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A note on “Quasi-analytical solution of two-dimensional Helmholtz equation”

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Abstract

The recent paper of Van Hirtum in this journal repeats a number of misconceptions about the use of conformal mappings in solving the two-dimensional Helmholtz equation. These are discussed, as is the fact that the numerical approach presented does not lead to accurate results. In general conformal mapping is not useful in solving Helmholtz's equation. Other, accurate, techniques are briefly reviewed.

Keywords: Helmholtz equation; conformal mapping; Bessel functions.

1. Introduction

The recent paper [1, hereafter VH17] in this journal repeats a number of misconceptions about the use of conformal mappings in solving the two-dimensional Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0. \tag{1}$$

In addition, the numerical procedure presented is quite misguided. This note points out these problems to reinforce the warnings of [2, hereafter M12] in which the focus is on scattering. The mathematical issues are related, although the interesting question of the analytic continuation to solutions of the Helmholtz equation discussed in [3], which is related to the Rayleigh hypothesis in scattering theory, does not arise here.

The derivation in VH17 follows that of [4, hereafter L82] in detail, although the text suggests that only the contour of integration is found in L82, rather than the derivation

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itself which is in fact the case. The key result is Equation (16) of VH17, which can be written as

$$\phi = \sum_{n=-\infty}^{\infty} a_n H_n(k|f(s)|) \left(\frac{f(s)}{|f(s)|} \right)^n , \tag{2}$$

where the conformal map $z = f(s)$ goes from a canonical plane $s = u + iv = \sigma e^{i\varphi}$ to the physical plane $z = x + iy = re^{i\theta}$. This is Equation (2.19) of [4], and can also be found as Equation (27) in [5] and Equation (24) in [6].

However the functions $J_n(kr)\mathrm{e}^{in\theta}$ and $Y_n(kr)\mathrm{e}^{in\theta}$ are solutions of the Helmholtz equation (1). Hence a solution can immediately be written down in the form

$$\phi = \sum_{n=-\infty}^{\infty} a_n H_n(kr)\mathrm{e}^{in\theta}. \tag{3}$$

This sum is in fact exactly (2) since $z = re^{i\theta} = f(s)$, and the derivation in VH17 and other references is unnecessarily complicated. This was pointed out in M12 which also discusses the convergence properties of such sums in the context of scattering.

2. Ellipses

The first warning is that the comment at the top of p. 296 of L82 that the function of $k|f(s)|$ in (2.17) and beyond "in the case of elliptical domains with elliptical coordinate systems [...] turns out to be the Mathieu functions" is wrong. The functions remain Bessel functions. A sum of Bessel functions may be used to approximate a Mathieu function, but that is a different issue.

The discussion of the ellipse is further marred by the fact that the conformal map presented is from the outside of the unit disc to the outside of the ellipse. This is not the same as the map from the inside of the unit disc to the inside of the ellipse, which is what is needed [7].

Mathieu functions may have been "cumbersome" to use in the past, but with modern software they are within easy reach. In fact they provide a nice example of the use of pseudospectral methods in [8]. The use of Mathieu functions in solving the Helmholtz equation in elliptical regions is discussed in [9, 10].

A true "quasi-analytic" way to solve the two-dimensional Helmholtz equation in an elliptical domain is given in [11], in which successively more accurate solutions are obtained.

3. Poor approximation

The second warning is that what VH17 is actually doing is finding eigenvalues of the Laplace operator in finite domains, which of course requires solving the Helmholtz equation. However, the approach presented just involves writing down the zeros of the individual Bessel function in the sum (3); see Equation (17) of VH17. This avoids the need to solve a zero-finding problem, but no attempt is made to apply the actual boundary condition. Obviously the results are exact for circles as in VH17 § 4.1. For other shapes, they can only be approximate. Worse, there is no way to control the error.

As a result, VH17 adds the factors β_{el} and β_{sq} to improve the agreement. These are basically geometric corrections to take into account the area of the domain and cannot increase the accuracy for higher eigenvalues.

4. Obtaining accurate results

Accurate results can be computed using the sum in (3). The most straightforward way is to use a Galerkin approximation on the boundary as in L82: one integrates (3) expressed in the auxiliary plane against $e^{-im\varphi}$ to obtain an infinite set of linear equations. The map $f(s)$ then provides the link between the physical coordinates of points on the boundary used in the computation of terms in (3) and the angle in the auxiliary plane φ.

Another approach is to use a collocation method, i.e. “point matching” in the language of [12]. As pointed in [12], one can pick other solutions to use in the sum, and the “method will work well or poorly according to the choice of the functions in the series.”

In both approaches, one is free to pick points on the boundary in the physical domain as desired. The conformal mapping gives one choice, but is not the only one and will become more and more difficult to use as the domain becomes more complicated. (In the elliptical case, while the conformal map given is not correct, it does nevertheless provide a mapping of the unit circle to the ellipse.)

While the collocation approach is effective for smooth domains, "when corners are present, functions must be included that have the behavior indicated [...] or the method will converge badly even for large values of N" [12]. Similar warnings hold for Galerkin methods.

Many other approaches exist, with classical methods being listed in [12]. Some notable recent methods include discretizing Boundary Integral Equations [13], the Method of Fundamental Solutions, discussed in [3, 14] and Radial Basis Functions [15]. Some of these correspond to existing software implementations, e.g. the Finite Element Method is implemented in the freely available PLTMG package and the proprietary MATLAB PDE toolbox.

5. Conclusion

The method of VH17 can only lead to poor results, whereas it is possible today to solve the Helmholtz equation accurately and reliably in a number of different geometries using a variety of approaches, some of which are essentially "off the shelf." Using sums of the form (3) is one starting point, but not following VH17. In particular conformal mapping is not necessary. The literature on the subject is enormous and the reader should have no trouble finding good review articles. Finally, it is not clear what is meant by "quasi-analytical", since all the algorithms that give accurate solutions of the Helmholtz equation are underpinned by a great deal of analysis. In mathematical modelling, one needs approaches that can be proved to be accurate and efficient.

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