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Mass Customization: Theories and Application

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Mass Customization: Theories and Application

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by

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ABSTRACT OF THE DISSERTATION

Mass Customization: Theories and Application

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ALI FATTAHI

Management - PhD

University of California, Los Angeles, 2019

Professor Reza H Ahmadi, Chair

As a general trend, firms are moving towards mass customization by allowing customers to configure products by selecting among options (features), e.g. automotive industry, consumer electronics, computers, furniture, and aircraft. This dissertation is motivated by the problems faced by a global auto manufacturer that offers 100-500 options for a car. These options can be combined in different ways resulting in 10^{25}-10^{40} different configurations (end-products). Since it is impossible to forecast the demand for configurations, firms forecast options' demand.

In this dissertation, we study three major problems. First, the current forecasting approach ignores the relationships between options and, as a result, the forecasts are frequently incorrect (inconsistent), which results in excess inventories, shortages, and customer dissatisfaction. We present an effective approach that verifies consistency of the forecast and finds the best consistent forecast in the case of inconsistency. The second problem is to determine how many units of each part is required over the planning horizon, known as parts' capacity planning problem. The firm signs contracts with parts' suppliers based on the predicted parts' capacities. We present a methodology for predicting parts' capacities. Last, we generalize and extend the methodology developed in the first problem and introduce a new variation of the Non-Negative Least-Squares (NNLS) problem that is defined as finding the Euclidean distance to a convex cone generated by a set of discrete points. Our new variation considers the cases where the discrete points are implicitly known and there are an exponentially large number of them. We present an effective approach for solving this new variation of NNLS, design a lower bound, and establish the convergence rate of the lower and upper bounds.

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2019

DEDICATION

For my dear mother, Golestan Fattahi

and

my dear father, Avaz Fattahi

and

my beloved wife, Yasaman Ahmadabadi.

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2. Fattahi, A, Dasu, S, Ahmadi, R (2019) The Weighted Non-Negative Least-Squares Problem with Implicitly Characterized Points. ***Operations Research***. Accepted.
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Chapter 1

Introduction

Firms in several industries including automotive, consumer electronics, aircraft, and furniture allow customers to configure products by selecting options. This dissertation is motivated by the problems faced by a global auto manufacturer (GAM) that offers 100-500 options for a car. A *configuration (end-product)* is comprised of a collection of options that satisfy *engineering (technological) constrains*—e.g., some options are mutually incompatible while, in other instances, selection of an option may require selection of another option. There are usually $10^{25} - 10^{40}$ different configurations for a car model.

Since it is impossible to forecast the demand at the level of configurations, firms forecast options' demand. In Chapter 2, we note that when firms forecast options' demand, they fail to consider all *engineering constraints* that define relationships between options, and as a result, the forecasts are frequently *inconsistent*. This has resulted in significant excess inventories, shortages, and customer dissatisfaction in the GAM. We study the problem of finding the best consistent forecast. This is a new and important problem. We formulate this problem as finding a point in the convex cone of feasible configurations that has the minimum Euclidean distance to a given forecast. Finding feasible configurations is itself a hard problem. We present an approach that sequentially constructs the convex cone and stops when it finds the best consistent forecast. We analyze the theoretical properties of our approach and establish its convergence rate. We apply our approach to a set of real instances from the GAM and observe that it effectively identifies inconsistent forecasts and finds the best consistent forecasts.

Firms use demand forecast to determine parts' capacities—i.e., the number of units of each part that is required over the planning horizon to satisfy demand. The GAM signs contracts with suppliers based on parts requirements. The total parts' cost in the GAM is around $90 billion per year and hence errors in parts' capacities can result in significant wastage. As stated above, due to the very large number of end products, the firm can only forecast demand at the option level. The challenge is that parts' requirement cannot be directly determined based on options' forecast since a large number of parts' requirements (up to 60%) is based on the combinations of options selected. The problem of determining parts' requirement in the context of mass customization is new and challenging. In Chapter 3, we study this problem and show that the current approach used by the GAM has a *mean absolute percentage error* of 35.9%. We develop an effective approach for solving large industrial instances of this problem and compare our approach to that of the current practice.

In Chapter 4, we introduce a new variation of the *Non-Negative Least-Squares* (NNLS) problem that is defined as finding the Euclidean distance to a convex cone generated by a set of discrete points. Existing works in the literature assume that the set of discrete points are *explicitly known*. In the new variation, the discrete points are *implicitly known* and there are an *exponentially large* number of them—e.g., the feasible solutions of an integer program. We design an effective solution approach for this new problem, present a lower bound, and establish the convergence rate of the lower and upper bounds. Chapter 4 in fact generalizes and extends the theory developed in Chapter 2 and it can have applications in manufacturing, machine learning, clustering, pattern recognition, and high-dimensional statistics.

Chapter 2

Forecasting Options' Penetration Rates Problem

2.1 Introduction

This chapter is based on the problems faced by large auto manufacturers (LAMs) that allow customers to configure their cars on the Internet. Automobiles consist of a number of modules such as engines, interiors, and suspensions, and each module has a number of different variants or options. A customer configures his or her product by choosing options that are compatible. The number of valid configurations can be extremely large. For example, the number of valid configurations for the Mercedes C-Class is in the order of 10^{21} (Kulber et al., 2010). Manufacturers need demand forecasts at the level of configurations for production planning, supplier contracts, and pricing decisions. Configurations drive the bill of materials. Unfortunately, the number of potential configurations and the difficulty in enumerating feasible configurations make it impossible to forecast configuration demand.

Most firms forecast at the option level. It is common practice to present this forecast in terms of a penetration statistic (PS). A PS consists of an assigned penetration rate to each option that is simply the fraction of cars that they believe will have that option over the planning horizon. In other words, if the penetration rate of an engine is 0.2, it is forecast that 20% of cars sold will have that engine. PSs are used to plan inventories and contract with suppliers. Errors in PSs can be extremely costly since inaccurate PS forecasts can result in excess inventories, shortages, and customer dissatisfaction.

Valid configurations conform to several restrictions between options that are called *rules* and represent marketing, manufacturing, process, and engineering constraints that allow products to be producible. The total demand of configurations and hence the forecast PS must satisfy these rules. A major drawback of forecasting PSs is the difficulty in ensuring their consistency with the rules. As a consequence, forecast PSs are frequently infeasible. LAMs face the major challenge of determining the feasibility of the forecast PS and modifying it in the case of infeasibility.

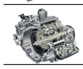
Rules can be divided into two general categories: *Family Cardinality Rules* (FCRs), which require selection of exactly/at most one option from a family of options, and *Option Implication Rules* (OIRs), which consist of logical relationships among options from different families. The following example illustrates these terms.

EXAMPLE 1. Features and specifications of the 2016 Hyundai Tucson[1] are grouped into families such as mechanical, exterior, and interior features. The four trim packages of this car are the SE, the Eco, the Sport, and the Limited. Fig. 2.1 compares the engine, transmission, and wheels of these trims. The SE's engine and transmission are `ENG1` and `TRN1`, respectively, while the Eco, Sport, and Limited's engine and transmission are `ENG2` and `TRN2`, respectively. The wheels of the SE and Eco are `WHL1`, and the wheels of the Sport and Limited are `WHL2`. Note that, in the Sport and Limited, the engine, transmission, and wheels are identical.

Assume for the sake of illustration that the 2016 Hyundai Tucson has only six options: `ENG1, ENG2, TRN1, TRN2, WHL1`, and `WHL2`. As shown in Fig. 2.1, there exist three FCRs, and two OIRs; exactly one option must be chosen from each family. According to the rule `ENG1`$\Longleftrightarrow$`TRN1`, `ENG1` is selected if and only if `TRN1` is selected. Also, `ENG1`$\Longrightarrow$`WHL1` means that if `ENG1` is chosen, then `WHL1` must be chosen as well. Note that engine FCR is also

[1]See hyundaiusa/tucson website for complete details. Notations: `ENG1` = Inline 4-cylinder, `ENG2` = Inline 4-cylinder turbocharged, `TRN1` = 6-speed automatic transmission with SHIFTRONIC, `TRN2` = 7-speed EcoShift Dual Clutch Transmission, `WHL1` = 17-inch alloy wheels with 225/60HR17 tires, and `WHL2` = 19-inch Sport alloy wheels with 245/45HR19 tires.

2016 HYUNDAI TUCSON
FEATURES & SPECIFICATIONS

		SE	ECO	SPORT/Limited
	ENGINE	ENG1	ENG2	ENG2
	TRANSMISSION	TRN1	TRN2	TRN2
	WHEELS	WHL1	WHL1	WHL2

FAMILIES: (exactly one option from each family)
Engine Family: {ENG1,ENG2}
Transmission Family: {TRN1,TRN2}
Wheels Family: {WHL1,WHL2}

OPTION IMPLICATION RULES:
ENG1⟺TRN1
ENG1⟹WHL1

Figure 2.1: A simplified version of the features and specifications of the 2016 Hyundai Tucson.

equivalent to ENG1⟺ ¬ ENG2. The number of OIRs may vary based on the format—e.g., one could write ENG1⟺TRN1 or as two OIRs ENG1⟸TRN1 and ENG1⟹TRN1.

Configurations that satisfy the FCRs and OIRs are: SE={ENG1, TRN1, WHL1}, ECO={ENG2, TRN2, WHL1}, and SPORT/LIMITED={ENG2, TRN2, WHL2}. The number of candidates for configurations is 2^6, while only three feasible configurations exist.

Suppose the rates of ENG1, ENG2, TRN1, TRN2, WHL1, and WHL2 are forecast as 0.6, 0.4, 0.6, 0.4, 0.3, and 0.7, respectively, meaning that 60% of 2016 Hyundai Tucson sales will have ENG1, 40% will have ENG2, and so on. These rates satisfy the FCRs (e.g., the rates of ENG1 and ENG2 add up to 1) but they are infeasible as they do not satisfy the OIRs. Due to the rule ENG1⟹WHL1, any configuration that includes ENG1 must also include WHL1; however, a configuration might have WHL1 but not ENG1. Thus, the rate of WHL1 must be greater than or equal to the rate of ENG1.

Consider Table 2.1 and assume that Hyundai has forecast a total of 1000 Tucsons to be sold in 2016 and procures engines, transmissions, and wheels accordingly (third row of Table 2.1). Although Hyundai has 600 ENG1 in inventory, they can use at most 300 ENG1 since there are only 300 WHL1 available. Unused quantities are found by subtracting the fourth

Table 2.1: Unused inventory and lost sales as a result of infeasible forecasting (total sale = 1000)

	ENG1	ENG2	TRN1	TRN2	WHL1	WHL2
Forecast PS	0.6	0.4	0.6	0.4	0.3	0.7
Inventory	600	400	600	400	300	700
Maximum usable quantities	300	400	300	400	300	400
Unused quantities	300	0	300	0	0	300
Shortage (lost sale)	$1000-(300+400)=300$					

row from the third row, and shortages are calculated by noting that they can sell at most 300 SE and 400 SPORT/LIMITED. Therefore, Hyundai incurs significant inventory costs due to unused inventory and lost profits as a result of lost sales.

Checking if a PS is feasible and finding the best feasible PS in the case of infeasibility is very important not only in the auto industry but also in any company producing configurable products. The LAM that we have been in contact with has around 400 options and 4000 rules for each car and the company's production planning group currently employs a manual process to detect some of the rule violations. Once a set of violations is detected, they send an error report to the regional marketing analysts who then seek appropriate modifications to the PS. These cycles are repeated until all evident violations are eliminated. However, this approach does not ensure that the PS is feasible. The current approach is time-consuming and fails to generate good alternatives when the forecast is infeasible.

In this chapter, we propose a formulation where the feasible region is the convex cone of feasible configurations and the objective function minimizes the Euclidean distance to the forecast PS. We develop an approach, by adapting the "fully corrective" variant of the Frank-Wolfe method (see, for example, Bach (2013); Clarkson (2010), and Jaggi (2013)), that sequentially constructs the feasible region. Our algorithm stops when it either finds a set of feasible configurations that are consistent with the given PS or when it finds the closest feasible PS, if the given PS is infeasible. Although we require solving an NP-hard subproblem at each iteration, we show that the optimality gap after k iterations will decrease with $\mathcal{O}(1/\sqrt{k})$. We present an upper bound on the convergence rate, under some mild assumptions, and establish the tightness of the bound. In addition, we show the connection between our subproblem and the maximum weighted satisfiability problem (Wahlstrom, 2008; Dahllof et al., 2005) and use existing heuristics to solve the subproblems. We test our methodology on a set of industrial instances and obtain very good solutions in a short time.

The remainder of this chapter is organized as follows. Section 2.2 presents related literature. Section 2.3 introduces our mathematical model and an alternative format for presenting rules. In section 2.4, we present our algorithm and discuss its theoretical properties. In section 2.5, we test the effectiveness of our algorithm and show its sensitivity to the number of options and rules. This is followed by conclusions.

2.2 Literature

In addition to the automotive industry, a number of others, including consumer electronics, computers, furniture, and aircraft (Feitzinger and Lee, 1997; Fohn et al., 1995; Kristianto et al., 2015; Rodriguez and Aydin, 2011) allow customers to configure products by selecting among options. While the use of modular design techniques and option-based product architecture (Ulrich, 1994; Siddique et al., 1998; Sanchez and Mahoney, 1996) increases the variety that can be made available, it also presents a number of challenges for customers (Franke and Piller, 2004; Huffman and Kahn, 1998; Chen and Wang, 2010; Walker and Bright, 2013) and producers (Woehler, 2011; Ostrosi et al., 2012).

Researchers have studied ways to price different options (Rodriguez and Aydin, 2011) and present assortments to consumers to learn about demand (Caro and Gallien, 2007; Balseiro et al., 2014; Kok et al., 2008). In the assortment planning literature, finished goods are presented to consumers and their choices are used to infer the relative attractiveness of different variants. Much of this literature is concerned with digital goods such as Internet advertisements where inventory of goods is not an issue. Assortment planning models have also been developed for retailers, but the focus is on the set of products to be presented to the customer. Our work is concerned with developing demand forecasts for configurable products. In this setting, parts needed for the configurable products have to be produced in advance or at least the production facilities and supply contracts have to be set up well ahead of demand realization. As previously stated, these decisions are based on PSs and not forecasts for individual configurations. We are concerned with determining whether the forecast PS is consistent with the rules that bind options. We develop a mathematical programming-based model, present an algorithm, and provide insights into its convergence rate.

The current format of the rules used by the LAM has been set up to facilitate activities such as design and manufacturing. Work by Barker et al. (1989); Roller and Kreuz (2003), and Amilhastre et al. (2002) is representative of the studies on representing rules related to configurable products. This literature is concerned with facilitating activities such as product design, engineering of production lines, and managing knowledge while we are concerned with exploring whether a forecast PS is consistent with the given rules. We represent rules in a manner that is simpler and more intuitive to mathematical programming.

At the core of our problem is a variant of the well-studied probabilistic satisfiability problem (PSAT) which is known to be NP-complete (Nilsson, 1994; Finger and De Bona, 2011; Georgakopoulos et al., 1988). PSAT consists of a set of clauses where each clause is the disjunction of one or more literals. A literal is either a Boolean variable or its negation. Given an assignment of probabilities to clauses, the question is to check the consistency of the given probabilities. PSAT is typically solved using a column generation technique (Kavvadias and Papadimitriou, 1990; Hansen et al., 1995) that converges slowly and there is no theoretical convergence rate guarantee. In our problem, rules can be written as a set of clauses and a penetration rate can be thought of as an assigned probability to a clause with one literal. Moreover, in our case, rules are all assigned probability 1, which simplifies the definition of the feasible set and allows us to design a specialized methodology to solve this problem. The approach we propose is similar to the Frank-Wolfe method that was originally developed to maximize a smooth concave function over a polytope (Frank and Wolfe, 1956) and then extended to more general problems (see, for example, Demyanov and Rubinov (1970)). Freund and Grigas (2014) discuss computational guarantees for various step-size rules within this algorithm. We apply this classical approach in a novel way to a problem with a non-convex objective function and a non-compact domain. We show that our algorithm has a faster running time on this more complicated problem. Additionally, we provide a lower bound on the optimum objective function value allowing the procedure to stop with a known optimality gap.

2.3 Problem Formulation

Given a set of options, $N = \{i | i = 1, \cdots, n\}$, with associated forecast PS, $\hat{\mathbf{p}} = (\hat{p}_1, \cdots, \hat{p}_n) \in \mathbb{R}^n$, and a set of rules, we have to determine whether $\hat{\mathbf{p}}$ satisfies the rules and, if not, find the best feasible PS (for the complete list of notations, see Appendix 2.8).

We next argue that our problem can be formulated as $\min_{\mathbf{p} \in \mathbb{A}} \|\mathbf{p} - \hat{\mathbf{p}}\|$, where $\|.\|$ denotes the Euclidean norm, the feasible set $\mathbb{A}$ can be expressed as $cone(\mathbb{P}) := \{\lambda \mathbf{p} | \mathbf{p} \in \mathbb{P}, \lambda \geq 0\}$, and $\mathbb{P}$ is the convex hull of feasible configurations (0-1 vectors). In this chapter, we assume $\mathbb{P} \neq \{\}$ and $\mathbb{P} \neq \{0\}$—i.e., there exists a nonzero feasible configuration. A point in $cone(\mathbb{P})$ is producible (feasible) since it can be represented as a nonnegative combination of feasible configurations. Therefore, a given penetration statistic $\mathbf{p} \in \mathbb{R}^n$ is feasible if and only if $\mathbf{p} \in cone(\mathbb{P})$. We note that, in comparison to PSAT, we use the simplification that all rules must be satisfied with probability 1, and hence, $\mathbb{P}$ becomes the convex hull of feasible configurations (i.e., we do not need to consider probabilities when characterizing $\mathbb{P}$).

We next show that Euclidean distance is the right objective measure to be considered. Let $\mathbf{q}_1, \cdots, \mathbf{q}_M \in \mathbb{R}^n$ denote a collection of forecasts from different entities within the LAM (this can include historical data). We define function $f : \mathbb{R}^n \to \mathbb{R}$ as $f(\mathbf{p}) = \sum_{m=1}^{M} (\mathbf{p} -$

$\mathbf{q}_m)^T(\mathbf{p} - \mathbf{q}_m)$, which returns sum of squared errors for a given $\mathbf{p}$. The LAM's forecast $\hat{\mathbf{p}}$ is obtained by minimizing f over all $\mathbf{p} \in \mathbb{R}^n$ without requiring $\mathbf{p} \in cone(\mathbb{P})$. $\hat{\mathbf{p}}$ is viewed as a consensus forecast among participating entities. We remark that $\hat{\mathbf{p}}$ is provided to us by the LAM and can be infeasible because it has not been verified against the feasibility constraints. We augment the LAM's approach by incorporating the feasibility constraints. The following lemma shows that finding the nearest point (Euclidean distance) to $\hat{\mathbf{p}}$ in the set $cone(\mathbb{P})$ solves the original problem $\min_{\mathbf{p} \in cone(\mathbb{P})} f(\mathbf{p})$.

Lemma 1. *Problem* $\min_{\mathbf{p} \in cone(\mathbb{P})} \|\mathbf{p} - \hat{\mathbf{p}}\|$ *is equivalent to* $\min_{\mathbf{p} \in cone(\mathbb{P})} f(\mathbf{p})$.

The proof is given in Appendix 2.9.1. Therefore, if $\hat{\mathbf{p}}$ is infeasible, we have to find a feasible PS such that it has the minimum Euclidean distance to $\hat{\mathbf{p}}$. The result validates the LAM's desire to find the closest feasible PS if the forecast $\hat{\mathbf{p}}$ is infeasible. In summary, our problem can be formulated as $\min_{\mathbf{p} \in \mathbb{P}, \lambda \geq 0} \|\lambda \mathbf{p} - \hat{\mathbf{p}}\|$. In subsections 2.3.1 and 2.3.2, we present a format for representing rules and an alternative formulation of $\mathbb{P}$.

2.3.1 Rules for Selecting Options

The LAM's current format for representing OIRs and FCRs consists of a set of well-defined propositional formulas (for details, see Appendix 2.9.2). It is known that these rules can be written as formulae in *conjunctive (or disjunctive) normal form* (CNF or DNF) (see, for example, Tseitin (1968); Wilson (1990); Chandru and Hooker (1999), and Yan and Hooker (1999)). The LAM's format is quite complicated and not easily conducive to modeling and theoretical analysis. Although our methodology for solving our problem (presented in section 2.4) can be applied to rules in any format, we develop a convenient format called ASR.

Format ASR: *Each option, say option `OP01`, has rules either in the form of `OP01`$\Longrightarrow$`RHS`, where `RHS` is the disjunction of some literals, or in the form of `OP01`$\Longleftarrow$`RHS`, where `RHS` is the conjunction of some literals (a literal is an option or its negation).*

The benefits of ASR compared to a CNF formula are as follows: (1) the ASR format is more intuitive because each rule defines the consequences of selecting an option in a straightforward manner, (2) the simple structure of rules in ASR format makes it easier for managers to understand the relationships among options, (3) rules that are written in ASR format lend themselves to more intuitive LP formulations (see subsection 2.3.2). The following lemma establishes the sufficiency of ASR (an illustrative example is given in Appendix 2.9.3). The proof is given in Appendix 2.9.4

Lemma 2. *Format ASR is sufficient to represent the LAM's rules for selecting options.*

2.3.2 Linear Programming Model Using ASR Format

We employ an approach similar to that of the probability consistency problem (Bertsimas and Tsitsiklis, 1997) to present an alternative formulation of $\mathbb{P}$ as a linear program. Let S denote a subset of options and define $\mathbb{S} := \{S|S \subseteq N\}$; hence, $|\mathbb{S}| = 2^n$. We define variables $x(S) \in [0,1]$ for all $S \in \mathbb{S}$. Consider Fig. 2.2 for illustration. Options A, B, and C divide the universal set into 8 mutually exclusive and collectively exhaustive regions with probability values $x(.)$. The probability of some intersecting options is equivalent to the sum of all $x(S)$'s where S includes such options. With some abuse of notations, we denote by $p(E) \in [0,1]$ the probability of event E.

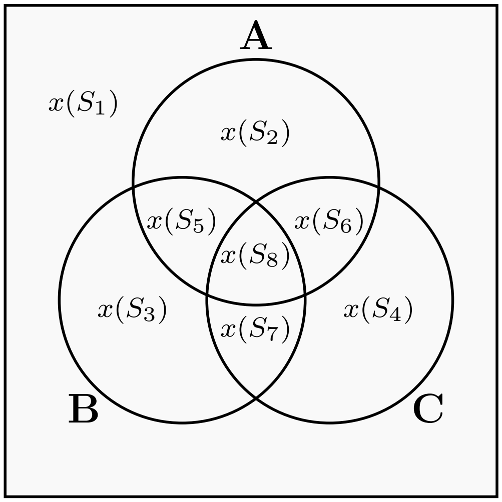
The probabilities of intersections are modeled as follows:

$$p\Big(\bigwedge_{j\in S} j\Big) = \sum_{\hat{S}:S\subset\hat{S}} x(\hat{S}), \qquad \forall S \in \mathbb{S}, \tag{2.1}$$

$$\sum_{S\in\mathbb{S}} x(S) = 1, \tag{2.2}$$

$$x(S) \geq 0, \qquad \forall S \in \mathbb{S}. \tag{2.3}$$

Using ASR format, there are two types of rules that we need to model. For the ease of presentation, in this subsection, we concentrate on cases where literals in ASR are options (not negations of options). Our results can simply be extended to the general case. Consider



$S_1 = \{\}$
$S_2 = \{A\}$
$S_3 = \{B\}$
$S_4 = \{C\}$
$S_5 = \{A, B\}$
$S_6 = \{A, C\}$
$S_7 = \{B, C\}$
$S_8 = \{A, B, C\}$
$p(A) = x_2 + x_5 + x_6 + x_8$
$p(B) = x_3 + x_5 + x_7 + x_8$
$p(C) = x_4 + x_6 + x_7 + x_8$
$p(A \wedge B) = x_5 + x_8$
$p(A \wedge C) = x_6 + x_8$
$p(B \wedge C) = x_7 + x_8$
$p(A \wedge B \wedge C) = x_8$

Figure 2.2: Graphical illustration: three options; eight subsets; probabilities of intersections. Note: e.g., $x_2 := x(S_2)$.

a rule in the form of $i \Rightarrow j_1 \vee \cdots \vee j_l$. This rule is formulated as $p(j_1 \vee \cdots \vee j_l | i) = 1$—i.e., given option i is selected, at least one of options $j_1, \cdots, j_l$ will be selected with probability 1. Using *Bayes theorem* and the *inclusion-exclusion* principle, $p(i) = p\big((i \wedge j_1) \vee (i \wedge j_2) \cdots \vee (i \wedge j_l)\big)$. Expanding the right-hand-side, we obtain:

$$\begin{aligned} p(i) - \sum_{m=1}^{l} p(i \wedge j_m) + \sum_{1 \le j_{m_1} < j_{m_2} \le l} p(i \wedge j_{m_1} \wedge j_{m_2}) - \sum_{1 \le j_{m_1} < j_{m_2} < j_{m_3} \le l} p(i \wedge j_{m_1} \wedge j_{m_2} \wedge j_{m_3}) \\ + \cdots + (-1)^{(l-1)} \sum_{1 \le j_{m_1} < \cdots < j_{m_{l-1}} \le l} p(i \wedge j_{m_1} \wedge \cdots \wedge j_{m_{l-1}}) \\ +(-1)^l p(i \wedge j_1 \wedge \cdots \wedge j_l) = 0, \qquad \forall \text{ rules} : i \Rightarrow j_1 \vee \cdots \vee j_l. \end{aligned} \tag{2.4}$$

Now consider a rule in the form of $i \Leftarrow j_1 \wedge \cdots \wedge j_l$. This rule is equivalent to $p(i | j_1 \wedge \cdots \wedge j_l) = 1$. Again, using *Bayes theorem*, we obtain:

$$p(j_1 \wedge \cdots \wedge j_l) - p(i \wedge j_1 \wedge \cdots \wedge j_l) = 0, \qquad \forall \text{ rules} : i \Leftarrow j_1 \wedge \cdots \wedge j_l. \tag{2.5}$$

Once rules are written using ASR format, Eqs. (2.4) and (2.5) are used to formulate them in terms of probabilities of intersections that are modeled in Eqs. (2.1), (2.2), and (2.3).

Thus, we formulate $\mathbb{P}$ as follows: $\mathbb{P} = \{\mathbf{p} \in \mathbb{R}^n | \ \mathbf{p}$ satisfies Eqs. (2.1), (2.2), (2.3), (2.4), and (2.5)$\}$, which is a polyhedron in the space of variables $\mathbf{p}$. In fact, $\mathbb{P}$ is the projection of some bigger polyhedron in the space of variables $\mathbf{p}$ and $\mathbf{x}$.

This formulation of $\mathbb{P}$ is intuitive because it is based on basic probability rules and Venn diagrams. Furthermore, due to the use of conditional probabilities, our formulation is easy to interpret. In existing literature, a feasible point of PSAT is represented as a convex combination of feasible configurations (for a detailed discussion, see Chandru and Hooker (1999)). We now show that our formulation of $\mathbb{P}$ is consistent with the classical formulation of PSAT. In particular, we show that there exists a 1-to-1 correspondence between the set of vertices of $\mathbb{P}$ and the set of feasible configurations. We begin with reformulating Eqs. (2.4) and (2.5).

Lemma 3. *Eq. (2.4) is equivalent to* $\sum_{\substack{S: i \in S, \\ \{j_1, \cdots, j_l\} \cap S = \emptyset}} x(S) = 0$*, for all rules* $i \Rightarrow j_1 \vee \cdots \vee j_l$*. Eq. (2.5) is equivalent to* $\sum_{\substack{S: \{j_1, \cdots, j_l\} \subset S \\ , i \notin S}} x(S) = 0$*, for all rules* $i \Leftarrow j_1 \wedge \cdots \wedge j_l$*.*

The first part follows from combining Eqs. (2.1) and (2.4) and the second part follows from combining Eqs. (2.1) and (2.5). We next show that the set $\mathbb{P}$ is a polyhedron with integer vertices. Let $V_{\mathbb{P}}$ be the set of vertices of $\mathbb{P}$. We define:

$$\begin{aligned} \mathbb{S}^0 := \big\{ S \in \mathbb{S} : \quad & \big(\exists \text{ rule } i \Leftarrow j_1 \wedge \cdots \wedge j_l \text{ s.t. } \{j_1, \cdots, j_l\} \subset S, i \notin S \big) \\ \vee \quad & \big(\exists \text{ rule } i \Rightarrow j_1 \vee \cdots \vee j_l \text{ s.t. } i \in S, \{j_1, \cdots, j_l\} \cap S = \emptyset \big) \big\}, \end{aligned}$$

and let $\mathbb{S}^1 := \mathbb{S} \backslash \mathbb{S}^0$.

Proposition 1. *The set* $\mathbb{P}$ *is a polyhedron with integer vertices and a one-to-one correspondence exists between* $\mathbb{S}^1$ *and* $V_{\mathbb{P}}$*. Moreover, for all* $S \in \mathbb{S}^1$*, there exists* $\mathbf{p} \in V_{\mathbb{P}}$ *such that* $p_i = 1$*, for all* $i \in S$*, and* $p_i = 0$*, for all* $i \notin S$*.*

The proof is given in Appendix 2.9.5. Let $\mathbb{Y}$ denote the set of feasible configurations. We formulate $\mathbb{Y}$ using binary variables and show that a one-to-one correspondence exists between

$\mathbb{Y}$, $\mathbb{S}^1$, and $V_{\mathbb{P}}$. Define binary variable y_i, which is 1 if option i is chosen and 0 otherwise. Hence, once rules are written using ASR format, they can be formulated as follows.[2]

$$y_i \leq y_{j_1} + y_{j_2} + \cdots + y_{j_l}, \qquad \forall \text{ rules} : i \Rightarrow j_1 \vee \cdots \vee j_l, \tag{2.6}$$

$$y_i + l - 1 \geq y_{j_1} + y_{j_2} + \cdots + y_{j_l}, \qquad \forall \text{ rules} : i \Leftarrow j_1 \wedge \cdots \wedge j_l, \tag{2.7}$$

$$y_i \in \{0, 1\}, \qquad \forall i \in N. \tag{2.8}$$

Hence, $\mathbb{Y} = \{\mathbf{y} = (y_1, \cdots, y_n) | \mathbf{y}$ satisfies Eqs. (2.6), (2.7), and (2.8)$\}$.

Corollary 1.1. *A one-to-one correspondence exists between $\mathbb{Y}$ and $V_{\mathbb{P}}$.*

The proof follows from noting that $\mathbb{S}^0$ is in fact the set of infeasible configurations. Appendix 2.9.6 provides an illustration of Proposition 1 and Corollary 1.1. In summary, in this subsection, we presented an alternative formulation of $\mathbb{P}$ as a linear program. Although the objective function of our problem is convex for a fixed λ, and the feasible region is a polyhedron, this problem is extremely difficult. In fact, it can be easily shown that our problem is NP-hard. Eqs (2.1) and (2.3), as well as the variable $x(.)$, are defined for all $S \in \mathbb{S}$ where $|\mathbb{S}| = 2^n$; hence, $\mathbb{P}$ is defined by more than 2^n variables and constraints. A possible approach to solving this problem without explicitly enumerating all variables and constraints is to use column generation (Kavvadias and Papadimitriou, 1990; Hansen et al., 1995), which has been the most popular technique for solving PSAT. Column generation consists of a master problem that is identical to the original problem but with only a small number of columns and a subproblem for determining the entering column. The subproblem can be expressed as optimizing a nonlinear 0-1 function and is solved using various techniques including algebraic methods, cutting-plane algorithms, enumerative algorithms and linearization methods (Hansen et al., 1993, 2000). An alternative is to use mixed integer packages such as CPLEX MIP (Hansen et al., 1999). Heuristics such as Tabu search can also

[2] Formulating logical propositions as linear constraints is also discussed in Yan and Hooker (1999) and Chandru and Hooker (1999).

be used (Glover, 1997). Finger and De Bona (2011) use column generation to solve PSAT problems with 200 variables, 100 probability assignments, and 800 disjunctive clauses, and Finger et al. (2013) study optimizing an objective function over all feasible solutions.

In general, column generation converges slowly, and existing literature does not include any theoretical bounds on the convergence rates. In the remainder, we present an algorithm and show that the optimality gap after k iterations will decrease with $\mathcal{O}(1/\sqrt{k})$. Each iteration runs very fast and our proposed algorithm finds very good solutions after a small number of iterations. We solve real instances of our problem that we have received from the LAM that has 400 options and 4000 rules in a reasonable computing time. Our algorithm, implemented on a personal computer, spends fewer than 5000 seconds to obtain solutions with a 1% error for real instances.

2.4 Solution Methodology

We present our solution approach for solving our problem by adapting the Frank-Wolfe (FW) method, also known as the *conditional gradient method* (Frank and Wolfe, 1956; Demyanov and Rubinov, 1970). At each iteration, the basic FW evaluates the gradient of the objective function at the current feasible point and maximizes the linear approximation of the objective function to find the next feasible point. Depending on the step-size, the method moves toward the obtained feasible point. The maximization subproblem at each iteration needs to be simpler than the original problem. The FW produces a sequence of feasible solutions. If the step-size is chosen appropriately, an $\mathcal{O}(\frac{1}{k})$ rate of convergence can be obtained (see, for example, Jaggi (2013) and Freund and Grigas (2014)).

Our approach adapts the "fully corrective" variant of the FW (FCFW) in which the iterates of the algorithm are chosen to be the best point in the convex combination of the points already found (Bach, 2013; Clarkson, 2010; Jaggi, 2013). The FCFW requires that the problem of finding the best point at each iteration be easy (see Yuan and Yan (2012) for heuristic approaches). In our approach, this step is performed in polynomial time.

The FW method can directly be applied to solve an optimization problem with a convex objective function over a compact convex domain (see, for example, Jaggi (2013)). Some researchers have recently extended the FW method to solve nonconvex optimization problems (Reddi et al., 2016; Lafond et al., 2015) and stochastic convex optimization problems (Hazan and Kale, 2012). We extend the application of the FW to solve a problem with a nonconvex objective function and noncompact domain (recall that our problem is $\min_{\mathbf{p}\in\mathbb{P},\lambda\geq 0} \|\lambda\mathbf{p}-\hat{\mathbf{p}}\|$). We do so by introducing a new problem with a convex objective function and show that the solution of the new problem provides the optimum solution to the original problem (see subsection 2.4.1).

The second complexity in our original problem is due to the unboundedness of the feasible region. Direct application of the FW algorithm can result in an unbounded solution during the maximization step. Although the feasible region continues to be unbounded in the new problem, in subsection 2.4.3, we show that the maximization step always produces a bounded solution.

In subsection 2.4.4 we derive a tight bound for the convergence rate when $\hat{\mathbf{p}}$ is feasible. Finally, in subsection 2.4.5, we provide a lower bound on the optimal value at each iteration, which helps us gauge the optimality gap.

2.4.1 Handling the Nonconvexity

Let $\mathcal{H}_{\hat{\mathbf{p}}} = \{\mathbf{p} \in \mathbb{R}^n | \hat{\mathbf{p}}^T\mathbf{p} = \hat{\mathbf{p}}^T\hat{\mathbf{p}}, \mathbf{p} \geq 0\}$—i.e., the nonnegative part of the hyperplane generated by the normal vector $\hat{\mathbf{p}}$ and the point $\hat{\mathbf{p}}$. We denote by $\mathcal{FH}_{\hat{\mathbf{p}}}$ the intersection of $\mathcal{H}_{\hat{\mathbf{p}}}$ and $cone(\mathbb{P})$—i.e., $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\mathbf{p} \in \mathcal{H}_{\hat{\mathbf{p}}} | \exists \bar{\mathbf{p}} \in \mathbb{P}, \lambda \geq 0 : \lambda\bar{\mathbf{p}} = \mathbf{p}\}$.

Let vectors $\mathbf{y}$ and θ denote feasible configurations and their images on $\mathcal{H}_{\hat{\mathbf{p}}}$, respectively. Note that, by "image," we mean $\theta = \lambda\mathbf{y} \in \mathcal{H}_{\hat{\mathbf{p}}}$, for some $\lambda \geq 0$. Fig. 2.3 provides an illustration using two scenarios. In scenario 1, the point $\theta =$"001" for instance, is the image of $\mathbf{y} = (0, 0, 1)$. Although factually $\theta = (0, 0, \frac{13}{12})$, for simplicity and demonstrating that this is the image of $\mathbf{y} = (0, 0, 1)$, we use the notation $\theta =$"001." An important difference between these scenarios is the boundedness of $\mathcal{H}_{\hat{\mathbf{p}}}$, which is a result of having/not having zero entries in $\hat{\mathbf{p}}$; in scenario 1, $\mathcal{H}_{\hat{\mathbf{p}}}$ is unbounded with an extreme direction $\mathbf{e}^1$, while $\mathcal{H}_{\hat{\mathbf{p}}}$ in scenario 2 is bounded. In the following lemmas, we present some of the interesting properties of the sets $\mathcal{H}_{\hat{\mathbf{p}}}$ and $\mathcal{FH}_{\hat{\mathbf{p}}}$ as well as the correspondence between the zero entries of $\hat{\mathbf{p}}$ and the extreme directions of the set $\mathcal{H}_{\hat{\mathbf{p}}}$. Define $I_\emptyset := \{i \in N | \hat{p}_i = 0\}$ and $I_1 := \{i \in N | \hat{p}_i > 0\}$.

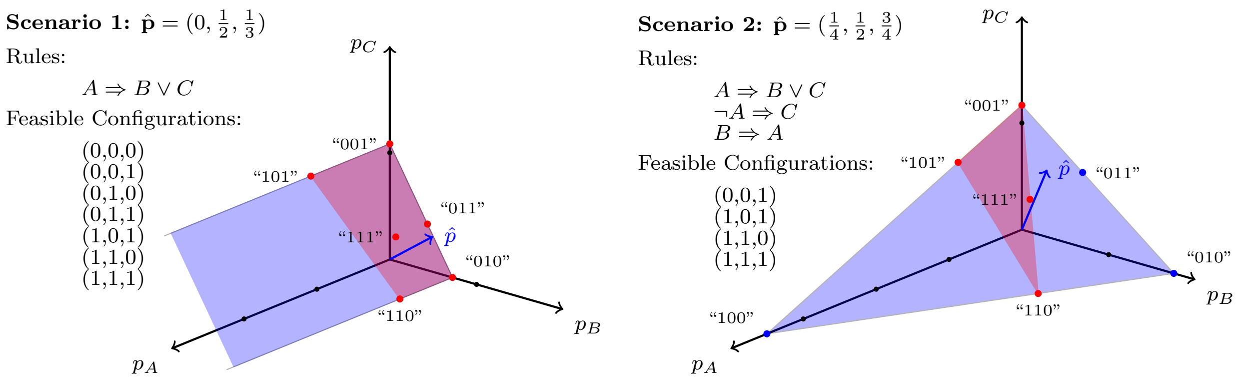


Figure 2.3: Graphical illustration of $\mathcal{H}_{\hat{\mathbf{p}}}$ and $\mathcal{FH}_{\hat{\mathbf{p}}}$.

Lemma 4. *If $\hat{\mathbf{p}} \in \mathbb{R}^n_{++}$, then $\mathcal{H}_{\hat{\mathbf{p}}}$ is a (n-1)-simplex with vertices $\frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_i}\mathbf{e}^i$, for all $i \in N$.*

Lemma 5. *If $\hat{\mathbf{p}} \in \mathbb{R}^n_+ \setminus \{0\}$, then the vertices of $\mathcal{H}_{\hat{\mathbf{p}}}$ are $\frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_i}\mathbf{e}^i$, for all $i \in I_1$, and the extreme directions of $\mathcal{H}_{\hat{\mathbf{p}}}$ are $\mathbf{e}^i$, for all $i \in I_\emptyset$.*

The proofs are given in Appendix 2.10.1-2.10.2. As a corollary, if $\hat{\mathbf{p}} \in \mathbb{R}^n_+ \setminus \{0\}$ and $\mathbf{p} \in \mathcal{H}_{\hat{\mathbf{p}}}$, then there exist $\gamma_i \geq 0$, for all $i \in I_1$, satisfying $\sum_{i \in I_1} \gamma_i = 1$, and $\xi_i \geq 0$, for all $i \in I_\emptyset$, such that $\mathbf{p} = \sum_{i \in I_1} \gamma_i \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_i}\mathbf{e}^i + \sum_{i \in I_\emptyset} \xi_i \mathbf{e}^i$.

Remark 1.1. $\hat{\mathbf{p}} \in cone(\mathbb{P})$ *if and only if* $\hat{\mathbf{p}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$, *for all* $\hat{\mathbf{p}} \in \mathbb{R}^n$.

Proposition 2. *Assume $\hat{\mathbf{p}} \in \mathbb{R}^n_+ \setminus \{0\}$. Let $\tilde{\mathbf{p}}$ be the vector generated by eliminating zero elements of $\hat{\mathbf{p}}$. Then, $\hat{\mathbf{p}} \in cone(\mathbb{P})$ if and only if $\tilde{\mathbf{p}} \in \mathcal{FH}_{\tilde{\mathbf{p}}}$.*

The proofs are given in Appendix 2.10.3-2.10.4. Proposition 2 may be used to simplify the problem if the objective is merely to determine feasibility and $\hat{\mathbf{p}}$ has 0 entries. In that case, it permits us to fix the options with a penetration rate of 0. Such simplifications are not possible in general since the objective is to determine the closest feasible solution. The reason is that if the given PS is infeasible, then a 0 penetration rate is also subject to change similar to all other penetration rates and the final value of such penetration rates could be different than 0.

Next, we establish the equivalence between $\min_{\mathbf{p}\in\mathcal{FH}_{\hat{\mathbf{p}}}} \|\mathbf{p}-\hat{\mathbf{p}}\|$ and $\min_{\mathbf{p}\in cone(\mathbb{P})} \|\mathbf{p}-\hat{\mathbf{p}}\|$.

Theorem 1. *Let* $\hat{\mathbf{p}} \in \mathbb{R}^n_+\backslash\{0\}$, $\mathbf{p}^{**} = \arg\min_{\mathbf{p}\in cone(\mathbb{P})} \|\mathbf{p}-\hat{\mathbf{p}}\|^2$, *and* $\mathbf{p}^{*} = \arg\min_{\mathbf{p}\in\mathcal{FH}_{\hat{\mathbf{p}}}} \|\mathbf{p}-\hat{\mathbf{p}}\|^2$. *Then:*

(a) $\mathbf{p}^{**} = 0$ *if and only if* $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\}$, *and*

(b) if $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$, *then* $\mathbf{p}^{**} = \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\mathbf{p}^{*T}\mathbf{p}^{*}}\mathbf{p}^{*}$.

The proof is given in Appendix 2.10.5. Thus, we aim to solve $\min_{\mathbf{p}\in\mathcal{FH}_{\hat{\mathbf{p}}}} \|\mathbf{p}-\hat{\mathbf{p}}\|$. The FCFW has two main steps at each iteration: *direction step* and *maximization step.* At iteration k, for all $k \geq 1$, the *direction step* finds vector β^k, and the *maximization step* maximizes $\beta^{kT}\theta$ over all $\theta \in \mathcal{FH}_{\hat{\mathbf{p}}}$ and obtains an optimal solution θ^{k+1}. We assume $\theta^1 \in \mathcal{FH}_{\hat{\mathbf{p}}}$ is given before the algorithm starts. If $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\}$, then $\mathbf{p}^{**} = 0$; hence, assume in the remainder that $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$ and θ^1 always exists. At iteration $k = 1$, the algorithm finds β^1 and maximizes $\beta^{1T}\theta$ to obtain θ^2. Next, we present each step in detail.

2.4.2 Direction Step

The improving direction is in fact the direction of the gradient; hence, we use $\beta^k = \hat{\mathbf{p}} - \mathbf{p}^{*k}$, where $\mathbf{p}^{*k}$ is the nearest penetration statistic to $\hat{\mathbf{p}}$ in the convex hull of $\{\theta^1, \cdots, \theta^k\}$. Then, $\mathbf{p}^{*k} = \sum_{i=1}^{k} \alpha_i^*\theta^i$, where $\alpha^* = (\alpha_1^*, \cdots, \alpha_k^*)$ is found by solving the following quadratic program.

$$\alpha^* = \arg\min_{\alpha} \quad \|\hat{\mathbf{p}} - \sum_{i=1}^{k} \alpha_i \theta^i\| \tag{2.9}$$

$$\text{subject to:} \quad \sum_{i=1}^{k} \alpha_i = 1 \tag{2.10}$$

$$\alpha_i \geq 0, \qquad \forall i = 1, \cdots, k. \tag{2.11}$$

This quadratic problem can be converted into a system of linear equations using the KKT conditions. Based on our experiment in section 2.5, solving this problem takes a negligible amount of time.

The distance between $\hat{\mathbf{p}}$ and $\mathbf{p}^{*k}$, which we refer to as *feasibility gap*, is denoted by $\mathcal{G}^k$. Then, $\mathcal{G}^k = \|\beta^k\|$, and the feasibility of $\hat{\mathbf{p}}$ is ascertained if $\mathcal{G}^k = 0$. In the following lemmas, we formally state this result and comment on the monotonicity of $\mathcal{G}^k$. The proofs are given in Appendix 2.11.1-2.11.2.

Lemma 6. *At iteration k of our algorithm, if $\mathcal{G}^k = 0$, then $\hat{\mathbf{p}} \in cone(\mathbb{P})$.*

Lemma 7. *During the execution of our algorithm, $\mathcal{G}^k$ is non-increasing in k.*

2.4.3 Maximization Step

In the *maximization step* at iteration k, we maximize $\beta^{kT}\theta$ to obtain θ^{k+1}: $\mathcal{M}(\beta^k) := \max_{\theta \in \mathcal{FH}_{\hat{\mathbf{p}}}} \beta^{kT}\theta$. We denote this problem by $\mathcal{M}(\beta^k)$ since β^k is the only parameter that changes from one iteration to the next. Note that the feasible region of $\mathcal{M}(\beta^k)$ can be unbounded; hence, we must show that $\mathcal{M}(\beta^k)$ has a bounded optimal solution, for all k.

Proposition 3. *Problem $\mathcal{M}(\beta^k)$ has a bounded optimal solution, for all k.*

The proof is given in Appendix 2.12.1. Problem $\mathcal{M}(\beta^k)$ is equivalent to the following

MILP problem.

$$\mathcal{M}(\beta^k) = \max_{\lambda,\theta,\mathbf{y}} \quad \beta^{kT}\theta \tag{2.12}$$

$$\text{subject to:} \quad \hat{\mathbf{p}}^T\theta = \hat{\mathbf{p}}^T\hat{\mathbf{p}}, \tag{2.13}$$

$$\lambda - M(1-y_i) \le \theta_i \le \lambda, \qquad \forall i \in N, \tag{2.14}$$

$$0 \le \theta_i \le My_i, \qquad \forall i \in N, \tag{2.15}$$

$$\lambda \ge 0, \tag{2.16}$$

$$\lambda \in \mathbb{R}, \theta \in \mathbb{R}^n, \mathbf{y} \in \mathbb{Y}. \tag{2.17}$$

Recall that $\mathbf{y}$ is a vector of zeros and ones that denotes a feasible configuration, and θ is the image of $\mathbf{y}$ on $\mathcal{H}_{\hat{\mathbf{p}}}$ through the coefficient $\lambda \ge 0$; hence, $\theta = \lambda\mathbf{y}$. Eq. (2.13) models $\theta \in \mathcal{H}_{\hat{\mathbf{p}}}$. Consider $\theta = \lambda\mathbf{y}$: if $y_i = 1$, then $\theta_i = \lambda$, and otherwise, if $y_i = 0$, then $\theta_i = 0$, for all $i \in N$. Based on this logic, $\theta = \lambda\mathbf{y}$ is represented by Eqs. (2.14)-(2.16), where M is a sufficiently large number. Finally, $\mathbf{y}$ must satisfy the rules for selecting options.

Proposition 4. *$\mathcal{M}(\beta^k)$ is NP-hard.*

The proof follows from noting that Satisfiability problem is imbedded in $\mathcal{M}(\beta^k)$. Clearly, $\mathcal{M}(\beta^k)$ is computationally complex; however, in the following, we show that this problem can be solved easily if the maximum weighted satisfiability (MWSAT) can be solved in polynomial time. MWSAT is defined as follows: given a CNF formula and a weight associated to each literal, find a truth assignment that satisfies the formula and maximizes the summation of weights for true literals (Wahlstrom, 2008; Dahllof et al., 2005). We show that solving $\mathcal{M}(\beta^k)$ is equivalent to finding the maximum value of $\omega \in \mathbb{R}$ for which the optimal value of MWSAT with weight vector $\beta^k + \omega\hat{\mathbf{p}}$ is zero. This condition is denoted by MWSAT$(\beta^k + \omega\hat{\mathbf{p}}) = 0$.

