorithm. The approach is appealing because of the elimination

of the "inner" problem. However, according to Luo *et al.* (1996), this transcription may result in an optimization problem with a constraint set that does not satisfy typical constraint qualifications. There exist special, more complicated, algorithms, described by Luo *et al.* (1996), for solving this transcribed problem. In any case, the transcribed problem requires second-order derivatives even for the solution of the first-order reliability approximating problem, which may be costly to compute.

Other researchers, such as Ng and Moses (1999) and Smilowitz and Madanat (2000) have used Markov models to describe the evolution of system performance in time, leading to design problems that are linear programs. Augusti et al. (1998) considered a probabilistic model for a highway network, which led to a discrete nonlinear optimization problem solvable by dynamic programming techniques. These formulations, though convenient from a computational standpoint, impose severe restrictions on the probabilistic models that can be used to describe structural behavior.

On the boundary between stochastic and deterministic optimization, we have a nontrivial literature on robust or worst-case scenario design, see, e.g., Gu *et al.* (2000), (2002)) and Li *et al.* (2001). However, no probabilistic characterization of uncertainties are employed in these approaches.

Kirjner-Neto *et al.* (1998) showed the equivalence between a special case of $\mathbf{P}_1$ involving multiple component failure probabilities defined by affine limit-state functions (see Chapter 2) and a semi-infinite optimization problem (see Chapter 3). The presented outer approximations algorithm converges to stationary points (see Chapter 3) of the problem at hand. Der Kiureghian and Polak (1998) and Polak *et al.* (2000) proved a similar equivalence for $\mathbf{P}_2$. To account for non-affine limit-state functions, Der Kiureghian and Polak (1998) parameterized the semi-infinite optimization problem and solved a sequence of such semi-infinite problems for a range of parameter values. The parameter values were determined by separate calculations of the failure probability. A first attempt to deal with series structural system in the spirit of Der Kiureghian and Polak (1998) was presented in Royset *et al.* (2001a), (2001b). Royset *et al.* (2001b) also contains an initial study on solution techniques for $\mathbf{P}_3$ based on a first-order reliability approximation.

Gasser and Schueller (1998) and Liaw and DeVries (2001) used the Response Surface Method to approximate the failure probability by a smooth function defined in the space of design variables. A standard nonlinear algorithm was then applied to the resulting approximate problem. This approach is numerically robust since the response surface is smooth and the difficulty related to non-differentiability of the failure probability is avoided. However, the overall efficiency of the method strongly depends on the accuracy of the response surface and the computational cost of establishing it, which tends to be high for problems with many design variables. In an earlier work, Marti (1997) used the Response Surface Method to solve optimization problems under uncertainty using stochastic approximation methods.

Approximations and decomposition strategies for large-scale optimal structural design problems in a multi-disciplinary environment have been considered by Sobieszczanski-Sobieski (1982), Jensen (1993), Sobieszczanski-Sobieski and Haftka (1997), Grandhi and Wang (1998), Oakley *et al.* (1998), Xiao *et al.* (1999), and Padmanabhan and Batill (2002). The multi-disciplinary environment is characterized by several teams, possibly geographically separated, working on different aspects of the structural shape, loading and response with conflicting design objectives in mind. Both hierarchical and nonhierarchical decomposition techniques, as well as response surface approximations have been proposed to address the organizational and computational challenges in the evaluation of the failure probability and other performance measures.

Other studies on approximation techniques for optimal design under uncertainty include Thanedar and Chamis (1990), Chen *et al.* (1997), and Abumeri *et al.* (2000), which considered applications to aerospace structures, and Reddy *et al.* (1994), Weiji and Li (1994), Li and Wang (1994), and Lee and Kwak (1995), which employ expansion techniques. As with most of the previously cited works, these studies do not address the issue of convergence of the algorithms used to solve the problem.

1.3 Proposed New Approach

In non-trivial cases, the failure probability of a structure cannot be evaluated exactly, and the failure probability, together with its approximations, is not known to be continuously differentiable with respect to the design variables. Hence, the only viable approach for solving reliability-based optimal structural design problems is to establish approximations to $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, which can be solved by existing or newly developed optimization algorithms.

We present a comprehensive treatment of the approach put forth in Royset *et al.* (2001a),(2001b) for solving $\mathbf{P}_1$ and $\mathbf{P}_2$ in the case of series structural system failure probabilities in the objective function or the constraint description. We construct approximations to $\mathbf{P}_1$ and $\mathbf{P}_2$ that are semi-infinite optimization problems solvable by discretization-type, or other semi-infinite optimization algorithms. The approximating problems use first-order approximations to the failure probability, and hence are first-order approximations to the original problem. Based on the convergence results for semi-infinite optimization algorithms, we are guaranteed to compute a stationary point of the first-order approximating problem. Such points are reasonably good approximations to the optimal design.

In a heuristic manner, we can improve on the first-order approximation by constructing a sequence of approximating problems. The first-order approximating problem is made to depend on a parameter related to the accuracy of the approximation. By adjusting the parameter based on separate, approximate calculations of the failure probability, we can solve a sequence of gradually more accurate approximating semi-infinite optimization problems. Hence, approximate solutions of $\mathbf{P}_1$ and $\mathbf{P}_2$ can be obtained by solving the sequence of approximating semi-infinite optimization problems.

In $\mathbf{P}_3$, the failure probability of the structural system and its components appear in both the objective function and the constraint set definition. There is no transparent approximating semi-infinite optimization problem available, as was the case for $\mathbf{P}_1$ and $\mathbf{P}_2$. Through a series of steps involving transcription and approximations, we

construct parameterized approximating problems that are solvable by either existing or new optimization algorithms. The approximating problems are parameterized and a procedure similar to the one described for $\mathbf{P}_1$ and $\mathbf{P}_2$ is adopted. Approximate solutions of $\mathbf{P}_3$ can be obtained by solving the approximating optimization problems for different values of the parameters. The parameters are adjusted based on separate calculations of the failure probability.

1.4 Organization of Report

Following this introductory chapter, we describe the fundamentals of structural reliability (Chapter 2) and optimization theory (Chapter 3). Chapter 2 gives a brief overview of the first-order reliability method (FORM), the second-order reliability method (SORM), and different versions of simulation methods. A discussion of time-variant reliability models is also included. Chapter 3 presents the well-known optimality conditions for nonlinear and semi-infinite optimization. We describe the Polak-He algorithm for constrained optimization involving finite max-functions, and its specialization for solving the optimization problem arising in the FORM. Chapter 3 ends with a section on algorithms for solving semi-infinite optimization problems.

Chapter 4 contains a collection of new algorithms for solving reliability-based optimal structural design problems. First, we consider the case with failure probabilities in either the objective function or the constraint set definition, i.e., $\mathbf{P}_1$ and $\mathbf{P}_2$. We derive first-order approximating problems in the form of semi-infinite optimization problems. In special cases, the approximating problems are identical to the original ones. We present several algorithms for the solution of the original design problems. Second, we consider the case with failure probabilities in both the objective and the constraint set definition, i.e., $\mathbf{P}_3$. The beginning part of this section addresses the special situation with only one failure mode. Transcription and approximations are performed to reach a first-order approximating problem that can be solved by existing algorithms or by the algorithm derived in Chapter 5. We describe algorithms that are based on a sequential solution of such approximating problems. The last part of the

section presents transcription and approximations that are applicable to the general problem with series structural system failure probabilities in both the objective and the constraint functions. We present two algorithms for the solution of the general problem, which are based on the sequential solution of the approximating problems.

In Chapter 5, we develop an implementable algorithm for the solution of a class of generalized semi-infinite min-max problems. Problems of this form arise in the solution of reliability-based optimal design problems, as formulated in Chapter 4, as well as in other applications. First we use exact penalties to convert a generalized semi-infinite min-max problem into a finite family of semi-infinite min-max-min problems. Second, the inner min-function is smoothed and the semi-infinite max part is approximated, using discretization, to obtain a three-parameter family of finite min-max problems. Under a calmness assumption, we show that when the penalty is sufficiently large the semi-infinite min-max-min problems have the same solutions as the original problem, and that when the smoothing and discretization parameters go to infinity the solutions of the finite min-max problems converge to solutions of the original problem, provided the penalty parameter is sufficiently large. Our algorithm combines tests for adjusting the penalty, the smoothing and discretization parameters, and makes use of a min-max algorithm as a subroutine. In effect, the min-max algorithm is applied to a sequence of gradually better-approximating min-max problems, with the penalty parameter eventually stopping to increase, but the smoothing and discretization parameters driven to infinity.

Chapter 6 presents a collection of numerical design examples from the field of structural engineering. The first example considers the design of a short column subject to axial forces and bending moments. Three cases are computed: Minimize the weight of the column subject to a constraint on the failure probability; minimize the failure probability subject to constraints on the weight of the column; and minimize the initial cost plus the expected cost of failure with a constraint on the failure probability. The second example considers the design of an offshore jacket platform for oil production subject to wave, wind, and service loads. The platform is designed for minimum failure probability subject to constraints on the total weight of

the structure. The first two examples consider one failure mode. The third example considers the design of a structural frame with three failure modes. We compute both the minimum weight with a system failure probability constraint and the minimum initial cost plus expect cost of failure with a system failure probability constraint. The fourth example considers a reinforced concrete girder in a highway bridge. We take four failure modes into account. First, we minimize the cost of the design subject to a system failure probability constraint. Second, we minimize the initial cost plus the expected cost of failure, with a constraint on the system failure probability. Third, we also include the effect of deterioration of the girder caused by corrosion. We minimize the initial cost plus the expected cost of failure based on the time-variant failure probability. Fourth, we find the initial design and the maintenance effort for the deteriorating girder that minimizes the life-cycle cost.

Chapter 7 summarizes the major findings in this report and points to future work.

Chapter 2

Theory of Structural Reliability

Structures are characterized by their high reliability and their great variety of types. Hence, there is an extremely limited amount of empirical data available about failures of specific types of structures. Consequently, the probabilistic model of a structure must be based on the physics of the situation using the framework of the theory of structural reliability, rather than empirical data.

This chapter presents the parts of the theory of structural reliability that are essential for the derivations of algorithms for reliability-based optimal design of structures. The overview is, to a large extent, based on Madsen *et al.* (1986) and Ditlevsen and Madsen (1996).

2.1 Time-invariant Reliability

Suppose that a time-invariant probabilistic model of a structure and its environment is defined in terms of an m-dimensional vector of random variables $\mathbf{V}$, with joint probability density function $f_{\mathbf{V}}(\cdot, \mathbf{x})$, where $\mathbf{x}$ is an n-dimensional vector of deterministic design variables. Failure of the structure is usually defined in terms of one or more limit-state functions $G_k : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $k \in \mathrm{K} \triangleq \{1, 2, ..., K\}$. As a function of $\mathbf{x}$ and realizations $\mathbf{v}$ of the random vector $\mathbf{V}$, the limit-state functions $G_k(\cdot, \cdot)$ describe the performance of the structure with respect to specific require-

ments. By convention, a limit-state function is non-positive whenever the associated performance requirement is violated. Note that some design variables may denote parameters in the distributions of the random variables $\mathbf{V}$, e.g., mean values of $\mathbf{V}$, and other design variables may denote parameters in the description of the structure, e.g., sizes of structural members.

As will be seen below, several computational reliability methods require a bijective transformation of realizations $\mathbf{v}$ of the random vector $\mathbf{V}$ into realizations $\mathbf{u}$ of a standard normal random vector $\mathbf{U}$. Such transformations can be defined under weak assumptions. For a given design vector $\mathbf{x}$, let $T_{\mathbf{x}} : \mathbb{R}^m \to \mathbb{R}^m$ be this transformation. Replacing $\mathbf{v}$ by $T_{\mathbf{x}}^{-1}(\mathbf{u})$, gives the equivalent limit-state functions $g_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $k \in \mathbf{K}$, defined by $g_k(\mathbf{x}, \mathbf{u}) \triangleq G_k(\mathbf{x}, T_{\mathbf{x}}^{-1}(\mathbf{u}))$.

A limit-state function $g_k(\cdot, \cdot)$, together with the rule that $g_k(\mathbf{x}, \mathbf{u}) \leq 0$ is defined as failure and $g_k(\mathbf{x}, \mathbf{u}) > 0$ is defined as safe, is referred to as a component. A component may or may not be associated with a physical component or a particular failure mode of the structure.

As a function of the design variables $\mathbf{x}$, we define the k-th component failure probability by

$$p_k(\mathbf{x}) \triangleq \int_{\Omega_k(\mathbf{x})} \varphi(\mathbf{u})\, d\mathbf{u}, \quad k \in \mathbf{K}, \tag{2.1.1}$$

where $\varphi(\cdot)$ is the m-dimensional standard normal probability density function and

$$\Omega_k(\mathbf{x}) \triangleq \{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) \leq 0\}. \tag{2.1.2}$$

Hence, $\Omega_k(\mathbf{x})$ is the k-th component failure domain for the structure, i.e., the domain in the outcome space of $\mathbf{U}$ where the performance requirement associated with the k-th limit-state function is violated. A realization $\mathbf{u}$ is a failure event for the k-th component whenever $\mathbf{u} \in \Omega_k(\mathbf{x})$.

We define the *critical failure component* for the design $\mathbf{x}$ to be the component with highest failure probability, i.e., the k'-th component, $k' \in \mathbf{K}$, is a critical failure component if and only if $p_{k'}(\mathbf{x}) \geq p_k(\mathbf{x})$ for all $k \in \mathbf{K}, k \neq k'$.

The collection of components, together with a rule saying what combinations of component failures constitute a system failure, is referred to as a structural system.

The system failure probability of the structure is defined by

$$p(\mathbf{x}) \triangleq \int_{\Omega(\mathbf{x})} \varphi(\mathbf{u})\, d\mathbf{u}, \tag{2.1.3}$$

where $\Omega(\mathbf{x}) \subset \mathbb{R}^m$ is the failure domain for the structure, i.e., the domain in the outcome space of $\mathbf{U}$ where performance requirements constituting structural system failure are violated. Hence the realization $\mathbf{u}$ is a failure event for the structure whenever $\mathbf{u} \in \Omega(\mathbf{x})$. We say the probabilistic model of the structure is a series structural system, whenever the failure domain is given by

$$\Omega(\mathbf{x}) \triangleq \bigcup_{k \in \mathbf{K}} \{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) \leq 0\}. \tag{2.1.4}$$

This report deals exclusively with series structural systems. If the right-hand side of (2.1.4) is replaced by

$$\bigcap_{k \in \mathbf{K}} \{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) \leq 0\}, \tag{2.1.5}$$

then we say the probabilistic model of the structure is a parallel structural system. The probabilistic model of a structure is a general structural system, if the definition of the failure domain involves both union an intersection operations.

2.1.1 First-Order and Second-Order Reliability Methods

A first-order approximation to $p_k(\mathbf{x}), k \in \mathbf{K}$, is obtained by linearizing the limit-state function $g_k(\mathbf{x}, \cdot)$ at the point in the set $\{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) = 0\}$ closest to origin. Let $\mathbf{u}_k^*(\mathbf{x})$ be such a closest point, i.e.,

$$\mathbf{u}_k^*(\mathbf{x}) \in \underset{\mathbf{u} \in \mathbb{R}^m}{\arg\min} \left\{ \frac{1}{2}\|\mathbf{u}\|^2 \mid g_k(\mathbf{x}, \mathbf{u}) \leq 0 \right\}. \tag{2.1.6}$$

Such closest points are referred to as *design points*. Note that in (2.1.6) we have assumed $g_k(\mathbf{x}, 0) > 0$ for all relevant design vectors $\mathbf{x}$. Effectively, this assumes that the failure probability $p_k(\mathbf{x})$ is smaller than about 0.5. Given the high reliability expected of structural components, this assumption does not impose a restriction on any practical problem.

The optimization problem in (2.1.6) is well-defined as long as $g_k(\mathbf{x}, \cdot)$ is continuous. However, (2.1.6) can only be solved efficiently when $g_k(\mathbf{x}, \cdot)$ is continuously differentiable. Under this condition, in addition to the Polak-He algorithm described in Chapter 3, (2.1.6) can be solved by standard nonlinear optimization algorithms such as NLPQL (Schittkowski 1985), DOT (Vanderplaats 1992), LANCELOT (Conn *et al.* 1992), CFSQP (Lawrence *et al.* 1997), NPSOL (Gill *et al.* 1998), SNOPT (Gill *et al.* 1998), and MINOS (Murtagh and Saunders 1998). The optimization problem in (2.1.6) has also been attempted solved by the iHLRF-algorithm Zhang and Der Kiureghian (1997).

It can be shown that the first-order approximation of the component failure probability takes the form

$$p_k(\mathbf{x}) \approx \Phi(-\beta_{1,k}(\mathbf{x})), \tag{2.1.7}$$

where

$$\beta_{1,k}(\mathbf{x}) \stackrel{\triangle}{=} \|\mathbf{u}_k^*(\mathbf{x})\| \tag{2.1.8}$$

is the first-order reliability index and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Equality holds in (2.1.7) when $g_k(\mathbf{x}, \cdot)$ is affine in $\mathbf{u}$.

Similarly, a first-order approximation can be defined for $p(\mathbf{x})$ in the case of series and parallel structural systems. For brevity, we will only give the expression for series structural systems, i.e., the case when (2.1.4) holds. Then,

$$p(\mathbf{x}) \approx 1 - \Phi_K(\mathbf{B}_1(\mathbf{x}), \mathbf{R}(\mathbf{x})), \tag{2.1.9}$$

where $\Phi_K(\cdot)$ is the K-dimensional standard normal cumulative distribution function, $\mathbf{B}_1(\mathbf{x}) \stackrel{\triangle}{=} (\beta_{1,1}(\mathbf{x}), ..., \beta_{1,K}(\mathbf{x}))$ is a K-dimensional vector, and $\mathbf{R}(\mathbf{x})$ is a $K \times K$ correlation matrix with elements

$$R_{k,l} = \langle\ \mathbf{u}_k^*(\mathbf{x})/\|\mathbf{u}_k^*(\mathbf{x})\|,\ \mathbf{u}_l^*(\mathbf{x})/\|\mathbf{u}_l^*(\mathbf{x})\|\ \rangle,\ k, l \in \mathbf{K}. \tag{2.1.10}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product. Bounding formulas for structural system reliability are also available (see Ditlevsen and Madsen (1996) and Song and Der Kiureghian (2002)).

A second-order approximation to the component failure probability $p_k(\mathbf{x}), k \in \mathbf{K}$, can be obtained by constructing a quadratic surface through the point $\mathbf{u}_k^*(\mathbf{x})$ defined in (2.1.6). The simplest formula based on such quadratic surfaces yields the approximation (Breitung 1984)

$$p_k(\mathbf{x}) \approx \Phi(\beta_{1,k}(\mathbf{x})) \prod_{j=1}^{m-1} (1 + \beta_{1,k}(\mathbf{x})\kappa_j(\mathbf{x}))^{-1/2}, \tag{2.1.11}$$

where $\kappa_j(\mathbf{x})$, $j = 1, ..., m-1$, are the main curvatures of the limit-state surface $\{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}) = 0\}$ at $\mathbf{u}_k^*(\mathbf{x})$. Other second-order approximations can be found in Hohenbichler *et al.* (1987), Tvedt (1983, Tvedt (1990), Der Kiureghian *et al.* (1987), and Der Kiureghian and De Stefano (1991).

2.1.2 Monte Carlo Simulation

The integrals defining $p_k(\mathbf{x})$ and $p(\mathbf{x})$ in (2.1.1) and (2.1.3), respectively, can be evaluated approximately by means of Monte Carlo Simulation. Let $\mathbf{u}_1, ..., \mathbf{u}_N$ be a family of simulated realizations of a collection of statistically independent standard normal random vectors $\mathbf{U}_1, ..., \mathbf{U}_N$. An approximation of $p(\mathbf{x})$ is given by

$$p(\mathbf{x}) \approx \frac{1}{N} \sum_{i=1}^{N} I_{\Omega(\mathbf{x})}(\mathbf{u}_i), \tag{2.1.12}$$

where $I_{\Omega(\mathbf{x})}(\mathbf{u}_i) = 1$ whenever $\mathbf{u}_i \in \Omega(\mathbf{x})$, and $I_{\Omega(\mathbf{x})}(\mathbf{u}_i) = 0$ otherwise. The coefficient of variation of the estimator $\hat{p}(\mathbf{x}) = \sum_{i=1}^{N} I_{\Omega(\mathbf{x})}(\mathbf{U}_i)/N$ is found to be equal to $\sqrt{(1 - p(\mathbf{x}))/(Np(\mathbf{x}))}$. Thus, a large number of simulations is necessary to accurately estimate small $p(\mathbf{x})$. A similar approximation of $p_k(\mathbf{x})$ can be obtained by replacing $\Omega(\mathbf{x})$ by $\Omega_k(\mathbf{x})$ in (2.1.12).

If the contributions to the integrals in (2.1.1) and (2.1.3) come from a known region in $\mathbb{R}^m$, the computational efficiency of the crude Monte Carlo Simulation can be improved by use of Importance Sampling. Let $h(\mathbf{w})$ be an m-dimensional joint probability density function, which is nonzero on $\Omega(\mathbf{x})$. Now, let $\mathbf{w}_1, ..., \mathbf{w}_N$ be a family of simulated realizations of a collection of statistically independent random

vectors $\mathbf{W}_1, ..., \mathbf{W}_N$ with probability density $h(\mathbf{w})$. An approximation of $p(\mathbf{x})$ is given by

$$p(\mathbf{x}) \approx \frac{1}{N} \sum_{i=1}^{N} I_{\Omega(\mathbf{x})}(\mathbf{w}_i) \frac{\varphi(\mathbf{w}_i)}{h(\mathbf{w}_i)}, \tag{2.1.13}$$

where $\varphi(\cdot)$ is the m-dimensional standard normal probability density function. If the main contribution to the integral (2.1.3) comes from a region in $\mathbb{R}^m$ where many realizations of the random vector with density $h(\cdot)$ occur, then it can be shown that fewer simulations are required compared to crude Monte Carlo Simulation. A similar approximation of $p_k(\mathbf{x})$ can be obtained by replacing $\Omega(\mathbf{x})$ by $\Omega_k(\mathbf{x})$ in (2.1.13). In practice, $h(\mathbf{w})$ is usually selected as a multi-variate normal probability density centered at the design point defined in (2.1.6). In case of multiple design points, a composite density formulated as a weighted sum of multi-variate normal densities centered at multiple design points can be used (Melchers 1989).

2.2 Time-variant Reliability

Most real-life structures experience changing material properties and load environment over time. It is beyond the scope of this report to discuss how such conditions can be modeled. However, we briefly describe a model used in the numerical examples in Chapter 6.

Suppose that a probabilistic model of a structure is defined in terms of an m-dimensional stochastic vector process $\mathbf{V}(t)$. Similar to Section 2.1, we can define the failure of the structure in terms of *point-in-time* limit-state functions $G_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $k \in \mathrm{K}$. Note that $G_k(\cdot, \cdot, \cdot)$ has three arguments: the design variables $\mathbf{x}$, realizations $\mathbf{v}(t)$ of the stochastic vector process $\mathbf{V}(t)$, and the time t. The point-in-time limit-state functions describe the performance of the structure with respect to specific requirements at a specific time.

As seen above, several computational reliability methods require a fixed time, bijective transformation of realizations $\mathbf{v}(t)$ of the stochastic vector process $\mathbf{V}(t)$ into realizations $\mathbf{u}(t)$ of the stochastic vector process $\mathrm{U}(t)$, which for each point in

time is standard normal. For a given design vector $\mathbf{x}$ and time t, let $T_{\mathbf{x},t} : \mathbb{R}^m \to \mathbb{R}^m$ be this transformation. Replacing $\mathbf{v}(t)$ by ${T_{\mathbf{x},t}}^{-1}(\mathbf{u}(t))$, gives the equivalent point-in-time limit-state function $g_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $k \in \mathbf{K}$, defined by $g_k(\mathbf{x}, \mathbf{u}(t), t) \stackrel{\triangle}{=} G_k(\mathbf{x}, {T_{\mathbf{x},t}}^{-1}(\mathbf{u}(t)), t)$.

As a function of the design variables $\mathbf{x}$ and time t, we define the k-th point-in-time component failure probability by

$$p_k(\mathbf{x}, t) \stackrel{\triangle}{=} \int_{\Omega_k(\mathbf{x},t)} \varphi(\mathbf{u})\, d\mathbf{u},\ k \in \mathbf{K}, \tag{2.2.1}$$

where $\varphi(\cdot)$ is the m-dimensional standard normal probability density function and

$$\Omega_k(\mathbf{x}, t) \stackrel{\triangle}{=} \{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}, t) \leq 0\}. \tag{2.2.2}$$

Hence, $\Omega_k(\mathbf{x}, t)$ is the k-th point-in-time component failure domain for the structure at time t, i.e., the domain in the outcome space of $\mathbf{U}(t)$, where the performance requirement associated with the k-th limit-state function is violated at time t. A realization $\mathbf{u}(t)$ is a failure event at t for the k-th component whenever $\mathbf{u}(t) \in \Omega_k(\mathbf{x}, t)$.

Similarly, the point-in-time system failure probability of the structure is defined by

$$p(\mathbf{x}, t) \stackrel{\triangle}{=} \int_{\Omega(\mathbf{x},t)} \varphi(\mathbf{u})\, d\mathbf{u}, \tag{2.2.3}$$

where $\Omega(\mathbf{x}, t) \subset \mathbb{R}^m$ is the failure domain for the structure at time t, i.e., the domain in the outcome space of $\mathbf{U}(t)$, where performance requirements constituting structural system failure are violated at time t. Hence the realization $\mathbf{u}(t)$ is a failure event at t for the structure whenever $\mathbf{u}(t) \in \Omega(\mathbf{x}, t)$. For series structural system, the point-in-time failure domain is given by

$$\Omega(\mathbf{x}, t) \stackrel{\triangle}{=} \bigcup_{k \in \mathbf{K}} \{\mathbf{u} \in \mathbb{R}^m \mid g_k(\mathbf{x}, \mathbf{u}, t) \leq 0\}. \tag{2.2.4}$$

For fixed time, the point-in-time failure probabilities become "time-invariant," and hence can be computed using any of the methods described in Section 2.1.

In many applications, the failure probability in a time-interval, $\mathbf{T} \stackrel{\triangle}{=} [t_1, t_2] \subset \mathbb{R}$, say, is of more interest than the point-in-time failure probabilities $p(\mathbf{x}, t)$ and $p_k(\mathbf{x}, t)$.

Let the *time-interval* component and series structural system failure probabilities be defined by

$$P_k(\mathbf{x}, \mathbf{T}) \triangleq Prob[\{\min_{t\in\mathbf{T}} g_k(\mathbf{x}, \mathbf{U}(t), t) \leq 0\}],\ k \in \mathbf{K}, \tag{2.2.5}$$

and

$$P(\mathbf{x}, \mathbf{T}) \triangleq Prob\left[\bigcup_{k\in\mathbf{K}} \{\min_{t\in\mathbf{T}} g_k(\mathbf{x}, \mathbf{U}(t), t) \leq 0\}\right], \tag{2.2.6}$$

respectively, where $Prob[A]$ denotes the probability of event A.

The time-interval failure probability is difficult to compute, but it can be estimated by various techniques, see, e.g., Breitung (1988), Li and Der Kiureghian (1995), Lutes and Sarkani (1997), Rackwitz (1998), and Der Kiureghian (2000). The time-interval component failure probability can be estimated by means of the outcrossing rate for the structure, where outcrossing is defined as the event that $g_k(\mathbf{x}, \mathbf{U}(t), t) > 0$ at time t and $g_k(\mathbf{x}, \mathbf{U}(t + dt), t + dt) \leq 0$ at time $t + dt$, with dt being an infinitesimal time increment. Hence, the mean outcrossing rate $\nu_k(t)$ for the k-th component is given by

$$\nu_k(t) \triangleq \lim_{\Delta t\to 0} \frac{Prob[\{g_k(\mathbf{x}, \mathbf{U}(t), t) > 0\} \bigcap \{g_k(\mathbf{x}, \mathbf{U}(t + \Delta t), t + \Delta t) \leq 0\}]}{\Delta t}. \tag{2.2.7}$$

By use of a Poisson model of outcrossing events, we obtain the approximation

$$P_k(\mathbf{x}, \mathbf{T}) \approx 1 - \exp\left(-\int_{t_1}^{t_2} \nu_k(t)dt\right) \tag{2.2.8}$$

and the upper bound

$$P_k(\mathbf{x}, \mathbf{T}) \leq \int_{t_1}^{t_2} \nu_k(t)dt. \tag{2.2.9}$$

A lower bound on the time-interval component failure probability can also be obtain by considering a series structural system of point-in-time events. The time-interval series structural system failure probability can be approximated by use of similar expressions to the ones in (2.2.8) and (2.2.9).

In some time-variant reliability problems, it is known that the limit-state function monotonically decreases with time. This may occur if the limit-state function is in terms of a damage measure that monotonically increases with time, e.g., crack size

in fatigue analysis. It may also occur if the limit-state function describes the performance of a decaying structure, e.g., a concrete structure in a corrosive environment. In such cases,

$$\min_{t\in\mathbf{T}} g_k(\mathbf{x},\mathbf{u}(t),t) = g_k(\mathbf{x},\mathbf{u}(t_2),t_2), \tag{2.2.10}$$

and the time-variant reliability problem converts to a time-invariant one, i.e., $P_k(\mathbf{x},\mathbf{T}) = p_k(\mathbf{x},t_2)$. Similarly, the time-interval series structural system failure probability is in such cases given by a point-in-time structural series system failure probability, i.e.,

$$P(\mathbf{x},\mathbf{T}) = p(\mathbf{x},t_2). \tag{2.2.11}$$

Chapter 3

Optimization Theory and Algorithms

This chapter defines the essential concepts in optimization theory, presents one algorithm for solving constrained optimization problems involving finite max-functions, and describes three algorithms for the solution of semi-infinite optimization problems. The presentation is based primarily on Polak (1997).

This chapter is self-contained, i.e., it does not use notation defined in other chapters. Any notation established in this chapter does not carry over to other chapters.

We first define some basic terms from mathematical analysis: (i) A set $\mathbf{A} \subset \mathbb{R}^n$ is open if and only if for every point $\mathbf{x}^* \in \mathbf{A}$ there exists a $\rho > 0$ such that $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}^*\| \leq \rho\} \subset \mathbf{A}$. (ii) A set $\mathbf{A} \subset \mathbb{R}^n$ is closed if and only if $\mathbf{A}^c$ is open, where ^{c} denotes the compliment. (iii) A set $\mathbf{A} \subset \mathbb{R}^n$ is bounded if and only if there exists a $\rho > 0$ such that $\mathbf{A} \subset \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq \rho\}$. (iv) A set $\mathbf{A} \subset \mathbb{R}^n$ is compact if and only if it is bounded and closed. (v) The interior of a set $\mathbf{A} \subset \mathbb{R}^n$ is equal to the union of all open sets contained in $\mathbf{A}$. (vi) A set $\mathbf{A} \subset \mathbb{R}^n$ has an interior if and only if the interior of $\mathbf{A}$ is non-empty. (vii) A set $\mathbf{A} \subset \mathbb{R}^n$ is convex if and only if for any $\mathbf{x}', \mathbf{x}'' \in \mathbf{A}$ and $\lambda \in [0, 1]$, $(\lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}'') \in \mathbf{A}$. (viii) Consider the set $\mathbf{A} \subset \mathbb{R}^n$. We say that conv $\mathbf{A}$ is the convex hull of $\mathbf{A}$ if it is the smallest convex set containing $\mathbf{A}$. If $\mathbf{A}$ is given in terms of a collection of points $\mathbf{v}(\mathbf{x}), \mathbf{x} \in \mathbf{X}$, i.e.,

$\mathbf{A} = \{\mathbf{w} \in \mathbb{R}^m \mid \mathbf{w} = \mathbf{v}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\}$, with $\mathbf{X}$ some set, then we simplify the notation by setting $\text{conv}\,\{\mathbf{w} \in \mathbb{R}^m \mid \mathbf{w} = \mathbf{v}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\} = \text{conv}_{\mathbf{x}\in\mathbf{X}}\,\mathbf{v}(\mathbf{x})$. (xi) If a subsequence of a sequence converges to a point, then we say that that point is an accumulation point of the sequence. (x) A function is continuously differentiable if it has continuous derivatives. (xi) A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on the set $\mathbf{X} \subset \mathbb{R}^n$ if and only if there exists an $L < \infty$ such that $\|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x}'')\| \leq L\|\mathbf{x}' - \mathbf{x}''\|$ for all $\mathbf{x}', \mathbf{x}'' \in \mathbf{A}$.

3.1 Optimality Conditions

Generally, optimization algorithms can only be shown to converge to stationary points, i.e., points satisfying an optimality condition, which, hopefully, will be local or global minimizers for the given problem. Standard first-order optimality conditions for inequality constrained and semi-infinite optimization are presented in Theorems 3.1.1 and 3.1.2 below. We refer to a point satisfying the constraints of an optimization problem as feasible with respect to that problem.

The inequality constrained problem has the form

$$\textbf{ICP} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \{f_0(\mathbf{x}) \mid f(\mathbf{x}) \leq 0\}, \tag{3.1.1}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ are defined by

$$f_0(\mathbf{x}) \stackrel{\triangle}{=} \max_{k\in\mathbf{p}} c_k(\mathbf{x}), \tag{3.1.2}$$

$$f(\mathbf{x}) \stackrel{\triangle}{=} \max_{j\in\mathbf{q}} f_j(\mathbf{x}), \tag{3.1.3}$$

with $\mathbf{p} \stackrel{\triangle}{=} \{1, 2, ..., p\}$ and $\mathbf{q} \stackrel{\triangle}{=} \{1, 2, ..., q\}$. The standard first-order necessary optimality conditions for **ICP**, also called Karush-Kuhn-Tucker conditions, are given in the next theorem.

Theorem 3.1.1. *Consider the problem* **ICP**. *Suppose that the functions* $f_j : \mathbb{R}^n \to \mathbb{R}, j \in \mathbf{q}$, *and the functions* $c_k : \mathbb{R}^n \to \mathbb{R}, k \in \mathbf{p}$, *are continuously differentiable. If*

$\mathbf{x}^*$ *is a local solution of* **ICP**, *then there exist multiplier vectors* $\mu = (\mu_0, \mu_1, ..., \mu_q)$, *with* $\mu_j \geq 0$, $j \in \{0, 1, ..., q\}$, *and* $\sum_{j=0}^{q} \mu_j = 1$, *and* $\nu = (\nu_1, ..., \nu_p)$, *with* $\nu_k \geq 0$, $k \in \mathbf{p}$, *and* $\sum_{k=1}^{p} \nu_k = 1$ *such that*

$$\mu_0 \left[\sum_{k=1}^{p} \nu_k \nabla c_k(\mathbf{x}^*) \right] + \sum_{j=1}^{q} \mu_j \nabla f_j(\mathbf{x}^*) = 0 \tag{3.1.4}$$

and

$$\mu_0 \left[\sum_{k=1}^{p} \nu_k [c_k(\mathbf{x}^*) - f_0(\mathbf{x}^*)] \right] + \sum_{j=1}^{q} \mu_j f_j(\mathbf{x}^*) = 0. \tag{3.1.5}$$

□

The semi-infinite optimization problem has the form

$$\textbf{SIP} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \{\psi_0(\mathbf{x}) \mid \psi(\mathbf{x}) \leq 0\}, \tag{3.1.6}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi(\mathbf{x}) \stackrel{\triangle}{=} \max_{j \in \mathbf{q}} \psi_j(\mathbf{x}), \tag{3.1.7}$$

and $\psi_j : \mathbb{R}^n \to \mathbb{R}, j = \{0, 1, ...q\}$, are given by

$$\psi_j(\mathbf{x}) \stackrel{\triangle}{=} \max_{\mathbf{y} \in \mathbf{Y}_j} \phi_j(\mathbf{x}, \mathbf{y}), \tag{3.1.8}$$

where $\phi_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{Y}_j \subset \mathbb{R}^m$ are sets that may have interiors. The reason such problems are called semi-infinite is that the design vector $\mathbf{x}$ is finite-dimensional, but there is an infinite number of functions $\phi_j(\cdot, \mathbf{y})$, determined by all the $\mathbf{y} \in \mathbf{Y}_j$. First-order necessary optimality conditions for **SIP** is given in Theorem 3.1.2 below. As seen from this theorem, a stationary point of **SIP** is a point $\mathbf{x}^*$ that makes the zero-vector to be contained in a specific convex hull ($\bar{G}(\mathbf{x}^*)$).

Theorem 3.1.2. *Consider the problem* **SIP**. *Suppose that the functions* $\phi_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, j \in \{0, 1, ..., q\}$, *are continuously differentiable, and that the subsets* $\mathbf{Y}_j$ *are compact. If* $\mathbf{x}^*$ *is a local solution of* **SIP**, *then*

$$\mathbf{0} \in \bar{G}(\mathrm{x}^*), \tag{3.1.9}$$

where $\bar{G}(\mathrm{x}^*) \subset \mathbb{R}^{n+1}$ *is defined by*

$$\bar{G}(\mathrm{x}^*) \stackrel{\triangle}{=} \operatorname*{conv}_{j\in\{0,1,\ldots,q\}} \bar{G}_j(\mathrm{x}^*), \tag{3.1.10}$$

with

$$\bar{G}_0(\mathrm{x}^*) \stackrel{\triangle}{=} \operatorname*{conv}_{\mathrm{y}\in\mathrm{Y}_0} \left\{ \begin{pmatrix} \psi_0(\mathrm{x}^*) - \phi_0(\mathrm{x}^*, \mathrm{y}) + \gamma\psi(\mathrm{x}^*)_+ \\ \nabla_{\mathrm{x}}\phi_0(\mathrm{x}^*, \mathrm{y}) \end{pmatrix} \right\}, \tag{3.1.11}$$

$\gamma > 0$ *an arbitrary parameter,* $\psi(\mathrm{x}^*)_+ \stackrel{\triangle}{=} \max\{0, \psi(\mathrm{x}^*)\}$, *and for* $j \in \mathbf{q}$,

$$\bar{G}_j(\mathrm{x}^*) \stackrel{\triangle}{=} \operatorname*{conv}_{\mathrm{y}\in\mathrm{Y}_j} \left\{ \begin{pmatrix} \psi(\mathrm{x}^*)_+ - \phi_j(\mathrm{x}^*, \mathrm{y}) \\ \nabla_{\mathrm{x}}\phi_j(\mathrm{x}^*, \mathrm{y}) \end{pmatrix} \right\}. \tag{3.1.12}$$

□

3.2 Polak-He Algorithm

In the following, we describe a Phase I - Phase II method of centers for solving ICP. We need the notation

$$F(\mathrm{x}', \mathrm{x}'') \stackrel{\triangle}{=} \max\{f_0(\mathrm{x}'') - f_0(\mathrm{x}') - \gamma f(\mathrm{x}')_+,\ f(\mathrm{x}'') - f(\mathrm{x}')_+\}, \tag{3.2.1}$$

where $\gamma > 0$ is a parameter and $f(\mathrm{x})_+ \stackrel{\triangle}{=} \max\{0, f(\mathrm{x})\}$, and

$$\Sigma_q^0 \stackrel{\triangle}{=} \{\mu = (\mu_0, \mu_1, ..., \mu_q) \in \mathbb{R}^{q+1} \mid \mu_j \geq 0, j \in \{0, 1, ..., q\},\ \sum_{j=0}^{q} \mu_j = 1\}, \tag{3.2.2}$$

$$\Sigma_p \stackrel{\triangle}{=} \{\nu = (\nu_1, ..., \nu_p) \in \mathbb{R}^{p+1} \mid \nu_k \geq 0, k \in \mathbf{p},\ \sum_{k=1}^{p} \nu_k = 1\}. \tag{3.2.3}$$

Polak-He Algorithm 3.2.1. (Solves ICP)

Parameters. $\alpha \in (0,1]$, $\beta \in (0,1)$, $\delta, \gamma > 0$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$.

Step 0. Set $i = 0$.

Step 1. Compute

$$\begin{aligned} \theta_i &= \theta(\mathbf{x}_i) \stackrel{\triangle}{=} - \min_{\mu \in \Sigma_q^0,\ \nu \in \Sigma_p} \left\{ \mu_0 \sum_{k=1}^{p} \nu_k [f_0(\mathbf{x}_i) - c_k(\mathbf{x}_i) + \gamma f(\mathbf{x}_i)_+] \right. \\ &+ \left. \sum_{j=1}^{q} \mu_j [f(\mathbf{x}_i)_+ - f_j(\mathbf{x}_i)] + \frac{1}{2\delta} \| \mu_0 \sum_{k=1}^{p} \nu_k \nabla c_k(\mathbf{x}_i) + \sum_{j=1}^{q} \mu_j \nabla f_j(\mathbf{x}_i) \|^2 \right\}, \end{aligned} \tag{3.2.4}$$

and

$$\mathbf{h}_i = -\frac{1}{\delta} \left\{ \mu_0^* \sum_{k=1}^{p} \nu_k^* \nabla c_k(\mathbf{x}_i) + \sum_{j=1}^{q} \mu_j^* \nabla f_j(\mathbf{x}_i) \right\}, \tag{3.2.5}$$

where (μ^*, ν^*) is any solution of (3.2.4).

Step 2. Compute the Armijo step-size

$$\lambda_i = \max_{s \in \mathbb{N}} \{ \beta^s \mid F(\mathbf{x}_i, \mathbf{x}_i + \beta^s \mathbf{h}_i) \leq \alpha \beta^s \theta_i \}. \tag{3.2.6}$$

Step 3. Set $\mathbf{x}_{i+1} = \mathbf{x}_i + \lambda_i \mathbf{h}_i$, replace i by $i+1$, and go to **Step 1**. □

Note that θ_i in (3.2.4) is a quadratic program in the variables (μ, ν) with linear constraints, and hence can be solved in a finite number of iterations.

Due to the nature of **ICP**, Polak-He Algorithm 3.2.1 may require, like any other optimization algorithm, an infinite number of iterations before it converges to a stationary points. Hence, in practical problem solving we need a stopping rule for terminating the calculations. We have not specified a stopping rule for Polak-He Algorithm 3.2.1. However, we recommend to use one of the standard stopping rules: terminate the calculations when $\|\mathbf{x}_{i+1} - \mathbf{x}_i\| \leq \epsilon$, $F(\mathbf{x}_i, \mathbf{x}_{i+1}) \geq -\epsilon$, and/or $\theta(\mathbf{x}_i) \geq -\epsilon$, where

$\epsilon > 0$ is a pre-defined parameter (e.g., 10^{-6}). The final iterate will then be an approximation to a stationary point, with ϵ determining the precision of the approximation.

The next theorem shows that points generated by Polak-He Algorithm 3.2.1 converge to a stationary point of the type defined in Theorem 3.1.1.

Theorem 3.2.1. *Consider the problem* **ICP**. *Suppose that the functions* $f_j, c_k : \mathbb{R}^n \to \mathbb{R}$, $j \in \mathbf{q}$, $k \in \mathbf{p}$, *are continuously differentiable. If* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is a sequence constructed by Polak-He Algorithm 3.2.1 in solving* **ICP**, *then any accumulation point* $\mathbf{x}^*$ *of the sequence* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *satisfies* $\theta(\mathbf{x}^*) = 0$, *defined in (3.2.4), and* $\mathbf{x}^*$ *is stationary, i.e.,* $\mathbf{x}^*$ *satisfies (3.1.4) and (3.1.5) for some* $\mu \in \Sigma_q^0, \nu \in \Sigma_p$. □

In the first-order reliability method (FORM), an optimization problem of the following form must be solved (see (2.1.6)):

$$\mathbf{P}_{FORM} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\mathbf{x}\|^2 \mid g(\mathbf{x}) \le 0 \right\}, \tag{3.2.7}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a limit-state function. For $\mathbf{P}_{FORM}$, Polak-He Algorithm 3.2.1 takes a simplified form: In (3.2.4), replace θ_i by

$$\begin{aligned} \theta_i \;=\; -\min_{0\le\mu\le 1} \;& \{\ \tfrac{1}{2}(\mathbf{x}_i^T\mathbf{x}_i - 2\mathbf{x}_i^T\nabla g(\mathbf{x}_i) + \nabla g(\mathbf{x}_i)^T\nabla g(\mathbf{x}_i))\mu^2 \\ & +(\mathbf{x}_i^T\nabla g(\mathbf{x}_i) - \nabla g(\mathbf{x}_i)^T\nabla g(\mathbf{x}_i) + g(\mathbf{x}_i))\mu \\ & +\nabla g(\mathbf{x}_i)^T\nabla g(\mathbf{x}_i) + g(\mathbf{x}_i)_+ - g(\mathbf{x}_i)\ \}, \end{aligned} \tag{3.2.8}$$

which has the solution

$$\mu^* = -\frac{\mathbf{x}_i^T\nabla g(\mathbf{x}_i) - \nabla g(\mathbf{x}_i)^T\nabla g(\mathbf{x}_i) + g(\mathbf{x}_i)}{\mathbf{x}_i^T\mathbf{x}_i - 2\mathbf{x}_i^T\nabla g(\mathbf{x}_i) + \nabla g(\mathbf{x}_i)^T\nabla g(\mathbf{x}_i)} \tag{3.2.9}$$

whenever the right-hand side of (3.2.9) is defined and has a value in $[0, 1]$. Otherwise, the solution of (3.2.8) is either $\mu^* = 0$ or $\mu^* = 1$, whichever yields the lowest value for the objective function in (3.2.8). Additionally, in (3.2.5), replace $\mathbf{h}_i$ by

$$\mathbf{h}_i = -\mu^*\mathbf{x}_i - (1-\mu^*)\nabla g(\mathbf{x}_i). \tag{3.2.10}$$

Finally, set the parameters $\gamma = \delta = 1$ in Polak-He Algorithm 3.2.1.

3.3 Semi-Infinite Optimization Algorithms

Algorithms for solving **SIP**s typically involve some form of discretization of the sets over which the maxima are computed. Subsections 3.3.1 and 3.3.2 describe such discretization methods. Subsection 3.3.3 presents an alternative algorithm for solving **SIP** for the special case with smooth constraints.

As was the case for **ICP**, **SIP** usually requires an infinite number of iterations of an optimization algorithm before convergence to a stationary points can be obtained. Hence, in practical problem solving we need a stopping rule for terminating the calculations. In the algorithms below, we have not specified a stopping rule. However, we recommend to use one of the standard stopping rules: terminate the calculations when $\|\mathbf{x}_{i+1} - \mathbf{x}_i\| \leq \epsilon$ and/or $i > i_\epsilon$, where $\epsilon > 0$ is a pre-defined parameter (e.g., 10^{-6}) and i_ϵ is the maximum number of iterations. The final iterate will then be an approximation to a stationary point.

3.3.1 Pre-defined Discretization Scheme

The following algorithm solves a sequence of **ICP**, which are gradually better approximations of **SIP**. We assume that one can construct subsets $\mathbf{Y}_{j,N} \subset \mathbf{Y}_j$, $j \in \{0, 1, ..., q\}$, with finite cardinality, which can be used in the construction of approximating problems.

Assumption 3.3.1 *We assume that there exist a strictly decreasing function* $\Delta : \mathbb{N} \to (0, \infty)$ *with the property that* $\Delta(N) \to 0$, *as* $N \to \infty$, *and constants* $N_0 \in \mathbb{N}$, $C < \infty$ *such that for every* $N \geq N_0$ *and* $\mathbf{y}_j \in \mathbf{Y}_j$, $j \in \{0, 1, ..., q\}$, *there exist* $\mathbf{y}'_j \in \mathbf{Y}_{j,N}$ *such that*

$$\|\mathbf{y}_j - \mathbf{y}'_j\| \leq C\Delta(N), \ j \in \{0, 1, ..., q\}. \tag{3.3.1}$$

□

For example, if $\mathbf{Y}_j$ is the unit cube in $\mathbb{R}^m$, i.e., $\mathbf{Y}_j = \mathbf{I}^m$, with $\mathbf{I} = [0,1]$, then we can define $\mathbf{Y}_{j,N} = \mathbf{I}_N^m$, where

$$\mathbf{I}_N = \{0, 1/a(N), 2/a(N), ..., (a(N)-1)/a(N), 1\}, \tag{3.3.2}$$

with $a(N) = 2^{N-N_0}N_0$. In this case, $\Delta(N) = 1/a(N)$ and $C = \frac{1}{2}m^{1/m}$.

