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# When Are Nash Equilibria Independent of the Distribution of Agents' Characteristics?

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We present some examples of Nash equilibria that are independent of the distribution of some parameter across the economic agents and describe a general theorem that characterizes this phenomenon.

In several Nash games to be described below the "outcome" of the game turns out to be independent of the distribution of some characteristic across the agents. The easiest way to describe what we mean is through some examples.

*Example* 1. *Cournot Equilibrium*

Let $y_i$ be the output of firm $i$, $c_i$ its constant marginal cost, $Y$ industry output, and $P(Y)$ the industry price. Then the first order conditions for a Cournot-Nash equilibrium can be written as:

$$P(Y)+P'(Y)y_i-c_i=0.$$

Summing these equations across the $n$ firms we have:

$$nP(Y)+P'(Y)Y=\sum_{i=1}^{n} c_i.$$

Making the weak assumption that the left hand side of this equation is downward sloping, there will be a unique $Y$ that solves this equation, which depends only on the *sum* of the marginal costs, not on its distribution across the firms. The observation that output and price in a Cournot industry are independent of the distribution of marginal costs has undoubtedly been noted and used several times in the literature. See for example, Dixit and Stern (1982) or Katz (1984).

We can show the precise way that individual behaviour changes when the distribution of costs changes. Simply note that when costs change by $(\Delta c_i)$, then if each firm changes its output by:

$$\Delta y_i=\frac{\Delta c_i}{P'(Y)}$$

it will continue to satisfy the appropriate first order conditions, assuming of course, that $y_i+\Delta y_i>0$, i.e., that we maintain an interior solution.

Note that the same independence result holds in any conjectural variations model, as long as all firms have the *same* conjectural variation and we consider only interior equilibria.

*Example* 2. *"Oil" igopoly*

Loury (1983) has presented an interesting model of oligopoly involving exhaustible resources. Suppose that we have $n$ firms that each own some amount $R_i$ of an exhaustible resource. Each firm acts as a Nash competitor. It turns out that in this model, if each firm supplies the resource in all of the periods, then the entire time path of the market price is independent of the distribution of $R_i$ across the firms. Loury (1983) establishes this result in a continuous time model and investigates the conditions under which this "interior" equilibrium will occur. Here we provide a somewhat different proof of his result in a discrete time model, and extend Loury's results by showing precisely which sorts of redistribution will result in an interior equilibrium and how each individual firm's behaviour will change.

Assuming zero extraction costs and an interior solution, the first order conditions for firm $i$ are:

$$\alpha^t[P(Y^t)+P'(Y^t)y_i^t]=P(Y^0)+P'(Y^0)(R_i-\textstyle\sum_{t=1}^T y_i^t) \quad \text{for } t=1,\ldots,T.$$

These conditions state that the discounted marginal revenue in each time period must equal the marginal revenue in period 0. Summing these equations over all firms for each time period $t$ we have:

$$\alpha^t[nP(Y^t)+P'(Y^t)Y^t]=nP(Y^0)+P'(Y^0)[\textstyle\sum_{i=1}^n R_i-\sum_{t=1}^T Y^t].$$

Under appropriate monotonicity assumptions, there will be a unique $Y^t$ that solves this equation for each $t$, and it depends only on the total endowment of resources.

We can also show how the output of each firm changes when the distribution of resources changes but the analysis is not quite as straightforward as in the previous example. Let $(\Delta R_i)$ be a change in resource ownership that keeps the total amount of the resource fixed, and let $(\Delta y_i^t)$ be the associated change in firm i's output in period $t$. Then in order to satisfy the first order conditions given above we must have:

$$\alpha^t[P(Y^t)+P'(Y^t)(y_i^t+\Delta y_i^t)]=P(Y^0)+P'(Y^0)(y_i^0+\Delta y_i^0) \quad \text{for } t=1,\ldots,T.$$

Solving for $\Delta y_i^t$ as a function of $\Delta y_0^t$ gives us:

$$\Delta y_i^t=P'(Y^0)\Delta y_i^0/\alpha^t P'(Y^t) \quad \text{for } t=1,\ldots,T.$$

Now sum these expressions for $\Delta y_i^t$ over $t=0,\ldots,T$ and set the result equal to $\Delta R_i$. Solving for $\Delta y_i^0$ gives us:

$$\Delta y_i^0=\Delta R_i/[1+P'(Y^0)]\textstyle\sum_{t=1}^T[1/\alpha^t P'(Y^t)]$$

Note that these equations are linear in $\Delta R_i$. Thus we can easily choose values of $\Delta R_i$ such that $y_i^t+\Delta y_i^t>0$ for $t=1,\ldots,T$ and $i=1,\ldots,n$ and $\sum_1^n \Delta R_i=0$.