Lemma 8. *(a) MWSAT$(\beta^k+\omega\hat{\mathbf{p}})$ is continuous and convex in ω, (b) MWSAT$(\beta^k+\omega\hat{\mathbf{p}}) \longrightarrow +\infty$ as $\omega \longrightarrow +\infty$, and (c) MWSAT$(\beta^k + \omega\hat{\mathbf{p}}) \le 0$ as $\omega \longrightarrow -\infty$.*

Proposition 5. $\mathcal{M}(\beta^k)$ *is equivalent to* $\max\left\{\omega \in \mathbb{R} | MWSAT(\beta^k + \omega\hat{\mathbf{p}}) = 0\right\}$.

See Appendix 2.12.2-2.12.3 for proofs. Solving $\mathcal{M}(\beta^k)$ is equivalent to finding the greatest root of MWSAT($\beta^k + \omega\hat{\mathbf{p}}$) that can be obtained by applying the well-known Newton's method. Hence, if MWSAT is easy, then $\mathcal{M}(\beta^k)$ is also easy. Wahlstrom (2008) studies counting MWSAT and presents effective algorithms with an upper bound on their running time. His algorithm for formulae with maximum of two variables per clause runs with an upper bound on its running time of $\mathcal{O}(1.2377^n)$. Wahlstrom (2008) also proves that his algorithm runs in polynomial time if, in addition, the degree of the formula is less than 3.

As a final remark, we note that we have to solve an NP-hard problem at each iteration of the FCFW. To reduce the computational time, we could adopt a heuristic approach to solve $\mathcal{M}(\beta^k)$. However, the problem of finding a feasible solution to $\mathcal{M}(\beta^k)$ is NP-complete and a heuristic approach can not guarantee finding a feasible solution in polynomial time. At iteration k, we solve $\mathcal{M}(\beta^k)$ approximately by allowing $\frac{\varphi}{\sqrt{k}}$ relative optimality error, where $\varphi \geq 0$. If φ increases, the algorithm runs faster, but the amount of improvement at each iteration may become smaller (see, for example, Jaggi (2013) for a similar approach). In section 2.5, we test our algorithm with $\varphi = 0, 1, 2, 3$. The average running time of $\mathcal{M}(\beta^k)$ is less than 100 seconds (on a personal computer) for all of the instances that we solve in this chapter including the real problems we received from the LAM.

2.4.4 Convergence rate

In this section, we establish the convergence rate of our algorithm given that it finds the worst possible point at each iteration. Recall that $\mathbf{p}^* := \arg\min_{\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}} \|\mathbf{p} - \hat{\mathbf{p}}\|$. It is well-known that, for each $k \geq 2$, the iterates $\mathbf{p}^{*k}$ satisfy $\|\mathbf{p}^{*k} - \hat{\mathbf{p}}\|^2 - \|\mathbf{p}^* - \hat{\mathbf{p}}\|^2 \leq \frac{4\mathcal{D}^2}{k+1}$, where $\mathcal{D}$ is the diameter of a subset of $\mathcal{FH}_{\hat{\mathbf{p}}}$ that contains the solutions of $\mathcal{M}(\beta^k)$ (Jaggi, 2013). In this section, we present a tight convergence rate guarantee for the case when $\hat{\mathbf{p}}$ is feasible—i.e., $\|\mathbf{p}^* - \hat{\mathbf{p}}\|^2 = 0$. The existing convergence rate guarantee in the literature for this case implies that $\|\mathbf{p}^{*k} - \hat{\mathbf{p}}\|^2 \leq \frac{4\mathcal{D}^2}{k+1}$, for all $k \geq 2$. Whereas we show $\|\mathbf{p}^{*k} - \hat{\mathbf{p}}\|^2 \leq \frac{1}{2}\frac{\mathcal{D}^2}{k}$, for all $k \geq 2$, and prove its tightness. This is a stronger result than that in the literature. To gain insights, we begin with a 3-dimensional example and then extend the result to the general case.

Example 2 (Fig. 2.4): Assumptions: (i) $\hat{\mathbf{p}}$ is feasible, (ii) $\mathcal{FH}_{\hat{\mathbf{p}}}$ is a 3-dimensional polyhedron, and $\mathcal{D} = \sqrt{2}$, and (iii) θ^1 is located at 1 unit Euclidean distance from $\hat{\mathbf{p}}$. Consider a 3-dimensional Cartesian coordinate system, with origin at $\hat{\mathbf{p}}$ and axis lines z_1, z_2, and z_3.

Let spr(†) denote the sphere with radius $\sqrt{2}$ which is centered at †. Because of assumption (i), θ^1 must belong to spr($\hat{\mathbf{p}}$) (Fig. 2.4(a)). Let θ_w^k denote the worst-case of θ^k, for all $k \geq 1$, such that it maximizes $\mathcal{G}^k$. Then θ_w^1 must belong to the boundary of spr($\hat{\mathbf{p}}$) that we denote by bspr($\hat{\mathbf{p}}$). According to assumption (iii), we let $\theta^1 = (1, 0, 0)$. As shown in Fig. 2.4(b), we obtain $\mathbf{p}^{*1} = \theta^1$ and $\mathcal{G}^1 = 1$.

Direction β^1 is shown in Fig. 2.4(c). Because of assumptions (i) and (ii), θ^2 must be within $\sqrt{2}$ Euclidean distance from $\hat{\mathbf{p}}$ and θ^1—i.e., $\theta^2 \in \text{spr}(\hat{\mathbf{p}}) \cap \text{spr}(\theta^1)$. In addition, because of assumption (i), we must have $\beta^{1T}\theta^2 \geq \beta^{1T}\hat{\mathbf{p}}$. Note that otherwise $\hat{\mathbf{p}}$ is separable from all feasible θ's and this contradicts assumption (i). We define half-space hsp(‡):=$\{\mathbf{z} \in \mathbb{R}^3 : ‡^T\mathbf{z} \geq ‡^T\hat{\mathbf{p}}\}$ and denote its boundary by bhsp(‡). Then $\theta^2 \in \text{spr}(\hat{\mathbf{p}}) \cap \text{spr}(\theta^1) \cap \text{hsp}(\beta^1)$. It is easy to see that $\theta_w^2 \in \text{bspr}(\theta^1) \cap \text{bhsp}(\beta^1)$. Therefore, take without loss of generality, $\theta^2 = (0, 1, 0)$.

As shown in Fig. 2.4(d), $\mathbf{p}^{*2}$ is the closest point in *conv*($\{\theta^1, \theta^2\}$) to $\hat{\mathbf{p}}$. We obtain $\mathbf{p}^{*2} = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\mathcal{G}^2 = \frac{1}{\sqrt{2}}$. Figs. 2.4(e) and 2.4(f) continue the analysis in a similar manner.

Recall that $\mathcal{G}^1 = 1$, $\mathcal{G}^2 = \frac{1}{\sqrt{2}}$, and $\mathcal{G}^3 = \frac{1}{\sqrt{3}}$. This convergence rate can be achieved as we show using the following example. Consider 4 options and assume that $\hat{\mathbf{p}} = (0, 0, 0, 1)$ and there are two rules as follows: option 4 must always be chosen and at most one option from the set $\{1, 2, 3\}$ can be selected. There then exists 4 feasible configurations (1,0,0,1), (0,1,0,1), (0,0,1,1), and (0,0,0,1). Note that $\mathcal{H}_{\hat{\mathbf{p}}} = \{\mathbf{p} \in \mathbb{R}^4 | \mathbf{p}_4 = 1, \mathbf{p} \geq 0\}$ and all feasible

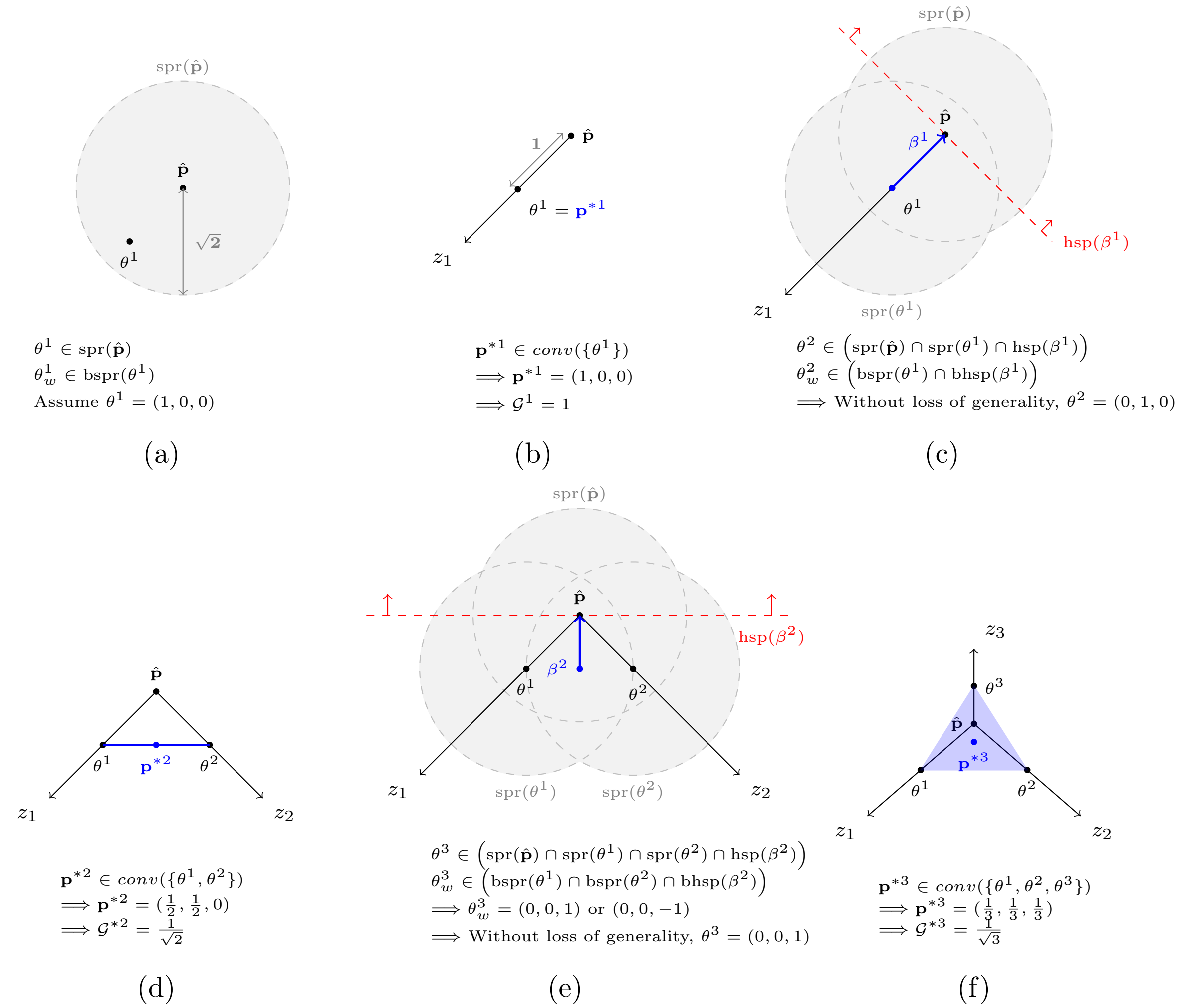


Figure 2.4: An illustrative example for the worst-case convergence of our algorithm.

configurations are in $\mathcal{H}_{\hat{\mathbf{p}}}$; hence, $\mathcal{FH}_{\hat{\mathbf{p}}}$ is a 3-dimensional polyhedron and it is the set of all convex combinations of (1,0,0,1), (0,1,0,1), (0,0,1,1), and (0,0,0,1). The diameter of $\mathcal{FH}_{\hat{\mathbf{p}}}$ is $\sqrt{2}$. Letting $\theta^1 = (1, 0, 0, 1)$, one could verify that $\mathcal{G}^k = \frac{1}{\sqrt{k}}$ for $1 \leq k \leq 3$.

In Example 2, if one moves θ^1 further away from $\hat{\mathbf{p}}$, the convergence of our algorithm will improve in the next iteration. For example, if $\theta^1 \in$bspr($\hat{\mathbf{p}}$), the algorithm converges in one iteration.

This convergence rate can be rigorously proven for a more general case in which $\hat{\mathbf{p}} \in cone(\mathbb{P})$, $\mathcal{G}^1 = \frac{\mathcal{D}}{\sqrt{2}}$, and $\theta^k = \theta^k_w$, for all $k \geq 2$. We next provide a formal definition of θ^k_w.

Define $\Theta^k = \{\theta \in \mathbb{R}^n : \|\theta - \theta^i\|^2 \leq \mathcal{D}^2, \forall i = 1, \cdots, k, \beta^{kT}(\theta - \hat{\mathbf{p}}) \geq 0\}$ and $\mathcal{A}^k = \{\alpha \in \mathbb{R}^{k+1} | \alpha \geq 0, \mathbf{1}_{k+1}^T \alpha = 1\}$. The following remark presents a way to obtain θ_w^{k+1}.

Remark 5.1. *If $\hat{\mathbf{p}} \in cone(\mathbb{P})$, then at iteration k of our algorithm, we have:*

$$\theta_w^{k+1} = \arg\max_{\theta \in \Theta^k} \left\{ \min_{\alpha \in \mathcal{A}^k} \left\| \sum_{i=1}^{k} \alpha_i \theta^i + \alpha_{k+1} \theta - \hat{\mathbf{p}} \right\| \right\}, \qquad \forall k \geq 1.$$

The proof is given in Appendix 2.13.1. For the general case that does not require feasibility of $\hat{\mathbf{p}}$, one needs to modify Θ^k as $\{\theta \in \mathbb{R}^n : \|\theta - \theta^i\|^2 \leq \mathcal{D}^2, \forall i = 1, \cdots, k\}$.

Theorem 2. *In our algorithm, if $\mathcal{G}^1 = \frac{\mathcal{D}}{\sqrt{2}}$, $\hat{\mathbf{p}} \in cone(\mathbb{P})$, and $\theta^k = \theta_w^k$, for all $k \geq 2$, then $\mathcal{G}^k \leq \frac{\mathcal{G}^1}{\sqrt{k}}$, for all $k = 1, 2, \cdots$. This is the tightest bound for $1 \leq k < n$.*

The proof is given in Appendix 2.13.2. In the remainder, we present a lower bound on $\mathcal{G}^k$ that we denote by $\mathcal{LB}^k$, for all $k \geq 1$. If $\hat{\mathbf{p}}$ is infeasible, according to the convergence rate of the FCFW, it is desirable to observe that $(\mathcal{G}^k - \mathcal{LB}^k)$ converges at a rate proportional to $\frac{1}{\sqrt{k}}$. In section 2.5, we show on a set of real instances that if $\hat{\mathbf{p}}$ is infeasible, this convergence rate is observed for $(\mathcal{G}^k - \mathcal{LB}^k)$.

2.4.5 Lower Bound on $\mathcal{G}^k$

Let $\mathcal{L}^k$ be a lower bound on the feasibility gap at iteration k and $\mathcal{U}^k$ be an upper bound on the possible improvement in the feasibility gap at iteration k. An advantage of having a lower bound is that by comparing it with $\mathcal{G}^k$, we can decide to terminate the algorithm if the difference is satisfactory.

Definition 1. Given $\mathcal{G}^k \neq 0, \mathcal{U}^k := \min\left\{\mathcal{G}^k, \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\}$, *and* $\mathcal{L}^k := \mathcal{G}^k - \mathcal{U}^k$, $k \geq 1$.

Since $\mathcal{G}^k \neq 0$, then $\beta^k \neq 0$. Let θ^{k+1} be the optimal solution of $\mathcal{M}(\beta^k)$ at iteration k. The orthogonal projection of $(\theta^{k+1} - \mathbf{p}^{*k})$ on the vector β^k is the vector $\frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^{k2}} \beta^k$ and its length is given by $\frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}$. The maximum improvement on $\mathcal{G}^k$ is $\frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}$ and this upper bound is useful if it is less than $\mathcal{G}^k$. Thus, $\mathcal{U}^k = \min\left\{\mathcal{G}^k, \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\}$. Obviously, once we have $\mathcal{U}^k$, the lower bound on the feasibility gap is the difference between $\mathcal{U}^k$ and $\mathcal{G}^k$; hence, $\mathcal{L}^k = \mathcal{G}^k - \mathcal{U}^k$. Appendix 2.14.1 provides a graphical explanation on how $\mathcal{G}^k$, $\mathcal{U}^k$, and $\mathcal{L}^k$ are obtained at each iteration. In the following proposition, we show that the infeasibility of $\hat{\mathbf{p}}$ is known once $\mathcal{L}^k > 0$ and that the best penetration statistic is obtained once $\mathcal{U}^k = 0$, for some k.

Proposition 6. *Given $\mathcal{G}^k \neq 0$ at iteration k of our algorithm: (i) if $\mathcal{L}^k > 0$, then $\hat{\mathbf{p}} \notin$ cone$(\mathbb{P})$, and (ii) if $\mathcal{U}^k = 0$, then $\mathbf{p}^{*k}$ is the nearest $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$.*

The proof is given in Appendix 2.14.2. Having shown how $\mathcal{U}^k$ and $\mathcal{L}^k$ are used in terminating the algorithm, in the following, we investigate the monotonicity of $\mathcal{U}^k$ and $\mathcal{L}^k$.

Lemma 9. *$\mathcal{L}^k$ and $\mathcal{U}^k$ are not necessarily monotone in k.*

The proof is given in Appendix 2.14.3. Since $\mathcal{L}^k$ is not monotone in k, one could use the maximum of all $\mathcal{L}^k$'s obtained. Let $\mathcal{LB}^k := \max_{i=1,\cdots,k} \{\mathcal{L}^i\}$. Obviously, $\mathcal{LB}^k$ is monotone non-decreasing in k, and $\mathcal{G}^k \geq \mathcal{LB}^k$, for all $k \geq 1$.

Our algorithm is given in Algorithm 1. The accuracy of this algorithm is formally stated in the following theorem. Our algorithm does not iterate forever but stops after a finite number of iterations.

Theorem 3. *Our algorithm finds the nearest $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$ and stops after finite iterations.*

The proof is given in Appendix 2.14.4. In summary, in this section, we started with a non-convex objective function and non-compact feasible region and showed how FCFW method can be applied. We derived an improved convergence rate guarantee, established its tightness, and provided a lower bound on the optimal value at each iteration.

2.5 Computational Experiments

We now report the results of our computational experiment for evaluating the effectiveness of our methodology and the sensitivity of our findings to the number of options and rules.

Algorithm 1

Input: Rules, $\hat{\mathbf{p}}$.

Output: Is $\hat{\mathbf{p}} \in cone(\mathbb{P})$? If no, find the nearest $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$.

θ^1 := an arbitrary point in $\mathcal{FH}_{\hat{\mathbf{p}}}$; ▷ `Assume` $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$.
δ :=0; ▷ $\delta = 1$ `means it is known` $\hat{\mathbf{p}} \notin cone(\mathbb{P})$, `and` $\delta = 0$ `means otherwise!`
for $k = 1, 2, 3, \cdots$ **do**
  $\mathbf{p}^{*k} := \sum_{i=1}^{k} \alpha_i^* \theta^i$, where α^* is obtained by solving Eqs. (2.9)-(2.11);
  $\beta^k := \hat{\mathbf{p}} - \mathbf{p}^{*k}$; $\mathcal{G}^k := \|\beta^k\|$;
  if $\mathcal{G}^k = 0$ **then**
    Report "$\hat{\mathbf{p}} \in cone(\mathbb{P})$"; Stop!
  end if
  Solve $\mathcal{M}(\beta^k)$; θ^{k+1} := the optimal value of θ;
  $\mathcal{U}^k := \min\left\{\mathcal{G}^k, \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\}$; $\mathcal{L}^k := \mathcal{G}^k - \mathcal{U}^k$;
  if $\mathcal{L}^k > 0$ & $\delta = 0$ **then**
    Report "$\hat{\mathbf{p}} \notin cone(\mathbb{P})$"; ▷ `Continue to find the nearest` $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ `to` $\hat{\mathbf{p}}$`!`
    $\delta = 1$; ▷ `To prevent reporting "`$\hat{\mathbf{p}} \notin cone(\mathbb{P})$`" in next iterations!`
  end if
  if $\mathcal{U}^k = 0$ **then**
    Report "$\mathbf{p}^{*k}$ is the nearest $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$"; Stop!
  end if
end for

We implement our algorithm in IBM ILOG CPLEX Optimization Studio 12.6.1 and use a PC with Processor Intel(R) Core(TM) i5-2520M CPU 2.50GHz, 4.00 GB of RAM, and 64-bit Operating System. We define error as $E^k = \mathcal{G}^k - \mathcal{LB}^k$. Because the largest possible Euclidean distance in an n-dimensional unit hypercube is $\sqrt{n}$, we normalize $\mathcal{G}^k$, $\mathcal{LB}^k$, and E^k by dividing by $\sqrt{n}$, which also eliminates scaling issues.

The LAM offers three general categories of cars: economy sedans, luxury sedans, and SUV/trucks. These products are offered in four regions of the world—North America (NA), Latin America (LA), Europe (EU), and Asia (AS). For each car, the LAM has a master problem that includes all options and rules. Depending on the region where the car is sold, some of the options are fixed to 0 or 1, which results in elimination of some of the rules. For example, typically a truck in Mexico has fewer options than it has in the U.S. The firm offers fewer luxury options, engines, and colors in Mexico. Red trucks are very rarely purchased in Mexico. As another example, consider Vietnam where luxury cars are rarely purchased and cars have few options. On the other hand, all options are available for the same car in Europe. When all options are available, penetration rates can be very small. Table 2.2 presents the number of options for different car categories and across different regions.

Table 2.2: The approximate number of options for different products and regions.

	NA	LA	EU	AS
Economy sedans	200	100	200	100
Luxury sedans	300	250	300	250
Trucks/SUVs	400	300	400	300

The computational complexity of our problem is primarily a function of the number of options. Generally, high variety leads to a large number of options with very small penetration rates, which makes the problem very difficult and increases the possibility of infeasibility of the forecast PS.

First, we focus on the impact of the number of options on the performance of our algorithm. We perform a sensitivity analysis on the number of options that are offered by the LAM, which ranges from 100 to 400. To show the performance of our algorithm in an industrial setting, we create instances that resemble the real problems encountered by the LAM. We use a master problem provided by the LAM that includes 410 options with assigned penetration rates and 4056 rules. We randomly generate instances with 100, 200, 300, and 400 options. For each size, we create 10 instances. We ignore the penetration rates of the options that are not included. Our method is applied to each instance for 1000 seconds and the average E^k is shown in Fig. 2.5 after 100, 500, and 1000 seconds. The four curves in each figure represent the performance of the FCFW when the subproblems are approximately solved. We consider exactly solving the subproblems and allowing an optimality error of $1/\sqrt{k}$, $2/\sqrt{k}$, and $3/\sqrt{k}$. Note that the range of the vertical axis varies to show the change in E^k for the different number of options. The average running time per iteration is shown in Fig. 2.6(a). Fig. 2.6(b) demonstrates the average number of iterations

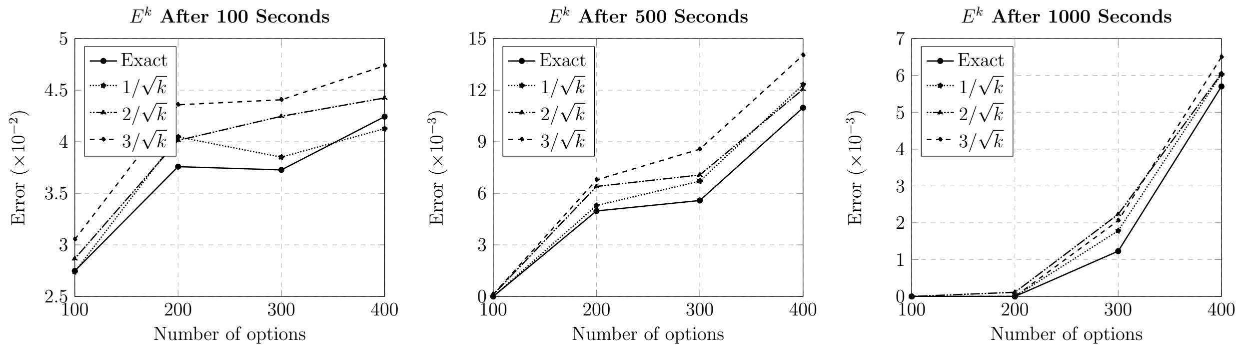


Figure 2.5: Normalized E^k for different number of options after 100, 500, and 1000 seconds.

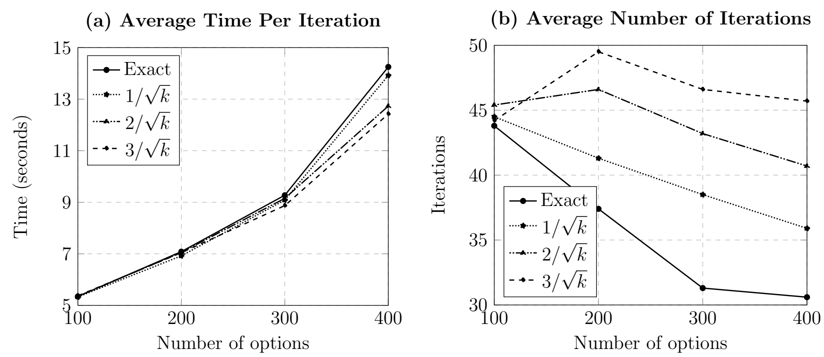


Figure 2.6: Average running time per iteration and average number of iterations.

that our method spends to determine the feasibility/infeasiblility of $\hat{\mathbf{p}}$. Using Figs. 2.5-2.6, we make the following observations:

1. The average error after 1000 seconds is close to 0 for instances with 100 options and small for other sizes. All four approaches appear to be very effective in solving large instances that arise in a real industrial setting. The four approaches find very good solutions in less than 1000 seconds. The exact approach almost always outperforms the other approaches.

2. Average time per iteration increases as the number of options increases. This is mainly due to $\mathcal{M}(\beta^k)$, which becomes more difficult as the number of options increases.

3. An increased number of options negatively affects the performance of our algorithm. The normalized E^k becomes larger as the number of options increases primarily due

to the increased solution time of $\mathcal{M}(\beta^k)$.

4. The infeasibility of $\hat{\mathbf{p}}$ is determined quite quickly. As expected, as the allowed optimality error of the heuristic increases, it takes more iterations to determine the infeasibility of $\hat{\mathbf{p}}$.

In the remainder, we examine the effects of changing the number of rules on the performance of our algorithm. The same master problem is used to generate instances with 1000, 2000, 3000, and 4000 rules. We create 10 instances for each size by randomly selecting the associated rules and relaxing the remaining rules. All instances have 410 options. We solve each instance for 1000 seconds, and the result is shown in Figs. 2.7-2.8. In Fig. 2.8(b), the majority of the instances with 1000 and 2000 rules are still feasible after 1000 seconds; hence, we use the total number of iterations in 1000 seconds. All of the instances with 3000 and 4000 rules are infeasible; hence, Fig. 2.8(b) shows the average number of iterations to determine the feasibility/infeasibility of these instances. We observe the following.

1. Our method finds very good solutions with low errors in all instances in fewer than 1000 seconds. As shown in Fig. 2.7, as the optimality error increases, the heuristic approach finds a smaller error for a small number of rules.

2. The average time per iteration decreases as the number of rules increases. In fact, rules strengthen $\mathcal{M}(\beta^k)$ by tightening the feasible region and help the CPLEX solver find the optimal solution very quickly. Eliminating some of the rules weakens $\mathcal{M}(\beta^k)$ and results in increased running time per iteration.

3. An increased number of rules enhances the performance of our method. A stronger $\mathcal{M}(\beta^k)$ is the only reason for this phenomenon.

Therefore, our algorithm is very effective in solving real problems and an increased number of options and rules affect its performance negatively and positively, respectively.

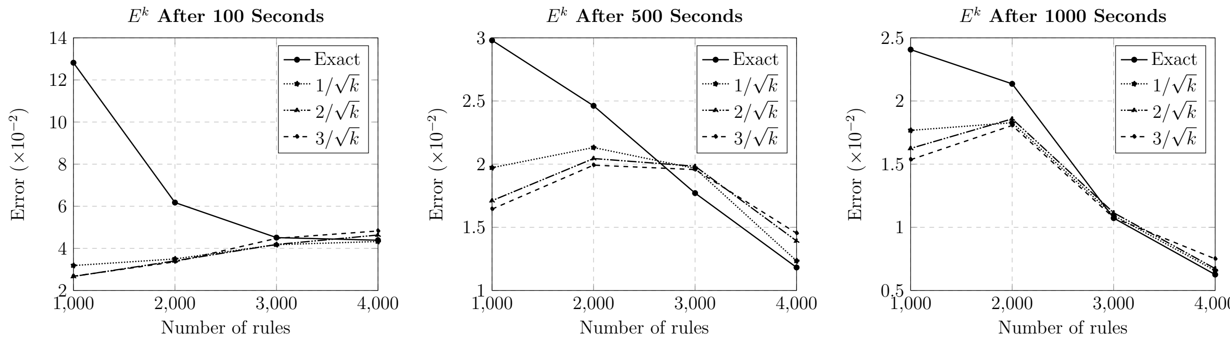


Figure 2.7: Normalized E^k for different numbers of rules after 100, 500, and 1000 seconds.

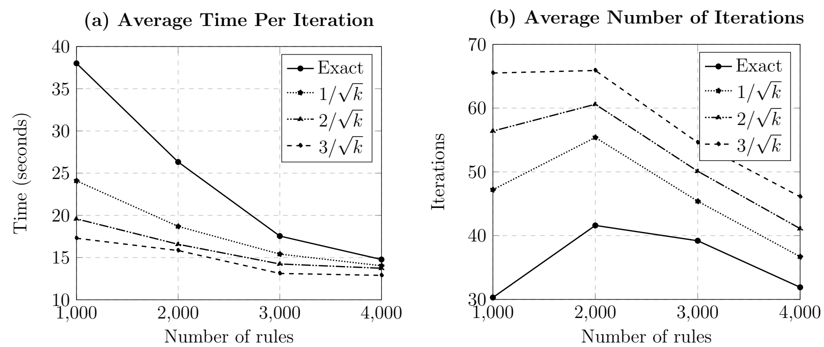


Figure 2.8: Average running time per iteration and average number of iterations.

2.6 Experiment with Industrial Data

In this section, we present the results of solving three real instances that we received from the LAM. The LAM has been one of the major companies in the global auto industry for more than 100 years and delivers approximately 10 million vehicles per year to more than 100 countries. The LAM forecasts PS every few months and approximately three years ahead; that is, PSs for 2019 model cars are forecast in 2016. Each car model is manufactured on one of 10-12 platforms. Each assignment to platform is based on the chassis size—e.g., subcompact sedans are manufactured on the same platform even though they are from different brands. On a given platform, the rules are always the same and they are usually in a simple and compact form.

The PSs are forecast by sales and marketing and sent to the engineering and finance departments for approval where some modifications are made to simplify the manufacturing

Table 2.3: The specifications of our industrial instances from the LAM.

	Options	OIRs	FCRs
Instance 1	415	3703	85
Instance 2	200	428	72
Instance 3	395	2111	97

process, reduce the cost of unnecessary variety, and enhance the economies of scale. The LAM has preprocessed the original set of options and rules to reduce them based on whether they are equal, opposite, or symmetric options. Equal options always appear together while for two opposite options, exactly one is always true. Options within a family that are symmetric have the same rules and are referenced by other rules in the same exact manner. For example, consider a vehicle's colors of white and green; if these colors appear in exactly the same rules, one could combine white and green into one color and decompose it later to regenerate the original colors.

We received three instances from the LAM with specifications listed in Table 2.3. Our exact method is applied to the three real instances, and normalized $\mathcal{G}^k$, $\mathcal{LB}^k$, and E^k over the running time of the algorithm and iterations are plotted in Fig. 2.9. Various ranges of axes for different instances are used to clearly show the behavior of the algorithm during the initial iterations. For example, while applying our method to instance 1, the big changes of $\mathcal{G}^k$ and $\mathcal{LB}^k$ occur from 0 to 5000 seconds; hence, we only show from 0 to 5000 seconds.

In all instances, the starting error is large and ranges from 30% in instance 1 to 60% in instance 3. The error reduces significantly in a few iterations. In instances 2 and 3, the error becomes approximately 1% after 10 iterations. The 1% error is observed in instance 1 after about 40 iterations. The time per iteration varies across instances; it is big for instance 1 and small for instances 2 and 3. The number of options and rules in instance 2 are smaller than in other instances, and our algorithm spends about 2 seconds per iteration. Instance 1 and 3 require approximately 100 and 10 seconds per iteration, respectively.

To show the implication of the convergence rate guarantee, we plot the function $\frac{E^1}{\sqrt{k}}$ and compare it with E^k. It is seen that, in these instances, the convergence of our method is significantly faster than $\frac{E^1}{\sqrt{k}}$.

All three instances are infeasible, which is determined as soon as $\mathcal{LB}^k$ becomes nonzero. This happens after 12, 3, and 2 iterations in instances 1, 2, and 3, respectively. Our method is then significantly fast in figuring out the infeasibility of $\hat{\mathbf{p}}$ in real instances.

Overall, the behavior of $\mathcal{G}^k$, $\mathcal{LB}^k$, and E^k is almost the same in all instances, and it can be seen that, especially in the initial iterations, the error decreases rapidly. Hence, our algorithm is very effective in finding a very good solution in a short period of time.

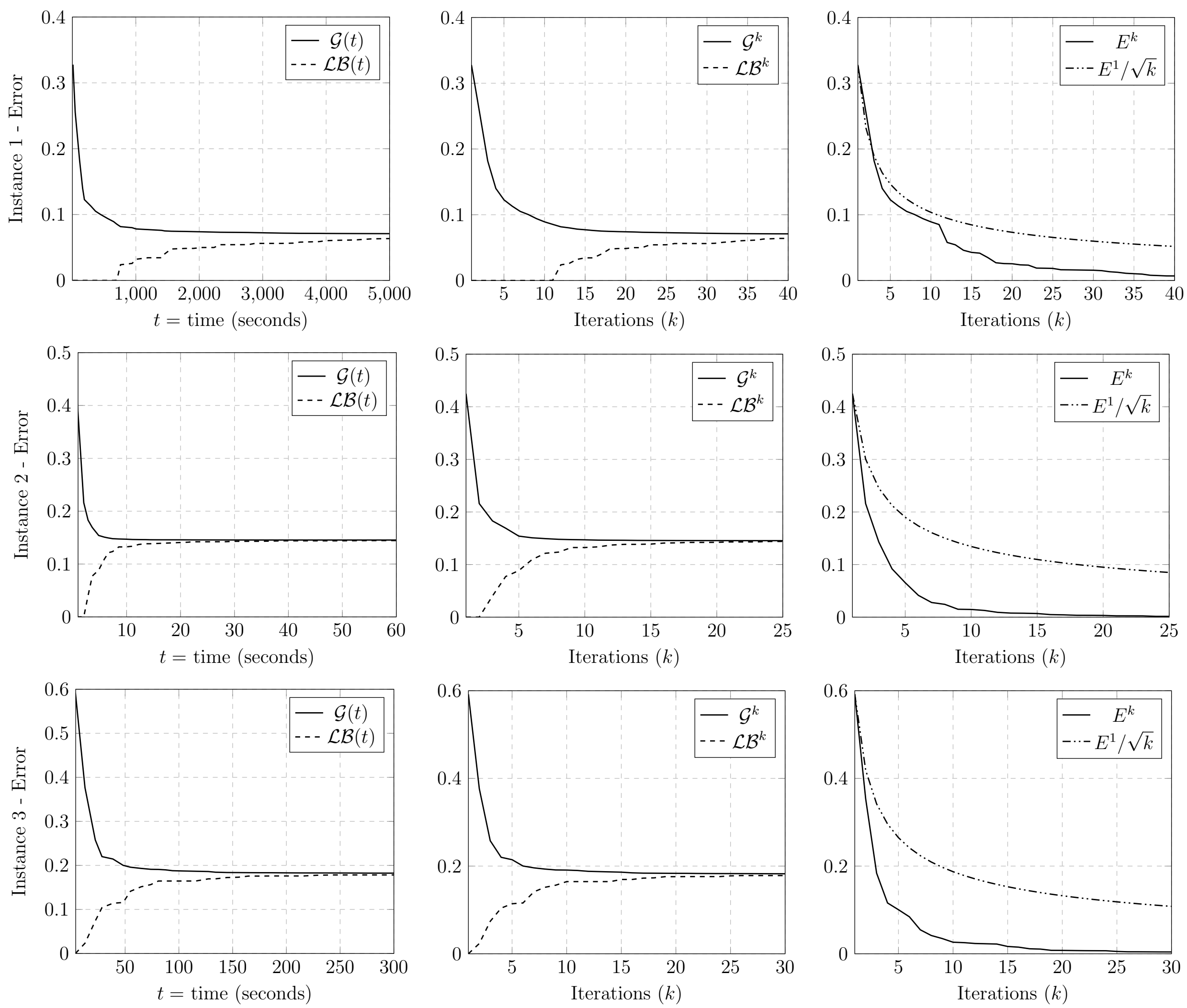


Figure 2.9: The performance of our exact method on real instances.

2.7 Conclusions

We study the problem of determining the feasibility of a forecast PS and finding the best feasible PS in the case of infeasibility. We present an approach that sequentially constructs the feasible region and stops when it either verifies the feasibility of the forecast PS or it finds the best feasible PS. We provide insights on the convergence rate of our algorithm and present a lower bound that enables early termination of our algorithm with a known optimality gap. We test our approach on a set of real instances and observe the following results: (i) increasing the number of options and/or decreasing the number of rules add to the difficulty of the problem, (ii) our approach finds very good solutions for real instances in a reasonable time, and (iii) the convergence rate of our approach is significantly faster than the worst-case that we prove in this chapter.

2.8 Appendix: Notations

Abbreviations:

ASR	Our new format for representing rules
CD	Conjunction of disjunctive clauses
CNF	Conjunctive normal form
DC	Disjunction of conjunctive clauses
FCR	Family cardinality rule
LAM	Large auto manufacturer
MWSAT	Maximum weighted satisfiability
OIR	Option implication rule
PSAT	Probabilistic satisfiability problem
PS	Penetration statistic

Notations:

n	Number of options
N	Set of options $N = \{i \mid i = 1, \cdots, n\}$
$p(.)$	Penetration rate or probability value
p_i	Penetration rate of option i, $\forall i \in N$
$\mathbf{p}$	A PS, $\mathbf{p} = (p_1, \cdots, p_n) \in \mathbb{R}^n$
$\mathbb{P}$	Convex hull of the set of feasible configurations
λ	Scalar, $\lambda \geq 0$
$cone(\mathbb{P})$	The cone generated by $\mathbb{P}$, $cone(\mathbb{P}) = \{\lambda\mathbf{p} \mid \mathbf{p} \in \mathbb{P}, \lambda \geq 0\}$
$\mathbf{q}_m$	A forecast from an entity within the LAM, $\forall m = 1, \cdots, M$
$f(.)$	A function that returns the sum of squared errors for a given PS
$\Vert.\Vert$	Euclidean norm

$\hat{\mathbf{p}}$	The forecasted PS
$\mathbf{p}^*$	The optimal solution of $\min_{\mathbf{p}\in\mathcal{FH}_{\hat{\mathbf{p}}}} \|\mathbf{p}-\hat{\mathbf{p}}\|^2$
$\mathbb{S}$	Set of all subsets of options, $\mathbb{S} := \{S \mid S \subseteq N\}$
$x(S)$	Variable for probability values, $x(S) \in [0,1], \forall S \in \mathbb{S}$.
$V_{\mathbb{P}}$	Set of vertices of $\mathbb{P}$
$\mathbb{S}^0$	Defined as $\mathbb{S}^0 := \Big\{S \in \mathbb{S} : \Big(\exists \text{ rule } i \Leftarrow j_1 \wedge \cdots \wedge j_l \text{ s.t. } \{j_1,\cdots,j_l\} \subset S, i \notin S\Big) \vee \Big(\exists \text{ rule } i \Rightarrow j_1 \vee \cdots \vee j_l \text{ s.t. } i \in S, \{j_1,\cdots,j_l\} \cap S = \emptyset\Big)\Big\}$
$\mathbb{S}^1$	Defined as $\mathbb{S}^1 := \mathbb{S} \setminus \mathbb{S}^0$
$\mathbb{Y}$	Set of feasible configurations
y_i	Binary variable which is 1 if option i is chosen, and 0 otherwise, $\forall i \in N$
$\mathbf{y}$	A vector of n binary variables, $\mathbf{y} = (y_1,\cdots,y_n)$
α	A vector of coefficients in the convex combination $\sum_{i=1}^k \alpha_i \theta^i$
α^*	The optimal value of α, or vector of coefficients to create $\mathbf{p}^{*k}$, i.e. $\mathbf{p}^{*k} = \sum_{i=1}^k \alpha_i^* \theta^i$
ω	The dual variable associated with $\hat{\mathbf{p}}^T\theta = \hat{\mathbf{p}}^T\hat{\mathbf{p}}$ in $\mathcal{M}(\beta^k)$, $\omega \in \mathbb{R}$
$\mathcal{L}^k$	The lower bound on the feasibility gap at iteration k
$\mathcal{U}^k$	The upper bound on the possible improvement in the feasibility gap at iteration k
$\mathcal{LB}^k$	The best lower bound found till iteration k, i.e. $\mathcal{LB}^k = \max_{i=1,\cdots,k}\left\{\mathcal{L}^k\right\}$
δ	1 if the infeasibility of $\hat{\mathbf{p}}$ has been reported in previous iterations, and 0 otherwise
k	Iteration counter, $k = 1, 2, \cdots$
$\mathbf{e}^i$	The vector of all zeros except for ith entry which is 1
φ	The coefficient in the optimality gap of the heuristic approach
$\mathcal{D}$	The diameter of a subset of $\mathcal{FH}_{\hat{\mathbf{p}}}$ that contains the solutions of $\mathcal{M}(\beta^k)$

2.9 Appendix for Problem Formulation

2.9.1 Proof of Lemma 1

The function f is convex and differentiable, with minimum occurring at $\hat{\mathbf{p}}$. Using $\frac{\partial f}{\partial \mathbf{p}} = 0$, we obtain $\hat{\mathbf{p}} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{q}_m$. We substitute $\hat{\mathbf{p}}$ by $\hat{\mathbf{p}} + \mathbf{\Delta}$, and it follows that:

$$\begin{aligned}
f(\hat{\mathbf{p}} + \mathbf{\Delta}) &= \sum_{m=1}^{M} (\hat{\mathbf{p}} - \mathbf{q}_m + \mathbf{\Delta})^T (\hat{\mathbf{p}} - \mathbf{q}_m + \mathbf{\Delta}) \\
&= \sum_{m=1}^{M} \Big((\hat{\mathbf{p}} - \mathbf{q}_m)^T (\hat{\mathbf{p}} - \mathbf{q}_m) + 2\mathbf{\Delta}^T (\hat{\mathbf{p}} - \mathbf{q}_m) + \|\mathbf{\Delta}\|^2 \Big) \\
&= f(\hat{\mathbf{p}}) + 2M\mathbf{\Delta}^T (\hat{\mathbf{p}} - \frac{1}{M}\sum_{m=1}^{M} \mathbf{q}_m) + M\|\mathbf{\Delta}\|^2 \\
&= f(\hat{\mathbf{p}}) + M\|\mathbf{\Delta}\|^2
\end{aligned}$$

Then, minimizing $f(\hat{\mathbf{p}} + \mathbf{\Delta})$ is equivalent to minimizing $\|\mathbf{\Delta}\|^2$.

2.9.2 LAM's Current Format for Representing Rules

In the LAM's current format, each OIR consists of a left-hand-side option, a set of positive right-hand-side options, and a set of negative right-hand-side options. The right-hand-side of an OIR is true if and only if all positive right-hand-side options are true and all negative right-hand-side options are false. There are 3 types of OIRs that we call `A`, `B`, and `R`. There is no logical difference between types `A` and `B`. An OIR implies a rule in the form of $\Longrightarrow$, regardless of its type; in addition, a type `R` rule implies a rule in the form of $\Longleftarrow$. For a left-hand-side option to be true, at least one of its associated right-hand-sides, regardless of the rule type, must be true; hence, if all right-hand-sides associated with a left-hand-side option are false, then the left-hand-side option must be false as well. If all right-hand-sides from type `R` OIRs are true, then the left-hand-side option must be true. An FCR can be of type E or L; exactly one option must be chosen from each type E family and at most one option can be selected from a type L family. We illustrate these rules using the following example.

Example. A hypothetical example is provided in Fig. 2.10 and includes 5 OIRs and 2 FCRs to represent the relationships among 17 options termed `OP01, OP02, ..., OP17`. In the first OIR, for instance, `OP01` is the left-hand-side option, the rule type is `A`, the positive right-hand-side options are `OP02` and `OP03`, and the negative right-hand-side option is `OP04`. The right-hand-side of the first OIR is true if and only if `OP02` and `OP03` are true and `OP04` is false. The first 3 OIRs have the same left-hand-side option; hence, if `OP01` is true then at least one of its associated right-hand-sides must be true. This logic can be written as: $\texttt{OP01} \Longrightarrow (\texttt{OP02}\wedge\texttt{OP03}\wedge\neg\texttt{OP4})\vee(\texttt{OP02}\wedge\texttt{OP03}\wedge\neg\texttt{OP05}\wedge\neg\texttt{OP06})\vee(\texttt{OP02}\wedge\texttt{OP03}\wedge\neg\texttt{OP06}\wedge\neg\texttt{OP07})$.

```
OIRs (Option Implication Rules):        FCRs (Family Cardinality Rules):

OP01, A {OP02, OP03} {OP04}             FM01, E {OP13, OP14, OP15}
OP01, B {OP02, OP03} {OP05, OP06}       FM02, L {OP16, OP17}
OP01, R {OP02, OP03} {OP06, OP07}
OP08, R {OP09} {OP10, OP11}
OP08, R {OP09} {OP10, OP12}
```

Figure 2.10: The LAM's default format for representing option implication and family cardinality rules.

Consider the OIRs 4 and 5, which are type `R` with the same left-hand-side option; hence, if both right-hand-sides in OIRs 4 and 5 are true, then option `OP08` must be true. This logic is written as $\texttt{OP08} \Longleftarrow (\texttt{OP09}\wedge\neg\texttt{OP10}\wedge\neg\texttt{OP11})\wedge(\texttt{OP09}\wedge\neg\texttt{OP10}\wedge\neg\texttt{OP12})$, which can be simplified as $\texttt{OP08} \Longleftarrow \texttt{OP09} \wedge \neg\texttt{OP10} \wedge \neg\texttt{OP11} \wedge \neg\texttt{OP12}$.

The second column of Fig. 2.10 shows two families, termed `FM01` and `FM02`, with types `E` and `L`, respectively. Based on these FCRs, exactly one of the options `OP13`, `OP14`, or `OP15`, and at most one of `OP16` or `OP17` must be chosen.

2.9.3 Converting Rules to ASR Format

In the example of Appendix 2.9.2, two rules are written for option OP01 as follows: OP01 $\Longrightarrow$ (OP02$\wedge$OP03$\wedge\neg$OP04)$\vee$(OP02$\wedge$OP03$\wedge\neg$OP05$\wedge\neg$OP06)$\vee$(OP02$\wedge$OP03$\wedge\neg$OP06$\wedge\neg$OP07) and OP01 $\Longleftarrow$ OP02 $\wedge$ OP03 $\wedge$ $\neg$OP06 $\wedge$ $\neg$OP07. The second rule is in ASR format. So, consider the first rule and note that its right-hand-side is written as a disjunction of conjunctive clauses (DC). Factoring out the greatest common factor—i.e., OP02$\wedge$OP03—and using the distributive property of union over intersection, we obtain:

OP01 $\Longrightarrow$ OP02 $\wedge$ OP03 $\wedge$ ($\neg$OP04 $\vee$ $\neg$OP05 $\vee$ $\neg$OP06) $\wedge$ ($\neg$OP04 $\vee$ $\neg$OP05 $\vee$ $\neg$OP07) $\wedge$ ($\neg$OP04 $\vee$ $\neg$OP06) $\wedge$ ($\neg$OP04 $\vee$ $\neg$OP06 $\vee$ $\neg$OP07).