Now, for $N = 1, 2, 3, ...$, we define approximations to (3.1.7) and (3.1.8) by

$$\psi_N(\mathbf{x}) \stackrel{\triangle}{=} \max_{j \in \mathbf{q}} \psi_{j,N}(\mathbf{x}), \tag{3.3.3}$$

$$\psi_{j,N}(\mathbf{x}) \stackrel{\triangle}{=} \max_{\mathbf{y} \in \mathbf{Y}_{j,N}} \phi_j(\mathbf{x}, \mathbf{y}), j \in \{0, 1, ..., q\}, \tag{3.3.4}$$

and, additionally, $\psi_N(\mathbf{x})_+ \stackrel{\triangle}{=} \max\{0,\ \psi_N(\mathbf{x})\}$. Similar to (3.2.1), for $N = 1, 2, 3, ...$, we define

$$F_N(\mathbf{x}', \mathbf{x}'') \stackrel{\triangle}{=} \max\{\psi_{0,N}(\mathbf{x}'') - \psi_{0,N}(\mathbf{x}') - \gamma\psi_N(\mathbf{x}')_+,\ \psi_N(\mathbf{x}'') - \psi_N(\mathbf{x}')_+\}, \tag{3.3.5}$$

where $\gamma > 0$ is a parameter. Finally, for $N = 1, 2, 3, ...$, and any $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$, we define

$$\tilde{\psi}_{j,N}(\mathbf{x}, \mathbf{x}+\mathbf{h}) \stackrel{\triangle}{=} \max_{\mathbf{y} \in \mathbf{Y}_{j,N}} \{\phi_j(\mathbf{x}, \mathbf{y}) + \langle \nabla_{\mathbf{x}} \phi_j(\mathbf{x}, \mathbf{y}),\ \mathbf{h}\rangle + \tfrac{1}{2}\|\mathbf{h}\|^2\}, j \in \{0, 1, ...q\}, \tag{3.3.6}$$

$$\tilde{\psi}_N(\mathbf{x}, \mathbf{x}+\mathbf{h}) \stackrel{\triangle}{=} \max_{j \in \mathbf{q}} \tilde{\psi}_{j,N}(\mathbf{x}, \mathbf{x}+\mathbf{h}), \tag{3.3.7}$$

$$\begin{aligned} \tilde{F}_N(\mathbf{x}, \mathbf{x}+\mathbf{h}) \stackrel{\triangle}{=} \max\{&\tilde{\psi}_{0,N}(\mathbf{x}, \mathbf{x}+\mathbf{h}) - \psi_{0,N}(\mathbf{x}) - \gamma\psi_N(\mathbf{x})_+, \\ &\tilde{\psi}_N(\mathbf{x}, \mathbf{x}+\mathbf{h}) - \psi_N(\mathbf{x})_+\}, \end{aligned} \tag{3.3.8}$$

$$\theta_N(\mathbf{x}) = \min_{\mathbf{h} \in \mathbb{R}^n} \tilde{F}_N(\mathbf{x}, \mathbf{x}+\mathbf{h}), \tag{3.3.9}$$

where $\gamma > 0$. It can be shown that $\theta_N(\mathbf{x})$ in (3.3.9) is of the same form as $\theta(\mathbf{x})$ in (3.2.4) (Theorem 2.2.8 in Polak (1997)), and hence it can be evaluated in a finite number of iterations.

Algorithm 3.3.1. (Solves SIP)

Parameters. $\alpha, \beta, \omega \in (0,1)$, $\gamma, \epsilon > 0$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0 \in \mathbb{N}$.

Step 0. Set $i = 0$.

Step 1. Enter inner loop.

Inner-Step 0. Set $N = N_i$.

Inner-Step 1. Compute $\theta_N(\mathbf{x}_i)$, defined in (3.3.9), and

$$\mathbf{h}_N(\mathbf{x}_i) = \arg \min_{\mathbf{h} \in \mathbb{R}^n} \tilde{F}_N(\mathbf{x}_i, \mathbf{x}_i + \mathbf{h}). \tag{3.3.10}$$

Inner-Step 2. If $\theta_N(\mathbf{x}_i) = 0$, set $\mathbf{x}^* = \mathbf{x}_i$, and go to **Inner-Step 4.**

Else, compute the step-size

$$\lambda_N(\mathbf{x}_i) = \max_{s \in \mathbb{N}} \{\beta^s \mid F_N(\mathbf{x}_i, \mathbf{x}_i + \beta^s h_N(\mathbf{x}_i)) \leq \alpha \beta^s \theta_N(\mathbf{x}_i)\}. \tag{3.3.11}$$

Inner-Step 3. Set $\mathbf{x}^* = \mathbf{x}_i + \lambda_N(\mathbf{x}_i)\mathbf{h}_N(\mathbf{x}_i)$.

Inner-Step 4. If $F_N(\mathbf{x}_i, \mathbf{x}^*) \leq -\epsilon \Delta(N)^\omega$, exit inner loop, and go to **Step 2**,

Else, replace N by $N + 1$, and go to **Inner-Step 1.**

Step 2. Set $\mathbf{x}_{i+1} = \mathbf{x}^*$, $N_{i+1} = N$, replace i by $i + 1$, and go to **Step 1.** □

The next theorem shows that points generated by Algorithm 3.3.1 converge to a stationary point of the type defined in Theorem 3.1.2.

Theorem 3.3.1. *Consider the problem* **SIP**. *Suppose that Assumption 3.3.1 holds, the functions* $\phi_j(\cdot,\cdot), j \in \{0, 1, ..., q\}$, *are continuously differentiable, the gradients* $\nabla_{\mathbf{x}} \phi_j(\cdot,\cdot), j \in \{0, 1, ..., q\}$, *are Lipschitz continuous on compact sets, and the sets* $\mathbf{Y}_j, j \in \{0, 1, ..., q\}$ *are compact. If* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is a sequence constructed by Algorithm*

3.3.1 in solving **SIP**, *then any accumulation point* $\hat{\mathbf{x}}$ *of the sequence* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is stationary, i.e.,* $0 \in \bar{G}(\hat{\mathbf{x}})$, *with* $\bar{G}(\hat{\mathbf{x}})$ *as defined in (3.1.9).* □

3.3.2 Method of Outer Approximations

The method of outer approximations solves **SIP** by constructing and solving a sequence of gradually more accurate approximating problems of the form **ICP**. As was the case in Subsection 3.3.1, the approximations are constructed by discretization of the sets $\mathbf{Y}_j, j \in \{0, 1, ..., q\}$. Instead of using a pre-defined sequence of sets $\mathbf{Y}_{j,N}, N = 1, 2, ...$, the method of outer approximations sequentially constructs the approximating sets as the algorithm progresses. The following algorithm also includes the "constraint-dropping" scheme from Gonzaga and Polak (1979), i.e., a scheme for reducing the cardinality of the sets $\mathbf{Y}_{j,N}$.

Outer Approximations Algorithm 3.3.2. (Solves SIP)

Parameters. $\alpha, \beta \in (0, 1)$.

Data. $N_0 \in \mathbb{N}$, $\mathbf{x}_{N_0} \in \mathbb{R}^n$, $\{\sigma_N\}_{N=N_0}^{\infty}$, with $\sigma_N \downarrow 0$, $\{\tau_N\}_{N=N_0}^{\infty}$, with $\tau_N \downarrow 0$, $\{\epsilon_{N,k}\}_{N=N_0, k\leq N}^{\infty}$, with $\epsilon_{N,N} = 0, \epsilon_{N,k} > 0$, for all $N, k < N$, $\epsilon_{N,k} \to \bar{\epsilon}_k$, as $N \to \infty$, uniformly in k, $\bar{\epsilon}_k > \epsilon_{N,k}$, for $N \geq k$, and $\bar{\epsilon}_k \to 0$, as $k \to \infty$.

Step 0. Set $N = N_0$.

Step 1. Compute approximate solutions $\mathbf{y}_{j,N} \in \mathbf{Y}_j$ of the problem

$$\max_{\mathbf{y}\in\mathbf{Y}_j} \psi_j(\mathbf{x}_N, \mathbf{y}) \tag{3.3.12}$$

by using Polak-He Algorithm 3.2.1. Terminate Polak-He Algorithm 3.2.1 when the left-hand side of (3.2.4) is greater than $-\sigma_N$.

Step 2. Set $\Gamma_N = \max\{0,\ \phi_j(\mathbf{x}_N, \mathbf{y}_{j,N})\ , j = 0, 1, ..., q\}$.

Step 3. For all $k \in \{N_0, 2, ..., N\}$ such that $\Gamma_k > \epsilon_{N,k}$, include $\mathbf{y}_{j,k}$ in $\mathbf{Y}_{j,N+1}$ for all $j \in \{0, 1, ..., q\}$ such that $\Gamma_k = \phi_j(\mathbf{x}_k, \mathbf{y}_{j,k})$.

Step 4. Use Polak-He Algorithm 3.2.1 to compute an approximate solution $\mathbf{x}_{N+1} \in \mathbb{R}^n$ of

$$\min_{\mathbf{x} \in \mathbb{R}^n} \{\psi_{0,N+1}(\mathbf{x}) \mid \psi_{N+1}(\mathbf{x}) \leq 0\}, \tag{3.3.13}$$

which satisfies

$$\theta_{N+1}(\mathbf{x}_{N+1}) \geq -\tau_N, \tag{3.3.14}$$

$$\psi_{N+1}(\mathbf{x}_{N+1}) \leq \tau_N, \tag{3.3.15}$$

with $\theta_{N+1}(\cdot)$ and $\psi_{N+1}(\cdot)$ defined in (3.3.9) and (3.3.3), respectively.

Step 5. Replace N by $N + 1$, and go to **Step 1**.

An example of the double indexed sequence $\{\epsilon_{N,k}\}_{N=N_0, k \leq N}^{\infty}$ is $\epsilon_{N,k} = \rho^k - \rho^N$, with $\rho \in (0, 1)$. It can be shown (Theorem 2.2.8 in Polak (1997)) that $\mathbf{x}$ satisfies $\theta_N(\mathbf{x}) = 0$ if and only if $\mathbf{x}$ is a stationary point for the problem $\min_{\mathbf{x} \in \mathbb{R}^n} \{\psi_{0,N}(\mathbf{x}) \mid \psi_N(\mathbf{x}) \leq 0\}$, i.e., $\mathbf{x}$ satisfies (3.1.4) and (3.1.5) with the appropriate change of notation. Hence, it is seen from Step 4 that Outer Approximation Algorithm 3.3.2 solves the sequence of approximation problems with gradually higher precision. The next theorem shows that points generated by Algorithm 3.3.2 converge to a point that is stationary, i.e., the point satisfies the optimality conditions in Theorem 3.1.2.

Theorem 3.3.2. *Consider the problem* **SIP**. *Suppose that the functions* $\phi_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, j \in \{0, 1, ..., q\}$, *are continuously differentiable, the subsets* $\mathbf{Y}_j, j \in \{0, 1, ..., q\}$, *are compact, and that the approximating solutions* $\mathbf{y}_{j,N} \to \mathbf{y}_j^*$, *as* $N \to \infty$, *with* $\mathbf{y}_j^*, j \in \{0, 1, ..., q\}$, *being a global solution of (3.3.12). If* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is a sequence constructed by Algorithm 3.3.2 in solving* **SIP**, *then any accumulation point* $\hat{\mathbf{x}}$ *of the sequence* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is stationary, i.e.,* $0 \in \bar{G}(\hat{\mathbf{x}})$, *with* $\bar{G}(\hat{\mathbf{x}})$ *as defined in (3.1.9).*□

3.3.3 Generalized Polak-He Algorithm

Consider the specialized case of **SIP**, with no semi-infinite constraints, given by

$$\textbf{SIP}' \qquad \min_{\mathbf{x}\in\mathbb{R}^n}\{\psi_0(\mathbf{x}) \mid f(\mathbf{x}) \leq 0\}, \tag{3.3.16}$$

where $\psi_0(\cdot)$ and $f(\cdot)$ are defined in (3.1.8) and (3.1.3), respectively.

Let

$$\begin{aligned}\Sigma_{n+1+q} \quad \triangleq \quad & \{\mu = (\mu_1, ..., \mu_{n+1+q}) \in \mathbb{R}^{n+1+q} \\ & \mid \mu_j \geq 0, j \in \{1, ..., n+1+q\}, \ \textstyle\sum_{j=1}^{n+1+q} \mu_j = 1\}.\end{aligned} \tag{3.3.17}$$

In Polak *et al.* (2000), we find the following generalization of Polak-He Algorithm 3.2.1.

Generalized Polak-He Algorithm 3.3.3. (Solves SIP′)

Parameters. $\alpha \in (0, 1]$, $\beta \in (0, 1)$, $\gamma > 0$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$.

Step 0. Set $i = 0$.

Step 1. Compute

$$\begin{aligned}\theta_i \ = \ & -\min_{\mu\in\Sigma_{n+1+q},\ \mathbf{y}_j\in\mathbf{Y}_0} \left\{ \sum_{j=1}^{n+1} \mu_j[\psi_0(\mathbf{x}_i) - \phi_0(\mathbf{x}_i, \mathbf{y}_j) + \gamma\phi(\mathbf{x}_i)_+] \right. \\ & + \sum_{j=1}^{q} \mu_{n+1+j}[\psi(\mathbf{x}_i)_+ - f_j(\mathbf{x}_i)] \\ & \left. + \frac{1}{2}\| \sum_{j=1}^{n+1} \mu_j \nabla_{\mathbf{x}}\phi_0(\mathbf{x}_i, \mathbf{y}_j) + \sum_{j=1}^{q} \mu_{n+1+j} \nabla f_j(\mathbf{x}_i)\|^2 \right\},\end{aligned} \tag{3.3.18}$$

and

$$\mathbf{h}_i = -\left\{ \sum_{j=1}^{n+1} \mu_j^* \nabla_{\mathbf{x}}\phi_0(\mathbf{x}_i, \mathbf{y}_j^*) + \sum_{j=1}^{q} \mu_{n+1+j}^* \nabla f_j(\mathbf{x}_i) \right\}, \tag{3.3.19}$$

where $(\mu^*, \mathbf{y}_1^*, \mathbf{y}_2^*, ..., \mathbf{y}_{n+1}^*)$, is any solution of (3.3.18).

Step 2. Compute the step-size

$$\lambda_i = \max_{s \in \mathbb{N}} \{\beta^s \mid F_G(\mathbf{x}_i, \mathbf{x}_i + \beta^s \mathbf{h}_i) \leq \alpha \beta^s \theta_i\} \tag{3.3.20}$$

where $F_G(\mathbf{x}', \mathbf{x}'') \stackrel{\triangle}{=} \max\{\psi_0(\mathbf{x}'') - \psi_0(\mathbf{x}') - \gamma f(\mathbf{x}')_+,\ f(\mathbf{x}'') - f(\mathbf{x}')_+\}$.

Step 3. Set $\mathbf{x}_{i+1} = \mathbf{x}_i + \lambda_i \mathbf{h}_i$, replace i by $i+1$, and go to **Step 1**. □

Contrary to the problem in (3.2.4), θ_i in (3.3.18) is *not* a quadratic program, but a standard nonlinear optimization problem. Hence, (3.3.18) may have non-unique solutions, even though the search direction $\mathbf{h}_i$ is uniquely defined by (3.3.19). Because of this, (3.3.18) can be ill-conditioned. Based on the observations in Polak *et al.* (2000) and Chapter 6, it appears that (3.3.18) can be solved with reasonable efficiently, at least for the case with $\mathbf{Y}_0$ being a ball, i.e., $\mathbf{Y}_0 = \{\mathbf{y} \in \mathbb{R}^m \mid \|\mathbf{y}\| \leq \rho\}$. The next theorem shows that points generated by Algorithm 3.3.3 converge to a point that is stationary, i.e., the point satisfies the optimality conditions in Theorem 3.1.2.

Theorem 3.3.3. *Consider the problem* **SIP**′. *Suppose that the functions* $\phi_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *and* $f_j : \mathbb{R}^n \to \mathbb{R}, j \in \mathbf{q}$, *are continuously differentiable, and the subset* $\mathbf{Y}_0$ *is compact. If* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is a sequence constructed by Algorithm 3.3.3 in solving* **SIP**′, *then any accumulation point* $\hat{\mathbf{x}}$ *of the sequence* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is stationary, i.e.,* $0 \in \bar{G}(\hat{\mathbf{x}})$, *with* $\bar{G}(\hat{\mathbf{x}})$ *as defined in (3.1.9).* □

Chapter 4

Algorithms for Reliability-Based Structural Design Optimization

This chapter defines three classes of reliability-based optimal structural design problems, which were introduced in Chapter 1, and derives algorithms for their solution. Since these problems are unsolvable by existing optimization algorithms, we derive tractable approximating problems that are solved iteratively. The problems are unsolvable due to the facts that the failure probability is not known to be continuously differentiable, and that it cannot be evaluated exactly in finite computational time.

The appearance of the failure probability in these problems dictates the solution strategy. Consequently, the problems are categorized according to where the failure probability appears. The first class of problems (P_1, $\mathrm{P}_{1,\mathrm{sys}}$) has a smooth, non-probabilistic objective function, but is subject to constraints expressed in terms of the failure probability. In the second class (P_2, $\mathrm{P}_{2,\mathrm{sys}}$), the objective function is given by the failure probability and the constraint set is non-probabilistic. The third class (P_3, $\mathrm{P}_{3,\mathrm{sys}}$) contains the failure probability in both the objective function and the constraint set description. The derivation of algorithms is based on the assumption that the system failure probability for the structure is given by a series structural system, as defined in Chapter 2. Additionally, it is assumed that the median point

of the random variables is not in the failure domain, defined in (2.1.2), for each component. In practice, this is usually the case due to the high reliability of engineering structures.

The algorithms in this chapter are described without a stopping rule for terminating the calculations. In practical problem solving, the calculations must end, obviously, at some point. We recommend to use one of the standard stopping rules: terminate the calculations when $\|\mathbf{x}_{i+1} - \mathbf{x}_i\| \leq \epsilon$ and/or $i > i_\epsilon$, where $\epsilon > 0$ is a pre-defined parameter and i_ϵ is the maximum number of iterations. Other stopping rules involving the objective and constraint functions can also be used.

4.1 Failure Probability in either Objective or Constraint Functions ($\mathbf{P}_1$, $\mathbf{P}_2$)

Let $\mathbf{x} \in \mathbb{R}^n$ be a vector of design variables, and let the component failure probabilities $p_k(\mathbf{x}), k \in \mathbf{K} = \{1, ..., K\}$, and the system failure probability $p(\mathbf{x})$ be as defined in Chapter 2. We define the first class of reliability-based optimal design problems by

$$\mathbf{P}_1 \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \{c_0(\mathbf{x}) \mid p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X}\}, \tag{4.1.1}$$

$$\mathbf{P}_{1,\text{sys}} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \{c_0(\mathbf{x}) \mid p(\mathbf{x}) \leq \hat{p},\ p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X}\}, \tag{4.1.2}$$

where $c_0 : \mathbb{R}^n \to \mathbb{R}$ is a smooth objective function describing the cost of the initial design, $\hat{p}_k, k \in \mathbf{K}$, and $\hat{p}$ are pre-defined probability bounds, and

$$\mathbf{X} \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid f_j(\mathbf{x}) \leq 0, j = 1, ..., q\}, \tag{4.1.3}$$

with $f_j : \mathbb{R}^n \to \mathbb{R}$ being smooth functions describing deterministic constraints. Note that $\mathbf{P}_1$ is different from $\mathbf{P}_{1,\text{sys}}$ by not including the system failure probability.

We define the second class of problems by

$$\mathbf{P}_2 \qquad \min_{\mathbf{x}\in\mathbb{R}^n}\{\max_{k\in\mathbf{K}} p_k(\mathbf{x}) \mid \mathbf{x}\in\mathbf{X}\}, \tag{4.1.4a}$$

$$\mathbf{P}_{2,\text{sys}} \qquad \min_{\mathbf{x}\in\mathbb{R}^n}\{p(\mathbf{x}) \mid \mathbf{x}\in\mathbf{X}\}. \tag{4.1.4b}$$

Note that $\mathbf{P}_2$ minimizes the component failure probability of the "least reliable" component, i.e., minimizes the critical failure component, while $\mathbf{P}_{2,\text{sys}}$ minimizes the failure probability of the system.

The problems $\mathbf{P}_1$, $\mathbf{P}_{1,\text{sys}}$, $\mathbf{P}_2$, and $\mathbf{P}_{2,\text{sys}}$ have a solution if $\mathbf{X}$ is compact and the functions $c_0(\cdot)$, $p(\cdot)$, $p_k(\cdot)$, $k\in\mathbf{K}$, are continuous. A sufficient condition for $p(\cdot)$, $p_k(\cdot)$, $k\in\mathbf{K}$, to be continuous is given below. This result is a trivial extension of Corollary 1 in Polak *et al.* (2000). First, we need the following assumption.

Assumption 4.1.1. We assume that

$$M(\{\mathbf{u}\in\mathbb{R}^m \mid g_k(\mathbf{x},\mathbf{u})=0\})=0, \tag{4.1.5a}$$

for all $k\in\mathbf{K}, \mathbf{x}\in\mathbf{X}$, where, for any set $\mathbf{S}\subset\mathbb{R}^m$, the measure

$$M(\mathbf{S}) \triangleq \int_{\mathbf{S}} \varphi(\mathbf{u})d\mathbf{u}, \tag{4.1.5b}$$

with $\varphi(\cdot)$ the standard normal probability density function. □

Essentially, Assumption 4.1.1 requires that the interval (for $m=1$), area (for $m=2$), volume (for $m=3$), etc. in which the limit-state function vanishes, have length, area, volume, etc equal to zero, respectively. This is normally satisfied in realistic design problems.

Theorem 4.1.1. *Suppose that Assumption 4.1.1 is satisfied and the limit-state functions $g_k(\cdot,\cdot), k\in\mathbf{K}$, are continuous. Then, the component failure probability $p_k(\cdot), k\in\mathbf{K}$, defined in (2.1.1), and the series structural system failure probability $p(\cdot)$, defined in (2.1.3) and (2.1.4), are continuous.* □

4.1.1 Approximating Problems

In Kirjner-Neto *et al.* (1998), Der Kiureghian and Polak (1998) and Polak *et al.* (2000) we find that approximating problems for $\mathbf{P}_1$ and $\mathbf{P}_2$ can be constructed by replacing the component failure probability terms $p_k(\cdot)$, $k \in \mathbf{K}$, appearing in (4.1.1) and (4.1.4a), by functions that denote the minima of the corresponding limit-state functions within balls of specified radii. We define a ball of radius $\rho > 0$ centered at $\hat{\mathbf{u}} \in \mathbb{R}^m$ by

$$\mathbb{B}(\hat{\mathbf{u}}, \rho) \stackrel{\triangle}{=} \{\mathbf{u} \in \mathbb{R}^m \mid \|\mathbf{u} - \hat{\mathbf{u}}\| \leq \rho\}. \tag{4.1.5c}$$

Hence, for any approximation parameter vector $\mathbf{s} = (s_1, ..., s_K)$, with $s_k > 0$, we obtain the following approximation to $\mathbf{P}_1$:

$$\mathbf{P}_{1,\mathbf{s}} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \{c_0(\mathbf{x}) \mid \psi_{k,s_k}(\mathbf{x}) \leq 0, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X}\}, \tag{4.1.6}$$

where for any $\alpha > 0$, $\psi_{k,\alpha} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_{k,\alpha}(\mathbf{x}) \stackrel{\triangle}{=} \max_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, \alpha)} \{-g_k(\mathbf{x}, \mathbf{u})\}. \tag{4.1.7}$$

Similarly, for any approximating parameter $r > 0$, we obtain the following approximation to $\mathbf{P}_2$:

$$\mathbf{P}_{2,r} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \{\max_{k \in \mathbf{K}} \psi_{k,r}(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}\}, \tag{4.1.8}$$

where $\psi_{k,r}(\cdot)$ is as defined in (4.1.7). Observe that $\mathbf{P}_{1,\mathbf{s}}$ and $\mathbf{P}_{2,r}$ are semi-infinite optimization problems, and hence can be solved by any of the appropriate algorithms described in Chapter 3.

Relations between $\mathbf{P}_1$ and $\mathbf{P}_{1,\mathbf{s}}$, and between $\mathbf{P}_2$ and $\mathbf{P}_{2,r}$ are stated in the following theorems. These results are minor extensions of the ones in Kirjner-Neto *et al.* (1998) and Der Kiureghian and Polak (1998). Similar extensions were derived independently in Polak *et al.* (2000). For the following exposition, it is convenient to denote the feasible set for $\mathbf{P}_1$ by

$$\mathbf{F}_1 \stackrel{\triangle}{=} \{\mathbf{x} \in \mathbb{R}^n \mid p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}\} \tag{4.1.9}$$

Theorem 4.1.2. *Consider* $\mathbf{P}_1$. *Suppose that the functions* $c_0(\cdot)$ *and* $g_k(\cdot,\cdot), k \in \mathbf{K}$, *are continuous, the feasible set* $\mathbf{F}_1$ *is compact, the limit-state functions* $g_k(\cdot,\cdot), k \in \mathbf{K}$, *are affine in their second argument, and that* $g_k(\mathbf{x},\mathbf{0}) > 0$ *for all* $\mathbf{x} \in \mathbf{F}_1, k \in \mathbf{K}$. *If* $s_k = -\Phi^{-1}(\hat{p}_k), k \in \mathbf{K}$, *with* $\Phi(\cdot)$ *being the standard normal cumulative distribution function, then* $\mathbf{x}^*$ *solves* $\mathbf{P}_1$ *if and only if it solves* $\mathbf{P}_{1,\mathbf{s}}$.

Proof. Since $g_k(\mathbf{x},\cdot)$ is affine for all $k \in \mathbf{K}$, there exist functions $a_k : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{b}_k : \mathbb{R}^n \to \mathbb{R}^m$ such that $g_k(\mathbf{x},\mathbf{u}) = a_k(\mathbf{x}) + \mathbf{b}_k(\mathbf{x})^T\mathbf{u}$. The fact that $g_k(\mathbf{x},\mathbf{0}) > 0$ for all $\mathbf{x} \in \mathbf{F}_1$, implies that we can assume without loss of generality that $g_k(\mathbf{x},\mathbf{u}) = 1 + \mathbf{c}_k(\mathbf{x})^T\mathbf{u}$, with $\mathbf{c}_k(\mathbf{x}) = \mathbf{b}_k(\mathbf{x})/a_k(\mathbf{x})$. Hence, it follows from (4.1.7) that

$$\psi_{k,s_k}(\mathbf{x}) = -1 + s_k\|\mathbf{c}_k(\mathbf{x})\|. \tag{4.1.10}$$

Consequently, $\mathbf{x}^* \in \mathbf{X}$ is a feasible point for $\mathbf{P}_{1,\mathbf{s}}$ if and only if

$$-1 + s_k\|\mathbf{c}_k(\mathbf{x}^*)\| \leq 0, \tag{4.1.11}$$

for all $k \in \mathbf{K}$. Hence, (4.1.11) holds if and only if

$$-\frac{1}{\|\mathbf{c}_k(\mathbf{x}^*)\|} \leq \Phi^{-1}(\hat{p}_k). \tag{4.1.12}$$

By the fact that $g_k(\mathbf{x},\cdot)$ is affine, $g_k(\mathbf{x},\mathbf{0}) > 0$ for all $\mathbf{x} \in \mathbf{F}_1$, and the standard normal probability density function is rotationally symmetric, we can infer from (2.1.1) that

$$p_k(\mathbf{x}) = \Phi(-\beta_{1,k}(\mathbf{x})), \tag{4.1.13}$$

with

$$\beta_{1,k}(\mathbf{x}) = \frac{1}{\|\mathbf{c}_k(\mathbf{x})\|}. \tag{4.1.14}$$

Since $\Phi(\cdot)$ is a strictly monotonically increasing function, it now follows from (4.1.13) and (4.1.14) that (4.1.12) holds if and only if

$$p_k(\mathbf{x}) \leq \hat{p}_k. \tag{4.1.15a}$$

This completes the proof. □

The consequence of Theorem 4.1.2 is that the approximating problem $\mathbf{P}_{1,\mathrm{s}}$ has solutions identical to those of $\mathbf{P}_1$ if the failure probability terms in the latter are expressed in terms of the first-order reliability approximation, see (2.1.8).

For non-affine limit-state functions, the situation is somewhat more complex. We see from (4.1.7) that an increase in the value of the parameter s_k in $\mathbf{P}_{1,\mathrm{s}}$ implies that the constraint $\psi_{k,s_k}(\mathbf{x}) \leq 0$ guarantees the non-negativity of $g_k(\mathbf{x},\cdot)$ in a larger ball than it would for a smaller s_k. The non-negativity of $g_k(\mathbf{x},\cdot)$ in a ball of radius s_k gives rise to an upper bound on $p_k(\mathbf{x})$, and this upper bound is closer to zero, i.e., it is tighter, for larger values of s_k than for smaller values. Hence, by selecting a sufficiently large value of s_k we can ensure that the constraint $p_k(\mathbf{x}) \leq \hat{p}_k$ is satisfied.

In the case of system failure probability constraints ($\mathbf{P}_{1,\mathrm{sys}}$), we can also use the parameter vector $\mathbf{s}$ to ensure that the constraint $p(\mathbf{x}) \leq \hat{p}$ is satisfied. From Ditlevsen and Madsen (1996), we have that

$$\max_{k\in\mathbf{K}} p_k(\mathbf{x}) \leq p(\mathbf{x}) \leq \sum_{k\in\mathbf{K}} p_k(\mathbf{x}). \tag{4.1.15b}$$

Hence, the failure probability of the system is closely related to the component failure probabilities and particularly the critical failure component. By selecting sufficiently large values of $s_k, k \in \mathbf{K}$, we can in view of (4.1.15b) and the discussion in the previous paragraph, ensure that the constraint $p(\mathbf{x}) \leq \hat{p}$ is satisfied.

Theorem 4.1.3. *Consider* $\mathbf{P}_2$. *Suppose that the functions* $g_k(\cdot,\cdot), k \in \mathbf{K}$, *are continuous, the feasible set* $\mathbf{X}$ *is compact, the limit-state functions* $g_k(\cdot,\cdot), k \in \mathbf{K}$, *are affine in their second argument, and that* $g_k(\mathbf{x},\mathbf{0}) > 0$, *for all* $\mathbf{x} \in \mathbf{X}, k \in \mathbf{K}$. *Then,* $\mathbf{x}^*$ *solves* $\mathbf{P}_2$ *if and only if it solves* $\mathbf{P}_{2,r}$, *with arbitrary* $r > 0$.

Proof. By the same arguments as in the proof of Theorem 4.1.2, we can assume without loss of generality that $g_k(\mathbf{x},\mathbf{u}) = 1 + \mathbf{c}_k(\mathbf{x})^T\mathbf{u}$. Hence, it follows from (4.1.7) that

$$\psi_{k,r}(\mathbf{x}) = -1 + r\|\mathbf{c}_k(\mathbf{x})\|, \tag{4.1.16}$$

and, therefore, $\mathbf{P}_{2,r}$ takes the form

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{\max_{k\in\mathbf{K}}\{-1+r\|\mathbf{c}_k(\mathbf{x})\|\}\mid \mathbf{x}\in\mathbf{X}\}. \tag{4.1.17}$$

By (4.1.13) and (4.1.14), and the fact that $\Phi(\cdot)$ is strictly increasing, we see that the new problem

$$\max_{\mathbf{x}\in\mathbb{R}^n}\left\{\min_{k\in\mathbf{K}}\frac{1}{\|\mathbf{c}_k(\mathbf{x})\|}\mid \mathbf{x}\in\mathbf{X}\right\} \tag{4.1.18}$$

is equivalent to $\mathbf{P}_2$, i.e., $\mathbf{x}^*$ is a solution of the problem in (4.1.18) if and only if it is a solution of $\mathbf{P}_2$. Furthermore, it follows by inspection that the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{\max_{k\in\mathbf{K}}\|\mathbf{c}_k(\mathbf{x})\|\mid \mathbf{x}\in\mathbf{X}\} \tag{4.1.19}$$

is also equivalent to $\mathbf{P}_2$. Since $r>0$ is a constant, the problem in (4.1.19) has the same solution as the one in (4.1.17). This completes the proof. □

The consequences of Theorem 4.1.3 are similar to the ones for Theorem 4.1.2. If the failure probability terms in $\mathbf{P}_2$ are expressed in terms of the first-order reliability indices $\beta_{1,k}(\mathbf{x})$, see (2.1.8), then the approximating problem $\mathbf{P}_{2,r}$ has solutions identical to those of $\mathbf{P}_2$ for arbitrary $r>0$.

A geometric interpretation of problems $\mathbf{P}_2$ and $\mathbf{P}_{2,r}$ using the first-order reliability approximation $p_k(\mathbf{x})\approx\Phi(-\beta_{1,k}(\mathbf{x}))$ helps us understand the situation for non-affine limit-state functions. Assuming that this approximation is exact, the distance from the origin in $\mathbb{R}^m$ to the nearest point in $\bigcup_{k\in\mathbf{K}}\Omega_k(\mathbf{x})$ is given by $\beta_{1,\hat{k}}(\mathbf{x})$, where $\hat{k}$ is the index of the critical failure component, see (2.1.8). In view of (4.1.13), we see that $\mathbf{P}_2$ finds the optimal design by maximizing this distance. In contrast, $\mathbf{P}_{2,r}$ finds the optimal design by minimizing $\max_{k\in\mathbf{K}}\psi_{k,r}(\cdot)$ or, equivalently, maximizing the minimum value of the limit-state functions within the ball of radius r. This minimum may occur at a point $\hat{\mathbf{u}}_{\hat{k}}(\mathbf{x})$, which, in general, is different from $\mathbf{u}^*_{\hat{k}}(\mathbf{x})$ as defined in (2.1.6), see Figure 4.1. From Theorem 4.1.3, it is clear that for affine limit-state functions these two approaches lead to identical results. For non-affine limit-state functions, we see from Figure 4.1 that the two approaches would produce identical designs if r is equal to $\beta_{1,\hat{k}}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is a solution of $\mathbf{P}_2$, and $p_{\hat{k}}(\mathbf{x}^*)=\Phi(-\beta_{1,\hat{k}}(\mathbf{x}^*))$. Furthermore,

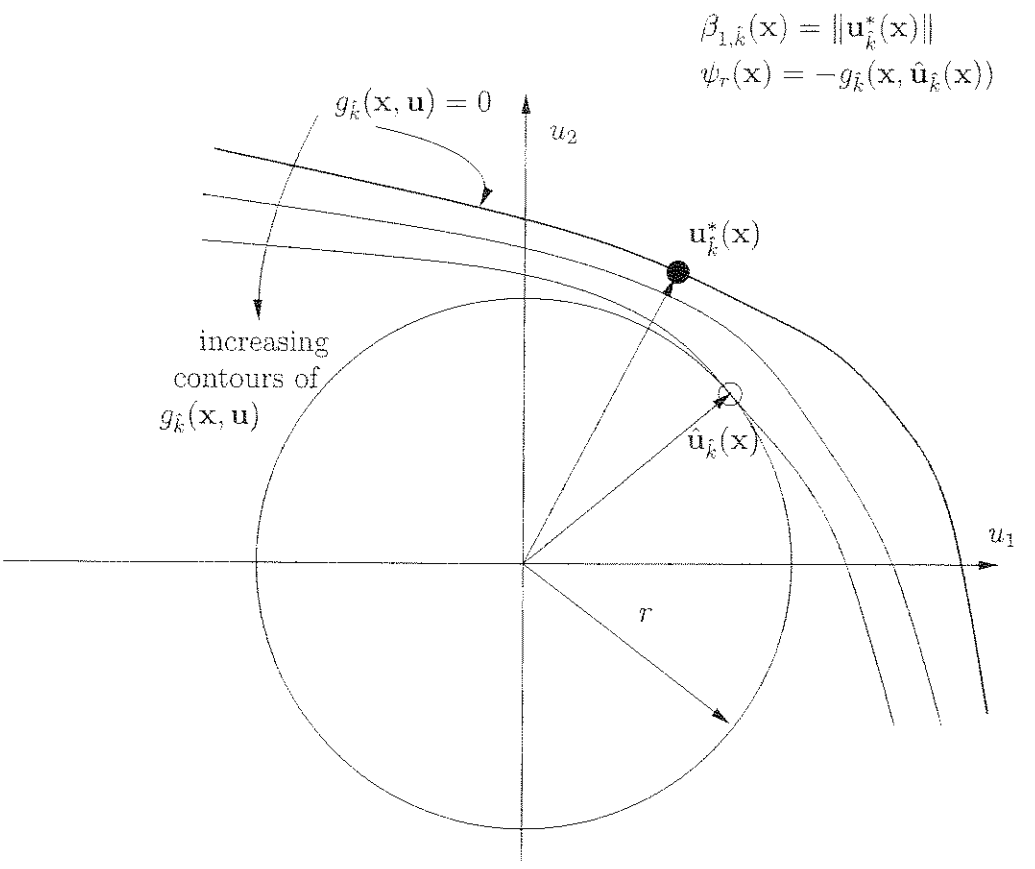

Figure 4.1: $\mathbf{P}_2$ and non-affine limit-state function, $m = 2$.

based on the geometric interpretation in Figure 4.1, one would expect the solution of $\mathbf{P}_{2,r}$ for a non-affine limit-state function to be close to the solution of $\mathbf{P}_2$ and to be insensitive to the value of r in the neighborhood of $\beta_{1,k}(\mathbf{x}^*)$. Consequently, we can also conclude in view of (4.1.15b) that $\mathbf{P}_{2,r}$, with r close to $-\Phi^{-1}(p(\mathbf{x}^*))$ ($\mathbf{x}^*$ solution of $\mathbf{P}_{2,\text{sys}}$), is a reasonably good approximation to $\mathbf{P}_{2,\text{sys}}$.

4.1.2 Algorithms

In view of Theorems 4.1.2 and 4.1.3, it is clear that when the limit-state functions $g_k(\cdot,\cdot), k \in \mathbf{K}$, are affine, $\mathbf{P}_1$ and $\mathbf{P}_2$ can be solved by applying one of the algorithms for semi-infinite optimization to $\mathbf{P}_{1,\text{s}}$ and $\mathbf{P}_{2,r}$. To obtain approximate solutions in the case of non-affine limit-state functions and/or problems involving series systems, we repeatedly solve the approximating problems $\mathbf{P}_{1,\text{s}}$ and $\mathbf{P}_{2,r}$ as described below. This approach was originally proposed by Der Kiureghian and Polak (1998) and Polak *et al.* (2000) for problems with component failure probabilities, i.e., $\mathbf{P}_1$ and $\mathbf{P}_2$. Here, we extend this approach to also address $\mathbf{P}_{1,\text{sys}}$ and $\mathbf{P}_{2,\text{sys}}$. Before we proceed, we need to define the following term.

Definition 4.1.1. We say that $\tilde{p}_k(\mathbf{x})$ is an *appropriate estimate* of the component failure probability $p_k(\mathbf{x})$ for the current run of Algorithm "A" if

(i) $\tilde{p}_k(\mathbf{x})$ can be computed in finite time,

(ii) $\tilde{p}_k(\mathbf{x})$ is computed using the same reliability method (e.g., FORM, SORM, Monte Carlo Simulation, see Chapter 2) as the one used to verify the final design obtained by the current run of Algorithm "A," and

(iii) $\tilde{p}_k(\mathbf{x})$ is computed with approximately the same accuracy as the one used when verifying the final design obtained by the current run of Algorithm "A."

We say that $\tilde{p}(\mathbf{x})$ is an *appropriate estimate* of the system failure probability $p(\mathbf{x})$ for the current run of Algorithm "A" if items (i), (ii) and (iii) hold with $\tilde{p}_k(\mathbf{x})$ replaced

by $\tilde{p}(\mathrm{x})$. □

Algorithm 4.1.1. (For P_1)

Data. $\mathrm{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \rightarrow \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $\mathrm{s}_0 = -(\Phi^{-1}(\hat{p}_1), \Phi^{-1}(\hat{p}_2), ..., \Phi^{-1}(\hat{p}_K))$.

Step 1. Set x_{i+1} to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\mathrm{P}_{1,\mathrm{s}_i}$, with initial point x_i.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathrm{x}_{i+1}), k \in \mathrm{K}$, of $p_k(\mathrm{x}_{i+1}), k \in \mathrm{K}$, for the current run of Algorithm 4.1.1, see Definition 4.1.1.

Step 3. Update the components of s_{i+1} by setting

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathrm{x}_{i+1}))}, k \in \mathrm{K}. \tag{4.1.20}$$

Step 4. Replace i by $i + 1$ and go to **Step 1**. □

Algorithm 4.1.2. (For $\mathrm{P}_{1,\mathrm{sys}}$)

Data. $\mathrm{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \rightarrow \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $\mathrm{s}_0 = -(\Phi^{-1}(\hat{p}_1), \Phi^{-1}(\hat{p}_2), ..., \Phi^{-1}(\hat{p}_K))$.

Step 1. Set x_{i+1} to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\mathrm{P}_{1,\mathrm{s}_i}$, with initial point x_i.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathrm{x}_{i+1}), k \in \mathrm{K}$, and $\tilde{p}(\mathrm{x}_{i+1})$ of $p_k(\mathrm{x}_{i+1}), k \in \mathrm{K}$, and $p(\mathrm{x}_{i+1})$ for the current run of Algorithm 4.1.2, see Definition 4.1.1.

Step 3. Update the components of s_{i+1} by setting

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p})}{\Phi^{-1}(\tilde{p}(\mathrm{x}_{i+1}))}, \ \forall k \in \arg\max_{k \in \mathrm{K}} \{\tilde{p}_k(\mathrm{x}_{i+1})\}, \tag{4.1.21}$$

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p}_k)}{\Phi^{-1}(\tilde{p}_k(\mathrm{x}_{i+1}))}, \ \forall k \notin \arg\max_{k \in \mathrm{K}} \{\tilde{p}_k(\mathrm{x}_{i+1})\}. \tag{4.1.22}$$

Step 4. Replace i by $i+1$ and go to **Step 1**. □

For the special case of $\mathbf{P}_{1,\text{sys}}$ with no component failure probability constraints, i.e.,

$$\mathbf{P}^*_{1,\text{sys}} \qquad \min_{\mathbf{x}\in\mathbb{R}^n}\{c_0(\mathbf{x}) \mid p(\mathbf{x}) \le \hat{p},\ \mathbf{x}\in\mathbf{X}\}, \tag{4.1.23}$$

we can apply the following algorithm.

Algorithm 4.1.3. (For $\mathbf{P}^*_{1,\text{sys}}$)

Parameter. $\gamma \ge 1$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $\mathbf{s}_0 = -\gamma(\Phi^{-1}(\hat{p}), \Phi^{-1}(\hat{p}), ..., \Phi^{-1}(\hat{p})) \in \mathbb{R}^K$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\mathbf{P}_{1,\mathbf{s}_i}$, with initial point $\mathbf{x}_i$.

Step 2. Compute an appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$ for the current run of Algorithm 4.1.3, see Definition 4.1.1.

Step 3. Update the components of $\mathbf{s}_{i+1}$ by setting

$$(s_k)_{i+1} = (s_k)_i \frac{\Phi^{-1}(\hat{p})}{\Phi^{-1}(\tilde{p}(\mathbf{x}_{i+1}))}. \tag{4.1.24}$$

Step 4. Replace i by $i+1$ and go to **Step 1**. □

The points $\mathbf{x}_1, \mathbf{x}_2, ...$ generated by Algorithm 4.1.1, 4.1.2 or 4.1.3 are approximate solutions of $\mathbf{P}_{1,\mathbf{s}_0}, \mathbf{P}_{1,\mathbf{s}_1}, ...$, and hence are first-order approximations to the solution of $\mathbf{P}_1$. It is expected and experienced, see Chapter 6, that by adjusting the approximation parameter vector $\mathbf{s}$, we are able to significantly improve on the first-order approximations. For series structural systems, $\mathbf{P}_{1,\mathbf{s}_0}, \mathbf{P}_{1,\mathbf{s}_1}, ...$ are also good approximations

to $\mathrm{P}_{1,\mathrm{sys}}$ because, as discussed following Theorem 4.1.2, the dominant contribution to the series system failure probability comes from the critical failure component.

The second class of problems can be solved by the following algorithm.

Algorithm 4.1.4. (For P_2 and $\mathrm{P}_{2,\mathrm{sys}}$)

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \rightarrow \infty$, $N_i \in \mathbb{N}$, $r_0, r_1, r_3, ...$, with $r_i > 0, i = 0, 1, 2, ...$

Step 0. Set $i = 0$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1, 3.3.2 or 3.3.3 on the problem P_{2,r_i}, with initial point $\mathbf{x}_i$.

Step 2. If considering P_2, compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$, for the current run of Algorithm 4.1.4, see Definition 4.1.1.

If considering $\mathrm{P}_{2,\mathrm{sys}}$, compute an appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$ for the current run of Algorithm 4.1.4, see Definition 4.1.1..

Step 3. Replace i by $i + 1$ and go to **Step 1.** □

The points $\mathbf{x}_1, \mathbf{x}_2, ...$ generated by Algorithm 4.1.4 are approximate solutions of $\mathrm{P}_{2,r_0}, \mathrm{P}_{2,r_1}, ...$, and hence are first-order approximations to the solution of P_2. It is expected and experienced, see Chapter 6 and the discussion following Theorem 4.1.3, that the design obtained from solving $\mathrm{P}_{2,r}$ for r-values close to $-\Phi^{-1}(p(\mathbf{x}^*))$ is close to $\mathbf{x}^*$, where $\mathbf{x}^*$ is the solution of the original problem P_2 or $\mathrm{P}_{2,\mathrm{sys}}$.

We cannot guarantee that any of the Algorithms 4.1.1, 4.1.2, 4.1.3 and 4.1.4 converges to the exact solution of the respective design problem under general conditions. However, since the design found is optimal with respect to a first-order approximation, we expect that results generated by Algorithms 4.1.1, 4.1.2, 4.1.3 or 4.1.4 will be sufficiently close to the true optimal solution for practical applications.

4.2 Failure Probability in both Objective and Constraint Functions ($\mathbf{P}_3$)

The third class of reliability-based optimal structural design problems consists of $\mathbf{P}_3$ and $\mathbf{P}_{3,\text{sys}}$. $\mathbf{P}_3$ has the form

$$\mathbf{P}_3 \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \mid p_k(\mathbf{x}) \le \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}, \tag{4.2.1}$$

with $c_k : \mathbb{R}^n \to \mathbb{R}, k \in \{0, 1, ..., K\}$, being smooth functions describing the initial cost $c_0(\cdot)$ and the cost $c_k(\cdot)$ associated with the failure of the k-th component. The objective function in $\mathbf{P}_3$ can be interpreted as the initial cost plus the expected cost of failure, when expected costs of failure of the components are additive. As above, the component failure probabilities $p_k(\cdot), k \in \mathbf{K}$, are defined in Chapter 2, $\hat{p}_k, k \in \mathbf{K}$, are pre-described bounds, and $\mathbf{X} \subset \mathbb{R}^n$ is the deterministic constraint set defined in (4.1.3).

Before we define $\mathbf{P}_{3,\text{sys}}$, we need to establish further notation. Consider the simultaneous design of L structures. Let $\mathbf{x} \in \mathbb{R}^n$ be the design vector containing all the design variables associated with all the structures. Let each structure be modeled as a series system and the corresponding failure probability $p^{(l)}(\mathbf{x}), l \in \mathbf{L} \triangleq \{1, ..., L\}$, of the l-th structure be defined by (2.1.3). The k-th limit-state function of the l-th structure is denoted $g_k^{(l)}(\cdot, \cdot)$, $k \in \mathbf{K}_l \triangleq \{1, ..., K_l\}, l \in \mathbf{L}$. Then we define

$$\mathbf{P}_{3,\text{sys}} \quad \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) p^{(l)}(\mathbf{x}) \mid p^{(l)}(\mathbf{x}) \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X} \right\}, \tag{4.2.2}$$

where $c_0^{(l)}, c^{(l)} : \mathbb{R}^n \to \mathbb{R}, l \in \mathbf{L}$, are smooth functions describing the initial cost and the cost associated with failure of the l-th structure, and $\hat{p}^{(l)}, l \in \mathbf{L}$, are pre-described bounds. This expanded definition of $\mathbf{P}_3$ is applicable to the optimal design of a portfolio of structures. An example would be the design of retrofit strategies for a portfolio of bridges.

The presence of failure probability terms in both the objective function and the

constraint set of $\mathbf{P}_3$ and $\mathbf{P}_{3,\text{sys}}$ makes the derivation of approximating problems more complicated. We proceed with two different approaches. The first approach is limited to $\mathbf{P}_3$, with $K = 1$, and results in an approximating problem of *generalized* semi-infinite type. This particular type of generalized semi-infinite optimization problems has in the past been unsolvable. We present a new algorithm for solving such problems in Chapter 5. The second approach addresses $\mathbf{P}_3$ and $\mathbf{P}_{3,\text{sys}}$ and results in an approximating problem that is a collection of semi-infinite optimization problems solvable by the algorithms in Chapter 3.

4.2.1 Algorithms for the Solution of Single Component $\mathbf{P}_3$

Approximating Problems

Consider the special case of $\mathbf{P}_3$ with $K = 1$. To simplify the notation, we let $p(\cdot) = p_1(\cdot)$, $g(\cdot,\cdot) = g_1(\cdot,\cdot)$, $\hat{p} = \hat{p}_1$, $\beta_1(\cdot) = \beta_{1,1}(\cdot)$, $\mathbf{u}^*(\cdot) = \mathbf{u}_1^*(\cdot)$, and $c(\cdot) = c_1(\cdot)$. The construction of an approximating problem consists of two steps. In the first step, we need the following auxiliary problem

$$\mathbf{P}_{3,r,s} \quad \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ c_0(\mathbf{x}) + c(\mathbf{x})\Phi\left(\frac{-rg(\mathbf{x},\mathbf{0})}{\psi_r(\mathbf{x}) + g(\mathbf{x},\mathbf{0})}\right) \mid \psi_s(\mathbf{x}) \leq 0,\ \mathbf{x}\in\mathbf{X}\right\}, \tag{4.2.3}$$

where $r, s > 0$ are approximation parameters, and for any $\alpha > 0$

$$\psi_\alpha(\mathbf{x}) \triangleq \max_{\mathbf{u}\in\mathbb{B}(\mathbf{0},\alpha)} \{-g(\mathbf{x},\mathbf{u})\}. \tag{4.2.4}$$

Theorem 4.2.1. *Consider* $\mathbf{P}_3$ *with* $K = 1$. *Suppose that the functions* $c_0(\cdot), c(\cdot)$ *and* $g(\cdot,\cdot)$, *are continuous, the feasible set* $\mathbf{F}_1$, *defined in (4.1.9), is compact, the limit-state function* $g(\cdot,\cdot)$, *is affine in its second argument, and that* $g(\mathbf{x},\mathbf{0}) > 0$ *for all* $\mathbf{x}\in\mathbf{F}_1$. *If* $r > 0$ *and* $s = -\Phi^{-1}(\hat{p})$, *with* $\Phi(\cdot)$ *being the standard normal cumulative distribution function, then* $\mathbf{x}^*$ *solves* $\mathbf{P}_3$ *if and only if it solves* $\mathbf{P}_{3,r,s}$.