*Example* 3. *Private provision of a public good*

Warr (1983) has recently presented an interesting neutrality result which has been extended in various directions by Bergstrom, Blume, and Varian (1983). Here we describe Warr's original result, using the setup of the latter paper. Suppose that each agent's utility depends on his private consumption $x_i$ and some public good, $G$. The amount of the public good is determined by private donations; each agent has some initial wealth level $w_i$ which he uses to finance his gift and his private consumption. Letting $g_i$ be the gift of agent $i$, $G$ be the sum of the gifts, and $G_{-i}$ be the sum of the gifts of all agents except for agent $i$ we can pose agent $i$'s maximization problem as:

$$\max u_i(x_i, g_i + G_{-i})$$
$$\text{s.t. } x_i + g_i = w_i$$
$$g_i \geqq 0$$

It follows from the structure of this maximization problem that an individual's optimal gift will take the form:

$$g_i = f_i(w_i + G_{-i}) - G_{-i}$$

where $f_i(\cdot)$ is the individual's unconstrained demand for the public good as a function of wealth.

Warr (1983) showed that in this sort of model redistributions of wealth among agents who actually contributed to the public good would not change the equilibrium level of the public good. It turns out that each dollar given to or taken from an individual results in a one-for-one increase or reduction in his or her contribution to the public good in equilibrium. Since the total dollars taken equal the total dollars given in a wealth redistribution, the aggregate amount of the public good will remain unchanged.

Kemp (1983) extended Warr's argument to the case of multiple public goods. Bergstrom, Blume, and Varian (1983) provide a somewhat different proof and examine other sorts of comparative statics in this class of models.

## A GENERAL RESULT

The examples given above are all members of a general class of Nash equilibria which we characterize in the following theorem.

**Theorem.** *Let $x_i$ be agent $i$'s choice, $X$ the sum of the choices, $c_i$ some characteristic of agent $i$, and $C$ the sum of the characteristics. Assume that the Nash equilibrium values of $x_i$ can be expressed as the solution to $n$ equations of the form:*

$$x_i = f_i(c_i, X) \quad \text{for } i = 1, \ldots, n$$

*where each $f_i$ is a continuous function. Then a sufficient condition for the equilibrium value of $X$ to be independent of the distribution of the agents' characteristics is that each $f_i$ has the form:*

$$f_i(c_i, X) = a_i(X) + b(X)c_i.$$

*If $n \geqq 3$ this is also necessary. Thus a change $(\Delta c_i)$ that keeps $C$ constant will result in an equilibrium response of $\Delta x_i = -b(X)\Delta c_i$.*

*Proof.* Sufficiency is obvious. For notational convenience we will prove necessity in the case of $n = 3$; the extension to larger $n$ follows trivially.

Fix $C$ and sum the equations to get:

$$X = f_1(c_1, X) + f_2(c_2, X) + f_3(C - c_1 - c_2, X).$$

As $c_1$ and $c_2$ change, $c_3$ adjusts to keep $C$ constant. By hypothesis, since $C$ is fixed, $X$ must be fixed. Hence this can be viewed as an equation in $c_1$ and $c_2$ alone. An equation of this form is known as a Pexider functional equation (Aczel (1966), p. 141). The only continuous solution to such an equation is of the form:

$$f_i(c_i, X) = a_i(X) + b(X)c_i$$

which is what we wanted to show. ||

It is not hard to see that the Cournot examples are of the required form. The public good example is a bit more obscure. Take the expression for $g_i$ and add $G_{-i}$ to each side to get:

$$G = f_i(w_i + G_{-i}).$$

Let $\phi_i(\cdot)$ be the inverse of $f_i(\cdot)$. Applying this inverse to each side of the above expression and subtracting $G$ from each side of the result gives us:

$$g_i = w_i + G - \phi_i(G)$$

which clearly has the required form. This example shows that some rearrangement of the "natural" equilibrium conditions may be necessary in order to apply our result.

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