The right-hand-side is a conjunction of disjunctive clauses (CD); hence, this rule is equivalent to rules 1-6 in Fig. 2.11. This reformatting is provable for the general case (see Lemma 2).

```
 Format ASR:
OP01 ⟹ OP02
OP01 ⟹ OP03
OP01 ⟹ ¬OP04 ∨ ¬OP05 ∨ ¬OP06
OP01 ⟹ ¬OP04 ∨ ¬OP05 ∨ ¬OP07
OP01 ⟹ ¬OP04 ∨ ¬OP06
OP01 ⟹ ¬OP04 ∨ ¬OP06 ∨ ¬OP07
OP01 ⟸ OP02 ∧ OP03 ∧ ¬OP06 ∧ ¬OP07
OP08 ⟹ OP09
OP08 ⟹ ¬OP10
OP08 ⟹ ¬OP11 ∨ ¬OP12
OP08 ⟸ OP09 ∧ ¬OP10 ∧ ¬OP11 ∧ ¬OP12
OP13 ⟹ ¬OP14
OP13 ⟹ ¬OP15
OP13 ⟸ ¬OP14 ∧ ¬OP15
OP14 ⟹ ¬OP15
OP16 ⟹ ¬OP17
```

Figure 2.11: The equivalent representation of Fig. 2.10 using ASR format.

2.9.4 Proof of Lemma 2

Consider an option, say option i, with at least one OIR and assume that none of these OIRs is of type R. Hence, by combining these OIRs (similar to Appendix 2.9.3), option i has exactly one rule in the form of $i \Longrightarrow$RHS. If RHS is always true, then the rule is redundant and hence is simply deleted. If RHS is always false, then option i must be false, and hence, the rule is equivalent to the following two rules in ASR format: $i \Longrightarrow j$ and $i \Longrightarrow \neg j$, for some option

$j \neq i$. We note that RHS can be written as a conjunction of some disjunctive clauses, and hence, it is easily seen that $i \Longrightarrow$RHS can be written as some rules in ASR format (similar to Appendix 2.9.3).

Consider an option, say option i, with at least one OIR of type R. Hence, by combining these OIRs (similar to Appendix 2.9.3), option i has exactly one rule in the form of $i \Longleftarrow$RHS. If RHS is always false, then the rule is redundant and hence is simply deleted. If RHS is always true, then option i must be true, and hence, the rule is equivalent to the following two rules in ASR format: $i \Longleftarrow j$ and $i \Longleftarrow \neg j$, for some option $j \neq i$. We note that RHS can be written as a conjunctive clause, and hence, $i \Longleftarrow$RHS is in ASR format.

A FCR of type L with one option is redundant and hence is simply deleted. A FCR of type E with one option, say option i, is equivalent to the following two rules in ASR format: $i \Longleftarrow j$ and $i \Longleftarrow \neg j$, for some option $j \neq i$. We next consider a FCR of type E that consists of options $i_1, i_2, \ldots, i_K$ where $K \geq 2$. This FCR is equivalent to Eqs. (2.18)-(2.19):

$$i_k \Longrightarrow \neg i_{k'}, \ \forall k = 1, \ldots, K-1, \ k' = k+1, \ldots, K, \tag{2.18}$$

$$i_1 \Longleftarrow \neg i_2 \wedge \cdots \wedge \neg i_K. \tag{2.19}$$

If the FCR is of type L, then it is equivalent to Eq. (2.18). Then, FCRs can be written in format ASR, and hence, the proof is complete.

2.9.5 Proof of Proportion 1

Using Lemma 3, Eq. (2.4) is equivalent to $\sum_{\substack{S: i \in S, \\ \{j_1, \cdots, j_l\} \cap S = \emptyset}} x(S) = 0$, for all rules in the form of $i \Rightarrow j_1 \vee \cdots \vee j_l$, which is equivalent to:

$$x(S) = 0, \quad \forall \text{ rules} : i \Rightarrow j_1 \vee \cdots \vee j_l, \text{ and } \forall S \text{ s.t. } i \in S, \{j_1, \cdots, j_l\} \cap S = \emptyset. \tag{2.20}$$

Similarly, using Lemma 3, Eq. (2.5) is equivalent to:

$$x(S) = 0, \quad \forall \text{ rules} : i \Leftarrow j_1 \wedge \cdots \wedge j_l, \text{ and } \forall S \text{ s.t. } \{j_1, \cdots, j_l\} \subset S, i \notin S. \tag{2.21}$$

Using Eqs. (2.20) and (2.21), and the definition of $\mathbb{S}^0$, we have $x(S) = 0$, for all $S \in \mathbb{S}^0$. Therefore, $\mathbb{P} = \{\mathbf{p} = (p_1, \cdots, p_n) |\ \mathbf{p}$ satisfies Eqs. (2.1), (2.2), (2.3), and $x(S) = 0, \forall S \in \mathbb{S}^0\}$. Eq. (2.1) does not impose any new constraint and is only used to calculate values of p_i's. Hence, consider solving Eqs. (2.2), (2.3), and $x(S) = 0, \forall S \in \mathbb{S}^0$. This system of equations include some nonnegative continuous variables of which some are fixed to zero and the rest must add up to 1. Thus, at any vertex of this system of equations, exactly one $x(.)$ where the corresponding S belongs to $\mathbb{S} \backslash \mathbb{S}^0$ is 1 and the rest are zero. Hence, the number of vertices are equivalent to $|\mathbb{S} \backslash \mathbb{S}^0|$. Moreover, using Eq. (2.1), we have $p_i = 0$ or 1 for all $i \in N$; therefore, the vertices of $\mathbb{P}$ are integer points.

Let us consider a particular vertex where $x(\hat{S}) = 1$ for exactly one $\hat{S}$ such that $\hat{S} \in \mathbb{S} \backslash \mathbb{S}^0$ and $x(S) = 0$, for all $S \neq \hat{S}, S \in \mathbb{S}$. Using Eq. (2.1), we have:

$$p_i = x(\hat{S})\mathbf{1}(i \in \hat{S}) + \sum_{S: i \in S, S \neq \hat{S}} x(S), \qquad \forall i \in N \tag{2.22}$$

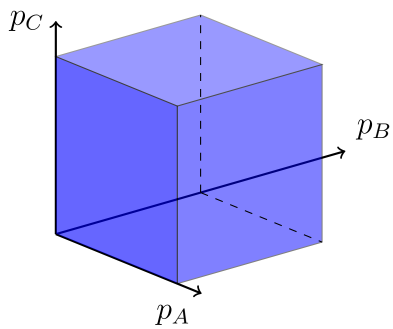
where $\mathbf{1}(.)$ is an indicator function that returns 1 if its argument is true, and 0 otherwise. But the second term in the right-hand-side of Eq. (2.22) is zero and the first term is $\mathbf{1}(i \in \hat{S})$; hence, $p_i = \mathbf{1}(i \in \hat{S})$, for all $i \in N$. Thus, the proof is complete.

2.9.6 Illustration of Proposition 1 and Corollary 1.1

Fig. 2.12 illustrates Proposition 1 and Corollary 1.1, using an example with three options: A, B, and C. Three scenarios are considered: no rules, three rules, and five rules. The set $\mathbb{P}$ and its vertices, sets $\mathbb{S}^0$ and $\mathbb{S}^1$, and the feasible configurations in each scenario are illustrated. One can see the existence of a one-to-one correspondence between $\mathbb{S}^1$, $\mathbb{Y}$, and $V_{\mathbb{P}}$.

Scenario 1: No rules; feasible region is $[0,1]^3$

$\mathbb{S}^0 = \{\}$
$\mathbb{S}^1 = \{\{\}, \{A\}, \{B\}, \{C\}, \{A,B\}, \{A,C\}$
$\quad , \{B,C\}, \{A,B,C\}\}$

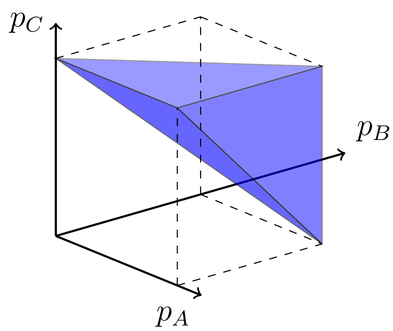


Feasible Configurations:

(0,0,0)
(0,0,1)
(0,1,0)
(0,1,1)
(1,0,0)
(1,0,1)
(1,1,0)
(1,1,1)

Scenario 2: Rules:
1) $A \Rightarrow B \vee C$
2) $\neg A \Rightarrow C$
3) $B \Rightarrow A$

$\mathbb{S}^0 = \{\{\}, \{A\}, \{B\}, \{B,C\}\}$
$\mathbb{S}^1 = \{\{C\}, \{A,B\}, \{A,C\}, \{A,B,C\}\}$

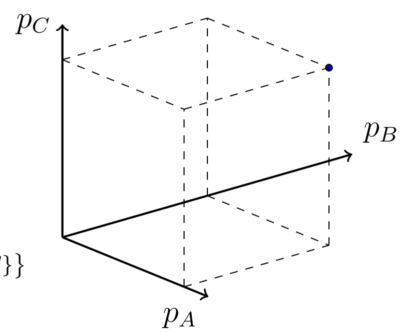


Feasible Configurations:

(0,0,1)
(1,0,1)
(1,1,0)
(1,1,1)

Scenario 3: Rules:
1) $A \Rightarrow B \vee C$
2) $\neg A \Rightarrow C$
3) $B \Rightarrow A$
4) $C \Rightarrow B$
5) $A \wedge B \Rightarrow C$

$\mathbb{S}^0 = \{\{\}, \{A\}, \{B\}, \{C\}, \{A,B\}, \{A,C\}, \{B,C\}\}$
$\mathbb{S}^1 = \{\{A,B,C\}\}$



Feasible Configurations:

(1,1,1)

Figure 2.12: Graphical illustration: existence of a one-to-one correspondence between $\mathbb{S}^1$, $\mathbb{Y}$, and $V_{\mathbb{P}}$.

2.10 Appendix for Handling the Nonconvexity

2.10.1 Proof of Lemma 4

$\mathcal{H}_{\hat{\mathbf{p}}}$ is defined by 1 equality and n inequalities; hence, the vertices of $\mathcal{H}_{\hat{\mathbf{p}}}$ are obtained by setting $n-1$ of inequalities to equalities and solving them together with the equation $\hat{\mathbf{p}}^T\mathbf{p} = \hat{\mathbf{p}}^T\hat{\mathbf{p}}$. Since $\hat{p}_i > 0$, for all $i \in N$, this results in vertices $\frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_i}\mathbf{e}^i$, for all $i \in N$. Moreover, since there are n affinely independent vertices, then $\mathcal{H}_{\hat{\mathbf{p}}}$ is a (n-1)-simplex.

2.10.2 Proof of Lemma 5

$\mathcal{H}_{\hat{\mathbf{p}}}$ is defined by 1 equality and n inequalities; hence, the vertices of $\mathcal{H}_{\hat{\mathbf{p}}}$ are obtained by setting $n-1$ of inequalities to equalities and solving them together with the equation $\hat{\mathbf{p}}^T\mathbf{p} = \hat{\mathbf{p}}^T\hat{\mathbf{p}}$. Consider i' such that $\hat{p}_{i'} > 0$. Setting $p_i = 0$, for all $i \neq i'$, we obtain $p_{i'} = \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_{i'}}$; hence, an extreme point $\frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_{i'}}\mathbf{e}^{i'}$ is found. Therefore, for each $i \in I_1$, there exists an extreme point $\frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\hat{p}_i}\mathbf{e}^i$.

Extreme directions of $\mathcal{H}_{\hat{\mathbf{p}}}$ are the extreme points of the set $\{\mathbf{d} \in \mathbb{R}^n | \hat{\mathbf{p}}^T\mathbf{d} = 0, \mathbf{d} \geq 0, \mathbf{1}^T\mathbf{d} = 1\}$, where $\mathbf{1}$ is the vector of appropriate size with all entries equal to 1. This results in the extreme directions $\mathbf{e}^i$, for all $i \in I_\emptyset$.

2.10.3 Proof of Remark 1.1

Using the definition of $\mathcal{FH}_{\hat{\mathbf{p}}}$, we have $\hat{\mathbf{p}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ if and only if the following two conditions hold: (i) $\hat{\mathbf{p}} \in \mathcal{H}_{\hat{\mathbf{p}}}$, and (ii) $\exists \bar{\mathbf{p}} \in \mathbb{P}, \lambda \geq 0$ s.t. $\lambda\bar{\mathbf{p}} = \hat{\mathbf{p}}$. Condition (i) is always satisfied and condition (ii) holds if and only if $\hat{\mathbf{p}} \in cone(\mathbb{P})$. Hence, the proof is complete.

2.10.4 Proof of Proposition 2

Let $\tilde{\mathbb{P}}$ denote the convex hull of all feasible configurations that do not contain options in $I_\emptyset$. Using remark 1.1, it suffices to show $\hat{\mathbf{p}} \in cone(\mathbb{P})$ if and only if $\tilde{\mathbf{p}} \in cone(\tilde{\mathbb{P}})$.

If $\hat{\mathbf{p}} \in cone(\mathbb{P})$, then $\hat{\mathbf{p}} = \lambda' \mathbf{p}'$ for some $\lambda' > 0$, $\mathbf{p}' \in \mathbb{P}$. There exist feasible configurations $\mathbf{y}^1, \ldots, \mathbf{y}^l$ and strictly positive scalars $\alpha_1, \ldots, \alpha_l$ satisfying $\sum_{l'=1}^{l} \alpha_{l'} = 1$ such that $\mathbf{p}' = \sum_{l'=1}^{l} \alpha_{l'} \mathbf{y}^{l'}$. Observe that in $\mathbf{y}^1, \ldots, \mathbf{y}^l$, the values corresponding to options in $I_\emptyset$ must be zero. Let $\tilde{\mathbf{y}}^1, \ldots, \tilde{\mathbf{y}}^l$ be the vectors generated by eliminating elements corresponding to options in $I_\emptyset$. Moreover, let $\mathbf{p}'' = \sum_{l'=1}^{l} \alpha_{l'} \tilde{\mathbf{y}}^{l'}$. Note that $\mathbf{p}'' \in \tilde{\mathbb{P}}$ and $\tilde{\mathbf{p}} = \lambda' \mathbf{p}''$; hence, $\tilde{\mathbf{p}} \in cone(\tilde{\mathbb{P}})$. The converse is proven in a similar fashion.

2.10.5 Proof of Theorem 1

(a, $\Rightarrow$) We first prove if $\mathbf{p}^{**} = 0$, then $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\}$, using a contradiction. Suppose that $\mathbf{p}^{**} = 0$ and $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$. Since $\mathcal{FH}_{\hat{\mathbf{p}}}$ is nonempty, then there exists $\bar{\mathbf{p}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$. Hence, using the definition of $\mathcal{FH}_{\hat{\mathbf{p}}}$, we have $\bar{\mathbf{p}} \in \mathbb{R}^n_+$, $\hat{\mathbf{p}}^T \bar{\mathbf{p}} = \hat{\mathbf{p}}^T \hat{\mathbf{p}}$ and $\bar{\mathbf{p}} \in cone(\mathbb{P})$. Moreover, because $\hat{\mathbf{p}}^T \hat{\mathbf{p}} > 0$, then $\bar{\mathbf{p}} \neq 0$.

Define $\bar{\bar{\mathbf{p}}} := \frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\bar{\mathbf{p}}^T \bar{\mathbf{p}}} \bar{\mathbf{p}}$. Since $\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\bar{\mathbf{p}}^T \bar{\mathbf{p}}} > 0$ and $\bar{\mathbf{p}} \in cone(\mathbb{P})$, then $\bar{\bar{\mathbf{p}}} \in cone(\mathbb{P})$ (because of definition of cone). In order to contradict the optimality of $\mathbf{p}^{**} = 0$, we want to show the objective value of $\bar{\bar{\mathbf{p}}}$ is strictly better than that of $\mathbf{0}$. We have: $\|\hat{\mathbf{p}} - \bar{\bar{\mathbf{p}}}\|^2 = \|\hat{\mathbf{p}}\|^2 + \|\bar{\bar{\mathbf{p}}}\|^2 - 2\hat{\mathbf{p}}^T \bar{\bar{\mathbf{p}}}$. Note that $\|\bar{\bar{\mathbf{p}}}\|^2 - 2\hat{\mathbf{p}}^T \bar{\bar{\mathbf{p}}} = (\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\bar{\mathbf{p}}^T \bar{\mathbf{p}}})^2 \bar{\mathbf{p}}^T \bar{\mathbf{p}} - 2(\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\bar{\mathbf{p}}^T \bar{\mathbf{p}}}) \hat{\mathbf{p}}^T \bar{\mathbf{p}} = -(\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\bar{\mathbf{p}}^T \bar{\mathbf{p}}}) \hat{\mathbf{p}}^T \hat{\mathbf{p}} < 0$ (because $\hat{\mathbf{p}}^T \bar{\mathbf{p}} = \hat{\mathbf{p}}^T \hat{\mathbf{p}}$). Therefore, $\|\hat{\mathbf{p}} - \bar{\bar{\mathbf{p}}}\|^2 < \|\hat{\mathbf{p}}\|^2 = \|\hat{\mathbf{p}} - \mathbf{0}\|^2$. This contradicts $\mathbf{p}^{**} = 0$ because $\bar{\bar{\mathbf{p}}}$ is feasible and has a strictly better objective value. Hence, $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\}$.

(a, $\Leftarrow$) We prove that if $\mathcal{FH}_{\hat{\mathbf{p}}} = \{\}$, then $\mathbf{p}^{**} = \mathbf{0}$. Recall that in this chapter we assume $\mathbb{P} \neq \{\}$, then $\mathbf{0} \in cone(\mathbb{P})$. We must show that $\mathbf{0}$ is optimal ($\mathbf{0}$ has the smallest objective value among all $\mathbf{p} \in cone(\mathbb{P})$).

We claim that if $\bar{\mathbf{p}} \in cone(\mathbb{P})$, then $\hat{\mathbf{p}}^T \bar{\mathbf{p}} \leq 0$. We prove this claim using a contradiction. Let $\bar{\mathbf{p}} \in cone(\mathbb{P})$ such that $\hat{\mathbf{p}}^T \bar{\mathbf{p}} > 0$. Define $\bar{\bar{\mathbf{p}}} := \frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\hat{\mathbf{p}}^T \bar{\mathbf{p}}} \bar{\mathbf{p}}$. Because $\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\hat{\mathbf{p}}^T \bar{\mathbf{p}}} > 0$ and $\bar{\mathbf{p}} \in cone(\mathbb{P})$, then $\bar{\bar{\mathbf{p}}} \in cone(\mathbb{P})$. Moreover, $\hat{\mathbf{p}}^T \bar{\bar{\mathbf{p}}} = \hat{\mathbf{p}}^T (\frac{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}{\hat{\mathbf{p}}^T \bar{\mathbf{p}}} \bar{\mathbf{p}}) = \hat{\mathbf{p}}^T \hat{\mathbf{p}}$. Therefore, $\bar{\bar{\mathbf{p}}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$, meaning that $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$. This is a contradiction; hence, $\hat{\mathbf{p}}^T \bar{\mathbf{p}} \leq 0$, for all $\bar{\mathbf{p}} \in cone(\mathbb{P})$.

Let $\bar{\mathbf{p}} \in cone(\mathbb{P})$. Define $\dot{\mathbf{p}} := (1 - \frac{\hat{\mathbf{p}}^T \bar{\mathbf{p}}}{\hat{\mathbf{p}}^T \hat{\mathbf{p}}}) \hat{\mathbf{p}}$ and $\ddot{\mathbf{p}} := (\frac{\hat{\mathbf{p}}^T \bar{\mathbf{p}}}{\hat{\mathbf{p}}^T \hat{\mathbf{p}}} \hat{\mathbf{p}} - \bar{\mathbf{p}})$. Note that $\dot{\mathbf{p}}$ and $\ddot{\mathbf{p}}$ are orthogonal because $\dot{\mathbf{p}}^T \ddot{\mathbf{p}} = (1 - \frac{\hat{\mathbf{p}}^T \bar{\mathbf{p}}}{\hat{\mathbf{p}}^T \hat{\mathbf{p}}})(\frac{\hat{\mathbf{p}}^T \bar{\mathbf{p}}}{\hat{\mathbf{p}}^T \hat{\mathbf{p}}} \hat{\mathbf{p}}^T \hat{\mathbf{p}} - \hat{\mathbf{p}}^T \bar{\mathbf{p}}) = 0$. Moreover, we have $\|\dot{\mathbf{p}}\|^2 \geq \|\hat{\mathbf{p}}\|^2$, because $(1 - \frac{\hat{\mathbf{p}}^T\bar{\mathbf{p}}}{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}) \geq 1$ (because $\hat{\mathbf{p}}^T\bar{\mathbf{p}} \leq 0$ and $\hat{\mathbf{p}}^T\hat{\mathbf{p}} > 0$). The objective value of $\bar{\mathbf{p}}$ is: $\|\hat{\mathbf{p}} - \bar{\mathbf{p}}\|^2 = \|\hat{\mathbf{p}} - \frac{\hat{\mathbf{p}}^T\bar{\mathbf{p}}}{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}\hat{\mathbf{p}} + \frac{\hat{\mathbf{p}}^T\bar{\mathbf{p}}}{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}\hat{\mathbf{p}} - \bar{\mathbf{p}}\|^2 = \|\dot{\mathbf{p}} - \ddot{\mathbf{p}}\|^2 = \|\dot{\mathbf{p}}\|^2 + \|\ddot{\mathbf{p}}\|^2 \geq \|\dot{\mathbf{p}}\|^2 \geq \|\hat{\mathbf{p}}\|^2 = \|\hat{\mathbf{p}} - \mathbf{0}\|^2$. Therefore, we proved that $\|\hat{\mathbf{p}} - \bar{\mathbf{p}}\|^2 \geq \|\hat{\mathbf{p}} - 0\|^2$, for all $\bar{\mathbf{p}} \in cone(\mathbb{P})$. Thus, $\mathbf{p}^{**} = 0$ is an optimal solution.

(b) Since $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$, using part (a), we have $\mathbf{p}^{**} \neq 0$. We first claim that $\mathbf{p}^{**T}\hat{\mathbf{p}} > 0$. Since $\mathbf{p}^{**},\ \hat{\mathbf{p}} \in \mathbb{R}^n_+$, then, $\mathbf{p}^{**T}\hat{\mathbf{p}} \geq 0$. Suppose to the contrary that $\mathbf{p}^{**T}\hat{\mathbf{p}} = 0$ (meaning that $\mathbf{p}^{**}$ and $\hat{\mathbf{p}}$ are orthogonal). Hence, $\|\hat{\mathbf{p}} - \mathbf{p}^{**}\|^2 = \|\hat{\mathbf{p}}\|^2 + \|\mathbf{p}^{**}\|^2 > \|\hat{\mathbf{p}}\|^2 = \|\hat{\mathbf{p}} - \mathbf{0}\|^2$. Since $\mathbf{0} \in cone(\mathbb{P})$ (see part (a)), this contradicts the optimality of $\mathbf{p}^{**}$, because $\mathbf{0}$ has a strictly better objective value.

We next show that $\mathbf{p}^{**}$ and $\mathbf{p}^{*}$ are unique. We prove this for $\mathbf{p}^{**}$ using a contradiction (the proof for $\mathbf{p}^{*}$ is similar). Suppose that $\dot{\mathbf{p}}, \ddot{\mathbf{p}} \in cone(\mathbb{P})$ such that $\dot{\mathbf{p}} \neq \ddot{\mathbf{p}}$ and both $\dot{\mathbf{p}}$ and $\ddot{\mathbf{p}}$ are optimal for problem $\min_{\mathbf{p} \in cone(\mathbb{P})} \|\mathbf{p} - \hat{\mathbf{p}}\|^2$; hence $\|\dot{\mathbf{p}} - \hat{\mathbf{p}}\|^2 = \|\ddot{\mathbf{p}} - \hat{\mathbf{p}}\|^2 \leq \|\mathbf{p} - \hat{\mathbf{p}}\|^2$, for all $\mathbf{p} \in cone(\mathbb{P})$. Define $\tilde{\mathbf{p}} := \frac{1}{2}(\dot{\mathbf{p}} + \ddot{\mathbf{p}})$. Note that $\tilde{\mathbf{p}} \in cone(\mathbb{P})$ because $cone(\mathbb{P})$ is a convex set. We show that the objective value of $\tilde{\mathbf{p}}$ is strictly better than that of $\dot{\mathbf{p}}$ (or $\ddot{\mathbf{p}}$). We have: $\|\tilde{\mathbf{p}} - \hat{\mathbf{p}}\|^2 = \|\frac{1}{2}(\dot{\mathbf{p}} + \ddot{\mathbf{p}}) - \hat{\mathbf{p}}\|^2 = \frac{1}{4}\|\dot{\mathbf{p}} - \hat{\mathbf{p}}\|^2 + \frac{1}{4}\|\ddot{\mathbf{p}} - \hat{\mathbf{p}}\|^2 + \frac{1}{2}(\dot{\mathbf{p}} - \hat{\mathbf{p}})^T(\ddot{\mathbf{p}} - \hat{\mathbf{p}}) \leq \|\dot{\mathbf{p}} - \hat{\mathbf{p}}\|^2$. The equality holds only if the angle between $(\dot{\mathbf{p}} - \hat{\mathbf{p}})$ and $(\ddot{\mathbf{p}} - \hat{\mathbf{p}})$ is 0, in which case we must have $\dot{\mathbf{p}} = \ddot{\mathbf{p}}$ (because the lengthes of vectors $(\dot{\mathbf{p}} - \hat{\mathbf{p}})$ and $(\ddot{\mathbf{p}} - \hat{\mathbf{p}})$ are equivalent). This contradicts $\dot{\mathbf{p}} \neq \ddot{\mathbf{p}}$; hence $\mathbf{p}^{**}$ is unique. As we use the convexity of the feasible region to prove the uniqueness of $\mathbf{p}^{**}$, the proof for $\mathbf{p}^{*}$ is almost identical (because $\mathcal{FH}_{\hat{\mathbf{p}}}$ is a convex set).

Define $\lambda^{**} := \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\mathbf{p}^{**T}\hat{\mathbf{p}}}$. Then, $\lambda^{**} > 0$, because $\hat{\mathbf{p}}^T\hat{\mathbf{p}} > 0$ and $\mathbf{p}^{**T}\hat{\mathbf{p}} > 0$. Moreover, $\lambda^{**}\mathbf{p}^{**} \in \mathcal{FH}_{\hat{\mathbf{p}}}$, because $\lambda^{**}\mathbf{p}^{**} \in cone(\mathbb{P})$ and $\hat{\mathbf{p}}^T(\lambda^{**}\mathbf{p}^{**}) = \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\mathbf{p}^{**T}\hat{\mathbf{p}}}\hat{\mathbf{p}}^T\mathbf{p}^{**} = \hat{\mathbf{p}}^T\hat{\mathbf{p}}$. Moreover, $\frac{\mathbf{p}^*}{\lambda^{**}} \in cone(\mathbb{P})$, because $\frac{1}{\lambda^{**}} > 0$ and $\mathbf{p}^* \in cone(\mathbb{P})$. Hence, in summary, we have $\mathbf{p}^*,\ \lambda^{**}\mathbf{p}^{**} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ and $\frac{\mathbf{p}^*}{\lambda^{**}},\ \mathbf{p}^{**} \in cone(\mathbb{P})$.

Finally, we prove that $\mathbf{p}^* = \lambda^{**}\mathbf{p}^{**}$ using a contradiction. Suppose to the contrary that

$\mathbf{p}^* \neq \lambda^{**}\mathbf{p}^{**}$ (or $\frac{1}{\lambda^{**}}\mathbf{p}^* \neq \mathbf{p}^{**}$). Since $\mathbf{p}^{**}$ is unique and $\frac{1}{\lambda^{**}}\mathbf{p}^* \in cone(\mathbb{P})$, then we have:

$$\begin{aligned}
&\|\mathbf{p}^{**} - \hat{\mathbf{p}}\|^2 < \|\frac{1}{\lambda^{**}}\mathbf{p}^* - \hat{\mathbf{p}}\|^2 \\
&\Longrightarrow \|\mathbf{p}^{**} - \frac{1}{\lambda^{**}}\hat{\mathbf{p}} - (1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}\|^2 < \|\frac{1}{\lambda^{**}}\mathbf{p}^* - \frac{1}{\lambda^{**}}\hat{\mathbf{p}} - (1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}\|^2 \\
&\Longrightarrow \|\mathbf{p}^{**} - \frac{1}{\lambda^{**}}\hat{\mathbf{p}}\|^2 + \|(1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}\|^2 < \|\frac{1}{\lambda^{**}}\mathbf{p}^* - \frac{1}{\lambda^{**}}\hat{\mathbf{p}}\|^2 + \|(1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}\|^2 \\
&\Longrightarrow \|\mathbf{p}^{**} - \frac{1}{\lambda^{**}}\hat{\mathbf{p}}\|^2 < \|\frac{1}{\lambda^{**}}\mathbf{p}^* - \frac{1}{\lambda^{**}}\hat{\mathbf{p}}\|^2 \\
&\Longrightarrow \|\lambda^{**}\mathbf{p}^{**} - \hat{\mathbf{p}}\|^2 < \|\mathbf{p}^* - \hat{\mathbf{p}}\|^2.
\end{aligned}$$

The third line follows from the fact that $(\mathbf{p}^{**} - \frac{1}{\lambda^{**}}\hat{\mathbf{p}})$ and $(1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}$ are orthogonal, and $(\frac{1}{\lambda^{**}}\mathbf{p}^* - \frac{1}{\lambda^{**}}\hat{\mathbf{p}})$ and $(1 - \frac{1}{\lambda^{**}})\hat{\mathbf{p}}$ are orthogonal. On the other hand, since $\mathbf{p}^*$ is unique and $\lambda^{**}\mathbf{p}^{**} \in \mathcal{FH}_{\hat{\mathbf{p}}}$, then we must have: $\|\mathbf{p}^* - \hat{\mathbf{p}}\|^2 < \|\lambda^{**}\mathbf{p}^{**} - \hat{\mathbf{p}}\|^2$. This is a contradiction; hence, $\mathbf{p}^* = \lambda^{**}\mathbf{p}^{**}$. Note that $\lambda^{**} = \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\mathbf{p}^{**T}\hat{\mathbf{p}}} = \frac{\mathbf{p}^{*T}\mathbf{p}^*}{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}$. Thus, we showed that $\mathbf{p}^{**} = \frac{1}{\lambda^{**}}\mathbf{p}^* = \frac{\hat{\mathbf{p}}^T\hat{\mathbf{p}}}{\mathbf{p}^{*T}\mathbf{p}^*}\mathbf{p}^*$, and hence the proof is complete.

2.11 Appendix for Direction Step

2.11.1 Proof of Lemma 6

$\mathcal{G}^k = 0$ if and only if $\beta^k = 0$; hence, $\hat{\mathbf{p}} - \mathbf{p}^{*k} = 0$ and it follows that $\hat{\mathbf{p}} = \sum_{i=1}^k \alpha_i^* \theta^i$ for some $\alpha_i^* \geq 0, \forall i = 1, \cdots, k$, satisfying $\sum_{i=1}^k \alpha_i^* = 1$. Moreover, $\theta^i \in \mathcal{FH}_{\hat{\mathbf{p}}}, \forall i = 1, \cdots, k$, and $\mathcal{FH}_{\hat{\mathbf{p}}}$ is a convex set. Thus, $\hat{\mathbf{p}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$, which is also equivalent to $\hat{\mathbf{p}} \in cone(\mathbb{P})$, once remark 1.1 is applied.

2.11.2 Proof of Lemma 7

Note that $\mathcal{G}^k = \|\beta^k\| = \|\hat{\mathbf{p}} - \mathbf{p}^{*k}\|$; hence, at iteration k, $\mathcal{G}^k$ is the optimal value of the problem given in Eqs. (2.9)-(2.11). At iteration $k+1$, fixing $\alpha_{k+1} = 0$, reduces the feasible region of the problem given in Eqs. (2.9)-(2.11), and the problem becomes identical to that at iteration k; hence, fixing $\alpha_{k+1} = 0$, we find $\mathcal{G}^k$, which is not smaller than $\mathcal{G}^{k+1}$. Therefore, $\mathcal{G}^k \geq \mathcal{G}^{k+1}$, $\forall k = 1, 2, \cdots$.

2.12 Appendix for Maximization Step

2.12.1 Proof of Proposition 3

Note that $\mathcal{FH}_{\hat{\mathbf{p}}} \subseteq \mathcal{H}_{\hat{\mathbf{p}}}$; hence, a direction for $\mathcal{FH}_{\hat{\mathbf{p}}}$ is also a direction for $\mathcal{H}_{\hat{\mathbf{p}}}$. Then, as a corollary of Lemma 5, a direction of $\mathcal{FH}_{\hat{\mathbf{p}}}$ can be represented as a non-negative combination of the extreme directions of $\mathcal{H}_{\hat{\mathbf{p}}}$.

We first show that $\mathcal{M}(\beta^k)$ does not have unbounded optimal value. Suppose to the contrary that the optimal value of $\mathcal{M}(\beta^k)$ is unbounded. Let $\bar{\mathbf{p}} \in \mathcal{FH}_{\hat{\mathbf{p}}}$ be arbitrary. There must exist a direction $\mathbf{d}$ such that: $(\bar{\mathbf{p}} + \eta\mathbf{d}) \in \mathcal{FH}_{\hat{\mathbf{p}}}$, for all $\eta \in \mathbb{R}_+$, and the objective value of $(\bar{\mathbf{p}} + \eta'\mathbf{d})$ is strictly greater than that of $(\bar{\mathbf{p}} + \eta''\mathbf{d})$, if $\eta' > \eta''$. Vector $\mathbf{d}$ can be written as a non-negative combination of $\mathbf{e}^i$'s, $i \in I_\emptyset$; hence, there exist $\xi_i \geq 0$, for all $i \in I_\emptyset$, such that: $\mathbf{d} = \sum_{i \in I_\emptyset} \xi_i \mathbf{e}^i$. Let us compute the difference between the objective values of $(\bar{\mathbf{p}} + \eta'\mathbf{d})$ and $(\bar{\mathbf{p}} + \eta''\mathbf{d})$:

$$
\begin{aligned}
(\hat{\mathbf{p}} - \mathbf{p}^{*k})^T((\bar{\mathbf{p}} + \eta'\mathbf{d}) - (\bar{\mathbf{p}} + \eta''\mathbf{d})) &= (\eta' - \eta'')(\hat{\mathbf{p}} - \mathbf{p}^{*k})^T \mathbf{d} \\
&= (\eta' - \eta'')(\hat{\mathbf{p}} - \mathbf{p}^{*k})^T \sum_{i \in I_\emptyset} \xi_i \mathbf{e}^i \\
&= -(\eta' - \eta'') \sum_{i \in I_\emptyset} \xi_i p_i^{*k} \leq 0,
\end{aligned}
$$

where the last line follows from $\hat{p}_i = 0$, for all $i \in I_\emptyset$ (also note that $\eta' > \eta''$). This is a contradiction because the objective value of $(\bar{\mathbf{p}} + \eta'\mathbf{d})$ is **not** strictly greater than that of $(\bar{\mathbf{p}} + \eta''\mathbf{d})$. Hence, $\mathcal{M}(\beta^k)$ has a bounded optimal value.

Additionally, because $\mathcal{FH}_{\hat{\mathbf{p}}}$ is a nonempty polyhedron and $\mathcal{FH}_{\hat{\mathbf{p}}} \subseteq \mathbb{R}^n_+$, then $\mathcal{FH}_{\hat{\mathbf{p}}}$ does not contain a line, and has some (at least one) extreme points. Therefore, since $\mathcal{FH}_{\hat{\mathbf{p}}}$ has at least one extreme point and does not contain lines, an extreme point of $\mathcal{FH}_{\hat{\mathbf{p}}}$ must be optimal. This extreme point is in the form of $\lambda\mathbf{y}$, where $0 < \lambda < +\infty$ and $\mathbf{y} \in \mathbb{Y}$. Note that there might be other alternative optimal solutions in the same form.

2.12.2 Proof of Lemma 8

(a) Note that $\text{MWSAT}(\beta^k + \omega\hat{\mathbf{p}}) = \max_{\mathbf{y}\in\mathbb{Y}}(\beta^k + \omega\hat{\mathbf{p}})^T\mathbf{y} = \max_{\mathbf{y}_{\text{rel}}\in conv(\mathbb{Y})}(\beta^k + \omega\hat{\mathbf{p}})^T\mathbf{y}_{\text{rel}}$ where $\mathbf{y}_{\text{rel}}$ is a n-vector of continuous variables and $conv(\mathbb{Y})$ is the convex hull of the feasible configurations. For each $\mathbf{y}_{\text{rel}} \in conv(\mathbb{Y})$, the objective function, $(\beta^k + \omega\hat{\mathbf{p}})^T\mathbf{y}_{\text{rel}}$, is convex in ω. Then, $\text{MWSAT}(\beta^k + \omega\hat{\mathbf{p}})$ is convex in ω (due to "pointwise maximum property"; see, for example, section 3.2.3 of [Boyd S, Vandenberghe L (2004) Convex optimization. *Cambridge university press.*]). In other words, for each $\mathbf{y}_{\text{rel}} \in conv(\mathbb{Y})$, the objective function, $(\beta^k + \omega\hat{\mathbf{p}})^T\mathbf{y}_{\text{rel}}$, is linear in ω; hence, $\text{MWSAT}(\beta^k + \omega\hat{\mathbf{p}})$ is a piecewise linear, convex, and continuous function in ω.

(b) Since $\mathcal{FH}_{\hat{\mathbf{p}}} \neq \{\}$, then there exists $\mathbf{y} \in \mathbb{Y}$ such that $\hat{\mathbf{p}}^T\mathbf{y} > 0$. Hence, the proof is complete.

(c) Note that:

$$\begin{aligned}
\text{MWSAT}(\beta^k + \omega\hat{\mathbf{p}}) &= \max_{\mathbf{y}\in\mathbb{Y}} (\hat{\mathbf{p}} - \mathbf{p}^{*k} + \omega\hat{\mathbf{p}})^T\mathbf{y} \\
&= \max_{\mathbf{y}\in\mathbb{Y}} (-\mathbf{p}^{*k} + (1+\omega)\hat{\mathbf{p}})^T\mathbf{y} \\
&= \max_{\mathbf{y}\in\mathbb{Y}} \; -\mathbf{p}^{*kT}\mathbf{y} + (1+\omega)\hat{\mathbf{p}}^T\mathbf{y}.
\end{aligned}$$

In addition, we have $\hat{\mathbf{p}}^T\mathbf{y} \geq 0$ and $\mathbf{p}^{*kT}\mathbf{y} \geq 0$, for all $\mathbf{y} \in \mathbb{Y}$. Therefore, $\text{MWSAT}(\beta^k + \omega\hat{\mathbf{p}}) \leq 0$ if $\omega \leq 1$.

2.12.3 Proof of Proposition 5

The proof consists of the following steps. First, we show that strong duality holds for $\mathcal{M}(\beta^k)$. Second, we derive the dual problem to show that $\mathcal{M}(\beta^k)$ is equivalent to:

$$\max \left\{ \omega \in \mathbb{R} | \mathrm{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) \leq 0 \right\}.$$

Finally, we use Lemma 8 to show that the condition holds as equality.

Step 1: Noting that Eqs. (2.14)-(2.16) are equivalent to $\theta = \lambda \mathbf{y}$. We substitute $\lambda \mathbf{y}$ for θ in $\mathcal{M}(\beta^k)$ and eliminate Eqs. (2.14)-(2.16). Hence,

$$\begin{aligned}
\mathcal{M}(\beta^k) \iff & \max_{\lambda \geq 0, \mathbf{y} \in \mathbb{Y}} \lambda \beta^{kT} \mathbf{y} \text{ s.t. } \lambda \hat{\mathbf{p}}^T \mathbf{y} = \hat{\mathbf{p}}^T \hat{\mathbf{p}} && (2.23) \\
\iff & \max_{\mathbf{y}_\lambda \in cone(\mathbb{Y})} \beta^{kT} \mathbf{y}_\lambda \text{ s.t. } \hat{\mathbf{p}}^T \mathbf{y}_\lambda = \hat{\mathbf{p}}^T \hat{\mathbf{p}} && (2.24) \\
\iff & \max_{\mathbf{y}_{\lambda,\mathrm{rel}} \in conv(cone(\mathbb{Y}))} \beta^{kT} \mathbf{y}_{\lambda,\mathrm{rel}} \text{ s.t. } \hat{\mathbf{p}}^T \mathbf{y}_{\lambda,\mathrm{rel}} = \hat{\mathbf{p}}^T \hat{\mathbf{p}}, && (2.25)
\end{aligned}$$

where $\mathbf{y}_\lambda, \mathbf{y}_{\lambda,\mathrm{rel}} \in \mathbb{R}^n$ are vectors of continuous variables, and $cone(.)$ and $conv(.)$ are respectively the cone and convex hull generated by a given set. Eq. (2.24) is obtained by defining variable $\mathbf{y}_\lambda \in \mathbb{R}^n$ and substituting it for $\lambda \mathbf{y}$; hence, $\mathbf{y}_\lambda = \lambda \mathbf{y} \in cone(\mathbb{Y})$. The feasible region of Eq. (2.24) consists of discrete points on hyperplane $\hat{\mathbf{p}}^T \mathbf{y}_\lambda = \hat{\mathbf{p}}^T \hat{\mathbf{p}}$ and objective function is linear in $\mathbf{y}_\lambda$; hence, Eq. (2.25) is obtained once we relax $\mathbf{y}_\lambda$ and use the convex hull of the feasible region. Since Eq. (2.25) is a convex optimization problem, then strong duality holds for $\mathcal{M}(\beta^k)$.

Step 2: Using Eq. (2.23), the dual of $\mathcal{M}(\beta^k)$ is written as follows (one could start by writing the dual of Eq. (2.25) which leads to the same result):

$$\min_{\omega} \max_{\lambda \geq 0, \mathbf{y} \in \mathbb{Y}} \lambda \beta^{kT} \mathbf{y} + \omega \lambda \hat{\mathbf{p}}^T \mathbf{y} - \omega \hat{\mathbf{p}}^T \hat{\mathbf{p}} \tag{2.26}$$

$$\iff \min_{\omega} \left\{ -\omega \hat{\mathbf{p}}^T \hat{\mathbf{p}} + \max_{\lambda \geq 0} \left\{ \max_{\mathbf{y} \in \mathbb{Y}} \lambda (\beta^k + \omega \hat{\mathbf{p}})^T \mathbf{y} \right\} \right\} \tag{2.27}$$

$$\iff \min_{\omega} \left\{ -\omega \hat{\mathbf{p}}^T \hat{\mathbf{p}} + \max_{\lambda \geq 0} \lambda \left\{ \max_{\mathbf{y} \in \mathbb{Y}} (\beta^k + \omega \hat{\mathbf{p}})^T \mathbf{y} \right\} \right\} \tag{2.28}$$

$$\iff \min_{\omega} \left\{ -\omega \hat{\mathbf{p}}^T \hat{\mathbf{p}} + \max_{\lambda \geq 0} \lambda \left(\text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) \right) \right\} \tag{2.29}$$

$$\iff \min_{\omega} -\omega \hat{\mathbf{p}}^T \hat{\mathbf{p}} \text{ s.t. } \text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) \leq 0 \tag{2.30}$$

$$\iff \max_{\omega} \omega \text{ s.t. } \text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) \leq 0. \tag{2.31}$$

Eq. (2.27) is obtained by rearranging terms. Since $\lambda \geq 0$, we take it outside of the maximization to obtain Eq. (2.28). The maximization problem $\max_{\mathbf{y} \in \mathbb{Y}} (\beta^k + \omega \hat{\mathbf{p}})^T \mathbf{y}$ is in fact MWSAT with weight vector $\beta^k + \omega \hat{\mathbf{p}}$. In Eq. (2.29), for a fixed ω, if the optimal value of MWSAT is strictly positive, then the optimal value of the maximization problem becomes $+\infty$; hence, we are interested in the values of ω for which the optimal value of MWSAT is non-positive (Eq. 2.30). Finally, note that $\hat{\mathbf{p}}^T \hat{\mathbf{p}} > 0$, which results in Eq. (2.31). Since strong duality holds, $\mathcal{M}(\beta^k)$ is equivalent to Eq. (2.31).

Step 3: Let $\omega^*_{\max}$ denote the greatest solution of $\text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) = 0$. Using lemma 8, $\omega^*_{\max} < +\infty$, and we have $\text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) > 0$ for all $\omega > \omega^*_{\max}$. Therefore, $\mathcal{M}(\beta^k) \iff \max \left\{ \omega \in \mathbb{R} | \text{MWSAT}(\beta^k + \omega \hat{\mathbf{p}}) = 0 \right\}$.

2.13 Appendix for Convergence Rate

2.13.1 Proof of Remark 5.1

Once θ is fixed, the minimization problem is to find the closest point in the convex hull of $\{\theta^1, \cdots, \theta^k, \theta\}$ to $\hat{\mathbf{p}}$. Since we are interested in the worst-case, we maximize over all θ's requiring that the following two constraints are satisfied. First, the Euclidean distance between any pair of feasible points must be less than or equal to the diameter of the set $\mathcal{FH}_{\hat{\mathbf{p}}}$. Second, $\beta^{kT}(\theta - \hat{\mathbf{p}}) \geq 0$, which is required to have $\hat{\mathbf{p}} \in cone(\mathbb{P})$ since otherwise a hyperplane exists that separates $\hat{\mathbf{p}}$ and $\mathcal{FH}_{\hat{\mathbf{p}}}$, and hence $\hat{\mathbf{p}}$ is infeasible.

2.13.2 Proof of Theorem 2

This theorem is proven in two steps. First, we show if $\hat{\mathbf{p}} \in cone(\mathbb{P})$, $\mathcal{G}^1 = \frac{\mathcal{D}}{\sqrt{2}}$, and $\theta^k = \theta_w^k$, for all $k \geq 2$, then: (1) $\mathcal{G}^k = \frac{\mathcal{G}^1}{\sqrt{k}}$, for all $1 \leq k \leq n$, and (2) $\mathcal{G}^k \leq \frac{\mathcal{G}^1}{\sqrt{k}}$, for all $k > n$. Second, we provide an example for which we have: $\mathcal{G}^k = \frac{\mathcal{G}^1}{\sqrt{k}}$, for all $1 \leq k \leq n-1$.

To prove the first step, we use an inductive argument to show if $\hat{\mathbf{p}} \in cone(\mathbb{P})$, $\mathcal{G}^1 = \frac{\mathcal{D}}{\sqrt{2}}$, and $\theta^k = \theta_w^k$, for all $k \geq 2$, then, without loss of generality, $\theta^k = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^k$, for all $k = 1, \cdots, n$, where $\mathbf{e}^k$ is the vector with all 0's except for a 1 in the kth coordinate—e.g., $\mathbf{e}^1 = (1, 0, 0, \cdots, 0)$.

Let $k = 1$ and θ^1 be given as we previously discussed. Then, $\mathbf{p}^{*1} = \theta^1$ (since $\mathbf{p}^{*k}$ is the closest point in the convex combination of the set $\{\theta^1, \cdots, \theta^k\}$ to $\hat{\mathbf{p}}$). Thus, $\frac{\mathcal{D}}{\sqrt{2}} = \mathcal{G}^1 = \|\beta^1\| = \|\hat{\mathbf{p}} - \mathbf{p}^{*1}\| = \|\hat{\mathbf{p}} - \theta^1\|$; hence, the Euclidean distance between $\hat{\mathbf{p}}$ and θ^1 is $\mathcal{G}^1$. Therefore, let $\theta^1 = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^1$, which is without loss of generality since it can always be achieved by a rotation about $\hat{\mathbf{p}}$.