Proof. Since the constraint sets in $\mathbf{P}_{3,r,s}$ and $\mathbf{P}_{1,s}$, with $K = 1$, are identical, we

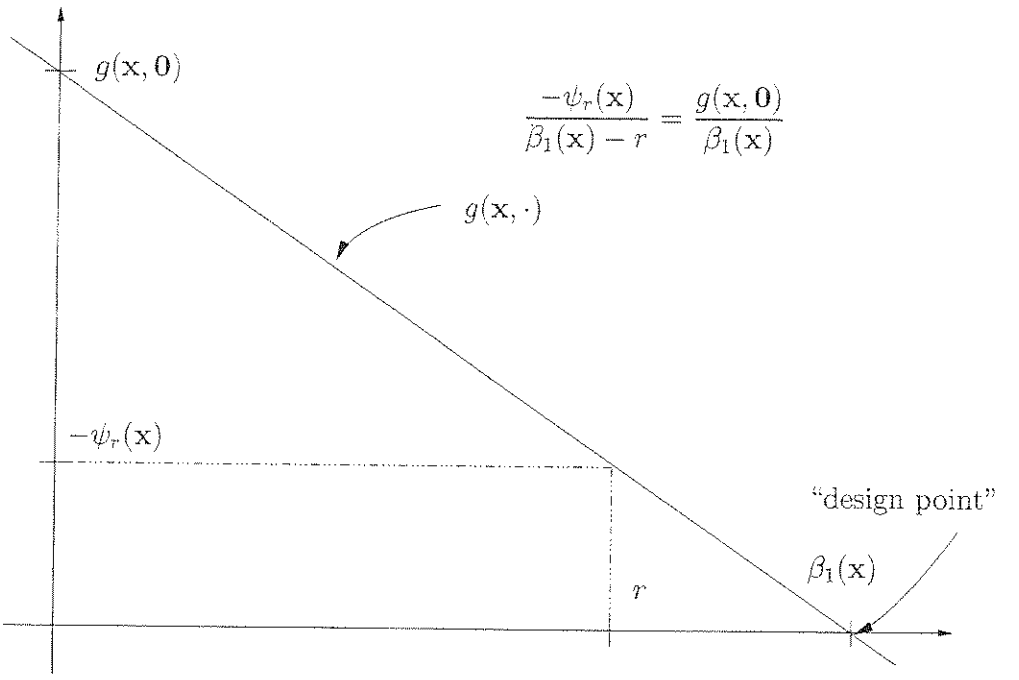

Figure 4.2: Affine limit-state function.

deduce from Theorem 4.1.2 that $\psi_s(\mathbf{x}) \leq 0, s = -\Phi^{-1}(\hat{p})$, if and only if $p(\mathbf{x}) \leq \hat{p}$. Since $g(\mathbf{x},\cdot)$ is affine, there exist functions $a : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(\mathbf{x},\mathbf{u}) = a(\mathbf{x}) + \mathbf{b}(\mathbf{x})^T\mathbf{u}$. The fact that $g(\mathbf{x},\mathbf{0}) > 0$ for all $\mathbf{x} \in \mathbf{F}_1$, defined in (4.1.9), implies that we can assume without loss of generality that $g(\mathbf{x},\mathbf{u}) = 1 + \mathbf{c}(\mathbf{x})^T\mathbf{u}$ with $\mathbf{c}(\mathbf{x}) = \mathbf{b}(\mathbf{x})/a(\mathbf{x})$. Hence, it follows from (4.2.4) that

$$\psi_r(\mathbf{x}) = -1 + r\|\mathbf{c}(\mathbf{x})\|. \tag{4.2.5}$$

Consequently, using (4.1.13) and (4.1.14), we obtain

$$\Phi\left(\frac{-rg(\mathbf{x},\mathbf{0})}{\psi_r(\mathbf{x}) + g(\mathbf{x},\mathbf{0})}\right) = \Phi\left(\frac{-r}{-1 + r\|\mathbf{c}(\mathbf{x})\| + 1}\right) = p(\mathbf{x}). \tag{4.2.6}$$

This completes the proof. □

Note that $\mathbf{P}_{3,r,s}$ replaces the failure probability in the objective function of $\mathbf{P}_3$, with $K = 1$, with an approximation based on geometric relations for affine limit-state

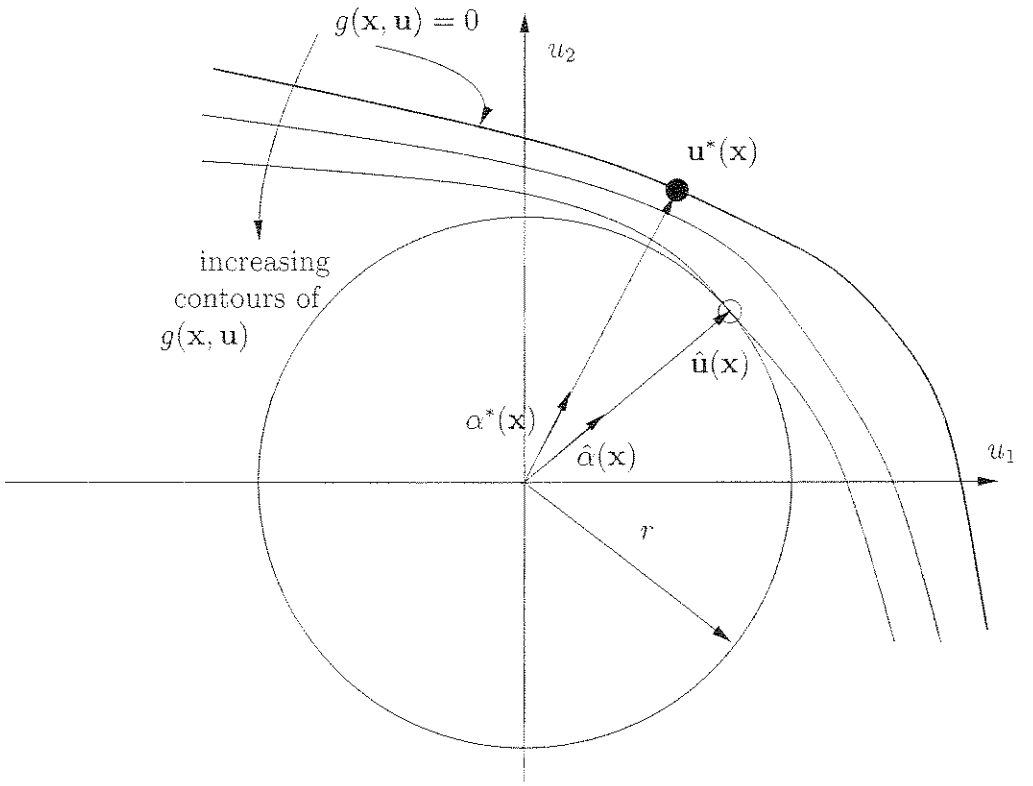

Figure 4.3: Non-affine limit-state function, $m = 2$.

functions, see Figure 4.2. The abscissa in Figure 4.2 is the line in $\mathbb{R}^m$ (the standard normal space) through the origin and the "design point" $\mathbf{u}^*(\mathbf{x})$ defined in (2.1.6), and the ordinate shows the corresponding values of the affine limit-state function for a fixed $\mathbf{x}$. For non-affine limit-state functions, the relation provides an approximation to the first-order reliability index $\beta_1(\mathbf{x})$, as illustrated in Figures 4.3, 4.4 and 4.5.

Figure 4.3 shows the locations of the "design point" $\mathbf{u}^*(\mathbf{x})$, defined in (2.1.6), and the maximizer $\hat{\mathbf{u}}(\mathbf{x}) = \arg\max_{\mathbf{u}\in\mathbb{B}(\mathbf{0},r)}\{-g(\mathbf{x},\mathbf{u})\}$ for a two-dimensional problem. The unit vectors $\alpha^*(\mathbf{x})$ and $\hat{\alpha}(\mathbf{x})$ denote the corresponding directions of these points. As illustrated, for a non-affine limit-state function the two points $\mathbf{u}^*(\mathbf{x})$ and $\hat{\mathbf{u}}(\mathbf{x})$ do not necessarily lie along the same line through the origin. Figure 4.4 shows the values of the limit-state function for a fixed $\mathbf{x}$ along the line through the origin and the "design point," and Figure 4.5 shows the values of the limit-state function for a fixed

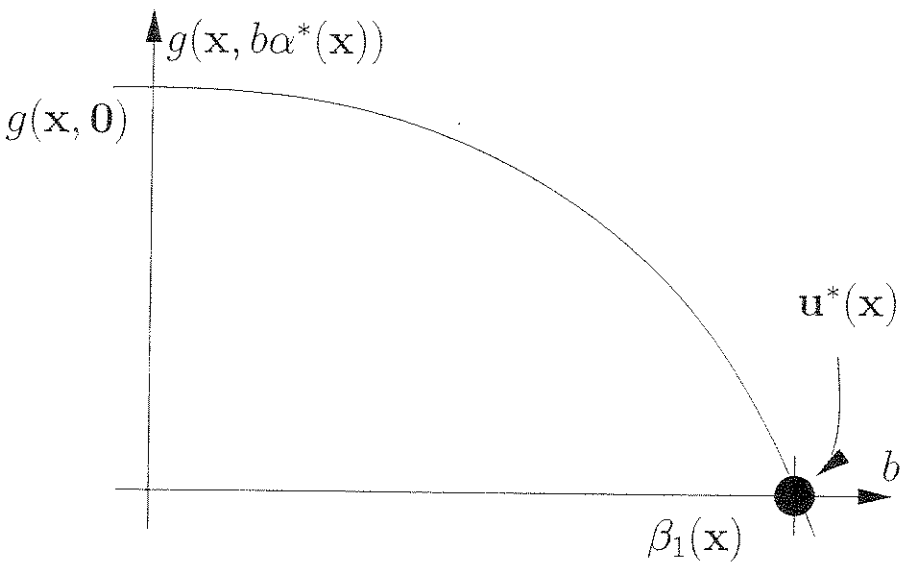

Figure 4.4: Non-affine limit-state function along $\alpha^*(\mathbf{x})$.

$\mathbf{x}$ along the line through the origin and the point $\hat{\mathbf{u}}(\mathbf{x})$. Obviously, the approximation in the objective function in $\mathrm{P}_{3,r,s}$ is exact when $\mathbf{u}^*(\mathbf{x})$ coincides with $\hat{\mathbf{u}}(\mathbf{x})$. Furthermore, when the first-order approximation in (2.1.7) is used, the approximation in the objective function in $\mathrm{P}_{3,r,s}$ is exact when $r = \beta_1(\mathbf{x})$.

Observe that $\mathrm{P}_{3,r,s}$ is not a semi-infinite optimization problem in the form described in Chapter 3. Hence, we proceed with the second step leading to a generalized semi-infinite optimization problem solvable under certain assumptions.

Essentially, the second step consists of moving the max-operator in the denominator of the objective function in $\mathrm{P}_{3,r,s}$, see (4.2.3), out in front of the expression. This operation can be performed if we restrict the values of $\mathbf{u}$ to a subset of $\mathbb{B}(\mathbf{0}, r)$ as rigorously stated in Theorem 4.2.2 below. The new approximating problem takes the form

$\mathrm{P}_{3,r,s,\gamma}$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \max_{\mathbf{u}\in\mathrm{Y}_{r\gamma}(\mathbf{x})} \left\{ c_0(\mathbf{x}) + c(\mathbf{x})\Phi\left(\frac{rg(\mathbf{x},\mathbf{0})}{g(\mathbf{x},\mathbf{u}) - g(\mathbf{x},\mathbf{0})}\right) \mid \psi_s(\mathbf{x}) \leq 0,\ \mathbf{x}\in\mathrm{X}\right\}, \quad (4.2.7)$$

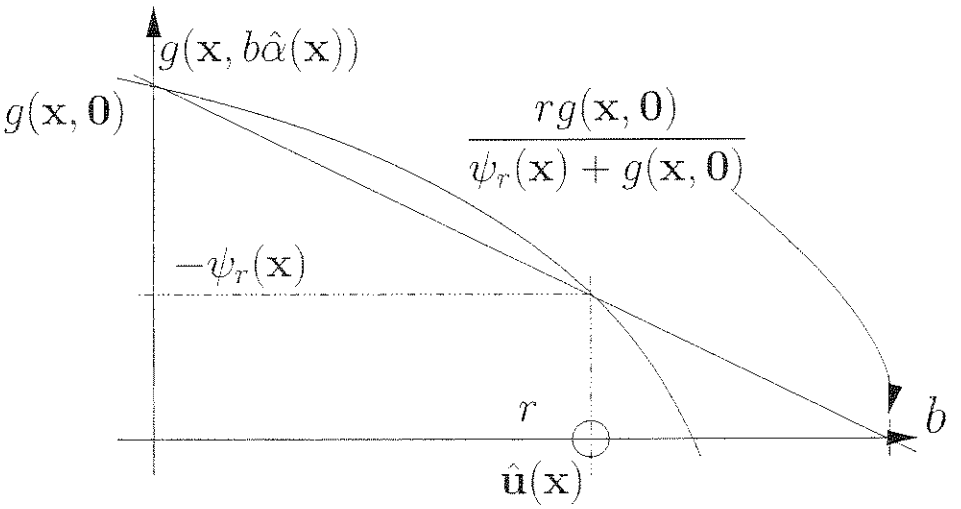

Figure 4.5: Non-affine limit-state function along $\hat{\alpha}(\mathbf{x})$.

where for $\gamma > 0$

$$\mathbf{Y}_{r\gamma}(\mathbf{x}) \stackrel{\triangle}{=} \{\mathbf{u} \in \mathbb{B}(\mathbf{0}, r) \mid g(\mathbf{x}, \mathbf{u}) - g(\mathbf{x}, \mathbf{0}) \leq -\gamma\}. \tag{4.2.8}$$

Let for any $s > 0$,

$$\mathbf{F}_{3,s} \stackrel{\triangle}{=} \{\mathbf{x} \in \mathbf{X} \mid \psi_s(\mathbf{x}) \leq 0\}. \tag{4.2.9}$$

This problem is related to $\mathbf{P}_{3,r,s}$ as stated in the next theorem.

Theorem 4.2.2. *Consider $\mathbf{P}_{3,r,s}$ with $r, s > 0$. Suppose that the functions $c_0(\cdot), c(\cdot)$ and $g(\cdot,\cdot)$ are continuous, the feasible set $\mathbf{F}_{3,s}$ is compact, $g(\mathbf{x}, \mathbf{0}) > 0$ for all $\mathbf{x} \in \mathbf{F}_{3,s}$, and that for some $\gamma > 0$, $\mathbf{Y}_{r\gamma}(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in \mathbf{F}_{3,s}$. Then $\mathbf{x}^*$ solves $\mathbf{P}_{3,r,s}$ if and only if it solves $\mathbf{P}_{3,r,s,\gamma}$.*

Proof. The constraint sets in $\mathbf{P}_{3,r,s}$ and $\mathbf{P}_{3,r,s,\gamma}$ are identical. Hence, we turn our attention to the objective functions. Since $\mathbf{Y}_{r\gamma}(\mathbf{x}) \subset \mathbb{B}(\mathbf{0}, r)$ for all $\mathbf{x} \in \mathbf{F}_{3,s}$, we must have

$$\min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, r)} g(\mathbf{x}, \mathbf{u}) \leq \min_{\mathbf{u} \in \mathbf{Y}_{r\gamma}(\mathbf{x})} g(\mathbf{x}, \mathbf{u}). \tag{4.2.10}$$

Suppose for the sake of a contradiction that strict inequality holds in (4.2.10). Then

there exists

$$\hat{\mathbf{u}} \in \arg \min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, r)} g(\mathbf{x}, \mathbf{u}) \tag{4.2.11}$$

such that $\hat{\mathbf{u}} \notin \mathbf{Y}_{r\gamma}(\mathbf{x})$. Hence,

$$g(\mathbf{x}, \hat{\mathbf{u}}) > g(\mathbf{x}, \mathbf{0}) - \gamma. \tag{4.2.12}$$

But since $\mathbf{Y}_{r\gamma}(\mathbf{x}) \neq \emptyset$, there must exist $\mathbf{u}^* \in \mathbf{Y}_{r\gamma}(\mathbf{x})$ such that

$$g(\mathbf{x}, \mathbf{u}^*) \leq g(\mathbf{x}, \mathbf{0}) - \gamma. \tag{4.2.13}$$

Hence, $\hat{\mathbf{u}}$ cannot be a minimizer of $g(\mathbf{x}, \mathbf{u})$ on $\mathbb{B}(\mathbf{0}, r)$, which is a contradiction. Consequently,

$$\min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, r)} g(\mathbf{x}, \mathbf{u}) = \min_{\mathbf{u} \in \mathbf{Y}_{r\gamma}(\mathbf{x})} g(\mathbf{x}, \mathbf{u}). \tag{4.2.14}$$

Next, for any compact set $\mathbf{X}' \subset \mathbb{R}^n$ and any continuous function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\max_{\mathbf{x} \in \mathbf{X}'} f(\mathbf{x}) = -\min_{\mathbf{x} \in \mathbf{X}'} \{-f(\mathbf{x})\}. \tag{4.2.15}$$

Furthermore, if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbf{X}'$, it follows by inspection that

$$\max_{\mathbf{x} \in \mathbf{X}'} \frac{1}{f(\mathbf{x})} = \frac{1}{\min_{\mathbf{x} \in \mathbf{X}'} f(\mathbf{x})}. \tag{4.2.16}$$

Hence, by (4.2.14), (4.2.15), (4.2.16), and the fact that $\Phi(\cdot)$ is strictly increasing, we obtain

$$\begin{aligned} \Phi\left(\frac{-rg(\mathbf{x}, \mathbf{0})}{\psi_r(\mathbf{x}) + g(\mathbf{x}, \mathbf{0})}\right) &= \Phi\left(\frac{-rg(\mathbf{x}, \mathbf{0})}{\max\{g(\mathbf{x}, \mathbf{u}) \mid \mathbf{u} \in \mathbb{B}(\mathbf{0}, r)\} + g(\mathbf{x}, \mathbf{0})}\right) \\ &= \Phi\left(\frac{rg(\mathbf{x}, \mathbf{0})}{\min\{g(\mathbf{x}, \mathbf{u}) - g(\mathbf{x}, \mathbf{0}) \mid \mathbf{u} \in \mathbf{Y}_{r\gamma}(\mathbf{x})\}}\right) \\ &= \max_{\mathbf{u} \in \mathbf{Y}_{r\gamma}(\mathbf{x})} \Phi\left(\frac{rg(\mathbf{x}, \mathbf{0})}{g(\mathbf{x}, \mathbf{u}) - g(\mathbf{x}, \mathbf{0})}\right). \end{aligned} \tag{4.2.17}$$

Since $c_0(\cdot)$ and $c(\cdot)$ are independent of $\mathbf{u}$, we see that the objective functions in $\mathbf{P}_{3,r,s}$ and $\mathbf{P}_{3,r,s,\gamma}$ are identical. Hence, the proof is complete. □

In view of Theorems 4.2.1 and 4.2.2, we see that $\mathbf{P}_3$, with $K = 1$, and $\mathbf{P}_{3,r,s,\gamma}$ have identical solutions when $g(\mathbf{x}, \cdot)$ is affine, $r > 0, s = -\Phi^{-1}(\hat{p})$, and $\gamma > 0$ is sufficiently

small. Furthermore, for non-affine limit-state functions, the solution $\mathbf{x}^*$ of $\mathbf{P}_{3,r,s,\gamma}$, with $r = s = -\Phi^{-1}(\hat{p})$, is correct to first-order approximation if $\psi_r(\mathbf{x}^*) = 0$, i.e., the constraint is active at $\mathbf{x}^*$. Otherwise, $\mathbf{P}_{3,r,s,\gamma}$ is a coarser approximation to $\mathbf{P}_3$.

The problem $\mathbf{P}_{3,r,s,\gamma}$ is a *generalized* semi-infinite optimization problem because the set $\mathbf{Y}_{r\gamma}(\mathbf{x})$ is a function of the design variables $\mathbf{x}$. Hence, the algorithms described for semi-infinite optimization problems in Chapter 3 are not applicable. In the following, we present two alternative strategies for solving $\mathbf{P}_{3,r,s,\gamma}$.

Algorithms

The first of two strategies for solving $\mathbf{P}_{3,r,s,\gamma}$ is based on the assumption that the set $\mathbf{Y}_{r\gamma}(\mathbf{x})$ can be replaced by a set independent of $\mathbf{x}$. This results in a semi-infinite optimization problem of the type described in Chapter 3, and hence, it can be solved by the algorithms described therein.

Assumption 4.2.1. We assume that there exists a subset $\mathbf{Y}'_{r,s} \subset \mathbb{B}(\mathbf{0}, r) \subset \mathbb{R}^m$ such that

(i) $g(\mathbf{x}, \mathbf{u}) - g(\mathbf{x}, \mathbf{0}) < 0$ for all $\mathbf{x} \in \mathbf{F}_{3,s}$ and $\mathbf{u} \in \mathbf{Y}'_{r,s}$, and

(ii) for all $\mathbf{x} \in \mathbf{F}_{3,s}$ (defined in (4.2.9))

$$\min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, r)} g(\mathbf{x}, \mathbf{u}) = \min_{\mathbf{u} \in \mathbf{Y}'_{r,s}} g(\mathbf{x}, \mathbf{u}). \tag{4.2.18}$$

□

Note that the set $\mathbf{Y}'_{r,s}$ depends on the parameter s because the set $\mathbf{F}_{3,s}$ depends on s. In a practical design problem, it can be difficult to construct a set $\mathbf{Y}'_{r,s}$. However, in some cases there is a clear interpretation of the components of the random vector $\mathbf{U}$, which may lead to a construction of $\mathbf{Y}'_{r,s}$. For example, $\mathbf{Y}'_{r,s}$ may be defined as a subset of a quadrant (m=2), octant (m = 3), etc., of $\mathbb{B}(\mathbf{0}, r)$ containing the minimizer in (4.2.11) for all $\mathbf{x} \in \mathbf{F}_{3,s}$. An initial first-order reliability analysis with

a well-chosen feasible $\mathbf{x}$ may help in determining an appropriate description of $\mathbf{Y}'_{r,s}$. The replacement of the set $\mathbf{Y}_{r\gamma}(\mathbf{x})$ by $\mathbf{Y}'_{r,s}$ results in the following problem.

$\mathbf{P}'_{3,r,s}$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \max_{\mathbf{u}\in\mathbf{Y}'_{r,s}} \left\{ c_0(\mathbf{x}) + c(\mathbf{x})\Phi\left(\frac{rg(\mathbf{x},\mathbf{0})}{g(\mathbf{x},\mathbf{u}) - g(\mathbf{x},\mathbf{0})}\right) \mid \psi_s(\mathbf{x}) \leq 0,\ \mathbf{x}\in\mathbf{X}\right\}, \qquad (4.2.19)$$

where $\mathbf{Y}'_{r,s}$ satisfies Assumption 4.2.1.

The following algorithms for $\mathbf{P}_3$ solve a sequence of $\mathbf{P}'_{3,r,s}$ problems under Assumption 4.2.1. The algorithms are similar to the ones described in Section 4.1.

Algorithm 4.2.1. (For $\mathbf{P}_3$, $K = 1$)

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$, $\mathbf{Y}'_{r,s} \subset \mathbb{B}(\mathbf{0}, r)$ satisfying Assumption 4.2.1 for all $r, s > 0$.

Step 0. Set $i = 0$, $r_0 = s_0 = -\Phi^{-1}(\hat{p})$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\mathbf{P}'_{3,r_i,s_i}$, defined in (4.2.19), with initial point $\mathbf{x}_i$.

Step 2. Compute an appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$ for the current run of Algorithm 4.2.1, see Definition 4.1.1.

Step 3. Set r_{i+1} so that it approximately satisfies

$$\Phi\left(\frac{-r_{i+1}g(\mathbf{x}_{i+1},\mathbf{0})}{\psi_{r_{i+1}}(\mathbf{x}_{i+1}) + g(\mathbf{x}_{i+1},\mathbf{0})}\right) = \tilde{p}(\mathbf{x}_{i+1}). \qquad (4.2.20)$$

Step 4. Set

$$s_{i+1} = s_i \frac{\Phi^{-1}(\hat{p})}{\Phi^{-1}(\tilde{p}(\mathbf{x}_{i+1}))}. \qquad (4.2.21a)$$

Step 5. Replace i by $i+1$ and go to **Step 1**. □

We may be interested in a first-order approximation to the structural reliability, i.e., in $\mathbf{P}_3$, we replace $p(\mathbf{x})$ by $\Phi(-\beta_1(\mathbf{x}))$, defined in (2.1.8). This leads to a slightly

simplified algorithm.

Algorithm 4.2.2. (For $\mathbf{P}_3$, $K = 1$, First-Order Reliability)

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$, $\mathbf{Y}'_{r,s} \subset \mathbb{B}(\mathbf{0}, r)$ satisfying Assumption 4.2.1 for all $r, s > 0$.

Step 0. Set $i = 0$, $r_0 = s = -\Phi^{-1}(\hat{p})$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\mathbf{P}'_{3,r_i,s}$, defined in (4.2.19), with initial point $\mathbf{x}_i$.

Step 2. Compute the first-order reliability index $\beta_1(\mathbf{x}_{i+1})$ to a level of accuracy that is considered acceptable.

Step 3. Set $r_{i+1} = \beta_1(\mathbf{x}_{i+1})$.

Step 4. Replace i by $i + 1$ and go to **Step 1**. □

The second strategy for solving $\mathbf{P}_{3,r,s,\gamma}$ is based on an algorithm to be described in Chapter 5. Prior to this study, there was no implementable algorithm for solving problems of the form $\mathbf{P}_{3,r,s,\gamma}$. The algorithm in Chapter 5 is the first implementable algorithm for problems in the form $\mathbf{P}_{3,r,s,\gamma}$, but it does not consider constraints. Hence, for the unconstrained version of $\mathbf{P}_3$ we can use the following algorithm. Let

$$\mathbf{P}_3^* \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \{c_0(\mathbf{x}) + c(\mathbf{x})p(\mathbf{x})\}. \qquad (4.2.21b)$$

Theoretically, a well-formulated objective function incorporating the cost of failure may make constraints superfluous. Hence, unconstrained problems of the form $\mathbf{P}_3^*$ can arise in practical structural design.

Algorithm 4.2.3. (For $\mathbf{P}_3^*$, $K = 1$)

Parameter $\gamma > 0$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $r_0 = s_0 = -\Phi^{-1}(\hat{p})$.

Step 1. Set $\mathbf{x}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 5.4.1 (see Chapter 5) on the problem $\mathbf{P}_{3,r_i,s_i,\gamma}$, defined in (4.2.19), with no constraints and initial point $\mathbf{x}_i$.

Step 2. Compute an appropriate estimate $\tilde{p}(\mathbf{x}_{i+1})$ of $p(\mathbf{x}_{i+1})$ for the current run of Algorithm 4.2.3, see Definition 4.1.1.

Step 3. Set r_{i+1} so that it approximately satisfies

$$\Phi\left(\frac{-r_{i+1}g(\mathbf{x}_{i+1},\mathbf{0})}{\psi_{r_{i+1}}(\mathbf{x}_{i+1}) + g(\mathbf{x}_{i+1},\mathbf{0})}\right) = \tilde{p}(\mathbf{x}_{i+1}). \tag{4.2.21c}$$

Step 4. Set

$$s_{i+1} = s_i \frac{\Phi^{-1}(\hat{p})}{\Phi^{-1}(\tilde{p}(\mathbf{x}_{i+1}))}. \tag{4.2.21d}$$

Step 5. Replace i by $i+1$ and go to **Step 1.** □

We cannot guarantee that any of the Algorithms 4.2.1, 4.2.2 and 4.2.3 converges to the solution of the respective design problem under general conditions. However, since the design found is approximately optimal with respect to a first-order reliability approximation, we expect that results generated by Algorithms 4.2.1, 4.2.2 or 4.2.3 will be sufficiently close to the true optimal solution for practical use.

4.2.2 Algorithm for the Solution of $\mathbf{P}_3$

Approximating Problems

We now return to the general form of $\mathbf{P}_3$ as defined in (4.2.1). We construct approximating problems for $\mathbf{P}_3$ by replacing the failure probabilities in the objective function of $\mathbf{P}_3$ with parameters. The parameters are included in an augmented design vector and, hence, their values are automatically determined by the optimization procedure.

Let $\overline{\mathbf{x}} = (\mathbf{x}, \mathbf{a}) \in \mathbb{R}^{n+K}$ be an augmented design vector, where $\mathbf{x} \in \mathbb{R}^n$ is the original design vector and $\mathbf{a} = (a_1, ..., a_K) \in \mathbb{R}^K$ is a vector of K parameters. We define the problem

$$\hat{\mathbf{P}}_3 \quad \min_{(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \mid p_k(\mathbf{x}) = a_k,\ 0 \le a_k \le \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \tag{4.2.22}$$

The equivalence between $\mathbf{P}_3$ and $\hat{\mathbf{P}}_3$ is clear from the following theorem:

Theorem 4.2.3. *Consider* $\mathbf{P}_3$. *Suppose that Assumption 4.1.1 is satisfied, the functions* $c_0(\cdot), c_k(\cdot), g_k(\cdot,\cdot), k \in \mathbf{K}$, *are continuous, and the set* $\mathbf{X}$ *is compact. Then,* $\mathbf{P}_3$ *and* $\hat{\mathbf{P}}_3$ *are equivalent in the sense that*

$$\begin{aligned} &\min_{(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \mid p_k(\mathbf{x}) = a_k,\ 0 \le a_k \le \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\} \\ &= \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \mid p_k(\mathbf{x}) \le \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \end{aligned} \tag{4.2.23}$$

Proof. By Theorem 4.2.1, $\mathbf{P}_3$ has a solution and the solution is a finite number. First observe that for any parameter vector $\mathbf{a} \in \mathbb{R}^K$, with $0 \le a_k \le \hat{p}_k, k \in \mathbf{K}$, the set

$$\hat{\mathbf{F}}(\mathbf{a}) \triangleq \{\mathbf{x} \in \mathbf{X} \mid p_k(\mathbf{x}) = a_k, k \in \mathbf{K}\}, \tag{4.2.24}$$

is contained in the feasible set

$$\mathbf{F}_3 \triangleq \{\mathbf{x} \in \mathbf{X} \mid p_k(\mathbf{x}) \le \hat{p}_k, k \in \mathbf{K}\}, \tag{4.2.25}$$

and that the objective functions in $\mathbf{P}_3$ and $\hat{\mathbf{P}}_3$ are identical on $\hat{\mathbf{F}}(\mathbf{a})$. Hence, for any $\mathbf{a} \in \mathbb{R}^K$, with $0 \le a_k \le \hat{p}_k, k \in \mathbf{K}$,

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \mid p_k(\mathbf{x}) \le \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\} \\ &\le \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \mid p_k(\mathbf{x}) = a_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}. \end{aligned} \tag{4.2.26}$$

Suppose $\mathbf{x}^*$ is a solution of $\mathbf{P}_3$. Set $a_k = p_k(\mathbf{x}^*), k \in \mathbf{K}$. By (4.2.26) and the fact that $\mathbf{x}^*$ is a solution of $\mathbf{P}_3$, we obtain

$$\begin{aligned}
& c_0(\mathbf{x}^*) + \sum_{k=1}^{K} c_k(\mathbf{x}^*) p_k(\mathbf{x}^*) \\
&= \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \mid p_k(\mathbf{x}) \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\} \\
&\leq \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}^*) \mid p_k(\mathbf{x}) = p_k(\mathbf{x}^*), k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\} \qquad (4.2.27) \\
&= \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) p_k(\mathbf{x}) \mid p_k(\mathbf{x}) = p_k(\mathbf{x}^*), k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\} \\
&= c_0(\mathbf{x}^*) + \sum_{k=1}^{K} c_k(\mathbf{x}^*) p_k(\mathbf{x}^*).
\end{aligned}$$

Hence, minimization over $\mathbf{a} \in \mathbb{R}^K$ satisfying $0 \leq a_k \leq \hat{p}_k, k \in \mathbf{K}$, of the right-hand side of (4.2.26) must yield (4.2.23). This completes the proof. □

It can be seen from (4.2.22) that $\hat{\mathbf{P}}_3$ is a minimization problem of a smooth objective function with failure probability *equality* constraints. This is similar to $\mathbf{P}_1$, but $\mathbf{P}_1$ contains *inequality* constraints. We proceed along the lines in Section 4.1 and construct an approximating problem with semi-infinite equality constraints. For any $\mathbf{t} = (t_1, ..., t_K) \in \mathbb{R}^K$, with $t_k > 0, k \in \mathbf{K}$, we define the approximating problem

$\hat{\mathbf{P}}'_{3,\mathbf{t}}$

$$\min_{\overline{\mathbf{x}}=(\mathbf{x},\mathbf{a}) \in \mathbb{R}^{n+K}} \left\{ c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k \mid \hat{\psi}_{k,t_k}(\overline{\mathbf{x}}) = 0,\ 0 \leq a_k \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X} \right\}, \qquad (4.2.28)$$

where, for any $\alpha > 0$, $k \in \mathbf{K}$, $\hat{\psi}_{k,\alpha} : \mathbb{R}^{n+K} \to \mathbb{R}$ is defined by

$$\hat{\psi}_{k,\alpha}(\overline{\mathbf{x}}) \triangleq \max_{\mathbf{u} \in \mathbb{B}(\mathbf{0},1)} \{-g_k(\mathbf{x}, -\Phi^{-1}(a_k)\alpha \mathbf{u})\}. \qquad (4.2.29)$$

The relation between $\hat{\mathbf{P}}_3$ and $\hat{\mathbf{P}}'_{3,\mathbf{t}}$ is given in the following theorem:

Theorem 4.2.4. *Suppose that Assumption 4.1.1 holds, the functions* $c_0(\cdot)$, $c_k(\cdot)$, $g_k(\cdot,\cdot)$, $k \in \mathbf{K}$, *are continuous, the set* $\mathbf{X}$ *is compact, the limit-state functions* $g_k(\cdot,\cdot), k \in \mathbf{K}$, *are affine in their second argument,* $\hat{p}_k < 0.5, k \in \mathbf{K}$, *and that* $g_k(\mathrm{x}, \mathbf{0}) > 0$ *for all* $\mathrm{x} \in \mathbf{X}, k \in \mathbf{K}$. *If* $t_k = 1, k \in \mathbf{K}$, *then* x^* *solves* $\hat{\mathbf{P}}_3$ *if and only if it solves* $\hat{\mathbf{P}}'_{3,\mathrm{t}}$.

Proof. By a linear transformation, we obtain that

$$\hat{\psi}_{k,\alpha}(\overline{\mathrm{x}}) = \max_{\mathbf{u} \in \mathbb{B}(0, -\Phi^{-1}(a_k)\alpha)} \{-g_k(\mathrm{x}, \mathbf{u})\}, \tag{4.2.30}$$

where $-\Phi^{-1}(a_k)\alpha$ is a positive number because of the assumption that $a_k \leq \hat{p}_k < 0.5, k \in \mathbf{K}$. It now follows by the same arguments as in the proof of Theorem 4.1.2 that $\hat{\psi}_{k,1}(\overline{\mathrm{x}}) = 0$ if and only if $p_k(\mathrm{x}) = a_k$, where $\overline{\mathrm{x}} = (\mathrm{x}, a)$. This completes the proof. □

Since $\hat{\mathbf{P}}'_{3,\mathrm{t}}$ contains semi-infinite equality constraints, we cannot directly use the algorithms described in Chapter 3, which are for inequality constraints. However, as the following theorem shows, the equalities in $\hat{\mathbf{P}}'_{3,\mathrm{t}}$ can be replaced by inequalities without altering the solution of $\hat{\mathbf{P}}'_{3,\mathrm{t}}$. For any $\mathrm{t} = (t_1, ..., t_K) \in \mathbb{R}^K$ with $t_k > 0, k \in \mathbf{K}$, we define the problem $\hat{\mathbf{P}}_{3,\mathrm{t}}$, which is identical to $\hat{\mathbf{P}}'_{3,\mathrm{t}}$ except that the equality constraints are replaced by inequalities.

$\hat{\mathbf{P}}_{3,\mathrm{t}}$

$$\min_{\overline{\mathrm{x}}=(\mathrm{x},\mathrm{a}) \in \mathbb{R}^{n+K}} \left\{ c_0(\mathrm{x}) + \sum_{k=1}^{K} c_k(\mathrm{x}) a_k \mid \hat{\psi}_{k,t_k}(\overline{\mathrm{x}}) \leq 0,\ 0 \leq a_k \leq \hat{p}_k, k \in \mathbf{K},\ \mathrm{x} \in \mathbf{X} \right\}. \tag{4.2.31}$$

Theorem 4.2.5. *Suppose that Assumption 4.1.1 holds, the functions* $c_0(\cdot)$, $c_k(\cdot)$, $g_k(\cdot,\cdot)$, $k \in \mathbf{K}$, *are continuous, the set* $\mathbf{X}$ *is compact,* $c_k(\mathrm{x}) > 0$ *for all* $k \in \mathbf{K}$ *and* $\mathrm{x} \in \mathbf{X}$, $\hat{p}_k < 0.5, k \in \mathbf{K}$, *and that for all* $k \in \mathbf{K}$ *and* $\mathrm{x} \in \mathbf{X}$ *there exists a* $\mathbf{u} \in \mathbb{R}^m$ *such that* $g_k(\mathrm{x}, \mathrm{u}) = 0$. *Then,* x^* *solves* $\hat{\mathbf{P}}_{3,\mathrm{t}}$ *if and only if it solves* $\hat{\mathbf{P}}'_{3,\mathrm{t}}$.

Proof. Suppose that $\overline{\mathrm{x}}^* = (\mathrm{x}^*, \mathrm{a}^*)$ is a solution of $\hat{\mathbf{P}}_{3,\mathrm{t}}$. Then, $\hat{\psi}_{k,t_k}(\overline{\mathrm{x}}^*) \leq 0,\ 0 \leq$

$a_k^* \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x}^* \in \mathbf{X}$, and

$$c_0(\mathbf{x}^*) + \sum_{k=1}^{K} c_k(\mathbf{x}^*) a_k^* \leq c_0(\mathbf{x}) + \sum_{k=1}^{K} c_k(\mathbf{x}) a_k, \tag{4.2.32a}$$

for all $\overline{\mathbf{x}} = (\mathbf{x}, \mathbf{a})$ satisfying $\hat{\psi}_{k,t_k}(\overline{\mathbf{x}}) \leq 0,\ 0 \leq a_k \leq \hat{p}_k, k \in \mathbf{K},\ \mathbf{x} \in \mathbf{X}$. Suppose, for the sake of a contradiction, that there exists a $\mathbf{K}^* \subset \mathbf{K}$ such that $\mathbf{K}^* \neq \emptyset$ and $\hat{\psi}_{k,t_k}(\overline{\mathbf{x}}^*) < 0$ for all $k \in \mathbf{K}^*$. Then we have from (4.2.30) that

$$\min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, -\Phi^{-1}(a_k^*) t_k)} g_k(\mathbf{x}^*, \mathbf{u}) > 0, \tag{4.2.32b}$$

for all $k \in \mathbf{K}^*$. By the assumption that $g_k(\mathbf{x}^*, \mathbf{u}) = 0$ for some $\mathbf{u}$ and the fact that $-\Phi^{-1}(\omega') > -\Phi^{-1}(\omega'')$ if and only if $\omega'' > \omega'$, there must exist for each $k \in \mathbf{K}^*$ an $a_k' \in (0, a_k^*)$ such that

$$\min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, -\Phi^{-1}(a_k') t_k)} g_k(\mathbf{x}^*, \mathbf{u}) = 0. \tag{4.2.32c}$$

Let $\mathbf{x}' = \mathbf{x}^*$, $a_k' = a_k^*, k \notin \mathbf{K}^*$, and let a_k' be as above for $k \in \mathbf{K}^*$. Furthermore, let $\mathbf{a}' = (a_1', a_2', ..., a_K')$ and $\overline{\mathbf{x}}' = (\mathbf{x}', \mathbf{a}')$. Clearly, $\overline{\mathbf{x}}'$ satisfies $\mathbf{x}' \in \mathbf{X}, 0 \leq a_k' \leq \hat{p}_k, k \in \mathbf{K}$, and $\hat{\psi}_{k,t_k}(\overline{\mathbf{x}}') = 0, k \notin \mathbf{K}^*, k \in \mathbf{K}$. Additionally, from (4.3.32c) we have that $\hat{\psi}_{k,t_k}(\overline{\mathbf{x}}') = 0, k \in \mathbf{K}^*$. Hence, $\overline{\mathbf{x}}'$ is a feasible point for $\hat{\mathbf{P}}_{3,t}$. Since $a_k' < a_k^*$ for all $k \in \mathbf{K}^* \neq \emptyset$ and $c_k(\mathbf{x}) > 0$ for all $k \in \mathbf{K}$ and $\mathbf{x} \in \mathbf{X}$, we have contradicted (4.2.32a). Consequently, every solution $\overline{\mathbf{x}}^* = (\mathbf{x}^*, \mathbf{a}^*)$ of $\hat{\mathbf{P}}_{3,t}$ must satisfy $\hat{\psi}_{k,t_k}(\overline{\mathbf{x}}^*) = 0, k \in \mathbf{K}$, and hence the conclusion follows. □

Note that the assumption that $c_k(\mathbf{x}) > 0$ for all $k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}$ in Theorem 4.2.5 is trivially satisfied in application because $c_k(\mathbf{x})$ is the monetary cost of failure of the k-th component for design $\mathbf{x}$. The assumption that $\hat{p}_k < 0.5, k \in \mathbf{K}$, is generally satisfied due to the high reliability of structural systems. Furthermore, the assumption that for all $k \in \mathbf{K}, \mathbf{x} \in \mathbf{X}$ there exists a $\mathbf{u} \in \mathbb{R}^m$ such that $g_k(\mathbf{x}, \mathbf{u}) = 0$ is usually satisfied. If there exists an $\mathbf{x} \in \mathbf{X}$ such that $g_k(\mathbf{x}, \mathbf{u}) \neq 0$ for all $\mathbf{u} \in \mathbb{R}^m$, then the corresponding component cannot fail for the given design vector $\mathbf{x}$. Such "absolutely safe" designs are unrealistic in applications. If a mathematical model should give rise to a limit-state function not satisfying the assumption, then the design space can be

limited to a subset $\mathbf{X}' \subset \mathbf{X}$ on which the assumption holds. Relevant designs *not* in $\mathbf{X}'$ are special cases, which typically give rise to trivial design problems.

Algorithm

In view of Theorems 4.2.3, 4.2.4 and 4.2.5, we can derive an algorithm for solving $\mathbf{P}_3$ based on the solution of a sequence of the problems $\hat{\mathbf{P}}_{3,\mathbf{t}}$ for varying values of the approximation parameters $\mathbf{t}$ and the use of the semi-infinite optimization algorithms in Chapter 3. The next algorithm is based on this idea.

Algorithm 4.2.4. (For $\mathbf{P}_3$)

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $\mathbf{a}_0 = (\hat{p}_1, ..., \hat{p}_K)$, $\mathbf{t}_0 = (1, ..., 1) \in \mathbb{R}^K$, $\overline{\mathbf{x}}_0 = (\mathbf{x}_0, \mathbf{a}_0)$.

Step 1. Set $\overline{\mathbf{x}}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\hat{\mathbf{P}}_{3,\mathbf{t}_i}$, defined in (4.2.31), with initial point $\overline{\mathbf{x}}_i$.

Step 2. Compute appropriate estimates $\tilde{p}_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$, of $p_k(\mathbf{x}_{i+1}), k \in \mathbf{K}$, for the current run of Algorithm 4.2.4, see Definition 4.1.1.

Step 3. For $k \in \mathbf{K}$, set

$$(t_k)_{i+1} = \frac{(t_k)_i \Phi^{-1}((a_k)_{i+1})}{\Phi^{-1}(\tilde{p}_k(\mathbf{x}_{i+1}))}, \tag{4.2.36}$$

where $\overline{\mathbf{x}}_{i+1} = (\mathbf{x}_{i+1}, \mathbf{a}_{i+1})$, with $\mathbf{a}_{i+1} = ((a_1)_{i+1}, ..., (a_K)_{i+1})$.

Step 4. Replace i by $i + 1$ and go to **Step 1**. □

We cannot guarantee that Algorithm 4.2.4 converges to the solution of $\mathbf{P}_3$. However, since the design found is approximately optimal with respect to a first-order approximation, we expect that results generated by Algorithm 4.2.4 will be sufficiently close to the true optimal solution for practical use. Typically, structural systems are reliable, and hence the failure probabilities tend to be small, i.e., $p(\mathbf{x})$ is small.

In implementation of Algorithm 4.2.4, numerical difficulties caused by the potential difference in the orders of magnitude of the components of $\mathbf{a}$ and $\mathbf{x}$ can be avoided by use of the transformation $a_k \triangleq \Phi(-b_k), k \in \mathbf{K}$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function. Then the optimization in Algorithm 4.2.4 is over the vector $(\mathbf{x}, \mathbf{b})$, where $\mathbf{b} = (b_1, ..., b_K)$.

4.2.3 Algorithm for the Solution of $\mathbf{P}_{3,\text{sys}}$

Consider the problem $\mathbf{P}_{3,\text{sys}}$ defined in (4.2.2). We follow the same approach as the one in Sub-section 4.2.2. Consequently, let

$\hat{\mathbf{P}}_{3,\text{sys}}$

$$\min_{(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) a_l \mid p^{(l)}(\mathbf{x}) = a_l,\ 0 \le a_l \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X} \right\}. \tag{4.2.37}$$

The equivalence between $\mathbf{P}_{3,\text{sys}}$ and $\hat{\mathbf{P}}_{3,\text{sys}}$ is clear from the next theorem, which follows by the same arguments as Theorem 4.2.3.

Theorem 4.2.7. *Consider* $\mathbf{P}_{3,\text{sys}}$. *Suppose that Assumption 4.1.1 is satisfied, the functions* $c_0^{(l)}(\cdot), c^{(l)}(\cdot), g_k^{(l)}(\cdot,\cdot), k \in \mathbf{K}_l, l \in \mathbf{L}$, *are continuous, and the set* $\mathbf{X}$ *is compact. Then,* $\mathbf{P}_{3,\text{sys}}$ *and* $\hat{\mathbf{P}}_{3,\text{sys}}$ *are equivalent in the sense that*

$$\begin{aligned} &\min_{(\mathbf{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) a_l \mid p^{(l)}(\mathbf{x}) = a_l,\ 0 \le a_l \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X} \right\} \\ &= \min_{\mathbf{x}\in\mathbb{R}^{n}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) p^{(l)}(\mathbf{x}) \mid p^{(l)}(\mathbf{x}) \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X} \right\}. \end{aligned} \tag{4.2.38}$$

□

We construct an approximating problem to $\hat{\mathbf{P}}'_{3,\text{sys}}$ with semi-infinite equality constraints. For any $\mathbf{t} = (t_1, ..., t_L) \in \mathbb{R}^L$, with $t_l > 0, l \in \mathbf{L}$, we define the approximating problem

$\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$

$$\min_{\overline{\mathrm{x}}=(\mathrm{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathrm{x}) + \sum_{l=1}^{L} c^{(l)}(\mathrm{x}) a_l \mid \hat{\psi}_{t_l}^{(l)}(\overline{\mathrm{x}}) = 0,\ 0 \le a_l \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathrm{x} \in \mathbf{X} \right\}, \tag{4.2.39}$$

where, for any $\alpha > 0$ and $l \in \mathbf{L}$, $\hat{\psi}_{\alpha}^{(l)} : \mathbb{R}^{n+L} \to \mathbb{R}$ is defined by

$$\hat{\psi}_{\alpha}^{(l)}(\overline{\mathrm{x}}) \triangleq \max_{k\in\mathbf{K}_l} \max_{\mathrm{u}\in\mathbb{B}(\mathbf{0},1)} \{-g_k^{(l)}(\mathrm{x}, -\Phi^{-1}(a_l)\alpha\mathrm{u})\}. \tag{4.2.40}$$

We are not able to prove an equivalence theorem of the same nature as Theorem 4.2.4 for the present case. However, if the limit-state functions $g_k^{(l)}(\cdot,\cdot), k \in \mathbf{K}_l, l \in \mathbf{L}$ were affine in their respective second arguments, then $\hat{\psi}_{t_l}^{(l)}(\overline{\mathrm{x}}) = 0$ implies that the critical failure component, say k'_l, of the l-th structure has failure probability $p_{k'_l}^{(l)}(\mathrm{x}) = \Phi(-\Phi^{-1}(a_l)t_l)$. Hence, when $t_l = 1$, $p_{k'_l}^{(l)}(\mathrm{x}) = a_l$. Due to the close relation between the component failure probabilities and the failure probability of the series system, see (4.1.15b), we can adjust t_l such that $p^{(l)}(\mathrm{x}) \approx a_l$ whenever $\hat{\psi}_{t_l}^{(l)}(\overline{\mathrm{x}}) = 0$. Hence, $\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$ is a good approximation to $\hat{\mathrm{P}}_{3,\mathrm{sys}}$ for a suitable selection of $\mathbf{t} = (t_1, ..., t_L)$, i.e., for all $l \in \mathbf{L}$,

$$\{\overline{\mathrm{x}} = (\mathrm{x}, \mathbf{a}) \in \mathbb{R}^{n+L} \mid \hat{\psi}_{t_l}^{(l)}(\overline{\mathrm{x}}) = 0\} \approx \{\overline{\mathrm{x}} = (\mathrm{x}, \mathbf{a}) \in \mathbb{R}^{n+L} \mid p^{(l)}(\mathrm{x}) = a_l\}. \tag{4.2.41}$$

Since $\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$ contains semi-infinite equality constraints, we cannot directly use the algorithms described in Chapter 3 for inequality constraints. However, we can proceed as in Sub-section 4.2.2. In the following theorem, we show that the equalities in $\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$ can be replaced by inequalities without altering the solution of $\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$. For any $\mathbf{t} = (t_1, ..., t_L) \in \mathbb{R}^L$, with $t_l > 0, l \in \mathbf{L}$, we define the problem $\hat{\mathrm{P}}_{3,\mathrm{sys},\mathbf{t}}$ which is identical to $\hat{\mathrm{P}}'_{3,\mathrm{sys},\mathbf{t}}$ except that the equality constraints are replaced by inequalities.