For $1 \leq k \leq n-1$, assume that $\theta^i = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^i$, for all $i = 1, \cdots, k$. In the following, we show $\theta^{k+1} = \theta_w^{k+1} = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^{k+1}$. Substituting $\theta^i = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^i$, for all $i = 1, \cdots, k$, in Remark

5.1, we have:

$$\begin{aligned}
\theta^{k+1} = \theta_w^{k+1} &= \arg\max_{\theta\in\Theta^k} \left\{ \min_{\alpha\in\mathcal{A}^k} \left\| \sum_{i=1}^{k} \alpha_i(\hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^i) + \alpha_{k+1}\theta - \hat{\mathbf{p}} \right\| \right\} \\
&= \arg\max_{\theta\in\Theta^k} \left\{ \min_{\alpha\in\mathcal{A}^k} \left\| \mathcal{G}^1 \sum_{i=1}^{k} \alpha_i\mathbf{e}^i + \alpha_{k+1}(\theta - \hat{\mathbf{p}}) \right\| \right\} \\
&= \arg\max_{\theta\in\Theta^k} \left\{ \min_{\alpha\in\mathcal{A}^k} \frac{1}{2} \left(\mathcal{G}^1 \sum_{i=1}^{k} \alpha_i\mathbf{e}^i + \alpha_{k+1}(\theta - \hat{\mathbf{p}}) \right)^T \left(\mathcal{G}^1 \sum_{i=1}^{k} \alpha_i\mathbf{e}^i + \alpha_{k+1}(\theta - \hat{\mathbf{p}}) \right) \right\} \\
&= \arg\max_{\theta\in\Theta^k} \left\{ \min_{\alpha\in\mathcal{A}^k} \frac{1}{2}\alpha^T Q\alpha \right\}, \qquad (2.32)
\end{aligned}$$

where $\Theta^k = \{\theta \in \mathbb{R}^n : \|\theta - \hat{\mathbf{p}} - \mathcal{G}^1\mathbf{e}^i\|^2 \leq \mathcal{D}^2$, for all $i = 1, \cdots, k, \mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} \leq 0\}$, $\mathcal{A}^k = \{\alpha \in \mathbb{R}^{k+1} : \alpha \geq 0, \mathbf{1}^{(k+1)T}\alpha = 1\}$, and

$$Q = \begin{pmatrix} \mathcal{G}^{1^2}\mathbf{I}_k & \mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)} \\ \mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)T} & (\theta - \hat{\mathbf{p}})^T(\theta - \hat{\mathbf{p}}) \end{pmatrix}$$

where $\mathbf{I}_k$ is a $k \times k$ identity matrix, $(\theta - \hat{\mathbf{p}})^{(k)}$ is a k-vector created by the first k entries of the vector $(\theta - \hat{\mathbf{p}})$, and Q is a $(k+1) \times (k+1)$ matrix.

Let $\mathscr{S}$ denote the Schur complement of the block $\mathcal{G}^{1^2}\mathbf{I}_k$ of the matrix Q; hence, $\mathscr{S} = (\theta - \hat{\mathbf{p}})^T(\theta - \hat{\mathbf{p}}) - \mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)T}(\mathcal{G}^{1^2}\mathbf{I}_k)^{-1}\mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)} = \sum_{i=k+1}^{n}(\theta_i - \hat{p}_i)^2 \geq 0$. The matrix Q is positive semi-definite, which can be shown by performing elementary row operations on Q and converting it to an upper diagonal matrix where the diagonal entries, and hence the eigenvalues, are $\mathcal{G}^{1^2}, \cdots, \mathcal{G}^{1^2}, \mathscr{S}$. For this, one needs to multiply rows $1, \cdots, k$ by $\frac{(\theta_i - \hat{p}_i)}{\mathcal{G}^1}$ and subtract their sum from row $k+1$. We have $\mathcal{G}^{1^2} > 0$; then, if $\mathscr{S} > 0$, Q is positive definite, and hence invertible. For now, we assume $\mathscr{S} > 0$, and hence Q is positive definite and invertible, which is confirmed by the solution. We will analyze $\mathscr{S} = 0$ later in this proof.

For a fixed θ, the problem $\min_{\alpha\in\mathcal{A}^k} \frac{1}{2}\alpha^T Q\alpha$ is convex since Q is positive definite and $\mathcal{A}^k$ is a convex set. The dual of this problem is $\max_{\vartheta\geq 0,\nu} g(\vartheta, \nu)$, where $g(\vartheta, \nu) = \inf_\alpha \mathscr{L}(\alpha, \vartheta, \nu)$,

$\mathscr{L}(\alpha, \vartheta, \nu) = \frac{1}{2}\alpha^T Q\alpha - (\vartheta - \nu\mathbf{1}^{(k+1)})^T\alpha - \nu$, $\vartheta \in \mathbb{R}^{k+1}$, and $\nu \in \mathbb{R}$. Since Q is positive definite, then $\mathscr{L}(\alpha, \vartheta, \nu)$ is convex in α. Since $\mathscr{L}(\alpha, \vartheta, \nu)$ is differentiable, the minimizer of $\mathscr{L}(\alpha, \vartheta, \nu)$ is found by $\nabla_\alpha \mathscr{L}(\alpha, \vartheta, \nu) = 0$, which results in $\alpha^* = Q^{-1}(\vartheta - \nu\mathbf{1}^{(k+1)})$, where:

$$Q^{-1} = \begin{pmatrix} \mathcal{G}^{1^{-2}}\mathbf{I}_k & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\mathcal{G}^{1^2}\mathscr{S}} \begin{pmatrix} (\theta - \hat{\mathbf{p}})^{(k)}(\theta - \hat{\mathbf{p}})^{(k)T} & -\mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)} \\ -\mathcal{G}^1(\theta - \hat{\mathbf{p}})^{(k)T} & \mathcal{G}^{1^2} \end{pmatrix}$$

Hence, using α^*, we obtain $g(\vartheta, \nu) = \inf_\alpha \mathscr{L}(\alpha, \vartheta, \nu) = -\frac{1}{2}(\vartheta - \nu\mathbf{1}^{(k+1)})^T Q^{-1}(\vartheta - \nu\mathbf{1}^{(k+1)}) - \nu$. The function $g(\vartheta, \nu)$ is concave in ϑ; hence, fixing ν, the maximum of $g(\vartheta, \nu)$ occurs in $\nabla_\vartheta g(\vartheta, \nu) = Q^{-1}(\vartheta - \nu\mathbf{1}^{(k+1)}) = 0$ or at the boundary $\vartheta = 0$. From $\nabla_\vartheta g(\vartheta, \nu) = 0$, it follows that $\vartheta = \nu\mathbf{1}^{(k+1)}$, and $\max_{\vartheta \geq 0, \nu} g(\vartheta, \nu) = \max_\nu -\nu = +\infty$, with $\nu = -\infty$; hence, $\vartheta < 0$ which is a contradiction. Therefore, optimal ϑ is 0; thus, $\max_{\vartheta \geq 0, \nu} g(\vartheta, \nu) = \max_\nu -\frac{1}{2}\nu^2 \mathbf{1}^{(k+1)T} Q^{-1}\mathbf{1}^{(k+1)} - \nu$. Since $\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)} > 0$, then this problem is maximized in $\nu = -1/(\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)})$. Substituting optimal ν, we obtain $\max_{\vartheta \geq 0, \nu} g(\vartheta, \nu) = 1/(\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)})$. Substituting this result in Eq. (2.32), we have $\theta^{k+1} = \underset{\theta \in \Theta^k}{\arg\max}\{1/(\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)})\} = \underset{\theta \in \Theta^k}{\arg\min}\{\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)}\}$. Moreover, using the definition of Q^{-1}, we have $\mathbf{1}^{(k+1)T}Q^{-1}\mathbf{1}^{(k+1)} = \frac{1}{\mathcal{G}^{1^2}}\left(k + \frac{1}{\mathscr{S}}\left(\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} - \mathcal{G}^1\right)^2\right)$; hence, since $\mathcal{G}^{1^2} > 0$ and k is constant, we have:

$$\theta^{k+1} = \theta_w^{k+1} = \underset{\theta \in \Theta^k}{\arg\min}\frac{1}{\mathscr{S}}\left(\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} - \mathcal{G}^1\right)^2 \tag{2.33}$$

Consider the constraint $\|\theta - \hat{\mathbf{p}} - \mathcal{G}^1\mathbf{e}^i\|^2 \leq \mathcal{D}^2$, for all $i = 1, \cdots, k$. This inequality can be written as $(\theta - \hat{\mathbf{p}})^T(\theta - \hat{\mathbf{p}}) - 2\mathcal{G}^1\mathbf{e}^{iT}(\theta - \hat{\mathbf{p}}) + \mathcal{G}^{1^2} \leq \mathcal{D}^2$, for all $i = 1, \cdots, k$. Noting that $(\theta - \hat{\mathbf{p}})^T(\theta - \hat{\mathbf{p}}) = \mathscr{S} + (\theta - \hat{\mathbf{p}})^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)}$, and $\mathcal{D}^2 = 2\mathcal{G}^{1^2}$, we have $\mathscr{S} + (\theta - \hat{\mathbf{p}})^{(k)T}(\theta -$

$\hat{\mathbf{p}})^{(k)} - 2\mathcal{G}^1\mathbf{e}^{iT}(\theta - \hat{\mathbf{p}}) \leq {\mathcal{G}^1}^2$, for all $i = 1, \cdots, k$. Thus, using Eq. (2.33), we have:

$$\theta^{k+1} = \theta_w^{k+1} = \arg\min_{\theta \in \mathbb{R}^n} \quad \frac{1}{\mathscr{S}}\left(\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} - \mathcal{G}^1\right)^2 \tag{2.34}$$

$$\text{s.t.} \quad \mathscr{S} + (\theta - \hat{\mathbf{p}})^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} - 2\mathcal{G}^1\mathbf{e}^{iT}(\theta - \hat{\mathbf{p}}) \leq {\mathcal{G}^1}^2, \forall i = 1, \cdots, k \tag{2.35}$$

$$\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} \leq 0. \tag{2.36}$$

Summing Eq. (2.35) over $i = 1, \cdots, k$, and dividing by k, we have $\mathscr{S} + (\theta - \hat{\mathbf{p}})^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} - \frac{2}{k}\mathcal{G}^1\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} \leq {\mathcal{G}^1}^2$. Noting that $\mathscr{S} > 0$, $(\theta - \hat{\mathbf{p}})^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} \geq 0$, and $\mathbf{1}^{(k)T}(\theta - \hat{\mathbf{p}})^{(k)} \leq 0$ (see Eq. 2.36), this problem is optimized if $(\theta - \hat{\mathbf{p}})^{(k)} = 0$ and $\mathscr{S} = {\mathcal{G}^1}^2$. Hence, the optimal θ must satisfy the following conditions: (i) the vector $(\theta - \hat{\mathbf{p}})$ must be orthogonal to $(\theta^1 - \hat{\mathbf{p}}), \cdots, (\theta^k - \hat{\mathbf{p}})$, and (ii) the Euclidean distance between the optimal θ and $\hat{\mathbf{p}}$ is equal to $\mathcal{G}^1$. Thus, although the optimal solution might not be unique, we let without loss of generality, $\theta^{k+1} = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^{k+1}$, which satisfies the above two conditions for optimality.

Using $\theta^{k+1} = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^{k+1}$, we have $Q = {\mathcal{G}^1}^2\mathbf{I}_{k+1}$, $Q^{-1} = {\mathcal{G}^1}^{-2}\mathbf{I}_{k+1}$, and $\nu = -\frac{{\mathcal{G}^1}^2}{k+1}$; hence, $\alpha^* = Q^{-1}(\vartheta - \nu\mathbf{1}^{(k+1)}) = \frac{1}{k+1}\mathbf{1}^{(k+1)}$.

Thus, we showed that $\theta^k = \hat{\mathbf{p}} + \mathcal{G}^1\mathbf{e}^k$ for all $1 \leq k \leq n$, and α^* at iteration k is $\frac{1}{k}\mathbf{1}^{(k)}$; hence, $\mathbf{p}^{k*} = \frac{1}{k}\sum_{i=1}^k \theta^i = \hat{\mathbf{p}} + \frac{\mathcal{G}^1}{k}\mathbf{1}^{(k)}$. Then, we have $\beta^k = \hat{\mathbf{p}} - \mathbf{p}^{*k} = -\frac{\mathcal{G}^1}{k}\mathbf{1}^{(k)}$, and it follows that $\mathcal{G}^k = \|\beta^k\| = \frac{\mathcal{G}^1}{\sqrt{k}}$.

Consider the case $\mathscr{S} = 0$, which happens for $k \geq n$ as well. As we previously discussed, the problem given in Eq. (2.32) is optimized if $(\theta - \hat{\mathbf{p}})$ is orthogonal to $(\theta^1 - \hat{\mathbf{p}}), \cdots, (\theta^k - \hat{\mathbf{p}})$; however, if $\mathscr{S} = 0$, then $(\theta - \hat{\mathbf{p}})$ is in the linear combination of $(\theta^1 - \hat{\mathbf{p}}), \cdots, (\theta^k - \hat{\mathbf{p}})$. Thus, $\mathscr{S} = 0$ leads to suboptimal solutions. Hence, our earlier assumption $\mathscr{S} > 0$ is valid. Moreover, $\mathcal{G}^k \leq \frac{\mathcal{G}^1}{\sqrt{k}}$ for $k > n$.

As the second step of the proof, we provide an example to show $\frac{\mathcal{G}^1}{\sqrt{k}}$ is the tightest upper bound on $\mathcal{G}^k$ for $1 \leq k \leq n-1$.

Example: Consider options $i = 1, \cdots, n$ and assume that $\hat{\mathbf{p}} = \mathbf{e}^n$ and two FCRs are given as follows: (i) an `E` type family that consists of only option n, and (ii) an `L` type family that consists of options $1, \cdots, n-1$—i.e., option n must always be chosen and at most one of options $1, \cdots, n-1$ can be selected. Then, there exists n feasible configurations as follows: $\mathbf{e}^1 + \mathbf{e}^n, \mathbf{e}^2 + \mathbf{e}^n, \cdots, \mathbf{e}^{n-1} + \mathbf{e}^n, \mathbf{e}^n$. Note that $\mathcal{H}_{\hat{\mathbf{p}}} = \{\mathbf{p} \in \mathbb{R}^n | p_n = 1, \mathbf{p} \geq 0\}$, and all feasible configurations are in $\mathcal{H}_{\hat{\mathbf{p}}}$; hence, $\mathcal{FH}_{\hat{\mathbf{p}}}$ is the convex combination of the n feasible configurations, which are the vertices of $\mathcal{FH}_{\hat{\mathbf{p}}}$. The diameter of $\mathcal{FH}_{\hat{p}}$ is $\mathcal{D} = \sqrt{2}$.

Let, without loss of generality, $\theta^1 = \mathbf{e}^1 + \mathbf{e}^n$:

Iteration k, for $1 \leq k \leq n-1$: $\mathbf{p}^{*k} = \frac{1}{k}(\mathbf{e}^1 + \cdots + \mathbf{e}^k) + \mathbf{e}^n$, $\beta^k = \hat{\mathbf{p}} - \mathbf{p}^{*k} = -\frac{1}{k}(\mathbf{e}^1 + \cdots + \mathbf{e}^k)$, and $\mathcal{G}^k = \|\beta^k\| = \frac{1}{\sqrt{k}}$. For $1 \leq k \leq n-2$, one could verify that all of the points in the set $\{\mathbf{e}^{k+1} + \mathbf{e}^n, \mathbf{e}^{k+2} + \mathbf{e}^n, \cdots, \mathbf{e}^{n-1} + \mathbf{e}^n, \mathbf{e}^n\}$ are optimal for the problem $\mathcal{M}(\beta^k)$; then, let, without loss of generality, $\theta^{k+1} = \mathbf{e}^{k+1} + \mathbf{e}^n$. For $k = n-1$, $\mathbf{e}^n$ is the only optimal solution for $\mathcal{M}(\beta^{n-1})$; hence, let $\theta^n = \mathbf{e}^n$.

Iteration n: $\mathbf{p}^{*n} = \mathbf{e}^n$, $\beta^n = \hat{\mathbf{p}} - \mathbf{p}^{*n} = 0$, and $\mathcal{G}^n = \|\beta^n\| = 0$. Then, our algorithm stops and reports that $\hat{\mathbf{p}} \in cone(\mathbb{P})$.

Thus, since in this example, we have $\mathcal{G}^k = \frac{\mathcal{G}^1}{\sqrt{k}}$, for all $1 \leq k \leq n-1$, we have proved that $\frac{\mathcal{G}^1}{\sqrt{k}}$ is the tightest upper bound on $\mathcal{G}^k$ for $1 \leq k \leq n-1$.

2.14 Appendix for Lower Bound

2.14.1 Illustration of Bounds

See Fig. 2.13. Consider 3 options A, B, and C, and assume there are only 3 feasible configurations {A,C}, {B,C}, and {A,B,C}, and the forecast rates of options are given by $\hat{\mathbf{p}} = (p_A, p_B, p_C) = (0, 0, 1)$. The set $\mathcal{FH}_{\hat{\mathbf{p}}}$ is shown in Fig. 2.13(a). Let the algorithm initialize at $\theta^1 = (1, 1, 1)$.

Iteration 1: $\mathbf{p}^{*1}$ is the closest point to $\hat{\mathbf{p}}$ in the convex hull of $\{\theta^1\}$; hence, $\mathbf{p}^{*\mathbf{1}} = \theta^1$. The vector $\beta^1 = \hat{\mathbf{p}} - \mathbf{p}^{*1}$ is shown and solving $\mathcal{M}(\beta^1)$ gives $\theta^2 = (0, 1, 1)$. The length of β^1 is $\mathcal{G}^1$ and the length of the projection of the vector $(\theta^2 - \mathbf{p}^{*1})$ on the vector β^1 is $\mathcal{U}^1$; moreover, $\mathcal{L}^1 = \mathcal{G}^1 - \mathcal{U}^1$.

Iteration 2: $\mathbf{p}^{*2}$ is the closest point to $\hat{\mathbf{p}}$ in the convex hull of $\{\theta^1, \theta^2\}$; hence, $\mathbf{p}^{*2} = \theta^2$. Finding β^2 and solving $\mathcal{M}(\beta^2)$, we obtain $\theta^3 = (1, 0, 1)$. The vector $(\theta^3 - \mathbf{p}^{*2})$ is shown. It is seen that $\mathcal{G}^2 = \mathcal{U}^2$, and $\mathcal{L}^2 = 0$.

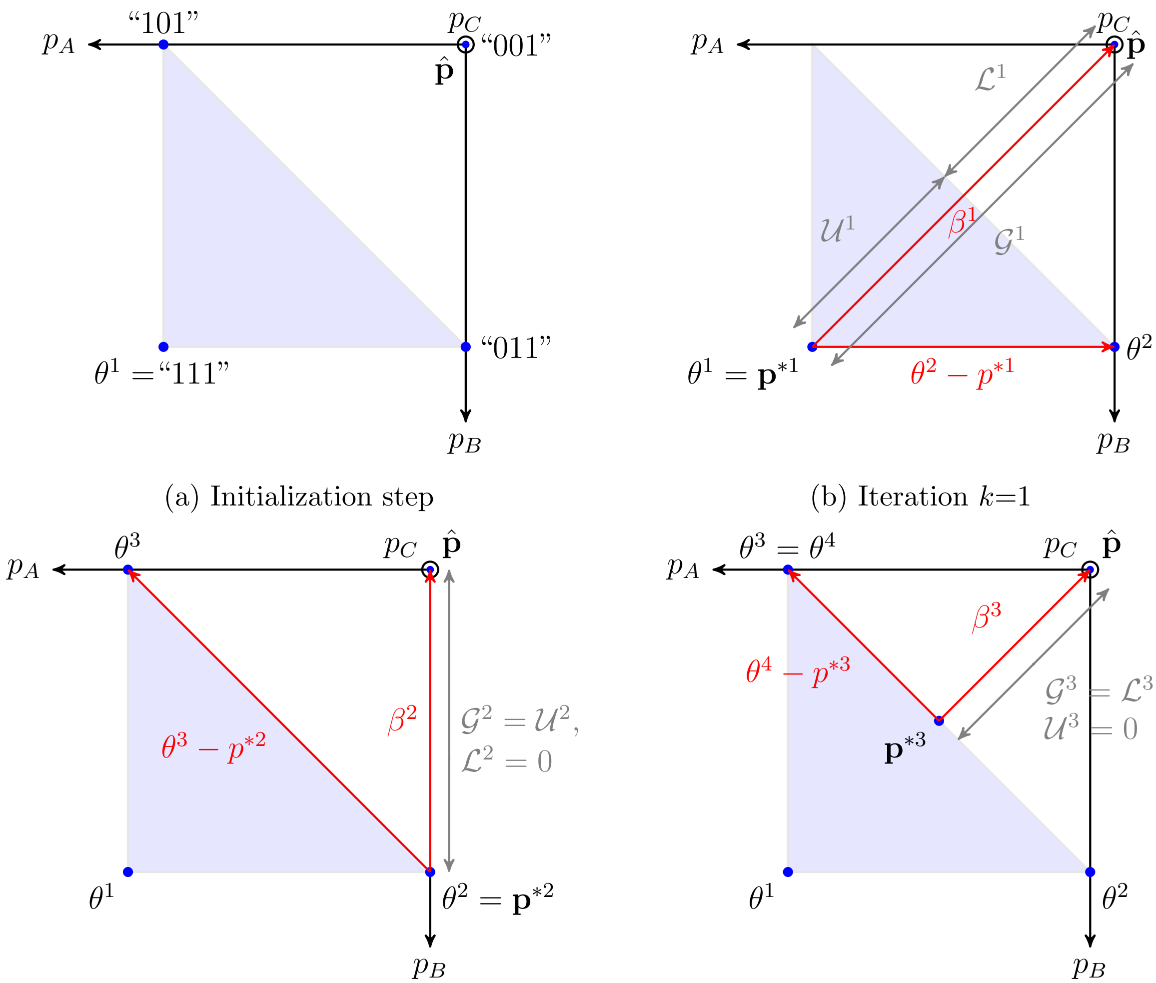
Iteration 3: $\mathbf{p}^{*3} = (0.5, 0.5, 1)$, and β^3 are shown. Solving $\mathcal{M}(\beta^3)$, we obtain $\theta^4 = \theta^3$.

In this example, we note that $\mathbf{p}^{*3}$ is the closest point in $\mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$; moreover, one could see that continuing the execution of the algorithm, iteration 3 will be repeated in all iterations $k > 3$; hence, a criteria is required to recognize this situation and terminate the algorithm.

2.14.2 Proof of Proposition 6

(i) Let $\mathcal{G}^k \neq 0$ and $\mathcal{L}^k > 0$. We have $\mathcal{L}^k = \mathcal{G}^k - \mathcal{U}^k = \mathcal{G}^k - \min\left\{\mathcal{G}^k, \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\} = \mathcal{G}^k + \max\left\{-\mathcal{G}^k, -\frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\} = \max\left\{0, \mathcal{G}^k - \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k}\right\}$; hence, since $\mathcal{L}^k > 0$, then $\mathcal{G}^k - \frac{\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k})}{\mathcal{G}^k} > 0$, and it follows that $\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k}) < \mathcal{G}^{k2} = \beta^{kT}\beta^k = \beta^{kT}(\hat{\mathbf{p}} - \mathbf{p}^{*k})$; thus, $\beta^{kT}\hat{\mathbf{p}} > \beta^{kT}\theta^{k+1}$. On the other hand, since θ^{k+1} is the maximizer of $\mathcal{M}(\beta^k)$, then we have $\beta^{kT}\mathbf{p} \leq \beta^{kT}\theta^{k+1}$, for all $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$. Therefore, we showed that $\beta^{kT}\mathbf{p} = \beta^{kT}\theta^{k+1}$ is a hyperplane that separates $\hat{\mathbf{p}}$ and $\mathcal{FH}_{\hat{\mathbf{p}}}$; hence, $\hat{\mathbf{p}} \notin \mathcal{FH}_{\hat{\mathbf{p}}}$, and it follows that $\hat{\mathbf{p}} \notin cone(\mathbb{P})$.

(ii) Let $\mathcal{G}^k \neq 0$ and $\mathcal{U}^k = 0$; by the definition of $\mathcal{U}^k$, we have $\beta^{kT}\theta^{k+1} = \beta^{kT}\mathbf{p}^{*k}$. Since θ^{k+1} is the maximizer of $\mathcal{M}(\beta^k)$, then we have $\beta^{kT}\mathbf{p} \leq \beta^{kT}\theta^{k+1}$, for all $\mathbf{p} \in \mathcal{FH}_{\hat{\mathbf{p}}}$. On the other hand, we have $0 < \mathcal{G}^{k2} = \beta^{kT}\beta^k = \beta^{kT}(\hat{\mathbf{p}} - \mathbf{p}^{*k})$, then $\beta^{kT}\hat{\mathbf{p}} > \beta^{kT}\mathbf{p}^{*k} = \beta^{kT}\theta^{k+1}$. Hence, we showed that $\beta^{kT}\mathbf{p} = \beta^{kT}\theta^{k+1}$ is a hyperplane that separates $\hat{\mathbf{p}}$ and $\mathcal{FH}_{\hat{\mathbf{p}}}$. Moreover, since $\mathbf{p}^{*k}$ belongs to $\mathcal{FH}_{\hat{\mathbf{p}}}$ and the separating hyperplane, and the vector $(\hat{\mathbf{p}} - \mathbf{p}^{*k})$ is orthogonal to the separating hyperplane, then $\mathbf{p}^{*k}$ is the nearest point in $\mathcal{FH}_{\hat{\mathbf{p}}}$ to $\hat{\mathbf{p}}$.



(a) Initialization step

(b) Iteration k=1

(c) Iteration k=2

(d) Iteration k=3

Figure 2.13: An example to illustrate $\mathcal{L}^k$ and $\mathcal{U}^k$. Note that $\mathcal{L}^k$ and $\mathcal{U}^k$ are not monotone in k.

2.14.3 Proof of Lemma 9

Fig. 2.13 provides a counter example for the monotonicity of $\mathcal{L}^k$ and $\mathcal{U}^k$. Note that $\mathcal{L}^1 = \frac{1}{\sqrt{2}}$, $\mathcal{L}^2 = 0$, $\mathcal{L}^3 = \frac{1}{\sqrt{2}}$. Hence, $\mathcal{L}^k$ is not necessarily monotone in k. The same is true for $\mathcal{U}^k$.

2.14.4 Proof of Theorem 3

The first part of the theorem follows from Lemma 6 and Proposition 6.

To prove our algorithm stops after finite iterations, note that if at iteration k, $\mathbf{p}^{*k} = \hat{\mathbf{p}}$

then $\mathcal{G}^k = 0$ and the algorithm stops; hence, assume $\mathbf{p}^{*k} \neq \hat{\mathbf{p}}$. We claim that: if solving $\mathcal{M}(\beta^k)$ returns θ^{k+1} such that $\theta^{k+1} \in \{\theta^1, \cdots, \theta^k\}$, then the algorithm stops. To prove this claim, note that at iteration k of our algorithm, the hyperplane $\beta^{kT}\mathbf{p} = \beta^{kT}\mathbf{p}^{*k}$ separates $\hat{\mathbf{p}}$ and $\{\theta^1, \cdots, \theta^k\}$; moreover, $\mathbf{p}^{*k}$ and at least one of $\theta^1, \cdots, \theta^k$, say θ^j, belong to this separating hyperplane. Suppose $\theta^{k+1} \in \{\theta^1, \cdots, \theta^k\}$, then it must also belong to the separating hyperplane; hence, without loss of generality assume $\theta^{k+1} = \theta^j$. Then, $\beta^{kT}(\theta^{k+1} - \mathbf{p}^{*k}) = \beta^{kT}(\theta^j - \mathbf{p}^{*k}) = 0$; hence, $\mathcal{U}^k = 0$, and the algorithm stops.

Thus, we showed that our algorithm either finds a new θ at each iteration or stops. Noting that the number of feasible configurations is finite, the algorithm stops after finite iterations.

2.15 References

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Chapter 3

Parts Capacity-Planning Problem

3.1 Introduction

Our problem is motivated by a global auto manufacturer (GAM) that allows customers to configure their cars online. Automobiles consist of a number of options such as engines, transmissions, chassis, electronic systems, interior and exterior designs, and suspensions. Customers configure end-products, also called *producible configurations*, by choosing options that are compatible (see for example Ulrich (1994); Walker and Bright (2013); Feitzinger and Lee (1997); Fohn et al. (1995)). For a general discussion on mass customization, configurable products, and their literature reviews, see Pine (1993); Da Silveira et al. (2001); Heiskala (2007); Sabin and Weigel (1998). The GAM offers 100-500 options for a *car model*, resulting in 10^{25}-10^{40} producible configurations (configurations that are *engineering*-wise producible).

Traditionally, manufacturers have relied on demand forecasts at the level of configurations for parts-capacity planning, production planning, supplier contracts, and pricing decisions (Hax and Candea, 1984; Orlicky, 1974; Whybark and Williams, 1976). In mass customization, it is impossible to forecast configurations' demand because of extremely large number of configurations. Typically, firms first forecast the cumulative demand—i.e., the total number of cars sold over the planning horizon. Second, they forecast options' demand in the form of an options-level penetration statistic (OPS). An OPS consists of a penetration rate for each of the options being offered (the penetration rate of an option represents the fraction of cars that they believe will have that option).

Our analysis on the data received from the GAM indicates that 30-60% of the parts required to produce a car model depend on the combination of options used (we refer to parts, components, and subassemblies as *parts*). For example, a specific combination of an engine and a transmission type generates a number of additional parts. For an industrial instance that has 433 options, Fig. 3.1 shows how many parts are defined by how many options. In this instance, 63% of parts' requirement depend on a single option and 37% of the parts' requirement depend on multiple options.

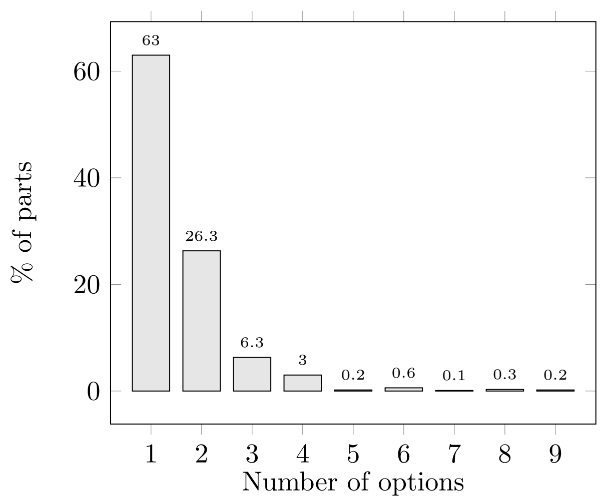

Figure 3.1: Percentage of parts that are defined based on different number of options.

In short, we need configurations-level forecast to estimate parts requirements. However, the relationship between an OPS and a configurations-level demand is a point-to-set mapping, as shown in the following example.

EXAMPLE 1. **(Point-to-Set Mapping)** Consider two options A' and A'' with penetration rates of 0.5 for both. Assume that a car can have both or one or none of these two options. This results in the following producible configurations: $\{\}$, $\{A'\}$, $\{A''\}$, and $\{A', A''\}$. Observe that the following configurations-level demands are consistent with the given OPS: (i) $0.5\{\}+0.5\{A', A''\}$, (ii) $0.5\{A'\}+0.5\{A''\}$, or (iii) $0.25\{\}+0.25\{A'\}+0.25\{A''\}+0.25\{A', A''\}$.

□

Similar to OPS, we let a configurations-level penetration statistic (CPS) consist of a penetration rate (fraction of the demand) for each producible configuration. A configuration is producible if it satisfies engineering constraints. Note that a CPS is a convex combination of producible configurations. A CPS is (*marketing*-wise) *consistent* if it satisfies the given OPS. As shown in Example 1, a multitude of CPSs are consistent with the given OPS. Each CPS maps to a specific quantity for a part's requirement. Thus, a given OPS maps to a range for a part's requirement. In this chapter, we will also express a part's requirement in terms of a requirement rate. A part's requirement is the ratio of the required quantity of that part to the total demand over the planning horizon.[1]

The GAM procures parts from approximately 20,000 suppliers and it is estimated that parts' cost constitutes approximately two-thirds of the total cost, or about $90 billion per year. An important element of the procurement contracts is the ranges of parts' requirement. Inaccurate calculation of these ranges has resulted in significant excess inventory and shortages of parts in the GAM. In addition, the wider the ranges, the costlier the parts' capacity acquisition (Bassok and Anupindi, 2008). Therefore, it is essential to find these ranges as precisely and as narrowly as possible.

In Example 1, we assumed that any combination of the two options can be selected. However, in general, this is not the case. A set of *engineering rules* exists that determines the compatibility of options. For example, some options are mutually incompatible while, in other instances, selection of an option may require selection of another option. These considerations result in a set of complex engineering constraints, which we refer to as *rules* (see Chapter 2). The following example illustrates the rules and is referred to throughout this chapter.

Example 2(a). (Fig. 3.2(a)) In this example, there are 10 options which we label as

[1] Note that, in a CPS, rates add up to 1; in an OPS, penetration rates are between 0 and 1 but they may not add up to 1; and, a part's requirement may be greater than 1, e.g. each car has 4 tiers.

Figure 3.2(a):
Set of options:
$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$.
Options with penetration rates:
$p(A_3) = 0.019$
$p(A_4) = 0.861$
$p(A_6) = 0.019$
$p(A_7) = 0.050$
$p(A_8) = 0.046$
$p(A_{10}) = 0.385$.
Rules for selecting options:
$A_1 \Rightarrow A_2 \vee A_5 \vee \neg A_{10}$
$A_1 \Rightarrow A_3 \vee \neg A_7 \vee \neg A_8$
$A_1 \Leftarrow A_7 \wedge A_6 \wedge \neg A_4$
$A_2 \Rightarrow A_3 \vee A_5 \vee A_6 \vee \neg A_7 \vee \neg A_8$
$A_2 \Leftarrow A_4 \wedge A_5 \wedge A_6 \wedge A_7$
$A_2 \Leftarrow \neg A_8 \wedge \neg A_9 \wedge \neg A_{10}$
$A_3 \Rightarrow A_1 \vee A_2 \vee \neg A_4 \vee \neg A_5$
$A_4 \Leftarrow A_1 \wedge A_2 \wedge A_3 \wedge A_5$
$A_5 \Rightarrow A_7 \vee A_8$
$A_5 \Rightarrow A_1 \vee A_2 \vee A_3 \vee A_4 \vee A_6 \vee A_9$
$A_6 \Rightarrow A_5$
$A_7 \Rightarrow A_1 \vee A_5 \vee A_4$
$A_8 \Rightarrow A_9 \vee A_{10}$
$A_8 \Rightarrow A_1$
$A_9 \Rightarrow A_{10}$
$A_9 \Rightarrow A_{10} \vee A_7$
$A_9 \Leftarrow A_1 \wedge A_2$
$A_{10} \Rightarrow A_1$
$A_{10} \Rightarrow A_2 \vee A_3 \vee A_4 \vee A_5$
$A_{10} \Leftarrow A_1 \wedge A_2$.

Figure 3.2(b):
Set of parts:
$\{R_1, R_2, R_3\}$.
Condition codes:
$A_1 \longrightarrow 1 \times R_1$
$(A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4) \longrightarrow 2 \times R_1$
$(\neg A_5 \wedge A_7) \longrightarrow 4 \times R_1$
$(A_3 \wedge A_4) \vee (A_3 \wedge A_6) \longrightarrow 1 \times R_2$
$(A_1 \wedge \neg A_2 \wedge A_3) \longrightarrow 2 \times R_3$
$A_3 \vee (A_4 \wedge \neg A_5 \wedge \neg A_6) \longrightarrow 1 \times R_3$.

Figure 3.2: Inputs of our problem for Example 2.

$A_1, \ldots, A_{10}$. These options can be engine types, transmissions, body types, sunroofs, audio systems, and so forth. Six of these ten options have assigned penetration rates (denoted by $p(A_3)$, $p(A_4)$, and so forth)—e.g, the penetration rate of A_3 is 0.019 meaning that it is believed that 1.9% of the planning horizon demand will have option A_3. There are 20 rules. Notations "$\neg$", "$\wedge$", "$\vee$", and "$\Rightarrow$" mean "*negation*," "*and*," "*or*," and "*implies*," respectively. For example, the first rule ($A_1 \Rightarrow A_2 \vee A_5 \vee \neg A_{10}$) means that if option A_1 is selected, then either A_2 must be selected or A_5 must be selected, or A_{10} must not be selected. □

A subset of options that satisfies all rules define a *producible configuration*. In the example above, $\{A_1, A_4, A_5, A_6, A_8, A_{10}\}$ and $\{A_2, A_4\}$ are examples of producible configurations. In general, finding a producible configuration is the well-known NP-complete Satisfiability problem (Garey and Johnson, 2002).

The relationship between configurations and parts' requirement is mediated by *condition codes*. Condition codes are stated in terms of associated options. Observe that in Fig. 3.2(b), there are 6 condition codes that define the requirement for parts R_1, R_2, and R_3. Notation "$\longrightarrow$" means that if the left hand side is true for a given configuration, then the number of parts specified on the right-hand side are required. For example, the third condition code ($(\neg A_5 \wedge A_7) \longrightarrow 4 \times R_1$) means that if option A_5 is not selected and option A_7 is selected, then we require 4 units of part R_1. The total part requirement is based on all condition codes that hold.

For the producible configuration $\{A_1, A_4, A_5, A_6, A_8, A_{10}\}$, the requirement for part R_1 is computed based on the three condition codes (Fig. 3.2(b)), and since only one holds, then one unit of R_1 is needed. Similarly, for the other producible configuration $\{A_2, A_4\}$, we need two units of R_1.

We define parts capacity-planning problem (PCPP) as the problem of finding the ranges for parts' requirement. This problem consists of the following main inputs: (i) forecast OPS, (ii) *rules* that define producible configurations, and (iii) *condition codes* that relate options to the parts' requirements. To the best of our knowledge, this problem is new and has not been studied in the literature. Our contributions are as follows.

1. Traditionally, parts' requirements have been computed using configurations-level demand forecast and bill of materials. We present a new problem and develop a model that finds ranges for parts' requirements where instead of configurations-level forecast and bill of materials, only *options-level penetration rates*, *rules*, and *condition codes* are available. Our formulation is easy to understand, as it intuitively captures the rules and condition codes.

2. PCPP is a challenging large-scale NP-hard problem. We develop a methodology that involves first reformulating PCPP with the following desirable properties: (i) the objective function is *convex*, and (ii) the feasible set is only the *convex hull of producible configurations*. Second, our methodology sequentially construct the feasible set until an optimal solution is found. We show that our approach effectively solves large industrial instances.

3. We compare our approach to the one used in GAM. The proposed approach provides accurate ranges for parts' requirements while our experiments on an industrial instance indicate that the existing practice has a *mean absolute percentage error* of 35.9%. This can result in substantial *shortage* and/or *excess* inventory of parts.

4. The range for a part's requirement, in principle, can be reduced if we can obtain

additional information on demand. We address the following questions. What if the firm can forecast the penetration rates for combination of options, which combination should it forecast? And what will be the impact of additional information on the range?

5. Finally, we extend our approach to consider the following cases: (i) options' penetration rates are given as ranges, and (ii) the firm requires to find a range for the joint requirement of multiple parts.

In short, we develop a new methodology that helps procurement managers in industries with highly configurable products. Our approach provides accurate estimates of the ranges of parts' requirements, which results in substantial reduction in parts' costs and wastage. We finally note that this problem is not limited to the automotive industry. In fact, other industries including, but not limited to, consumer electronics, aircraft, and computer industries offer configurable products (Feitzinger and Lee, 1997; Fohn et al., 1995; Kristianto et al., 2015; Rodriguez and Aydin, 2011).

The remainder of this chapter is organized as follows. In Section 3.2, we develop a simpler representation of condition codes and formulate the PCPP as a mixed-integer linear programming (MILP) problem. Section 3.3 presents our methodology for solving industrial instances in a reasonable amount of time and a lower bound on the optimal range. In Section 3.4, we show how and which additional information regarding penetration rates of single and joint options can assist in narrowing the ranges and then propose a heuristic guideline for selecting which additional information to acquire. In Section 3.5, we apply our methodology to an industrial instance given to us by the GAM, provide a detailed description of the approach currently used by the GAM, and present a numerical comparative analysis. Section 3.6 extends and generalizes our methodology to solve two important variations of PCPP. Finally, Section 3.7 summarises the proposed methodology, discusses some of the limitations, and provides directions for future research.

3.2 Parts capacity-planning problem (PCPP) formulation

In this section, we formulate the PCPP as a mixed-integer linear programming (MILP) problem that intuitively captures the engineering rules, options' forecast, and condition codes. As stated earlier, rules determine whether a potential configuration (a subset of options) is producible. The problem of determining if there exists a producible configuration is the well-known satisfiability problem. Therefore, it is not practical to explicitly compute all producible configurations. Instead, we implicitly characterize the set of all CPSs that satisfy the given OPS. We refer to CPSs that satisfy the given OPS as *consistent* CPSs. We then map these consistent CPSs to the parts requirements through condition codes. Each CPS maps to a value for each part's requirement. Since there is a set of consistent CPSs (as opposed to a single CPS), we obtain a range for each part's requirement. To find the minimum (respectively, maximum) value of the range, we formulate a problem that, among the set of all consistent CPSs, finds a CPS that minimizes (respectively, maximizes) the part's requirement. In short, this problem has two sets of constraints: (i) configurations have to be producible meaning that they have to satisfy the *rules*, which we refer to as *engineering constraints*, and (ii) CPSs have to satisfy the given OPS, which we refer to as *marketing constraints*.

We use letter A to denote options, letter R to denote parts, and letter F to denote propositional formulas (propositional formulas are used in the representation of rules and condition codes). Let $\mathcal{N}$, $\mathcal{R}$, and $\mathcal{F}$ denote the set of all options, parts, and propositional formulas, respectively. We also use index $i = 1, \ldots, n$ for options, where $n := |\mathcal{N}|$.

Condition codes determine part requirements for configurations and are stated in the form of $F \longrightarrow \alpha R$, meaning that if the propositional formula F is "true," we then require α units of part R. We define $\mathscr{C}(R) := \{(F, \alpha) \in \mathcal{F} \times \mathbb{Z}_{++} | F \longrightarrow \alpha R\}$, as the set of all condition codes for part R. For example, for part R_2 in Fig. 3.2, we have: $\mathscr{C}(R_2) = \{((A_3 \wedge A_4) \vee (A_3 \wedge A_6), 1)\}$.

Let $\mathbf{y}$ denote a generic producible configuration and $\mathbb{Y}$ denote the set of all producible configurations. We show how to determine a part's requirement for configuration $\mathbf{y}$ using the condition codes. Recall that associated with each configuration is a set of options. To build configuration $\mathbf{y}$, we require $\sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha v_{\mathbf{y}}(F)$ units of part R, where $v_{\mathbf{y}}(F)$ is the value of formula F in configuration $\mathbf{y}$, i.e., if $\mathbf{y}$ satisfies F, then $v_{\mathbf{y}}(F) = 1$, and otherwise $v_{\mathbf{y}}(F) = 0$. For example in Fig. 3.2, the number of units of part R_1 that is required to produce configuration $\mathbf{y}$ is equal to: $v_{\mathbf{y}}(A_1) + 2v_{\mathbf{y}}((A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4)) + 4v_{\mathbf{y}}(\neg A_5 \wedge A_7)$. Observe that the subset $\{A_2, A_4\}$ is a producible configuration. Part R_1's requirement for this configuration is equal to $0 + 2 \times 1 + 4 \times 0$; hence, producing this configuration requires 2 units of part R_1.

Let $p(A)$ denote the penetration rate of option A and $p(F)$ denote the penetration rate of formula F (note that penetration rates of formulas will be used in characterizing parts' requirement). For example, if $F = A_1 \wedge A_2$, then $p(A_1 \wedge A_2)$ is the joint penetration rate of $A_1 \wedge A_2$ and it shows the fraction of cars sold over the planning horizon that include both options A_1 and A_2. We sometimes use p_A instead of $p(A)$ for ease of notation.

Recall that a convex combination of producible configurations represents a CPS. Thus, a CPS consists of a coefficient $a_{\mathbf{y}}$ for each producible configuration $\mathbf{y} \in \mathbb{Y}$ such that these coefficients satisfy: $a_{\mathbf{y}} \geq 0$, for all $\mathbf{y} \in \mathbb{Y}$, and $\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} = 1$. A CPS maps to a specific value for a part's requirement. Let Q_R denote the requirement of part R for a given CPS. We have:

$$Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha v_{\mathbf{y}}(F) = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F) = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha p(F).$$

The first equality follows from the definition of Q_R. Note that $\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F)$ is the fraction of cars that satisfy formula F; hence, the last equality holds because $p(F) = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F)$.

Since there are many consistent CPSs and each map into a specific value for part R's requirement, a range is obtained for Q_R. This range is determined by minimizing and maximizing Q_R over all consistent CPSs. Hence, we use $Q_R = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha p(F)$ as the objective function of our MILP model. The difficulty, however, is that some of the F's in this equation may be complex. Thus, we perform the following simplification. If $(F,\alpha) \in \mathscr{C}(R)$ and F is complex, then we define a new *artificial* option A' and replace (F,α) with (A',α) in the set $\mathscr{C}(R)$, while adding $F \Leftrightarrow A'$ to the set of rules. For example, if $(A_3 \wedge A_4, 2) \in \mathscr{C}(R)$, then we introduce a new option A', replace $(A_3 \wedge A_4, 2)$ with $(A', 2)$, and add $A_3 \wedge A_4 \Leftrightarrow A'$ to the set of rules. Let $\tilde{\mathscr{C}}(R)$ denote the simplified version of the set $\mathscr{C}(R)$. Hence, our objective function becomes $Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i$.

We next formulate the feasible region of our problem. We start by formulating the rules. The GAM's *engineering* rules are a set of well-defined propositional formulas and can be written in *conjunctive normal form* (CNF) (see, for example, Tseitin (1968) and Wilson (1990)). A CNF formula is in the form of $C_1 \wedge \cdots \wedge C_\ell$ where $C_1, \ldots, C_\ell$ are disjunctive clauses. A clause is called "disjunctive" if it is the disjunction of some literals (a literal is either an option or its negation). Although our methodology can be easily extended to rules in any format, we assume rules are in CNF.

We write one constraint for each clause.[2] As an example, consider clause $A_1 \vee \neg A_2$. This clause is formulated as $y_{A_1} + (1 - y_{A_2}) \geq 1$, where binary variables y are defined as follows: $y_A = 1$ if option A is selected, and $y_A = 0$ otherwise. This constraint means that either $y_{A_1} = 1$ or $y_{A_2} = 0$ or both. Hence, we can formulate the set of all producible configurations $\mathbb{Y}$ as follows:

$$\mathbb{Y} := \{\mathbf{y} \in \{0,1\}^n | \quad \sum_{i\in PO_C} y_i + \sum_{i\in NO_C} (1 - y_i) \geq 1, \ \forall C\},$$

where, $\mathbf{y} := (y_1, \ldots, y_n)$, and PO_C (respectively, NO_C) is the set of positive (respectively, negative) options in clause C. For example, if $C = \neg A_3 \vee A_4$, then $PO_C = \{A_4\}$ and

[2]Formulating logical propositions as linear constraints is also discussed in Yan and Hooker (1999) and Chandru and Hooker (1999).

$NO_C = \{A_3\}$. In this chapter, we assume $\mathbf{0} \in \mathbb{Y}$.

An OPS consists of a penetration rate $p_i \in [0,1]$ for each option $i = 1, \ldots, n$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ denote an OPS. This OPS satisfies the engineering rules if there exists a CPS, $a_{\mathbf{y}}$'s, such that $\sum_{\mathbf{y} \in \mathbb{Y}} a_{\mathbf{y}} y_i = p_i$, for all i. In other words, an OPS $\mathbf{p}$ satisfies the engineering constraints if $\mathbf{p} \in conv(\mathbb{Y})$, where $conv(\mathbb{Y})$ denotes the convex hull of $\mathbb{Y}$. We refer to the condition $\mathbf{p} \in conv(\mathbb{Y})$ as the *engineering constraints*. Finally, we note that the *marketing* constraints (forecast OPS) are formulated as: $p_i = \hat{p}_i$, for all $i \in \mathcal{N}_1$, where $\hat{p}_i$ denotes the forecast penetration rate of option i, and $\mathcal{N}_1$ denotes the set of options for which forecast penetration rates exist. For example in Fig. 3.2, $\mathcal{N}_1 = \{A_3, A_4, A_6, A_7, A_8, A_{10}\}$. Thus, we formulate our problem, which we refer to as (P1), as follows:

$$\text{(P1):} \quad \min / \max \qquad Q_R = \sum_{(i,\alpha) \in \tilde{\mathscr{C}}(R)} \alpha p_i \tag{3.1}$$

$$\text{s.t.} \qquad p_i = \hat{p}_i, \ \forall i \in \mathcal{N}_1, \tag{3.2}$$

$$\mathbf{p} \in conv(\mathbb{Y}). \tag{3.3}$$

Let $Q^*_{R,L}$ and $Q^*_{R,U}$ respectively denote the optimal values of the minimization (P1min) and maximization (P1max) problems. The detailed MILP formulation of (P1) along with an example is given in Appendix 3.9.1. Last, we establish the NP-hardness of (P1).