$\hat{\mathrm{P}}_{3,\mathrm{sys},\mathbf{t}}$

$$\min_{\overline{\mathrm{x}}=(\mathrm{x},\mathbf{a})\in\mathbb{R}^{n+L}} \left\{ \sum_{l=1}^{L} c_0^{(l)}(\mathrm{x}) + \sum_{l=1}^{L} c^{(l)}(\mathrm{x}) a_l \mid \hat{\psi}_{t_l}^{(l)}(\overline{\mathrm{x}}) \le 0,\ 0 \le a_l \le \hat{p}^{(l)}, l \in \mathbf{L},\ \mathrm{x} \in \mathbf{X} \right\}, \tag{4.2.42}$$

Theorem 4.2.8. *Suppose that Assumption 4.1.1 holds, the functions $c_0^{(l)}(\cdot)$, $c^{(l)}(\cdot)$, $g_k^{(l)}(\cdot,\cdot)$, $k \in \mathbf{K}_l, l \in \mathbf{L}$, are continuous, the set $\mathbf{X}$ is compact, $c^{(l)}(\mathrm{x}) > 0$ for all $l \in \mathbf{L}$*

and $\mathbf{x} \in \mathbf{X}$, $\hat{p}^{(l)} < 0.5, l \in \mathbf{L}$, *and that for all* $l \in \mathbf{L}$ *and* $\mathbf{x} \in \mathbf{X}$ *there exist a* $\mathbf{u} \in \mathbb{R}^m$ *and a* $k \in \mathbf{K}_l$ *such that* $g_k^{(l)}(\mathbf{x}, \mathbf{u}) = 0$. *Then,* $\mathbf{x}^*$ *solves* $\hat{\mathbf{P}}_{3,\mathrm{sys,t}}$ *if and only if it solves* $\hat{\mathbf{P}}'_{3,\mathrm{sys,t}}$.

Proof. Suppose that $\overline{\mathbf{x}}^* = (\mathbf{x}^*, \mathbf{a}^*)$ is a solution of $\hat{\mathbf{P}}_{3,\mathrm{sys,t}}$. Then, $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}^*) \leq 0,\ 0 \leq a_l^* \leq \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x}^* \in \mathbf{X}$, and

$$\sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}^*) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}^*) a_l^* \leq \sum_{l=1}^{L} c_0^{(l)}(\mathbf{x}) + \sum_{l=1}^{L} c^{(l)}(\mathbf{x}) a_k, \tag{4.2.43a}$$

for all $\overline{\mathbf{x}} = (\mathbf{x}, \mathbf{a})$ satisfying $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}) \leq 0,\ 0 \leq a_l \leq \hat{p}^{(l)}, l \in \mathbf{L},\ \mathbf{x} \in \mathbf{X}$. Suppose for the sake of a contradiction that there exists an $\mathbf{L}^* \subset \mathbf{L}$ such that $\mathbf{L}^* \neq \emptyset$ and $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}^*) < 0$, for all $l \in \mathbf{L}^*$. Then by (4.2.40) and a linear transformation, we obtain that

$$\min_{k \in \mathbf{K}_l} \min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, -\Phi^{-1}(a_l^*) t_l)} g_k^{(l)}(\mathbf{x}^*, \mathbf{u}) > 0, \tag{4.2.43b}$$

for all $l \in \mathbf{L}^*$. By the assumption that $g_k^{(l)}(\mathbf{x}^*, \mathbf{u}) = 0$ for some $\mathbf{u}$ and k, and the fact that $-\Phi^{-1}(\omega') > -\Phi^{-1}(\omega'')$ if and only if $\omega'' > \omega'$, there must exist for each $l \in \mathbf{L}^*$ an $a_l' \in (0, a_l^*)$ such that

$$\min_{k \in \mathbf{K}_l} \min_{\mathbf{u} \in \mathbb{B}(\mathbf{0}, -\Phi^{-1}(a_l') t_l)} g_k^{(l)}(\mathbf{x}^*, \mathbf{u}) = 0, \tag{4.2.43c}$$

Let $\mathbf{x}' = \mathbf{x}^*$, $a_l' = a_l^*, l \notin \mathbf{L}^*, l \in \mathbf{L}$, $\mathbf{a}' = (a_1', a_2', ..., a_L')$, and $\overline{\mathbf{x}}' = (\mathbf{x}', \mathbf{a}')$. Clearly, $\overline{\mathbf{x}}'$ satisfies $\mathbf{x}' \in \mathbf{X}, 0 \leq a_l' \leq \hat{p}^{(l)}, l \in \mathbf{L}$, and $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}') = 0, l \notin \mathbf{L}^*, l \in \mathbf{L}$. Additionally, from (4.3.43c) we have that $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}') = 0, l \in \mathbf{L}^*$. Hence, $\overline{\mathbf{x}}'$ is a feasible point for $\hat{\mathbf{P}}_{3,\mathrm{sys,t}}$. Since $a_l' < a_l^*$ for all $l \in \mathbf{L}^* \neq \emptyset$ and $c^{(l)}(\mathbf{x}) > 0$ for all $l \in \mathbf{L}$ and $\mathbf{x} \in \mathbf{X}$, we have contradicted (4.2.43a). Consequently, every solution $\overline{\mathbf{x}}^* = (\mathbf{x}^*, \mathbf{a}^*)$ of $\hat{\mathbf{P}}_{3,\mathrm{sys,t}}$ must satisfy $\hat{\psi}_{t_l}^{(l)}(\overline{\mathbf{x}}^*) = 0, l \in \mathbf{L}$, and hence the conclusion follows. □

For remarks on the assumptions in Theorem 4.2.8, see the paragraph after Theorem 4.2.5.

Algorithm

In view of Theorems 4.2.7 and 4.2.8, and the comments after (4.2.40), we can derive an algorithm for solving $\mathbf{P}_{3,\mathrm{sys}}$ based on the solution of a sequence of the prob-

lems $\hat{\mathbf{P}}_{3,\mathrm{sys},\mathbf{t}}$ for varying values of the approximation parameters $\mathbf{t}$ and the use of the semi-infinite optimization algorithms in Chapter 3. The next algorithm is based on this idea and it is quite similar to Algorithm 4.2.4.

Algorithm 4.2.5. (For $\mathbf{P}_{3,\mathrm{sys}}$)

Data. $\mathbf{x}_0 \in \mathbb{R}^n$, $N_0, N_1, N_3, ...$, with $N_i \uparrow \infty$, as $i \to \infty$, $N_i \in \mathbb{N}$.

Step 0. Set $i = 0$, $\mathbf{a}_0 = (\hat{p}_1, ..., \hat{p}_L)$, $\mathbf{t}_0 = (1, ..., 1) \in \mathbb{R}^L$, $\overline{\mathbf{x}}_0 = (\mathbf{x}_0, \mathbf{a}_0)$.

Step 1. Set $\overline{\mathbf{x}}_{i+1}$ to be the last iterate after N_i iterations of Algorithm 3.3.1 or 3.3.2 on the problem $\hat{\mathbf{P}}_{3,\mathbf{t}_i}$, defined in (4.2.31), with initial point $\overline{\mathbf{x}}_i$.

Step 2. Compute appropriate estimates $\tilde{p}^{(l)}(\mathbf{x}_{i+1}), l \in \mathbf{L}$, of $p^{(l)}(\mathbf{x}_{i+1}), l \in \mathbf{L}$, for the current run of Algorithm 4.2.5, see Definition 4.1.1.

Step 3. For $l \in \mathbf{L}$, set

$$(t_l)_{i+1} = \frac{(t_l)_i \Phi^{-1}((a_l)_{i+1})}{\Phi^{-1}(\tilde{p}^{(l)}(\mathbf{x}_{i+1}))}, \tag{4.2.47}$$

where $\overline{\mathbf{x}}_{i+1} = (\mathbf{x}_{i+1}, \mathbf{a}_{i+1})$, with $\mathbf{a}_{i+1} = ((a_1)_{i+1}, ..., (a_L)_{i+1})$.

Step 4. Replace i by $i + 1$ and go to **Step 1**. □

We cannot guarantee that Algorithm 4.2.5 converges to the solution of $\mathbf{P}_{3,\mathrm{sys}}$. However, since the design found is approximately optimal with respect to a first-order approximation, we expect that results generated by Algorithm 4.2.5 will be sufficiently close to the true optimal solution for practical use.

Typically, structural systems are reliable, and hence the failure probabilities tend to be small, i.e., $p(\mathbf{x})$ is small. In implementation of Algorithm 4.2.5, numerical difficulties caused by the potential difference in the orders of magnitude of the components of $\mathbf{a}$ and $\mathbf{x}$ can be avoided by use of the transformation $a_l \stackrel{\triangle}{=} \Phi(-b_l), l \in \mathbf{L}$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function. Then the optimization in Algorithm 4.2.5 is over the vector $(\mathbf{x}, \mathbf{b})$, where $\mathbf{b} = (b_1, ..., b_L)$.

Chapter 5

An Algorithm for Generalized Semi-Infinite Min-Max Problems

As seen in Section 4.2, there is a need for solving generalized semi-infinite optimization problems of the form (4.2.7). There is no implementable algorithm for (4.2.7) or its unconstrained version in the literature. This chapter derives an algorithm that can be used to solve (4.2.7) in the case of no constraints.

This chapter is self-contained, i.e., it does not use notation defined in other chapters. Any notation established in this chapter does not carry over to other chapters.

5.1 Introduction

We consider the class of generalized semi-infinite min-max problems in the form

$$\mathrm{P} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \psi(\mathbf{x}), \tag{5.1.1}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi(\mathbf{x}) \triangleq \max_{\mathbf{y}\in\mathbf{Y}} \{\phi(\mathbf{x},\mathbf{y}) \mid \mathbf{f}(\mathbf{x},\mathbf{y}) \leq 0\}, \tag{5.1.2}$$

with $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{r_1}$, $\mathbf{Y} \triangleq \{\mathbf{y} \in \mathbb{R}^m | \mathbf{g}(\mathbf{y}) \leq 0\}$, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^{r_2}$, and $v \leq 0$ meaning $v_1 \leq 0$, ..., $v_q \leq 0$, for any $\mathbf{v} = (v_1, ..., v_q) \in \mathbb{R}^q$. It can be

seen that (4.2.7) without constraints is of the form (5.1.1).

In addition to be of practical interest in the field of reliability-based optimal structural design, this class of generalized semi-infinite min-max problems is also theoretical interest. There is a nontrivial literature dealing with existence of and formulas for directional derivatives of generalized max-functions, such as the one in (5.1.2), (e.g., Bonnans and Shapiro (2000) and Rockafellar and Wets (1997)), and with first-order optimality conditions for generalized semi-infinite optimization problems of the form

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{f_0(\mathbf{x}) \mid \psi(\mathbf{x}) \leq 0\}, \tag{5.1.6}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ is smooth and $\psi(\cdot)$ is as in (5.1.2) (Jongen *et al.* (1998),Ruckmann and Shapiro (1999), Stein (2001), Stein and Still (2000), Still (1999), Weber (1999)). Just as we do in our Assumption 5.2.6 below, Weber (1999), also, assumes that the linear independence constraint qualification for the "inner problem" (5.1.2) is satisfied. Under this assumption, Weber (1999) shows that the problem in (5.1.6) is equivalent to a standard semi-infinite optimization problem, i.e., a problem in the form $\min_{\mathbf{x}\in\mathbb{R}^n}\{f_0(\mathbf{x}) \mid \phi(\mathbf{x},\omega) \leq 0,\ \omega \in \Omega\}$, with $\phi(\cdot,\cdot)$ smooth and Ω of infinite cardinality. However, it is not clear how to implement a procedure for constructing the equivalent problem.

There are only a few papers dealing with numerical methods for problems in the form (5.1.6). In Graettinger and Krogh (1988), an algorithm is presented, without a convergence proof, for a special class of problems, with $f_0(\mathbf{x}) = x \in \mathbb{R}$, arising in the evaluation of the acceleration radius of manipulator positioning systems. Other basic ideas for solving problems of the form (5.1.6) in robotics (maneuverability problems), can be found in Hettich and Still (1991). A special case of (5.1.6) arising in robotics and minimum time optimal control problems is considered in Kaplan and Tichatschke (1997), where $y \in \mathbb{R}, m = 1$. In Kaplan and Tichatschke (1998), we find an algorithm for the solution of the special case with $\phi(\mathbf{x},\mathbf{y}) = \frac{1}{2}\langle \mathbf{y}, G\mathbf{y}\rangle + \langle a, \mathbf{y}\rangle + \langle \mathbf{y}, Hx\rangle$, G, H matrices, $f_k(\mathbf{x},\mathbf{y}) = \langle p_k, \mathbf{y}\rangle + q_k(\mathbf{x})$, $a, p_k \in \mathbb{R}^m$ and convex functions q_k. In Still (1999) and Still (2001) we find a conceptual algorithm for solving the problem

(5.1.6). In these papers it is assumed that the LICQ, second-order sufficient conditions, and strict complementary slackness, for the "inner-problem," in (5.1.2) hold. The algorithm in Still (1999) and Still (2001) applies a globally convergent Newton-type method to the Karush-Kuhn-Tucker system for a locally reduced problem. In addition, a conceptual algorithm, based on discretization, is presented in Still (2001). In the still unpublished paper Levitin (2001), Levitin employs a differentiable penalty function to remove the constraints $\mathbf{f}(\mathbf{x}, \mathbf{y}) \leq 0$, and shows that the sequence of global solutions of the penalized problem converges to a global solution of (5.1.6), as the penalty goes to infinity. Thus, in spirit, his approach is close to ours. To the authors' knowledge there exists no implementable algorithm for solving general forms of $\mathbf{P}$.

In this chapter we present an implementable algorithm for solving general forms of $\mathbf{P}$ under a calmness assumption. We use an exact penalty function to eliminate the inequalities in (5.1.2) that depend on $\mathbf{x}$, i.e., $\mathbf{f}(\mathbf{x}, \mathbf{y}) \leq 0$, and as a result convert the generalized semi-infinite min-max problem into a standard semi-infinite min-max problem with an unknown penalty parameter. In principle, we could have picked any one of the existing exact penalty or augmented Lagrangian functions for this purpose, see, e.g., Di Pillo (1994) and Polak (1997). However, the use of augmented Lagrangians together with differentiable multiplier estimates as in Glad and Polak (1979), is unattractive because it would require a second-order sufficient condition to hold at solutions of the "inner problem" in (5.1.2), evaluation of second-order derivatives even by a first-order algorithm, and the linear independence assumption on the gradients $\nabla_{\mathbf{y}} f_k(\mathbf{x}, \mathbf{y})$ and $\nabla g_k(\mathbf{y})$ at every $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbf{Y}$. Hence we opted for a standard nondifferentiable exact penalty function, which avoids the need for an assumption about a second-order sufficient condition, second-order derivative evaluations, and requires only the linear independence assumption on the gradients $\nabla_{\mathbf{y}} f_k(\mathbf{x}, \mathbf{y})$ and $\nabla g_k(\mathbf{y})$ at points $\mathbf{y} \in \mathbf{Y}$, which are solutions to the "inner problem" (5.1.2). The selected approach leads to an algorithm that generates sequences converging to weaker stationary points than the ones given in Stein (2001), see Section 5.7. It is unknown whether a different penalty function would have resulted in an algorithm converging to stronger stationary points.

Since a penalty function of the form $\phi(\mathbf{x},\mathbf{y}) - \pi\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty$ is in fact a min-function, use of a nondifferentiable exact penalty function results in a semi-infinite min-max-min problem with an unknown penalty parameter. This problem can be approximated by a finite min-max problem obtained by discretizing the semi-infinite part and smoothing the min-function. This adds two more parameters to the resulting min-max problem. In view of this, our algorithm combines tests for adjusting the three parameters with the Pironneau-Polak-Pshenichnyi min-max algorithm Polak (1997) and Pshenichnyi and Danilin (1975). Under mild assumptions, we show that if the algorithm generates a bounded sequence, then the penalty parameter remains bounded and that there exists an accumulation point which satisfies a first-order optimality condition.

In Section 5.2 we define the penalized problem and establish its relation to $\mathbf{P}$. In the process we obtain a new first-order optimality condition for $\mathbf{P}$. Approximations for the solution of the penalized problem are defined in Section 5.3. Section 5.4 presents the algorithm and the proof of its convergence. The chapter ends with a numerical example, concluding remarks, an additional result regarding optimality conditions.

5.2 Exact Penalization

As described in the introduction, we introduce exact penalization for the violation of the constraints $\mathbf{f}(\mathbf{x},\mathbf{y}) \leq 0$ in (5.1.2). Let π denote this penalty. Hence, for any $\pi > 0$ we define a family of related problems by

$$\mathbf{P}_\pi \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \psi_\pi(\mathbf{x}), \tag{5.2.1}$$

where $\psi_\pi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_\pi(\mathbf{x}) \stackrel{\triangle}{=} \max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) - \pi\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty\}. \tag{5.2.2}$$

with $\|\mathbf{v}_+\|_\infty \stackrel{\triangle}{=} \max\{\max\{v_1,0\},...,\max\{v_q,0\}\}$.

At first glance (5.2.1) looks like an ordinary min-max problem. However, $\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty$ is a max-function, and hence we see that, with $\mathbf{r} \stackrel{\triangle}{=} \{1,...,r\}$ and $r \stackrel{\triangle}{=} r_1 +$

1,

$$\psi_\pi(\mathbf{x}) = \max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) - \pi\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty\} = \max_{\mathbf{y}\in\mathbf{Y}}\min_{k\in\mathbf{r}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}), \tag{5.2.3a}$$

where

$$\phi_{k,\pi}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} \phi(\mathbf{x},\mathbf{y}) - \pi f_k(\mathbf{x},\mathbf{y}), \quad k \in \mathbf{r}_1 \stackrel{\triangle}{=} \{1,...,r_1\}, \tag{5.2.3b}$$

$$\phi_{r,\pi}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} \phi(\mathbf{x},\mathbf{y}). \tag{5.2.3c}$$

We need the following notation: Let $\mathbb{B}(\mathbf{x},\rho) \stackrel{\triangle}{=} \{\mathbf{x}' \in \mathbb{R}^n | \|\mathbf{x}-\mathbf{x}'\| \le \rho\}$, and let $\omega_\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\hat{\mathbf{Y}} : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ and $\hat{\mathbf{Y}}_\pi : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be defined by

$$\omega_\pi(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} \min_{k\in r}\phi_{k,\pi}(\mathbf{x},\mathbf{y}), \tag{5.2.3d}$$

$$\hat{\mathbf{Y}}(\mathbf{x}) \stackrel{\triangle}{=} \arg\max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) \mid \mathbf{f}(\mathbf{x},\mathbf{y}) \le 0\}, \tag{5.2.4a}$$

$$\hat{\mathbf{Y}}_\pi(\mathbf{x}) \stackrel{\triangle}{=} \arg\max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) - \pi\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty\} = \{\mathbf{y}\in\mathbf{Y} \mid \omega_\pi(\mathbf{x},\mathbf{y}) = \psi_\pi(\mathbf{x})\}. \tag{5.2.4b}$$

Note that (5.2.3a,d) imply that $\psi_\pi(\mathbf{x}) = \max_{\mathbf{y}\in\mathbf{Y}}\omega_\pi(\mathbf{x},\mathbf{y})$.

Assumption 5.2.1. We assume that

(i) $\phi(\cdot,\cdot), f_k(\cdot,\cdot), k \in \mathbf{r}_1 = \{1,...,r_1\}$, and $g_k(\cdot), k \in \mathbf{r}_2 \stackrel{\triangle}{=} \{1,...,r_2\}$, are continuously differentiable, and

(ii) $\mathbf{Y} \subset \mathbb{R}^m$ is compact, and $\{\mathbf{y}\in\mathbf{Y} | \mathbf{f}(\mathbf{x},\mathbf{y}) \le 0\} \neq \emptyset$ for all $\mathbf{x} \in \mathbb{R}^n$. □

The notion of calmness, see Clarke (1983, Burke (1991), can be used to show the local equivalence of $\mathbf{P}_\pi$ and $\mathbf{P}$ for π sufficiently large. For any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^{r_1}$, consider the perturbed "inner-problem," see (5.1.2), defined by

$$\mathbf{IP}(\mathbf{x},\mathbf{u}) \qquad \max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) \mid \mathbf{f}(\mathbf{x},\mathbf{y}) \le \mathbf{u}\}. \tag{5.2.5}$$

Let the value function $v : \mathbb{R}^n \times \mathbb{R}^{r_1} \to \mathbb{R}\bigcup\{-\infty\}$ of $\mathbf{IP}(\mathbf{x},\mathbf{u})$ be defined by

$$v(\mathbf{x},\mathbf{u}) \stackrel{\Delta}{=} \max_{\mathbf{y}\in\mathbf{Y}}\{\phi(\mathbf{x},\mathbf{y}) \mid \mathbf{f}(\mathbf{x},\mathbf{y}) \leq \mathbf{u}\}, \tag{5.2.6}$$

where $v(\mathbf{x},\mathbf{u}) = -\infty$ if $\mathbf{f}(\mathbf{x},\mathbf{y}) > \mathbf{u}$ for all $\mathbf{y} \in \mathbf{Y}$.

We now define local calmness. A sufficient condition for local calmness will be given at the end of the section.

Definition 5.2.2. *We say that* $\mathbf{IP}(\hat{\mathbf{x}},0)$ *is* locally calm *at* $\hat{\mathbf{x}} \in \mathbb{R}^n$, *if there exist* $\hat{\rho} > 0$ *and* $\hat{\alpha} < \infty$ *such that*

$$v(\mathbf{x},\mathbf{u}) - v(\mathbf{x},0) \leq \hat{\alpha}\|\mathbf{u}\|_\infty, \tag{5.2.7}$$

for every $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}},\hat{\rho})$ *and* $\mathbf{u} \in \mathbb{R}^{r_1}$. □

Theorem 5.2.3. *Suppose that Assumption 5.2.1 holds and that* $\mathbf{IP}(\hat{\mathbf{x}},0)$ *is locally calm at* $\hat{\mathbf{x}} \in \mathbb{R}^n$. *Then, there exist a* $\hat{\pi} < \infty$ *and a* $\hat{\rho} > 0$ *such that* $\psi(\mathbf{x}) = \psi_{\hat{\pi}}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}},\hat{\rho})$, *and hence* $\hat{\mathbf{x}}$ *is a local minimizer for* $\mathbf{P}$ *if and only if* $\hat{\mathbf{x}}$ *is a local minimizer for* $\mathbf{P}_{\hat{\pi}}$.

Proof. Let $\hat{\rho} > 0$ and $\hat{\alpha} < \infty$ be as in Definition 5.2.2. Now, let $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}},\hat{\rho})$ and $\mathbf{y} \in \hat{\mathbf{Y}}(\mathbf{x})$ be arbitrary. We will show that $\mathbf{y} \in \hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x})$, with $\hat{\pi} = \hat{\alpha}$. For the sake of a contradiction, suppose that $\mathbf{y} \notin \hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x})$. Then there exists $\mathbf{y}' \in \mathbf{Y}$ such that

$$\phi(\mathbf{x},\mathbf{y}') - \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y}')_+\|_\infty > \phi(\mathbf{x},\mathbf{y}) - \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty. \tag{5.2.8a}$$

Hence,

$$\begin{aligned} \phi(\mathbf{x},\mathbf{y}') - \phi(\mathbf{x},\mathbf{y}) &> \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y}')_+\|_\infty - \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty \\ &= \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y}')_+\|_\infty. \end{aligned} \tag{5.2.8b}$$

Next, $\phi(\mathbf{x},\mathbf{y}') \leq v(\mathbf{x},\mathbf{f}(\mathbf{x},\mathbf{y}')_+)$ and $\phi(\mathbf{x},\mathbf{y}) = v(\mathbf{x},0)$. Hence, by (5.2.7)

$$\begin{aligned} \phi(\mathbf{x},\mathbf{y}') - \phi(\mathbf{x},\mathbf{y}) &\leq v(\mathbf{x},\mathbf{f}(\mathbf{x},\mathbf{y}')_+) - v(\mathbf{x},0) \\ &\leq \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y}')_+\|_\infty, \end{aligned} \tag{5.2.8c}$$

which is a contradiction. Hence, $\mathbf{y} \in \hat{\mathbf{Y}}(\mathbf{x})$, $\mathbf{y} \in \hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x})$, and

$$\begin{aligned} \psi_{\hat{\pi}}(\mathbf{x}) &= \phi(\mathbf{x},\mathbf{y}) - \hat{\pi}\|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty \\ &= \psi(\mathbf{x}). \end{aligned} \tag{5.2.8d}$$

Hence $\psi(\mathbf{x}) = \psi_{\hat{\pi}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}},\hat{\rho})$, and the result follows. □

Optimality conditions can be expressed in terms of continuous, non-positive valued optimality functions, which vanish at local minimizers, see Polak (1997).

Theorem 5.2.4. *Suppose that Assumption 5.2.1 holds, and for any $\pi > 0$, let $\theta_\pi : \mathbb{R}^n \to \mathbb{R}$ be an optimality function defined by*

$$\theta_\pi(\mathbf{x}) \stackrel{\triangle}{=} -\min_{\bar{\zeta} \in \bar{G}\psi_\pi(\mathbf{x})} \zeta_{-1} + \zeta_0 + \frac{1}{2}\|\zeta\|^2, \tag{5.2.9}$$

$$\bar{G}\psi_\pi(\hat{\mathbf{x}}) \stackrel{\triangle}{=} \operatorname*{conv}_{\mathbf{y}\in\mathbf{Y}} \operatorname*{conv}_{k\in\mathbf{r}} \left\{ \begin{pmatrix} \phi_{k,\pi}(\hat{\mathbf{x}},\mathbf{y}) - \omega_\pi(\hat{\mathbf{x}},\mathbf{y}) \\ \psi_\pi(\hat{\mathbf{x}}) - \omega_\pi(\hat{\mathbf{x}},\mathbf{y}) \\ \nabla_{\mathbf{x}}\phi_{k,\pi}(\hat{\mathbf{x}},\mathbf{y}) \end{pmatrix} \right\}, \tag{5.2.10}$$

where elements of $\bar{G}\psi_\pi(\hat{\mathbf{x}}) \subset \mathbb{R}^{n+2}$ are denoted by $\bar{\zeta} = (\zeta_{-1},\zeta_0,\zeta)$, with $\zeta \in \mathbb{R}^n$. Then, **(i)** *$\theta_\pi(\cdot)$ is continuous and non-positive valued, and* **(ii)** *if $\hat{\mathbf{x}}$ is a local minimizer for $\mathbf{P}_\pi$, then $\theta_\pi(\hat{\mathbf{x}}) = 0$.*

Proof. **(i)** By Corollary 5.3.9 and 5.4.2 in Polak (1997), $\theta_\pi(\cdot)$ is continuous and non-positive valued.

(ii) If $\hat{\mathbf{x}}$ is a local minimizer for $\mathbf{P}_\pi$, then

$$d_-\psi_\pi(\mathbf{x};\mathbf{h}) \geq 0, \quad \forall\ \mathbf{h} \in \mathbb{R}^n, \tag{5.2.11a}$$

where $d_-\psi_\pi(\mathbf{x};\mathbf{h})$ is the lower Dini directional derivatives of $\psi_\pi(\cdot)$ at a point $\mathbf{x}$, in a direction $\mathbf{h}$, i.e.,

$$d_-\psi_\pi(\mathbf{x};\mathbf{h}) \triangleq \liminf_{t\downarrow 0} \frac{\psi_\pi(\mathbf{x}+t\mathbf{h})-\psi_\pi(\mathbf{x})}{t}. \tag{5.2.11b}$$

Next, for any $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbf{Y}$, let $\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y}) \triangleq \{k\in\mathbf{r} \mid \phi_{k,\pi}(\mathbf{x},\mathbf{y})=\omega_\pi(\mathbf{x},\mathbf{y})\}$. By using (5.2.4b), the facts that for any $\mathbf{y}\in\mathbf{Y}$, $-\psi_\pi(\mathbf{x})\le -\omega_\pi(\mathbf{x},\mathbf{y})$, that $\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y})\subset\mathbf{r}$, and the definition of $\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y})$, we obtain that for any $\mathbf{x},\mathbf{h}\in\mathbb{R}^n$ and $t>0$,

$$\begin{aligned}
\frac{\psi_\pi(\mathbf{x}+t\mathbf{h})-\psi_\pi(\mathbf{x})}{t} &= \max_{\mathbf{y}\in\hat{\mathbf{Y}}_\pi(\mathbf{x}+t\mathbf{h})} \min_{k\in\mathbf{r}} \frac{\phi_{k,\pi}(\mathbf{x}+t\mathbf{h},\mathbf{y})-\psi_\pi(\mathbf{x})}{t} \\
&\le \max_{\mathbf{y}\in\hat{\mathbf{Y}}_\pi(\mathbf{x}+t\mathbf{h})} \min_{k\in\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y})} \frac{\phi_{k,\pi}(\mathbf{x}+t\mathbf{h},\mathbf{y})-\omega_\pi(\mathbf{x},\mathbf{y})}{t} \qquad (5.2.11c)\\
&= \max_{\mathbf{y}\in\hat{\mathbf{Y}}_\pi(\mathbf{x}+t\mathbf{h})} \min_{k\in\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y})} \langle \nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x}+sth,\mathbf{y}),\mathbf{h}\rangle,
\end{aligned}$$

where $s\in[0,1]$. Hence, since $\hat{\mathbf{Y}}_\pi(\cdot)$ is outer semicontinuous in the sense of Kuratowski-Painlevé, see Rockafellar and Wets (1997) and Polak (1997), we have that

$$\liminf_{t\downarrow 0} \frac{\psi_\pi(\mathbf{x}+t\mathbf{h})-\psi_\pi(\mathbf{x})}{t} \le \max_{\mathbf{y}\in\hat{\mathbf{Y}}_\pi(\mathbf{x})} \min_{k\in\hat{\mathbf{r}}_\pi(\mathbf{x},\mathbf{y})} \langle \nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}),\mathbf{h}\rangle. \tag{5.2.11d}$$

Next, we proceed by contraposition. Suppose that $0\notin\bar{G}\psi_\pi(\hat{\mathbf{x}})$. Then there exists a nonzero vector $\mathbf{h}\in\mathbb{R}^n$ such that $\langle\nabla_{\mathbf{x}}\phi_{k,\pi}(\hat{\mathbf{x}},\mathbf{y}),\mathbf{h}\rangle<0$ for all $\mathbf{y}\in\hat{\mathbf{Y}}_\pi(\hat{\mathbf{x}})$ and all $k\in\hat{\mathbf{r}}_\pi(\hat{\mathbf{x}},\mathbf{y})$. Hence by (5.2.11d), $d_-\psi_\pi(\hat{\mathbf{x}};\mathbf{h})<0$. Therefore, (5.2.11a) implies $0\in\bar{G}\psi_\pi(\hat{\mathbf{x}})$ and $\theta_\pi(\hat{\mathbf{x}})=0$. □

In view of Theorem 5.2.3, we can formulate the following optimality condition for P.

Theorem 5.2.5. *Suppose that Assumption 5.2.1 holds and that* $\mathbf{IP}(\hat{\mathbf{x}},0)$ *is locally calm at* $\hat{\mathbf{x}}\in\mathbb{R}^n$. *If* $\hat{\mathbf{x}}$ *is a local minimizer for* P, *then there exists a* $\hat{\pi}<\infty$ *such that* $\theta_{\hat{\pi}}(\hat{\mathbf{x}})=0$ *and* $\psi(\hat{\mathbf{x}})=\psi_{\hat{\pi}}(\hat{\mathbf{x}})$. □

The optimality condition for P in Theorem 5.2.5 can be related to an optimality condition in Stein (2001), see Section 5.7.

In the remainder of the section, we derive results leading to the conclusion that Assumption 5.2.1, together with Assumption 5.2.6 below, are sufficient conditions for local calmness.

Assumption 5.2.6. *We assume that for any* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \hat{\mathbf{Y}}(\mathbf{x})$, *the vectors* $\nabla_{\mathbf{y}} f_k(\mathbf{x},\mathbf{y}), k \in \mathbf{r}_1^*(\mathbf{x},\mathbf{y})$, *together with the vectors* $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$, *are linearly independent, where* $\mathbf{r_1} = \{1,...,r_1\}$, $\mathbf{r_2} = \{1,...,r_2\}$, *and*

$$\mathbf{r}_1^*(\mathbf{x},\mathbf{y}) \triangleq \{k \in \mathbf{r_1} | f_k(\mathbf{x},\mathbf{y}) - \|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty = 0\}, \tag{5.2.12a}$$

$$\mathbf{r}_2^*(\mathbf{y}) \triangleq \{k \in \mathbf{r_2} | g_k(\mathbf{y}) = 0\}. \tag{5.2.12b}$$

□

Next, we will define a test function, which plays a crucial role in determining the value of the penalty π that is sufficiently large to ensure the local equivalence between $\mathbf{P}$ and $\mathbf{P}_\pi$ near a point $\hat{\mathbf{x}} \in \mathbb{R}^n$. We need the following building blocks: Let

$$\mathbf{A}(\mathbf{x},\mathbf{y}) \triangleq \begin{pmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{x},\mathbf{y}) \\ \mathbf{g}_{\mathbf{y}}(\mathbf{y}) \end{pmatrix} \tag{5.2.13a}$$

be an $(r_1 + r_2) \times m$ matrix with

$$\mathbf{f}_{\mathbf{y}}(\mathbf{x},\mathbf{y}) \triangleq (\nabla_{\mathbf{y}} f_1(\mathbf{x},\mathbf{y}),...,\nabla_{\mathbf{y}} f_{r_1}(\mathbf{x},\mathbf{y}))^T, \tag{5.2.13b}$$

$$\mathbf{g}_{\mathbf{y}}(\mathbf{y}) \triangleq (\nabla g_1(\mathbf{y}),...,\nabla g_{r_2}(\mathbf{y}))^T, \tag{5.2.13c}$$

and

$$\mathbf{B}(\mathbf{x},\mathbf{y}) \triangleq \mathrm{diag}(\mathbf{B}_1(\mathbf{x},\mathbf{y}),\mathbf{B}_2(\mathbf{y})) \tag{5.2.13d}$$

be an $(r_1+r_2)\times(r_1+r_2)$ diagonal matrix defined in terms of the two diagonal matrices

$$\mathbf{B}_1(\mathbf{x},\mathbf{y}) \triangleq \mathrm{diag}([f_1(\mathbf{x},\mathbf{y}) - \|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty]^2,...,[f_{r_1}(\mathbf{x},\mathbf{y}) - \|\mathbf{f}(\mathbf{x},\mathbf{y})_+\|_\infty]^2), \tag{5.2.13e}$$

$$\mathbf{B}_2(\mathbf{y}) \triangleq \mathrm{diag}([g_1(\mathbf{y})]^2,...,[g_{r_2}(\mathbf{y})]^2). \tag{5.2.13f}$$

Furthermore, let $\mathbf{z} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{r_1+r_2}$ be defined by

$$\mathbf{z}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} (\eta(\mathbf{x},\mathbf{y}),\xi(\mathbf{x},\mathbf{y}))^T \stackrel{\triangle}{=} [\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})]^+\mathbf{A}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\phi(\mathbf{x},\mathbf{y}), \tag{5.2.13g}$$

where $\eta(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{r_1}$, $\xi(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{r_2}$ and $\mathbf{M}^+$ denotes the pseudoinverse[1] of the matrix $\mathbf{M}$.

Using a similar construction as in Glad and Polak (1979), we define for any $\pi > 0$ the test function $t_\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$t_\pi(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} -\pi + \sigma \sum_{k=1}^{r_1} |\eta_k(\mathbf{x},\mathbf{y})| \tag{5.2.13h}$$

where $\sigma > 1$.

The function $\eta(\cdot,\cdot)$ has the following properties, which will ensure that the test function in (5.2.13h) is well-defined for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, and it is continuous whenever $\eta(\cdot,\cdot)$ is continuous. Note that $\eta(\mathbf{x},\mathbf{y})$ is under certain assumptions related to the multipliers of the "inner-problem" in (5.1.2), see the proof of Lemma 5.2.7.

Lemma 5.2.7. *Suppose Assumption 5.2.1 holds and $\sigma > 1$ in (5.2.13h).*

(i) Then, $\eta(\cdot,\cdot)$ is well defined for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

(ii) If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ are such that $\nabla_{\mathbf{y}} f_k(\mathbf{x},\mathbf{y}), k \in \mathbf{r}_1^*(\mathbf{x},\mathbf{y})$, (see (5.2.12a)), together with $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$, (see (5.2.12b)), are linearly independent, then $\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})$, see (5.2.13a,d), is positive definite, and $\eta(\cdot,\cdot)$ is continuous at $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

(iii) If $\mathbf{x} \in \mathbb{R}^n$ and $\pi > 0$ is such that $t_\pi(\mathbf{x},\mathbf{y}_\mathbf{x}) \leq 0$ for some $\mathbf{y}_\mathbf{x} \in \hat{\mathbf{Y}}_\pi(\mathbf{x})$, and $\nabla_{\mathbf{y}} f_k(\mathbf{x},\mathbf{y}_\mathbf{x}), k \in \mathbf{r}_1^*(\mathbf{x},\mathbf{y}_\mathbf{x})$, (see (5.2.12a)), together with $\nabla g_k(\mathbf{y}_\mathbf{x}), k \in \mathbf{r}_2^*(\mathbf{y}_\mathbf{x})$, (see (5.2.12b)), are linearly independent, then $\mathbf{y}_\mathbf{x} \in \hat{\mathbf{Y}}(\mathbf{x})$ and $\psi(\mathbf{x}) = \psi_\pi(\mathbf{x})$.

[1]The pseudoinverse of a real matrix $\mathbf{M}$ is obtained by first taking a singular-value decomposition $\mathbf{M} = \mathbf{PDQ}$, with $\mathbf{P}$ and $\mathbf{Q}$ unitary matrices, and $\mathbf{D}$ diagonal, and then setting $\mathbf{M}^+ = \mathbf{Q}^T\mathbf{D}^+\mathbf{P}^T$. The pseudoinverse of a diagonal matrix is obtained by replacing the i-th diagonal term d_{ii} with $1/d_{ii}$ whenever $d_{ii} \neq 0$, otherwise with 0, see Kincaid and Cheney (1996).

Proof. (i) By Theorem 4 in Section 5.4 in Kincaid and Cheney (1996), the pseudoinverse is unique, and hence $\eta(\cdot,\cdot)$ is uniquely defined for all $\mathrm{x} \in \mathbb{R}^n$ and $\mathrm{y} \in \mathbb{R}^m$.

(ii) Let $\mathrm{x} \in \mathbb{R}^n$ and $\mathrm{y} \in \mathbb{R}^m$ be such that $\nabla_{\mathrm{y}} f_k(\mathrm{x},\mathrm{y}), k \in \mathrm{r}_1^*(\mathrm{x},\mathrm{y})$, (see (5.2.12a)), together with $\nabla g_k(\mathrm{y}), k \in \mathrm{r}_2^*(\mathrm{y})$, (see (5.2.12b)), are linearly independent. By the definition in (5.2.13g), $\mathrm{z}(\mathrm{x},\mathrm{y})$ satisfies the equation

$$[\mathbf{A}(\mathrm{x},\mathrm{y})\mathbf{A}(\mathrm{x},\mathrm{y})^T + \mathbf{B}(\mathrm{x},\mathrm{y})]\mathrm{z}(\mathrm{x},\mathrm{y}) - \mathbf{A}(\mathrm{x},\mathrm{y})\nabla_{\mathrm{y}}\phi(\mathrm{x},\mathrm{y}) = 0, \tag{5.2.14a}$$

which is also the first-order necessary optimality condition for the unconstrained convex quadratic optimization problem

$$\min_{\mathrm{z}\in\mathbb{R}^{r_1+r_2}} \{\| -\nabla_{\mathrm{y}}\phi(\mathrm{x},\mathrm{y}) + \mathbf{A}(\mathrm{x},\mathrm{y})^T\mathrm{z}\|^2 + \langle \mathrm{z}, \mathbf{B}(\mathrm{x},\mathrm{y})\mathrm{z}\rangle\}. \tag{5.2.14b}$$

We will first show that $\mathrm{z}(\mathrm{x},\mathrm{y})$ is the unique solution of (5.2.14b). Since (5.2.14b) is a quadratic problem, we only need to show that the quadratic function being minimized is positive definite. Clearly, this function is positive semi-definite. Let $\mathrm{z} = (\eta, \xi)$. Then, the quadratic part of the cost function in (5.2.14b) can be written as follows:

$$\begin{aligned} &\langle \mathrm{z}, [\mathbf{A}(\mathrm{x},\mathrm{y})\mathbf{A}(\mathrm{x},\mathrm{y})^T + \mathbf{B}(\mathrm{x},\mathrm{y})]\mathrm{z}\rangle \\ &= \|\mathrm{f}_{\mathrm{y}}(\mathrm{x},\mathrm{y})^T\eta + \mathrm{g}_{\mathrm{y}}(\mathrm{y})^T\xi\|^2 + \langle \eta, \mathbf{B}_1(\mathrm{x},\mathrm{y})\eta\rangle + \langle \xi, \mathbf{B}_2(\mathrm{x},\mathrm{y})\xi\rangle. \end{aligned} \tag{5.2.15a}$$

Hence, the quadratic function in (5.2.14b) is positive definite if and only if

$$\|\mathrm{f}_{\mathrm{y}}(\mathrm{x},\mathrm{y})^T\eta + \mathrm{g}_{\mathrm{y}}(\mathrm{y})^T\xi\|^2 + \langle \eta, \mathbf{B}_1(\mathrm{x},\mathrm{y})\eta\rangle + \langle \xi, \mathbf{B}_2(\mathrm{x},\mathrm{y})\xi\rangle = 0 \tag{5.2.15b}$$

implies $\eta = 0$ and $\xi = 0$. Now, when (5.2.15b) holds, we must have that $\eta_k = 0$ for all $k \notin \mathrm{r}_1^*(\mathrm{x},\mathrm{y})$, and $\xi_k = 0$ for all $k \notin \mathrm{r}_2^*(\mathrm{y})$. Hence, (5.2.15b) implies that

$$\sum_{k\in\mathrm{r}_1^*(\mathrm{x},\mathrm{y})} \eta_k \nabla_{\mathrm{y}} f_k(\mathrm{x},\mathrm{y}) + \sum_{k\in\mathrm{r}_2^*(\mathrm{y})} \xi_k \nabla g_k(\mathrm{y}) = 0. \tag{5.2.15c}$$

It now follows from the linear independence hypothesis that (5.2.15c), and hence also (5.2.15b), hold if and only if $\eta = 0$ and $\xi = 0$. This shows that $[\mathbf{A}(\mathrm{x},\mathrm{y})\mathbf{A}(\mathrm{x},\mathrm{y})^T + \mathbf{B}(\mathrm{x},\mathrm{y})]$ is positive definite, and hence $\mathrm{z}(\mathrm{x},\mathrm{y})$ is the unique solution of (5.2.14b).

Next, since there is an unique solution to (5.2.14b), it follows that $[\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})]$ is invertible, and the inverse is identical to the pseudoinverse. Hence,

$$\mathbf{z}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} [\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})]^{-1}\mathbf{A}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\phi(\mathbf{x},\mathbf{y}) \tag{5.2.15d}$$

Since $[\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})]$ is positive definite, there exists $\epsilon > 0$ such that $[\mathbf{A}(\mathbf{x}',\mathbf{y}')\mathbf{A}(\mathbf{x}',\mathbf{y}')^T + \mathbf{B}(\mathbf{x}',\mathbf{y}')]$ is positive definite for all $(\mathbf{x}',\mathbf{y}') \in \mathbb{B}((\mathbf{x},\mathbf{y}),\epsilon)$. Hence, (5.2.15d) holds, with $\mathbf{x} = \mathbf{x}'$ and $\mathbf{y} = \mathbf{y}'$, for all $(\mathbf{x}',\mathbf{y}') \in \mathbb{B}((\mathbf{x},\mathbf{y}),\epsilon)$, which implies that $\mathbf{z}(\cdot,\cdot)$ is continuous at $(\mathbf{x},\mathbf{y})$.

(iii) Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y}_\mathbf{x} \in \mathbb{R}^m$ and $\pi > 0$ be such that $t_\pi(\mathbf{x},\mathbf{y}_\mathbf{x}) \leq 0$, $\mathbf{y}_\mathbf{x} \in \hat{\mathbf{Y}}_\pi(\mathbf{x})$, and $\nabla_{\mathbf{y}} f_k(\mathbf{x},\mathbf{y}_\mathbf{x})$, $k \in \mathbf{r}_1^*(\mathbf{x},\mathbf{y}_\mathbf{x})$, together with $\nabla g_k(\mathbf{y}_\mathbf{x}), k \in \mathbf{r}_2^*(\mathbf{y}_\mathbf{x})$, are linearly independent at $(\mathbf{x},\mathbf{y}_\mathbf{x})$. Then $\mathbf{y}_\mathbf{x}$ is a minimizer for the problem, see (5.2.3a) and (5.2.4b),

$$\min_{\mathbf{y}\in\mathbf{Y}} \max_{k\in\mathbf{r}} \{-\phi_{k,\pi}(\mathbf{x},\mathbf{y})\}, \tag{5.2.15e}$$

and it follows from first-order optimality conditions, see Polak (1997), that there exist multipliers $\nu \in \mathbb{R}^r$, with $\nu_k \geq 0, k \in \mathbf{r}, \sum_{k=1}^r \nu_k = 1$, and $\mu \in \mathbb{R}^{r_2+1}$, with $\mu_k \geq 0, k \in \{0,1,...,r_2\}, \sum_{k=0}^{r_2} \mu_k = 1$, such that

$$\mu_0 \left[\sum_{k=1}^{r} -\nu_k \nabla_{\mathbf{y}} \phi_{k,\pi}(\mathbf{x},\mathbf{y}_\mathbf{x})\right] + \sum_{k=1}^{r_2} \mu_k \nabla g_k(\mathbf{y}_\mathbf{x}) = 0, \tag{5.2.15f}$$

$$\mu_0 \left[\sum_{k=1}^{r} \nu_k(-\phi_{k,\pi}(\mathbf{x},\mathbf{y}_\mathbf{x}) + \omega_\pi(\mathbf{x},\mathbf{y}_\mathbf{x}))\right] + \sum_{k=1}^{r_2} \mu_k g_k(\mathbf{y}_\mathbf{x}) = 0. \tag{5.2.15g}$$

By the linear independence hypothesis, $\mu_0 > 0$. Using (5.2.3b) and (5.2.3c), (5.2.15f) can be rewritten as

$$-\nabla_{\mathbf{y}}\phi(\mathbf{x},\mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_1} \nu_k \pi \nabla_{\mathbf{y}} f_k(\mathbf{x},\mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_2} \frac{\mu_k}{\mu_0} \nabla g_k(\mathbf{y}_\mathbf{x}) = 0, \tag{5.2.15h}$$

and, also using the fact that each term in (5.2.15g) must be non-negative, (5.2.15g) can be rewritten as

$$\nu_k \pi (f_k(\mathbf{x},\mathbf{y}_\mathbf{x}) - \|\mathbf{f}(\mathbf{x},\mathbf{y}_\mathbf{x})_+\|_\infty) = 0, \quad k \in \mathbf{r}_1, \tag{5.2.15i}$$

$$\nu_r \pi \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty = 0, \tag{5.2.15j}$$

$$\frac{\mu_k}{\mu_0} g_k(\mathbf{y}_\mathbf{x}) = 0, \quad k \in \mathbf{r_2}. \tag{5.2.15k}$$

Then, we see from (5.2.14b), and the definitions (5.2.13a) and (5.2.13d) that

$$\begin{aligned} &\min_{\eta \in \mathbb{R}^{r_1}, \xi \in \mathbb{R}^{r_2}} \{\| - \nabla_\mathbf{y} \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_1} \eta_k \nabla_\mathbf{y} f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_2} \xi_k \nabla g_k(\mathbf{y}_\mathbf{x})\|^2 \\ &+ \sum_{k=1}^{r_1} [\eta_k (f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) - \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty)]^2 + \sum_{k=1}^{r_2} [\xi_k g_k(\mathbf{y}_\mathbf{x})]^2\} \quad \geq 0. \end{aligned} \tag{5.2.15l}$$

Since the cost function in (5.2.15l) is non-negative for all vectors $\eta \in \mathbb{R}^{r_1}$ and $\xi \in \mathbb{R}^{r_2}$, it follows, by taking $\eta = (\nu_1 \pi, ..., \nu_{r_1} \pi)$ and $\xi = (\mu_1/\mu_0, ..., \mu_{r_2}/\mu_0)$ and (5.2.15h,i,k), that

$$\begin{aligned} &\min_{\eta \in \mathbb{R}^{r_1}, \xi \in \mathbb{R}^{r_2}} \{\| - \nabla_\mathbf{y} \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_1} \eta_k \nabla_\mathbf{y} f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_2} \xi_k \nabla g_k(\mathbf{y}_\mathbf{x})\|^2 \\ &+ \sum_{k=1}^{r_1} [\eta_k (f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) - \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty)]^2 + \sum_{k=1}^{r_2} [\xi_k g_k(\mathbf{y}_\mathbf{x})]^2\} \\ \leq \; &\| - \nabla_\mathbf{y} \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_1} \nu_k \pi \nabla_\mathbf{y} f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) + \sum_{k=1}^{r_2} \frac{\mu_k}{\mu_0} \nabla g_k(\mathbf{y}_\mathbf{x})\|^2 \\ &+ \sum_{k=1}^{r_1} [\nu_k \pi (f_k(\mathbf{x}, \mathbf{y}_\mathbf{x}) - \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty)]^2 + \sum_{k=1}^{r_2} \left[\frac{\mu_k}{\mu_0} g_k(\mathbf{y}_\mathbf{x})\right]^2 = 0. \end{aligned} \tag{5.2.15m}$$

Since the linear independence property holds at $(\mathbf{x}, \mathbf{y}_\mathbf{x})$, it follows from the proof of part (ii) that (5.2.14b) has an unique solution. Hence, in view of (5.2.15l,m), $\eta(\mathbf{x}, \mathbf{y}_\mathbf{x}) = (\nu_1 \pi, ..., \nu_{r_1} \pi)$.

Suppose that $\|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty > 0$. Then by (5.2.15j), $\nu_r = 0$ and hence $\sum_{k=1}^{r_1} \nu_k = 1$. Now, since $t_\pi(\mathbf{x}, \mathbf{y}_\mathbf{x}) \leq 0$ by assumption, see (5.2.13h),

$$\pi \geq \sigma \sum_{k=1}^{r_1} |\eta_k(\mathbf{x}, \mathbf{y}_\mathbf{x})| = \sigma \sum_{k=1}^{r_1} \nu_k \pi = \sigma \pi. \tag{5.2.15n}$$

But this is a contradiction because $\sigma > 1$. Hence $\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x}) \leq 0$. Since $\mathbf{y}_\mathbf{x} \in \hat{\mathbf{Y}}_\pi(\mathbf{x})$, we

have that for every $\mathbf{y}' \in \mathbf{Y}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}') \leq 0$,

$$\begin{aligned} \phi(\mathbf{x}, \mathbf{y}') &= \phi(\mathbf{x}, \mathbf{y}') - \pi \|\mathbf{f}(\mathbf{x}, \mathbf{y}')_+\|_\infty \\ &\leq \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}) - \pi \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty \\ &= \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}). \end{aligned} \tag{5.2.15o}$$

Hence, $\mathbf{y}_\mathbf{x} \in \hat{\mathbf{Y}}(\mathbf{x})$, and

$$\begin{aligned} \psi_\pi(\mathbf{x}) &= \phi(\mathbf{x}, \mathbf{y}_\mathbf{x}) - \pi \|\mathbf{f}(\mathbf{x}, \mathbf{y}_\mathbf{x})_+\|_\infty \\ &= \psi(\mathbf{x}). \end{aligned} \tag{5.2.15p}$$

This completes the proof. □

In the following, for any $\mathbf{S} \subset \mathbb{R}^m$ and $\rho > 0$, let $\mathbf{S} + \mathbb{B}_\rho \triangleq \{\mathbf{y} \in \mathbb{R}^m | \|\mathbf{y} - \mathbf{y}'\| \leq \rho, \mathbf{y}' \in \mathbf{S}\}$. Furthermore, we denote the convergence of an infinite (sub)sequence $\{\mathbf{x}_i\}_{i \in K}, K \in \mathbb{N}$, to a point $\mathbf{x}$, by $\mathbf{x}_i \to^K \mathbf{x}$.

Lemma 5.2.8. *Suppose Assumptions 5.2.1 and 5.2.6 hold. Then, for every $\hat{\mathbf{x}} \in \mathbb{R}^n$ there exist a compact set $\Omega(\hat{\mathbf{x}}) \subset \mathbb{R}^m$, and a scalar $\rho_{\hat{\mathbf{x}}} > 0$ such that*

(i) for every $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})$ and $\mathbf{y} \in \Omega(\hat{\mathbf{x}})$, $\mathbf{A}(\mathbf{x}, \mathbf{y})\mathbf{A}(\mathbf{x}, \mathbf{y})^T + \mathbf{B}(\mathbf{x}, \mathbf{y})$, see (5.2.13a,d), is positive definite, and hence $\nabla_\mathbf{y} f_k(\mathbf{x}, \mathbf{y}), k \in \mathbf{r}_1^*(\mathbf{x}, \mathbf{y})$, (see (5.2.12a)), together with $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$, (see (5.2.12b)), are linearly independent, and

(ii) $\hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\rho_{\hat{\mathbf{x}}}} \subset \Omega(\hat{\mathbf{x}})$.

Proof. Let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be arbitrary. By Assumption 5.2.6, $\nabla_\mathbf{y} f_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}), k \in \mathbf{r}_1^*(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, together with $\nabla g_k(\hat{\mathbf{y}}), k \in \mathbf{r}_2^*(\hat{\mathbf{y}})$, are linearly independent for any $\hat{\mathbf{y}} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$. It follows from Lemma 5.2.7(ii) that $\mathbf{A}(\hat{\mathbf{x}}, \hat{\mathbf{y}})\mathbf{A}(\hat{\mathbf{x}}, \hat{\mathbf{y}})^T + \mathbf{B}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is positive definite. Thus, by continuity of $\mathbf{A}(\cdot, \cdot)$ and $\mathbf{B}(\cdot, \cdot)$, there exist a compact set $\Omega(\hat{\mathbf{x}}) \subset \mathbb{R}^m$ and a $\rho_{\hat{\mathbf{x}}} > 0$ such that $\mathbf{A}(\mathbf{x}, \mathbf{y})\mathbf{A}(\mathbf{x}, \mathbf{y})^T + \mathbf{B}(\mathbf{x}, \mathbf{y})$ is positive definite for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})$ and $\mathbf{y} \in \Omega(\hat{\mathbf{x}})$, and $\hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\rho_{\hat{\mathbf{x}}}} \subset \Omega(\hat{\mathbf{x}})$.

By positive definiteness, both sides of (5.2.15a) are strictly positive for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})$, $\mathbf{y} \in \Omega(\hat{\mathbf{x}})$ and $\mathbf{z} = (\eta, \xi) \neq 0$, with $\eta \in \mathbb{R}^{r_1}$ and $\xi \in \mathbb{R}^{r_2}$. Hence, (5.2.15b) must imply that (η, ξ) in (5.2.15b) is zero for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})$ and $\mathbf{y} \in \Omega(\hat{\mathbf{x}})$. But then, $\nabla_y f_k(\mathbf{x}, \mathbf{y}), k \in \mathbf{r}_1^*(\mathbf{x}, \mathbf{y})$, together with $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$ must be linearly independent for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})$ and $\mathbf{y} \in \Omega(\hat{\mathbf{x}})$. Because if that was not true, we may have (5.2.15b) satisfied for $(\eta, \xi) \neq 0$. This completes the proof. □

Lemma 5.2.9. *Suppose that Assumption 5.2.1 holds. Then, for every $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\pi > 0$, $\rho > 0$ and $\epsilon > 0$, there exist $\hat{\pi} \in [\pi, \infty)$ and $\hat{\rho} \in (0, \rho]$ such that $\hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_\epsilon$ for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \hat{\rho})$.*

Proof. Let $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\pi > 0$, $\rho > 0$ and $\epsilon > 0$ be arbitrary. To prove the desired result, we will show that **(i)** there exists a $\hat{\pi} \in [\pi, \infty)$ such that $\hat{\mathbf{Y}}_{\hat{\pi}}(\hat{\mathbf{x}}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\epsilon/2}$, and **(ii)** there exists a $\hat{\rho} \in (0, \rho]$ such that $\hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}) \subset \hat{\mathbf{Y}}_{\hat{\pi}}(\hat{\mathbf{x}}) + \mathbb{B}_{\epsilon/2}$ for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \hat{\rho})$.

(i) Let the set-valued function $\Gamma : [0, \infty) \to 2^{\mathbb{R}^m}$ be defined by

$$\Gamma(s) \stackrel{\triangle}{=} \arg\max_{y \in \mathbf{Y}} \phi'(s, \mathbf{y}), \tag{5.2.16a}$$

where $\phi' : [0, \infty) \times \mathbb{R}^m \to \mathbb{R} \bigcup \{-\infty\}$ is defined by

$$\phi'(s, \mathbf{y}) \stackrel{\triangle}{=} \begin{cases} \phi(\hat{\mathbf{x}}, \mathbf{y}) - \dfrac{1}{s}\|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y})_+\|_\infty, & s > 0 \\ \phi(\hat{\mathbf{x}}, \mathbf{y}), & s = 0, \|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y})_+\|_\infty = 0 \\ -\infty, & s = 0, \|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y})_+\|_\infty > 0 \end{cases} . \tag{5.2.16b}$$

First, we show that $\Gamma(\cdot)$ is outer semicontinuous at $s = 0$ in the sense of Kuratowski-Painlevé, see Rockafellar and Wets (1997) and Polak (1997). By Theorem 5.3.7 in Polak (1997), we only need to show that the outer limit of $\{\Gamma(s_i)\}_{i=0}^\infty$ is contained in $\Gamma(0)$ for any sequence $\{s_i\}_{i=0}^\infty \subset [0, \infty)$ such that $s_i \to 0$. Let $\{s_i\}_{i=0}^\infty \subset [0, \infty)$, be such that $s_i \to 0$, and let $\hat{\mathbf{y}} \in \mathbb{R}^m$ be a point in the outer limit of $\{\Gamma(s_i)\}_{i=0}^\infty$. Then there exists a sequence $\{\mathbf{y}_i\}_{i=0}^\infty$ such that $\mathbf{y}_i \in \Gamma(s_i)$ for all $i \in \mathbb{N}$, and $\mathbf{y}_i \to \hat{\mathbf{y}}$, as $i \to \infty$.

Now, consider the hypo-graphs, see Rockafellar and Wets (1997) and Polak (1997), of the problems $\max_{\mathbf{y}\in\mathbf{Y}} \phi'(s_i, \mathbf{y})$ given by

$$E_i \stackrel{\triangle}{=} \{(\mathrm{y}_0, \mathbf{y}) \in \mathbb{R}^{m+1} \mid \mathbf{y} \in \mathbf{Y},\ \mathrm{y}_0 \leq \phi'(s_i, \mathbf{y})\}, \tag{5.2.16c}$$

and of the problem $\max_{\mathbf{y}\in\mathbf{Y}} \phi'(0, \mathbf{y})$ given by

$$E \stackrel{\triangle}{=} \{(\mathrm{y}_0, \mathbf{y}) \in \mathbb{R}^{m+1} \mid \mathbf{y} \in \mathbf{Y},\ \mathrm{y}_0 \leq \phi'(0, \mathbf{y})\}. \tag{5.2.16d}$$

By Theorem 3.3.2 in Polak (1997), the sequence of sets $\{E_i\}_{i=0}^{\infty}$ converges to E in the Kuratowski-Painlevé sense (see Rockafellar and Wets (1997) and Polak (1997)) if and only if **(a)** for any $\mathbf{y}' \in \mathbf{Y}$, $\liminf_{i\to\infty} \phi'(s_i, \mathbf{y}') \geq \phi'(0, \mathbf{y}')$, and **(b)** for any infinite sequence $\{\mathbf{y}'_i\}_{i\in K} \subset \mathbf{Y}$, $K \subset \mathbb{N}$, such that $\mathbf{y}'_i \to^K \mathbf{y}'$, as $i \to \infty$, $\limsup_{i\to^K\infty} \phi'(s_i, \mathbf{y}') \leq \phi'(0, \mathbf{y}')$.

First, we consider **(a)**. Suppose $\mathbf{y}' \in \mathbf{Y}$. Then, we have directly from (5.2.16b) that $\lim_{i\to\infty} \phi'(s_i, \mathbf{y}') = \phi'(0, \mathbf{y}')$.

Second, we consider **(b)**. Let $\{\mathbf{y}'_i\}_{i\in K} \subset \mathbf{Y}$ be an infinite sequence, $K \subset \mathbb{N}$, such that $\mathbf{y}'_i \to^K \mathbf{y}'$, as $i \to \infty$. Without loss of generality, we assume that $\mathbf{y}'_i \to \mathbf{y}'$, as $i \to \infty$. Now, we have two cases.

Case I: Suppose $\|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}')_+\|_\infty > \delta$, for some $\delta > 0$. Then by continuity of $\mathbf{f}(\cdot,\cdot)$, there exists an $i_0 \in \mathbb{N}$ such that $\|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}'_i)_+\|_\infty \geq \delta/2$ for all $i > i_0$. Hence for all $i > i_0$, such that $s_i > 0$, $\phi'(s_i, \mathbf{y}'_i) = \phi(\hat{\mathbf{x}}, \mathbf{y}'_i) - \|\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}'_i)_+\|_\infty / s_i \leq \phi(\hat{\mathbf{x}}, \mathbf{y}'_i) - \delta/(2s_i)$, and for all $i > i_0$, such that $s_i = 0$, $\phi'(s_i, \mathbf{y}'_i) = -\infty$. Since $s_i \to 0$, we have that $\lim_{i\to\infty} \phi'(s_i, \mathbf{y}'_i) = \phi'(0, \mathbf{y}') = -\infty$.

Case II: Suppose $\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}') \leq 0$. Then by (5.2.16b), $\limsup_{i\to\infty} \phi'(s_i, \mathbf{y}'_i) \leq \phi(\hat{\mathbf{x}}, \mathbf{y}') = \phi'(0, \mathbf{y}')$.

Hence by Theorem 3.3.2 in Polak (1997), $\{E_i\}_{i=0}^{\infty}$ converges to E. As a consequence of the convergence of $\{E_i\}_{i=0}^{\infty}$ to E, Theorem 3.3.3 in Polak (1997) states that any accumulation point of a sequence of global maximizers of $\max_{\mathbf{y}\in\mathbf{Y}} \phi'(s_i, \mathbf{y})$ is a global maximizer of $\max_{\mathbf{y}\in\mathbf{Y}} \phi'(0, \mathbf{y})$. Hence, $\hat{\mathbf{y}} \in \Gamma(0)$, which is a contradiction. Hence, we have that $\Gamma(\cdot)$ is outer semicontinuous at $s = 0$.

Next, let $\mathbf{y}^* \in \Gamma(0)$. It follows from (5.2.16b) and Assumption 5.2.1(ii), that $\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}^*) \leq 0$ and $\mathbf{y}^* \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$. Hence, $\Gamma(0) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}})$.

Finally, by outer semicontinuity of $\Gamma(\cdot)$ at $s = 0$, there exists a $\hat{\pi} \in [\pi, \infty)$ such that $\hat{\mathbf{Y}}_{\hat{\pi}}(\hat{\mathbf{x}}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\epsilon/2}$, and (i) holds.

(ii) Let $\hat{\pi}$ be as in (i). First, we show that $\hat{\mathbf{Y}}_{\hat{\pi}}(\cdot)$ is outer semicontinuous at $\hat{\mathbf{x}}$. By Theorem 5.3.7 in Polak (1997), we only need to show that the outer limit of $\{\hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}_i)\}_{i=0}^{\infty}$ is contained in $\hat{\mathbf{Y}}_{\hat{\pi}}(\hat{\mathbf{x}})$ for any sequence $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ such that $\mathbf{x}_i \to \hat{\mathbf{x}}$. Let $\{\mathbf{y}_i\}_{i=0}^{\infty}$ be an arbitrary sequence such that $\mathbf{y}_i \in \hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}_i)$ and $\mathbf{y}_i \to \hat{\mathbf{y}}$. Then,

$$\phi(\mathbf{x}_i, \mathbf{y}_i) - \hat{\pi}\|\mathbf{f}(\mathbf{x}_i, \mathbf{y}_i)_+\|_\infty \geq \phi(\mathbf{x}_i, \mathbf{y}) - \hat{\pi}\|\mathbf{f}(\mathbf{x}_i, \mathbf{y})_+\|_\infty, \tag{5.2.16e}$$

for all $i \in \mathbb{N}$ and $\mathbf{y} \in \mathbf{Y}$. Hence, by adding $\phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \hat{\pi}\|\mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{y}})_+\|_\infty$ to both sides of (5.2.16e), and rearranging terms, we obtain that

$$\begin{aligned}\phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \hat{\pi}\|\mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{y}})_+\|_\infty \quad &\geq \max_{\mathbf{y} \in \mathbf{Y}} \{\phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \hat{\pi}\|\mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{y}})_+\|_\infty - \phi(\mathbf{x}_i, \mathbf{y}_i) \\ &\quad + \hat{\pi}\|\mathbf{f}(\mathbf{x}_i, \mathbf{y}_i)_+\|_\infty + \phi(\mathbf{x}_i, \mathbf{y}) - \hat{\pi}\|\mathbf{f}(\mathbf{x}_i, \mathbf{y})_+\|_\infty\}.\end{aligned} \tag{5.2.16f}$$

It now follows by the continuity of the right-hand side of (5.2.16f) that $\phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \hat{\pi}\|\mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{y}})_+\|_\infty \geq \psi_{\hat{\pi}}(\hat{\mathbf{x}})$, and hence that $\hat{\mathbf{y}} \in \hat{\mathbf{Y}}_{\hat{\pi}}(\hat{\mathbf{x}})$. Hence $\hat{\mathbf{Y}}_{\hat{\pi}}(\cdot)$ is outer semicontinuous at $\hat{\mathbf{x}}$, which implies that (ii) holds. Now, the conclusion follows directly from (i) and (ii). □

Theorem 5.2.10. *Suppose Assumptions 5.2.1 and 5.2.6 hold. Then, for any* $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\mathbf{IP}(\hat{\mathbf{x}}, 0)$ *is locally calm at* $\hat{\mathbf{x}}$.

Proof. Let $\hat{\mathbf{x}} \in \mathbb{R}^n$. By Lemmas 5.2.7(ii) and 5.2.8, there exist a compact set $\Omega(\hat{\mathbf{x}}) \subset \mathbb{R}^m$ and $\rho_{\hat{\mathbf{x}}} > 0$ such that $\eta(\cdot, \cdot)$ is continuous on $\mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}}) \times \Omega(\hat{\mathbf{x}})$ and $\hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\rho_{\hat{\mathbf{x}}}} \subset \Omega(\hat{\mathbf{x}})$. Hence,

$$\pi^* \triangleq \max_{\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho_{\hat{\mathbf{x}}})} \max_{\mathbf{y} \in \Omega(\hat{\mathbf{x}})} \sigma \sum_{k=1}^{r_1} |\eta_k(\mathbf{x}, \mathbf{y})|, \tag{5.2.17a}$$

with $\sigma > 1$, is well-defined. By Lemma 5.2.9, there exist $\hat{\rho} \in (0, \rho_{\hat{\mathbf{x}}}]$ and $\hat{\pi} \geq \pi^*$ such that $\hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_{\rho_{\hat{\mathbf{x}}}}$ for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \hat{\rho})$. Let $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \hat{\rho})$ be arbitrary. Then, $\hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x}) \subset \Omega(\hat{\mathbf{x}})$, and hence for any $\mathbf{y}_{\mathbf{x}} \in \hat{\mathbf{Y}}_{\hat{\pi}}(\mathbf{x})$, we have by (5.2.17a) and (5.2.13h) that $t_{\hat{\pi}}(\mathbf{x}, \mathbf{y}_{\mathbf{x}}) \leq 0$, and, by Lemma 5.2.8 that $\nabla_{\mathbf{y}} f_k(\mathbf{x}, \mathbf{y}_{\mathbf{x}}), k \in \mathbf{r}_1^*(\mathbf{x}, \mathbf{y}_{\mathbf{x}})$, together with $\nabla g_k(\mathbf{y}_{\mathbf{x}}), k \in \mathbf{r}_2^*(\mathbf{y}_{\mathbf{x}})$ are linearly independent. Hence, by Lemma 5.2.7(iii), $\mathbf{y}_{\mathbf{x}} \in \hat{\mathbf{Y}}(\mathbf{x})$. Next, let $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{u} \in \mathbb{R}^{r_1}$ be such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) \leq \mathbf{u}$. Then,

$$\begin{aligned} \phi(\mathbf{x}, \mathbf{y}_{\mathbf{x}}) &= \phi(\mathbf{x}, \mathbf{y}_{\mathbf{x}}) - \hat{\pi}\|\mathbf{f}(\mathbf{x}, \mathbf{y}_{\mathbf{x}})_+\|_\infty \\ &\geq \phi(\mathbf{x}, \mathbf{y}) - \hat{\pi}\|\mathbf{f}(\mathbf{x}, \mathbf{y})_+\|_\infty \\ &\geq \phi(\mathbf{x}, \mathbf{y}) - \hat{\pi}\|(\mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{u})_+\|_\infty - \hat{\pi}\|\mathbf{u}\|_\infty \\ &= \phi(\mathbf{x}, \mathbf{y}) - \hat{\pi}\|\mathbf{u}\|_\infty. \end{aligned} \tag{5.2.17b}$$

By (5.2.6), $v(\mathbf{x}, 0) = \phi(\mathbf{x}, \mathbf{y}_{\mathbf{x}})$. For every $\mathbf{u} \in \mathbb{R}^{r_1}$ such that $v(\mathbf{x}, \mathbf{u}) > -\infty$, there exists $\mathbf{y}'_{\mathbf{u}} \in \mathbf{Y}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}'_{\mathbf{u}}) \leq \mathbf{u}$ and $\phi(\mathbf{x}, \mathbf{y}'_{\mathbf{u}}) = v(\mathbf{x}, \mathbf{u})$. Hence, by (5.2.17b), for every $\mathbf{u} \in \mathbb{R}^{r_1}$ such that $v(\mathbf{x}, \mathbf{u}) > -\infty$, we have that

$$v(\mathbf{x}, \mathbf{u}) - v(\mathbf{x}, 0) \leq \hat{\pi}\|\mathbf{u}\|_\infty. \tag{5.2.17c}$$

Since (5.2.17c) also holds for $\mathbf{u} \in \mathbb{R}^{r_1}$ such that $v(\mathbf{x}, \mathbf{u}) = -\infty$, we have that (5.2.17c) holds for every $\mathbf{u} \in \mathbb{R}^{r_1}$. Finally, because $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \hat{\rho})$ was assumed arbitrary, the conclusion follows with $\hat{\alpha} = \hat{\pi}$. □

5.3 Approximations to $\mathbf{P}_\pi$

In view of Theorem 5.2.3, $\mathbf{P}$ can be solved by solving $\mathbf{P}_\pi$ for a sufficiently large $\pi > 0$. To facilitate the solution of $\mathbf{P}_\pi$, we introduce two approximations. First, for any set $\mathbf{Y}_N \subset \mathbf{Y}$, $N \in \mathbb{N} \triangleq \{1, 2, ...\}$, of finite cardinality and $\pi > 0$, we define the approximation $\psi_{\pi,N} : \mathbb{R}^n \to \mathbb{R}$ to the function $\psi_\pi(\cdot)$, see (5.2.3a), by

$$\psi_{\pi,N}(\mathbf{x}) \triangleq \max_{\mathbf{y} \in \mathbf{Y}_N} \omega_\pi(\mathbf{x}, \mathbf{y}), \tag{5.3.1a}$$

with $\omega_\pi(\cdot,\cdot)$ as in (5.2.3d).

Second, we introduce a smoothing technique that can be found in Bertsekas (1982, Li (1997). For any $\pi > 0$ and $p > 0$, let $\omega_{\pi,p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the smooth approximation to $\omega_\pi(\cdot,\cdot)$ defined by

$$\omega_{\pi,p}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} -\frac{1}{p}\ln(\sum_{k=1}^{r} e^{-p\phi_{k,\pi}(\mathbf{x},\mathbf{y})}). \tag{5.3.1b}$$

Hence, for any $\pi > 0$, $N \in \mathbb{N}$ and $p > 0$ we define a family of min-max approximations to $\mathbf{P}_\pi$ by

$$\mathbf{P}_{\pi,N,p} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \psi_{\pi,N,p}(\mathbf{x}), \tag{5.3.1c}$$

where $\psi_{\pi,N,p} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_{\pi,N,p}(\mathbf{x}) \stackrel{\triangle}{=} \max_{\mathbf{y}\in\mathbf{Y}_N} \omega_{\pi,p}(\mathbf{x},\mathbf{y}), \tag{5.3.2a}$$

Referring to Section 5.2.1 in Polak (1997), we find that a continuous optimality function, $\theta_{\pi,N,p} : \mathbb{R}^n \to \mathbb{R}$, for the problem $\mathbf{P}_{\pi,N,p}$ is given by

$$\theta_{\pi,N,p}(\mathbf{x}) \stackrel{\triangle}{=} -\min_{\bar{\xi}\in\bar{G}\psi_{\pi,N,p}(\mathbf{x})} \xi_0 + \frac{1}{2}\|\xi\|^2, \tag{5.3.2b}$$

where $\bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{n+1}$, with $\xi \in \mathbb{R}^n$, and

$$\bar{G}\psi_{\pi,N,p}(\mathbf{x}) \stackrel{\triangle}{=} \operatorname*{conv}_{\mathbf{y}\in\mathbf{Y}_N} \left\{ \begin{pmatrix} \psi_{\pi,N,p}(\mathbf{x}) - \omega_{\pi,p}(\mathbf{x},\mathbf{y}) \\ \nabla_{\mathbf{x}}\omega_{\pi,p}(\mathbf{x},\mathbf{y}) \end{pmatrix} \right\} \tag{5.3.2c}$$

We require that the error associated with the discretization of the set $\mathbf{Y}$ satisfies a certain relation as specified in Assumption 5.3.1(ii) below. Note also that Assumption 5.3.1(iv), below, is a Mangasarian-Fromowitz-type constraint qualification.

Assumption 5.3.1. *We assume that*

(i) *$\phi(\cdot,\cdot), f_k(\cdot,\cdot), k \in \mathbf{r_1}$, and $g_k(\cdot), k \in \mathbf{r_2}$, are twice continuously differentiable,*

(ii) *there exist a strictly decreasing function* $\Delta : \mathbb{N} \bigcup \{\infty\} \to [0, \infty)$, *with the property that* $\Delta(N) \to 0$, *as* $N \to \infty$, *and* $\Delta(\infty) \stackrel{\triangle}{=} 0$, *and constants* $N_0 \in \mathbb{N}$, $C < \infty$ *such that for every* $N \geq N_0$ *and* $\mathbf{y} \in \mathbf{Y}$, *there exists a* $\mathbf{y}' \in \mathbf{Y}_N$ *such that*

$$\|\mathbf{y} - \mathbf{y}'\| \leq C\Delta(N), \tag{5.3.3a}$$

(iii) *for every* $N \geq N_0$ *and* $\mathbf{x} \in \mathbb{R}^n$

$$\{\mathbf{y} \in \mathbf{Y}_N \mid \mathbf{f}(\mathbf{x}, \mathbf{y}) \leq 0\} \neq \emptyset, \tag{5.3.3b}$$

(iv) *for any* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbf{Y}$ *there exists an* $\mathbf{h} \in \mathbb{R}^m$ *and* $\hat{u} > 0$ *such that for all* $k \in \mathbf{r_1}$ *satisfying* $f_k(\mathbf{x}, \mathbf{y}) = 0$, $\langle \nabla_{\mathbf{y}} f_k(\mathbf{x}, \mathbf{y}), \mathbf{h} \rangle < 0$, *and for all* $u \in (0, \hat{u}]$, $\mathbf{g}(\mathbf{y} + u\mathbf{h}) \leq 0$.

□

For example, if $\mathbf{Y}$ is the unit cube in $\mathbb{R}^m$, i.e., $\mathbf{Y} = \mathbf{I}^m$, with $\mathbf{I} = [0, 1]$, then we can define $\mathbf{Y}_N = \mathbf{I}_N^m$, where

$$\mathbf{I}_N = \{0, 1/a(N), 2/a(N), ..., (a(N) - 1)/a(N), 1\}, \tag{5.3.3c}$$

with $a(N) = 2^{N-N_0}N_0$. In this case, $\Delta(N) = 1/a(N)$ and $C = \frac{1}{2}m^{1/2}$.

We need the following notation: Let $\hat{\mathbf{Y}}_N : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, $\hat{\mathbf{Y}}_{\pi,N} : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ and be defined by

$$\hat{\mathbf{Y}}_N(\mathbf{x}) \stackrel{\triangle}{=} \arg\max_{\mathbf{y} \in \mathbf{Y}_N} \{\phi(\mathbf{x}, \mathbf{y}) | \mathbf{f}(\mathbf{x}, \mathbf{y}) \leq 0\}, \tag{5.3.4a}$$

$$\hat{\mathbf{Y}}_{\pi,N}(\mathbf{x}) \stackrel{\triangle}{=} \{\mathbf{y} \in \mathbf{Y}_N \mid \omega_\pi(\mathbf{x}, \mathbf{y}) = \psi_{\pi,N}(\mathbf{x})\} = \arg\max_{\mathbf{y} \in \mathbf{Y}_N} \{\phi(\mathbf{x}, \mathbf{y}) - \pi \|\mathbf{f}(\mathbf{x}, \mathbf{y})_+\|_\infty\}. \tag{5.3.4b}$$

Lemma 5.3.2. *Suppose Assumptions 5.2.1 and 5.3.1 hold and* $\pi > 0$. *If* $\mathbf{x}_i \to \hat{\mathbf{x}}$, $\mathbf{y}_i \to \hat{\mathbf{y}}$, $N_i \to \infty$, *as* $i \to \infty$, *with* $\mathbf{y}_i \in \hat{\mathbf{Y}}_{\pi,N_i}(\mathbf{x}_i)$ *and* $N_i \in \mathbb{N}$, *for all* $i \in \mathbb{N}$, *then*

$\hat{\mathrm{y}} \in \hat{\mathbf{Y}}_\pi(\hat{\mathrm{x}})$.

Proof. Let $\pi > 0$, $\{\mathrm{x}_i\}_{i=0}^\infty \subset \mathbb{R}^n$, $\{\mathrm{y}_i\}_{i=0}^\infty \subset \mathbb{R}^m$, $\{N_i\}_{i=0}^\infty \subset \mathbb{N}$, $\hat{\mathrm{x}} \in \mathbb{R}^n$ and $\hat{\mathrm{y}} \in \mathbf{Y}$ be such that $\mathrm{y}_i \in \hat{\mathbf{Y}}_{\pi,N_i}(\mathrm{x}_i)$, $\mathrm{x}_i \to \hat{\mathrm{x}}$, $\mathrm{y}_i \to \hat{\mathrm{y}}$, and $N_i \to \infty$. Then,

$$\phi(\mathrm{x}_i, \mathrm{y}_i) - \pi\|\mathbf{f}(\mathrm{x}_i, \mathrm{y}_i)_+\|_\infty \geq \phi(\mathrm{x}_i, \mathrm{y}) - \pi\|\mathbf{f}(\mathrm{x}_i, \mathrm{y})_+\|_\infty, \tag{5.3.5a}$$

for all $i \in \mathbb{N}$ and $\mathrm{y} \in \mathbf{Y}_{N_i}$. Hence, by adding $\phi(\hat{\mathrm{x}}, \hat{\mathrm{y}}) - \pi\|\mathbf{f}(\hat{\mathrm{x}}, \hat{\mathrm{y}})_+\|$ to both sides of (5.3.5a), and rearranging terms, we obtain that

$$\begin{aligned} \phi(\hat{\mathrm{x}}, \hat{\mathrm{y}}) - \pi\|\mathbf{f}(\hat{\mathrm{x}}, \hat{\mathrm{y}})_+\|_\infty \; \geq \max_{\mathrm{y} \in \mathbf{Y}_{N_i}} \{ & \phi(\hat{\mathrm{x}}, \hat{\mathrm{y}}) - \pi\|\mathbf{f}(\hat{\mathrm{x}}, \hat{\mathrm{y}})_+\|_\infty - \phi(\mathrm{x}_i, \mathrm{y}_i) \\ & + \pi\|\mathbf{f}(\mathrm{x}_i, \mathrm{y}_i)_+\|_\infty + \phi(\mathrm{x}_i, \mathrm{y}) - \pi\|\mathbf{f}(\mathrm{x}_i, \mathrm{y})_+\|_\infty \}. \end{aligned} \tag{5.3.5b}$$

It now follows from the continuity of the right-hand side of (5.3.5b), see Corollary 5.4.2 in Polak (1997), and Assumption 5.3.1(ii) that $\phi(\hat{\mathrm{x}}, \hat{\mathrm{y}}) - \pi\|\mathbf{f}(\hat{\mathrm{x}}, \hat{\mathrm{y}})_+\|_\infty \geq \psi_\pi(\hat{\mathrm{x}})$, and hence that $\hat{\mathrm{y}} \in \hat{\mathbf{Y}}_\pi(\hat{\mathrm{x}})$. □

Lemma 5.3.3. *Suppose Assumptions 5.2.1 and 5.3.1 hold. Then,*

(i) for every $\hat{\mathrm{x}} \in \mathbb{R}^n$ and $\epsilon > 0$, there exist $\hat{\pi} < \infty$, $\hat{N} < \infty$ and $\hat{\rho} > 0$, such that $\hat{\mathbf{Y}}_{\pi,N}(\mathrm{x}) \subset \hat{\mathbf{Y}}(\hat{\mathrm{x}}) + \mathbb{B}_\epsilon$, for all $\mathrm{x} \in \mathbb{B}(\hat{\mathrm{x}}, \hat{\rho})$, $\pi \geq \hat{\pi}$ and $N \geq \hat{N}$, $N \in \mathbb{N}$, and

(ii) for every $\hat{\mathrm{x}} \in \mathbb{R}^n$, $N \in \mathbb{N}$, $N \geq N_0$, with N_0 as in Assumption 5.3.1(ii), and $\epsilon > 0$, there exist $\hat{\pi} < \infty$ and $\hat{\rho} > 0$, such that $\hat{\mathbf{Y}}_{\pi,N}(\mathrm{x}) \subset \hat{\mathbf{Y}}_N(\hat{\mathrm{x}}) + \mathbb{B}_\epsilon$, for all $\mathrm{x} \in \mathbb{B}(\hat{\mathrm{x}}, \hat{\rho})$ and $\pi \geq \hat{\pi}$.

Proof. (i) For any $s \geq 0$, $t \in T \triangleq \{t \in (0,1] \mid 1/t \in \mathbb{N}\} \bigcup \{0\}$ and $\mathrm{x} \in \mathbb{R}^n$, let $w = (s, t, \mathrm{x}) \in \mathbb{R}^{n+2}$. We define the set-valued function $W : [0, \infty) \times T \times \mathbb{R}^n \to 2^{\mathbb{R}^m}$ by

$$W(w) \triangleq \arg\max_{\mathrm{y} \in \mathbf{Y}_t^*} \tilde{\phi}(s, \mathrm{x}, \mathrm{y}), \tag{5.3.6a}$$

where $\mathbf{Y}_t^* \stackrel{\triangle}{=} \mathbf{Y}_N$, with $N = 1/t$, for $t \in T, t > 0$, $\mathbf{Y}_t^* \stackrel{\triangle}{=} \mathbf{Y}$ for $t = 0$, and $\tilde{\phi} : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \bigcup \{-\infty\}$ is defined by

$$\tilde{\phi}(s, \mathbf{x}, \mathbf{y}) \stackrel{\triangle}{=} \begin{cases} \phi(\mathbf{x}, \mathbf{y}) - \frac{1}{s}\|\mathbf{f}(\mathbf{x}, \mathbf{y})_+\|_\infty, & s > 0 \\ \phi(\mathbf{x}, \mathbf{y}), & s = 0, \|\mathbf{f}(\mathbf{x}, \mathbf{y})_+\|_\infty = 0 \\ -\infty, & s = 0, \|\mathbf{f}(\mathbf{x}, \mathbf{y})_+\|_\infty > 0 \end{cases} . \qquad (5.3.6b)$$

Let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be arbitrary. We will first show that $W(\cdot)$ is outer semicontinuous at $\hat{w} = (0, 0, \hat{\mathbf{x}})$ in the sense of Kuratowski-Painlevé, see Rockafellar and Wets (1997) and Polak (1997). By Theorem 5.3.7 in Polak (1997), we only need to show that the outer limit of $\{W(w_i)\}_{i=0}^\infty$ is contained in $W(\hat{w})$ for any sequence $\{w_i\}_{i=0}^\infty \subset [0, \infty) \times T \times \mathbb{R}^n$ such that $w_i \to \hat{w}$. Let $\{w_i\}_{i=0}^\infty$, with $w_i = (s_i, t_i, \mathbf{x}_i) \in [0, \infty) \times T \times \mathbb{R}^n$, be such that $w_i \to \hat{w}$, and let $\hat{\mathbf{y}} \in \mathbb{R}^m$ be a point in the outer limit of $\{W(w_i)\}_{i=0}^\infty$. Then there exists a sequence $\{\mathbf{y}_i\}_{i=0}^\infty$ such that $\mathbf{y}_i \in W(w_i)$ for all $i \in \mathbb{N}$, and $\mathbf{y}_i \to \hat{\mathbf{y}}$, as $i \to \infty$.

Now, consider the hype-graphs, see Rockafellar and Wets (1997) and Polak (1997), of the problems $\max_{\mathbf{y} \in \mathbf{Y}_{t_i}^*} \tilde{\phi}(s_i, \mathbf{x}_i, \mathbf{y})$ given by

$$E_i \stackrel{\triangle}{=} \{(\mathbf{y}_0, \mathbf{y}) \in \mathbb{R}^{m+1} \mid \mathbf{y} \in \mathbf{Y}_{t_i}^*, \ \mathbf{y}_0 \leq \tilde{\phi}(s_i, \mathbf{x}_i, \mathbf{y})\}, \qquad (5.3.6c)$$

and of the problem $\max_{\mathbf{y} \in \mathbf{Y}} \tilde{\phi}(0, \hat{\mathbf{x}}, \mathbf{y})$ given by

$$E \stackrel{\triangle}{=} \{(\mathbf{y}_0, \mathbf{y}) \in \mathbb{R}^{m+1} \mid \mathbf{y} \in \mathbf{Y}, \ \mathbf{y}_0 \leq \tilde{\phi}(0, \hat{\mathbf{x}}, \mathbf{y})\}. \qquad (5.3.6d)$$

By Theorem 3.3.2 in Polak (1997), the sequence of sets $\{E_i\}_{i=0}^\infty$ converges to E in the Kuratowski-Painlevé sense (see Rockafellar and Wets (1997) and Polak (1997)) if and only if **(a)** for any $\tilde{\mathbf{y}} \in \mathbf{Y}$, there exists a sequence $\{\tilde{\mathbf{y}}_i\}_{i=0}^\infty$, with $\tilde{\mathbf{y}}_i \in \mathbf{Y}_{t_i}^*$, such that $\tilde{\mathbf{y}}_i \to \tilde{\mathbf{y}}$, as $i \to \infty$, and $\liminf_{i \to \infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) \geq \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}})$, and **(b)** for every infinite sequence $\{\tilde{\mathbf{y}}_i\}_{i \in K}$, with $K \subset \mathbb{N}$, such that $\tilde{\mathbf{y}}_i \in \mathbf{Y}_{t_i}^*$ for all $i \in K$, and $\tilde{\mathbf{y}}_i \stackrel{K}{\to} \tilde{\mathbf{y}}$, as $i \to \infty$, $\limsup_{i \stackrel{K}{\to} \infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) \leq \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}})$.

First, consider **(a)**. Suppose that $\tilde{\mathbf{y}} \in \mathbf{Y}$. Now, we have two cases.

Case I: Suppose $\|\mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{y}})_+\|_\infty > 0$. Then, $\tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}}) = -\infty$, and hence by Assumption

5.3.1(ii) there exists a sequence $\{\tilde{\mathbf{y}}_i\}_{i=0}^{\infty} \subset \mathbf{Y}$ such that $\tilde{\mathbf{y}}_i \in \mathbf{Y}_{t_i}^*$, for all $i \in \mathbb{N}$, $\tilde{\mathbf{y}}_i \to \tilde{\mathbf{y}}$, as $i \to \infty$, and $\liminf_{i\to\infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) \geq \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}})$.

Case II: Suppose $\mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{y}}) \leq 0$. We infer from Assumption 5.3.1(iv) that there exist $\mathbf{h} \in \mathbb{R}^m$, $\delta > 0$ and $u^* > 0$ such that for all $u \in (0, u^*]$

$$f_k(\hat{\mathbf{x}}, \tilde{\mathbf{y}} + u\mathbf{h}) \leq -\delta u, \qquad \forall \ k \in \mathbf{r_1}, \tag{5.3.6e}$$

$$\mathbf{g}(\tilde{\mathbf{y}} + u\mathbf{h}) \leq 0. \tag{5.3.6f}$$

Let $L < \infty$ be a Lipschitz constant for $f_k(\cdot,\cdot), k \in \mathbf{r_1}$, on $\mathbb{B}(\hat{\mathbf{x}}, 1) \times \mathbb{B}(\tilde{\mathbf{y}}, u^*\|\mathbf{h}\|)$. Hence by (5.3.6e), for all $u \in (0, u^*/2]$, $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, 1)$ and $\mathbf{y} \in \mathbb{B}(\tilde{\mathbf{y}} + u\mathbf{h}, u^*\|\mathbf{h}\|/2)$, we have that

$$f_k(\mathbf{x}, \mathbf{y}) \leq -\delta u + L(\|\mathbf{x} - \hat{\mathbf{x}}\| + \|\tilde{\mathbf{y}} + u\mathbf{h} - \mathbf{y}\|), \qquad \forall \ k \in \mathbf{r_1}, \tag{5.3.6g}$$

Let $\alpha = \delta/(2L)$. Then, there exits $u^{**} \in (0, u^*/2]$ such that for all $u \in (0, u^{**}]$, $\alpha u \leq \min\{1,\ u^*\|\mathbf{h}\|/2\}$. Let $u \in (0, u^{**}]$. Then for all $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \alpha u)$ and $\mathbf{y} \in \mathbb{B}(\tilde{\mathbf{y}} + u\mathbf{h}, \alpha u)$,

$$f_k(\mathbf{x}, \mathbf{y}) \leq 0, \qquad \forall \ k \in \mathbf{r_1}, \tag{5.3.6h}$$

Since $\mathbf{x}_i \to \hat{\mathbf{x}}$ and $t_i \to 0$, as $i \to \infty$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $\|\mathbf{x}_i \to \hat{\mathbf{x}}\| \leq \alpha u^{**}$ and $C\Delta(1/t_i) \leq \alpha u^{**}$, see Assumption 5.3.1(ii). For all $i \geq i_0$, we define $u_i = \max\{\|\mathbf{x}_i - \hat{\mathbf{x}}\|,\ C\Delta(1/t_i)\}/\alpha$, and $\mathbf{y}_i' = \tilde{\mathbf{y}} + u_i\mathbf{h}$. By (5.3.6f), $\mathbf{g}(\mathbf{y}_i') \leq 0$, and hence $\mathbf{y}_i' \in \mathbf{Y}$. Then by Assumption 5.3.1(ii), for every $i \geq i_0$ there exists $\tilde{\mathbf{y}}_i \in \mathbf{Y}_{t_i}^*$ such that $\|\mathbf{y}_i' - \tilde{\mathbf{y}}_i\| \leq C\Delta(1/t_i)$. It now follows by construction that $\{\tilde{\mathbf{y}}_i\}_{i=i_0}^{\infty}$ is such that $\tilde{\mathbf{y}}_i \to \tilde{\mathbf{y}}$, as $i \to \infty$, and by (5.3.6h) that $\|\mathbf{f}(\mathbf{x}_i, \tilde{\mathbf{y}}_i)_+\|_\infty = 0$ for all $i \geq i_0$. Hence by continuity of $\phi(\cdot,\cdot)$, $\lim_{i\to\infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) = \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}})$.

Second, consider **(b)**. Let $\{\tilde{\mathbf{y}}_i\}_{i\in K}$ be an infinite sequence, $K \subset \mathbb{N}$, such that $\tilde{\mathbf{y}}_i \in \mathbf{Y}_{t_i}^*$, for all $i \in K$, $\tilde{\mathbf{y}}_i \to^K \tilde{\mathbf{y}}$, as $i \to \infty$. Without loss of generality, we assume that $\tilde{\mathbf{y}}_i \to \tilde{\mathbf{y}}$, as $i \to \infty$. Now, we have two cases.

Case I: Suppose $\|\mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{y}})_+\|_\infty > \delta$, for some $\delta > 0$. Then by continuity of $\mathbf{f}(\cdot,\cdot)$, there exists an $i_0 \in \mathbb{N}$ such that $\|\mathbf{f}(\mathbf{x}_i, \tilde{\mathbf{y}}_i)_+\|_\infty \geq \delta/2$ for all $i > i_0$. Hence for all $i > i_0$,

such that $s_i > 0$, $\tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) = \phi(\mathbf{x}_i, \tilde{\mathbf{y}}_i) - \|\mathbf{f}(\mathbf{x}_i, \tilde{\mathbf{y}}_i)_+\|_\infty / s_i \leq \phi(\mathbf{x}_i, \tilde{\mathbf{y}}_i) - \delta/(2s_i)$, and for all $i > i_0$, such that $s_i = 0$, $\tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) = -\infty$. Since $w_i \to \hat{w}$, we have that $s_i \to 0$, and hence $\lim_{i\to\infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) = \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}}) = -\infty$.

Case II: Suppose $\mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{y}}) \leq 0$. Then it follows directly from (5.3.6b) that $\limsup_{i\to\infty} \tilde{\phi}(s_i, \mathbf{x}_i, \tilde{\mathbf{y}}_i) \leq \phi(\hat{\mathbf{x}}, \tilde{\mathbf{y}}) = \tilde{\phi}(0, \hat{\mathbf{x}}, \tilde{\mathbf{y}})$.

Hence by Theorem 3.3.2 in Polak (1997), $\{E_i\}_{i=0}^{\infty}$ converges to E. As a consequence of the convergence of $\{E_i\}_{i=0}^{\infty}$ to E, Theorem 3.3.3 in Polak (1997) states that any accumulation point of a sequence of global maximizers of $\max_{\mathbf{y}\in\mathbf{Y}_{i_i}^*} \tilde{\phi}(s_i, \mathbf{x}_i, \mathbf{y})$ is a global maximizer of $\max_{\mathbf{y}\in\mathbf{Y}} \tilde{\phi}(0, \hat{\mathbf{x}}, \mathbf{y})$. Hence, $\hat{\mathbf{y}} \in W(\hat{w})$. So we have that $W(\cdot)$ is outer semicontinuous at $\hat{w} = (0, 0, \hat{\mathbf{x}})$.

Next, let $\mathbf{y}^* \in W(\hat{w})$, with $\hat{w} = (0, 0, \hat{\mathbf{x}})$. It follows from Assumption 5.2.1(ii) and (5.3.6b) that $\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}^*) \leq 0$ and $\mathbf{y}^* \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$. Hence, $W(\hat{w}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}})$.

Next let $\epsilon > 0$. Then, by outer semicontinuity of $W(\cdot)$ at $\hat{w} = (0, 0, \hat{\mathbf{x}})$, there exists $\rho > 0$ such that $W(w) \subset W(\hat{w}) + \mathbb{B}_\epsilon$ for all $w \in [0, \infty) \times T \times \mathbb{R}^n$ with $\|w - \hat{w}\|_\infty \leq \rho$. Hence, for all $\pi \geq 1/\rho$, $N \geq 1/\rho$, $N \in \mathbb{N}$, and $\mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \rho)$, $\hat{\mathbf{Y}}_{\pi,N}(\mathbf{x}) \subset \hat{\mathbf{Y}}(\hat{\mathbf{x}}) + \mathbb{B}_\epsilon$.

(ii) Using the same arguments as in (i), we obtain (ii). This completes the proof. □

The approximating, smooth functions in (5.3.1b) have the property, see Bertsekas (1982, Li (1997), that

$$0 \leq \omega_\pi(\mathbf{x}, \mathbf{y}) - \omega_{\pi,p}(\mathbf{x}, \mathbf{y}) \leq \frac{1}{p} \ln r \tag{5.3.7}$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbf{Y}$ and $\pi > 0$. Hence, for all $\mathbf{x} \in \mathbb{R}^n$ and $\pi > 0$

$$\begin{aligned} \psi_\pi(\mathbf{x}) &= \max_{\mathbf{y}\in\mathbf{Y}} \omega_\pi(\mathbf{x}, \mathbf{y}) \\ &\leq \max_{\mathbf{y}\in\mathbf{Y}} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}) + \frac{1}{p} \ln r \\ &= \psi_{\pi,p}(\mathbf{x}) + \frac{1}{p} \ln r, \end{aligned} \tag{5.3.8a}$$

with

$$\psi_{\pi,p}(\mathbf{x}) \stackrel{\Delta}{=} \max_{\mathbf{y}\in\mathbf{Y}} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}). \tag{5.3.8b}$$

Next, it also follows from (5.3.7) that for all $\mathbf{x} \in \mathbb{R}^n$ and $\pi > 0$

$$\begin{aligned} \psi(\mathbf{x}) &= \max_{\mathbf{y}\in\mathbf{Y}} \omega_\pi(\mathbf{x},\mathbf{y}) \\ &\geq \max_{\mathbf{y}\in\mathbf{Y}} \omega_{\pi,p}(\mathbf{x},\mathbf{y}) \\ &= \psi_{\pi,p}(\mathbf{x}). \end{aligned} \tag{5.3.8c}$$

By the same arguments leading to (5.3.8a) and (5.3.8c), we have that

$$0 \leq \psi_{\pi,N}(\mathbf{x}) - \psi_{\pi,N,p}(\mathbf{x}) \leq \frac{1}{p} \ln r, \tag{5.3.9}$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\pi > 0$ and $N \in \mathbb{N}$.