Proposition 7 (NP-Hardness)**.** *Problem (P1) is NP-hard.*

The proof simply follows from noting that satisfiability problem (known to be NP-complete; see, e.g., Garey and Johnson (2002)) is imbedded in (P1).

3.3 Solution approach

PCPP is a large-scale optimization problem and existing approaches for solving the MILP representation of this problem are not effective. Therefore, we need to develop a specialized solution approach. In this section, we propose a methodology that involves "dualizing" the *marketing* constraints so that the new formulation is equivalent to the original one and has the following features: (i) the feasible region is only $conv(\mathbb{Y})$, which is a polytope and its extreme points are producible configurations, and (ii) the objective function is piecewise linear and convex. We sequentially construct the feasible set until an optimal solution is found. This feature of our solution approach is similar to a variant of the Frank-Wolfe method, which has been shown to be effective for solving a class of large convex optimization problems (Bach, 2013; Clarkson, 2010; Jaggi, 2013). Finally, we present a procedure to find a bound on the optimal range which can assist in obtaining an optimality gap for the early termination of the algorithm.

3.3.1 An equivalent formulation of PCPP

Recall that (P1) minimizes (respectively, maximizes) $Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i$ subject to *engineering* and *marketing* constraints. To work with a more manageable feasible region, we relax the *marketing* constraints and add penalty terms for violation of these constraints to the objective function. This changes the objective function to $\sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \pm \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i|$, where M is a sufficiently large number. In this subsection, we show that this objective function for the minimization and maximization cases can be transformed to $\|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p} - \hat{\mathbf{p}}_{R,L})\|_1$ and $\|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p} - \hat{\mathbf{p}}_{R,U})\|_1$, respectively, for carefully chosen vectors $\mathbf{w}_R$, $\hat{\mathbf{p}}_{R,L}$, and $\hat{\mathbf{p}}_{R,U}$, where $\|.\|_1$ indicates the first norm, $\mathbf{Diag}(\mathbf{w}_R)$ is a diagonal matrix with $\mathbf{w}_R$ as the diagonal elements, and $\mathbf{p}$ is the vector of decision variables (all vectors are in $\mathbb{R}^n$). One could view $\hat{\mathbf{p}}_{R,L}$ and $\hat{\mathbf{p}}_{R,U}$ as the target points for $\mathbf{p}$ where the objective is to push $\mathbf{p}$ as close as possible to these target points. Moreover, $\mathbf{w}_R$ could be viewed as a weight vector that determines proportional importance for different options—i.e., the larger the weight of an option, the more effort should be made for matching the corresponding decision variable p with its target value.

Define vector $\mathbf{w}_R = (w_{R,1}, \ldots, w_{R,n}) \in \mathbb{R}^n$ as follows: for all $i \in \mathcal{N}$, if $i \in \mathcal{N}_1$, then let

$w_{R,i} := M$; otherwise, let $w_{R,i} := \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha$. Define vectors $\hat{\mathbf{p}}_{R,L},\ \hat{\mathbf{p}}_{R,U} \in \mathbb{R}^n$ as follows: for all $i \in \mathcal{N}$, if $i \in \mathcal{N}_1$, then let $\hat{p}_{R,L,i} := \hat{p}_i$ and $\hat{p}_{R,U,i} := \hat{p}_i$; otherwise, if $i \in \mathcal{N}\backslash\mathcal{N}_1$, then let $\hat{p}_{R,L,i} := 0$ and $\hat{p}_{R,U,i} := 1$. For example, for part R_1 in Example 2, we have: $\mathbf{w}_{R_1} = (1, 0, M, M, 0, M, M, M, 0, M, 2, 4)$, $\hat{\mathbf{p}}_{R_1,L} = (0, 0, 0.019, 0.861, 0, 0.019, 0.050, 0.046, 0, 0.385, 0, 0)$, and $\hat{\mathbf{p}}_{R_1,U} = (1, 1, 0.019, 0.861, 1, 0.019, 0.050,\ 0.046, 1, 0.385, 1, 1)$.

Our equivalent formulation can be expressed as the following problems, referred to as (P2L) and (P2U) (note that (P2) refers to both problems (P2L) and (P2U)):

$$\begin{aligned}
\text{(P2L)} \quad &: \quad \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p} - \hat{\mathbf{p}}_{R,L})\|_1, \\
\text{(P2U)} \quad &: \quad \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p} - \hat{\mathbf{p}}_{R,U})\|_1.
\end{aligned}$$

Theorem 4 shows that solving (P2) provides an optimal solution to (P1).

Theorem 4 (Nonlinear Programming Equivalence)**.** *If M is sufficiently big, then an optimal solution of problem (P2) is also an optimal solution for problem (P1).*

Proof is given in Appendix 3.10.1. Problem (P2) has the following features: (i) the feasible region is the convex hull of $\mathbb{Y}$, (ii) the objective is to minimize the sum of weighted absolute differences between p_i's and $\hat{p}_{R,L,i}$'s (or $\hat{p}_{R,U,i}$'s), and (iii) the objective functions are piecewise linear and convex. Finally, we remark that our approach is different from the classical *Lagrangian Relaxation* method in two aspects: (a) M is a scalar and not a vector of Lagrange multipliers, and (b) this approach does not result in any optimality gap as shown in Theorem 4. In the remainder, we propose an approach for solving problem (P2).

3.3.2 Applying Frank-Wolfe for solving problem (P2)

We now discuss how to solve problem (P2L), and the same approach can be used for solving problem (P2U). Our approach is given in Algorithm 2, which is known as the "fully corrective" variant of the Frank-Wolfe method (Bach, 2013; Clarkson, 2010; Jaggi, 2013). Note that the index for the parts (R) is dropped in the remainder of this subsection for ease of notation.

At each iteration $k \geq 1$, we find the best known solution $\mathbf{p}^{(k)}$ by minimizing the objective function over the convex hull of the set $\{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\}$, as shown in Line 3 of Algorithm 2. This is an easy optimization problem with linear constraints and a piecewise linear and convex objective function.

Since the objective function of (P2L) is piecewise linear and convex, we use the subgradient vector $\mathbf{g}^{(k)} \in \mathbb{R}^n$ at iteration $k \geq 1$, which is defined as follows:

$$g_i^{(k)} := \begin{cases} w_i & , \text{ if } p_i^{(k)} > \hat{p}_{L,i} \\ -w_i & , \text{ if } p_i^{(k)} < \hat{p}_{L,i} \\ \sim \text{Uniform}[-w_i, w_i] & , \text{ if } p_i^{(k)} = \hat{p}_{L,i}, \end{cases} \tag{3.4}$$

for all $i \in \mathcal{N}$. Note that, if $p_i^{(k)} = \hat{p}_{L,i}$, we can set $g_i^{(k)}$ to any value in $[-w_i, w_i]$; hence, we generate a number using a uniform distribution to allow for finding new points in different directions as the algorithm iterates.

In Line 5, we find a new extreme point of the feasible region $conv(\mathbb{Y})$ by minimizing a linear function $\mathbf{g}^{(k)T}\mathbf{y}$. We exclude the previously found solutions using *no-good* constraints (Hooker, 2000). For example, if $\mathbf{y}^{(0)} = (y_1^{(0)}, \ldots, y_n^{(0)})$, a no-good constraint to exclude $\mathbf{y}^{(0)}$ is:

$$\sum_{i:y_i^{(0)}=1} y_i - \sum_{i:y_i^{(0)}=0} y_i \leq \left(\sum_{i:y_i^{(0)}=1} 1 \right) - 1.$$

Note that although $\mathbf{y}^{(0)}$ is not feasible for this constraint, any $\mathbf{y} \neq \mathbf{y}^{(0)}$ is feasible. If we generate exactly one extreme point of $conv(\mathbb{Y})$ at each iteration, then the optimization problem in Line 5 has exactly k no-good constraints at iteration $k \geq 1$. Using no-good constraints has the following advantages: (i) Algorithm 2 is guaranteed to generate a new extreme point at each iteration, and (ii) if $|\mathbb{Y}|$ is small, Algorithm 2 can explore all points in $\mathbb{Y}$, in which case the problem in Line 5 becomes infeasible and $\mathbf{p}^{(k)}$ is a guaranteed optimal solution of (P2L).

Algorithm 2 Solving problem (P2L)

Input: Rules, $\mathbf{w}$, $\hat{\mathbf{p}}_L$. ▷ "Rules" characterize $\mathbb{Y}$.
Output: $\mathbf{p}_L^*$. ▷ $\mathbf{p}_L^*$ is the optimal solution of (P2L).
1: Find $\mathbf{y}^{(0)} \in \mathbb{Y}$; ▷ A producible configuration for initialization.
2: **for** $k = 1, 2, 3, \cdots$ **do**
3: Solve $\mathbf{p}^{(k)} := \arg\min_{\mathbf{p} \in conv(\{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1$;
4: Define $\mathbf{g}^{(k)}$ as in Eq. (3.4);
5: Solve $\mathbf{y}^{(k)} := \arg\min_{\mathbf{y} \in \mathbb{Y} \setminus \{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\}} \mathbf{g}^{(k)T}\mathbf{y}$; **if** infeasible **then** let $\mathbf{p}_L^* := \mathbf{p}^{(k)}$; Stop!
6: **end for**.

The Frank-Wolfe algorithm converges with $\mathcal{O}(1/k)$ if the feasible region is a compact convex subset of any vector space and the objective function is convex and continuously differentiable (Dunn and Harshbarger, 1978; Jones, 1992; Patriksson, 1993; Clarkson, 2010; Jaggi, 2013). This convergence rate exists because of the existence of a finite *curvature constant* for the objective function (see for example Jaggi (2013)). However, the objective function of problem (P2) does not have a finite curvature constant; hence, the convergence rate of Algorithm 2 cannot be guaranteed. Nonetheless if problem (P2) is approximated by using $\|.\|_2^2$ instead of $\|.\|_1$, the convergence rate $\mathcal{O}(1/k)$ is achieved. Note that using $\|.\|_2^2$ results in finding the optimal solution if $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_U$ are the optimal solutions of (P2L) and (P2U), respectively. However, this is not true in general.

Our algorithm starts with a producible configuration $\mathbf{y}^{(0)}$ (see Line 1). Moreover, note that any set of producible configurations can be used (instead of $\mathbf{y}^{(0)}$) for initialization. Hence, as a warm start, we use the set of producible configurations that is found by applying the algorithm of Chapter 2. That algorithm finds a set of producible configurations such that the forecast OPS belongs to their convex cone. We will show the effectiveness of our approach for solving industrial instances in section 3.5.

3.3.3 Bound on the optimal parts' requirement

Our problem, if solved to optimality, provides the tightest range for a part's requirement. A bound on the range may be needed when the algorithm is terminated prior to completion. Recall that our problem has two sets of constraints: *Engineering* and *Marketing*. An intuitive approach for obtaining a bound is to focus on the options that are present in the condition codes of the part under consideration and drop the *marketing* constraints for the other options. Since the number of options that are present in the condition codes of a part is less than 10, we enumerate all producible *scenarios* for those options and develop a *scenario* based formulation that can be solved in a negligible amount of time.

Let OP_R denote the set of options that are present in the condition codes of part R, e.g. in Fig. 3.2, $\text{OP}_{R_2} = \{A_3,\ A_4,\ A_6\}$. We enumerate all scenarios for the options in OP_R (this is possible since the size of OP_R is usually less than 10, e.g. see Fig. 3.1). These scenarios are denoted by 0-1 vectors, e.g., $(1, 0, 1)$ means only A_3 and A_6 are selected. For each scenario, we determine whether the rules are satisfiable (all rules and not only the rules that are related to OP_R). Let SC_R denote the set of satisfiable scenarios, also referred to as *sub-configurations*. With some abuse of notation, we denote sub-configurations by $\mathbf{y} \in \text{SC}_R$.

Let $\tilde{\hat{\mathbf{p}}}$ denote the vector of penetration rates for the options in OP_R. For each sub-configuration $\mathbf{y} \in \text{SC}_R$, let $\varsigma_{\mathbf{y}}$ denote the number of units of part R required to produce one unit of that sub-configuration. We aim to find a convex combination of the sub-configurations that satisfy the penetration rates for the options in OP_R and minimize (respectively, maximize) the part R's requirement. This is achieved by solving the following linear programming problem, referred to as (PB).

$$\begin{aligned}
\text{(PB):} \quad \min / \max \quad & Q_R^\circ = \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} \varsigma_{\mathbf{y}} \\
\text{s.t.} \quad & \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} \mathbf{y} = \tilde{\hat{\mathbf{p}}}, \\
& \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathrm{SC}_R.
\end{aligned}$$

Let $Q_{R,L}^\circ$ and $Q_{R,U}^\circ$ denote the optimal values of the minimization and maximization cases of problem (PB), respectively. In the following, we show $Q_{R,L}^\circ \leq Q_{R,L}^*$ and $Q_{R,U}^* \leq Q_{R,U}^\circ$ (recall that $Q_{R,L}^*$ and $Q_{R,U}^*$ respectively denote the optimal values of (P1min) and (P1max)).

Proposition 8 (Bound on the Optimal Range)**.** *The optimal values of problems (P1) and (PB) satisfy:* $Q_{R,L}^\circ \leq Q_{R,L}^*$ *and* $Q_{R,U}^* \leq Q_{R,U}^\circ$.

The proof is given in Appendix 3.10.2. We note that the quality of this bound can be improved by incorporating additional options in OP_R. The more options included in OP_R the closer the sub-configurations to full configurations.[3] To determine which additional option from $\mathcal{N}_1 \backslash \mathrm{OP}_R$ to incorporate, a measure that captures the relative association (e.g. number of times that they appear in the same rules) with the options that are already in OP_R can be used. In section 3.5, we show the performance of this bound and how it can be improved by considering more options.

3.4 Value of additional marketing information

In mass customization, one of the major challenges in parts-capacity planning is the need for narrow ranges for parts' requirement. The GAM's contracts with their suppliers are based on these ranges, and the wider the ranges the costlier the parts-capacity acquisition.

[3]We note that the number of sub-configurations is exponential in the size of OP_R and hence a limited number of additional options can be incorporated.

The ranges that we obtain for parts' requirements in Section 3.3 are provably the tightest ranges that one can find using the available information (forecast OPS, rules, and condition codes). These ranges are sometimes wide mainly because the set of configurations-level demand that satisfies the given options-level forecast is very large. Thus, we have to collect additional information to narrow the obtained ranges. The additional information acts as a new constraint that shrinks the feasible set and improves the ranges. We also note that acquiring additional information is costly, and different information sets are obtained at different costs. In this section, we propose an interactive approach that adds one (or more) information at a time until the desired width of a range is obtained. Specifically, we aim to answer the following questions. What is the impact of additional information (e.g., on the penetration rates of single and joint options) on the ranges of parts' requirement? And, which information set is more likely to have the highest impact? For example, what is the impact of the joint penetration rate of engine `XYZ` and transmission `UVW` on reducing the range of a part's requirement?

3.4.1 Incorporating new information in the PCPP

We present a method for incorporating new information and ensuring the consistency of the new information with the rules and forecast OPS. We call the constraint $p(F) = \gamma$ (penetration rate of formula F is equal to γ) an *information set*, where F is a propositional formula and $0 \leq \gamma \leq 1$. Note that $p(F) = \gamma$ is equivalent to $p_{A'} = \gamma$ and $A' \Leftrightarrow F$, where A' is an *artificial* option. In other words, to add the information set $p(F) = \gamma$ to PCPP, we define a new *artificial* option A', set the penetration rate of option A' to γ, and add $A' \Leftrightarrow F$ to the set of rules.

Note that, it is necessary for $p(F) = \gamma$ to be *consistent* with the engineering and marketing constraints, meaning that adding this information to the PCPP does not make the problem infeasible. In the following lemma, we determine a range for γ such that the new information $p(F) = \gamma$ is consistent.

Lemma 10 (Consistency of a New Information Set)**.** *Let* $a_{L,\mathbf{y}}$ *and* $a_{U,\mathbf{y}}$ *denote the optimal coefficients of configuration* $\mathbf{y} \in \mathbb{Y}$ *in (P1min) and (P1max), respectively. Define* $\gamma_{L,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ *and* $\gamma_{U,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$*. The information set* $p(F) = \gamma$ *is consistent if* $\min\{\gamma_{L,F}, \gamma_{U,F}\} \leq \gamma \leq \max\{\gamma_{L,F}, \gamma_{U,F}\}$.

The proof is in Appendix 3.11.1. The condition in Lemma 10 is sufficient but not necessary. We next generalize this notion and present a method for ensuring the consistency of adding a collection of new information sets.

Lemma 11 (Consistency of a Collection of New Information Sets)**.** *Let* $a_{L,\mathbf{y}}$ *and* $a_{U,\mathbf{y}}$ *denote the optimal coefficients of configuration* $\mathbf{y} \in \mathbb{Y}$ *in (P1min) and (P1max), respectively. Let* $\tilde{\mathcal{F}} \subseteq \mathcal{F}$*. Define* $\gamma_{L,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ *and* $\gamma_{U,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$*, for all* $F \in \tilde{\mathcal{F}}$*. Let* $0 \leq \bar{\gamma} \leq 1$*. The following set of constraints is consistent:*

$$p(F) = \gamma_{L,F} + \bar{\gamma}(\gamma_{U,F} - \gamma_{L,F}), \quad \forall F \in \tilde{\mathcal{F}}.$$

We skip the proof as it is similar to the proof of Lemma 10. The condition of Lemma 11 is sufficient but not necessary. To illustrate the application of Lemma 11, assume that the allowable ranges for the information on F' and F'', obtained using Lemma 10, are $[0.2, 0.4]$ and $[0.5, 0.9]$, respectively. We can simultaneously add the following two information sets to PCPP: $p(F') = 0.3$ and $p(F'') = 0.7$ (because these values are the midpoints of the allowable ranges, these constraints are consistent).

3.4.2 Values of different information sets

Having new information sets on different formulas can result in different reductions in the range of a part's requirement. For example, is information set $p(A_1 \wedge A_2)$ better/worse than $p(A_1 \wedge A_2 \wedge A_3)$? What about $p(A_1 \vee A_2)$ versus $p(A_1 \vee A_2 \vee A_3)$? In this subsection, we show that, in general, it is not theoretically possible to establish ordering. Consequently, we present a heuristic guideline for comparing the values of different information sets.

Proposition 9 (Ordering Different Information Sets)**.** *Let $F', F'' \in \mathcal{F}$ and $0 \leq \gamma \leq 1$ such that each of the following information sets is consistent: $p(F') = \gamma$, $p(F' \wedge F'') = \gamma$, and $p(F' \vee F'') = \gamma$. Let $Q^{*'}_{R,L}$ and $Q^{*'}_{R,U}$ denote the optimal values of PCPP if $p(F') = \gamma$ is added, $Q^{*''}_{R,L}$ and $Q^{*''}_{R,U}$ denote the optimal values of PCPP if $p(F' \wedge F'') = \gamma$ is added, and $Q^{*'''}_{R,L}$ and $Q^{*'''}_{R,U}$ denote the optimal values of PCPP if $p(F' \vee F'') = \gamma$ is added. Then,*

(a) if $\gamma = 0$, then $Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L} \leq Q_R \leq Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L}$,

(b) if $\gamma = 1$, then $Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L} \leq Q_R \leq Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L}$.

Proof is in Appendix 3.11.2. To illustrate Proposition 9, let $F' = A_1 \wedge A_2$ and $F'' = A_3$. Hence, we are interested in comparing the values of information sets $p(A_1 \wedge A_2) = \gamma$ and $p(A_1 \wedge A_2 \wedge A_3) = \gamma$. Note that the former means $100\gamma\%$ of the cars sold over the planning horizon are forecast to have options A_1 and A_2, while the latter means $100\gamma\%$ of the cars sold are forecast to have options A_1, A_2, and A_3. Dependent on the value of γ, one information set can weakly dominate the other.

Proposition 9 implies that if $\gamma = 0$ (or small), then *pooling* of information ($\vee$) is preferred, while if $\gamma = 1$ (or close to 1), then *refinement* of information ($\wedge$) is preferred. However, the value of γ is not known a priori, and hence, the ordering cannot be established in advance.

We next propose a heuristic guideline for comparing two information sets. We prefer F' over F'' if $|\gamma_{U,F'} - \gamma_{L,F'}| > |\gamma_{U,F''} - \gamma_{L,F''}|$, where $\gamma_{U,F'}$, $\gamma_{L,F'}$, $\gamma_{U,F''}$, and $\gamma_{L,F''}$ are calculated as described in Lemma 10.

Information Ordering Criteria. *Information on F' is preferred to information on F'' if $|\gamma_{U,F'} - \gamma_{L,F'}| > |\gamma_{U,F''} - \gamma_{L,F''}|$.*

The intuition for this criteria is as follows: if $p(F')$ changes more than $p(F'')$, it can imply a higher influence on the range, and hence, we expect to see a higher reduction in the range by fixing the value of $p(F')$ compared to $p(F'')$. Our experimental analysis verifies the effectiveness of this criteria. We use this criteria in the design of our interactive approach.

3.4.3 Interactive approach for reducing the parts' requirement ranges

We develop an interactive approach that consists of an "expert" and a "system." See Appendix 3.11.3 for details and the flowchart. The system solves the PCPP and offers a list of candidate propositional formulas to the expert who then selects one or more of the candidates and determines their penetration rates. The system incorporates this information set and solves the PCPP again. This cycle is repeated until a pre-defined criteria is met. For example, a common criteria used in the GAM is that the width of the range should not be bigger than 20% of the minimum value of the range. The justification is that the supplier can plan for the minimum value of the range and increase the production by working overtime at most 20% of the regular time. An illustrative example is provided in Appendix 3.11.3. We will show how additional information helps narrowing the ranges for parts' requirement on an industrial instance in section 3.5.

3.5 Industrial Applications and Computational Experiment

In this section, we first show the computational effectiveness of our approach and the proposed bound on an industrial instance provided by the GAM. Next, we describe the method that is currently used in practice, compare our approach to that of the current practice, and discuss the advantages of our proposed methodology. We finally show the value of additional information on narrowing the parts' requirement ranges.

3.5.1 Performance of our solution approach on an industrial instance

We show the performance of our algorithm on an industrial instance that we have received from the GAM. This instance has 433 options and 171 rules.[4] In our analysis, we focus on three parts, which we denote by $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$.

[4]We note that, in real instances, some options exist without forecast penetration rates—options that are not customer facing but are options available for internal operations.

We apply Algorithm 1 for solving problem (P2) to obtain ranges on the requirement of the three parts. We implement our algorithm in IBM ILOG CPLEX Optimization Studio 12.6.1 and use a PC with Processor Intel(R) Core(TM) i5-2520M CPU 2.50GHz, 4.00 GB of RAM, and 64-bit Operating System. We run the algorithm for 10,000 seconds and report the results in Fig. 3.3 for different parts.

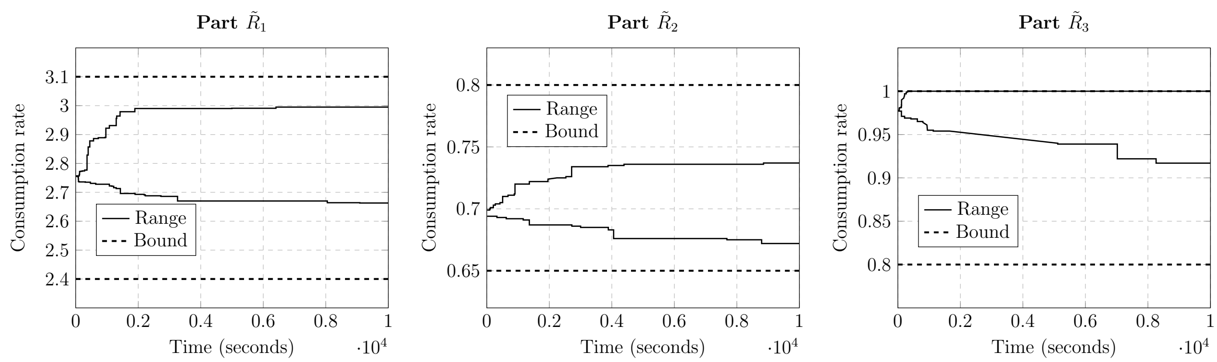

Figure 3.3: Computing the range using Algorithm 2 and performance of the proposed bound.

We note that major changes occur initially and then the minimum and maximum values of the ranges become approximately flat (except for the minimum value of the range for part $\tilde{R}_3$ that demonstrates large changes around 7,000 seconds). The optimal ranges for parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$ are respectively 12.47%, 9.67%, and 9.05% of the minimum values of the optimal ranges (based on the results after 10,000 seconds).

We also apply our approach presented in subsection 3.3.3 to obtain bounds on the ranges. This results in bounds (2.399,3.100), (0.650,0.800), and (0.800,1.000) for parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$, respectively. Fig. 3.3 shows these bounds using thick dashed lines. The error associated with the minimum values of the bounds are 9.91%, 3.27%, and 12.76%, and with the maximum values of the bounds are 3.5%, 8.55%, and 0%, respectively.

Improving bound on the range

As discussed in subsection 3.3.3, the bound on the ranges can be improved by incorporating additional options in the set OP_R. We next show how such options can be selected and how

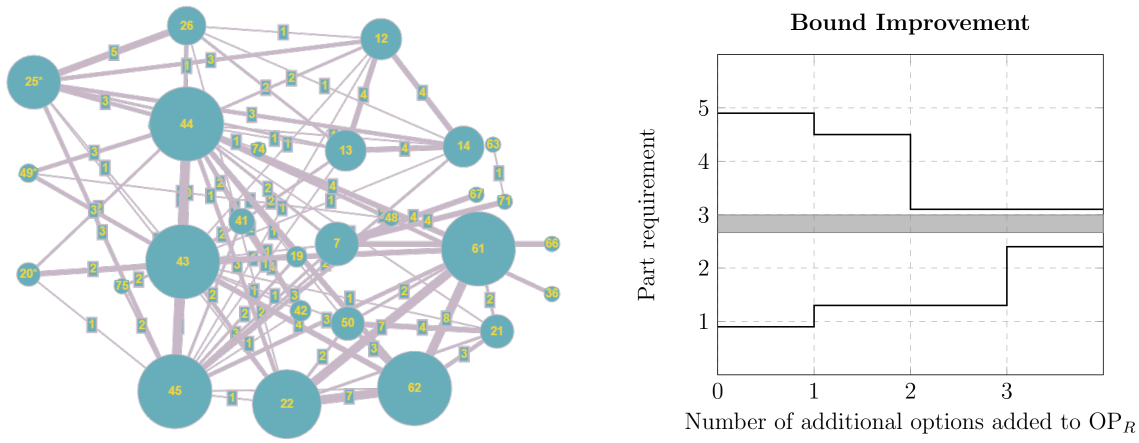

Figure 3.4: Improving bound by adding new options to $\text{OP}_{\tilde{R}}$.

they impact the bound. We focus on a part $\tilde{R}$ that has 6 options in its condition codes, i.e., $\text{OP}_{\tilde{R}}$ contains 6 options. We construct a graph of *options-relations* where the nodes are the options and there exist an arc between two options if they appear in a same rule. Moreover, the weight of an arc shows the number of times that two options appear in the same rules. Fig. 3.4 shows a portion of this graph. The nodes with a star (in the left side of the graph: options 25, 49, and 20) are the options that belong to $\text{OP}_{\tilde{R}}$. Among the set of options that are not included in $\text{OP}_{\tilde{R}}$, we choose the one with highest incident weight. In this example, options 26, 44, and 72 are sequentially added to $\text{OP}_{\tilde{R}}$ (note that option 72 is not shown in the graph). Fig. 3.4 shows improvements on the bound by these additions. The optimal range for this part is also shown by the shaded area. The error of the minimum value of the bound improves from 66.2% to 9.9% and the error of the maximum value improves from 63.6% to 3.5% by adding the three options.

3.5.2 Description and comparison to current practice

In this subsection, we describe, evaluate, and compare our results to that of the current practice.

Description of the current practice

The GAM's approach for determining the requirement of a part R involves generating all producible sub-configurations that include the options that are present in the condition codes of that part. Then, a convex combination of these sub-configurations is found that is as close as possible to the penetration rates of the associated options. This is used to find a point and range estimates for the requirement of part R. Their approach can be formally presented as follows.

Step 1: Generating sub-configurations. Let OP_R and SC_R be constructed as described in subsection 3.3.3.

Step 2: Least-squares fit. Recall that $\tilde{\hat{\mathbf{p}}}$ denotes the vector of penetration rates for the options in OP_R. Determine the sub-configurations' coefficients by solving the following quadratic problem:

$$
\begin{aligned}
&\min \left\| \sum_{\mathbf{y} \in \mathrm{SC}_R} a_{\mathbf{y}} \mathbf{y} - \tilde{\hat{\mathbf{p}}} \right\|_2^2 \\
&\text{s.t.} \sum_{\mathbf{y} \in \mathrm{SC}_R} a_{\mathbf{y}} = 1, \\
&a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathrm{SC}_R.
\end{aligned}
$$

Let $a^*_{\mathbf{y}}$'s denote the optimal solution. In fact, this problem finds a point in the convex hull of SC_R that has the minimum Euclidean distance to $\tilde{\hat{\mathbf{p}}}$. The optimal point is $\sum_{\mathbf{y} \in \mathrm{SC}_R} a^*_{\mathbf{y}} \mathbf{y}$ that is represented as a convex combination of the sub-configurations.

Step 3: Estimating part R's requirement. Recall that $\varsigma_{\mathbf{y}}$ denotes the number of units of part R required to produce one unit of sub-configuration $\mathbf{y}$. A point estimate of part R's requirement is calculated as $Q^{\diamond}_R := \sum_{\mathbf{y} \in \mathrm{SC}_R} a^*_{\mathbf{y}} \varsigma_{\mathbf{y}}$. A range is then considered for the requirement of part R with a center at $Q^{\diamond}_R$ and a pre-specified radius, e.g., if $Q^{\diamond}_R > 0$, the radius may be specified as 10% of $Q^{\diamond}_R$.

In fact, the optimal point $\sum_{\mathbf{y} \in \mathrm{SC}_R} a^*_{\mathbf{y}} \mathbf{y}$ in step 2 represents a demand realization on sub-configurations, i.e., $a^*_{\mathbf{y}}$ is the fraction of the total demand that includes sub-configuration $\mathbf{y}$. This demand realization maps to a point estimate for the part's requirement using the condition codes.

There are two major issues with this approach. First, it may find an infeasible point/range estimate (Appendix 3.12.1 provides an example for this case). Second, it obtains a point estimate using an arbitrary demand realization while there are usually many consistent demand realizations (as alternative optimal solutions $a^*_{\mathbf{y}}$'s in step 2).

Comparison to current practice

Fig. 3.5 provides a comparison between our approach and the current practice using a smaller instance that has 78 options' penetration rates. We consider different number of penetration rates denoted by n_1 (varies from 10 to 78) and focus on three parts denoted by $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$. For each n_1, we find a point estimate by applying the current approach described in subsection 3.5.2, and then, a range is created with 10% radius around the point estimate. "Practice" and "New" indicate the ranges found by applying the current approach and our method, respectively (the shaded area indicates the range found by our approach). We make the following observations:

- For small and medium n_1, the range found by their approach is usually a subset of the optimal range found by our approach which may result in shortage/excess of parts.
- Their approach sometimes (for medium and large n_1) finds a range that includes points outside of the optimal range that is found by our approach. In such cases, their approach suggests requirement that may never happen, i.e., they have excess capacity.
- For parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$, the *mean absolute percentage errors* (MAPE) for the minimum values of the ranges obtained by their approach are respectively 30.95%, 17.13%, and 13.78%, and for the maximum values of those ranges are 8.76%, 25.04%, and 12.04%, respectively (compared to the minimum and maximum values of the optimal

ranges found by our approach). On the average, the minimum and maximum values of the ranges found by their approach have MAPEs of 20.62% and 15.28%, respectively. Consequently, the average total error of the ranges, which is the sum of these two errors, is 35.9%.

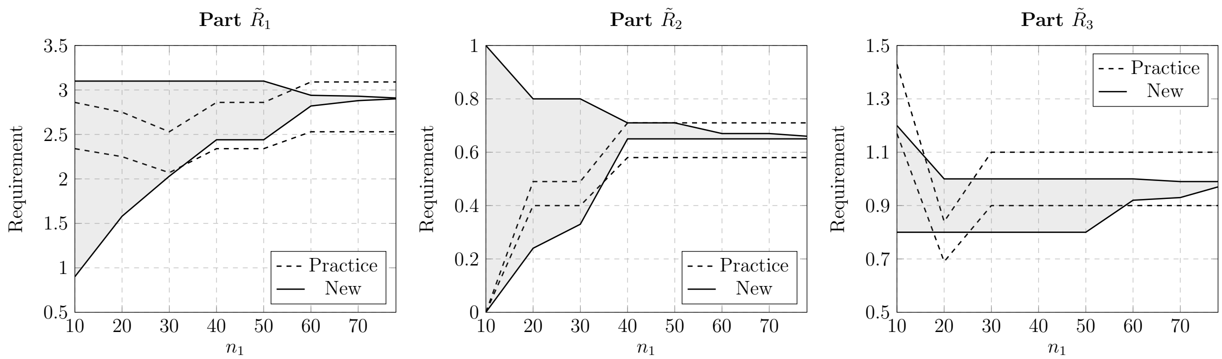

Figure 3.5: Our approach versus the current practice (the shaded areas have been determined by our approach).

3.5.3 Impact of additional information on the obtained ranges

Recall that the joint penetration rates of options can be incorporated as an additional penetration rate of a hypothetical option. Since we do not have the actual joint penetration rates, we use a subset of the given penetration rates as surrogate for additional information. For example, when we move from $n_1 = 10$ to $n_1 = 20$, the additional 10 penetration rates can be considered as additional information. Fig. 3.5 shows the effects of additional information on the obtained ranges by our method and the current approach.

- The width of the range found by our approach reduces as a result of incorporating new penetration rates, and for large n_1, the ranges are very tight for all parts. Whereas, in the current approach, since the radius of the range is always pre-specified, adding new information may not result in tightening the ranges. Furthermore, to obtain a satisfactory range using our approach, one could start with the given set of penetration rates and acquire new information as needed. For example, if 40 options initially have penetration rates, the range for parts $\tilde{R}_1$ and $\tilde{R}_2$ may be satisfactory while the range for part $\tilde{R}_3$ may be considered too wide. Hence, we may ask for more penetration rates to tighten the range for part $\tilde{R}_3$. Obviously, the cost for obtaining the additional penetration rates must be taken into consideration.

- In our approach, the minimum (respectively, maximum) value of the range is non-increasing (respectively, non-decreasing) in the information level n_1. This indicates that the obtained range for any information level n_1 always include the *actual value*—by the *actual value* we mean the requirement that correspond to the case of *full information* on the penetration rates of all options and joint penetration rates of options. However, the minimum and maximum values of the range produced by the current approach are not necessarily monotone and the obtained ranges may exclude the *actual value*. For example, for part $\tilde{R}_1$, when $n_1 \leq 50$ the range does not include the *actual value* that is found by our approach for large n_1. Similar error happens for parts $\tilde{R}_2$ and $\tilde{R}_3$ when $n_1 \leq 30$ and $n_1 \leq 20$, respectively.

- In the case of high information level (large n_1), our approach finds very tight ranges, and hence, we conclude that our approach is able to effectively absorb all available information and provide the tightest possible range for any information level n_1. In contrast, the range obtained by the current practice approach is unnecessarily wide for large n_1 indicating that the current approach is unable to utilize the given information and provides a range estimate based on a small subset of the penetration rates only.

In summary, the current approach may provide a range that excludes the *actual value* or a range that is unnecessarily wide. Whereas, our approach always utilizes all available information and finds the tightest possible range that includes the *actual value*.

3.6 Extensions

In this section, we extend our methodology in the following directions. First, we consider cases where options' penetration rates are given as points, ranges, or a combination of both. Second, based on the current practice in the GAM, usually a group of parts, with similar manufacturing requirements, are contracted to a single supplier. In these cases, the contract negotiation is also based on the total volume of the parts that are being subcontracted. Hence, in addition to providing ranges on individual parts' requirement, we provide a range estimate on the group of parts.

3.6.1 When ranges on options' penetration rates are given

Currently in the GAM, the forecast penetration rates of options are single values denoted by $\hat{p}_i$'s. What if options' penetration rates are given as ranges for some/all options or a combination of points for some options and ranges for some other options? In this subsection, we present a general approach that incorporates such scenarios. Let $p_{L,i}$ and $p_{U,i}$ denote the given lower and upper bounds on the penetration rate of option i satisfying $0 \leq p_{L,i} \leq p_{U,i} \leq 1$. Thus, if an option has a single value forecast, then, $p_{L,i} = p_{U,i}$, and if an option has not been assigned any forecast, then, $p_{L,i} = 0$ and $p_{U,i} = 1$. This generalization of problem (P1) is formulated as:

$$\text{(PG):} \quad \min / \max \quad Q_R = \sum_{(i,\alpha) \in \tilde{\mathscr{C}}(R)} \alpha p_i \tag{3.5}$$

$$\text{s.t.} \quad p_{L,i} \leq p_i \leq p_{U,i}, \ \forall i \in \mathcal{N}, \tag{3.6}$$

$$\mathbf{p} \in conv(\mathbb{Y}). \tag{3.7}$$

Problem (PG) can be formulated as a mixed-integer linear program similar to Appendix 3.9.1. To solve industrial instances of (PG), based on a similar motivation that we discussed in section 3.3, we present an equivalent nonlinear program:

$$\text{(PG}'\text{):} \quad \min / \max \quad \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \pm M \sum_{i=1}^{n} \max\{p_{L,i} - p_i, 0, p_i - p_{U,i}\} \tag{3.8}$$

$$\text{s.t.} \quad \mathbf{p} \in conv(\mathbb{Y}), \tag{3.9}$$

where, for sufficiently large M, solving (PG$'$) provides an optimal solution to (PG). Theorem 5 states this equivalence.

Theorem 5 (Generalization of Nonlinear Programming Equivalence)**.** *An optimal solution of problem (PG$'$) is an optimal solution to problem (PG) if*

$$M > \left(\sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha \right) \left(1 + \frac{2n+1}{\sqrt{\lambda_{\min}(n)}} \right),$$

where $\lambda_{\min}(n)$ denotes the smallest eigenvalue of all matrices in the form of $B^\top B$ where B is an invertible matrix of size $(2n+1)\times(2n+1)$ with all entries being a member of $\{-1,0,1\}$.

The proof is given in Appendix 3.13.1. This theorem simply means that for sufficiently large M, one can solve (PG$'$) to obtain an optimal solution to (PG). In addition, it specifies a lower bound on M as a function of α's, n, and $\lambda_{\min}(n)$. An increase in the values of α's and/or n require using larger values for M. Note that $\lambda_{\min}(n)$ is itself a function of n and it exists for a fixed n (because there is a finite number of possibilities for matrix B and the smallest eigenvalue for each possibility can be computed). Wolkowicz and Styan (1980) and Merikoski and Virtanen (1997) present lower bounds on the smallest eigenvalue of a symmetric positive definite matrix.

We remark that the minimization (respectively, maximization) case of problem (PG$'$) has a piecewise linear convex (respectively, concave) objective function, and hence, its subgradient vector can be determined at any given point. In addition, the feasible set of (PG$'$) is similar to that of (P2) and therefore a similar algorithm presented in subsection 3.3.2 can be applied.

3.6.2 Obtaining a range for a group of parts

We characterize the requirement of a *group of parts* that require a similar manufacturing processes and are contracted to the same supplier—i.e., $\sum_{R\in\tilde{\mathcal{R}}} Q_R$, for some $\tilde{\mathcal{R}} \subseteq \mathcal{R}$. Clearly, if we have unique values for Q_R's (not ranges), we can then determine the sum by summing up those values. However, finding a range for $\sum_{R\in\tilde{\mathcal{R}}} Q_R$ is complicated. Our approach is based on creating a hypothetical part which incorporates all condition codes for the parts that are included in the group. By finding a range on the requirement of this hypothetical part, we obtain a range on the requirement of the *group of parts*. The following proposition presents our approach.

Proposition 10 (Requirement for a Group of Parts)**.** *Let* $\tilde{\mathcal{R}} \subseteq \mathcal{R}$*,* $\tilde{\mathcal{R}} \neq \{\}$*. Then,* $\sum_{R\in\tilde{\mathcal{R}}} Q_R = Q_{\hat{R}}$*, where* $\hat{R}$ *is a hypothetical part with the following set of condition codes:*

$$\mathscr{C}(\hat{R}) = \{(F, \alpha')|F \in \bigcup_{R\in\tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(R), \alpha' = \sum_{R\in\tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha\},$$

and $\mathscr{C}\mathscr{F}(R) := \{F|\exists(F, \alpha) \in \mathscr{C}(R)\}$*.*

The proof is given in Appendix 3.13.2. Therefore, using proposition 10, the requirement of the group of parts is given by $Q_{\hat{R}} = \sum_{(F,\alpha)\in\mathscr{C}(\hat{R})} \alpha p(F)$. Appendix 3.13.3 provides an illustration of Proposition 10.

Next, we extend our result to obtain a range on the requirement of a *resource* that is used by the supplier (not the manufacturer) to produce a group of parts. A *resource* consumption can be represented as a nonnegative combination of the parts' requirement, where the coefficients indicate the unit *resource* usage for producing parts. Let $\tilde{\mathcal{R}}_{\mathrm{X}} \subseteq \mathcal{R}$ denote the subset (group) of parts that require resource X. We focus on a single resource and our analysis must be repeated for each resource. For ease of notation, in the remainder, we drop index X and refer to resource X simply as "the resource." The following proposition presents our approach.

Proposition 11 (Consumption Rate of a Resource)**.** *Let* $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, $\tilde{\mathcal{R}} \neq \{\}$, *and* $\eta_R \in \mathbb{R}_{++}$, *for all* $R \in \tilde{\mathcal{R}}$, *be given. Then,* $\sum_{R\in\tilde{\mathcal{R}}} \eta_R Q_R = Q_{\hat{R}}$, *where* $\hat{R}$ *is a hypothetical part with the following set of condition codes:*

$$\mathscr{C}(\hat{R}) = \{(F, \alpha')|F \in \bigcup_{R\in\tilde{\mathcal{R}}} \mathscr{CF}(R), \alpha' = \sum_{R\in\tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha\eta_R\},$$

and $\mathscr{CF}(R) := \{F|\exists(F, \alpha) \in \mathscr{C}(R)\}$.

The proof is similar to Proposition 10 and hence is skipped. Thus, the resource consumption rate is given by $Q_{\hat{R}} = \sum_{(F,\alpha)\in\mathscr{C}(\hat{R})} \alpha p(F)$. We finally note that our approach generates ranges for all resources that are used to produce a group of parts, which is useful for both the supplier and manufacturer in contract negotiations.

3.7 Conclusions

We study a parts-capacity planning problem in the context of mass customization. We develop a new methodology for finding the ranges on the parts' requirement when the demand forecast on configurations does not exist and the manufacturers forecast a demand on options. We present a formulation that enables us to generate a range on the parts' requirement and propose an effective methodology to solve it. We show how additional information on the joint options' penetration rates impacts the parts' requirement ranges and design an interactive method for achieving a desired target range. We apply our method to an industrial instance provided by the GAM and compare our results to the current approach. We finally extend our methodology to the following cases: (i) when ranges on the options' penetration rates are given, and (ii) finding a range on the requirement of a group of parts.

Our algorithm for solving problem (P2) has the following shortcoming. First, since the objective function is not smooth, the algorithm may converge slowly and lacks a convergence rate guarantee. However, since we have a bound on the optimal range, we can assess the optimality gap when the algorithm is terminated prior to completion. Further research on improving the speed of our algorithm, obtaining a convergence rate guarantee, and designing more effective bounds would be helpful.

Finally, we propose the following directions for future research. First, in parts' capacity management, range on a part's requirement provides one dimension of the variability. Another dimension of variability is the standard deviation which can be used to truncate the ranges for parts' requirements. Developing such measures would support parts' capacity management. Second, in this chapter, we consider an strategic parts' capacity planning that is performed prior to launching the product. This is an static model that determines a range for each part's requirement. Once the product is launched, parts' requirement can dynamically change based on demand evolution. The dynamic parts' capacity planning is a critical problem to be considered.

3.8 Appendix: Notations

Abbreviations:

CPS	Configurations-level penetration statistic
GAM	Global auto manufacturer
MILP	Mixed-integer linear programming
OPS	Options-level penetration statistic
P1, P1min, P1max	Our MILP problems
P2, P2L, P2U	Our NLP problems
PCPP	Parts capacity-planning problem (refers to (P1) and/or (P2))

Notations:

$A, A_1, A_2, \cdots \in \mathcal{N}$	Options ($\mathcal{N}$ is the set of all options)
$R, R_1, R_2, \cdots \in \mathcal{R}$	Parts ($\mathcal{R}$ is the set of all parts)
$F, F_1, F_2, \cdots \in \mathcal{F}$	Propositional formulas ($\mathcal{F}$ is the set of all propositional formulas)
$\mathscr{C}(R)$, $\tilde{\mathscr{C}}(R)$	The set of original and simplified condition codes for part R, respectively
$v_{\mathbf{y}}(F)$	The value of formula F in configuration $\mathbf{y}$
p	Penetration rate (defined for both formulas and options)
Q_R	Requirement of part R
$i = 1, \dots, n$	Index for options
$\mathbf{y} = (y_1, \dots, y_n)$	A producible configuration
$a_{\mathbf{y}}$	Penetration rate of configuration $\mathbf{y}$
$\mathbb{Y}$, $conv(\mathbb{Y})$	The set of all producible configurations and its convex hull
M	A sufficiently big number
$\mathbf{w}_R$, $\mathbf{Diag}(\mathbf{w}_R)$	The weight vector and the corresponding diagonal matrix in (P2)
$\hat{\mathbf{p}}_{R,L},\ \hat{\mathbf{p}}_{R,U}$	Parameters defined and used in formulating (P2)
$Q^*_{R,L},\ Q^*_{R,U}$	Optimal values of the PCPP (min. and max., respectively)
k	Iteration counter in our algorithm
$\mathbf{y}^{(k)}$	The new extreme point of $conv(\mathbb{Y})$ generated at iteration k
$\mathbf{p}^{(k)}$	The best known solution at iteration k of our algorithm
$\mathbf{g}^{(k)}$	The subgradient at iteration k of our algorithm
$\mathcal{N}_1$, n_1	The set and the number of options with forecast penetration rates, respectively
γ	The right-hand-side value of an information $p(F) = \gamma$

3.9 Appendix for PCPP formulation

3.9.1 Mixed-integer linear programming formulation

Note that $conv(\mathbb{Y})$ is bounded by a unit hypercube in $\mathbb{R}^n$. Moreover, a point inside a polytope (bounded polyhedron) in $\mathbb{R}^n$ can be represented as a convex combination of at most $n+1$ extreme points (Bertsimas and Tsitsiklis, 1997). Hence, if $\mathbf{p} \in conv(\mathbb{Y})$, then $\mathbf{p}$ can be represented as a convex combination of at most $n+1$ extreme points of $conv(\mathbb{Y})$. Therefore, there exist $\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n, \mathbf{y}^{n+1} \in \mathbb{Y}$ and $\mathbf{a} = (a_1, \ldots, a_n, a_{n+1}) \in \mathbb{R}_+^{n+1}$ such that $\mathbf{p} = \sum_{j=1}^{n+1} a_j \mathbf{y}^j$ and $\sum_{j=1}^{n+1} a_j = 1$. Thus, $\mathbf{p} \in conv(\mathbb{Y})$ can be formulated as the problem of finding, at most, $n+1$ producible configurations such that $\mathbf{p}$ belongs to their convex hull. Thus, we formulate our MILP problem as follows:

$$\text{(P1):} \quad \min / \max \quad Q_R = \sum_{(i,\alpha) \in \widetilde{\mathscr{C}}(R)} \alpha p_i \tag{3.10}$$

$$\text{s.t.} \quad p_i = \hat{p}_i, \ \forall i \in \mathcal{N}_1, \tag{3.11}$$

$$z_i^j \leq y_i^j, \ \forall i, j, \tag{3.12}$$

$$y_i^j - 1 \leq z_i^j - a_j \leq 1 - y_i^j, \ \forall i, j \tag{3.13}$$

$$p_i = \sum_{j=1}^{n+1} z_i^j, \ \forall i, \tag{3.14}$$

$$\sum_{j=1}^{n+1} a_j = 1, \tag{3.15}$$

$$a_j \geq 0, \ z_i^j \geq 0, \ \mathbf{y}^j \in \mathbb{Y}, \ \forall i, j. \tag{3.16}$$

Eqs. (3.12)-(3.14) are equivalent to the constraints $p_i = \sum_{j=1}^{n+1} a_j y_i^j$, for all $i = 1, \ldots, n$. Note that these constraints are not linear because of the multiplication of variables a_j and y_i^j. Eqs. (3.12)-(3.14) provide an approach to formulating these constraints using linear (in)equalities. We discuss this in more detail. For all i, j, we define new variables $z_i^j \geq 0$ such that $z_i^j = a_j y_i^j$. If $y_i^j = 0$, then $z_i^j = 0$, and if $y_i^j = 1$, then $z_i^j = a_j$. This is guaranteed by

Eqs. (3.12)-(3.14). Eq. (3.15) ensures that the coefficients of the producible configurations in the convex combination add up to 1.