Lemma 5.3.5. *Suppose Assumptions 5.2.1 and 5.3.1(ii) hold. Then for every bounded set $\mathbf{S} \subset \mathbb{R}^n$ and $\pi > 0$, there exists a constant $K < \infty$ such that for all $N \geq N_0$, with N_0 as in Assumption 5.3.1(ii), $p > 0$ and $\mathbf{x} \in \mathbf{S}$*

$$0 \leq \psi_\pi(\mathbf{x}) - \psi_{\pi,N}(\mathbf{x}) \leq K\Delta(N) \tag{5.3.10a}$$

$$0 \leq \psi_{\pi,p}(\mathbf{x}) - \psi_{\pi,N,p}(\mathbf{x}) \leq K\Delta(N). \tag{5.3.10b}$$

Proof. Since $\phi_{k,\pi}(\cdot,\cdot), k \in \mathbf{r}$, are continuously differentiable, they are Lipschitz continuous on bounded sets. Hence, $\omega_\pi(\cdot,\cdot)$ is also Lipschitz continuous on bounded sets. First, because $\mathbf{Y}_N \subset \mathbf{Y}$, we always have that $\psi_{\pi,N}(\mathbf{x}) \leq \psi_\pi(\mathbf{x})$. Second, let $\mathbf{S} \subset \mathbb{R}^n$ be a bounded set, and let $L < \infty$ be a Lipschitz constant for $\omega_\pi(\cdot,\cdot)$ on $\mathbf{S}$. For any $\mathbf{x} \in \mathbf{S}$, there must exists a $\mathbf{y}_\mathbf{x} \in \mathbf{Y}$ such that $\psi_\pi(\mathbf{x}) = \omega_\pi(\mathbf{x},\mathbf{y}_\mathbf{x})$. By Assumption 5.3.1(ii), there exists $\mathbf{y}'_\mathbf{x} \in \mathbf{Y}_N$ such that $\|\mathbf{y}'_\mathbf{x} - \mathbf{y}_\mathbf{x}\| \leq C\Delta(N)$. Hence,

$$\psi_{\pi,N}(\mathbf{x}) \geq \omega_\pi(\mathbf{x},\mathbf{y}'_\mathbf{x}) \geq \omega_\pi(\mathbf{x},\mathbf{y}_\mathbf{x}) - LC\Delta(N) = \psi_\pi(\mathbf{x}) - LC\Delta(N). \tag{5.3.10c}$$

Hence (5.3.10a) holds with $K = LC$.

Next, $\omega_{\pi,p}(\cdot,\cdot)$, defined in (5.3.1b), has gradient with respect to $\mathbf{y}$

$$\nabla_{\mathbf{y}}\omega_{\pi,p}(\mathbf{x},\mathbf{y}) \triangleq \sum_{k=1}^{r} \mu_{k,\pi,p}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}), \tag{5.3.10d}$$

where, for any $k^* \in \mathbf{r}$,

$$\mu_{k^*,\pi,p}(\mathbf{x},\mathbf{y}) \triangleq \frac{\exp[-p\phi_{k^*,\pi}(\mathbf{x},\mathbf{y})]}{\sum_{k\in\mathbf{r}}\exp[-p\phi_{k,\pi}(\mathbf{x},\mathbf{y})]}. \tag{5.3.10e}$$

Hence, by the Mean Value Theorem and (5.3.10d,e), we have that for all $p > 0$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$

$$|\omega_{\pi,p}(\mathbf{x},\mathbf{y}') - \omega_{\pi,p}(\mathbf{x},\mathbf{y})| \leq r\sum_{k=1}^{r} \|\nabla_{\mathbf{y}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}+s(\mathbf{y}'-\mathbf{y}))\|\ \|\mathbf{y}'-\mathbf{y}\|, \tag{5.3.10f}$$

for some $s \in [0,1]$. Hence, $\omega_{\pi,p}(\cdot,\cdot)$ is Lipschitz continuous on bounded sets with a Lipschitz constant independent of p. The result now follows by the same arguments as for (5.3.10a). □

Lemma 5.3.6. *Suppose that Assumptions 5.2.1 and 5.3.1(i) hold. Then for every bounded set $\mathbf{S} \subset \mathbb{R}^n$ and $\pi > 0$, there exists an $L < \infty$ such that*

$$\left\langle \mathbf{v}, \frac{\partial^2\omega_{\pi,p}(\mathbf{x},\mathbf{y})}{\partial\mathbf{x}^2}\mathbf{v} \right\rangle \leq pL\|\mathbf{v}\|^2, \tag{5.3.11a}$$

for all $\mathbf{y} \in \mathbf{Y}$, $\mathbf{x} \in \mathbf{S}, \mathbf{v} \in \mathbb{R}^n$ and $p \geq 1$. □

Proof. Let $\pi > 0$ be arbitrary. By Assumption 5.3.1(i), $\omega_{\pi,p}(\cdot,\mathbf{y})$, $\mathbf{y} \in \mathbf{Y}$, is twice differentiable with gradient

$$\nabla_{\mathbf{x}}\omega_{\pi,p}(\mathbf{x},\mathbf{y}) \triangleq \sum_{k=1}^{r} \mu_{k,\pi,p}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}), \tag{5.3.11b}$$

where $\mu_{k,\pi,p}(\mathbf{x},\mathbf{y})$ is given by (5.3.10e), and Hessian matrix

$$\frac{\partial^2\omega_{\pi,p}(\mathbf{x},\mathbf{y})}{\partial\mathbf{x}^2} \triangleq \sum_{k=1}^{r}\left[\nabla_{\mathbf{x}}\mu_{k,\pi,p}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y})^T + \mu_{k,\pi,p}(\mathbf{x},\mathbf{y})\frac{\partial^2\phi_{k,\pi}(\mathbf{x},\mathbf{y})}{\partial\mathbf{x}^2}\right], \tag{5.3.11c}$$

where, for any $k^* \in \mathbf{r}$,

$$\nabla_{\mathbf{x}}\mu_{k^*,\pi,p}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} p\mu_{k^*,\pi,p}(\mathbf{x},\mathbf{y})\sum_{k=1}^{r}\mu_{k,\pi,p}(\mathbf{x},\mathbf{y})(\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}) - \nabla\phi_{k^*,\pi}(\mathbf{x},\mathbf{y})). \tag{5.3.11d}$$

Let $\mathbf{S} \subset \mathbb{R}^n$ be bounded. Then by continuity, there exists a $K < \infty$ such that $\|\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y})\| \leq K$ and $\langle \mathbf{v}, \partial^2\phi_{k,\pi}(\mathbf{x},\mathbf{y})/\partial\mathbf{x}^2\mathbf{v}\rangle \leq K\|\mathbf{v}\|^2$ for all $\mathbf{x} \in \mathbf{S}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbf{Y}$ and $k \in \mathbf{r}$. Then, for all $\mathbf{x} \in \mathbf{S}$,

$$\|\nabla_{\mathbf{x}}\mu_{k^*,\pi,p}(\mathbf{x},\mathbf{y})\| \leq p\sum_{k=1}^{r}\|\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\phi_{k^*,\pi}(\mathbf{x},\mathbf{y})\| \leq 2prK. \tag{5.3.11e}$$

Hence, there exists $K_1 < \infty$ such that

$$\langle\ \mathbf{v}, \sum_{k\in\mathbf{r}}\nabla_{\mathbf{x}}\mu_{k,\pi,p}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\phi_{k,\pi}(\mathbf{x},\mathbf{y})^T\mathbf{v}\ \rangle \leq pK_1\|\mathbf{v}\|^2, \tag{5.3.11f}$$

for all $\mathbf{x} \in \mathbf{S}$, $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbf{Y}$. By inspection, $0 < \mu_{k,\pi,p}(\mathbf{x},\mathbf{y}) < 1$, for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbf{Y}$, $k \in \mathbf{r}$ and $p > 0$. Hence, for $p \geq 1$, $\mathbf{x} \in \mathbf{S}$, $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{v} \in \mathbb{R}^n$

$$\begin{aligned}\left\langle\ \mathbf{v}, \frac{\partial^2\omega_{\pi,p}(\mathbf{x},\mathbf{y})}{\partial\mathbf{x}^2}\mathbf{v}\ \right\rangle &\leq pK_1\|\mathbf{v}\|^2 + rK\|\mathbf{v}\|^2 \\ &\leq p(K_1 + rK)\|\mathbf{v}\|^2.\end{aligned} \tag{5.3.11g}$$

Hence, $L = K_1 + rK$. This completes the proof. □

Lemma 5.3.7. *Suppose that Assumptions 5.2.1 and 5.3.1(ii) hold, and that the sequences* $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$, $\{N_i\}_{i=0}^{\infty} \subset \mathbb{N}$, *and* $\{p_i\}_{i=0}^{\infty} \subset (0,\infty)$ *are such that* $\mathbf{x}_i \to \hat{\mathbf{x}}$, $p_i \to \infty$ *and* $N_i \to \infty$, *as* $i \to \infty$. *Then for any* $\pi > 0$, $\limsup \theta_{\pi,N_i,p_i}(\mathbf{x}_i) \leq \theta_\pi(\hat{\mathbf{x}})$. □

Proof. For every i, let

$$\bar{\xi}_i \stackrel{\triangle}{=} (\xi_{0,i}, \xi_i) \in \underset{\bar{\xi}\in\bar{G}\psi_{\pi,N_i,p_i}(\mathbf{x}_i)}{\arg\min}\ \xi_0 + \frac{1}{2}\|\xi\|^2. \tag{5.3.12a}$$

Then there exist multipliers $\nu_{i,j} \geq 0$, a set $\mathbf{J}_i \subset \mathbb{N}$, and $\mathbf{y}_{i,j} \in \mathbf{Y}_{N_i}$, such that $\sum_{j\in\mathbf{J}_i}\nu_{i,j} = 1$,

$$\xi_{0,i} = \sum_{j\in\mathbf{J}_i}\nu_{i,j}[\psi_{\pi,N_i,p_i}(\mathbf{x}_i) - \omega_{\pi,p_i}(\mathbf{x}_i,\mathbf{y}_{i,j})], \tag{5.3.12b}$$

and

$$\xi_i = \sum_{j \in \mathbf{J}_i} \nu_{i,j} \nabla_{\mathbf{x}} \omega_{\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}). \tag{5.3.12c}$$

In view of (5.3.10e), we have that all $\mu_{k,\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}) \geq 0$, and $\sum_{k \in \mathbf{r}} \mu_{k,\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}) = 1$. Hence, we obtain the following expression for $\bar{\xi}_i$

$$\bar{\xi}_i = \sum_{j \in \mathbf{J}_i} \nu_{i,j} \sum_{k \in \mathbf{r}} \mu_{k,\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}) \begin{pmatrix} \psi_{\pi,N_i,p_i}(\mathbf{x}_i) - \omega_{\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}) \\ \nabla_{\mathbf{x}} \phi_{k,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) \end{pmatrix}. \tag{5.3.12d}$$

Now let

$$\bar{\zeta} \stackrel{\triangle}{=} (\zeta_{-1}, \zeta_0, \zeta) = \sum_{j \in \mathbf{J}_i} \nu_{i,j} \sum_{k \in \mathbf{r}} \mu_{k,\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j}) \begin{pmatrix} \phi_{k,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) \\ \psi_{\pi}(\mathbf{x}_i) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) \\ \nabla_{\mathbf{x}} \phi_{k,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) \end{pmatrix}. \tag{5.3.12e}$$

Then, by inspection $\bar{\zeta} \in \bar{G}\psi_{\pi}(\mathbf{x}_p)$, and hence

$$\begin{aligned} -\theta_{\pi}(\mathbf{x}_i) &\leq \zeta_{-1} + \zeta_0 + \tfrac{1}{2}\|\zeta\|^2 \\ &= \sum_{j \in \mathbf{J}_i} \nu_{i,j} \{[\psi_{\pi}(\mathbf{x}_i) - \psi_{\pi,N_i,p_i}(\mathbf{x}_i)] - [\omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j})]\} \\ &\quad + \zeta_{-1} - \theta_{\pi,N_i,p_i}(\mathbf{x}_i). \end{aligned} \tag{5.3.12f}$$

Now, for any $j \in \mathbf{J}_i$ and $k* \in \mathbf{r}$,

$$\begin{aligned} & \mu_{k*\pi,p_i}(\mathbf{x}_i, \mathbf{y}_{i,j})[\phi_{k*,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j})] \\ = & \frac{\phi_{k*,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j})}{\sum_{k \in \mathbf{r}} \exp\{p_i[(\phi_{k*,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \phi_{k,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j})]\}} \\ \leq & \frac{\phi_{k*,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j})}{\exp\{p_i[\phi_{k*,\pi}(\mathbf{x}_i, \mathbf{y}_{i,j}) - \omega_{\pi}(\mathbf{x}_i, \mathbf{y}_{i,j})]\}} \\ \leq & \frac{1}{p_i[\exp(1)]}. \end{aligned} \tag{5.3.12g}$$

It now follows from (5.3.7), (5.3.9), (5.3.10a), (5.3.12g) and (5.3.12f) that

$$-\theta_{\pi}(\mathbf{x}_i) \leq -\theta_{\pi,N_i,p_i}(\mathbf{x}_i) + \frac{r}{p_i[\exp(1)]} + \frac{1}{p_i} \ln r + K\Delta(N_i), \tag{5.3.12h}$$

with $K < \infty$ as in (5.3.10a). By continuity of $\theta_\pi(\cdot)$ and (5.3.12h), the result follows directly. □

5.4 Algorithm for P

In view of Theorem 5.2.5, P can be solved by solving $\mathbf{P}_\pi$ for a sufficiency large penalty $\pi > 0$. However, a priori the size of the penalty is unknown. Hence, in the following algorithm we use the test function defined in (5.2.13h) to control the penalty π.

As shown in Section 3, $\psi_\pi(\cdot)$ can be approximated by $\psi_{\pi,N,p}(\cdot)$. Hence, $\mathbf{P}_\pi$ can be approximately solved by solving $\mathbf{P}_{\pi,N,p}$. The following algorithm adaptively increases the precision parameters N and p, based on a series of tests, such that for all $\mathbf{x} \in \mathbb{R}^n$, $\psi_{\pi,N,p}(\mathbf{x})$ converges to $\psi_\pi(\mathbf{x})$. For given π, N and p, the algorithm calls the Pironneau-Polak-Pshenichnyi min-max algorithm (see Polak (1997) and Pshenichnyi and Danilin (1975)) as a subroutine to perform one iteration on $\mathbf{P}_{\pi,N,p}$.

In the algorithm below, let $\Delta : \mathbb{N} \to \mathbb{R}$, $N_0 \in \mathbb{N}$ be as in Assumption 5.3.1, and let $\mathbf{Y}_N \subset \mathbf{Y}, N \geq N_0$ be the finite-cardinality subsets of $\mathbf{Y}$ in the definition of $\mathbf{P}_{\pi,N,p}$.

Algorithm 5.4.1.

Parameters. $\alpha, \beta, \mu, \rho \in (0,1)$; $\tau_1 \geq \ln(r_1 + 1)$; $\sigma_{-1}, \tau_2 > 0$; $\sigma, \kappa > 1$; $\gamma >> 1$; $\pi_{-1} > 0$; $p_0 \geq 1$; $\hat{p} \geq p_0$, $\hat{p} >> 1$; $\zeta \in \mathbb{N}$, $\zeta \geq 2$; $N_0 \in \mathbb{N}$.

Data. $\mathbf{x}_0 \in \mathbb{R}^n$.

Step 0. Set $i = 0$, $j = 0$, $k = 0$, and $\delta = 1$.

Step 1. Compute $\mathbf{y}_i \in \hat{\mathbf{Y}}_{\pi_{i-1},N_i}(\mathbf{x}_i)$, and the smallest eigenvalue $\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i)$ of the matrix $[\mathbf{A}(\mathbf{x}_i, \mathbf{y}_i)\mathbf{A}(\mathbf{x}_i, \mathbf{y}_i)^T + \mathbf{B}(\mathbf{x}_i, \mathbf{y}_i)]$, see (5.2.13a,d), and set $\pi = \pi_{i-1}$.

Step 2. If $\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) \geq \sigma_{i-1}$, set $\sigma_i = \sigma_{i-1}$, and go to **Step 3**.

Else, set $\sigma_i = \mu\sigma_{i-1}$, and go to **Step 10**.

Step 3. If $t_\pi(\mathbf{x}_i, \mathbf{y}_i) \leq 0$, see (5.2.13h), set $\pi_i = \pi$, and go to **Step 4**.

Else, go to **Step 10**.

Step 4. Compute $\theta_{\pi_i,N_i,p_i}(\mathbf{x}_i)$ and the augmented search direction, see (5.3.2b),

$$(\mathbf{h}_{0,\pi_i,N_i,p_i}(\mathbf{x}_i), \mathbf{h}_{\pi_i,N_i,p_i}(\mathbf{x}_i)) = -\underset{\xi\in\bar{G}\psi_{\pi_i,N_i,p_i}(\mathbf{x}_i)}{\arg\min}\ \xi_0 + \frac{1}{2}\|\xi\|^2. \tag{5.4.1a}$$

Step 5. Compute $\mathbf{x}_{i+1} = \mathbf{x}_i + \lambda_{\pi_i,N_i,p_i}(\mathbf{x}_i)\mathbf{h}_{\pi_i,N_i,p_i}(\mathbf{x}_i)$, where the Armijo step-size

$$\lambda_{\pi_i,N_i,p_i}(\mathbf{x}_i) =$$
$$\max_{s\in\mathbb{N}}\{\beta^s \mid \psi_{\pi_i,N_i,p_i}(\mathbf{x}_i + \beta^s\mathbf{h}_{\pi_i,N_i,p_i}(\mathbf{x}_i)) - \psi_{\pi_i,N_i,p_i}(\mathbf{x}_i) \le \alpha\beta^s\theta_{\pi_i,N_i,p_i}(\mathbf{x}_i)\}. \tag{5.4.1b}$$

Step 6. If

$$\Delta\psi_i \stackrel{\Delta}{=} \psi_{\pi_i,N_i,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N_i,p_i}(\mathbf{x}_i) \ge -\frac{\tau_1}{p_i} - \tau_2\Delta(N_i), \tag{5.4.1c}$$

go to **Step 7**.

Else, set $N_{i+1} = N_i$ and $p_{i+1} = p_i$, replace i by $i+1$, and go to **Step 1**.

Step 7. Set $N_{i+1} \in \mathbb{N}$ equal to the smallest integer satisfying

$$\Delta(N_{i+1}) \le \min\left\{\max\left\{\frac{1-\rho}{\tau_2}|\Delta\psi_i|,\ \Delta(\zeta N_i)\right\},\ \frac{\gamma-1}{\gamma}\Delta(N_i)\right\}. \tag{5.4.1d}$$

Step 8. If (initial stage)

$$\max\left\{\frac{\tau_1}{\rho|\Delta\psi_i|},\ \frac{\gamma p_i}{\gamma-1}\right\} \le \hat{p} \ \ and \ \ \delta = 1, \tag{5.4.1e}$$

set $p_{i+1} = \max\{\tau_1/(\rho|\Delta\psi_i|)\ ,\ \gamma p_i/(\gamma-1)\}$, replace k by $k+1$, i by $i+1$, and go to **Step 1**.

Elseif (switch stage)

$$\max\left\{\frac{\tau_1}{\rho|\Delta\psi_i|},\ \frac{\gamma p_i}{\gamma-1}\right\} > \hat{p} \ \ and \ \ \delta = 1, \tag{5.4.1f}$$

set $\delta = \max\{2\ ,\ \gamma\hat{p}/((\gamma-1)(k+1))\}$, $p_{i+1} = \delta(k+2)$, replace k by $k+1$, i by $i+1$, and go to **Step 1**.

Else (final stage) go to **Step 9**.

Step 9. **If** $\delta(i+2) < \gamma p_i/(\gamma-1)$, set $p_{i+1} = p_i$, replace i by $i+1$, and go to **Step 1**.

Else find the smallest $k^* \in \mathbb{N}$ such that $k \leq k^* \leq i$ and $\delta(k^*+2) \geq \gamma p_i/(\gamma-1)$, and set $p_{i+1} = \delta(k^*+2)$, replace k by k^*+1, i by $i+1$, and go to **Step 1**.

Step 10. Set $\mathrm{x}_j^* = \mathrm{x}_i$, $\pi = \kappa^{j+1}\pi_{-1}$, replace j by $j+1$, and go to **Step 3**. □

Lemma 5.4.2. *Suppose that Algorithm 5.4.1 has generated a sequence* $\{p_i\}_{i=0}^{\infty}$. *Then, the following hold:*

(i) *If the test in (5.4.1c) is satisfied an infinite number of times, then there exists an* $i^* \in \mathbb{N}$ *such that* p_{i^*+1} *is set in the "switch stage" of Step 8.*

(ii) *If there exists an* $i^* \in \mathbb{N}$ *such that* p_{i^*+1} *is set in the "switch stage" of Step 8, then for all* $i < i^*$ *such that (5.4.1c) is satisfied,* p_{i+1} *is set in the "initial stage," and for all* $i > i^*$ *such that (5.4.1c) is satisfied,* p_{i+1} *is set in the "final stage."*

Proof. **(ii)** Suppose that there exists an $i^* \in \mathbb{N}$ such that p_{i^*+1} is set in the "switch stage." Then, Algorithm 5.4.1 sets $\delta = \max\{2, \gamma\hat{p}/((\gamma-1)(k+1))\} \geq 2 > 1$ in iteration i^*. Hence, (5.4.1e,f) cannot hold for $i > i^*$, and p_{i+1} must be set in the "final stage" of Step 8 for all $i > i^*$ such that (5.4.1c) holds. Hence, p_{i+1} must be set in the "initial stage" of Step 8 for all $i < i^*$ such that (5.4.1c) holds.

(i) Suppose for the sake of a contradiction that for all $i \in \mathbb{N}$, p_{i+1} is not set in the "switch stage" of Step 8. Then, $\delta = 1$ for all $i \in \mathbb{N}$ because $\delta = 1$ for $i = 0$, and δ is only changed in the "switch stage" of Step 8. Hence, because $\delta = 1$ for all $i \in \mathbb{N}$ and the hypothesis that p_{i+1} is not set in the "switch stage", p_{i+1} is set by the "initial stage" of Step 8 for all $i \in \mathbb{N}$ such that (5.4.1c) is satisfied. Hence, $p_{i+1} \geq \gamma p_i/(\gamma-1)$ for all $i \in \mathbb{N}$ such that (5.4.1c) is satisfied. Since $p_{i+1} \geq \gamma p_i/(\gamma-1)$ an infinite number of times, there must exists an $i^{**} \in \mathbb{N}$ such

that $\max\{\tau_1/(\rho|\Delta\psi_{i^{**}}|),\ \gamma p_{i^{**}}/(\gamma-1)\} > \hat{p}$, see the "initial stage" of Step 8. Hence (5.4.1f) is satisfied for $i = i^{**}$, but (5.4.1e) is not. This is a contradiction, which completes the proof. □

The mechanisms in Algorithm 5.4.1 can by described as follows. Step 2 ensures that the linear independence property of Assumption 5.2.6 is eventually satisfied at $(\mathbf{x}_i, \mathbf{y}_i)$, and Step 3 ensures that the test function remains non-positive. In view of Lemma 5.2.7(iii), we see that Steps 2 and 3 increase the penalty π to a sufficiently large value that ensures local equivalence between $\mathbf{P}$ and $\mathbf{P}_\pi$.

Suppose that π^* is sufficiently large, i.e., there exists an $i^* \in \mathbb{N}$ such that $\pi_i = \pi^*$ for all $i > i^*$. Then, Algorithm 5.4.1 solves the sequence of approximating problem $\{\mathbf{P}_{\pi^*,N_i,p_i}\}_{i=i*}^{\infty}$, associate with a sequence of monotonically increasing precision parameters N_i, p_i that diverge to infinity. At a given precision level, N', p', say, Algorithm 5.4.1 computes iterates that approach a stationary point of the approximating problem $\mathbf{P}_{\pi^*,N',p'}$. When the current iterate is sufficiently close to a stationary point for $\mathbf{P}_{\pi^*,N',p'}$, as determined by the test in (5.4.1c), the precision level is increased from N', p' to N'', p'', say. Algorithm 5.4.1 then continues by computing iterates that are approaching a stationary point of $\mathbf{P}_{\pi^*,N'',p''}$ until the test in (5.4.1c) again determines that the precision level has to be increased. The last iteration of the previous precision level is used as a "warm start" for calculations on the next precision level.

It becomes gradually harder and harder to satisfy (5.4.1c) as $N_i, p_i \to \infty$. Hence, as the precision level is increased, the iterates generated by Algorithm 5.4.1 gradually get closer and closer to a stationary point of the current approximating problem before the precision level is increased. Thus, Algorithm 5.4.1 computes approximate solutions to a sequence of approximating problems $\{\mathbf{P}_{\pi^*,N_i,p_i}\}_{i=i^*}^{\infty}$ with higher and higher precision as the number of iterations increases.

The sequences of precision parameters $\{N_i\}_{i=0}^{\infty}$ and $\{p_i\}_{i=0}^{\infty}$ are not determined a priori but is constructed by Algorithm 5.4.1. When (5.4.1c) is satisfied, the precision level is increased by an amount determined by Steps 7, 8 and 9.

In the "early" iterations, i.e., before the test in (5.4.1e) fails, the smoothing pre-

cision parameter is increased by an amount related to the value of the cost-decrease $\Delta\psi_i$. When $|\Delta\psi_i|$ is large, p_{i+1} tends to be only marginally larger than p_i, with a minimum increase of $p_i/(\gamma - 1)$. On the other hand, when $|\Delta\psi_i|$ is small, p_{i+1} tends to be augmented by a considerable amount.

When the test in (5.4.1e) fails, δ is set to be larger than 1 in the "switch stage" of Step 8. Hence for all subsequent iterations, the increase of the precision parameter will be determined by the "final stage", i.e, Step 8, of Algorithm 5.4.1. In the "final stage", the precision parameter is augmented by a multiple of δ whenever $p_{i+1} > p_i$.

The increase of the precision parameters N_i, p_i are motivated by the following considerations: **(i)** Suppose that the algorithm parameter $\tau_1 = \ln(r_1 + 1)$, where $r_1 \in \mathbb{N}$ is as in (5.2.3b), $\tau_2 = K$, where $K < \infty$ is as in Lemma 5.3.5, $\psi(\mathbf{x}_{i+1}) = \psi_{\pi^*}(\mathbf{x}_{i+1})$, and $\psi(\mathbf{x}_i) = \psi_{\pi^*}(\mathbf{x}_i)$, then we have by (5.3.8a), (5.3.8c), (5.3.9) and Lemma 5.3.5 that

$$\psi(\mathbf{x}_{i+1}) - \psi(\mathbf{x}_i) \leq \Delta\psi_i + \frac{\tau_1}{p_i} + \tau_2\Delta(N_i). \tag{5.4.2}$$

Hence, $\psi(\mathbf{x}_{i+1}) - \psi(\mathbf{x}_i) < 0$ whenever the test in (5.4.1c) fails, i.e., the precision is not increased as long as the new iterate guarantees a decrease in the cost function $\psi(\cdot)$. The constant K in Lemma 5.3.5 may seldom be known. In absence of any information about K, we recommend setting $\tau_2 = 1$. Note that larger values for τ_2 will drive N to infinity faster. **(ii)** When (5.4.1c) is satisfied, we can no longer guarantee that $\psi(\mathbf{x}_{i+1}) - \psi(\mathbf{x}_i) < 0$, and we set N_{i+1} and p_{i+1} to be larger than N_i and p_i, which, hopefully, will ensure that $\psi(\mathbf{x}_{i+2}) - \psi(\mathbf{x}_{i+1}) < 0$ will hold. **(iii)** If the current iterate is very close to a stationary point of the approximating problem, $|\Delta\psi_i|$ tends to become extremely small. Hence, the factor ζ, see (5.4.1d), and the fixed increase of p_i in the "final stage" is introduced to prevent N_i, p_i to become very large prematurely. **(iv)** Lemma 5.4.3 below must hold.

Let $t > 0$ be the desired tolerance on the solution. Then every $p_i >> \ln(r_1 + 1)/t$ is associated with an error, see (5.3.7), $\ln(r_1 + 1)/p_i << t$. Hence we recommend that the algorithm parameter $\hat{p}$, used to decide when to switch from the "initial stage" to the "final stage", be set equal to $\ln(r_1 + 1)/t$. Furthermore, we recommend to set γ equal to a large number, e.g., 10^5, to avoid any practical influence on the

determination of p_{i+1}.

The parameter $\rho \in (0,1)$ controls how the error associated with the discretization of $\mathbf{Y}$ compares with the error associated with the smoothing of $\omega_\pi(\cdot,\cdot)$. When ρ is close to unity, the error associated with the discretization tends to be "small" and the error associated with smoothing tends to be "large". When ρ is close to zero, the situation is reversed. Since a fine discretization implies a high computational cost, it can be efficient to bias the approximation error towards the smoothing error by selecting ρ close to 0.

Algorithm 5.4.1 is quite insensitive to the selection of the parameters $\sigma_{-1} > 0$ and $\mu \in (0,1)$ used in Step 2. However, note that larger values of σ_{-1} and μ will cause the penalty π to increase faster. We recommend setting $\sigma_{-1} = 10^{-5}$ and $\mu = 0.5$.

Lemma 5.4.3. *Suppose that Assumption 5.2.1 holds and that the sequences* $\{\mathbf{x}_i\}_{i=0}^{\infty}$, $\{N_i\}_{i=0}^{\infty}$ *and* $\{p_i\}_{i=0}^{\infty}$ *are generated by Algorithm 5.4.1. Then the following hold:*

(i) *The sequences* $\{N_i\}_{i=0}^{\infty}$ *and* $\{p_i\}_{i=0}^{\infty}$ *are monotonically increasing, and, if* $p_{i+1} > p_i$, *then* $p_{i+1} \geq \gamma p_i/(\gamma-1)$, *and, if* $N_{i+1} > N_i$, *then* $\Delta(N_{i+1}) \leq (\gamma-1)\Delta(N_i)/\gamma$, *with* γ *as in Algorithm 5.4.1.*

(ii) *If* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *has an accumulation point, then* $N_i \to \infty$, $p_i \to \infty$, *and* $\sum_{i=0}^{\infty} 1/p_i = \infty$.

Proof. **(i)** If the test in (5.4.1c) fails, then $N_{i+1} = N_i$ and $p_{i+1} = p_i$. If the test in (5.4.1c) is satisfied, then, according to Step 7 of Algorithm 5.4.1 (see (5.4.1d)), $\Delta(N_{i+1}) \leq (\gamma-1)\Delta(N_i)/\gamma$. Next, consider the construction of $\{p_i\}_{i=0}^{\infty}$. If the test in (5.4.1c) is satisfied, then we have three cases corresponding to the "initial," "switch" and "final" stages of Step 8.

Case I: Suppose that p_{i+1} is defined as in the "initial stage" of Step 8 in Algorithm 5.4.1. Then, $p_{i+1} \geq \gamma p_i/(\gamma-1)$.

Case II: Suppose that p_{i-1} is defined as in the "switch stage" of Step 8 in Algorithm 5.4.1. Then,

$$\begin{aligned} p_{i+1} &= \max\left\{2, \frac{\gamma}{\gamma-1}\frac{\hat{p}}{k+1}\right\}(k+2) \\ &\geq \frac{\gamma}{\gamma-1}\hat{p}. \end{aligned} \tag{5.4.3a}$$

If $i > 0$, then, by Lemma 5.4.2(ii), p_i was constructed according to the "initial stage" of Step 8. Hence, it follows from the definition of p_i and (5.4.1e) that $p_i \leq \hat{p}$. Hence, by (5.4.3a) we have that $p_{i+1} \geq \gamma p_i/(\gamma-1)$. If $i = 0$, then $p_{i+1} \geq \gamma p_i/(\gamma-1)$ because $\hat{p} \geq p_0$.

Case III: Suppose that p_{i+1} is defined as in the "final stage" of Step 8, see Step 9. Then, $p_{i+1} \geq \gamma p_i/(\gamma-1)$ whenever $p_{i+1} > p_i$. Hence, (i) holds.

(ii) Suppose that Algorithm 5.4.1 has generated the sequence $\{\mathbf{x}_i\}_{i=0}^{\infty}$ with accumulation point $\hat{\mathbf{x}}$, and that at least one of the sequences $\{N_i\}_{i=0}^{\infty}$, $\{p_i\}_{i=0}^{\infty}$ are bounded from above. Now, we have three cases.

Case I: Suppose that both $\{N_i\}_{i=0}^{\infty}$ and $\{p_i\}_{i=0}^{\infty}$ are bounded. Then the test in (5.4.1c) can only be satisfied a finite number of times, because otherwise (5.4.1d) would have caused $\{N_i\}_{i=0}^{\infty}$ to diverge to infinity. Hence, there must exist an $i^* \in \mathbb{N}$, an $N^* < \infty$ and a $p^* < \infty$ such that for all $i > i^*$, $N_i = N^*$, $p_i = p^*$, and

$$\psi_{\pi_i,N^*,p^*}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N^*,p^*}(\mathbf{x}_i) < -\frac{\tau_1}{p^*} - \tau_2\Delta(N^*). \tag{5.4.3b}$$

By inspection, $\psi_{\pi'',N}(\mathbf{x}) - \psi_{\pi',N}(\mathbf{x}) \leq 0$ for all $\pi'' \geq \pi'$, $N \in \mathbb{N}$, and $\mathbf{x} \in \mathbb{R}^n$. Hence, by (5.3.9) and (5.4.3b), we have that for all $i > i^*$

$$\begin{aligned} & \psi_{\pi_{i+1},N^*}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N^*}(\mathbf{x}_i) \\ = \;& \psi_{\pi_{i+1},N^*}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N^*}(\mathbf{x}_{i+1}) + \psi_{\pi_i,N^*}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N^*}(\mathbf{x}_i) \\ \leq \;& 0 + \psi_{\pi_i,N^*,p^*}(\mathbf{x}_{i+1}) - \psi_{\pi_i,N^*,p^*}(\mathbf{x}_i) + \frac{1}{p^*}\ln(r_1+1) \\ < \;& -\frac{\tau_1}{p^*} - \tau_2\Delta(N^*) + \frac{1}{p^*}\ln(r_1+1) \\ \leq \;& -\tau_2\Delta(N^*). \end{aligned} \tag{5.4.3c}$$

Thus, $\psi_{\pi_i,N^*}(\mathbf{x}_i) \to -\infty$, as $i \to \infty$. But there exists an infinite subset $K \subset \mathbb{N}$ such that $\mathbf{x}_i \to^K \hat{\mathbf{x}}$, as $i \to \infty$. If $\{\pi_i\}_{i=0}^{\infty}$ is bounded, then there exists an $i^{**} \geq i^*$ such that

$\pi_i = \pi^*$ for all $i > i^{**}$, and hence by continuity, $\psi_{\pi^*,N^*}(\mathbf{x}_i) \to^K \psi_{\pi^*,N^*}(\mathbf{x}^*)$, as $i \to \infty$. If $\pi_i \to \infty$, then we can infer from Lemma 5.3.3(ii) that $\psi_{\pi_i,N^*}(\mathbf{x}_i) \to^K \psi_{N^*}(\mathbf{x}^*)$, as $i \to \infty$. This is a contradiction.

Case II: Suppose that $\{N_i\}_{i=0}^{\infty}$ is bounded, but $\{p_i\}_{i=0}^{\infty}$ diverges to infinity. Then the test in (5.4.1c) can only be satisfied a finite number of times, because otherwise (5.4.1d) would have caused $\{N_i\}_{i=0}^{\infty}$ to diverge to infinity. Since $p_{i+1} = p_i$ whenever the test in (5.4.1c) fails, it follows that $p_{i+1} > p_i$ only a finite number of times. Hence, $\{p_i\}_{i=0}^{\infty}$ has to be bounded, which is a contradiction.

Case III: Suppose that $\{N_i\}_{i=0}^{\infty}$ diverges to infinity, but $\{p_i\}_{i=0}^{\infty}$ is bounded from above. Then the test in (5.4.1c) must be satisfied an infinite number of times, because otherwise $\{N_i\}_{i=0}^{\infty}$ would not have diverged to infinity. Hence, Algorithm 5.4.1 enters Step 8 an infinite number of times. By Lemma 5.4.2, there exists an $i^* \in \mathbb{N}$ such that p_{i+1} is set by the "final stage" for all $i > i^*$ such that (5.4.1c) is satisfied. Since $\{p_i\}_{i=0}^{\infty}$ is bounded from above and Step 8 is entered a infinite number of times, we must have that, see Step 9,

$$\delta(i+2) < \frac{\gamma p_i}{\gamma - 1} \tag{5.4.3d}$$

for an infinite number of iterations. But since there exists an $p^* < \infty$ such that $p_i \leq p^*$ for all $i \in \mathbb{N}$, (5.4.3d) cannot be satisfied for an infinite number of iterations which is a contradiction.

Hence, $N_i \to \infty$ and $p_i \to \infty$, as $i \to \infty$. Next, we prove that $\sum_{i=0}^{\infty} 1/p_i = \infty$. Since $p_{i+1} > p_i$ only if (5.4.1c) is satisfied, and $p_i \to \infty$, as $i \to \infty$, the test in (5.4.1c) must be satisfied an infinite number of times. Hence, by Lemma 5.4.2, there exists an $i^* \in \mathbb{N}$ such that for all $i > i^*$, p_{i+1} is set by the "final stage" of Step 8 whenever (5.4.1c) is satisfied. Hence, for all $i > i^*$, $p_{i+1} = p_i$ or $p_{i+1} \leq \delta(i+2)$, see Step 9. The final result now follows from the fact that $\sum_{i=0}^{\infty} 1/i = +\infty$. □

Lemma 5.4.4. *Suppose that Assumptions 5.2.1 and 5.3.1(i) hold. For every bounded set* $\mathbf{S} \subset \mathbb{R}^n$, $\pi > 0$ *and* $\alpha, \beta \in (0,1)$, *there exists a* $K_{\mathbf{S}} < \infty$ *such that for all* $p \geq 1$,

$N \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{S}$

$$\psi_{\pi,N,p}(\mathbf{x} + \lambda_{\pi,N,p}(\mathbf{x})\mathbf{h}_{\pi,N,p}(\mathbf{x})) - \psi_{\pi,N,p}(\mathbf{x}) \leq \alpha \frac{K_{\mathbf{S}}}{p} \theta_{\pi,N,p}(\mathbf{x}), \tag{5.4.4a}$$

where $\lambda_{\pi,N,p}(\mathbf{x})$ and $\mathbf{h}_{\pi,N,p}(\mathbf{x})$ are the step size and search direction of Algorithm 5.4.1, see (5.4.1a,b).

Proof. In Section 5.2.1 in Polak (1997), we find the following equivalent form of $\theta_{\pi,N,p}(\cdot)$, see (5.3.2b),

$$\theta_{\pi,N,p}(\mathbf{x}) = \min_{\mathbf{h} \in \mathbb{R}^n} \max_{\mathbf{y} \in \mathbf{Y}_N} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}) - \psi_{\pi,N,p}(\mathbf{x}) + \langle \nabla_{\mathbf{x}} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}), \mathbf{h} \rangle + \frac{1}{2}\|\mathbf{h}\|^2. \tag{5.4.4b}$$

Let $\mathbf{S} \subset \mathbb{R}^n$ be bounded. It follows from Assumption 5.2.1 and (5.3.2b,c) that there exists a constant $M < \infty$ such that $\|\mathbf{h}_{\pi,N,p}(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in \mathbf{S}$, $N \in \mathbb{N}$ and $p > 0$. Next, let $\mathbf{S}_B \subset \mathbb{R}^n \stackrel{\Delta}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}'\| \leq M, \mathbf{x}' \in \mathbf{S}\}$ and let $L \in [1, \infty)$ be the constant corresponding to $\mathbf{S}_B$ such that (5.3.11a) holds for all $\mathbf{x} \in \mathbf{S}_B$, $\mathbf{y} \in \mathbf{Y}$, $v \in \mathbb{R}^n$ and $p \geq 1$. Then for all $\lambda \in (0, 1]$, $\mathbf{x} \in \mathbf{S}$, $N \in \mathbb{N}$ and $p \geq 1$, we have by expansion, Lemma 5.3.6 and (5.4.4b) that for some $s \in [0, 1]$

$$\begin{aligned}
& \psi_{\pi,N,p}(\mathbf{x} + \lambda \mathbf{h}_{\pi,N,p}(\mathbf{x})) - \psi_{\pi,N,p}(\mathbf{x}) \\
= \; & \max_{\mathbf{y} \in \mathbf{Y}_N} \{\omega_{\pi,p}(\mathbf{x} + \lambda \mathbf{h}_{\pi,N,p}(\mathbf{x}), \mathbf{y}) - \psi_{\pi,N,p}(\mathbf{x})\} \\
= \; & \max_{\mathbf{y} \in \mathbf{Y}_N} \{\omega_{\pi,p}(\mathbf{x}, \mathbf{y}) - \psi_{\pi,N,p}(\mathbf{x}) + \lambda \langle \nabla_{\mathbf{x}} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}), \mathbf{h}_{\pi,N,p}(\mathbf{x}) \rangle \\
& + \frac{\lambda^2}{2} \left\langle \mathbf{h}_{\pi,N,p}(\mathbf{x}), \frac{\partial^2 \omega_{\pi,p}(\mathbf{x} + s\lambda \mathbf{h}_{\pi,N,p}(\mathbf{x}), \mathbf{y})}{\partial \mathbf{x}^2} \mathbf{h}_{\pi,N,p}(\mathbf{x}) \right\rangle \Big\} \\
\leq \; & \lambda \max_{\mathbf{y} \in \mathbf{Y}_N} \{\omega_{\pi,p}(\mathbf{x}, \mathbf{y}) - \psi_{\pi,N,p}(\mathbf{x}) + \langle \nabla_{\mathbf{x}} \omega_{\pi,p}(\mathbf{x}, \mathbf{y}), \mathbf{h}_{\pi,N,p}(\mathbf{x}) \rangle + \frac{\lambda}{2} pL \|\mathbf{h}_{\pi,N,p}(\mathbf{x})\|^2\} \\
= \; & \lambda(\theta_{\pi,N,p}(\mathbf{x}) + \frac{1}{2}(\lambda pL - 1)\|\mathbf{h}_{\pi,N,p}(\mathbf{x})\|^2).
\end{aligned} \tag{5.4.4c}$$

Hence, for all $\lambda \in (0, 1/(pL)]$

$$\begin{aligned}
& \psi_{\pi,N,p}(\mathbf{x} + \lambda \mathbf{h}_{\pi,N,p}(\mathbf{x})) - \psi_{\pi,N,p}(\mathbf{x}) - \alpha\lambda\theta_{\pi,N,p}(\mathbf{x}) \\
\leq \; & \lambda(1 - \alpha)\theta_{\pi,N,p}(\mathbf{x}) \leq 0.
\end{aligned} \tag{5.4.4d}$$

Now it follows from (5.4.4d) and the step-size rule in (5.4.1b) that

$$\lambda_{\pi,N,p}(\mathbf{x}) \geq \frac{\beta}{pL} \tag{5.4.4e}$$

for all $\mathbf{x} \in \mathbf{S}$, $N \in \mathbb{N}$ and $p \geq 1$. Hence the conclusion follow with $K_{\mathbf{S}} = \beta/L$. This completes the proof. □

Theorem 5.4.5. *Suppose that Assumptions 5.2.1 and 5.3.1 hold and that Algorithm 5.4.1 has generated a bounded sequence* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *and a finite sequence* $\{\mathbf{x}_j^*\}_{j=0}^{j^*}$. *Then, there exist an infinite subset* $K \subset \mathbb{N}$ *and an* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *such that* $\mathbf{x}_i \stackrel{K}{\longrightarrow} \hat{\mathbf{x}}$ *and* $\theta_{\pi^*}(\hat{\mathbf{x}}) = 0$, *where* $\pi^* = \kappa^{j^*+1}\pi_{-1}$, *with* κ, π_{-1} *as in Algorithm 5.4.1.*

Proof. Since $\{\mathbf{x}_j^*\}_{j=0}^{j^*}$ is a finite sequence, there exists an $i^* \in \mathbb{N}$ such that $\pi_i = \pi^* \stackrel{\Delta}{=} \kappa^{j^*+1}\pi_{-1}$ for all $i > i^*$. For the sake of a contradiction, suppose that there exists an $\epsilon > 0$ such that

$$\limsup_{i \to \infty} \theta_{\pi^*,N_i,p_i}(\mathbf{x}_i) \leq -\epsilon. \tag{5.4.5a}$$

Since $\{\mathbf{x}_i\}_{i=0}^{\infty}$ is a bounded sequence, it has at least one accumulation point. Hence by Lemma 5.4.3(ii), $p_i, N_i \to \infty$, as $i \to \infty$. Next, by Lemma 5.4.4 there exist an $M < \infty$ such that

$$\psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \leq \alpha \frac{M}{p_i} \theta_{\pi^*,N_i,p_i}(\mathbf{x}_i), \tag{5.4.5b}$$

for all $i > i^*$. Now, for all $N \in \mathbb{N}$ and $p > 0$, let

$$\tilde{\psi}_{\pi^*,N,p}(\mathbf{x}) \stackrel{\Delta}{=} \psi_{\pi^*,N,p}(\mathbf{x}) + \frac{\gamma}{p} \ln r + \gamma K \Delta(N), \tag{5.4.5c}$$

where $\gamma > 1$ is as in Algorithm 5.4.1 and $K < \infty$ as in Lemma 5.3.5. Now we have three cases corresponding to whether p and N were increased or not in Step 7 and 8 of Algorithm 5.4.1:

Case I. Suppose that $p_i < p_{i+1}$ and $N_i < N_{i+1}$. Then by Lemma 5.4.3(i)

$$p_{i+1} \geq \frac{\gamma}{\gamma - 1} p_i, \tag{5.4.5d}$$

$$\Delta(N_{i+1}) \leq \frac{\gamma - 1}{\gamma}\Delta(N_i), \tag{5.4.5e}$$

and we have that for all $i > i^*$

$$\begin{aligned}
& \tilde{\psi}_{\pi^*,N_{i+1},p_{i+1}}(\mathbf{x}_{i+1}) - \tilde{\psi}_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
= \; & \psi_{\pi^*,N_{i+1},p_{i+1}}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
& + \left(\frac{\gamma}{p_{i+1}} - \frac{\gamma}{p_i}\right)\ln r + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
= \; & \psi_{\pi^*,N_{i+1},p_{i+1}}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) + \psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
& + \left(\frac{\gamma}{p_{i+1}} - \frac{\gamma}{p_i}\right)\ln r + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
\leq \; & \psi_{\pi^*,N_{i+1}}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i}(\mathbf{x}_{i+1}) + \frac{1}{p_i}\ln r + \alpha\frac{M}{p_i}\theta_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
& + \left(\frac{\gamma}{p_{i+1}} - \frac{\gamma}{p_i}\right)\ln r + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
\leq \; & \psi_{\pi^*}(\mathbf{x}_{i+1}) - \psi_{\pi^*}(\mathbf{x}_{i+1}) + \frac{1}{p_i}\ln r + K\Delta(N_i) + \alpha\frac{M}{p_i}\theta_{\pi^*,N_i,p_i}(\mathbf{x}_i) \qquad (5.4.5f) \\
& + \left(\frac{\gamma}{p_{i+1}} - \frac{\gamma}{p_i}\right)\ln r + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
\leq \; & \alpha\frac{M}{p_i}\theta_{\pi^*,N_i,p_i}(\mathbf{x}_i).
\end{aligned}$$

Case II. Suppose that $p_i = p_{i+1}$ and $N_i < N_{i+1}$. Then (5.4.5e) holds, and we have that for all $i > i^*$

$$\begin{aligned}
& \tilde{\psi}_{\pi^*,N_{i+1},p_{i+1}}(\mathbf{x}_{i+1}) - \tilde{\psi}_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
= \; & \psi_{\pi^*,N_{i+1},p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
= \; & \psi_{\pi^*,N_{i+1},p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) + \psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
& + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \qquad (5.4.5g) \\
\leq \; & \psi_{\pi^*,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,p_i}(\mathbf{x}_{i+1}) + K\Delta(N_i) + \alpha\frac{M}{p_i}\theta_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\
& + \gamma K(\Delta(N_{i+1}) - \Delta(N_i)) \\
\leq \; & \alpha\frac{M}{p_i}\theta_{\pi^*,N_i,p_i}(\mathbf{x}_i).
\end{aligned}$$

Case III. Suppose that $p_i = p_{i+1}$ and $N_i = N_{i+1}$. Then we have that for all $i > i^*$

$$\begin{aligned} & \tilde{\psi}_{\pi^*,N_{i+1},p_{i+1}}(\mathbf{x}_{i+1}) - \tilde{\psi}_{\pi^*,N_i,p_i}(\mathbf{x}_i) \\ = \; & \psi_{\pi^*,N_i,p_i}(\mathbf{x}_{i+1}) - \psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \qquad (5.4.5h) \\ \leq \; & \alpha \frac{M}{p_i} \theta_{\pi^*,N_i,p_i}(\mathbf{x}_i). \end{aligned}$$

By Lemma 5.4.3(ii), $\sum_{i=0}^{\infty} 1/p_i = +\infty$. Hence by (5.4.5a) and (5.4.5f,g,h), $\tilde{\psi}_{\pi^*,N_i,p_i}(\mathbf{x}_i) \to -\infty$, as $i \to \infty$. Then we also must have $\psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) \to -\infty$, as $i \to \infty$. Let $\mathbf{x}^*$ be an accumulation point of $\{\mathbf{x}_i\}_{i=0}^{\infty}$. Then, there exists as infinite subset $K^* \subset \mathbb{N}$ such that $\mathbf{x}_i \to^{K^*} \mathbf{x}^*$, and by (5.3.9) and (5.3.10a) $|\psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) - \psi_{\pi^*}(\mathbf{x}^*)| \leq |\psi_{\pi^*,N_i,p_i}(\mathbf{x}_i) - \psi_{\pi^*,N_i}(\mathbf{x}_i)| + |\psi_{\pi^*,N_i}(\mathbf{x}_i) - \psi_{\pi^*}(\mathbf{x}_i)| + |\psi_{\pi^*}(\mathbf{x}_i) - \psi_{\pi^*}(\mathbf{x}^*)| \to^{K^*} 0$, as $i \to \infty$, which is a contradiction. Thus,

$$\limsup_{i \to \infty} \theta_{\pi^*,N_i,p_i}(\mathbf{x}_i) = 0. \qquad (5.4.5i)$$

Hence by Lemma 5.3.7, and (5.4.5i), there have to exist an infinite subset $K \subset \mathbb{N}$ and an $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{x}_i \to^K \hat{\mathbf{x}}$ and $\theta_{\pi^*}(\hat{\mathbf{x}}) = 0$. This completes the proof. □

Lemma 5.4.6. *Suppose that Assumptions 5.2.1 and 5.2.6 hold. Then, the smallest eigenvalue* $\sigma_{\min}(\cdot,\cdot)$ *of the matrix-valued function* $[\mathbf{A}(\cdot,\cdot)\mathbf{A}(\cdot,\cdot)^T + \mathbf{B}(\cdot,\cdot)]$, *see (5.2.13a,d), is continuous, and for every compact set* $\mathbf{S} \subset \mathbb{R}^n$,

$$\min_{\mathbf{x} \in \mathbf{S}} \min_{\mathbf{y} \in \hat{\mathbf{Y}}(\mathbf{x})} \sigma_{\min}(\mathbf{x},\mathbf{y}) > 0. \qquad (5.4.6)$$

Proof. For any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, let $\mathbf{C}(\mathbf{x},\mathbf{y}) = \mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{A}(\mathbf{x},\mathbf{y})^T + \mathbf{B}(\mathbf{x},\mathbf{y})$, with the smallest eigenvalue $\sigma_{\min}(\mathbf{x},\mathbf{y})$. Since $\sigma_{\min}(\mathbf{x},\mathbf{y}) = \min_{\|\mathbf{v}\|=1}\langle \mathbf{v}, \mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{v}\rangle$ and $\mathbf{C}(\cdot,\cdot)$ is continuous, it follows from Corollary 5.4.2 in Polak (1997) that $\sigma_{\min}(\cdot,\cdot)$ is continuous.