Fig. 3.6 shows the solution of our MILP problem for part R_1 in Example 2. We obtain the range $2.155 \leq Q_{R_1} \leq 2.585$ and the width of this range is 0.43 (20% of the minimum value). Note that some of the $\mathbf{y}^j$'s are equal; e.g., $\mathbf{y}^8 = \mathbf{y}^{11}$ in the minimization problem. This means that the optimal solution can be represented as a convex combination of less than $n+1$ extreme points of $conv(\mathbb{Y})$.

Minimization problem:

Optimal value: 2.155

$\mathbf{p} = (0.385, 0.904, 0.019, 0.861, 0.096, 0.019, 0.05, 0.046, 0.289, 0.385, 0.885, 0)$

$\mathbf{a} = (0, 0.024, 0.572, 0, 0.019, 0, 0.027, 0.019, 0.05, 0.289, 0, 0, 0)$

$\mathbf{y}^1 = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^2 = (0,1,0,0,0,0,0,0,0,0,1,0)$
$\mathbf{y}^3 = (0,1,0,1,0,0,0,0,0,0,1,0)$
$\mathbf{y}^4 = (0,1,0,0,0,0,0,0,0,0,1,0)$
$\mathbf{y}^5 = (0,1,1,0,0,0,0,0,0,0,0,0)$
$\mathbf{y}^6 = (0,1,1,0,1,0,1,0,0,0,0,0)$
$\mathbf{y}^7 = (1,0,0,0,1,0,0,1,0,1,0,0)$
$\mathbf{y}^8 = (1,0,0,0,1,1,0,1,0,1,0,0)$
$\mathbf{y}^9 = (1,0,0,0,1,0,1,0,0,1,0,0)$
$\mathbf{y}^{10} = (1,1,0,1,0,0,0,0,1,1,1,0)$
$\mathbf{y}^{11} = (1,0,0,0,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{12} = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{13} = (1,0,0,1,1,1,0,1,0,1,0,0)$.

Maximization problem:

Optimal value: 2.585

$\mathbf{p} = (0.385, 1, 0.019, 0.861, 0.019, 0.019, 0.05, 0.046, 0.385, 0.385, 1, 0.05)$

$\mathbf{a} = (0, 0, 0.05, 0, 0.227, 0.062, 0.027, 0, 0, 0, 0.615, 0.019, 0)$

$\mathbf{y}^1 = (1,1,0,1,0,0,1,0,1,1,1,1)$
$\mathbf{y}^2 = (1,1,1,1,0,0,1,1,1,1,1,1)$
$\mathbf{y}^3 = (1,1,0,0,0,0,1,0,1,1,1,1)$
$\mathbf{y}^4 = (1,1,0,0,0,0,1,0,1,1,1,1)$
$\mathbf{y}^5 = (1,1,0,1,0,0,0,0,1,1,1,0)$
$\mathbf{y}^6 = (1,1,0,0,0,0,0,0,1,1,1,0)$
$\mathbf{y}^7 = (1,1,0,0,0,0,0,1,1,1,1,0)$
$\mathbf{y}^8 = (1,0,0,0,1,1,1,0,0,1,0,0)$
$\mathbf{y}^9 = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{10} = (1,0,1,0,1,0,1,1,1,1,0,0)$
$\mathbf{y}^{11} = (0,1,0,1,0,0,0,0,0,0,1,0)$
$\mathbf{y}^{12} = (1,1,1,1,1,1,0,1,1,1,1,0)$
$\mathbf{y}^{13} = (1,1,1,1,1,1,0,1,1,1,1,0)$.

Hence, the obtained range for Q_{R_1} is:
$2.155 \leq Q_{R_1} \leq 2.585$.

Similarly, for parts R_2 and R_3, we find:
$0 \leq Q_{R_2} \leq 0.019$
$0.765 \leq Q_{R_3} \leq 0.918$.

Figure 3.6: The solution of our MILP problem for parts R_1, R_2, and R_3 (also see Example 2, Fig. 3.2).

3.10 Appendix for solution approach

3.10.1 Proof of Theorem 4: Nonlinear Programming Equivalence

We need the following definitions in this proof. Let $\mathcal{N}_3$ denote the set of *artificial* options defined to simplify the complex condition codes for part R, and define $n_3 := |\mathcal{N}_3|$. Moreover, define $\mathcal{N}_2 := \mathcal{N} \backslash \{\mathcal{N}_1 \cup \mathcal{N}_3\}$, and $n_2 := |\mathcal{N}_2|$. Define $\mathscr{C}_1(R) := \{(i,\alpha) \in \mathscr{C}(R) | i \in \mathcal{N}_1\}$, $\mathscr{C}_2(R) := \{(i,\alpha) \in \mathscr{C}(R) | i \in \mathcal{N}_2\}$, and $\mathscr{C}_3(R) := \{(i,\alpha) | i \in \mathcal{N}_3, \exists (F,\alpha) \in \mathscr{C}(R) : F \Leftrightarrow i\}$. Hence, $\tilde{\mathscr{C}}(R) = \mathscr{C}_1(R) \cup \mathscr{C}_2(R) \cup \mathscr{C}_3(R)$.

In the remainder of this proof, we sometimes drop index R for the ease of notation.

Define vector $\beta \in \mathbb{R}^n$ as follows: for all $i = 1, \dots, n$, let $\beta_i := \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha$. Moreover, let $\mathbf{p}_L^*$ and $\mathbf{p}_U^*$ denote the optimal solutions of (P2L) and (P2U), respectively.

Recall that the objective function of problem (P1) is $Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i$. Since $\tilde{\mathscr{C}}(R) = \bigcup_{\ell=1}^{3} \mathscr{C}_\ell(R)$ and $\mathscr{C}_\ell(R)$'s are mutually exclusive, then the objective function of problem (P1) can be written as $Q_R = \sum_{\ell=1}^{3} \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i$. Thus, problem (P1) is equivalent to:

$$\min / \max \quad \sum_{\ell=1}^{3} \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \tag{3.17}$$

$$\text{s.t.} \quad p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1, \; \mathbf{p} \in conv(\mathbb{Y}). \tag{3.18}$$

We propose a relaxation of this problem by relaxing the equality constraints $p_i = \hat{p}_i$, for all $i \in \mathcal{N}_1$, and adding (respectively, subtracting) $\sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i|$ to the objective function of (P1min) (respectively, (P1max)). This results in the following problems:

$$\text{(P1Rmin):} \qquad \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\},$$

$$\text{(P1Rmax):} \qquad \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}.$$

By (P1R), we refer to both problems (P1Rmin) and (P1Rmax). In the following lemma, we show that, if M is sufficiently big, then the optimal solution of the relaxation problem (P1R) is also an optimal solution for (P1).

Lemma 12. *For a fixed β, there exists M such that solving problem (P1R) provides an optimal solution for (P1).*

Proof. Note that this lemma is an special case of Theorem 5 that we will state and prove in subsection 3.6.1 and Appendix 3.13.1. Theorem 5 proves the existence of M for the case where the *Marketing Constraints* are replaced with:

$$p_{L,i} \le p_i \le p_{U,i}, \quad \forall i \in \mathcal{N},$$

where $p_{L,i}$ and $p_{U,i}$ are the forecast lower and upper bounds on the penetration rate of option i. In other words, the forecast penetration rate of option i is given as a range $[p_{L,i}, p_{U,i}]$ rather than a single point. On the other hand, recall that the *Marketing Constraints* in (P1) are:

$$p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1.$$

Since $\mathcal{N}_1 \subseteq \mathcal{N}$, and each penetration rate is always between 0 and 1, then the *Marketing Constraints* in (P1) can be equivalently written as:

$$\hat{p}_i \leq p_i \leq \hat{p}_i, \quad \forall i \in \mathcal{N}_1,$$
$$0 \leq p_i \leq 1, \quad \forall i \in \mathcal{N} \backslash \mathcal{N}_1.$$

Hence, by applying Theorem 5 the proof is complete. □

Recall that $Q^*_{R,L}$ and $Q^*_{R,U}$ denote the optimal values of problems (P1min) and (P1max), respectively. We first prove the theorem for problem (P1min) by showing $Q^*_{R,L} = \beta^T \mathbf{p}^*_L$.

Using the equivalence between (P1min) and (P1Rmin), we have:

$$Q^*_{R,L} = \sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha\hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}. \quad (3.19)$$

Note that, based on the definition of β, we have: $\sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha\hat{p}_i = \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i$. Moreover, we have: $\sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i = \sum_{i\in\mathcal{N}_\ell} \beta_i p_i$, for all $\ell = 2,\ 3$. Hence, Eq. (3.19) can be written as:

$$Q^*_{R,L} = \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{i\in\mathcal{N}_\ell} \beta_i p_i \Big\} + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}. \quad (3.20)$$

Since $p_i \geq 0$, for all $i \in \mathcal{N}$, we have: $\sum_{i\in\mathcal{N}_\ell} \beta_i p_i = \sum_{i\in\mathcal{N}_\ell} \beta_i |p_i - 0|$, for all $\ell = 2,\ 3$. Therefore, using the definition of $\mathbf{w}$ and $\hat{\mathbf{p}}_L$, the objective function of the minimization problem in Eq. (3.20) is equal to $\sum_{i\in\mathcal{N}} w_i|p_i - \hat{p}_{L,i}|$. Thus,

$$Q^*_{R,L} \;=\; \sum_{i\in\mathcal{N}_1} \beta_i \hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1. \tag{3.21}$$

Define $\mathbf{p}^*_L := \arg\min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1$. Using Lemma 12, we have $p^*_{L,i} = \hat{p}_i$, for all $i \in \mathcal{N}_1$. Therefore, using Eq. (3.20), we have:

$$Q^*_{R,L} = \sum_{i\in\mathcal{N}_1} \beta_i p^*_{L,i} + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i p^*_{L,i} = \beta^T \mathbf{p}^*_L,$$

and hence, the proof of $Q^*_{R,L} = \beta^T\mathbf{p}^*_L$ is complete. The proof for $Q^*_{R,U} = \beta^T\mathbf{p}^*_U$ has similar steps that we summarize as follows:

$$\begin{aligned}
Q^*_{R,U} &= \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha \hat{p}_i + \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i \hat{p}_i + \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i p_i - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i \hat{p}_i + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i - \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i(1 - p_i) + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i \hat{p}_i + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i - \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_U)\|_1 \\
&= \beta^T \mathbf{p}^*_{\mathbf{U}},
\end{aligned}$$

where, $\mathbf{p}^*_{\mathbf{U}} := \arg\min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_U)\|_1$. To obtain the fourth line, we add and subtract $\sum_{\ell=2}^{3}\sum_{i\in\mathcal{N}_\ell} \beta_i$ to the objective function of the maximization problem in the third line. Then, since $+\sum_{\ell=2}^{3}\sum_{i\in\mathcal{N}_\ell} \beta_i$ is constant, we bring it outside of the maximization problem. Moreover, we multiply the objective function by -1, which results in a minimization problem. To obtain the fifth line, we note that $1 - p_i \geq 0$, for all i; hence, $1 - p_i = |p_i - 1|$, for all i. We then apply the definition of $\hat{\mathbf{p}}_U$. The last line follows by applying the optimal solution $\mathbf{p}^*_{\mathbf{U}}$ to the third line and noting that $\sum_{i\in\mathcal{N}_1} M|p^*_{U,i} - \hat{p}_i| = 0$ (because of Lemma 12).

3.10.2 Proof of Proposition 8: Bound on the Optimal Range

Recall that problem (P1min) (respectively, (P1max)) finds a point in $conv(\mathbb{Y})$ that satisfies the marketing constrains and minimizes (respectively, maximizes) the requirement of part R. Recall that $a_{\mathbf{y}}$ denotes the coefficient of configuration $\mathbf{y} \in \mathbb{Y}$ in a CPS. Moreover, with some abuse of notation, let $\varsigma_{\mathbf{y}}$ denote the number of units of part R required to produce one unit of configuration $\mathbf{y} \in \mathbb{Y}$. In fact, problem (P1) is equivalent to:

$$
\begin{aligned}
(\text{P1}'): \quad \min / \max \qquad & Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}}\varsigma_{\mathbf{y}} \\
\text{s.t.} \qquad & \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} y_i = \hat{p}_i, \ \forall i \in \mathcal{N}_1, \\
& \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathbb{Y}.
\end{aligned}
$$

Let us reorder the options such that $\mathbf{y} \in \mathbb{Y}$ can be written in the form of $\mathbf{y} = \binom{\mathbf{y}_1}{\mathbf{y}_2}$ where $\mathbf{y}_1$ contains the values of options in OP_R and $\mathbf{y}_2$ contains the values of options not in OP_R. Observe that $\varsigma_{\mathbf{y}_1} = \varsigma_{\mathbf{y}}$, for all $\mathbf{y} = \binom{\mathbf{y}_1}{\mathbf{y}_2} \in \mathbb{Y}$. Thus, the objective function of (P1′) can be written as:

$$
Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}}\varsigma_{\mathbf{y}} = \sum_{\mathbf{y}_1\in\text{SC}_R} \varsigma_{\mathbf{y}_1} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}}.
$$

Similarly, the marketing constraints can be written as:

$$
\sum_{\mathbf{y}_1\in\text{SC}_R} y_i \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = \hat{p}_i, \ \forall i \in \mathcal{N}_1 \cap \text{OP}_R, \tag{3.22}
$$

$$
\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} y_i = \hat{p}_i, \ \forall i \in \mathcal{N}_1 \backslash \text{OP}_R. \tag{3.23}
$$

Consider a relaxation of (P1′) by dropping constraints (3.23):

$$
\begin{aligned}
(\mathrm{P}1''):\quad \min/\max \quad & Q''_R = \sum_{\mathbf{y}_1 \in \mathrm{SC}_R} \varsigma_{\mathbf{y}_1} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} \\
\text{s.t.} \quad & \sum_{\mathbf{y}_1 \in \mathrm{SC}_R} y_i \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = \hat{p}_i,\ \forall i \in \mathcal{N}_1 \cap \mathrm{OP}_R, \\
& \sum_{\mathbf{y}_1 \in \mathrm{SC}_R} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathbb{Y}.
\end{aligned}
$$

Let $Q''_{R,L}$ and $Q''_{R,U}$ denote the optimal values of the minimization and maximization cases of problem (P1″), respectively. We next show that given a feasible solution of (P1″), there exists a feasible solution of (PB) with the same objective value. Consider a feasible solution $a_{\mathbf{y}}$'s of problem (P1″). Define $\bar{a}_{\mathbf{y}_1} := \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}}$, for all $\mathbf{y}_1 \in \mathrm{SC}_R$. Note that $\bar{a}_{\mathbf{y}_1}$'s satisfy all constraints of (PB) and the corresponding objective value is equal to that of $a_{\mathbf{y}}$'s in problem (P1″). This implies that: $Q^{\circ}_{R,L} \leq Q''_{R,L}$ and $Q''_{R,U} \leq Q^{\circ}_{R,U}$. In addition, since (P1″) is a relaxation of (P1), we have: $Q''_{R,L} \leq Q^{*}_{R,L}$ and $Q^{*}_{R,U} \leq Q''_{R,U}$. By combining these inequalities, we obtain that the optimal values of problems (P1) and (PB) satisfy: $Q^{\circ}_{R,L} \leq Q^{*}_{R,L}$ and $Q^{*}_{R,U} \leq Q^{\circ}_{R,U}$. Hence, the proof is complete.

3.11 Appendix for value of additional Information

3.11.1 Proof of Lemma 10: Consistency of a New Information Set

Consider adding information $p(F) = \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$. This is equivalent to adding $p_{A'} = \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ and $A' \Leftrightarrow F$, where A' is an artificial option. Note that the updated set of producible configurations after adding option A' is as follows:

$$
\mathbb{Y}' := \left\{ \begin{pmatrix} \mathbf{y} \\ y_{A'} \end{pmatrix} \middle| \mathbf{y} \in \mathbb{Y}, y_{A'} = v_{\mathbf{y}}(F) \right\}.
$$

Moreover, define $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} := a_{L,\mathbf{y}}$, for all $\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'$. Note that $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}}$'s are feasible coefficients for the updated producible configurations in the new problem. Hence, we only need to show that:

$$p_{A'} = \sum_{\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'} \tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} y_{A'} = \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F).$$

Note that this equation holds because $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} := a_{L,\mathbf{y}}$, for all $\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'$, and $y_{A'} = v_{\mathbf{y}}(F)$.

Similarly, we can show that adding information $p(F) = \sum_{\mathbf{y}\in\mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$ is also feasible. Finally, since $p(F) = \gamma_{L,F}$ and $p(F) = \gamma_{U,F}$ are consistent, then $p(F) = \gamma$ is consistent for all γ such that $\min\{\gamma_{L,F}, \gamma_{U,F}\} \leq \gamma \leq \max\{\gamma_{L,F}, \gamma_{U,F}\}$.

3.11.2 Proof of Proposition 9: Ordering Different Information Sets

Define A' as a new artificial option, and consider the following three problems.

$$\begin{aligned}
\text{Problem (i):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.} \quad & p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}), \\
& p_{A'} = \gamma,\ A' \Leftrightarrow F', \\
\text{Problem (ii):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.} \quad & p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}), \\
& p_{A'} = \gamma,\ A' \Leftrightarrow F' \wedge F'', \\
\text{Problem (iii):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.} \quad & p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}), \\
& p_{A'} = \gamma,\ A' \Leftrightarrow F' \vee F''.
\end{aligned}$$

In all problems, there are $n+1$ binary variables that can be shown as a (n+1)-dimensional binary vector $\tilde{\mathbf{y}} = \binom{\mathbf{y}}{y_{A'}}$. Moreover, let $\tilde{\mathbf{p}} = \binom{\mathbf{p}}{p_{A'}}$. Finally, note that a feasible $\tilde{\mathbf{p}}$ can be written as a convex combination of some producible configurations. Consider the following cases:

(a) $\gamma = 0$: In this case $p_{A'} = 0$, meaning that in all problems, we must have $y_{A'} = 0$ for all producible configurations that have positive coefficients in the convex combination. In other words, if $\tilde{\mathbf{p}} = \binom{\mathbf{p}}{0}$, then $\tilde{\mathbf{p}}$ can be written as a convex combination of some producible configurations in the form of $\binom{\mathbf{y}}{0}$. Hence, we can restrict ourselves to $\tilde{\mathbf{y}}$'s with the last entry 0. In this case, in problem (i), we must have $(v(F'), v(F'')) \in \{(0,0),(0,1)\}$; in problem (ii), we must have $(v(F'), v(F'')) \in \{(0,0),(0,1),(1,0)\}$; and in problem (iii), we must have $(v(F'), v(F'')) \in \{(0,0)\}$. Thus, any $\binom{\mathbf{y}}{0}$ that is feasible for (iii) is also feasible for (i), and any $\binom{\mathbf{y}}{0}$ that is feasible for (i) is also feasible for (ii). Thus, in this case, the optimal solution of (iii) is feasible for (i) and the optimal solution of (i) is feasible for (ii), meaning that:

$$Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L} \leq Q_R \leq Q^{*'''}_{R,U} \leq Q^{*'}_{R,U} \leq Q^{*''}_{R,U}.$$

(b) $\gamma = 1$: In this case we can restrict ourselves to $\tilde{\mathbf{y}}$'s with the last entry equal to 1. Hence, in problem (i), we must have $(v(F'), v(F'')) \in \{(1,0),(1,1)\}$; in problem (ii), we must have $(v(F'), v(F'')) \in \{(1,1)\}$; and in problem (iii), we must have $(v(F'), v(F'')) \in \{(0,1),(1,0),(1,1)\}$. Thus, any $\binom{\mathbf{y}}{1}$ that is feasible for (ii) is also feasible for (i), and any $\binom{\mathbf{y}}{1}$ that is feasible for (i) is also feasible for (iii). Thus, in this case, the optimal solution of (ii) is feasible for (i), and the optimal solution of (i) is feasible for (iii), meaning that:

$$Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L} \leq Q_R \leq Q^{*''}_{R,U} \leq Q^{*'}_{R,U} \leq Q^{*'''}_{R,U}.$$

3.11.3 An interactive approach for reducing a part's requirement range

Fig. 3.7 shows the flowchart of our proposed interactive method. The flowchart consists of two components: the expert's decisions and the system's decisions. The "expert" provides new information, and the "system" solves PCPP and performs some systematic operations.

First, the expert determines a subset of formulas $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ for which new information can be found. The set $\tilde{\mathcal{F}}$ is usually large but enumerable. For example, $\tilde{\mathcal{F}}$ can be the set of all conjunctions of one or two positive options.

Second, the system solves PCPP and checks whether the obtained range is acceptable. This decision is made by a pre-defined criteria. For example, a common criteria is that the width of the range should not be bigger than 20% of the minimum value.

Third, if the range is not acceptable, the system calculates $|\gamma_{U,F} - \gamma_{L,F}|$ for all $F \in \tilde{\mathcal{F}}$, and creates a short list by choosing some of the formulas with the biggest nonzero values for $|\gamma_{U,F} - \gamma_{L,F}|$. The number of formulas in the short list is usually pre-defined—e.g., the short list consists of 10 formulas at most.

Last, the expert chooses a formula from the short list and determines its penetration rate. The system adds this new information set to it and solves the problem again. This process is repeated until an acceptable range is found.

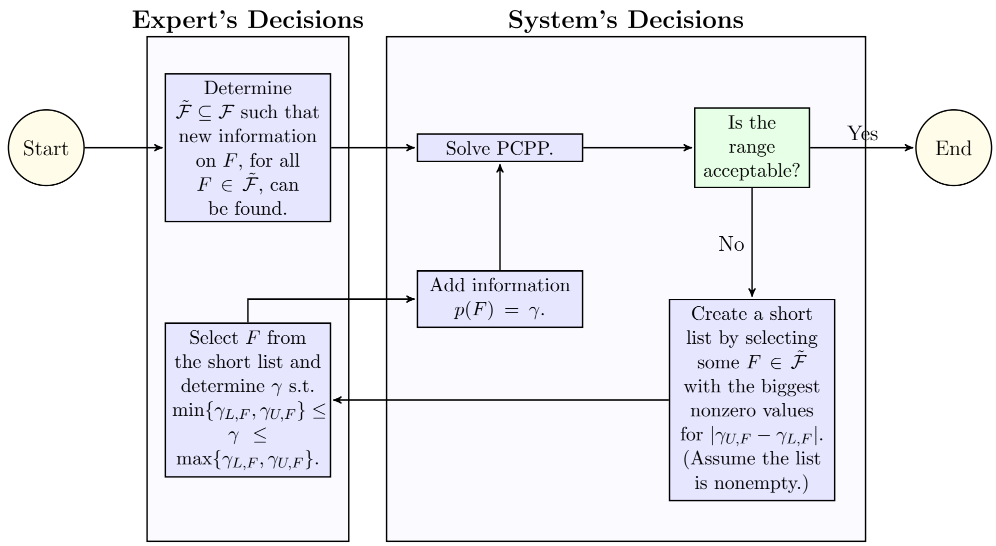

Figure 3.7: An interactive method for obtaining an acceptable range for a part's requirement.

Fig 3.8 illustrates our interactive method on part R_1 that we introduced in Example 2. We assume that the obtained range is acceptable if the width of the range is less than 10% of the minimum value. The short list (in steps (4) and (8)) consists of 4-5 formulas, with the allowable interval for the formulas' penetration rates. For example, in step (4), $0.246 \leq p(A_1 \wedge A_4) \leq 0.347$ means that the expert can assign a value to the penetration rate of formula $A_1 \wedge A_4$ in the interval $[0.246, 0.347]$. The short lists are sorted in the non-increasing order of the width of the allowable intervals. We assume that the expert selects a formula and assigns the midpoint of its allowable interval to its penetration rate. It is seen that for part R_1, after two rounds of adding new information set and resolving the problem, the obtained range satisfies our criteria.

Step-by-step execution of our interactive method for part R_1

(1) The expert determines $\tilde{\mathcal{F}}$ as the set of all conjunctions of one or two positive options.

(2) Solving PCPP results in:

$$2.155 \leq Q_{R_1} \leq 2.585$$

The width of this range is 19.95% of the minimum value.

(3) The range is not acceptable.

(4) Short list:

$$\begin{array}{lclcl} 0.246 & \leq & p(A_1 \wedge A_4) & \leq & 0.347 \\ 0.289 & \leq & p(A_1 \wedge A_2) & \leq & 0.385 \\ 0.904 & \leq & p(A_2) & \leq & 1 \\ 0.289 & \leq & p(A_2 \wedge A_9) & \leq & 0.385. \end{array}$$

(5) The expert adds the following information set:

$$p(A_2) = 0.952.$$

(6) Solving PCPP results in:

$$2.251 \leq Q_{R_1} \leq 2.481.$$

The width of this range is 10.21% of the minimum value.

(7) The range is not acceptable.

(8) Short list:

$$\begin{array}{lclcl} 0.021 & \leq & p(A_5 \wedge A_9) & \leq & 0.096 \\ 0.027 & \leq & p(A_4 \wedge A_5) & \leq & 0.077 \\ 0.048 & \leq & p(A_1 \wedge A_5) & \leq & 0.096 \\ 0 & \leq & p(A_2 \wedge A_5) & \leq & 0.048 \\ 0.048 & \leq & p(A_5) & \leq & 0.096 \end{array}$$

(9) The expert adds the following information set:

$$p(A_5 \wedge A_9) = 0.0585.$$

In order to add this information set, we define an artificial option A' such that $p_{A'} = 0.0585$. Moreover, we add the following rule to the problem:

$$A' \Leftrightarrow A_5 \wedge A_9.$$

(10) Solving the PCPP results in:

$$2.251 \leq Q_{R_1} \leq 2.439.$$

The width of this range is 8.35% of the minimum value.

(11) The range is acceptable, and hence we stop!

Figure 3.8: Illustrating our interactive method (also see Example 2 and Fig. 3.2).

3.12 Appendix for industrial applications

3.12.1 An example for the infeasibility of the point/range estimate found by the current approach

Consider options: A', A'', and A'''. Rules: $A''' \Rightarrow A' \vee A''$, $A' \Rightarrow A'''$, $A'' \Rightarrow A'''$. Condition code: $A' \wedge A'' \rightarrow R$. Penetration rates: $p_{A'} = \frac{1}{2}$, $p_{A''} = \frac{1}{2}$, and $p_{A'''} = \frac{4}{5}$.

The options that are related to part R are A' and A''. Producible sub-configurations: $\mathbf{y} = (y_{A'}, y_{A''}) \in \{(0,0), (1,0), (0,1), (1,1)\}$. The penetration rates $(p_{A'}, p_{A''}) = (\frac{1}{2}, \frac{1}{2})$ map into a set of points in the space of sub-configurations that has the following two extreme points: $(0, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0, \frac{1}{2})$. These extreme points map into requirement 0 and $\frac{1}{2}$, respectively. Thus, the current approach finds a point estimate in $[0, \frac{1}{2}]$ and then creates a range estimate using the 10% rule.

We next show that the obtained range $[0, \frac{1}{2}]$ contains only one feasible point as the requirement of R and any other point in this range is infeasible. Note that the set of rules can be summarized as $A''' \Leftrightarrow A' \vee A''$, which implies that $p_{A'''} = p(A' \vee A'') = \frac{4}{5}$. On the other hand, we have: $Q_R = p(A' \wedge A'') = p_{A'} + p_{A''} - p(A' \vee A'') = \frac{1}{5}$. Therefore, the only feasible point in the range $[0, \frac{1}{2}]$ is $\frac{1}{5}$. In addition, if the current approach determines, for example, $\frac{1}{2}$ as the point estimate, then the range estimate using the 10% rule will be $[0.45, 0.55]$. It is seen that the range estimate found by the current approach does not include the correct requirement of $\frac{1}{5}$.

3.13 Appendix for extensions

3.13.1 Proof of Theorem 5: Generalization of Nonlinear Programming Equivalence

Define vector $\beta = (\beta_1, \ldots, \beta_n)$ as follows: $\beta_i := \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha$, for all $i = 1, \ldots, n$. Note that $\beta \in \mathbb{R}_+^n$ and recall that in this chapter we assume $\mathbf{0} \in \mathbb{Y}$. Note that our problem (PG) can be equivalently formulated as follows:

$$\begin{aligned}
\text{(PG):} \quad \min/\max \quad & \beta^\top Y\mathbf{x} && (3.24)\\
\text{s.t.} \quad & \mathbf{p}_L \le Y\mathbf{x} \le \mathbf{p}_U, && (3.25)\\
& \mathbf{1}^\top\mathbf{x} = 1,\ \mathbf{x} \in \mathbb{R}_+^m, && (3.26)
\end{aligned}$$

where $Y := [\mathbf{y}^1|\mathbf{y}^2|\ldots|\mathbf{y}^m]$, $m := |\mathbb{Y}|$, $\mathbf{p}_L := (p_{L,1}, \ldots, p_{L,n})$, and $\mathbf{p}_U := (p_{U,1}, \ldots, p_{U,n})$. We want to prove that, for sufficiently large M, solving the following problem provides an optimal solution to problem (PG).

$$
\begin{aligned}
(\text{PG}'):\quad \min/\max \quad & \beta^\top Y\mathbf{x} \pm M\sum_{i=1}^{n}\max\{p_{L,i}-Y_i\mathbf{x}, 0, Y_i\mathbf{x}-p_{U,i}\} && (3.27)\\
\text{s.t.} \quad & \mathbf{1}^\top\mathbf{x}=1,\ \mathbf{x}\in\mathbb{R}^m_+, && (3.28)
\end{aligned}
$$

where Y_i denotes the ith row of matrix Y.

Let $\lambda_{\min}(n)$ denote the smallest eigenvalue of all matrices in the form of $B^\top B$ where B is an invertible matrix of size $(2n+1)\times(2n+1)$ with all entries being a member of $\{-1,0,1\}$. Since $B^\top B$ is symmetric and positive definite and there are a finite number of possibilities for B (and, consequently, for $B^\top B$), then $\lambda_{\min}(n)$ exists and $\lambda_{\min}(n)>0$. Note that there are at most $3^{(2n+1)^2}$ possibilities for B (because $B\in\{-1,0,1\}^{(2n+1)\times(2n+1)}$ and B must be invertible); hence, one needs to find the eigenvalues of $B^\top B$ for each possibility, which results in at most $(2n+1)3^{(2n+1)^2}$ different values, and finally, select the smallest value.

We aim to prove that solving (PG$'$) provides an optimal solution to (PG) if:

$$
M>\|\beta\|_1\left(1+\frac{2n+1}{\sqrt{\lambda_{\min}(n)}}\right).
$$

Our proof is based on analyzing the impact of changing $\mathbf{p}_L$ and $\mathbf{p}_U$ on the optimal value of problem (PG). We only consider vectors $\mathbf{p}_L$ and $\mathbf{p}_U$ for which the feasible region of (PG) is nonempty. Define $\mathbb{P}:=\{(\mathbf{p}_L,\mathbf{p}_U)\in[0,1]^{2n}|\ \text{Problem (PG) has a feasible solution}\}$. Hence, for all $(\mathbf{p}_L,\mathbf{p}_U)\in\mathbb{P}$, problem (PG) has a bounded optimal value since: $0\le\beta^\top\mathbf{p}_L\le\beta^\top Y\mathbf{x}\le\beta^\top\mathbf{p}_U\le\|\beta\|_1$ (also because $\beta\in\mathbb{R}^n_+$). Let the function $z(\mathbf{p}_L,\mathbf{p}_U)$, with domain $\mathbb{P}$, denote the optimal value of (PG). Thus, we have:

$$
0\le z(\mathbf{p}_L,\mathbf{p}_U)\le\|\beta\|_1,\qquad\forall(\mathbf{p}_L,\mathbf{p}_U)\in\mathbb{P}.
$$

We first focus on the maximization case of problem (PG). Dual of the maximization problem is as follows:

$$
\begin{aligned}
\text{(Dmax):} \quad \min \quad & \mathbf{e_1}^\top \mathbf{p}_U - \mathbf{e_2}^\top \mathbf{p}_L + e_3 && (3.29)\\
\text{s.t.} \quad & Y^\top \mathbf{e_1} - Y^\top \mathbf{e_2} + e_3 \mathbf{1} \geq Y^\top \beta, && (3.30)\\
& \mathbf{e_1},\ \mathbf{e_2} \in \mathbb{R}^n_+, && (3.31)
\end{aligned}
$$

where $\mathbf{e_1}$, $\mathbf{e_2}$, and e_3 are the dual variables (note that $e_3 \in \mathbb{R}$). The following lemma presents a characterization of the extreme points of the feasible set of the dual problem.

Lemma 13. *Let $\mathbb{E}$ denote the set of extreme points of the feasible set of problem (Dmax). Then, (a) $1 \leq |\mathbb{E}| < \infty$ and (b) $\|\mathbf{e}\|_1 \leq \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$, for all $\mathbf{e} \in \mathbb{E}$.*

PROOF. (a) Eq. (3.30) consists of a constraint associated with each row of matrix $Y^\top$. Thus, since $\mathbf{0} \in \mathbb{Y}$, there exists a constraint in the form of $e_3 \geq 0$. Therefore, all dual variables are nonnegative. Define $\mathbf{e} := (\mathbf{e_1}, \mathbf{e_2}, e_3) \in \mathbb{R}^{2n+1}_+$ as the vector of dual variables. Note that the feasible set of (Dmax) does not contain a line because it is a subset of $\mathbb{R}^{2n+1}_+$. Moreover, the feasible set of (Dmax) is nonempty because (PG) has a bounded optimal value for all $(\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}$ and $\mathbb{P}$ is nonempty (note that $(\mathbf{0}, \mathbf{1}) \in \mathbb{P}$). Thus, the feasible set of (Dmax) has at least one extreme point (due to Proposition 2.1.2 of Bertsekas (2009)). Additionally, a polyhedron (with at least one extreme point) that is defined by a finite number of (in)equalities have a finite number of extreme points.

(b) Note that the feasible set of (Dmax) is defined by $2n+1$ variables and $2n+m$ inequalities. Hence, an extreme point corresponds to $2n+1$ linearly independent binding (active) constraints. Therefore, an extreme point is obtained by finding $2n+1$ linearly independent constraints, setting them as equality, and solving the $2n+1$ equations which results in a unique solution $\mathbf{e}$; if $\mathbf{e}$ is feasible for problem (Dmax), then it is an extreme point. Let n' denote the number of linearly independent rows (or columns) of $[Y^\top \ \ -Y^\top \ \ \mathbf{1}]$ and note that $n' \leq n+1$. Let us select $2n+1$ linearly independent constraints and set them as equality. This includes n'' constraints from Eq. (3.30), such that $n'' \leq n'$, and $2n+1-n''$ constraints from Eq. (3.31). Denote the resulting system as $\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \mathbf{e} = \begin{bmatrix} \bar{Y}_1 \\ \mathbf{0} \end{bmatrix} \binom{\beta}{\mathbf{0}}$ where $\bar{Y}_1$ and $\bar{Y}_2$ are matrices of size $n'' \times (2n+1)$ and $(2n+1-n'') \times (2n+1)$ that correspond to the constraints that are selected from Eqs. (3.30) and (3.31), respectively, and $\mathbf{0}$ is a matrix (or vector) of appropriate size with all entries equal to zero. Note that:

$$\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \mathbf{e} = \begin{bmatrix} \bar{Y}_1 \\ \mathbf{0} \end{bmatrix} \binom{\beta}{\mathbf{0}} + \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \binom{\beta}{\mathbf{0}} - \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \binom{\beta}{\mathbf{0}} = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \binom{\beta}{\mathbf{0}} - \begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \end{bmatrix} \binom{\beta}{\mathbf{0}}.$$

Additionally, since $\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix}$ is invertible, we define $\bar{\bar{Y}} := -\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix}^{-1}$; hence, we have:

$$\begin{aligned}
\mathbf{e} &= \binom{\beta}{\mathbf{0}} + \bar{\bar{Y}} \begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \binom{\beta}{\mathbf{0}} \end{bmatrix} \\
&\leq \binom{\beta}{\mathbf{0}} + \left\| \bar{\bar{Y}} \begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \binom{\beta}{\mathbf{0}} \end{bmatrix} \right\|_2 \mathbf{1} \\
&= \binom{\beta}{\mathbf{0}} + \left(\begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \binom{\beta}{\mathbf{0}} \end{bmatrix}^\top \bar{\bar{Y}}^\top \bar{\bar{Y}} \begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \binom{\beta}{\mathbf{0}} \end{bmatrix} \right)^{1/2} \mathbf{1} \\
&\leq \binom{\beta}{\mathbf{0}} + \sqrt{1/\lambda_{\min}(n)} \left\| \begin{bmatrix} \mathbf{0} \\ \bar{Y}_2 \binom{\beta}{\mathbf{0}} \end{bmatrix} \right\|_2 \mathbf{1} \\
&\leq \binom{\beta}{\mathbf{0}} + \sqrt{1/\lambda_{\min}(n)} \, \|\beta\|_1 \, \mathbf{1}.
\end{aligned}$$

Note that the fourth line follows because $1/\lambda_{\min}(n)$ is an upper bound on the eigenvalues of $\bar{\bar{Y}}^\top \bar{\bar{Y}}$. Additionally, since $\mathbf{e} \geq \mathbf{0}$, then we have:

$$\|\mathbf{e}\|_1 \;\;\leq\;\; \|\beta\|_1 \left(1+(2n+1)\sqrt{1/\lambda_{\min}(n)}\right),$$

and hence the proof is complete. □

Using Lemma 13, the feasible set of (Dmax) has at least one extreme point. Moreover, since the optimal value of (Dmax) is attained for all $(\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}$, then an extreme point of the feasible set of (Dmax) is optimal (due to Proposition 2.4.1 of Bertsekas (2009)). Therefore,

$$z(\mathbf{p}_L, \mathbf{p}_U) = \min_{\mathbf{e}\in\mathbb{E}} \; \mathbf{e}^\top(\mathbf{p}_U, -\mathbf{p}_L, 1), \qquad \forall(\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}.$$

It is easy to see that z is a piecewise linear and concave function over $\mathbb{P}$. Consider changing $(\mathbf{p}_L, \mathbf{p}_U)$ to $(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) \in \mathbb{P}$ such that $\boldsymbol{\Delta_1}$ and $\boldsymbol{\Delta_2}$ are arbitrary vectors in $\mathbb{R}^n_+$. Since z is concave, then:

$$z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) \leq z(\mathbf{p}_L, \mathbf{p}_U) + \nabla z(\mathbf{p}_L, \mathbf{p}_U)^\top(-\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2}),$$

where ∇z denotes the subgradient of z, which is unique and equal to $(-\mathbf{e}_2, \mathbf{e}_1)$ for some $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ if z is linear at $(\mathbf{p}_L, \mathbf{p}_U)$. Note that if there are more than one $\mathbf{e} \in \mathbb{E}$ that achieve minimum at $(\mathbf{p}_L, \mathbf{p}_U)$, then z is not linear and its subgradient is the convex hull of all such $(-\mathbf{e}_2, \mathbf{e}_1)$'s. In addition, the above inequality holds for all $(-\mathbf{e}_2, \mathbf{e}_1)$ where $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ achieves the minimum at $(\mathbf{p}_L, \mathbf{p}_U)$. Thus, there exists $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ such that:

$$\begin{aligned}
z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) &\leq z(\mathbf{p}_L, \mathbf{p}_U) + (-\mathbf{e}_2, \mathbf{e}_1)^\top(-\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2}) \\
&\leq z(\mathbf{p}_L, \mathbf{p}_U) + \|\mathbf{e}\|_1 \|(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})\|_1 \\
&\leq z(\mathbf{p}_L, \mathbf{p}_U) + \|(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})\|_1 \, \|\beta\|_1 \left(1+(2n+1)\sqrt{1/\lambda_{\min}(n)}\right).
\end{aligned}$$

Let $\mathbf{x}$ denote an arbitrary feasible solution of problem (PG$'$). Define $\Delta_{1,i} := \max\{0, p_{L,i} - Y_i\mathbf{x}\}$ and $\Delta_{2,i} := \max\{0, Y_i\mathbf{x} - p_{U,i}\}$, for all i, and let $\boldsymbol{\Delta_1} := (\Delta_{1,1}, \ldots, \Delta_{1,n})$ and $\boldsymbol{\Delta_2} := (\Delta_{2,1}, \ldots, \Delta_{2,n})$. In addition, assume that at least one of $\boldsymbol{\Delta_1}$ or $\boldsymbol{\Delta_2}$ is nonzero implying that $\mathbf{x}$ is an infeasible point for problem (PG). We have:

$$z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) - \|(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})\|_1 \, \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right) \;\; \leq \;\; z(\mathbf{p}_L, \mathbf{p}_U).$$

Let $z'(\mathbf{x})$ denote the objective value of problem (PG$'$) for the feasible solution $\mathbf{x}$. Let $\mathbf{x}^*$ denote the optimal solution of problem (PG). Note that $\mathbf{x}^*$ is feasible for problem (PG$'$). We next show that $z'(\mathbf{x}) < z'(\mathbf{x}^*)$, which implies that $\mathbf{x}^*$ is an optimal solution of (PG$'$).

$$\begin{aligned}
z'(\mathbf{x}) &= \beta^\top Y\mathbf{x} - M\sum_{i=1}^{n} \max\{p_{L,i} - Y_i\mathbf{x}, 0, Y_i\mathbf{x} - p_{U,i}\} \\
&\leq z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) - M\|(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})\|_1 \\
&< z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2}) - \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right) \|(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})\|_1 \\
&\leq z(\mathbf{p}_L, \mathbf{p}_U).
\end{aligned}$$

The second line follows because $\beta^\top Y\mathbf{x} \leq z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2})$ (here, note that $\mathbf{x}$ is a feasible solution of (PG) after we change $(\mathbf{p}_L, \mathbf{p}_U)$ to $(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2})$ while $z(\mathbf{p}_L - \boldsymbol{\Delta_1}, \mathbf{p}_U + \boldsymbol{\Delta_2})$ is the optimal value after this change). The third line follows because we assume $M > \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$. Finally, we observe that:

$$z'(\mathbf{x}) \;\; < \;\; z(\mathbf{p}_L, \mathbf{p}_U) = z'(\mathbf{x}^*),$$

which completes the proof for the maximization case of problem (PG). The proof for the minimization case is very similar. Dual of the minimization problem is as follows:

$$\begin{aligned}
\text{(Dmin):} \quad \max \quad & -\mathbf{e_1}^\top \mathbf{p}_U + \mathbf{e_2}^\top \mathbf{p}_L - e_3 && (3.32)\\
\text{s.t.} \quad & -Y^\top \mathbf{e_1} + Y^\top \mathbf{e_2} - e_3 \mathbf{1} \le Y^\top \beta, && (3.33)\\
& \mathbf{e_1},\ \mathbf{e_2} \in \mathbb{R}^n_+, && (3.34)
\end{aligned}$$

The following lemma presents a characterization of the extreme points of (Dmin).

Lemma 14. *Let* $\mathbb{E}_{\min}$ *denote the set of extreme points of the feasible set of problem (Dmin). Then, (a)* $1 \le |\mathbb{E}_{\min}| < \infty$ *and (b)* $\|\mathbf{e}\|_1 \le \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$*, for all* $\mathbf{e} \in \mathbb{E}_{\min}$.

Proof. The proof follows the same steps as the proof of Lemma 13. The main difference is that in part (b), when we select $2n+1$ linearly independent constraints, the resulting system is denoted as $\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \mathbf{e} = \begin{bmatrix} \bar{Y}_1 \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \beta \\ 0 \end{bmatrix}$. The rest of the proof is similar and hence is skipped. □

Using Lemma 14, we can formulate the minimization case of (PG) as:

$$z_{\min}(\mathbf{p}_L, \mathbf{p}_U) = \max_{\mathbf{e} \in \mathbb{E}_{\min}} \mathbf{e}^\top(-\mathbf{p}_U, \mathbf{p}_L, -1), \qquad \forall (\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P},$$

where $z_{\min}(\mathbf{p}_L, \mathbf{p}_U)$ denotes the optimal value of the minimization case of (PG) and is a piecewise linear and convex function over $\mathbb{P}$. Consider changing $(\mathbf{p}_L, \mathbf{p}_U)$ to $(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) \in \mathbb{P}$ such that $\mathbf{\Delta_1}$ and $\mathbf{\Delta_2}$ are arbitrary vectors in $\mathbb{R}^n_+$. Since z is convex, then there exists $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}_{\min}$ such that:

$$\begin{aligned}
z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) \ &\ge\ z(\mathbf{p}_L, \mathbf{p}_U) + (\mathbf{e}_2, -\mathbf{e}_1)^\top(-\mathbf{\Delta_1}, \mathbf{\Delta_2})\\
&\ge\ z(\mathbf{p}_L, \mathbf{p}_U) - \|\mathbf{e}\|_1 \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1\\
&\ge\ z(\mathbf{p}_L, \mathbf{p}_U) - \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right).
\end{aligned}$$

The rest of the proof for the minimization case of (PG) is very similar to the maximization case that we showed above, and hence, is skipped. Thus, the proof is complete.

3.13.2 Proof of Proposition 10: Requirement for a Group of Parts

We have:

$$\begin{aligned}
\sum_{R\in\tilde{\mathcal{R}}} Q_R &= \sum_{R\in\tilde{\mathcal{R}}} \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha p(F) \\
&= \sum_{R\in\tilde{\mathcal{R}}} \sum_{F\in\mathscr{C}\mathscr{F}(R)} p(F) \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha \\
&= \sum_{R\in\tilde{\mathcal{R}}} \sum_{F\in\bigcup_{\bar{R}\in\tilde{\mathcal{R}}}\mathscr{C}\mathscr{F}(\bar{R})} p(F) \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha \\
&= \sum_{F\in\bigcup_{\bar{R}\in\tilde{\mathcal{R}}}\mathscr{C}\mathscr{F}(\bar{R})} p(F) \sum_{R\in\tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha \\
&= \sum_{\substack{F\in\bigcup_{\bar{R}\in\tilde{\mathcal{R}}}\mathscr{C}\mathscr{F}(\bar{R}) \\ \alpha'=\sum_{R\in\tilde{\mathcal{R}}}\sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)}\alpha}} \alpha' p(F) \\
&= \sum_{(F,\alpha')\in\mathscr{C}(\hat{R})} \alpha' p(F),
\end{aligned}$$

where, $\mathscr{C}(\hat{R}) = \{(F,\alpha')|F \in \bigcup_{R\in\tilde{\mathcal{R}}}\mathscr{C}\mathscr{F}(R), \alpha' = \sum_{R\in\tilde{\mathcal{R}}}\sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)}\alpha\}$. Note that the third line follows because if $F \notin \mathscr{C}\mathscr{F}(R)$, then $\sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)}\alpha = 0$. Hence, the proof is complete.

3.13.3 Illustration of finding a range for a group of parts

We illustrate Proposition 10 on parts R_1, R_2, and R_3, that we introduced in Example 2. Our approach is graphically shown in Fig. 3.9.

To show the effectiveness of the range obtained using Proposition 10, we compare it to a simple range on the requirement of a group of parts that is obtained by simply summing the minimum and maximum values on the ranges for individual parts. This simple approach

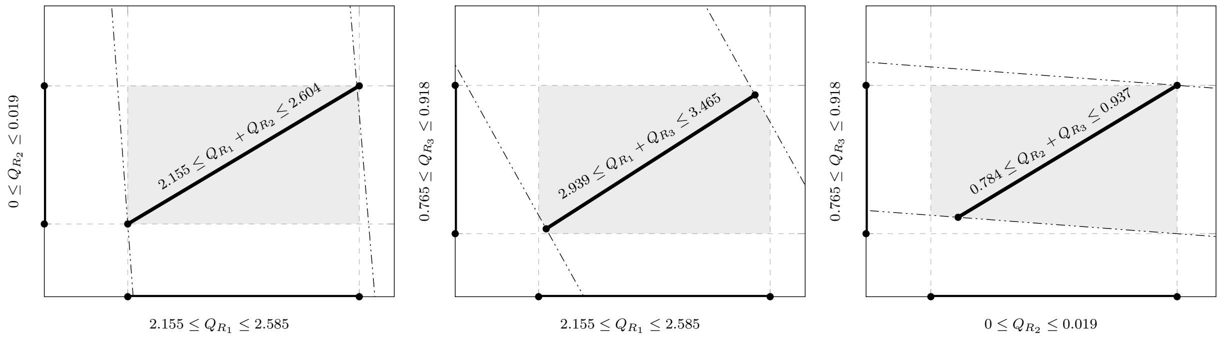

Figure 3.9: Graphical illustration of finding concurrent ranges for parts.

results in the following ranges:

$$2.155 \leq Q_{R_1} + Q_{R_2} \leq 2.604$$

$$2.92 \leq Q_{R_1} + Q_{R_3} \leq 3.503$$

$$0.765 \leq Q_{R_2} + Q_{R_3} \leq 0.937$$

$$2.92 \leq Q_{R_1} + Q_{R_2} + Q_{R_3} \leq 3.522.$$

We next obtain these ranges using Proposition 10. Consider finding a range for $(Q_{R_1} + Q_{R_2})$. Define a hypothetical part $\hat{R}$ with the following set of condition codes:

$$\begin{aligned} \mathscr{C}(\hat{R}) = \quad & \{(A_1, 1),\ ((A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4), 2), \\ & ((\neg A_5 \wedge A_7), 4),\ ((A_3 \wedge A_4) \vee (A_3 \wedge A_6), 1)\}. \end{aligned}$$

Note that in this example, $\mathscr{C}(\hat{R})$ is simply the union of $\mathscr{C}(R_1)$ and $\mathscr{C}(R_2)$. We obtain the following range for $Q_{\hat{R}}$, which is in fact a range for $(Q_{R_1} + Q_{R_2})$.

$$2.155 \leq Q_{\hat{R}} \leq 2.604.$$

Thus, by using Proposition 10, and by repeating the above procedure for other combinations, we obtain the following ranges:

$$2.155 \leq Q_{R_1} + Q_{R_2} \leq 2.604$$
$$2.939 \leq Q_{R_1} + Q_{R_3} \leq 3.465$$
$$0.784 \leq Q_{R_2} + Q_{R_3} \leq 0.937$$
$$2.939 \leq Q_{R_1} + Q_{R_2} + Q_{R_3} \leq 3.484.$$

In this example, the range on the group of parts is about 90% of the range obtained by summing the ranges for individual parts.

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Chapter 4

The Weighted Non-Negative Least-Squares Problem with Implicitly Characterized Points

4.1 Introduction

The non-negative least-squares (NNLS) problem is defined as finding the Euclidean distance between a target point and a convex cone generated by a set of discrete points in $\mathbb{R}^n$ (Lawson and Hanson, 1995; Franc et al., 2005). We study NNLS in which the set of discrete points are not a priori given but are implicitly known (e.g., the set of integer feasible solutions of a mixed-integer program). In particular, we are interested in the instances where the number of discrete points is exponential in n—e.g., when the discrete points are the feasible solutions of a binary program, the number of solutions is $\mathcal{O}(2^n)$.

Our problem is primarily motivated by an application in a large auto manufacturer, in which n represents the number of options (e.g., engine E1, engine E2, transmission T1, and so on) that define the car configurations (end-products). In this setting, the configurations have to satisfy a set of feasibility constraints and hence the number of feasible configurations can be $\mathcal{O}(2^n)$, which are only implicitly defined. The target point is an estimation of future demand of options over the planning horizon—e.g., the number of cars sold over the planning horizon with engine E1. The convex cone generated by the feasible configurations constitutes the set of feasible demand over the planning horizon (see Chapters 2 and 3). The problem is to find a point in the convex cone that has the minimum weighted Euclidean distance to the target point (weights may reflect the importance or the price of options). In addition, this problem can generally be found in the manufacturing systems where products are configured based on a set of options available for the customers. Other applications include variants of classical clustering problems. In some clustering problems, the objective is to find the Euclidean distance between a point and the convex hull of a set of points in $\mathbb{R}^n$, called a cluster, whereas our problem is to find the Euclidean distance between a target point and the convex cone of a set of implicitly known points in $\mathbb{R}^n$.

Although the NNLS is a convex optimization problem and solved efficiently (see, for example, Boutsidis and Drineas (2009) and Potluru (2012)), NNLS with implicit points cannot be solved using the existing approaches. Therefore, we first develop a surrogate problem where by solving it we find the optimal solution to our problem. Our surrogate problem enables us to design a Frank-Wolfe based approach (Frank and Wolfe, 1956) for finding the optimal solution. We generate a feasible solution (upper bound) and a lower bound at each iteration k of our proposed approach. We show that both the upper and lower bounds converge to the optimal value at $\mathcal{O}(1/k)$. We numerically investigate the effectiveness of our approach on a set of instances. Finally, we remark that the computational complexity of our approach depends on the computational complexity of the subproblem that we solve at each iteration. If the subproblem is polynomially solved, then the overall computational complexity of our approach is polynomial in n.

We also note that a special case of this problem (non-weighted), where the discrete points are the solutions of a satisfiability problem, is studied in Chapter 2. This Chapter extends and generalizes those results by allowing the discrete points to be the integer solutions of a mixed-integer program, providing a detailed convergence rate guarantee on the upper bound and establishing the convergence rate of the lower bound.

This chapter is organized as follows. Section 4.2 formally defines our problem. In section 4.3, we present the surrogate problem and propose our algorithm for solving it. We establish the convergence rate of the upper and lower bounds in section 4.4. Section 4.5 presents our numerical results. Section 4.6 concludes.

4.2 Problem Description

Consider a finite set $\mathbb{Y} = \{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^m\} \subseteq \mathbb{R}^n$, a *target point* $\hat{\mathbf{z}} \in \mathbb{R}^n$, and an $n \times n$ *invertible* matrix W, satisfying the following assumptions: (i) ${\mathbf{y}'}^T W^T W \mathbf{y}'' \geq 0$, for all $\mathbf{y}'$, $\mathbf{y}'' \in \mathbb{Y}$,[1] (ii) there exists $\mathbf{y} \in \mathbb{Y}$ such that $\mathbf{y} \neq 0$, and (iii) $\hat{\mathbf{z}} \neq 0$ (note that assumptions (ii) and (iii) are used to eliminate the trivial cases). A complete list of notations is available in Appendix 4.7.

In particular, we are interested in cases where $\mathbb{Y}$ is only implicitly known and m is exponential in n. Let $\mathcal{CC}(\mathbb{Y})$ denote the convex cone generated by the set $\mathbb{Y}$. In fact, $\mathcal{CC}(\mathbb{Y})$ contains all points $\mathbf{z}$ such that $\mathbf{z} = \sum_{j=1}^{m} \xi_j \mathbf{y}^j$ for some non-negative coefficients ξ_j's. We are interested in the problems of the form:

$$\begin{aligned} \text{(P1):} \quad \min \quad & h(\mathbf{z}) = \|W(\mathbf{z} - \hat{\mathbf{z}})\|_2^2 \\ \text{s.t.} \quad & \mathbf{z} \in \mathcal{CC}(\mathbb{Y}). \end{aligned}$$

In fact, (P1) finds a point $\mathbf{z}^* \in \mathcal{CC}(\mathbb{Y})$ that has the closest weighted Euclidean distance to $\hat{\mathbf{z}}$. Fig. 4.1 provides an example in $\mathbb{R}^3_+$. For the ease of illustration, we assume $\mathbb{Y}$ contains five points that are as shown in Fig. 4.1. Given vectors $\mathbf{y}^1, \ldots, \mathbf{y}^5$, we want to find the

[1] We acknowledge that, in general, Assumption (i) is restrictive and difficult to check and guarantee. In the application that motivated this chapter, all discrete points are in the same orthant and W is diagonal, which satisfies Assumption (i). Furthermore, in many similar practical applications, this assumption is simply satisfied. Theoretically, one needs to solve $\min_{\mathbf{y}', \mathbf{y}'' \in \mathbb{Y}} {\mathbf{y}'}^T W^T W \mathbf{y}''$ in order to check Assumption (i). We note that this problem has a nonlinear objective function because it contains components that are the product of two discrete variables, e.g., $y'_{i'} y''_{i''}$. Since y's are discrete, then $y'_{i'} y''_{i''}$ can be linearized, which increases the number of variables polynomially. The resulting problem may still be significantly difficult to solve.

Euclidean distance between $\hat{\mathbf{z}}$ and the convex cone generated by $\mathbf{y}^1, \ldots, \mathbf{y}^5$. We assume, in this example, W is a 3×3 identity matrix.

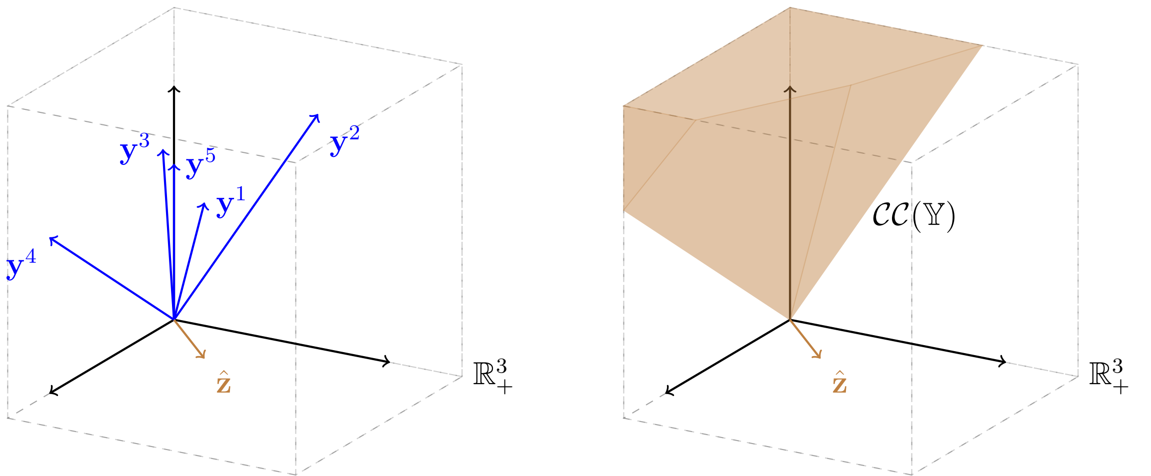


Figure 4.1: An illustrative example of (P1) where $\mathbf{y}^1 = (1, 1, 2)$, $\mathbf{y}^2 = (0, 2, 3)$, $\mathbf{y}^3 = (2, 1, 3)$, $\mathbf{y}^4 = (3, 0, 2)$, $\mathbf{y}^5 = (0, 0, 2)$, and $\hat{\mathbf{z}} = (1, 1, 0)$.

Problem (P1) has a strongly convex objective function. Appendix 4.8 presents two alternative formulations of (P1) and discusses their drawbacks. Problem (P1) is very difficult, and general purpose solvers cannot solve medium and large instances. In the remainder, we propose an effective approach for solving (P1).

4.3 Solution Methodology

We are particularly interested in special cases of (P1) where $\mathbb{Y}$ is implicitly known, but we can solve $\min_{\mathbf{y}\in\mathbb{Y}} \mathbf{c}^T\mathbf{y}$ in a reasonable time[2] for a given coefficient vector $\mathbf{c} \in \mathbb{R}^n$.[3] For example, $\mathbb{Y}$ can be the set of feasible solutions of a satisfiability problem, binary program, or mixed-integer program. Since we can solve $\min_{\mathbf{y}\in\mathbb{Y}} \mathbf{c}^T\mathbf{y}$ in a reasonable time, then we can sequentially construct $\mathbb{Y}$. We note that the optimal solution $\mathbf{z}^*$ can be represented as

[2] "Reasonable time" does not imply polynomial time; rather, it means that the problem is solved fast.

[3] Note that we use $\min_{\mathbf{y}\in\mathbb{Y}} \mathbf{c}^T\mathbf{y}$ to motivate our approach since this problem has basically similar objective function and constraints to the subproblem that we will introduce in subsection 4.3.2.

a positive combination of at most n points in $\mathbb{Y}$; hence, a cleverly designed algorithm may find the optimal solution to (P1) by solving $\min_{\mathbf{y}\in\mathbb{Y}} \mathbf{c}^T\mathbf{y}$ at most $\mathcal{O}(n)$ times.

We design our solution approach based on this observation. Our approach belongs to the class of methods that sequentially construct the feasible region and stop when a good solution is found. One such approach is the well-known Frank-Wolfe algorithm (Frank and Wolfe, 1956; Demyanov and Rubinov, 1970), also known as the *conditional gradient method.* In the literature, the Frank-Wolfe algorithm is used to minimize a convex objective function over a compact convex domain (Jaggi, 2013; Lacoste-Julien and Jaggi, 2015). Reddi et al. (2016); Lafond et al. (2015); Hazan and Kale (2012) have recently extended the application of the Frank-Wolfe to minimizing a non-convex objective function and stochastic optimization.

In subsection 4.3.1, we present our surrogate problem. We propose our Frank-Wolfe based algorithm, discuss our lower bounding procedure, and provide an illustrative example in subsection 4.3.2. We analyze the subproblem that has to be solved at each iteration of the algorithm and present a modeling framework for it in subsection 4.3.3.

4.3.1 Surrogate Problem for Problem (P1)

We note that (P1) has a non-compact feasible region and the Frank-Wolfe type approaches cannot be directly applied to it; hence, as a critical step of our approach, we first present a surrogate problem (P2) that enables us to employ our solution method. We will show that one can determine the optimal solution of (P1) in polynomial time if the optimal solution of (P2) is known. Problem (P2) is given as follows:

$$\begin{aligned}
\text{(P2):}\ \min \quad & f(\mathbf{z}) = \|\mathbf{z} - W\hat{\mathbf{z}}\|_2^2 \\
\text{s.t.} \quad & \mathbf{z} \in \mathcal{CC}(\tilde{\mathbb{Y}}),\ \hat{\mathbf{z}}^T W^T \mathbf{z} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}},
\end{aligned}$$

where $\tilde{\mathbb{Y}} := \{W\mathbf{y}^1, \ldots, W\mathbf{y}^m\}$. Define $\mathcal{H} := \{\mathbf{z} \in \mathbb{R}^n | \hat{\mathbf{z}}^T W^T \mathbf{z} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\}$. In fact, $\mathcal{H}$ is the set of points in $\mathbb{R}^n$ that belong to the hyperplane $\hat{\mathbf{z}}^T W^T \mathbf{z} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}$. Moreover, let

$\mathcal{FH}$ denote the set of feasible solutions to (P2)—i.e., $\mathcal{FH} := \{\mathbf{z} \in \mathcal{H} | \mathbf{z} \in \mathcal{CC}(\tilde{\mathbb{Y}})\}$.

We provide a brief intuition for forming our surrogate problem (P2). In fact, we aim to generate a new problem that has a compact feasible region. Therefore, we concentrate on the intersection of the cone $\mathcal{CC}(\tilde{\mathbb{Y}})$ and the hyperplane $\mathcal{H}$. Although the intersection is not compact in general, we will show in subsection 4.3.3 that there exists a compact subset of the intersection that contains the iterates of our algorithm.

The sets $\mathcal{H}$ and $\mathcal{FH}$ are illustrated in Fig. 4.2 for the example introduced in Fig. 4.1 (assuming that W is a 3×3 identity matrix).

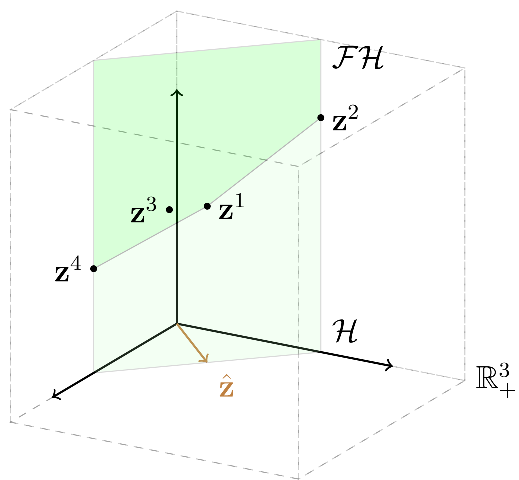


Figure 4.2: Illustration of $\mathcal{H}$ (entire shaded region) and $\mathcal{FH}$ (darker region) (assumption: W is an identity matrix). Note that there is no $\mathbf{z}^5$ since $\mathbf{y}^5$ in Fig. 4.1 is orthogonal to $\hat{\mathbf{z}}$.

The following theorem shows that by solving (P2), one can obtain an optimal solution for (P1) in polynomial time.

Theorem 6. *(a) $\mathbf{z}^* = 0$ solves (P1) if and only if (P2) is infeasible and (b) if (P2) has a feasible solution, then (P1) and (P2) have unique optimal solutions, which we denote by $\mathbf{z}^*$ and $\mathbf{z}^{**}$, respectively, and $\mathbf{z}^* = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{**T}\mathbf{z}^{**}} W^{-1}\mathbf{z}^{**}$.*

PROOF. (a, $\Rightarrow$) We use a contradiction to show if $\mathbf{z}^* = 0$, then $\mathcal{FH} = \{\}$. Assume that $\mathbf{z}^* = 0$ and $\mathcal{FH} \neq \{\}$. Then, there exists $\bar{\mathbf{z}} \in \mathcal{FH}$. Note that $\bar{\mathbf{z}} \neq 0$ (using the definition of

$\mathcal{FH}$, we have $\hat{\mathbf{z}}^T W^T \bar{\mathbf{z}} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} > 0$ because $W^T W$ is a positive definite matrix (because W is invertible) and $\hat{\mathbf{z}} \neq 0$ (because of our assumption (iii) in section 4.2)).

Define $\bar{\bar{\mathbf{z}}} := \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} W^{-1} \bar{\mathbf{z}} \neq 0$ (note that $\bar{\bar{\mathbf{z}}}$ is well-defined because $\bar{\mathbf{z}} \neq 0$). We show that $\bar{\bar{\mathbf{z}}}$ is feasible for (P1) and has a strictly better objective value than $\mathbf{z}^* = 0$. To prove the feasibility of $\bar{\bar{\mathbf{z}}}$, note the following. First, since $\bar{\mathbf{z}} \in \mathcal{CC}(\tilde{\mathbb{Y}})$, then $W^{-1}\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$. Second, since $\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} > 0$ and $W^{-1}\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$, then $\bar{\bar{\mathbf{z}}} \in \mathcal{CC}(\mathbb{Y})$ (because of the definition of cone). Thus, $\bar{\bar{\mathbf{z}}}$ is feasible for (P1). Next, note that:

$$
\begin{aligned}
& -\frac{(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}})^2}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} < 0 \\
\Rightarrow \quad & \frac{(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}})^2}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} - 2\frac{(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}})^2}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} < 0 \\
\Rightarrow \quad & \frac{(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}})^2}{(\bar{\mathbf{z}}^T \bar{\mathbf{z}})^2} \bar{\mathbf{z}}^T \bar{\mathbf{z}} - 2\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} < 0 \\
\Rightarrow \quad & \frac{(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}})^2}{(\bar{\mathbf{z}}^T \bar{\mathbf{z}})^2} \bar{\mathbf{z}}^T \bar{\mathbf{z}} - 2\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \hat{\mathbf{z}}^T W^T \bar{\mathbf{z}} + \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} < \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} \\
\Rightarrow \quad & \Big(\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \bar{\mathbf{z}} - W\hat{\mathbf{z}}\Big)^T \Big(\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \bar{\mathbf{z}} - W\hat{\mathbf{z}}\Big) < (W\hat{\mathbf{z}})^T (W\hat{\mathbf{z}}) \\
\Rightarrow \quad & h(\bar{\bar{\mathbf{z}}}) < h(0),
\end{aligned}
$$

and hence $\bar{\bar{\mathbf{z}}}$ has a strictly better objective value than $\mathbf{z}^* = 0$, which contradicts the optimality of $\mathbf{z}^* = 0$. Thus, the proof of (a, $\Rightarrow$) is complete.

(a, $\Leftarrow$) We first show (using a contradiction) that if $\mathcal{FH} = \{\}$, then $\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}} \leq 0$, for all $\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$. Assume that there exists $\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$ such that $\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}} > 0$. Define $\bar{\bar{\mathbf{z}}} := \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}} W \bar{\mathbf{z}}$. Because $\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}} > 0$ and $W\bar{\mathbf{z}} \in \mathcal{CC}(\tilde{\mathbb{Y}})$, then $\bar{\bar{\mathbf{z}}} \in \mathcal{CC}(\tilde{\mathbb{Y}})$. Moreover, $\hat{\mathbf{z}}^T W^T \bar{\bar{\mathbf{z}}} = \hat{\mathbf{z}}^T W^T (\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}} W \bar{\mathbf{z}}) = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}$. Therefore, $\bar{\bar{\mathbf{z}}} \in \mathcal{FH}$, meaning that $\mathcal{FH} \neq \{\}$. This is a contradiction, and hence, we proved that: if $\mathcal{FH} = \{\}$, then $\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}} \leq 0$, for all $\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$.

Let $\bar{\mathbf{z}} \in \mathcal{CC}(\mathbb{Y})$ be arbitrary. We show that the objective value of 0 is at least as good as the objective value of $\bar{\mathbf{z}}$. We note that:

$$
\begin{aligned}
h(\bar{\mathbf{z}}) &= \|W(\hat{\mathbf{z}}-\bar{\mathbf{z}})\|_2^2 \\
&= \left\|W\Big(\hat{\mathbf{z}} - \frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\hat{\mathbf{z}} + \frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\hat{\mathbf{z}} - \bar{\mathbf{z}}\Big)\right\|_2^2 \\
&= \left\|\Big(1 - \frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\Big)W\hat{\mathbf{z}} + \Big(\frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}W\hat{\mathbf{z}} - W\bar{\mathbf{z}}\Big)\right\|_2^2 \\
&= \left\|\Big(1 - \frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\Big)W\hat{\mathbf{z}}\right\|_2^2 + \left\|\frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}W\hat{\mathbf{z}} - W\bar{\mathbf{z}}\right\|_2^2 \\
&\geq \left\|\Big(1 - \frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\Big)W\hat{\mathbf{z}}\right\|_2^2 \\
&\geq \|W(\hat{\mathbf{z}}-0)\|_2^2 \\
&= h(0),
\end{aligned}
$$

meaning that the objective value of $\bar{\mathbf{z}}$ is greater than or equal to the objective value of 0. In the second line, we add and subtract $\frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}\hat{\mathbf{z}}$ to the expression inside the parentheses. To obtain the fourth line, we note that $W\hat{\mathbf{z}}$ and $(\frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}W\hat{\mathbf{z}} - W\bar{\mathbf{z}})$ are orthogonal—i.e., $(W\hat{\mathbf{z}})^T(\frac{\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}W\hat{\mathbf{z}} - W\bar{\mathbf{z}}) = 0$. To obtain the sixth line, we use the fact that $\hat{\mathbf{z}}^T W^T W \bar{\mathbf{z}} \leq 0$, which we proved earlier in this part. Since $\bar{\mathbf{z}}$ is an arbitrary feasible solution of (P1), then $\mathbf{z}^* = 0$ solves (P1). Thus, the proof of (a, $\Leftarrow$) is complete.

(b) We first show (using a contradiction) that if $\mathcal{FH} \neq \{\}$, then $\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} > 0$. Assume that $\mathcal{FH} \neq \{\}$ and $\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} \leq 0$. Then, we have: $h(\mathbf{z}^*) = \|W(\hat{\mathbf{z}} - \mathbf{z}^*)\|_2^2 = \|W\hat{\mathbf{z}}\|_2^2 + \|W\mathbf{z}^*\|_2^2 - 2\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} \geq \|W\hat{\mathbf{z}}\|_2^2 + \|W\mathbf{z}^*\|_2^2 > \|W\hat{\mathbf{z}}\|_2^2 = \|W(\hat{\mathbf{z}} - 0)\|_2^2 = h(0)$ (note that $-2\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} \geq 0$, and $W\mathbf{z}^* \neq 0$ because W is an invertible matrix, and, in part (a), we showed that if $\mathcal{FH} \neq \{\}$, then $\mathbf{z}^* \neq 0$). This contradicts the optimality of $\mathbf{z}^*$ because 0 has a strictly better objective value (note that, using part (a), we must have $\mathbf{z}^* \neq 0$ because we assume $\mathcal{FH} \neq \{\}$). Hence, we proved that: if $\mathcal{FH} \neq \{\}$, then $\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} > 0$.

We note that in (P1) and (P2) the objective functions are strongly convex (due to invertibility of W) and the feasible regions are nonempty and convex sets; hence, (P1) and (P2) have unique optimal solutions, which we denote by $\mathbf{z}^*$ and $\mathbf{z}^{**}$, respectively.

Define $\lambda := \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{*T} W^T W \hat{\mathbf{z}}}$. Note that $\lambda W \mathbf{z}^* \in \mathcal{FH}$ for the following reasons. First, $\lambda > 0$ because $\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} > 0$ and $\mathbf{z}^{*T} W^T W \hat{\mathbf{z}} > 0$. Second, since $W\mathbf{z}^* \in \mathcal{CC}(\tilde{\mathbb{Y}})$, then $\lambda W \mathbf{z}^* \in \mathcal{CC}(\tilde{\mathbb{Y}})$. Finally, we note that:

$$\hat{\mathbf{z}}^T W^T (\lambda W \mathbf{z}^*) = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{*T} W^T W \hat{\mathbf{z}}} \hat{\mathbf{z}}^T W^T W \mathbf{z}^* = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}},$$

and hence, we proved that $\lambda W \mathbf{z}^* \in \mathcal{FH}$.

We also note that $\frac{1}{\lambda} W^{-1} \mathbf{z}^{**} \in \mathcal{CC}(\mathbb{Y})$ for the following reasons. First, $\frac{1}{\lambda} > 0$. Second, $W^{-1}\mathbf{z}^{**} \in \mathcal{CC}(\mathbb{Y})$. Hence, in summary, we have shown that: $\frac{1}{\lambda} W^{-1} \mathbf{z}^{**}$ and $\mathbf{z}^*$ are feasible for (P1), and $\mathbf{z}^{**}$ and $\lambda W \mathbf{z}^*$ are feasible for (P2).

To complete the proof of part (b), we show that $\mathbf{z}^{**} = \lambda W \mathbf{z}^*$ using a contradiction. Suppose to the contrary that $\mathbf{z}^{**} \neq \lambda W \mathbf{z}^*$ (or $\frac{1}{\lambda} W^{-1} \mathbf{z}^{**} \neq \mathbf{z}^*$). Since $\mathbf{z}^*$ is unique and $\frac{1}{\lambda} W^{-1} \mathbf{z}^{**}$ is feasible for (P1), then we have:

$$\begin{aligned}
& h(\mathbf{z}^*) < h(\frac{1}{\lambda} W^{-1} \mathbf{z}^{**}) \\
\Rightarrow & \left\| W\left(\mathbf{z}^* - \frac{1}{\lambda}\hat{\mathbf{z}}\right) - \left(1 - \frac{1}{\lambda}\right) W\hat{\mathbf{z}} \right\|_2^2 < \left\| W\left(\frac{1}{\lambda} W^{-1}\mathbf{z}^{**} - \frac{1}{\lambda}\hat{\mathbf{z}}\right) - \left(1 - \frac{1}{\lambda}\right) W\hat{\mathbf{z}} \right\|_2^2 \\
\Rightarrow & \left\| W\left(\mathbf{z}^* - \frac{1}{\lambda}\hat{\mathbf{z}}\right) \right\|_2^2 + \left\| \left(1 - \frac{1}{\lambda}\right) W\hat{\mathbf{z}} \right\|_2^2 < \left\| W\left(\frac{1}{\lambda} W^{-1}\mathbf{z}^{**} - \frac{1}{\lambda}\hat{\mathbf{z}}\right) \right\|_2^2 + \left\| \left(1 - \frac{1}{\lambda}\right) W\hat{\mathbf{z}} \right\|_2^2 \\
\Rightarrow & \left\| W\left(\mathbf{z}^* - \frac{1}{\lambda}\hat{\mathbf{z}}\right) \right\|_2^2 < \left\| W\left(\frac{1}{\lambda} W^{-1}\mathbf{z}^{**} - \frac{1}{\lambda}\hat{\mathbf{z}}\right) \right\|_2^2 \\
\Rightarrow & \| W(\lambda \mathbf{z}^* - \hat{\mathbf{z}}) \|_2^2 < \| W(W^{-1}\mathbf{z}^{**} - \hat{\mathbf{z}}) \|_2^2 \\
\Rightarrow & f(\lambda W \mathbf{z}^*) < f(\mathbf{z}^{**}).
\end{aligned}$$

The third line follows from the fact that $W(\mathbf{z}^* - \frac{1}{\lambda}\hat{\mathbf{z}})$ and $(1 - \frac{1}{\lambda}) W\hat{\mathbf{z}}$ are orthogonal, and $W(\frac{1}{\lambda} W^{-1}\mathbf{z}^{**} - \frac{1}{\lambda}\hat{\mathbf{z}})$ and $(1 - \frac{1}{\lambda}) W\hat{\mathbf{z}}$ are orthogonal (by feasibility of z^{**} on the hyperplane in (P2)). In the last line, note that $\lambda W \mathbf{z}^*$ is feasible for (P2) and has a strictly better objective value than $\mathbf{z}^{**}$. This contradicts the optimality of $\mathbf{z}^{**}$ for (P2). Hence, $\mathbf{z}^{**} = \lambda W \mathbf{z}^*$ or $\mathbf{z}^* = \frac{1}{\lambda} W^{-1} \mathbf{z}^{**}$.

So far, we have shown that $\mathbf{z}^* = \hat{\lambda} W^{-1}\mathbf{z}^{**}$, where $\hat{\lambda} := \frac{1}{\bar{\lambda}} > 0$. To complete the proof, we have to determine $\hat{\lambda}$ (note that we must show $\hat{\lambda} = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{**T}\mathbf{z}^{**}}$). We can find $\hat{\lambda}$ by solving the following optimization problem (because we already know that $\hat{\lambda} W^{-1}\mathbf{z}^{**}$ is the optimal solution of (P1)).

$$\min_{\bar{\lambda}>0} h(\bar{\lambda} W^{-1}\mathbf{z}^{**}) = \|W(\bar{\lambda} W^{-1}\mathbf{z}^{**} - \hat{\mathbf{z}})\|_2^2.$$

The objective function is convex in $\bar{\lambda}$. Using first order conditions, we obtain $\hat{\lambda} = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{**T}\mathbf{z}^{**}} > 0$. Therefore, we have:

$$\mathbf{z}^* = \hat{\lambda} W^{-1}\mathbf{z}^{**} = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{**T}\mathbf{z}^{**}} W^{-1}\mathbf{z}^{**},$$

and hence the proof is complete. □

Based on Theorem 6, we note the following. First, (P1) always has a finite optimal solution, which we denote by $\mathbf{z}^*$. Second, (P2) can be infeasible or it has a bounded optimal solution, which we denote by $\mathbf{z}^{**}$. Finally, if (P2) is infeasible, then $\mathbf{z}^* = 0$ solves (P1); otherwise, the optimal solution of (P1) is found using equations $\mathbf{z}^* = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\mathbf{z}^{**T}\mathbf{z}^{**}} W^{-1}\mathbf{z}^{**}$ in polynomial time. Therefore, we need to solve problem (P2) to determine the optimal solution to (P1).

4.3.2 An Algorithm for Solving the Surrogate Problem

We present our method for solving (P2), which is also shown in Algorithm 3 (assuming $\mathcal{FH} \neq \emptyset$). An initial feasible point is found that we denote by $\mathbf{z}^{(0)} \in \mathcal{FH}$ (note that since $\mathcal{FH} \neq \emptyset$, then there exists $\mathbf{y}^\circ \in \mathbb{Y}$ such that $\hat{\mathbf{z}}^T W^T W \mathbf{y}^\circ > 0$). We denote by $\mathcal{L}^{(k)}$ a lower bound on the optimal value of (P2) at iteration k. We will discuss determining $\mathcal{L}^{(k)}$ later in this subsection. This lower bound is initially set to 0. At each iteration $k = 0, 1, 2, \ldots,$ we first determine the best known point, denoted by $\mathbf{z}^{*(k)}$, by minimizing the objective function of (P2) over the convex hull of all feasible points known so far (Line (3) of Algorithm 3).

Algorithm 3 Solving problem (P2) (Assumption: $\mathcal{FH} \neq \emptyset$)

1: Let $\mathbf{y}^\circ \in \mathbb{Y}$ such that $\hat{\mathbf{z}}^T W^T W \mathbf{y}^\circ > 0$; Define $\mathbf{z}^{(0)} := \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \mathbf{y}^\circ} W \mathbf{y}^\circ \in \mathcal{FH}$; Define $\mathcal{L}^{(0)} := 0$;
2: **for** $k = 0, 1, 2, \dots$ **do**
3: Solve $\mathbf{z}^{*(k)} = \arg\min_{\mathbf{z} \in \mathcal{CH}(\{\mathbf{z}^{(0)}, \dots, \mathbf{z}^{(k)}\})} f(\mathbf{z})$;
4: **if** $f(\mathbf{z}^{*(k)}) \leq \mathcal{L}^{(k)} + \varepsilon$ **then**[4]
5: Return $\mathbf{z}^{*(k)}$;
6: **end if**
7: Solve $\mathcal{M}(k)$ and denote the optimal solution by $\mathbf{z}^{(k+1)}$;
8: Define $\mathcal{L}^{(k+1)} := \max\Big\{\mathcal{L}^{(k)}, \Big(\max\Big\{0, \frac{(\mathbf{z}^{(k+1)} - W\hat{\mathbf{z}})^T (\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2\Big\}$;
9: **if** $f(\mathbf{z}^{*(k)}) \leq \mathcal{L}^{(k+1)} + \varepsilon$ **then**
10: Return $\mathbf{z}^{*(k)}$;
11: **end if**
12: **end for**

If the difference between the objective value of $\mathbf{z}^{*(k)}$ and $\mathcal{L}^{(k)}$ is less than or equal to an acceptable error ε, we stop (Line (4) of Algorithm 3). Otherwise, we minimize the linear approximation of the objective function at $\mathbf{z}^{*(k)}$ over the feasible region of (P2). In fact, we solve the following minimization problem:

$$\mathcal{M}(k): \min_{\mathbf{z} \in \mathcal{FH}} \quad \mathbf{z}^T (\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}),$$

and denote the optimal solution by $\mathbf{z}^{(k+1)}$ (see Line (7) of Algorithm 3).

An important feature of Algorithm 3 is the presentation of a lower bound at each iteration, which is computed as:

$$\mathcal{L}^{(k+1)} := \max\Big\{\mathcal{L}^{(k)}, \Big(\max\Big\{0, \frac{(\mathbf{z}^{(k+1)} - W\hat{\mathbf{z}})^T (\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2\Big\},$$

for all $k \geq 0$. Note that our lower bound is always well-defined because the denominator is non-zero (since if $\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2 = 0$ then $f(\mathbf{z}^{*(k)}) = 0$, and hence the algorithm must have stopped in Line (4)). The validity of this lower bound is proven in the following proposition.

Proposition 12. *In Algorithm 3, $f(\mathbf{z}^{**}) \geq \mathcal{L}^{(k)}$, for all $k \geq 0$.*

Proof. First, note that $\mathcal{L}^{(0)} = 0$, which is valid; hence, we focus on proving the validity of $\mathcal{L}^{(k)}$ for $k \geq 1$. Note that we can represent the lower bound as:

$$\mathcal{L}^{(k+1)} := \max\Big\{0, \max_{k'=0,1,\ldots,k}\Big(\max\Big\{0, \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2\Big\},$$

for all $k \geq 0$. Thus, it suffices to prove:

$$f(\mathbf{z}^{**}) \geq \Big(\max\Big\{0, \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2,$$

for all $k' \geq 0$. We use a contradiction. Suppose that there exist k' and $\bar{\mathbf{z}} \in \mathcal{FH}$ such that:

$$f(\bar{\mathbf{z}}) < \Big(\max\Big\{0, \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2.$$

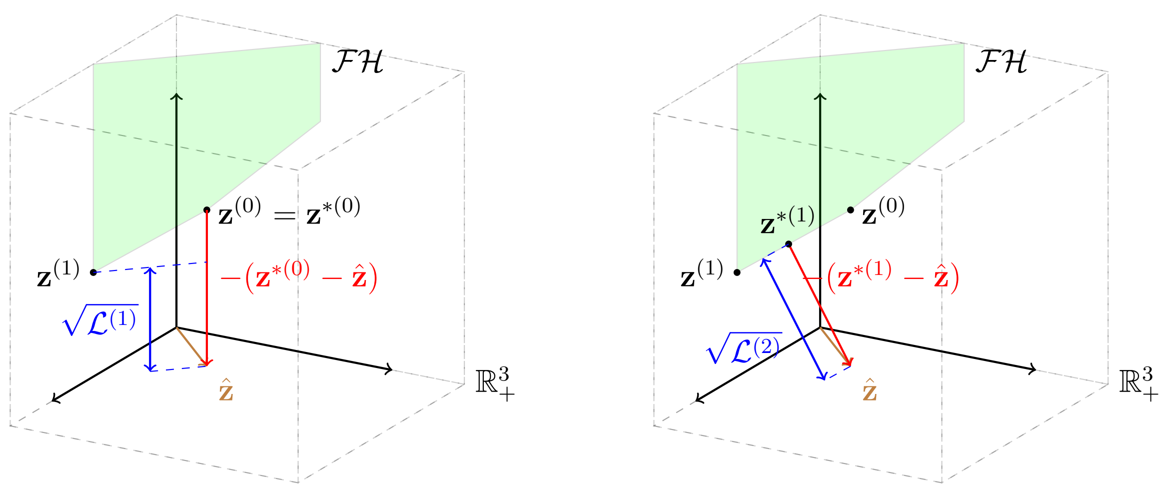
Note that if $\big((\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})\big)/\big(\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2\big) \leq 0$, then we have $f(\bar{\mathbf{z}}) < 0$ which is a contradiction; hence, assume that $\big((\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})\big)/\big(\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2\big) > 0$. Thus, we have:

$$\begin{aligned}
& f(\bar{\mathbf{z}}) < \Big(\frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big)^2 \\
\Rightarrow \quad & \|\bar{\mathbf{z}} - W\hat{\mathbf{z}}\|_2 < \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2} \\
\Rightarrow \quad & \|\bar{\mathbf{z}} - W\hat{\mathbf{z}}\|_2\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2 < (\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}) \\
\Rightarrow \quad & (\bar{\mathbf{z}} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}) \leq \|\bar{\mathbf{z}} - W\hat{\mathbf{z}}\|_2\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2 < (\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}) \\
\Rightarrow \quad & \bar{\mathbf{z}}^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}) < \mathbf{z}^{(k'+1)T}(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}).
\end{aligned}$$

The fourth line follows from the Cauchy-Schwartz inequality. Note that the last line contradicts the optimality of $\mathbf{z}^{(k'+1)}$ for problem $\mathcal{M}(k')$. Hence, the proof is complete. □

We note that $\mathcal{L}^{(k)}$ is nondecreasing in k due to the way it is defined. We stop Algorithm 3 if the difference between the current objective value and the lower bound is less than or equal to an acceptable error ε (see Lines (9)-(11) of Algorithm 3). Our experiment shows that $\mathcal{L}^{(k)}$ is significantly effective in terminating the execution of Algorithm 3.

We graphically show the steps of applying Algorithm 3 to the example presented in Figs. 4.1 and 4.2. Recall that this example assumes W is a 3×3 identity matrix. For the ease of illustration, we assume $\varepsilon = 0$. Let $\mathbf{z}^{(0)} \in \mathcal{FH}$ be as shown in Fig. 4.3(a), and $\mathcal{L}^{(0)} = 0$. In iteration 0, we find $\mathbf{z}^{*(0)} = \mathbf{z}^{(0)}$. We then minimize $\mathbf{z}^T(\mathbf{z}^{*(0)} - \hat{\mathbf{z}})$ over the feasible region $\mathcal{FH}$ and find $\mathbf{z}^{(1)}$. Using the obtained $\mathbf{z}^{(1)}$, we compute $\mathcal{L}^{(1)}$, which is shown in Fig. 4.3(a). The termination criteria is not satisfied because $\mathcal{L}^{(1)} < f(\mathbf{z}^{*(0)}) = \|\mathbf{z}^{*(0)} - \hat{\mathbf{z}}\|_2^2$. Thus, we continue to iteration $k = 1$ (see Fig. 4.3(b)). We first find $\mathbf{z}^{*(1)}$ as the closest point to $\hat{\mathbf{z}}$ in the convex hull of $\{\mathbf{z}^{(0)}, \mathbf{z}^{(1)}\}$. Then, solving $\mathcal{M}(1)$, we find $\mathbf{z}^{(1)}$ (or $\mathbf{z}^{(0)}$). We then compute $\mathcal{L}^{(2)}$ and note that the termination criteria is met because $\mathcal{L}^{(2)} = f(\mathbf{z}^{*(1)}) = \|\mathbf{z}^{*(1)} - \hat{\mathbf{z}}\|_2^2$. Thus, we stop and $\mathbf{z}^{*(1)}$ is the optimal solution of (P2)—i.e., $\mathbf{z}^{**} = \mathbf{z}^{*(1)}$. We remark that if Algorithm 3 is not equipped with $\mathcal{L}^{(k)}$, it iterates forever and cannot guarantee the optimality of $\mathbf{z}^{*(1)}$.



(a) Iteration 0: $\mathbf{z}^{(1)}$ minimizes $\mathcal{M}(0)$; $\mathcal{L}^{(1)}$ is found and $\mathcal{L}^{(1)} < f(\mathbf{z}^{*(0)})$.

(b) Iteration 1: $\mathbf{z}^{(1)}$ (or $\mathbf{z}^{(0)}$) minimizes $\mathcal{M}(1)$; $\mathcal{L}^{(2)}$ is found and $\mathcal{L}^{(2)} = f(\mathbf{z}^{*(1)})$; Stop!

Figure 4.3: Graphical illustration of Algorithm 3 on the example presented in Figs. 4.1 and 4.2.

4.3.3 Analysis of the Subproblem $\mathcal{M}(k)$

The applicability of Algorithm 3 depends on finding a bounded optimal solution $\mathbf{z}^{(k+1)}$ by solving $\mathcal{M}(k)$. This is guaranteed if, for example, the feasible region of $\mathcal{M}(k)$ is compact. We remark that $\mathcal{FH}$ is a closed and convex polyhedron, but not necessarily compact because it can be unbounded (see, for example, Fig. 4.2). We must ensure there exists a nonempty and compact polyhedron that contains the optimal solutions of $\mathcal{M}(k)$. This result is not obvious as illustrated in Fig. 4.4. Suppose $\mathcal{FH}$, $W\hat{\mathbf{z}}$, and $\mathbf{z}^{*(0)}$ are as shown in Fig. 4.4, and $\mathcal{FH}$ is unbounded with extreme directions $\mathbf{d}^1$ and $\mathbf{d}^2$. The objective value of $\mathcal{M}(0)$ improves in the direction of $-(\mathbf{z}^{*(0)} - W\hat{\mathbf{z}})$. Hence, the objective value approaches to $-\infty$ in the direction of $\mathbf{d}^2$, and there exists no optimal solution. If this case happens, we cannot generate $\mathbf{z}^{(1)}$ and hence cannot proceed. A situation similar to Fig. 4.4 never happens as we show in the following theorem.

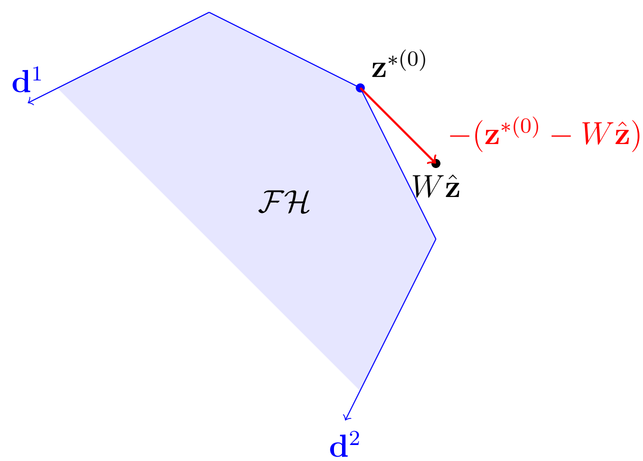


Figure 4.4: An example to highlight the importance of Theorem 7.

Theorem 7. *If $\mathcal{FH} \neq \{\}$, then, for all $k \geq 0$, $\mathcal{M}(k)$ has a bounded optimal solution, which is an extreme point of $\mathcal{FH}$ and in the form of $\lambda W\mathbf{y}$, where $0 < \lambda < +\infty$ and $\mathbf{y} \in \mathbb{Y}$.*

Proof. Note that if for all $W\mathbf{y} \in \tilde{\mathbb{Y}}$, there exists $0 < \lambda < +\infty$ such that $\hat{\mathbf{z}}^T W^T(\lambda W\mathbf{y}) = \hat{\mathbf{z}}^T W^T W\hat{\mathbf{z}}$, then $\mathcal{FH}$ is a bounded set (also because $\mathbb{Y}$ has a finite number of elements). This case happens if $\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0$, for all $\mathbf{y} \in \mathbb{Y}$. On the other hand, if there exists $\mathbf{y} \in \mathbb{Y}$ such that $\hat{\mathbf{z}}^T W^T W \mathbf{y} \leq 0$, then $\mathcal{FH}$ can be unbounded.[5]

We first show that $\mathcal{M}(k)$ has a bounded optimal value. Note that for all $\mathbf{z} \in \mathcal{FH}$ we have: $\mathbf{z}^T W \hat{\mathbf{z}} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} = \text{II}$, where II is a constant. Then, for all $\mathbf{z} \in \mathcal{FH}$, the objective value of $\mathcal{M}(k)$ is equal to $\mathbf{z}^T \mathbf{z}^{*(k)} - \text{II}$. On the other hand, since both $\mathbf{z}$ and $\mathbf{z}^{*(k)}$ are some nonnegative combinations of $W\mathbf{y}$'s, by Assumption (i), the objective is lower bounded by $-\text{II}$.

We next show that $\mathcal{FH}$ has at least one extreme point and does not contain lines ($\mathcal{FH}$ contains a line if there exist $\mathbf{z}',\ \mathbf{z}'' \in \mathcal{FH}$, $\mathbf{z}' \neq \mathbf{z}''$, such that $\theta \mathbf{z}' + (1-\theta)\mathbf{z}'' \in \mathcal{FH}$, for all $\theta \in \mathbb{R}$). First, $\mathcal{FH}$ is a nonempty polyhedron because of the assumption of the theorem. Second, note that $\mathcal{CC}(\tilde{\mathbb{Y}})$ does not contain lines because, for all $\mathbf{z}',\ \mathbf{z}'' \in \mathcal{CC}(\tilde{\mathbb{Y}})$, we have:

$$\mathbf{z}'^T \mathbf{z}'' = \left(\sum_{j_1=1}^{m} \xi'_{j_1} W \mathbf{y}^{j_1} \right)^T \left(\sum_{j_2=1}^{m} \xi''_{j_2} W \mathbf{y}^{j_2} \right) = \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \xi'_{j_1} \xi''_{j_2} \mathbf{y}^{j_1 T} W^T W \mathbf{y}^{j_2} \geq 0,$$

due to our assumption (i) in section 4.2; hence, $\mathcal{FH}$ does not contain lines because $\mathcal{FH} \subseteq \mathcal{CC}(\tilde{\mathbb{Y}})$. In addition, since $\mathcal{FH}$ is closed and nonempty, then $\mathcal{FH}$ has at least one extreme point (due to Proposition 2.1.2 of Bertsekas (2009)).