Next let $\mathbf{S} \subset \mathbb{R}^n$ be a compact set. By Theorem 5.4.3 in Polak (1997), $\hat{\mathbf{Y}}(\cdot)$, see (5.2.4a), is outer semicontinuous and compact-valued. Hence, by Theorem 5.4.1 in Polak (1997), $\min_{\mathbf{y} \in \hat{\mathbf{Y}}(\cdot)} \sigma_{\min}(\cdot,\mathbf{y})$ is lower semicontinuous. Since the infimum of

a lower semicontinuous function over a compact set is attained, (5.4.6) follows from Lemma 5.2.7(ii) and Assumption 5.2.6. □

Theorem 5.4.7. *Suppose that Assumptions 5.2.1, 5.2.6 and 5.3.1 hold, and that* $\{\mathbf{x}_i\}_{i=0}^{\infty}$ *is a bounded sequence generated by Algorithm 5.4.1. Then, there exist an* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *and an infinite subset* $K \subset \mathbb{N}$ *such that* $\mathbf{x}_i \rightarrow^K \hat{\mathbf{x}}$, *as* $i \rightarrow \infty$, *and* $\hat{\mathbf{x}}$ *is a stationary point for* P.

Proof. Let $\{\mathbf{x}_j^*\}$ be the sequence generated by Algorithm 5.4.1 in Step 10. We will show that $\{\mathbf{x}_j^*\}$ must be a finite sequence. For the sake of a contradiction, suppose that $\{\mathbf{x}_j^*\}_{j=0}^{\infty}$ is an infinite sequence. Since $\{\mathbf{x}_i\}_{i=0}^{\infty}$ is a bounded sequence, $\{\mathbf{x}_j^*\}_{j=0}^{\infty}$ is bounded, and hence, there must exist an infinite subset $K^* \subset \mathbb{N}$ and $\mathbf{x}^{**} \in \mathbb{R}^n$ such that $\mathbf{x}_j^* \rightarrow^{K^*} \mathbf{x}^{**}$, as $i \rightarrow \infty$.

By Lemmas 5.2.7(ii) and 5.2.8, there exist a compact set $\Omega(\mathbf{x}^{**})$ and a $\rho_{\mathbf{x}^{**}}$ such that $\eta(\cdot,\cdot)$ is continuous on $\mathbb{B}(\mathbf{x}^{**}, \rho_{\mathbf{x}^{**}}) \times \Omega(\mathbf{x}^{**})$, and

$$\hat{\mathbf{Y}}(\mathbf{x}^{**}) + \mathbb{B}_{\rho_{\mathbf{x}^{**}}} \subset \Omega(\mathbf{x}^{**}). \tag{5.4.7a}$$

Hence,

$$\pi^{**} \triangleq \max_{\mathbf{x} \in \mathbb{B}(\mathbf{x}^{**}, \rho_{\mathbf{x}^{**}})} \max_{\mathbf{y} \in \Omega(\mathbf{x}^{**})} \sigma \sum_{k=1}^{\tau_1} |\eta_k(\mathbf{x}, \mathbf{y})| \tag{5.4.7b}$$

is well-defined, and therefore, $t_\pi(\mathbf{x}, \mathbf{y}) \leq 0$ for all $\pi \geq \pi^{**}$, $\mathbf{x} \in \mathbb{B}(\mathbf{x}^{**}, \rho_{\mathbf{x}^{**}})$ and $\mathbf{y} \in \Omega(\mathbf{x}^{**})$. Since $\{\mathbf{x}_j^*\}$ is an infinite sequence, $\pi_i \rightarrow \infty$, as $i \rightarrow \infty$. Hence, there exists $i_0 \in \mathbb{N}$ such that $\pi_i > \pi^{**}$ for all $i \geq i_0$. Hence, $t_{\pi_{i-1}}(\mathbf{x}_i, \mathbf{y}) \leq 0$ for all $i > i_0$, $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_{\mathbf{x}^{**}})$ and $\mathbf{y} \in \Omega(\mathbf{x}^{**})$. By Lemma 5.4.3(ii), $N_i \rightarrow \infty$, as $i \rightarrow \infty$. Let $\{\mathbf{y}_i\}_{i=0}^{\infty}$ be the sequence generated by Algorithm 5.4.1 in Step 1. Then, by Lemma 5.3.3(i) and (5.4.7a) there exist $i_1 \geq i_0$ and $\rho_1 \in (0, \rho_{\mathbf{x}^{**}}]$ such that for all $i > i_1$ with $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_1)$,

$$\mathbf{y}_i \in \hat{\mathbf{Y}}_{\pi_{i-1}, N_i}(\mathbf{x}_i) \subset \hat{\mathbf{Y}}(\mathbf{x}^{**}) + \mathbb{B}_{\rho_{\mathbf{x}^{**}}} \subset \Omega(\mathbf{x}^{**}). \tag{5.4.7c}$$

Therefore, for all $i > i_1$ and $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_1)$, $t_{\pi_{i-1}}(\mathbf{x}_i, \mathbf{y}_i) \leq 0$.

Next, by Lemma 5.4.6 there exists $\epsilon > 0$ such that

$$2\epsilon = \min_{\mathbf{x} \in \mathbb{B}(\mathbf{x}^{**}, \rho_1)} \min_{\mathbf{y} \in \hat{\mathbf{Y}}(\mathbf{x})} \sigma_{\min}(\mathbf{x}, \mathbf{y}). \tag{5.4.7d}$$

Moreover, $\sigma_{\min}(\cdot, \cdot)$ is continuous, and hence uniformly continuous on $\mathbb{B}(\mathbf{x}^{**}, \rho_1) \times \Omega(\mathbf{x}^{**})$. Hence there exists $\rho_2 \in (0, \rho_1]$ such that

$$|\sigma_{\min}(\mathbf{x}', \mathbf{y}') - \sigma_{\min}(\mathbf{x}'', \mathbf{y}'')| \leq \epsilon \tag{5.4.7e}$$

for all $\mathbf{x}', \mathbf{x}'' \in \mathbb{B}(\mathbf{x}^{**}, \rho_1)$ and $\mathbf{y}', \mathbf{y}'' \in \Omega(\mathbf{x}^{**})$, with $\|\mathbf{x}' - \mathbf{x}''\| \leq \rho_2$ and $\|\mathbf{y}' - \mathbf{y}''\| \leq \rho_2$. By Lemma 5.3.3(i), there exist $\rho_3 \in (0, \rho_2]$ and $i_2 > i_1$ such that for all $i > i_2$ with $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_3)$,

$$\mathbf{y}_i \in \hat{\mathbf{Y}}_{\pi_{i-1}, N_i}(\mathbf{x}_i) \subset \hat{\mathbf{Y}}(\mathbf{x}^{**}) + \mathbb{B}_{\rho_2} \subset \Omega(\mathbf{x}^{**}). \tag{5.4.7f}$$

Consequently, for all $i > i_2$ with $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_3)$, there exists $\mathbf{y}_i' \in \hat{\mathbf{Y}}(\mathbf{x}^{**})$ such that $\|\mathbf{y}_i' - \mathbf{y}_i\| \leq \rho_2$. Hence by (5.4.7d) and (5.4.7e), we have that for all $i > i_2$ with $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_3)$,

$$\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) = \sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) - \sigma_{\min}(\mathbf{x}^{**}, \mathbf{y}_i') + \sigma_{\min}(\mathbf{x}^{**}, \mathbf{y}_i') \geq -\epsilon + 2\epsilon = \epsilon. \tag{5.4.7g}$$

Since for all $i > i_1$ and $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_1)$, $t_{\pi_{i-1}}(\mathbf{x}_i, \mathbf{y}_i) \leq 0$, i.e., the test in Step 3 is satisfied for all $\pi \geq \pi_{i-1}$. Since $\mathbf{x}_j^* \xrightarrow{K^*} \mathbf{x}^{**}$, there must exist an infinite set $K^{**} \subset \mathbb{N}$, with elements diverging to infinity, such that $\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) < \sigma_{i-1}$ for all $i \in K^{**}$, i.e., the test in Step 2 fails an infinite number of times. Hence, $\sigma_i \to 0$, as $i \to \infty$. Therefore, there exists an $i_3 > i_2$ such that for all $i > i_3$, $\sigma_{i-1} < \epsilon$. Now, we have that for all $i > i_3$ with $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_3)$, $\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) \geq \sigma_{i-1}$ and $t_\pi(\mathbf{x}_i, \mathbf{y}_i) \leq 0$ for all $\pi \geq \pi_{i-1}$. Consequently, no $\mathbf{x}_i \in \mathbb{B}(\mathbf{x}^{**}, \rho_3)$ is converted into $\mathbf{x}_j^*$ after i_3, which is a contradiction. Hence $\{\mathbf{x}_j^*\}_{j=0}^{j^*}$ is a finite sequence with $j^* < \infty$.

It follows from Theorem 5.4.5 there exist an infinite subset $K \subset \mathbb{N}$ and an $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{x}_i \xrightarrow{K} \hat{\mathbf{x}}$ and $\theta_{\pi^*}(\hat{\mathbf{x}}) = 0$, with $\pi^* \triangleq \kappa^{j^*+1}\pi_{-1}$. Furthermore, there exists $i^* \in \mathbb{N}$ such that for all $i \geq i^*$ the test in Step 2 is satisfied and the test in Step 3

is satisfied with $\pi = \pi_{i-1}$, i.e., there exists $\sigma^* > 0$ such that for all $i \geq i^*$, $\pi_i = \pi^*$, $\sigma_{\min}(\mathbf{x}_i, \mathbf{y}_i) \geq \sigma^*$, and

$$t_{\pi^*}(\mathbf{x}_i, \mathbf{y}_i) \leq 0. \tag{5.4.7h}$$

Now, $\{\mathbf{y}_i\}_{i=0}^{\infty} \subset \mathbf{Y}$, which is compact. Hence there exist $L \subset K$ and $\hat{\mathbf{y}} \in \mathbf{Y}$ such that $\mathbf{y}_i \xrightarrow{L} \hat{\mathbf{y}}$, as $i \to \infty$. By Lemma 5.4.3(ii), $N_i \to \infty$. Hence, Lemma 5.3.2 gives that $\hat{\mathbf{y}} \in \hat{\mathbf{Y}}_{\pi^*}(\hat{\mathbf{x}})$. By Lemma 5.4.6, $\sigma_{\min}(\cdot, \cdot)$ is continuous, and hence by continuity, $\sigma_{\min}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \geq \sigma^*$. This implies that $\mathbf{A}(\hat{\mathbf{x}}, \hat{\mathbf{y}})\mathbf{A}(\hat{\mathbf{x}}, \hat{\mathbf{y}})^T + \mathbf{B}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is positive definite, and hence by Lemma 5.2.8(i), $\nabla_{\mathbf{y}} f_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}), k \in \mathbf{r}_1^*(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, together with $\nabla g_k(\hat{\mathbf{y}}), k \in \mathbf{r}_2^*(\hat{\mathbf{y}})$, are linearly independent. By Lemma 5.2.7(ii), $t_{\pi^*}(\cdot, \cdot)$ is continuous at $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. It follows from (5.4.7h) that $t_{\pi^*}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \leq 0$. Hence, by Lemma 5.2.7(iii),

$$\psi(\hat{\mathbf{x}}) = \psi_{\pi^*}(\hat{\mathbf{x}}). \tag{5.4.7i}$$

It now follows from Theorems 5.2.5 and 5.2.10 that $\hat{\mathbf{x}}$ is stationary for $\mathbf{P}$. This completes the proof. □

5.5 Numerical Example

We illustrate Algorithm 5.4.1 by a numerical example computed on a 500 MHz PC running Matlab Mathworks (1999). Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $y \in \mathbb{R}$, and

$$\phi(\mathbf{x}, y) = 3(x_1 - y)^2 + (2 - y)x_2^2 + 5(x_3 + y)^2 + 2x_1 + 3x_2 - x_3 + e^{4y^2}, \tag{5.5.1a}$$

$$f(\mathbf{x}, y) = \frac{1}{4}\sin(x_1 x_2) + y - \frac{1}{2}, \tag{5.5.1b}$$

$$g_1(y) = -y, \tag{5.5.1c}$$

$$g_2(y) = y - 1, \tag{5.5.1d}$$

i.e., $r_1 = 1$, $r_2 = 2$ and $\mathbf{Y} = [0, 1] \subset \mathbb{R}$.

Based on the reasoning in the paragraphs following Lemma 5.4.2, we take $\tau_1 = \ln 2$, $\tau_2 = 1$, $\sigma = \kappa = 2$, $\rho = 0.001$, $\mu = 0.5$, $\sigma_{-1} = 10^{-5}$, $\hat{p} = 5 \cdot 10^4$, $\gamma = 10^5$ and $\zeta = 2$. Furthermore, we set the step-size parameters to be $\alpha = 0.5$ and $\beta = 0.8$. The

discretization scheme is such that $\mathbf{Y}_N$ contains $N+1$ equally spaced numbers in $[0,1]$, i.e., $\mathbf{Y}_1 = \{0,1\}, \mathbf{Y}_2 = \{0, 0.5, 1\}, \mathbf{Y}_3 = \{0, 0.333, 0.667, 1\}$, etc, and $\Delta(N) = 1/N$. The approximation parameters are set to be $p_0 = 1$, $N_0 = 1$ and $\pi_{-1} = 1$, which give a coarse approximation.

Using the starting point $\mathbf{x}_0 = (2, 1, 0)$, we obtain the local minimizer $\hat{\mathbf{x}} = (-0.0033, -1.0002, -0.3928)$, with $\psi(\hat{\mathbf{x}}) = 2.4100$. The penalty π was increased to 1024 in the first iteration, and remained constant after that throughout the rest of the computation. The parameter N and p reach 1113 and $1 \cdot 10^5$, respectively, before the calculations are terminated. It is observed that the precision parameters N and p stay low until the iterate is close to a local minimizer. The initially coarse approximations reduce ill-conditioning potentially caused by a high smoothing precision parameter, see Polak and Royset Polak *et al.* (2002) for an examination of such effects, and computational cost caused by high discretization.

5.6 Concluding Remarks

We have developed an implementable algorithm for a class of generalized semi-infinite min-max problems based on a sequential solution of gradually better-approximating finite min-max problems. The approximating problems are obtained by exact penalization, discretization and smoothing. The penalty, discretization and smoothing parameters are automatically adjusted by using a series of tests. Under mild assumptions, we have shown that if the algorithm generates a bounded sequence, then the penalty parameter remains bounded and there exists an accumulation point which satisfies a first-order optimality condition.

Clearly, discretization is a computationally expensive technique in high-dimensional spaces, and hence the proposed algorithm will be computationally inefficient for problems with a high-dimensional semi-infinite part, i.e., large m. In spite of this, we used a discretization technique because of the need for global maximizers of the inner problem of the min-max-min problem. Obviously, other global optimization techniques could have been used, but we have not evaluated the relative merits of alternative

techniques.

5.7 Relations between Optimality Conditions

The optimality condition for **P** derived in Theorem 5.2.5 (see also Theorem 5.2.10) can be related to the following optimality condition deduced from Theorem 3.3 in Stein (2001).

Theorem 5.7.1. *Suppose that* $\hat{\mathbf{x}}$ *is a local minimizer for* **P**, *that Assumption 5.2.1 holds, and that the vectors* $\nabla_{\mathbf{y}} f_k(\hat{\mathbf{x}}, \mathbf{y}), k \in \mathbf{r}_1^*(\hat{\mathbf{x}}, \mathbf{y})$, *together with the vectors* $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$, *are linearly independent for all* $\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$. *Then,*

$$0 \in \operatorname*{conv}_{\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})} \{\nabla_{\mathbf{x}} \phi(\hat{\mathbf{x}}, \mathbf{y}) - \mathbf{f}_{\mathbf{x}}(\hat{\mathbf{x}}, \mathbf{y})^T \alpha(\hat{\mathbf{x}}, \mathbf{y})\}, \tag{5.7.1a}$$

where $\alpha(\hat{\mathbf{x}}, \mathbf{y}) \in \mathbb{R}^{r_1}$, *together with* $\beta(\hat{\mathbf{x}}, \mathbf{y}) \in \mathbb{R}^{r_2}$ *(not used here), are the unique Karush-Kuhn-Tucker multipliers for the "inner-problem" (5.1.2) at the point* $\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$, *i.e.,* $(\alpha(\hat{\mathbf{x}}, \mathbf{y}), \beta(\hat{\mathbf{x}}, \mathbf{y}))$ *satisfy*

$$\nabla_{\mathbf{y}} \phi(\hat{\mathbf{x}}, \mathbf{y}) - \mathbf{f}_{\mathbf{y}}(\hat{\mathbf{x}}, \mathbf{y})^T \alpha(\hat{\mathbf{x}}, \mathbf{y}) - \mathbf{g}_{\mathbf{y}}(\mathbf{y})^T \beta(\hat{\mathbf{x}}, \mathbf{y}) = 0, \tag{5.7.1b}$$

$$\alpha(\hat{\mathbf{x}}, \mathbf{y})^T \mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}) + \beta(\hat{\mathbf{x}}, \mathbf{y})^T \mathbf{g}(\mathbf{y}) = 0, \tag{5.7.1c}$$

$$\alpha(\hat{\mathbf{x}}, \mathbf{y}) \geq 0, \ \beta(\hat{\mathbf{x}}, \mathbf{y}) \geq 0, \tag{5.7.1d}$$

$$\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}) \leq 0, \ \mathbf{g}(\mathbf{y}) \leq 0. \tag{5.7.1e}$$

□

We can then show that the stationary condition in Theorem 5.7.1. relates to the stationary condition in Theorem 5.2.5.

Theorem 5.7.2. *Suppose that Assumption 5.2.1 holds, that* $\hat{\mathbf{x}}$ *satisfies (5.7.1a), and that the vectors* $\nabla_{\mathbf{y}} f_k(\hat{\mathbf{x}}, \mathbf{y}), k \in \mathbf{r}_1^*(\hat{\mathbf{x}}, \mathbf{y})$, *together with the vectors* $\nabla g_k(\mathbf{y}), k \in \mathbf{r}_2^*(\mathbf{y})$,

are linearly independent for all $\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$. *If* $\pi > 0$ *is such that* $\psi(\hat{\mathbf{x}}) = \psi_\pi(\hat{\mathbf{x}})$, *and for all* $\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$

$$\pi \geq \sum_{k=1}^{r_1} |\eta_k(\hat{\mathbf{x}}, \mathbf{y})|, \tag{5.7.2a}$$

with $\eta(\cdot,\cdot)$ *as in (5.2.13g), then* $0 \in \bar{G}\psi_\pi(\hat{\mathbf{x}})$.

Proof. By Caratheodory's Theorem, see, e.g., Theorem 5.2.5 in Polak (1997), (5.7.1a) holds if and only if there exist $\hat{\mathbf{y}}_i \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$, $i \in \{1, ..., n+1\}$, and a multiplier vector $\hat{\mu} \in \Sigma_{n+1} \stackrel{\triangle}{=} \{\mu \in \mathbb{R}^{n+1} | \ \mu_i \geq 0, i \in \{1, ..., n+1\}, \sum_{i=1}^{n+1} \mu_i = 1\}$ such that

$$0 = \sum_{i=1}^{n+1} \hat{\mu}_i \nabla_{\mathbf{x}} \phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) - \sum_{i=1}^{n+1} \sum_{k=1}^{r_1} \hat{\mu}_i \alpha_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) \nabla_{\mathbf{x}} f_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i). \tag{5.7.2b}$$

We will now construct multipliers such that $0 \in \bar{G}\psi_\pi(\hat{\mathbf{x}})$. Let $\pi > 0$ satisfy (5.7.2a) for all $\mathbf{y} \in \hat{\mathbf{Y}}(\hat{\mathbf{x}})$,

$$\zeta_{k,i} \stackrel{\triangle}{=} \frac{1}{\pi} \alpha_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i), \ k \in \mathbf{r_1}, \tag{5.7.2c}$$

$$\zeta_{0,i} \stackrel{\triangle}{=} 1 - \sum_{k=1}^{r_1} \zeta_{k,i}, \tag{5.7.2d}$$

$$\mu_i \stackrel{\triangle}{=} \hat{\mu}_i, \ i \in \{1, ..., n+1\}, \tag{5.7.2e}$$

and $\mathbf{y}_i \stackrel{\triangle}{=} \hat{\mathbf{y}}_i, i \in \{1, ..., n+1\}$. Trivially, $\mu \in \Sigma_{n+1}, \mathbf{y}_i \in \mathbf{Y}$, and $\mathbf{f}(\hat{\mathbf{x}}, \mathbf{y}_i) \leq 0$, for all $i \in \{1, ..., n+1\}$. Furthermore, for all $k \in \mathbf{r_1}$ and $i \in \{1, ..., n+1\}$

$$\begin{aligned} \mu_i \zeta_{k,i} [\phi_{k,\pi}(\hat{\mathbf{x}}, \mathbf{y}_i) - \omega_\pi(\hat{\mathbf{x}}, \mathbf{y}_i)] &= -\pi \mu_i \zeta_{k,i} f_k(\hat{\mathbf{x}}, \mathbf{y}_i) \\ &= -\hat{\mu}_i \alpha_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) f_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) = 0, \end{aligned} \tag{5.7.2f}$$

because from (5.7.1c,d,e), $\alpha_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) f_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i) = 0$, for all $k \in \mathbf{r_1}$ and $i \in \{1, ..., n+1\}$. Also, $\phi_{r,\pi}(\hat{\mathbf{x}}, \mathbf{y}_i) - \omega_\pi(\hat{\mathbf{x}}, \mathbf{y}_i) = 0$ for all $i \in \{1, ..., n+1\}$. Next, by (5.7.2b)

$$\begin{aligned}\sum_{i=1}^{n+1}\sum_{k=1}^{r}\mu_i\zeta_{k,i}\nabla_{\mathbf{x}}\phi_{k,\pi}(\hat{\mathbf{x}},\mathbf{y}_i) &= \sum_{i=1}^{n+1}\mu_i\nabla_{\mathbf{x}}\phi(\hat{\mathbf{x}},\mathbf{y}_i)-\sum_{i=1}^{n+1}\sum_{k=1}^{r_1}\mu_i\zeta_{k,i}\pi\nabla_{\mathbf{x}}f_k(\hat{\mathbf{x}},\mathbf{y}_i)\\ &= \sum_{i=1}^{n+1}\hat{\mu}_i\nabla_{\mathbf{x}}\phi(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)-\sum_{i=1}^{n+1}\sum_{k=1}^{r_1}\hat{\mu}_i\alpha_k(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)\nabla_{\mathbf{x}}f_k(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)=0.\end{aligned} \tag{5.7.2g}$$

It now remains to show that $\zeta_{0,i}\geq 0$ for all $i\in\{1,...,n+1\}$. It follows by inspection that the unique multipliers $\alpha(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)$ and $\beta(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)$, see (5.7.1b-e), solve the minimization problem in (5.2.14b) with $\mathbf{x}$ and $\mathbf{y}$ replaced by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}_i$, respectively. Hence, $\alpha(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)$ and $\beta(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)$ also satisfy the necessary optimality conditions for (5.2.14b) given in (5.2.14a). Since the solution of (5.2.14a) is unique under the linear independence assumption, see the proof of Lemma 5.2.7(ii), we have by definition of $\eta(\cdot,\cdot)$, see (5.2.13g), that $\eta(\hat{\mathbf{x}},\mathbf{y}_i)=\alpha(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)$, for all $i\in\{1,...,n+1\}$. Hence,

$$\begin{aligned}\zeta_{0,i} &= 1-\sum_{k=1}^{r_1}\zeta_{k,i}=1-\frac{1}{\pi}\sum_{k=1}^{r_1}\alpha_k(\hat{\mathbf{x}},\hat{\mathbf{y}}_i)\\ &= 1-\frac{1}{\pi}\sum_{k=1}^{r_1}\eta_k(\hat{\mathbf{x}},\mathbf{y}_i)\geq 1-\frac{1}{\pi}\pi=0.\end{aligned} \tag{5.7.2h}$$

This completes the proof. □

Chapter 6

Numerical Design Examples

In this chapter, we present four examples demonstrating applications of the proposed approach to reliability-based optimal structural design. The first example considers a rectangular column and optimal design problems in the form of $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ with one failure component, as defined in Chapter 4. This example is sufficiently simple to allow reproduction of the results by the interested reader. The second example concerns the selection of member sizes for an offshore platform and the optimal design problem $\mathbf{P}_2$, with one failure component. This example demonstrates an application where the limit-state function is not available in a closed form and finite element analysis is required to evaluate the limit-state function and its gradient. The third example considers the design of a one-bay frame that has several failure mechanisms. This example illustrates the optimal design of a series structural system in the form of $\mathbf{P}_{1,\text{sys}}$. The fourth example concerns the design of a reinforced concrete girder in a highway bridge. This example demonstrates application of the proposed optimal design algorithm with system failure probabilities in both the objective function and the constraint set definition. The resulting problem is in the form of $\mathbf{P}_{3,\text{sys}}$.

For the purpose of these applications, the optimization algorithm and reliability methods were programmed in Mathworks (1999), with the exception of the system reliability analysis in the third example, where the program CalREL (Liu *et al.* 1989) was used. Since computational efficiency was not the focus in this study, no effort

Table 6.1: Statistics of lognormal random variables in short column example.

Variable	Mean	c.o.v.
M_1	250 kNm	0.30
M_2	125 kNm	0.30
P_a	2500 kN	0.20
Y	40 MPa	0.10

was made to select an ideal optimization algorithm for the problem, or optimize the computer code with respect to computation time. Finally, we note that the sizes of structural members in actual practice are constrained in more ways than we consider in these examples, e.g., the availability of standard sizes. Thus, these examples should be considered as illustrative of the algorithms developed in this study rather than actual design exercises.

6.1 Short Rectangular Column

Consider a short column with a rectangular cross section of dimensions b and h and material yield strength Y, which is subjected to bi-axial bending moments M_1 and M_2 and axial force P_a. Assuming an elastic-perfectly plastic material, the reliability of the column is defined by the limit-state function

$$G(\mathbf{x}, \mathbf{v}) = 1 - \frac{4m_1}{bh^2y} - \frac{4m_2}{b^2hy} - \left(\frac{p_a}{bhy}\right)^2, \tag{6.1.1}$$

where $\mathbf{v} = (m_1, m_2, p_a, y) \in \mathbb{R}^4$ denotes a realization of the random vector $\mathbf{V} = (M_1, M_2, P_a, Y)$ and $\mathbf{x} = (b, h) \in \mathbb{R}^2$ denotes the vector of design variables. Since there is only one failure component in this example, we simplify the notation from Chapter 4 by dropping the subscript k associated with the failure component. We assume that M_1, M_2, P_a and Y are statistically independent lognormal random variables with the means and coefficients of variation (c.o.v.) listed in Table 6.1.

Table 6.2: Results for $\mathbf{P}_1$ design of short column.

Iter. i	$\mathbf{x}_i$	$c_0(\mathbf{x}_i)$	$p(\mathbf{x}_i)$	$\beta_1(\mathbf{x}_i)$	s_i
0	(0.500 m, 0.500 m)	0.250 m^2	-	-	3.00
1	(0.346 m, 0.553 m)	0.191 m^2	0.002401	3.00	3.19
2	(0.334 m, 0.586 m)	0.195 m^2	0.001350	3.19	-

6.1.1 Single Component $\mathbf{P}_1$

Suppose that the column is to be designed for minimum cross-sectional area $A = bh$, subject to failure probability $p(\mathbf{x}) \leq 0.001350$, $b, h \geq 0$, and $0.5 \leq b/h \leq 2$, i.e., $\mathbf{P}_1$, with $K = 1$, as defined in Chapter 4. The last constraint is imposed to bound the aspect ratio of the cross section. We use Algorithm 4.1.1 with Algorithm 3.3.1 to obtain the results summarized in Table 6.2. Included are the iterates $\mathbf{x}_i$, the objective function $c_0(\mathbf{x}_i)$, the failure probability $p(\mathbf{x}_i)$, the first-order reliability index $\beta_1(\mathbf{x}_i)$ (see (2.1.8)), and the parameter s_i for iterations 0, 1, and 2. The failure probability $p(\cdot)$ was computed by Monte Carlo Simulation with a coefficient of variation 0.05. Note that the design after the first iteration satisfies a constraint $\beta_1(\mathbf{x}_1) \leq 3 = -\Phi^{-1}(0.001350)$. Hence, the design $\mathbf{x}_1$ gives a sufficiently safe structure if the reliability constraint is to be satisfied in first-order approximation. In view of the comments after Theorem 4.1.2, this was expected.

6.1.2 Single Component $\mathbf{P}_2$

Now suppose we wish to design the column for minimum failure probability, subject to $b, h \geq 0$, $0.5 \leq b/h \leq 2$, and, additionally, $bh \leq 0.1875$ m^2, i.e., $\mathbf{P}_2$, with $K = 1$. We use Algorithm 4.1.4 with Algorithm 3.3.3 to obtain the results summarized in Table 6.3. Included are the values of the parameter r_i, iterates $\mathbf{x}_i$, the objective function (the failure probability) $p(\mathbf{x}_i)$, and the first-order reliability index $\beta_1(\mathbf{x}_i)$. The failure probability $p(\cdot)$ was computed by Monte Carlo Simulation with coefficient of variation 0.05.

Note that the range of values of parameter r encompasses the value $-\Phi^{-1}(p(\mathbf{x}_i))$ = 2.70, as recommended in the discussion after Algorithm 4.1.4. It is observed that

Table 6.3: Results for $\mathbf{P}_2$ design of short column.

Iter. i	$\mathbf{x}_i$	$p(\mathbf{x}_i)$	$\beta_1(\mathbf{x}_i)$	$\Phi(-\beta_1(\mathbf{x}_i))$	r_i
0	(0.500 m, 0.500 m)	-	-	-	1.5
1	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	2.0
2	(0.309 m, 0.606 m)	0.003467	2.90	0.001866	2.5
3	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	2.8
4	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	2.9
5	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	3.0
6	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	3.5
7	(0.310 m, 0.605 m)	0.003467	2.90	0.001866	4.0
8	(0.309 m, 0.607 m)	0.003467	2.90	0.001866	-

the optimal solution is virtually invariant of r. Furthermore, the first-order approximation of the failure probability, $\Phi(-\beta_1(\mathbf{x}_i))$, is significantly different from $p(\mathbf{x}_i)$, thus indicating that the limit-state surface is strongly non-affine. We know from Theorem 4.1.3 that for an affine limit-state function the assumed value of r is immaterial. The results in the Table 6.3 indicate that this property also approximately holds for non-affine limit-state functions. Based on the results in Table 6.3, the optimal dimensions of the column in this case are 0.310 m and 0.605 m with the optimal failure probability 0.003467.

6.1.3 Single Component $\mathbf{P}_3$

Suppose that the column is to be designed for minimum cross-sectional area $c_0(\mathbf{x}) = bh$ plus expected cost of failure, where the cost of failure is assumed to be 100 times the cost of the cross section, i.e., $c(\mathbf{x}) = 100bh$, subject to $p(\mathbf{x}) \leq 0.001350$, $b, h \geq 0$, and $0.5 \leq b/h \leq 2$. This problem is of form $\mathbf{P}_3$ with $K = 1$.

Let $T_{\mathbf{x}}(\cdot)$ be the transformation of realizations $\mathbf{v}$ of $\mathbf{V}$ into realizations $\mathbf{u}$ of a standard normal random vector $\mathbf{U}$, as described in Section 2.1. In this particular case, $T_{\mathbf{x}}(\cdot)$ is independent of $\mathbf{x}$, and hence we write $T(\cdot) = T_{\mathbf{x}}(\cdot)$. Generally, we can attempt to construct a subset of $\mathbb{R}^m$ satisfying Assumption 4.2.1 by considering "larger-than-average" realizations of random variables associated with loads and "less-than-average" realizations of random variables associated with the strength of the

Table 6.4: Results for $\mathbf{P}_3$ design of short column, first-order reliability.

Iter. i	$\mathbf{x}_i$	$\beta_1(\mathbf{x}_i)$	Obj. of $\mathbf{P}'_{3,r_i,s}$	Obj. of $\mathbf{P}_3$	r_i	s
0	(0.500 m, 0.500 m)	-	-	-	3.00	3.00
1	(0.314 m, 0.625 m)	3.22	0.205	0.208	3.22	3.00
2	(0.349 m, 0.572 m)	3.33	0.207	0.208	3.33	3.00
3	(0.349 m, 0.572 m)	3.33	0.208	0.208	-	-

Table 6.5: Results for $\mathbf{P}_3$ design of short column.

i	$\mathbf{x}_i$	$p(\mathbf{x}_i)$	Obj. of $\mathbf{P}'_{3,r_i,s}$	Obj. of $\mathbf{P}_3$	r_i	s_i
0	(0.500 m, 0.500 m)	-	-	-	3.00	3.00
1	(0.314 m, 0.625 m)	0.001223	0.205	0.220	3.71	2.97
2	(0.370 m, 0.556 m)	0.000466	0.212	0.215	3.94	2.67
3	(0.370 m, 0.561 m)	0.000362	0.214	0.215	4.01	2.39
4	(0.372 m, 0.559 m)	0.000362	0.215	0.215	-	-

structure. Since M_1, M_2 and P_a are random variables associated with loads and Y is associated with the strength of the column, we can show that the set

$$\begin{aligned} \mathbf{Y}'_{r,s} \;=\; & \{\mathbf{u} = (T(m_1), T(m_2), T(p_a), T(y)) \in \mathbb{R}^4 \\ & \mid \mathbf{u} \in \mathbb{B}(\mathbf{0}, r),\ T(m_1) \geq 0,\ T(m_2) \geq 0,\ T(p_a) \geq 0,\ T(y) \leq 0\} \end{aligned} \tag{6.1.2}$$

satisfies Assumption 4.2.1 for $r, s > 0$. Suppose we were interested in a first-order reliability model, i.e., we accept the approximation $p(\mathbf{x}) \approx \Phi(-\beta_1(\mathbf{x}))$. Then, we can solve the design problem by using Algorithm 4.2.2 with Algorithm 3.3.1. The results are summarized in Table 6.4. Included are iterates $\mathbf{x}_i$, the first-order reliability index $\beta_1(\mathbf{x}_i)$, the objective function of $\mathbf{P}'_{3,r_i,s_i}$ (see 4.2.19), the objective function $c_0(\mathbf{x}_i) + c(\mathbf{x}_i)p(\mathbf{x}_i)$, and the value of the parameters r_i and s_i. Note that the constraint $p(\mathbf{x}) \leq 0.001350$ is equivalent to $\beta_1(\mathbf{x}) \geq 3$ in first-order approximation.

We see from Table 6.4 that by construction we satisfy the reliability constraint $\beta_1(\mathbf{x}_i) \geq 3$ for each iteration. However, the objective function in $\mathbf{P}'_{3,r_i,s}$ is not accurate, i.e., there is a discrepancy between column 4 and 5 in Table 6.4. In accordance to Algorithm 4.2.2, we update the parameter r to improve the accuracy of the objective function in $\mathbf{P}'_{3,r_i,s}$.

Now suppose we do not accept the approximation $p(\mathbf{x}) \approx \Phi(-\beta_1(\mathbf{x}))$, but want to

Table 6.6: Distributions of random variables in offshore jacket structure.

Variable	Distribution	Mean	c.o.v.
H	Gumbel	70 kN	0.35
P	Gumbel	2940 kN	0.10
W	Gumbel	20 kN	0.10
E	Lognormal	210 GPa	0.12
k_s	Lognormal	50 kN/m	0.30

use "exact" reliability calculations obtained by Monte Carlo simulation (with c.o.v. 0.05). Then we can solve the problem by using Algorithm 4.2.1 with Algorithm 3.3.1. The results are summarized in Table 6.5. As can be seen, the failure probability constraint is satisfied, but the objective function in $\mathbf{P}'_{3,r_i,s_i}$ is not accurate until we reach the fourth iteration.

6.2 Offshore Jacket Structure (P_2)

Consider the idealized offshore jacket structure shown in Figure 6.1, which is modeled as a plain truss with linear elastic members and supported by linear elastic springs representing the flexibility of the foundation. The structure is subjected to combined wave and wind loads of magnitudes proportional to H, and gravity loads P and W, all applied at the nodes of the truss. The load magnitudes as well as the elastic modulus of the material, E, and the stiffnesses of the supporting springs, k_s, are considered to be statistically independent random variables with the distributions listed in Table 6.6.

The structure has six different member types, each type having a circular tubular cross section with an outside radius R_i and wall thickness $t_i, i = 1, ..., 6$. The ratio of the wall thickness to the outside radius of each member is assumed to be a constant, $t_i/R_i = 0.05$. The task is to optimize the radii R_i to achieve minimum probability for the horizontal displacement at the top right node of the structure to exceed a threshold of $D_0 = 0.20$ m. This is a single component reliability-based optimal design

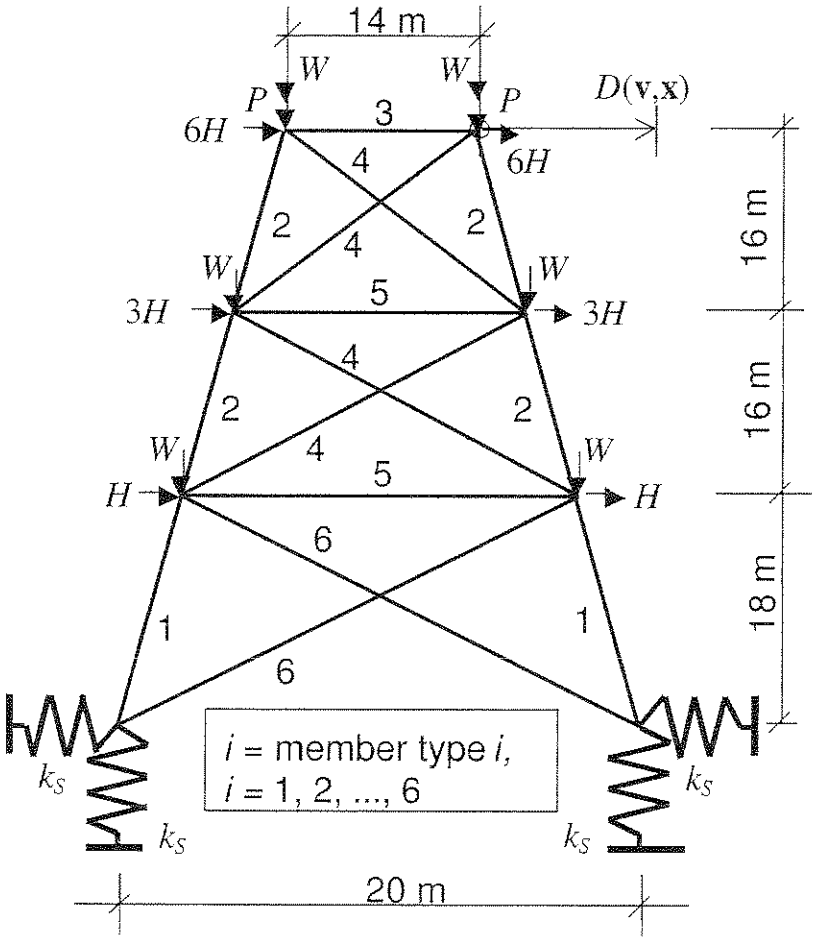

Figure 6.1: Offshore jacket structure.

problem of the form $\mathbf{P}_2$. The limit-state function is defined as

$$G(\mathbf{x}, \mathbf{v}) = D_0 - D(\mathbf{v}, \mathbf{x}), \tag{6.2.1}$$

where $\mathbf{v} \in \mathbb{R}^5$ is a realization of the random vector $\mathbf{V} = (H, P, W, E, k_s)$, $\mathbf{x} = (R_1, ..., R_6) \in \mathbb{R}^6$ is the vector of design variables, and $D(\mathbf{v}, \mathbf{x})$ is the horizontal displacement of the structure at the top right node for given $\mathbf{v}$ and $\mathbf{x}$. Since there is only one failure component in this example, we can in this section simplify the notation from Chapter 4 by dropping the subscript k associated with the component. The function $D(\cdot, \cdot)$ is not available in an explicit form and its evaluation requires structural analysis by means of, e.g., a finite element code. A Matlab structural analysis program was written for the present application to compute $D(\cdot, \cdot)$ and its gradients with respect to $\mathbf{v}$ and $\mathbf{x}$. The jacket structure is designed for minimum failure probability, subject to the total volume of the material not exceeding 6.25 m^3. This implies the constraint function

$$f_1(\mathbf{x}) = \sum_{i=1}^{6} \sum_{j=1}^{N_i} L_{i,j} \pi (R_i^2 - (R_i - t_i)^2) - 6.25, \tag{6.2.2}$$

where $L_{i,j}$ is the length of the j-th member of type i and N_i is the number of members of type i. To prevent failure of the members by buckling, each member of the truss is constrained to have a slenderness ratio (length divided by the radius of gyration of the cross section) not exceeding 130. For the circular tubular cross section, this implies $2L_{i,j}/\sqrt{R_i^2 + (R_i - t_i)^2} \leq 130$. Let L_i be the maximum length of members of type i. Using $t_i = 0.05 R_i$, the above inequality leads to the constraint functions

$$f_{i+1}(\mathbf{x}) = \frac{2L_i}{130\sqrt{1 + 0.95^2}} - R_i, \ i = 1, 2, ..., 6, \tag{6.2.3}$$

which are equivalent to the following constraints on the individual radii: $R_1 \geq 0.201$ m, $R_2 \geq 0.179$ m, $R_3 \geq 0.156$ m, $R_4 \geq 0.259$ m, $R_5 \geq 0.199$ m, and $R_6 \geq 0.291$ m.

We design the jacket structure by solving problem $\mathbf{P}_2$, with $K = 1$ and the constraint set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^6 \mid f_i(\mathbf{x}) \leq 0, i = 1, 2, ..., 7\}$, using Algorithm 4.1.4 with Algorithm 3.3.3. The results are summarized in Table 6.7. Included are iterates $\mathbf{x}_i$,

Table 6.7: Results for $\mathbf{P}_2$ design of offshore jacket structure.

Iter. i	$\mathbf{x}_i$ (m)	$p(\mathbf{x}_i)$	$\beta_1(\mathbf{x}_i)$	r_i
0	$(0.300, 0.300, 0.300, 0.300, 0.300, 0.300)$	-	-	2.5
1	$(0.354, 0.240, 0.156, 0.259, 0.199, 0.291)$	$9.574 \cdot 10^{-5}$	2.72	3.0
2	$(0.354, 0.240, 0.156, 0.259, 0.199, 0.291)$	$9.574 \cdot 10^{-5}$	2.72	3.5
3	$(0.354, 0.240, 0.156, 0.259, 0.199, 0.291)$	$9.574 \cdot 10^{-5}$	2.72	4.0
4	$(0.354, 0.240, 0.156, 0.259, 0.199, 0.291)$	$9.574 \cdot 10^{-5}$	2.72	4.5
5	$(0.354, 0.240, 0.156, 0.259, 0.199, 0.291)$	$9.574 \cdot 10^{-5}$	2.72	-

the objective function (the failure probability) $p(\mathbf{x}_i)$, the first-order reliability index $\beta_1(\mathbf{x}_i)$, the values of the parameter r_i. The failure probability $p(\cdot)$ was computed by Monte Carlo Simulation with a coefficient of variation of 0.01.

It is observed in Table 6.7 that the optimal solution as well as the failure probabilities are virtually invariant of the assumed value of r. In view of Theorem 4.1.3, this result is as expected. It is noted that the first-order failure probability approximation, $p(\mathbf{x}_i) \approx \Phi(-\beta_1(\mathbf{x}_i)) = 9.9611 \cdot 10^{-5}$, is quite close to the "exact" failure probability computed by Monte Carlo simulation. This indicates that the limit-state surface for the problem is most probably nearly affine. From the solution given in Table 6.7, it is clear that the constraint functions $f_j(\cdot), j = 1, 4, 5, 6, 7$, are active at the optimal design point.

As mentioned above, the transformed limit-state function $g(\mathbf{x}, \cdot) = G(\mathbf{x}, T_{\mathbf{x}}^{-1}(\cdot))$, with $T_{\mathbf{x}}(\cdot)$ as in Chapter 2, for the example under consideration appears to be nearly affine. One way to impose greater deviation from an affine limit-state function is to use more strongly non-normal distributions and larger variances for the random variables $\mathbf{V}$. To achieve this, we consider each random variable to be independently uniformly distributed with the bounds, means and c.o.v.'s listed in Table 6.8. We realize that the assumed distributions or the range of variations may not be realistic for an offshore jacket structure. Nevertheless, we use these values to check the robustness of the proposed optimal design algorithm in terms of its sensitivity to the assumed value of r with a non-affine limit-state function. Table 6.9 summarizes the results of the design of the jacket structure for the new set of random variables. We observe that now there

Table 6.8: Uniformly distributed random variables for modified jacket example.

Variable	Range	Mean	c.o.v.
H	(-20 kN, 160 kN)	70 kN	0.74
P	(1880 kN, 4000 kN)	2940 kN	0.21
W	(-10 kN, 50 kN)	20 kN	0.87
E	(130 GPa, 290 GPa)	210 GPa	0.22
k_s	(25 kN/m, 75 kN/m)	50 kN/m	0.29

Table 6.9: Results for $\mathbf{P}_2$ design of modified offshore jacket structure.

Iter. i	$\mathbf{x}_i$ (m)	$p(\mathbf{x}_i)$	$\beta_1(\mathbf{x}_i)$	r_i
0	$(0.300, 0.300, 0.300, 0.300, 0.300, 0.300)$	-	-	2.5
1	$(0.354, 0.241, 0.156, 0.259, 0.199, 0.291)$	$7.532 \cdot 10^{-5}$	3.52	3.0
2	$(0.354, 0.241, 0.156, 0.259, 0.199, 0.291)$	$7.235 \cdot 10^{-5}$	3.53	3.5
3	$(0.354, 0.241, 0.156, 0.259, 0.199, 0.291)$	$7.235 \cdot 10^{-5}$	3.53	4.0
4	$(0.354, 0.241, 0.156, 0.259, 0.199, 0.291)$	$7.235 \cdot 10^{-5}$	3.53	4.5
5	$(0.354, 0.241, 0.156, 0.259, 0.199, 0.291)$	$7.235 \cdot 10^{-5}$	3.53	-

is a significant difference between $p(\mathbf{x}_i)$ and $\Phi(-\beta_1(\mathbf{x}_i)) = 2.078 \cdot 10^{-4}$, indicating that the limit-state function is non-affine. Nevertheless, the optimal solution and the failure probabilities still remain practically in-variant to the assumed value of r. This is a confirmation of our earlier conjecture that the solution of problem $\mathbf{P}_{2,r}$ for a non-affine limit-state function is insensitive to the value of r in a broad neighborhood of the value of the first-order reliability index at a solution. The design solution in this case is nearly the same as the solution for the previous case. Again, the constraint functions $f_j(\cdot), j = 1, 4, 5, 6, 7$ are active at the solution point.