Thus, since $\mathcal{FH}$ is a closed subset of $\mathbb{R}^n$ with at least one extreme point and since the optimal value of $\mathcal{M}(k)$ is attained over $\mathcal{FH}$, then an extreme point of $\mathcal{FH}$ must be optimal (due to Proposition 2.4.1 of Bertsekas (2009)). This extreme point is in the form of $\lambda W \mathbf{y}$, where $0 < \lambda < +\infty$ and $\mathbf{y} \in \mathbb{Y}$ (there might be other alternative optimal solutions). Let $\mathcal{CFH}$ denote the convex hull of all $\lambda W \mathbf{y}$'s, where $0 < \lambda < +\infty$ and $\mathbf{y} \in \mathbb{Y}$. Thus, we proved that the optimal solution of $\mathcal{M}(k)$ belongs to $\mathcal{CFH}$. Moreover, $\mathcal{CFH}$ is a nonempty and compact polyhedron. □

The following important results are noted based on Theorem 7. First, $\mathcal{M}(k)$ has a bounded optimal solution, which is an extreme point of $\mathcal{FH}$, and hence we can limit our

[5]Note that if there exists $\check{\mathbf{y}} \in \mathbb{Y}$ such that $\hat{\mathbf{z}}^T W^T W \check{\mathbf{y}} \leq 0$, then there exists no $\lambda > 0$ satisfying $\lambda W \check{\mathbf{y}} \in \mathcal{FH}$.

search region to the extreme points of $\mathcal{FH}$ in the form of $\lambda W\mathbf{y}$ where $0 < \lambda < +\infty$ and $\mathbf{y} \in \mathbb{Y}$. Second, solving $\mathcal{M}(k)$ with the limited search region guarantees that an obtained optimal solution is always bounded. Thus, we reformulate $\mathcal{M}(k)$ based on these results as follows.

$$\begin{aligned}
\mathcal{M}(k):\ \min \quad & (\lambda W\mathbf{y})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) \\
\text{s.t.} \quad & \mathbf{y} \in \mathbb{Y}, \lambda \geq 0 \\
& \hat{\mathbf{z}}^T W^T(\lambda W\mathbf{y}) = \hat{\mathbf{z}}^T W^T W\hat{\mathbf{z}}.
\end{aligned}$$

Note that $\lambda \neq 0$ because $\hat{\mathbf{z}}^T W^T W\hat{\mathbf{z}} > 0$. By $\mathcal{M}(k)$, we always refer to this new formulation in the rest of this chapter. Moreover, we denote by $\mathbf{z}^{(k+1)}$ the optimal value of $\lambda W\mathbf{y}$.

We note that the effectiveness of Algorithm 3 depends on how the subproblem $\mathcal{M}(k)$ is modeled and solved at each iteration. We present a general framework for formulating $\mathcal{M}(k)$ when $\mathbb{Y}$ is the set of integer feasible solutions of mixed-integer programs. Before we present our framework, we remark that if $\mathbb{Y}$ is given explicitly, then the solution of $\mathcal{M}(k)$ is obtained by computing:

$$\min_{\substack{\mathbf{y}\in\mathbb{Y}:\\ \hat{\mathbf{z}}^T W^T W\mathbf{y} > 0}} \left\{ \left(\left(\frac{\hat{\mathbf{z}}^T W^T W\hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W\mathbf{y}} \right) W\mathbf{y} \right)^T (\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) \right\},$$

which can be performed in a time that is polynomial in m.

We next present our modeling framework. We assume that $\mathbf{y} \in \mathbb{Y}$ is in the form of $\mathbf{y} = (y_1, \ldots, y_n)$ such that $1 - 2^U \leq y_i \leq 2^U$, for all $i = 1, \ldots, n$, where $(1 - 2^U)$ and 2^U are known lower and upper bounds on the values of y_i's. For example, if $U = 3$, then it is known that all integer variables are between -7 and 8. We note that U can be selected sufficiently big so that the lower and upper bounds are valid for all y_i's, but for achieving the most effective model, we must set U to the smallest possible integer value. We formulate

$\mathcal{M}(k)$ as the following mixed-integer linear programming problem:

$$\mathcal{M}(k): \min \quad (W\mathbf{x})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) \tag{4.1}$$

$$\text{s.t.} \quad \hat{\mathbf{z}}^T W^T W \mathbf{x} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}} \tag{4.2}$$

$$y_i = (1 - 2^U) + \sum_{u=0}^{U} 2^u v_{iu}, \ \forall i \tag{4.3}$$

$$x_i = (1 - 2^U)\lambda + \sum_{u=0}^{U} 2^u s_{iu}, \ \forall i \tag{4.4}$$

$$0 \leq s_{iu} \leq M v_{iu}, \ \forall i, u \tag{4.5}$$

$$M(v_{iu} - 1) \leq s_{iu} - \lambda \leq M(1 - v_{iu}), \ \forall i, u \tag{4.6}$$

$$\mathbf{y} \in \mathbb{Y}, \ v_{iu} \in \{0, 1\}, \ \forall i, u, \ \lambda \geq 0 \tag{4.7}$$

where x_i, y_i, v_{iu}, s_{iu}, and λ are decision variables, $i = 1, \ldots, n$, $u = 0, \ldots, U$, and M is a sufficiently big number (an upper bound on the value of λ). For each y_i, we define $U + 1$ binary variables v_{iu}. The values of v_{iu}'s are determined through Eq. (4.3). For each v_{iu}, we define a continuous variable s_{iu} such that $s_{iu} = \lambda v_{iu}$. Eqs. (4.5) and (4.6) ensure that $s_{iu} = \lambda v_{iu}$, for all i, u. Finally, we define continuous variables x_i such that $x_i = \lambda y_i$, which is guaranteed by Eqs. (4.1), (4.2), and (4.4).

Our formulation of $\mathcal{M}(k)$ requires n integer variables, $n(U + 2) + 1$ continuous variables, and $n(U + 1)$ binary variables. Note that this is a general formulation and, for example, by setting $U = 0$, we achieve a model for cases where $\mathbb{Y}$ is the set of feasible solutions to satisfiability problems and binary programs.

We finally remark that the computational complexity of Algorithm 1 is the product of the *number of iterations* that it takes to find a solution with a pre-specified optimality gap and the *computational effort in each iteration*. If the computational requirement of solving $\mathcal{M}(k)$ is polynomially bounded, then the overall computational complexity of Algorithm 1 is polynomial in n. On the other hand, if $\mathcal{M}(k)$ is NP-hard, then the overall computational complexity is exponential in n. We also emphasize that the computational complexity of $\mathcal{M}(k)$ is application-dependent.

4.4 Convergence Rate Guarantee

At each iteration k of Algorithm 3, we obtain a lower bound ($\mathcal{L}^{(k+1)}$) and an upper bound ($f(\mathbf{z}^{*(k)})$) on the optimal value ($f(\mathbf{z}^{**})$). In this section, we show that both $f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**})$ and $f(\mathbf{z}^{**}) - \mathcal{L}^{(k+1)}$ are upper bounded by $\mathcal{O}(1/k)$—i.e., both upper bound and lower bound converge to the optimal value at $\mathcal{O}(1/k)$. This also shows that $f(\mathbf{z}^{*(k)}) - \mathcal{L}^{(k+1)}$ converges to 0 at the same rate, which guarantees that the algorithm will stop after a finite number of iterations for a given $\varepsilon > 0$.

4.4.1 Upper Bound Convergence

It is generally known that in the Frank-Wolfe methods, after $k \geq 1$ iterations, $f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**})$ is upper bounded by $\frac{2C_f}{k+2}$ where C_f is the *curvature constant* of the objective function over the domain (see, for example, Frank and Wolfe (1956), and Jaggi (2013)). As one of our contributions, we further specify a detailed upper bound on C_f. The following theorem states the convergence rate of our upper bound procedure.

Theorem 8 (Upper Bound Convergence)**.** *In Algorithm 3, for $k \geq 1$, we have:*

$$f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**}) \leq \frac{8}{k+2}\big(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0,\ \mathbf{y} \in \mathbb{Y}} \frac{\mathbf{y}^T W^T W \mathbf{y}}{(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2}.$$

In addition, if W is an $n \times n$ identity matrix, $\mathbb{Y} \subseteq \{0,1\}^n$, and $0 \leq \hat{z}_i \leq 1$, for all i, then we have:

$$f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**}) \leq \frac{8}{k+2}\Big(\frac{\hat{\mathbf{z}}^T \hat{\mathbf{z}}}{\hat{z}_{\min,nz}}\Big)^2 (1 + \Delta),$$

where $\hat{z}_{\min,nz}$ is the smallest nonzero entry of $\hat{\mathbf{z}}$ and Δ is the number of zero entries of $\hat{\mathbf{z}}$.

Note that the upper bounds in Theorem 8 can be arbitrarily large since, for example in the second result, one can construct an instance such that $\hat{z}_{\min,nz}$ is arbitrarily small. However, given that $\hat{z}_{\min,nz} > 0$, the upper bound always exists and is finite. The existence of a finite value is all we need to establish the convergence rate of our algorithm.

In Theorem 7, we showed that if $\mathcal{FH} \neq \{\}$, then the optimal solution of $\mathcal{M}(k)$ belongs to $\mathcal{CFH}$, which is a nonempty and compact polyhedron. Note that in this section, we always assume $\mathcal{FH} \neq \{\}$ (which was also assumed in the presentation of Algorithm 3). Our analysis of the convergence rate depends on the existence of a finite *curvature constant*, which is defined as:

$$C_f := \sup_{\mathbf{z}^1, \mathbf{z}^2 \in \mathcal{CFH}, 0 \leq \gamma \leq 1} \frac{2}{\gamma^2} \big(f(\mathbf{z}^1 + \gamma(\mathbf{z}^2 - \mathbf{z}^1)) - f(\mathbf{z}^1) - \gamma(\mathbf{z}^2 - \mathbf{z}^1) \nabla f(\mathbf{z}^1) \big).$$

It is well-known that the curvature constant for $\frac{1}{2}\|\mathbf{z}\|_2^2$ is the squared Euclidean diameter of the domain (see, for example, Jaggi (2013)). We provide a detailed upper bound on C_f using the structure of our problem.

Lemma 15. *In (P2), we have:* $C_f \leq 4\big(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0,\ \mathbf{y} \in \mathbb{Y}} \frac{\mathbf{y}^T W^T W \mathbf{y}}{(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2}$.

Proof. Using Theorem 7, $\mathcal{CFH}$ is the convex hull of all $\lambda W \mathbf{y}$'s such that $\mathbf{y} \in \mathbb{Y}$, $\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0$, and $\lambda = \frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \mathbf{y}}$. Thus, noting that C_f is equal to 2 times the squared Euclidean diameter of the domain $\mathcal{CFH}$ (this can be shown by substituting f and ∇f in the definition of C_f), we have:

$$\begin{aligned}
C_f &= 2 \max_{\mathbf{z}^1,\mathbf{z}^2 \in \mathcal{CFH}} \|\mathbf{z}^1 - \mathbf{z}^2\|_2^2 \\
&= 2 \max_{\substack{\hat{\mathbf{z}}^T W^T(\lambda_i W\mathbf{y}^i) = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}},\ i=1,2,\\ 0<\lambda_1,\lambda_2<+\infty,\ \mathbf{y}^1,\mathbf{y}^2\in\mathbb{Y}}} \|\lambda_1 W\mathbf{y}^1 - \lambda_2 W\mathbf{y}^2\|_2^2 \\
&= 2 \max_{\substack{\hat{\mathbf{z}}^T W^T(\lambda_i W\mathbf{y}^i) = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}},\ i=1,2,\\ 0<\lambda_1,\lambda_2<+\infty,\ \mathbf{y}^1,\mathbf{y}^2\in\mathbb{Y}}} \lambda_1^2 \mathbf{y}^{1T} W^T W \mathbf{y}^1 + \lambda_2^2 \mathbf{y}^{2T} W^T W \mathbf{y}^2 - 2\lambda_1\lambda_2 \mathbf{y}^{1T} W^T W \mathbf{y}^2 \\
&\leq 4 \max_{\substack{\hat{\mathbf{z}}^T W^T(\lambda W\mathbf{y}) = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}},\\ 0<\lambda<+\infty,\ \mathbf{y}\in\mathbb{Y}}} \lambda^2 \mathbf{y}^T W^T W \mathbf{y} \\
&= 4 \max_{\hat{\mathbf{z}}^T W^T W\mathbf{y}>0,\ \mathbf{y}\in\mathbb{Y}} \Big(\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \mathbf{y}}\Big)^2 \mathbf{y}^T W^T W \mathbf{y} \\
&= 4\big(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T W^T W\mathbf{y}>0,\ \mathbf{y}\in\mathbb{Y}} \frac{\mathbf{y}^T W^T W \mathbf{y}}{(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2}.
\end{aligned}$$

In the second line, we use the fact that $\mathbf{z}^1$ and $\mathbf{z}^2$ must be extreme points of $\mathcal{CFH}$; hence, they should be in the form of $\mathbf{z}^i = \lambda_i W \mathbf{y}^i$ where $0 < \lambda_i < +\infty$, $\mathbf{y}^i \in \mathbb{Y}$, and $\lambda_i W\mathbf{y}^i$ must belong to the hyperplane $\hat{\mathbf{z}}^T W^T \mathbf{z} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}$. In the third line, we eliminate $-2\lambda_1\lambda_2 \mathbf{y}^{1T} W^T W \mathbf{y}^2$ because it is always non-positive (see our assumption (i) in section 4.2). As a result, the problem decomposes into two identical problems (fourth line). In the fifth line, we substitute λ with $\frac{\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}}{\hat{\mathbf{z}}^T W^T W \mathbf{y}}$, but we have to ensure that the denominator is positive. Hence, the proof is complete. $\square$

In Lemma 15, the upper bound is finite because for all $\mathbf{y} \in \mathbb{Y}$, the numerator ($\mathbf{y}^T W^T W \mathbf{y}$) is finite and the denominator ($(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2$) is nonzero. This upper bound can be conveniently simplified for some special cases of our problem. We discuss an important special case where W is an $n \times n$ identity matrix, $\mathbb{Y} \subseteq \{0,1\}^n$, and $0 \leq \hat{z}_i \leq 1$, for all i.

Lemma 16. *In (P2), if W is an $n \times n$ identity matrix, $\mathbb{Y} \subseteq \{0,1\}^n$, and $0 \leq \hat{z}_i \leq 1$, for all i, then we have: $C_f \leq 4\big(\frac{\hat{\mathbf{z}}^T\hat{\mathbf{z}}}{\hat{z}_{\min,nz}}\big)^2(1+\Delta)$ where $\hat{z}_{\min,nz}$ is the smallest nonzero entry of $\hat{\mathbf{z}}$ and Δ is the number of zero entries of $\hat{\mathbf{z}}$.*

PROOF. We first show the following result: Let $0 \leq a_1 \leq a_2 \leq \cdots \leq 1$. Then, for all $\ell \geq 1$, such that $a_\ell > 0$, we have:

$$\frac{\ell}{(a_1 + \cdots + a_\ell)^2} \geq \frac{\ell + 1}{(a_1 + \cdots + a_{\ell+1})^2}.$$

To prove this result, we note that $1 \leq \frac{\ell+1}{\ell}$, for all $\ell \geq 1$, and then:

$$\begin{aligned}
\frac{(a_1 + \cdots + a_\ell)^2}{\ell} &\leq (\frac{\ell+1}{\ell})\frac{(a_1 + \cdots + a_\ell)^2}{\ell} \\
&= \frac{(\ell^2 + 2\ell + 1)(a_1 + \cdots + a_\ell)^2}{\ell^2(\ell+1)} \\
&= \frac{(\ell^2 + 1)(a_1 + \cdots + a_\ell)^2}{\ell^2(\ell+1)} + \frac{(2\ell)(a_1 + \cdots + a_\ell)^2}{\ell^2(\ell+1)} \\
&= \frac{(a_1 + \cdots + a_\ell)^2 + \frac{(a_1+\cdots+a_\ell)^2}{\ell^2}}{(\ell+1)} + \frac{2(a_1 + \cdots + a_\ell)(a_1 + \cdots + a_\ell)}{\ell(\ell+1)} \\
&\leq \frac{(a_1 + \cdots + a_\ell)^2 + a_{\ell+1}^2}{(\ell+1)} + \frac{2(a_1 + \cdots + a_\ell)(a_{\ell+1})}{(\ell+1)} \\
&= \frac{(a_1 + \cdots + a_{\ell+1})^2}{\ell+1},
\end{aligned}$$

noting that $\frac{a_1+\cdots+a_\ell}{\ell} \leq a_{\ell+1}$.

On the other hand, using Lemma 15, and noting that $\mathbb{Y} \subseteq \{0,1\}^n$, we have:

$$\begin{aligned}
C_f &\leq 4\big(\hat{\mathbf{z}}^T\hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T\mathbf{y}>0,\ \mathbf{y}\in\{0,1\}^n} \frac{\mathbf{y}^T\mathbf{y}}{(\hat{\mathbf{z}}^T\mathbf{y})^2} \\
&\leq 4\big(\frac{\hat{\mathbf{z}}^T\hat{\mathbf{z}}}{\hat{z}_{\min,nz}}\big)^2(1+\Delta).
\end{aligned}$$

Note that $\mathbf{y}^T\mathbf{y}$ is the number of ones in $\mathbf{y}$, and the denominator $\hat{\mathbf{z}}^T\mathbf{y}$ must be positive. Assume that $\hat{z}_i$'s are sorted in a non-decreasing order. Using the result that we showed earlier in this proof, the fraction is maximized if we select all zero $\hat{z}_i$'s and exactly one smallest nonzero ($\hat{z}_{\min,nz}$). Thus, the proof is complete. □

4.4.2 Lower Bound Convergence

Establishing that $\mathcal{L}^{(k+1)}$ converges to $f(\mathbf{z}^{**})$ is important since it guarantees that the algorithm will stop after a finite number of iterations for a given $\varepsilon > 0$. The following theorem states the convergence rate of our lower bounding procedure.

Theorem 9 (Lower Bound Convergence)**.** *In Algorithm 3, there exists K^* such that, for $k \geq K^*$, we have:*

$$f(\mathbf{z}^{**}) - \mathcal{L}^{(k+1)} \leq \frac{8\beta}{k+2}\big(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0,\ \mathbf{y} \in \mathbb{Y}} \frac{\mathbf{y}^T W^T W \mathbf{y}}{(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2},$$

where $\beta = 3.375$. In addition, if W is an $n \times n$ identity matrix, $\mathbb{Y} \subseteq \{0,1\}^n$, and $0 \leq \hat{z}_i \leq 1$, for all i, then, for $k \geq K^*$, we have:

$$f(\mathbf{z}^{**}) - \mathcal{L}^{(k+1)} \leq \frac{8\beta}{k+2}\Big(\frac{\hat{\mathbf{z}}^T \hat{\mathbf{z}}}{\hat{z}_{\min,nz}}\Big)^2 (1+\Delta),$$

where $\hat{z}_{\min,nz}$ is the smallest nonzero entry of $\hat{\mathbf{z}}$ and Δ is the number of zero entries of $\hat{\mathbf{z}}$.

We first present three Lemmas 17-19 that show the steps of the proof. We will then show how Theorem 9 is proven using these three lemmas.

Lemma 17. $(\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) \geq \|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2$*, for all $\mathbf{z} \in \mathcal{CFH}$.*

Proof. First, note that if $\mathbf{z}^{**} = W\hat{\mathbf{z}}$, then both sides of the inequality become 0; hence, we assume $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$. It is enough to show $\mathbf{z}^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) \geq \mathbf{z}^{**T}(\mathbf{z}^{**} - W\hat{\mathbf{z}})$, for all $\mathbf{z} \in \mathcal{CFH}$. We use a contradiction. Assume there exists $\bar{\mathbf{z}} \in \mathcal{CFH}$ such that $\bar{\mathbf{z}}^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) < \mathbf{z}^{**T}(\mathbf{z}^{**} - W\hat{\mathbf{z}})$. Obviously, $\bar{\mathbf{z}} \neq \mathbf{z}^{**}$. Consider the following problem:

$$\min_{0 \leq \alpha \leq 1} \|\alpha\bar{\mathbf{z}} + (1-\alpha)\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2,$$

which has a strongly convex objective function in α; hence, this problem has a unique optimal solution. Using first order condition, the unique optimal value of α is equal to $-\frac{(\bar{\mathbf{z}}-\mathbf{z}^{**})^T(\mathbf{z}^{**}-W\hat{\mathbf{z}})}{(\bar{\mathbf{z}}-\mathbf{z}^{**})^T(\bar{\mathbf{z}}-\mathbf{z}^{**})} > 0$. Thus, there exists $\bar{\bar{\mathbf{z}}} \in \mathcal{CFH}$ such that $\bar{\bar{\mathbf{z}}} \neq \mathbf{z}^{**}$ and $\bar{\bar{\mathbf{z}}}$ has a strictly better objective value than $\mathbf{z}^{**}$. This contradicts the optimality of $\mathbf{z}^{**}$. Hence, the proof is complete. □

Lemma 18. *If $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$, then there exists K such that $(\mathbf{z}-W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)}-W\hat{\mathbf{z}}) > 0$, for all $\mathbf{z} \in \mathcal{CFH}$ and $k \geq K$.*

Proof. We first show that: for $\epsilon > 0$, there exists K such that if $k \geq K$, then $\|\mathbf{z}^{*(k)}-\mathbf{z}^{**}\|_2 < \epsilon$. Let $\epsilon > 0$ be given and define $K := \max\left\{1, \left\lceil \frac{2C_f}{\epsilon^2} - 2\right\rceil + 1\right\}$. Note that because of the definition of K, we have $\frac{2C_f}{K+2} < \epsilon^2$. Then, for $k \geq K$ we have:

$$\begin{aligned}
\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2^2 &= \|(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) - (\mathbf{z}^{**} - W\hat{\mathbf{z}})\|_2^2 \\
&= \|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2^2 + \|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2 - 2(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) \\
&\leq \|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2^2 - \|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2 \\
&= f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**}) \leq \frac{2C_f}{k+2} \leq \frac{2C_f}{K+2} < \epsilon^2,
\end{aligned}$$

and hence $\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 < \epsilon$, for $k \geq K$. Note that we use the result of Lemma 17 to obtain the third line, i.e., $(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) \geq \|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2$. In the last line, we use the upper bound convergence result that was discussed in subsection 4.4.1, i.e., $f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**}) \leq \frac{2C_f}{k+2}$, for $k \geq 1$.

Define $Q := \max_{\mathbf{z}\in\mathcal{CFH}} \|\mathbf{z} - W\hat{\mathbf{z}}\|_2$ and note that $0 < Q < \infty$. Moreover, let $\epsilon := \frac{1}{Q}\|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2 > 0$. Therefore, there exists K such that:

$$\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 < \frac{1}{Q}\|\mathbf{z}^{**} - W\hat{\mathbf{z}}\|_2^2, \ \forall k \geq K.$$

Combining this result with Lemma 17, we obtain:

$$
\begin{aligned}
& (\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{**} - W\hat{\mathbf{z}}) > Q\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 \\
\Rightarrow \quad & (\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{**} - \mathbf{z}^{*(k)} + \mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) > Q\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 \\
\Rightarrow \quad & (\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) > (W\hat{\mathbf{z}} - \mathbf{z})^T(\mathbf{z}^{**} - \mathbf{z}^{*(k)}) + Q\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 \\
& \qquad\qquad\qquad\qquad\qquad \geq -\|W\hat{\mathbf{z}} - \mathbf{z}\|_2\|\mathbf{z}^{**} - \mathbf{z}^{*(k)}\|_2 + Q\|\mathbf{z}^{*(k)} - \mathbf{z}^{**}\|_2 \\
\Rightarrow \quad & (\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) > (Q - \|W\hat{\mathbf{z}} - \mathbf{z}\|_2)\|\mathbf{z}^{**} - \mathbf{z}^{*(k)}\|_2 \geq 0 \\
\Rightarrow \quad & (\mathbf{z} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) > 0,
\end{aligned}
$$

for all $\mathbf{z} \in \mathcal{CFH}$ and $k \geq K$. Note that the fourth line follows because $(W\hat{\mathbf{z}}-\mathbf{z})^T(\mathbf{z}^{**}-\mathbf{z}^{*(k)}) \geq -\|W\hat{\mathbf{z}}-\mathbf{z}\|_2\|\mathbf{z}^{**}-\mathbf{z}^{*(k)}\|_2$, and the fifth line follows because $Q \geq \|W\hat{\mathbf{z}}-\mathbf{z}\|_2$, for all $\mathbf{z} \in \mathcal{CFH}$ (due to the definition of Q). Hence, the proof is complete. □

Lemma 19. *If $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$, then:*

$$
f(\mathbf{z}^{**}) - \mathcal{LB}^{(k+1)} = g(\mathbf{z}^{*(k)}) - \big(f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**})\big) - \frac{1}{4}\frac{g^2(\mathbf{z}^{*(k)})}{f(\mathbf{z}^{*(k)})},
$$

for all $k \geq 0$, where $\mathcal{LB}^{(k+1)} := \big(\frac{(\mathbf{z}^{(k+1)}-W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)}-W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)}-W\hat{\mathbf{z}}\|_2}\big)^2$, and $g(\mathbf{z}) := \max_{\bar{\mathbf{z}}\in\mathcal{CFH}}(\mathbf{z} - \bar{\mathbf{z}})^T\nabla f(\mathbf{z})$.

Proof. Note that since $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$, then $\mathbf{z}^{*(k)} \neq W\hat{\mathbf{z}}$, for all $k \geq 0$. Thus, $f(\mathbf{z}^{*(k)}) > 0$ and

$\mathcal{LB}^{(k+1)}$ is well-defined, for all $k \geq 0$. We have:

$$
\begin{aligned}
\mathcal{LB}^{(k+1)} &= \Big(\frac{(\mathbf{z}^{(k+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2}\Big)^2 \\
&= \Big(\frac{(\mathbf{z}^{(k+1)} - \mathbf{z}^{*(k)} + \mathbf{z}^{*(k)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2}\Big)^2 \\
&= \Big(\frac{(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2} - \frac{(\mathbf{z}^{*(k)} - \mathbf{z}^{(k+1)})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}\|_2}\Big)^2 \\
&= \Big(\sqrt{f(\mathbf{z}^{*(k)})} - \frac{1}{2}\frac{g(\mathbf{z}^{*(k)})}{\sqrt{f(\mathbf{z}^{*(k)})}}\Big)^2 \\
&= -g(\mathbf{z}^{*(k)}) + f(\mathbf{z}^{*(k)}) + \frac{1}{4}\frac{g^2(\mathbf{z}^{*(k)})}{f(\mathbf{z}^{*(k)})}.
\end{aligned}
$$

To obtain the fourth line, we note that $g(\mathbf{z}^{*(k)}) = \max_{\bar{\mathbf{z}} \in \mathcal{CFH}}(\mathbf{z}^{*(k)} - \bar{\mathbf{z}})^T \nabla f(\mathbf{z}^{*(k)}) = 2(\mathbf{z}^{*(k)} - \mathbf{z}^{(k+1)})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}})$. Finally, the result follows by multiplying the obtained equation by -1 and adding $f(\mathbf{z}^{**})$ to both sides. □

Completing the Proof of Theorem 9. Using Lemma 18, if $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$, then there exists K such that $(\mathbf{z}^{(k+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k)} - W\hat{\mathbf{z}}) > 0$, for all $k \geq K$. Similar to our discussion in the proof of Proposition 12, we have:

$$
\begin{aligned}
f(\mathbf{z}^{**}) - \mathcal{L}^{(k+1)} &= f(\mathbf{z}^{**}) - \max\Big\{0, \max_{k'=0,1,\ldots,k}\Big(\max\Big\{0, \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2\Big\} \\
&\leq f(\mathbf{z}^{**}) - \max_{k'=K,K+1,\ldots,k}\Big(\max\Big\{0, \frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big\}\Big)^2 \\
&= f(\mathbf{z}^{**}) - \max_{k'=K,K+1,\ldots,k}\Big(\frac{(\mathbf{z}^{(k'+1)} - W\hat{\mathbf{z}})^T(\mathbf{z}^{*(k')} - W\hat{\mathbf{z}})}{\|\mathbf{z}^{*(k')} - W\hat{\mathbf{z}}\|_2}\Big)^2 \\
&= \min_{k'=K,K+1,\ldots,k}\big\{f(\mathbf{z}^{**}) - \mathcal{LB}^{(k'+1)}\big\} \\
&\leq \min_{k'=K,K+1,\ldots,k} g(\mathbf{z}^{*(k')}),
\end{aligned}
$$

for all $k \geq K$. The last line follows because of Lemma 19 and noting that $f(\mathbf{z}^{**}) - \mathcal{LB}^{(k+1)} \leq g(\mathbf{z}^{*(k)})$, for all $k \geq 0$ (because $f(\mathbf{z}^{*(k)}) - f(\mathbf{z}^{**}) \geq 0$, $g^2(\mathbf{z}^{*(k)}) \geq 0$, and $f(\mathbf{z}^{*(k)}) > 0$, for all

$k \geq 0$). Jaggi (2013) shows that if $k \geq 2$, then:

$$\min_{k' \in \{\lceil \frac{2}{3}(k+2) \rceil - 2, \ldots, k\}} g(\mathbf{z}^{*(k')}) \leq \frac{2C_f \beta}{k+2},$$

where $\beta = 3.375$. Let K^* be such that: (i) $K^* \geq K$, and (ii) if $k \geq K^*$ then $\lceil \frac{2}{3}(k+2) \rceil - 2 \geq K$. Thus, for $k \geq K^*$, we have:

$$\begin{aligned} f(\mathbf{z}^{**}) - \mathcal{L}^{(k+1)} &\leq \min_{k'=K,K+1,\ldots,k} g(\mathbf{z}^{*(k')}) \\ &\leq \min_{k'=\lceil \frac{2}{3}(k+2) \rceil - 2, \ldots, k} g(\mathbf{z}^{*(k')}) \\ &\leq \frac{2C_f \beta}{k+2}. \end{aligned}$$

We complete the proof by applying the results that we showed in Lemmas 15 and 16. Moreover, note that our above proof is for the case $\mathbf{z}^{**} \neq W\hat{\mathbf{z}}$; however, it can be easily verified that if $\mathbf{z}^{**} = W\hat{\mathbf{z}}$, then $f(\mathbf{z}^{**}) = 0$ and hence the upper bounds of Theorem 9 hold. □

We finally note that one can obtain the following lower bound on $\mathcal{L}^{(k+1)}$ using Theorem 9 (and noting that $\mathcal{L}^{(k)} \geq 0$, for all $k \geq 0$). In Algorithm 3, there exists K^* such that, for $k \geq K^*$, we have:

$$\mathcal{L}^{(k+1)} \geq \max\left\{0, f(\mathbf{z}^{**}) - \frac{8\beta}{k+2}\big(\hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\big)^2 \max_{\hat{\mathbf{z}}^T W^T W \mathbf{y} > 0,\ \mathbf{y} \in \mathbb{Y}} \frac{\mathbf{y}^T W^T W \mathbf{y}}{(\hat{\mathbf{z}}^T W^T W \mathbf{y})^2}\right\}.$$

In addition, if W is an $n \times n$ identity matrix, $\mathbb{Y} \subseteq \{0,1\}^n$, and $0 \leq \hat{z}_i \leq 1$, for all i, then, for $k \geq K^*$, we have:

$$\mathcal{L}^{(k+1)} \geq \max\left\{0, f(\mathbf{z}^{**}) - \frac{8\beta}{k+2}\Big(\frac{\hat{\mathbf{z}}^T \hat{\mathbf{z}}}{\hat{z}_{\min,nz}}\Big)^2 (1 + \Delta)\right\},$$

where $\hat{z}_{\min,nz}$ is the smallest nonzero entry of $\hat{\mathbf{z}}$ and Δ is the number of zero entries of $\hat{\mathbf{z}}$.

4.5 Numerical Results

In this section, we first present the result of testing our algorithm on a limited set of randomly generated instances to generally show the quality of the upper and lower bounds and the actual and theoretical convergence behavior.[6] In these instances, the implicit points are characterized by mixed-integer linear programs (MILPs). Second, we look at cases where the discrete points are clustered and show how this impacts the performance of our algorithm. Finally, we apply our algorithm to a set of industrial instances provided to us by a global auto manufacturer.

4.5.1 When the Implicit Points are Characterized by MILPs

In our instances, we assume that $\mathbb{Y}$ is the set of integer feasible solutions of $A\mathbf{y} + B\chi \leq \mathbf{b}$ where $\mathbf{y} \in \mathbb{Z}_+^n$, $\chi \in \mathbb{R}_+^q$, A and B are $n \times p$ and $q \times p$ matrices, respectively, and $\mathbf{b} \in \mathbb{R}^n$. Each entry of A and B is 0 with probability π and generated using a uniform distribution between -10 and 10, with probability $1 - \pi$. Each entry of $\mathbf{b}$ is generated uniformly between 0 and 100. Each entry of $\hat{\mathbf{z}}$ is generated uniformly between 0 and 1. Matrix W is assumed to be diagonal where each diagonal entry is generated uniformly between 0 and 10. Table 4.1 shows the parameter setting and results for 9 sets of problems solved. For each set, we solve 10 instances and report the average statistics.

Fig. 4.5 shows the average convergence of the lower and upper bounds for the problem sets 2, 5, and 8 as well as the theoretical lower and upper bounds. Note that the lower and upper bounds converge in a reasonable number of iterations. Furthermore, we note that the actual convergence rates are much faster than the theoretical ones. Moreover, although the theoretical lower bounds are zero in the first 100 iterations, the actual lower bounds (dashed red curves) rapidly converge to the optimal value. Table 4.1 summarizes the results of the 9

[6] Note that the existing approaches for solving NNLS cannot be applied for the NNLS with implicit points since enumerating all feasible points is not practical. Therefore, our method cannot be compared to the existing methods.

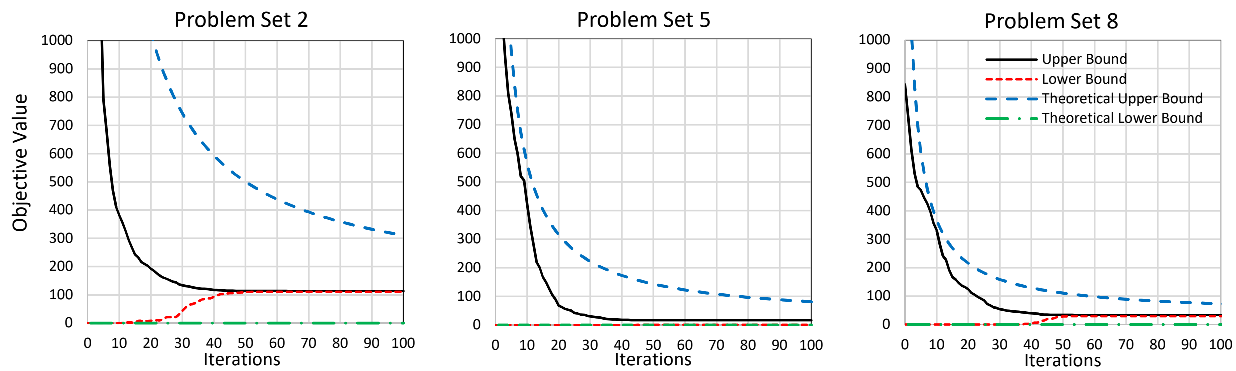


Figure 4.5: Randomly generated instances.

Table 4.1: Summary statistic (The algorithm is implemented in IBM ILOG CPLEX Optimization Studio 12.6.1 on a PC with Processor Intel(R) Core(TM) i5-2520M CPU 2.50GHz, 4.00 GB of RAM, and 64-bit Operating System.)

Problem Set	n	p	q	π	Integers' Upper Bounds	Avg. number of iterations	Avg. time per iteration (seconds)
1	50	200	0	0.9	1	43.20	0.20
2	50	200	0	0.9	3	49.11	4.78
3	50	200	0	0.9	8	34.50	36.45
4	50	100	0	0.8	1	61.00	0.37
5	50	100	0	0.8	3	44.20	40.13
6	50	100	0	0.8	8	26.20	76.91
7	50	200	10	0.9	1	46.70	0.24
8	50	200	10	0.9	3	48.30	0.63
9	50	200	10	0.9	8	42.20	11.04

problem sets.

4.5.2 When the Implicit Points Form Clusters

In this experiment, we set the number of implicit points, $|\mathbb{Y}|$, to 1000, and let n, the dimension of $\mathbb{R}^n$, vary between 100 and 1000 with the increments of 100. For the ease of experimentation, we let W be an identity matrix. We consider 1, 2, and 5 clusters for each problem size. Let NC denote the number of clusters and CF denote cluster coefficient—a measure of the closeness of the $\mathbf{y}$'s in a cluster. If $CF = 0$, then all $\mathbf{y}$'s are randomly

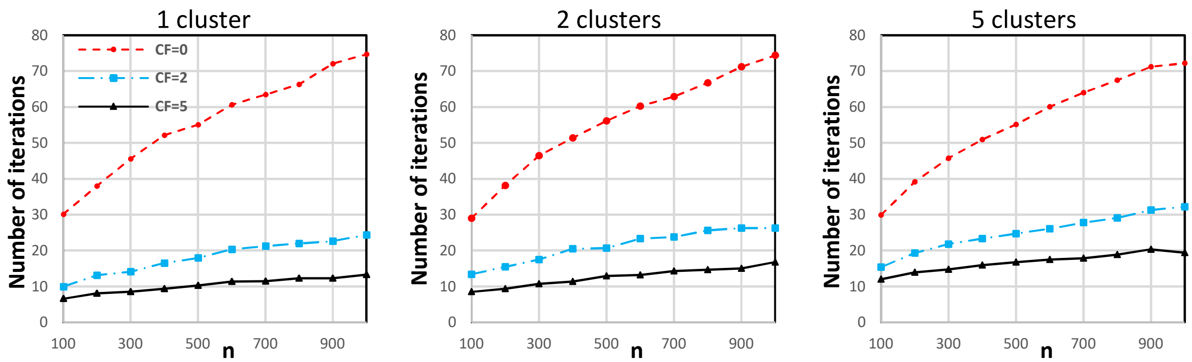


Figure 4.6: Effect of n, number of clusters, and cluster coefficient (CF) on the number of iterations.

distributed in $\mathbb{R}^n_+$. In this case, we create each $\mathbf{y}$ by uniformly generating n nonnegative numbers. If $CF > 0$, then we assume that the data points from $(\frac{(\phi-1)|\mathbb{Y}|}{NC}+1)$ to $(\frac{\phi|\mathbb{Y}|}{NC})$ are in the ϕ'th cluster and the $(\frac{(\phi-1)|\mathbb{Y}|}{NC}+1)$'th data point is the center of the ϕ'th cluster, for all $\phi = 1, \ldots, NC$. The center of cluster ϕ, denoted by $\mathbf{y}^c_\phi$, is created randomly. Then, a new point is generated in cluster ϕ using the formula $(CF\mathbf{y}^c_\phi + \check{\mathbf{y}})$ where $\check{\mathbf{y}}$ is generated randomly in $\mathbb{R}^n_+$. Obviously, if $CF = 0$, then there is no clustering, and as CF increases, the angle between the points and the center of the cluster becomes smaller. Note that the lengths of $\mathbf{y}$'s are not important as we are interested in the rays generated by $\mathbf{y}$'s.

Fig. 4.6 summarizes the result of applying our approach for 1, 2, and 5 clusters, where the horizontal and vertical axis show the number of dimensions and the number of iterations. Each point indicates the average of the results of 50 randomly generated instances. Fig. 4.6 shows that, as expected, the number of iterations (almost linearly) increases in n. As CF increases, the number of iterations decreases. Finally, the number of iterations increases in the number of clusters.

4.5.3 Industrial Instances

We next test our methodology on 2 industrial instances. In each of these instances, n is the number of options (features, as we discuss in the introduction) and the constraint set

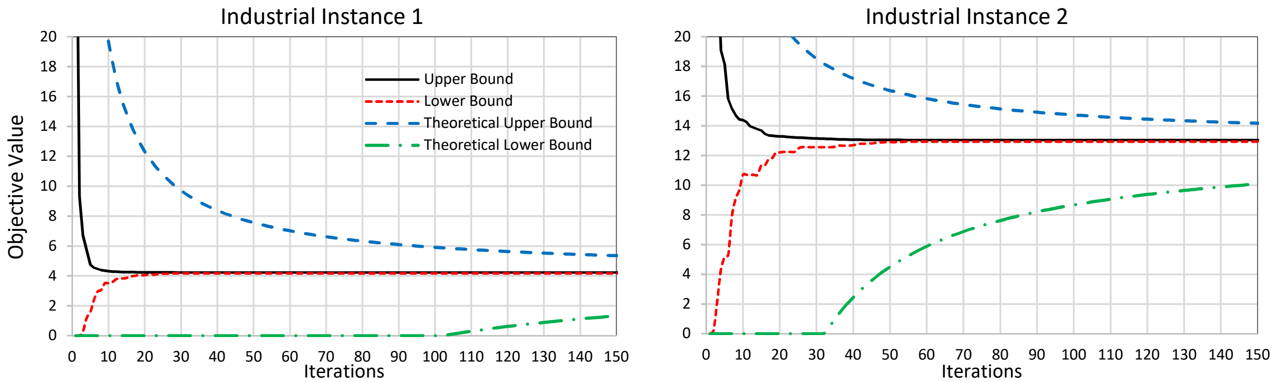


Figure 4.7: Industrial instances.

for defining $\mathbb{Y}$ is similar to satisfiability constraints. These two instances have 200 and 395 options, and 500 and 2,208 constraints, respectively. Fig. 4.7 shows our results. Note that the theoretical lower bounds become positive after about 100 iterations, and the actual lower bounds are very effective.

In summary, our computational experiment verifies the performance of our proposed methodology for solving problem (P1) and the effectiveness of the lower and upper bounding procedures.

4.6 Conclusions

In this chapter, we present a new methodology for solving NNLS when the discrete points are implicitly characterized. This class of problems are found in manufacturing systems, clustering, machine learning, and statistics. We first define a surrogate problem that enables us to apply our approach, which is similar to a variant of the Frank-Wolfe algorithm. At each iteration, we find a lower bound and an upper bound and show that their difference converges to 0 at a rate of $\mathcal{O}(1/k)$. We perform an experiment on a set of numerical examples to illustrate the effectiveness of our approach when the discrete points are the solutions of MILPs and when they form clusters. We finally show the results of applying our method to two industrial instances.

4.7 Appendix: Notations

Abbreviations:

NNLS	Non-negative least squares problem
P1	Our formulation of NNLS with implicit points
P2	Our surrogate formulation of NNLS with implicit points

Notations:

$\mathbb{Y} = \{\mathbf{y}^1, \dots, \mathbf{y}^m\}$	The set of discrete points in $\mathbb{R}^n$.
m	The number of points in $\mathbb{Y}$.
n	Dimension of $\mathbb{R}^n$.
$\hat{\mathbf{z}}$	The target point.
$\mathcal{CC}(\mathbb{Y})$	The convex cone of the set $\mathbb{Y}$.
$\mathcal{CH}(.)$	The convex hull of a given set.
$\xi_1, \dots, \xi_m$	The non-negative coefficients of $\mathbf{y}$'s.
W	An $n \times n$ invertible matrix.
$h(\mathbf{z}) = \|W(\mathbf{z} - \hat{\mathbf{z}})\|_2^2$	The objective function of (P1).
$f(\mathbf{z}) = \|\mathbf{z} - W\hat{\mathbf{z}}\|_2^2$	The objective function of (P2).
$\mathbf{z}^*,\ \mathbf{z}^{**}$	The optimal solutions of (P1) and (P2), respectively.
$\mathbf{c}$	A coefficient vector in $\mathbb{R}^n$.
$\tilde{\mathbb{Y}}$	The set $\{W\mathbf{y}^1, \dots, W\mathbf{y}^m\}$.
$\mathcal{H}$	The hyperplane defined as $\mathcal{H} := \{\mathbf{z} \in \mathbb{R}^n \mid \hat{\mathbf{z}}^T W^T \mathbf{z} = \hat{\mathbf{z}}^T W^T W \hat{\mathbf{z}}\}$.
$\mathcal{FH}$	The set of feasible solutions of (P2).
k	The iteration counter in Algorithm 3.
$\mathcal{M}(k)$	The subproblem that has to be solved at each iteration of Algorithm 3.

$\mathbf{z}^{(k+1)}$	The new point found at iteration $k \geq 0$, by solving $\mathcal{M}(k)$.
$\mathbf{z}^{*(k)}$	The best found solution at iteration $k \geq 0$.
$\mathcal{L}^{(k+1)}$	The lower bound that is determined at iteration $k \geq 0$.
ε	An acceptable error.
$\lambda,\ x_i,\ v_{iu},\ s_{iu}$	The variables used in the formulation of $\mathcal{M}(k)$.
$1-2^U,\ 2^U$	The known lower and upper bounds on the values of y_i's.
$\mathcal{CFH}$	The compact subset of $\mathcal{FH}$ that includes the optimal solution of $\mathcal{M}(k)$.
C_f	The curvature constant of f over the domain $\mathcal{CFH}$.
Δ	The number of zero entries of $\hat{\mathbf{z}}$.

4.8 Appendix: Alternative Formulations of NNLS with Implicit Points

A possible approach is to formulate the feasible region of (P1) as a set of mixed-discrete nonlinear programming constraints, as we discuss below. The optimal solution $\mathbf{z}^*$ may be in the interior or boundary of $\mathcal{CC}(\mathbb{Y})$; hence, $\mathbf{z}^*$ can be represented as a non-negative combination of at most n points in $\mathbb{Y}$. Thus, there exist $\xi_1, \ldots, \xi_n \in \mathbb{R}_+$ and $\mathbf{y}^1, \ldots, \mathbf{y}^n \in \mathbb{Y}$ such that $\mathbf{z}^* = \sum_{i=1}^n \xi_i \mathbf{y}^i$. Therefore, (P1) can be formulated as:

$$\begin{aligned} \min \quad & h(\mathbf{z}) = \|W(\mathbf{z} - \hat{\mathbf{z}})\|_2^2 \\ \text{s.t.} \quad & \mathbf{z} = \sum_{i=1}^n \xi_i \mathbf{y}^i,\ \xi_1, \ldots, \xi_n \in \mathbb{R}_+,\ \mathbf{y}^1, \ldots, \mathbf{y}^n \in \mathbb{Y}, \end{aligned}$$

which is a significantly difficult problem because of the nonlinearity in the objective function and constraints, and the existence of $\mathcal{O}(n)$ continuous and $\mathcal{O}(n^2)$ discrete variables (note that nonlinearity is due to the fact that both ξ and $\mathbf{y}$'s are decision variables). An special case of our problem is when $\mathbb{Y}$ is the set of feasible solutions to a binary linear program. In this special case, the feasible region of (P1) can be defined through linear constraints by using $\mathcal{O}(n^2)$ continuous and $\mathcal{O}(n^2)$ binary variables. This problem is still very difficult because if, for example, $n = 100$, then (P1) is defined using 10,000 binary variables.

Another possible approach is to formulate (P1) as follows:

$$\begin{aligned} (\text{P1}') : \ \min \quad & \tilde{h}(\mathbf{z}, \lambda) = \|W(\lambda \mathbf{z} - \hat{\mathbf{z}})\|_2^2 \\ \text{s.t.} \quad & \mathbf{z} \in \mathcal{CH}(\mathbb{Y}), \ \lambda \geq 0, \end{aligned}$$

where $\mathcal{CH}(\mathbb{Y})$ is the convex hull of $\mathbb{Y}$. This alternative formulation is not advantageous because the objective function is not necessarily convex. This is shown in the following lemma.

Lemma 20. $\tilde{h}(\mathbf{z}, \lambda) = \|W(\lambda \mathbf{z} - \hat{\mathbf{z}})\|_2^2$ *is not necessarily convex over the domain* $\mathbf{z} \in \mathcal{CH}(\mathbb{Y}), \ \lambda \geq 0$.

Proof. Consider an instance of (P1′) with $n = 1$, $\mathbb{Y} = \{0, 1\}$, $\hat{z} = 0.5$, and $W = 1$. For this instance, (P1′) can be written as $\min_{0 \leq z \leq 1, \ \lambda \geq 0} (\lambda z - 0.5)^2$. Fig. 4.8 shows the graph of $(\lambda z - 0.5)^2$ over the domain $0 \leq z \leq 1, \ 0 \leq \lambda \leq 1$. Note that the objective function is not convex (consider, for example, the diagonal from (0,0) to (1,1)). □

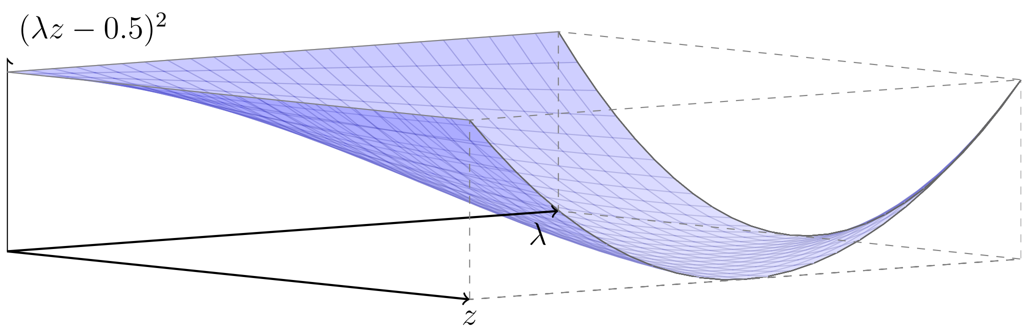


Figure 4.8: The graph of $(\lambda z - 0.5)^2$ over the domain $0 \leq z \leq 1, \ 0 \leq \lambda \leq 1$

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