6.3 Structural Frame ($\mathbf{P}_{1,\text{sys}}$)

Consider the design of a one-bay frame subject to random loads H in the horizontal direction and V in the vertical direction, as shown in Figure 6.2. The columns of the frame have rectangular cross sections of width b and depth h_1 and the beam has a rectangular cross section of width b and depth h_2. The material of the frame is elasto-plastic with yield stress Y, which is also considered to be a random variable. Plastic

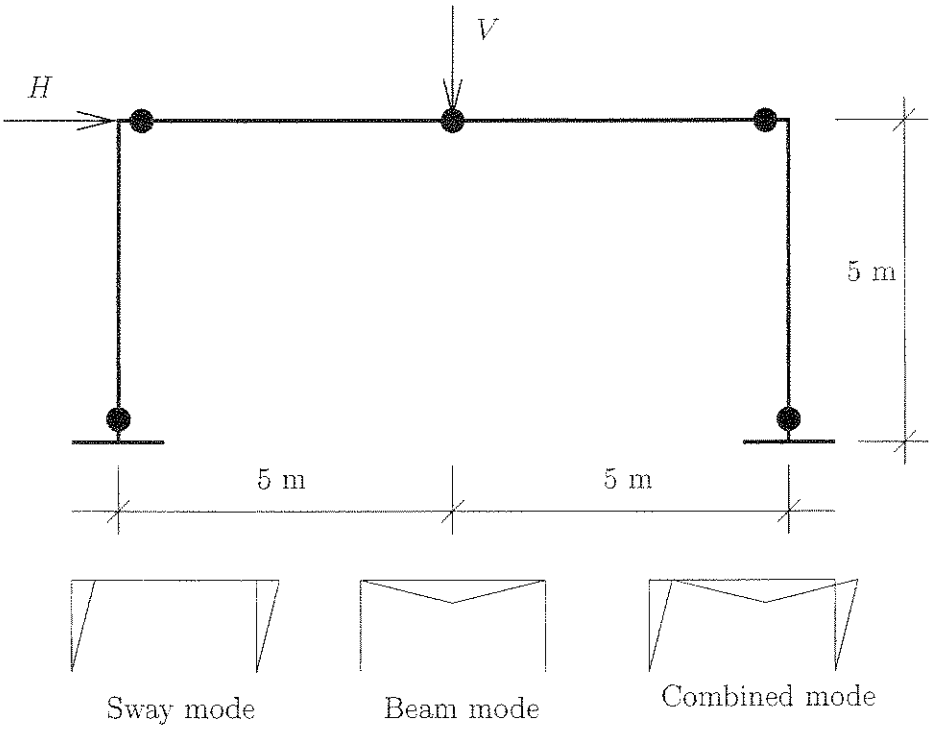

Figure 6.2: One-bay frame example.

hinges may form in the lower ends of the columns and in the beam. Under the applied loads, the frame may collapse in any of the "sway," "beam" or "combined" failure mechanisms shown in the lower portion of Figure 6.2. Thus, the frame constitutes a series structural system. The objective is to determine the optimal dimensions (b, h_1, h_2) for minimum material volume, subject to a constraint on the system failure probability of the frame.

Using the method of virtual work, limit-state functions defining the three failure modes of the structural system are derived as

$$G_1(\mathbf{x}, \mathbf{v}) = 0.5bh_1^2y + 0.5bh_2^2y - 5h, \tag{6.3.1}$$

$$G_2(\mathbf{x}, \mathbf{v}) = bh_2^2y - 5v, \tag{6.3.2}$$

$$G_3(\mathbf{x}, \mathbf{v}) = 0.5bh_1^2y + bh_2^2y - 5h - 5v, \tag{6.3.3}$$

where $\mathbf{v} = (h, v, y) \in \mathbb{R}^3$ is a realization of the random vector $\mathbf{V} = (H, V, Y)$ and

Table 6.10: Distributions of random variables in frame example.

Variable	Distribution	Mean	c.o.v.
H	Gumbel	50 kN	0.30
V	Gumbel	60 kN	0.20
Y	Lognormal	25 MPa	0.10

$\mathbf{x} = (b, h_1, h_2) \in \mathbb{R}^3$ is the vector of design variables. We assume H, V, and Y are statistically independent random variables with the distributions listed in Table 6.10. The objective is to minimize the material volume $c_0(\mathbf{x}) = 10(bh_1 + bh_2)$, subject to the system failure probability constraint $p(\mathbf{x}) \leq 0.006210$. We also impose a minimum of 0.2 m on each member dimension, a maximum aspect ratio of 2 on each cross section, and a "strong-column-weak-beam" design requirement. These requirements result in the constraint functions $f_1(\mathbf{x}) = 0.2 - h_1$, $f_2(\mathbf{x}) = 0.2 - h_2$, $f_3(\mathbf{x}) = 0.2 - b$, $f_4(\mathbf{x}) = h_1 - 2b$, and $f_5(\mathbf{x}) = h_2 - h_1 + 0.0001$. Note that the requirement $h_1 > h_2$ forces the formation of hinges in the beam instead of the upper ends of the columns.

This problem is of type $\mathbf{P}_{1,\text{sys}}$, with $K = 3$ and the constraint set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^3 \mid f_j(\mathbf{x}) \leq 0, j = 1, 2, ..., 5\}$, which we solve by using Algorithm 4.1.3 with Algorithm 3.3.1 and parameter $\gamma = 1.072$. This parameter was determined by computing the first-order reliability index for the most critical failure mode of an initial design of the frame. This initial point was obtained by solving a problem of the type $\mathbf{P}_1$ with $K = 3$. Note that the value of γ is not critical, but a well-selected value results in convergence of Algorithm 4.1.3 in fewer iterations. The results are summarized in Table 6.11, where the design vector $\mathbf{x}_i$, the objective $c_0(\mathbf{x}_i)$, the system failure probability $p(\mathbf{x}_i)$, and the parameter $\mathbf{s}_i$ are listed. The system failure probability is evaluated using Monte Carlo simulation with a coefficient of variation of 0.02. From Table 6.11, it is not clear whether the "strong-column-weak-beam" design requirement is satisfied after the second iteration. However, an examination of the omitted digits reveals that the constraint is satisfied.

Table 6.11: Results for $\mathbf{P}_{1,\text{sys}}$ design of structural frame.

Iter. i	$\mathbf{x}_i$ (m)	$c_0(\mathbf{x}_i)$	$p(\mathbf{x}_i)$	$\mathbf{s}_i$
0	$(0.201, 0.348, 0.348)$	-	-	(2.80,2.80,2.80)
1	$(0.200, 0.350, 0.350)$	1.40	0.007143	(2.86,2.86,2.86)
2	$(0.200, 0.352, 0.352)$	1.41	0.006210	-

6.4 Reinforced Concrete Girder

Consider a highway bridge with reinforced concrete girders of the type shown in Figures 6.3 and 6.4. In this example, we design one such girder using the material and load data from Lin and Frangopol (1996) and Frangopol *et al.* (1997). The design variables are collected in the vector

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3) \in \mathbb{R}^9, \tag{6.4.1}$$

where A_s is the area of the tension steel reinforcement, b is the width of the flange, h_f is the thickness of the flange, b_w is the width of the web, h_w is the height of the web, A_v is the area of the shear reinforcement (twice the cross-section area of a stirrup), and S_1, S_2 and S_3 are the spacings of shear reinforcements in intervals 1, 2, and 3, respectively, see Figure 6.4. The random variables describing the loading and material properties are collected in the vector

$$\mathbf{V} = (f_y, f'_c, P_D, M_L, P_{S1}, P_{S2}, P_{S3}, W) \in \mathbb{R}^8, \tag{6.4.2}$$

where f_y is the yield strength of the reinforcement, f'_c is the compressive strength of concrete, P_D is the dead load excluding the weight of the girder, M_L is the live load moment, P_{S1}, P_{S2} and P_{S3} are the live load shear forces in intervals 1, 2 and 3, respectively, see Figure 6.4, and W is the unit weight of concrete. Following Lin and Frangopol (1996), all the random variables are considered to be independent and normally distributed with the means and coefficients of variation as listed in Table 6.12. In the remainder of this section, the random variables and their realizations are denoted with the same symbol. Let the girder length be $L_g = 18.30$ m, and the distance from the bottom fiber to the centroid of the tension reinforcement be $\alpha = 0.1$ m, see Figure 6.3.

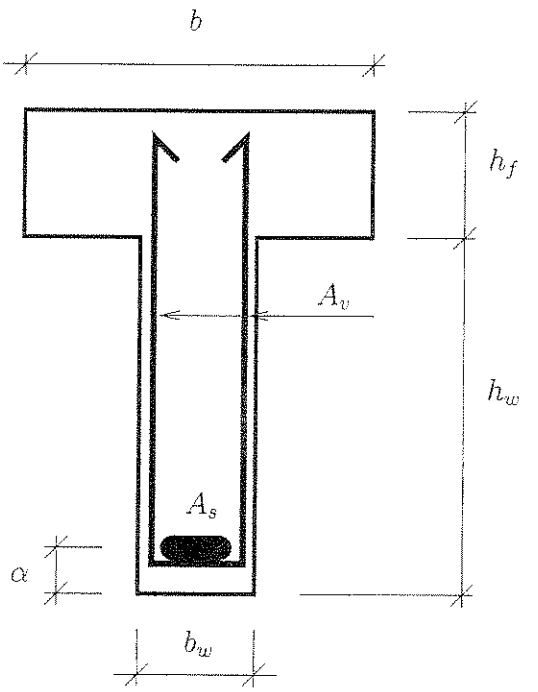

Figure 6.3: Cross-section of reinforced concrete girder.

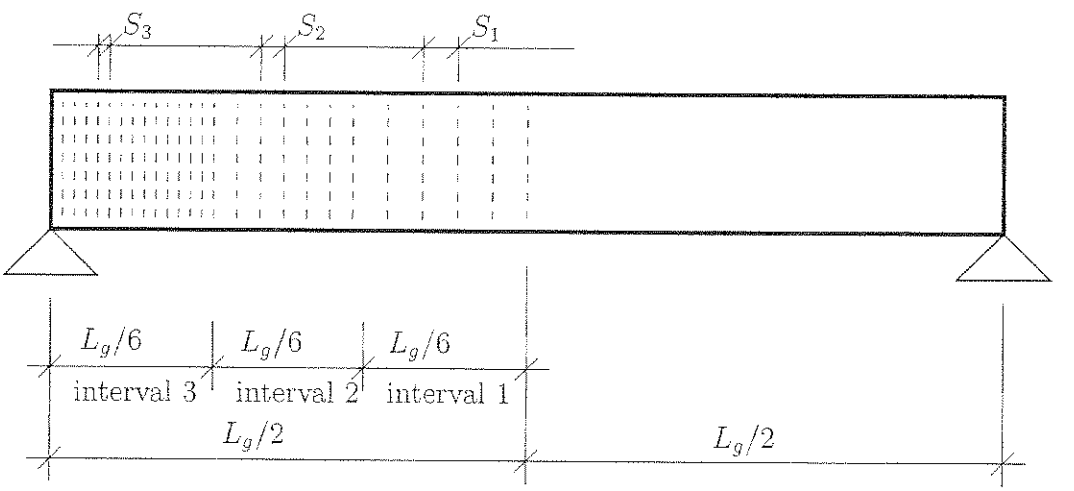

Figure 6.4: Reinforced concrete girder with shear reinforcement.

Table 6.12: Statistics of normal random variables in girder example.

Variable	Description	Mean	c.o.v.
f_y	Yield strength of reinforcement	$413.4 \cdot 10^6$ Pa	0.15
f'_c	Compressive strength of concrete	$27.56 \cdot 10^6$ Pa	0.15
P_D	Dead load excluding girder	$13.57 \cdot 10^3$ N/m	0.20
M_L	Live load moment	$929 \cdot 10^3$ Nm	0.243
P_{S1}	Live load shear in interval 1	$138.31 \cdot 10^3$ N	0.243
P_{S2}	Live load shear in interval 2	$183.39 \cdot 10^3$ N	0.243
P_{S3}	Live load shear in interval 3	$228.51 \cdot 10^3$ N	0.243
W	Unit weight of concrete	$22.74 \cdot 10^3$ N/m^3	0.10

The objective is to design the girder according to the specifications in AASHTO (1992). However, these specifications do not lead to problems of the form $\mathbf{P}_1$, $\mathbf{P}_2$ or $\mathbf{P}_3$, as defined in Chapter 4. In fact, the resulting optimal design problems are not well-defined for two reasons. First, some of the constraints are not continuous functions, but of the form $f(\mathbf{x}) \leq 1$ whenever $h(\mathbf{x}) \leq 0$ and otherwise $f(\mathbf{x}) \leq 2$, where $f(\cdot)$ and $h(\cdot)$ are continuous functions. Second, $h(\cdot)$ may also depend on the random variables of the problem. In the following, the first difficulty is overcome by considering different cases. For example, Case 1 has the constraints $f(\mathbf{x}) \leq 1$ and $h(\mathbf{x}) \leq 0$, while Case 2 has the constraints $f(\mathbf{x}) \leq 2$ and $h(\mathbf{x}) \geq 0$. The optimal design for each case is found independently, and the design with the smallest value of the objective function is our solution. The second difficulty is overcome by replacing any random variables in the definition of $h(\cdot)$ by their mean values. Consequently, we proceed by first defining four cases that must be solved independently.

Case 1

Case 1 corresponds to the situation where the force in the tension reinforcement can be balanced by a compression force in the flange, i.e.,

$$0.85 f'_c b h_f \geq f_y A_s, \tag{6.4.3}$$

and the shear capacity in the shear reinforcement is less than or equal to a value related to the cross-section area and the strength of concrete, i.e.,

$$\frac{A_v f_y}{S_1} \leq 4\sqrt{\gamma f_c'} b_w, \tag{6.4.4}$$

where $\gamma = 6.89 \cdot 10^3$ and the variables are given in SI units (i.e., meter, Newton, etc). Hence, (6.4.3) and (6.4.4), with f_c' and f_y replaced by their mean values, are the constraints for Case 1. Consequently, we have the following deterministic constraints (all variables in SI units):

$$-0.85\overline{f'}_c b h_f + \overline{f}_y A_s \leq 0, \tag{6.4.5}$$

$$\frac{A_v \overline{f}_y}{S_1} - 4\sqrt{\gamma \overline{f'}_c} b_w \leq 0, \tag{6.4.6}$$

where $\overline{f}_y$ and $\overline{f'}_c$ are the mean values of f_y and f_c', respectively, see Table 6.12, and

$$S_1 - \frac{A_v \overline{f}_y}{50\gamma b_w} \leq 0, \tag{6.4.7}$$

$$S_1 - (h_f + h_w - \alpha)/2 \leq 0, \tag{6.4.8}$$

$$S_1 - 0.6096 \leq 0, \tag{6.4.9}$$

$$S_2 - \frac{A_v \overline{f}_y}{50\gamma b_w} \leq 0, \tag{6.4.10}$$

$$S_2 - (h_f + h_w - \alpha)/2 \leq 0, \tag{6.4.11}$$

$$S_2 - 0.6096 \leq 0, \tag{6.4.12}$$

$$S_3 - \frac{A_v \overline{f}_y}{50\gamma b_w} \leq 0, \tag{6.4.13}$$

$$S_3 - (h_f + h_w - \alpha)/2 \leq 0, \tag{6.4.14}$$

$$S_3 - 0.6096 \leq 0, \tag{6.4.15}$$

$$b_w/2 - h_f \leq 0, \tag{6.4.16}$$

$$b - 4b_w \leq 0, \tag{6.4.17}$$

$$b_w - b \leq 0, \tag{6.4.18}$$

$$1 - A_s/0.001 \leq 0, \tag{6.4.19}$$

$$b - 1.22 \leq 0, \tag{6.4.20}$$

$$0.15 - h_f \leq 0, \tag{6.4.21}$$

$$0.15 - b_w \leq 0, \tag{6.4.22}$$

$$h_w/b_w - 4 \leq 0, \tag{6.4.23}$$

$$1 - A_v/0.0001 \leq 0, \tag{6.4.24}$$

$$-h_w \leq 0, \tag{6.4.25}$$

$$-S_1 \leq 0, \tag{6.4.26}$$

$$-S_2 \leq 0, \tag{6.4.27}$$

$$-S_3 \leq 0, \tag{6.4.28}$$

$$h_f + h_w - 1.2, \leq 0, \tag{6.4.29}$$

$$\frac{A_v \overline{f}_y}{2\gamma S_3 b_w \sqrt{\overline{f'_c}/\gamma}} - 4 \leq 0, \tag{6.4.30}$$

$$\rho(\mathbf{x}) - 0.75\rho_b(\mathbf{x}) \leq 0, \tag{6.4.31}$$

where

$$\rho(\mathbf{x}) = \frac{A_s}{b(h_f + h_w - \alpha)}, \tag{6.4.32}$$

$$\rho_b(\mathbf{x}) = \frac{0.85^2 \overline{f'}_c}{\overline{f}_y} \frac{87000}{87000 + \overline{f}_y/\gamma}, \tag{6.4.33}$$

and

$$\rho_0 - \rho(\mathbf{x}) \leq 0, \tag{6.4.34}$$

with $\rho_0 = 200\gamma/\overline{f}_y$. The 28 inequalities in (6.4.5)-(6.4.34) define the constraint set $\mathbf{X}$ in (4.1.3). The reader should consult Lin and Frangopol (1996) regarding background information on the above constraints, which result from AASHTO (1992) rules.

The girder is considered a series structural system with four components defined as follows: The failure in flexure is specified by the limit-state function

$$G_1(\mathbf{x},\mathbf{v}) = 1 - \frac{M_L}{\omega(\mathbf{x},\mathbf{v})} - \frac{P_D L_g^2}{8\omega(\mathbf{x},\mathbf{v})} - \frac{(bh_f + b_w h_w)WL_g^2}{8\omega(\mathbf{x},\mathbf{v})}, \tag{6.4.35}$$

where

$$\omega(\mathbf{x},\mathbf{v}) = A_s f_y \left(h_f + h_w - \alpha - \frac{\eta(\mathbf{x},\mathbf{v})}{2} \right) \tag{6.4.36}$$

and $\eta(\mathbf{x},\mathbf{v}) = A_s f_y/(0.85 f_c' b)$. Failure in shear in interval 1 is defined by the limit-state function

$$G_2(\mathbf{x},\mathbf{v}) = 1 - \frac{P_{S1}}{\kappa_1(\mathbf{x},\mathbf{v})} - \frac{P_D L_g}{6\kappa_1(\mathbf{x},\mathbf{v})} - \frac{(bh_f + b_w h_w)WL_g}{6\kappa_1(\mathbf{x},\mathbf{v})}, \tag{6.4.37}$$

where $\kappa_1(\mathbf{x},\mathbf{v}) = 8.45 b_w (h_f + h_w - \alpha)\sqrt{f_c'/\gamma}/0.0254^2 + A_v f_y (h_f + h_w - \alpha)/S_1$, with all variables in SI units. Failure in shear in interval 2 is defined by the limit-state function

$$G_3(\mathbf{x},\mathbf{v}) = 1 - \frac{P_{S2}}{\kappa_2(\mathbf{x},\mathbf{v})} - \frac{P_D L_g}{3\kappa_2(\mathbf{x},\mathbf{v})} - \frac{(bh_f + b_w h_w)WL_g}{3\kappa_2(\mathbf{x},\mathbf{v})}, \tag{6.4.38}$$

where $\kappa_2(\mathbf{x},\mathbf{v}) = 8.45 b_w (h_f + h_w - \alpha)\sqrt{f_c'/\gamma}/0.0254^2 + A_v f_y (h_f + h_w - \alpha)/S_2$, with all variables in SI units. Failure in shear in interval 3 is defined by the limit-state function

$$G_4(\mathbf{x},\mathbf{v}) = 1 - \frac{P_{S3}}{\kappa_3(\mathbf{x},\mathbf{v})} - \frac{P_D L_g}{2\kappa_3(\mathbf{x},\mathbf{v})} - \frac{(bh_f + b_w h_w)WL)g}{2\kappa_3(\mathbf{x},\mathbf{v})}, \tag{6.4.39}$$

where $\kappa_3(\mathbf{x},\mathbf{v}) = 8.45 b_w (h_f + h_w - \alpha)\sqrt{f_c'/\gamma}/0.0254^2 + A_v f_y (h_f + h_w - \alpha)/S_3$, with all variables in SI units. The reader should consult Lin and Frangopol (1996) regarding background information on the above limit-state functions, which originate from AASHTO (1992) rules.

Case 2

Case 2 corresponds to the situation where the force in the tension reinforcement cannot be balanced by a compression force in the flange, i.e.,

$$0.85 f_c' b h_f \leq f_y A_s, \tag{6.4.40}$$

and the shear capacity in the shear reinforcement is less than or equal to a value related to the cross-section area and the strength of concrete, i.e., (6.4.4) holds. Hence, in Case 2, the constraint set $\mathbf{X}$ is defined by the inequality

$$0.85\overline{f'}_c b h_f - \overline{f}_y A_s \leq 0, \tag{6.4.41}$$

the inequalities (6.4.6)-(6.4.30), and the inequalities (6.4.31) and (6.4.34) but now with

$$\rho(\mathbf{x}) = \frac{A_s}{b_w(h_f + h_w - \alpha)}, \tag{6.4.42}$$

$$\rho_b(\mathbf{x}) = \left(\frac{0.85^2 \overline{f'}_c}{\overline{f}_y} \frac{87000}{87000 + \overline{f}_y/\gamma} + \frac{0.85\overline{f'}_c(b - b_w)h_f}{b_w \overline{f}_y (h_f + h_w - \alpha)} \right) \frac{b_w}{b}. \tag{6.4.43}$$

The limit-state functions are defined by (6.4.35), (6.4.37), (6.4.38) and (6.4.39), but now with

$$\omega(\mathbf{x}, \mathbf{v}) = f_y[A_s(h_f + h_w - \alpha - \eta(\mathbf{x}, \mathbf{v})/2) + \xi(\mathbf{x}, \mathbf{v})(\eta(\mathbf{x}, \mathbf{v}) - h_f)/2], \tag{6.4.44}$$

where $\xi(\mathbf{x}, \mathbf{v}) = 0.85 f'_c(b - b_w)h_f/f_y$ and $\eta(\mathbf{x}, \mathbf{v}) = (A_s - \xi(\mathbf{x}, \mathbf{v}))f_y/(0.85 f'_c b_w)$, all variables in SI units.

Case 3

Case 3 corresponds to the situation where the force in the tension reinforcement can be balanced by a compression force in the flange, i.e., (6.4.3) holds, and the shear capacity in the shear reinforcement is greater than or equal to a value related to the cross-section area and the strength of concrete, i.e.,

$$\frac{A_v f_y}{S_1} \geq 4\sqrt{\gamma f'_c} b_w, \tag{6.4.45}$$

Hence, in Case 3, the constraint set $\mathbf{X}$ is defined by the inequalities

$$-\frac{A_v \overline{f}_y}{S_1} + 4\sqrt{\gamma \overline{f'}_c} b_w \leq 0, \tag{6.4.46}$$

(6.4.5), the modified inequalities

$$S_1 - \frac{A_v \overline{f}_y}{100\gamma b_w} \leq 0, \tag{6.4.47}$$

$$S_1 - (h_f + h_w - \alpha)/4 \leq 0, \tag{6.4.48}$$

$$S_1 - 0.3048 \leq 0, \tag{6.4.49}$$

$$S_2 - \frac{A_v \overline{f}_y}{100\gamma b_w} \leq 0, \tag{6.4.50}$$

$$S_2 - (h_f + h_w - \alpha)/4 \leq 0, \tag{6.4.51}$$

$$S_2 - 0.3048 \leq 0, \tag{6.4.52}$$

$$S_3 - \frac{A_v \overline{f}_y}{100\gamma b_w} \leq 0, \tag{6.4.53}$$

$$S_3 - (h_f + h_w - \alpha)/4 \leq 0, \tag{6.4.54}$$

$$S_3 - 0.3048 \leq 0, \tag{6.4.55}$$

and the inequalities (6.4.16)-(6.4.34). The limit-state functions are given by (6.4.35)-(6.4.39).

Case 4

Case 4 corresponds to the situation where the force in the tension reinforcement cannot be balanced by a compression force in the flange, i.e., (6.4.40) holds, and the shear capacity in the shear reinforcement is greater than or equal to a value related to the cross-section area and the strength of concrete, i.e., (6.4.45) holds. Hence in Case 4, the constraint set $\mathbf{X}$ is given by the inequalities (6.4.41), (6.4.46)-(6.4.55), (6.4.16)-(6.4.30), (6.4.31) and (6.4.34), with $\rho(\mathbf{x})$ and $\rho_b(\mathbf{x})$ given by (6.4.42) and (6.4.43), respectively.

6.4.1 Design for Minimum Initial Cost ($\mathbf{P}_{1,\text{sys}}$)

Suppose that the objective is to minimize the material cost of the reinforced concrete girder subject to a constraint on the system failure probability, i.e, a design problem of the type $\mathbf{P}_{1,\text{sys}}$. Let $C_s = 50$ and $C_c = 1$ be the unit costs of steel reinforcement and concrete per cubic meter, respectively. As in Lin and Frangopol (1996), we define the objective function to be

$$c_0(\mathbf{x}) = 0.75 C_s L_g A_s + C_s n_S A_v (h_f + h_w - \alpha + 0.5 b_w) + C_c L_g (b h_f + b_w h_w), \tag{6.4.56}$$

Table 6.13: Results for $\mathbf{P}_{1,\text{sys}}$ design of reinforced concrete girder.

	Iteration 25
A_s	0.009832 m^2
b	0.418 m
h_f	0.415 m
b_w	0.196 m
h_w	0.785 m
A_v	0.0001859 m^2
S_1	0.508 m
S_2	0.224 m
S_3	0.140 m
$c_0(\mathbf{x}_i)$	13.664
$p(\mathbf{x}_i)$	0.001310

where $n_S = L_g(1/S_1 + 1/S_2 + 1/S_3)/3$ is the total number of stirrups. In (6.4.56), the first term represents the cost of the bending reinforcement. The factor 0.75 appears due to the assumption that the total amount of bending reinforcement is placed only within a length $L_g/2$ centered at the middle point of the girder, and the remaining part is reinforced with $0.5A_s$. The second and third terms in (6.4.56) represent the costs of shear reinforcement and concrete, respectively. Let the constraint on the system failure probability be $p(\mathbf{x}) \leq 0.001350$.

This problem is of type $\mathbf{P}^*_{1,\text{sys}}$, see (4.1.23), which we solve by using Algorithm 4.1.3 with Algorithm 3.3.2 and parameter $\gamma = 1$. Case 1 defined above yields the lowest cost, and the result for this case after 25 iterations of Algorithm 4.1.3 are given in Table 6.13, where the design vector $\mathbf{x}_i$, the objective $c_0(\mathbf{x}_i)$, and the system failure probability $p(\mathbf{x}_i)$ are listed. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01.

A direct comparison with Lin and Frangopol (1996) is not possible because of different assumptions regarding α (see Figure 6.3) and the fact that we have introduce additional constraints ((6.4.20), (6.4.21), (6.4.22) and (6.4.29)) to eliminate the possibility of an unrealistic geometric shape of the girder. However, the design in Table 6.13, with a cost of 13.6, appears to be better than the one reported in Lin and Frangopol (1996), which has a cost of 16.7.

Table 6.14: Results for $\mathbf{P}_{3,\text{sys}}$ design of reinforced concrete girder.

	Iteration 25
A_s	0.01157 m^2
b	0.492 m
h_f	0.415 m
b_w	0.196 m
h_w	0.785 m
A_v	0.0002266 m^2
S_1	0.502 m
S_2	0.226 m
S_3	0.142 m
a_{25}	0.0001799
Objective	17.017
$p(\mathbf{x}_{25})$	0.0001875

6.4.2 Design for Minimum Life-Cycle Cost ($\mathbf{P}_{3,\text{sys}}$)

Suppose that the objective is to minimize the initial cost plus the expected cost of failure of the reinforced concrete girder, subject to a constraint on the system failure probability, i.e, a design problem of the type $\mathbf{P}_{3,\text{sys}}$. Since we consider only one structure, we have $L = 1$ in the definition of $\mathbf{P}_{3,\text{sys}}$, see (4.2.2). Hence in the following we drop the superscript (l) when $L = 1$. Let $C_s = 50$ and $C_c = 1$ be the unit costs of the steel reinforcement and concrete per cubic meter, as before. We define the initial cost $c_0(\cdot)$ to be as in (6.4.56). The cost of failure is assumed to be $c(\mathbf{x}) = 500c_0(\mathbf{x})$. Let the constraint on the system failure probability $p(\mathbf{x}) \leq 0.001350$.

We solve $\mathbf{P}_{3,\text{sys}}$ by using Algorithm 4.2.5 with Algorithm 3.3.2. Case 1 defined above yields the lowest cost, and the result for this case after 25 iterations of Algorithm 4.2.5 is given in Table 6.14, where the design vector $\mathbf{x}_{25}$, the auxiliary design variable a_{25}, see (4.2.37), the objective $c_0(\mathbf{x}_{25})+c(\mathbf{x}_{25})p(\mathbf{x}_{25})$, and the system failure probability $p(\mathbf{x}_{25})$ are listed. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01.

We see in Table 6.14 that there is a relatively small discrepancy between a_{25} and $p(\mathbf{x}_{25})$. This is caused by the approximation in (4.2.41). As the computations progress in Algorithm 4.2.5, the parameters $\mathbf{t}$ are automatically modified in a way

that the error in the approximation in (4.2.41) is reduced. Note that the system failure probability constraint is not active in this example.

6.4.3 Design for Minimum Cost of Deteriorating Girder ($P_{3,sys}$)

Suppose that the girder is subject to corrosion of its longitudinal reinforcement. We adopt a corrosion model similar to that used in Frangopol *et al.* (1997), where the diameter $D_b(t)$ of a longitudinal reinforcement bar at time t is given by

$$D_b(t) \triangleq \begin{cases} D_{b0} - 2\nu(t - T_I), & t > T_I \\ D_{b0}, & \text{otherwise} \end{cases} \tag{6.4.57}$$

with D_{b0} being the initial diameter, ν being the corrosion rate, and T_I being the corrosion initiation time. The factor 2 in (6.4.57) takes into account that the reinforcement bar is subject to corrosion from all sides. We assume $T_I \triangleq A + Bc_a$, where A is a lognormal random variable with mean 5 years and c.o.v. equal to 0.20, representing the time it takes to initiate corrosion with a 10 mm concrete cover, B is a lognormal random variable with mean 300 years/m and c.o.v. equal to 0.20, representing the additional time it takes to initiate corrosion per meter additional concrete cover, and c_a is the concrete cover in meters in addition to the 10 mm minimum cover. The additional concrete cover c_a is considered a design variable and is included in the design vector $\mathbf{x}$, i.e.,

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3, c_a) \in \mathbb{R}^{10}, \tag{6.4.58}$$

We assume that the corrosion rate ν is lognormally distributed with mean $4.0 \cdot 10^{-5}$ m/years and coefficient of variation 0.30. All the random variables are assumed to be statistically independent, with the distribution parameters as listed in Table 6.15.

As seen from (6.4.57), the area of bending reinforcement is reduced over time. The remaining bending reinforcement area after time t is

$$A_s'(t) = n_b \pi D_b(t)^2/4 \tag{6.4.59}$$

Table 6.15: Statistics of lognormal random variables describing corrosion.

Variable	Mean	c.o.v.
A	5 years	0.20
B	300 years/m	0.20
ν	$4.0 \cdot 10^{-5}$ m/years	0.30

where n_b is the number of reinforcing bars and $D_b(t)$ is given in (6.4.57). We assume that the initial diameter of all the reinforcing bars is 0.025 m. Then, we obtain that

$$A_s'(t) = A_s R_c(t), \tag{6.4.60}$$

where the reduction factor

$$R_c(t) = 1 - 4\nu(t - T_I)/0.025 + 4\nu^2(t - T_I)^2/0.025^2. \tag{6.4.61}$$

The reinforced concrete girder is now a time-varying structure with A_s replaced by $A_s'(t)$ in the definitions of the limit-state functions $G_1(\cdot), G_2(\cdot), G_3(\cdot)$ and $G_4(\cdot)$, see (6.4.35), (6.4.37), (6.4.38) and (6.4.39), respectively. Hence, the reliability of the girder can be analyzed according to the framework in Section 2.2. Let $T_L = 50$ years be the lifetime of the girder. We assume the system failure probability in the time interval $[0, T_L]$ (see (2.2.6)) is equal to the point-in-time system failure probability at T_L, $p(\mathbf{x}, T_L)$ (see (2.2.3)), i.e., (2.2.11) holds. This results in an optimal design problem of the form $\mathbf{P}_{3,\text{sys}}$, with $L = 1$, where the system failure probability $p(\mathbf{x}) = p(\mathbf{x}, T_L)$, the initial cost is

$$c_0(\mathbf{x}) = 0.75 C_s L_g A_s + C_s n_S A_v (h_f + h_w - \alpha + 0.5 b_w) + C_c L_g (b h_f + b_w h_w) + C_c L_g b_w c_a, \tag{6.4.62}$$

the cost of failure is $c(\mathbf{x}) = 500 c_0(\mathbf{x})$, and the deterministic constraints defining $\mathbf{X}$ are as above with two changes. First, A_s is replaced by $\overline{A'}_s(T_L) = A_s \overline{R}_c(T_L)$ in (6.4.5), (6.4.19), (6.4.32), (6.4.41) and (6.4.42), where $\overline{R}_c(T_L)$ is equal to $R_c(T_L)$ with ν, A and B replaced by their respective mean values, see Table 6.15. Second, we also include the following two additional constraints bounding the new design variable c_a:

$$c_a/0.05 - 1 \leq 0, \tag{6.4.63}$$

Table 6.16: Results for $\mathbf{P}_{3,sys}$ design of deteriorating girder.

	Iteration 25
A_s	0.01460 m^2
b	0.511 m
h_f	0.415 m
b_w	0.196 m
h_w	0.785 m
A_v	0.0002291 m^2
S_1	0.502 m
S_2	0.226 m
S_3	0.142 m
c_a	0.050 m
a_{25}	0.0001869
Objective	19.712
$p(\mathbf{x}_{25})$	0.0001909

$$-c_a \leq 0. \tag{6.4.64}$$

The first of these constraints imposes an upper limit of 0.05 m on c_a.

We ignore the effect of the small additional load caused by the weight of the additional concrete cover. As above, let $C_s = 50$ and $C_c = 1$ be the unit costs of the reinforcement and concrete per cubic meter, respectively, and let the system failure probability constraint be $p(\mathbf{x}) \leq 0.001350$. We solve $\mathbf{P}_{3,sys}$ by using Algorithm 4.2.5 with Algorithm 3.3.2. Case 1 defined above yields the lowest cost, and the result for this case after 25 iterations of Algorithm 4.2.5 is given in Table 6.16, where the design vector $\mathbf{x}_{25}$, the auxiliary design variable a_{25}, the objective $c_0(\mathbf{x}_{25}) + c(\mathbf{x}_{25})p(\mathbf{x}_{25})$, and the system failure probability $p(\mathbf{x}_{25})$ are listed. The system failure probability is evaluated using Monte Carlo simulation with c.o.v. 0.01.

Similar to the situation in Sub-section 6.4.2, we see from Table 6.16 that there is a discrepancy between a_{25} and $p(\mathbf{x}_{25})$. This is caused by the approximation in (4.2.41). As the computations progress in Algorithm 4.2.5, the parameters $\mathbf{t}$ are modified in a way to reduce the error in the approximation in (4.2.41). Note that the constraint (6.4.63) is active, i.e., the use of maximum concrete cover is most cost efficient.

6.4.4 Design of Maintenance Plan for Deteriorating Girder ($\mathrm{P}_{3,\mathrm{sys}}$)

As can be seen from comparing Tables 6.14 and 6.16, the presence of the corrosion threat results in a significant increase in optimal bending reinforcement. Hence, it can be beneficial to consider maintaining the girder during its lifetime.

Suppose that the lifetime of the girder is $T_L = 60$ years, and that it is decided to maintain the structure in intervals of 20 years, i.e., at 20 and 40 years after its construction. The time of maintenance can be incorporated as a design variable, but in this example we have fixed those times for simplicity. Let $m_1 \in [0,1]$ and $m_2 \in [0,1]$ be two design variables characterizing the maintenance effort at 20 years and 40 years, respectively. Let $m_i = 0$ denote no maintenance, and $m_i = 1$ denote full maintenance, i.e., restoration to the state of the structure at the beginning of the i-th time interval. Furthermore, we consider $1 - m_1$ as the fraction of the aging of the structure from initial construction ($t = 0$) to the first maintenance action ($t = 20$ years), which is *not* restored to its initial condition. Thus, $40 - 20m_1$ years is the effective age of the structure before the second maintenance action at $t = 40$ years. Similarly, $1 - m_2$ is the fraction of the aging of the structure from the first maintenance action ($t = 20$ years) to the second maintenance action ($t = 40$ years), which is *not* mitigated by the second maintenance effort, i.e., $60 - 20m_1 - 20m_2$ years is the effective age of the structure at $t = T_L = 60$ years. We add the two variables m_1 and m_2 to the vector of design variables, i.e.,

$$\mathbf{x} = (A_s, b, h_f, b_w, h_w, A_v, S_1, S_2, S_3, c_a, m_1, m_2) \in \mathbb{R}^{12}. \tag{6.4.65}$$

We ensure the safety of the girder by imposing the constraints that the system failure probabilities in the three time intervals $[0, 20]$ years, $[20, 40]$ years, and $[40, 60]$ years are each less than 0.001350. As in Sub-section 6.4.3, we assume that the time-interval system failure probability can be approximated by the point-in-time system failure probability $p'(\mathbf{x}, T)$ at the end of the time interval T. To comply with the notation in the definition of $\mathrm{P}_{3,\mathrm{sys}}$, see (4.2.2), we think of the girder in the three time

intervals as "different" structures. Hence, we have the three probability constraints

$$p^{(1)}(\mathbf{x}) = p(\mathbf{x}, 20) \leq 0.001350, \tag{6.4.66}$$

$$p^{(2)}(\mathbf{x}) = p(\mathbf{x}, 40 - 20m_1) \leq 0.001350, \tag{6.4.67}$$

$$p^{(3)}(\mathbf{x}) = p(\mathbf{x}, 60 - 20m_1 - 20m_2) \leq 0.001350, \tag{6.4.68}$$

where, for any t, $p(\mathbf{x}, t)$ is as in Sub-section 6.4.3. The constraint in (6.4.66) limits the system failure probability of the 20-year-old structure at $t = 20$ years. The constraint in (6.4.67) limits the system failure probability of the effectively $(40 - 20m_1)$-year-old structure at $t = 40$ years. Finally, the constraint in (6.4.68) limits the system failure probability of the effectively $(60 - 20m_1 - 20m_2)$-year-old structure at $t = 60$ years.

The three "different" structures are subject to the same deterministic constraints defining $\mathbf{X}$ as in Sub-section 6.4.1 with the additional constraints in (6.4.63) and (6.4.64) and

$$m_j - 1 \leq 0, j = 1, 2, \tag{6.4.69}$$

$$-m_j \leq 0, j = 1, 2. \tag{6.4.70}$$

For the sake of this example, we construct the following cost-model. Let the initial cost of the structure be as in (6.4.62), the cost of maintenance be equal to $0.1 \exp(1/(1 - m_j))$, $j = 1, 2$, and the cost of failure be equal to the initial cost times 500 in each of the three time intervals. Hence in the notation of $\mathbf{P}_{3,\text{sys}}$, see (4.2.2), we have

$$c_0^{(1)}(\mathbf{x}) = 0.75C_sL_gA_s + C_sn_SA_v(h_f + h_w - \alpha + 0.5b_w) + C_cL_g(bh_f + b_wh_w) + C_cL_gb_wc_a, \tag{6.4.71}$$

$$c_0^{(1+j)}(\mathbf{x}) = 0.1e^{1/(1-m_j)}, j = 1, 2, \tag{6.4.72}$$

$$c^{(j)}(\mathbf{x}) = 500c_0^{(1)}(\mathbf{x}), j = 1, 2, 3. \tag{6.4.73}$$

As above, let $C_s = 50$ and $C_c = 1$ be the unit costs of the reinforcement and concrete per cubic meter, respectively. We solve $\mathbf{P}_{3,\text{sys}}$, with $L = 3$, by using Algorithm 4.2.5 with Algorithm 3.3.2. Case 1 defined above yields the lowest cost, and the result for this case after 25 iterations of Algorithm 4.2.5 is given in Table 6.17,

Table 6.17: Results for $\mathbf{P}_{3,\text{sys}}$ design of maintenance plan.

	Iteration 25
A_s	0.01320 m^2
b	0.561 m
h_f	0.415 m
b_w	0.211 m
h_w	0.785 m
A_v	0.0002582 m^2
S_1	0.544 m
S_2	0.236 m
S_3	0.145 m
c_a	0.050 m
m_1	0.4684
m_2	0.5316
Objective	21.121
$p^{(1)}(\mathbf{x}_{25})$	0.00004752
$p^{(2)}(\mathbf{x}_{25})$	0.00005945
$p^{(3)}(\mathbf{x}_{25})$	0.00010041

where the design vector $\mathbf{x}_{25}$, the objective $\sum_{l=1}^{3} c_0^{(l)}(\mathbf{x}_{25}) + \sum_{l=1}^{3} c^{(l)}(\mathbf{x}_{25}) p^{(l)}(\mathbf{x}_{25})$, and the system failure probabilities $p^{(l)}(\mathbf{x}_{25})$ are listed. The system failure probabilities are evaluated using Monte Carlo simulation with c.o.v. 0.01.

We observe from Table 6.17 that the system failure probability is smallest in the first time interval and largest in the last time interval. Hence, with the present cost-model, it is not cost efficient to maintain the initial safety level of the structure. This result is expected from the form of (6.4.72), where the cost of maintenance goes to infinite when the structure is restored to its initial state. We also see from Table 6.17 that it is most cost efficient to have a 12 percent more intense maintenance effort at 40 years than at 20 years.

Chapter 7

Conclusions

7.1 Summary of Major Findings

We have developed a collection of algorithms for solving three classes of reliability-based optimal structural design problems. The first class of problems is to minimize the cost of the design, subject to failure probability and structural constraints. The second class is to minimize the failure probability of the design, subject to cost and structural constraints. The third class of problems is to minimize the initial cost plus the expected cost of failure, subject to failure probability and structural constraints. The failure probabilities can describe component failures or series structural system failures.

In their original form, the three classes of problems appear to be intractable by any rigorous approach for at least two reasons. First, the failure probability is defined in terms of a high-dimensional integral over a domain that depends on the design variables. The failure probability function, or its approximations, is not known to be continuously differentiable, and hence standard nonlinear optimization algorithms are not applicable. Second, even if the failure probability were to be smooth, computationally it would be extremely expensive to approximately evaluate the failure probability and its gradient in the optimal design problem.

Based on a first-order approximation to the failure probability, we have con-

structed approximating problems that can be solved repeatedly to obtain an approximation to a solution of the original design problems. By the use of higher-order reliability methods in the iterative scheme, e.g., second-order or Monte Carlo simulation, the approximating solution can be made to satisfy failure probability constraints in the sense of any computational reliability method. The approximating problems are either semi-infinite optimization problems that can be solved using algorithms from the literature, or they are generalized semi-infinite optimization problems that can be solved using a newly developed algorithm.

The newly developed algorithm is based on exact penalties, which convert the generalized semi-infinite min-max problem into a finite family of semi-infinite min-max-min problems. Furthermore, the inner min-function is smoothed and the semi-infinite max part is approximated, using discretization, to obtain a three-parameter family of finite min-max problems. Under a calmness assumption, we have shown that when the penalty is sufficiently large the semi-infinite min-max-min problems have the same solutions as the original problem, and that when the smoothing and discretization parameters go to infinity the solutions of the finite min-max problems converge to solutions of the original problem, provided the penalty parameter is sufficiently large. The new algorithm combines tests for adjusting the penalty, the smoothing and the discretization parameters and makes use of a min-max algorithm as a subroutine. In effect, the min-max algorithm is applied to a sequence of gradually better-approximating min-max problems, with the penalty parameter eventually stopping to increase, but the smoothing and discretization parameters driven to infinity. The algorithm is found to perform well on a test example. The newly developed algorithm is applicable to a wide range of problems arising in engineering and elsewhere.

The approximating problems to the original optimal design problems are made dependent on a set of parameters that can be adjusted to improve the accuracy of the first-order approximations. The adjustment of the parameters is based on separate, approximate evaluation of the failure probability, including by means of Monte Carlo simulation. In special cases, we have shown that the approximating problems are

identical to the original ones. We have developed a set of parameter-adjustment rules that, together with subroutines for solving the approximating problems, composes the collection of new algorithms for solving reliability-based optimal structural design problems. Effectively, the new algorithms solve a sequence of first-order approximating problems that are constructed as the computations progress. It is observed that the parameter-adjustment rules are efficient tools for improving the accuracy of the first-order approximations.

A significant advantage of the new algorithms for solving reliability-based optimal structural design problems is that the failure probabilities are only evaluated as part of the parameter-adjustment rules. This gives the user flexibility in the selection of the method for approximately computing the failure probability. The first-order or second-order reliability method, Monte Carlo Simulation, or any other computational reliability method can be employed.

The report shows application of the new algorithms in a collection of numerical design examples from the field of structural engineering. The first example considers the design of a short column subject to axial forces and bending moments. Three cases were computed: Minimize the weight of the column subject to a constraint on the failure probability; minimize the failure probability subject to a constraint on the weight of the column; and minimize the initial cost plus the expected cost of failure with a constraint on the failure probability. The new algorithms converged to an approximate solution in few iterations. In fact in the second case, it was necessary to perform only one iteration to obtain an approximate solution of the original design problem. Hence, the first-order approximating problem is an excellent approximation of the original problem.

The second example considers the design of an offshore jacket platform for oil production subject to wave, wind and service loads. The platform is designed for minimum failure probability subject to a constraint on the total weight of the structure. As in the first example, the new algorithm converged in only one iteration.

The third example considers the design of a structural frame with three failure modes. We computed both the minimum weight with a system failure probability

constraint and the minimum initial cost plus expect cost of failure with a system failure probability constraint. It is observed that the new algorithms converge in few iterations, even for this case with a system failure probability constraint.

The fourth example considers a reinforced concrete girder in a highway bridge. We took four failure modes into account. First, we minimized the cost of the design subject to a system failure probability constraint. Second, we minimized the initial cost plus the expected cost of failure, with a constraint on the system failure probability. Third, we also included the effect of deterioration of the girder caused by corrosion. We minimized the initial cost plus the expected cost of failure based on a time-variant failure probability. Fourth, we found the initial design and the maintenance effort for the deteriorating girder, which together minimize the life-cycle cost. This example demonstrates that the new algorithms can also be used in maintenance planning and in cases with time-variant failure probabilities.

The reliability-based optimal structural design algorithms developed in this study present significant departures from the state-of-the-art. In particular, careful attention is given to the underlying assumptions and approximations to ensure a rigorous mathematical foundation for the algorithms. This, together with the fact that first-, or second-order reliability methods, Monte Carlo Simulation, or any other computational reliability method can be employed, makes the algorithms efficient, robust and versatile tools for solving reliability-based optimal structural design problems.

7.2 Further Studies

Typically, the behavior of real-life structures is modeled using ordinary (ODE) or partial (PDE) differential equations. Such equations are usually solved using numerical integration methods, such as the Runge-Kutta or Finite Element Methods. Hence, any objective or constraint function associated with ODEs or PDEs can be evaluated only approximately, with the mesh size acting as a precision parameter. In this report, we have assumed that the behavior of the structure can be evaluated exactly in finite computing time. Effectively, this means that we have assumed that

the structure is discretized in time and space prior to application of the optimization algorithms. Hence, we compute the optimal design of a discretized structure that may or may not represent the real structure accurately.

A more sophisticated way to deal with ODEs and PDEs is to incorporate the discretization into the optimization algorithms. The difficulty associated with such discretization of ODEs and PDEs can be managed, at least in principle, using the theory of consistent approximations, which provides a framework for adaptively increasing the precision as computations progress. This approach tremendously increases the utilization efficiency of appropriate numerical methods, as well as providing a means for establishing proof of convergence to a solution. However, in a complex structure, many integrations may be required, and the precision of each may have to be controlled independently. At present there is no multi-parameter adaptive precision scheme in the literature, and this will have to be developed to obtain high accuracy, efficient numerical methods for the solution of large classes of optimal design problems. A first effort to use consistent approximations in reliability analysis can be found in Royset *et al.* (2002).

Failure probabilities of structures are defined in terms of a high-dimensional integral over a domain that depends on the design variables. Such functions are not known to be continuously differentiable, and even if they are, the evaluation of the failure probabilities and their gradients will be extremely costly. In this report, a particular scheme is used to approximate the failure probability. The major disadvantages of this approximation scheme are its limitation to series structural systems and to reliability models with a constant or monotonic failure probability in time. Future efforts should focus on developing smooth approximations to the failure probability of general systems with gradually improved accuracy as precision parameters are driven to infinity. A preliminary study of optimization problems with non-smooth failure probability functions can be found in Polak and Royset (2002). Furthermore, efforts should be directed towards problems where the reliability is not monotonic in time, but is defined in terms of out-crossing rates.

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