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Independence Friendly Dynamic Semantics: Integrating Exceptional Scope, Anaphora and their Interactions

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Independence Friendly Dynamic Semantics: Integrating Exceptional Scope, Anaphora and their Interactions

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Contents

Abstract

Karl DeVries

Independence Friendly Dynamic Semantics: Integrating Exceptional Scope, Anaphora and their Interactions

The goal of this dissertation is to provide a semantic account for exceptional scope indefinites in terms of independence friendly reasoning. I take the view that an indefinite takes exceptional scope when its witness is required not to vary with the value of a variable introduced by a syntactically higher quantifier. This dissertation shows that a straightforward implementation of this view in a static logic results in a system that assigns truth conditions to sentences containing wide scope indefinites that are too strong. I show, surprisingly, that a better implementation of this intuition requires dynamic logic. While using a dynamic logic is a necessary ingredient in the analysis of wide scope indefinites in terms of independence, it is not a sufficient one. I survey a number of recent dynamic systems, examine possible definitions of maximization, and show that only some of these permit the proposed analysis of wide scope indefinites. I show that a system of dynamic plural logic (DPlL) with unselective maximization can be modified to fully account for wide scope indefinites in terms of independent witness choice.âĂŃ

To my parents, Ken and Shirley

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Chapter 1

Introduction

This dissertation considers two empirical phenomena. The first, examined in chapter 2, is the treatment of bare cardinal partitives like that in (1).

(1) That book could belong to one of three people. (Ladusaw, 1982)

The second, more widely studied phenomena, is the semantics of exceptional scope indefinites. Uniting these two phenomena is a series of logics in which formulas are interpreted relative to sets of assignment functions instead of single assignment functions.

This dissertation consists of four contentful chapters. Chapter 2 discusses the semantics of partitives. This chapter offers an in depth description of this construction and isolates four uses to which it can be put, only two of which have been previously described in the literature. It also presents new data that suggest logical similarities between the readings of bare cardinal partitives and the adjectives *same* and *different*. Uniting this chapter with the subsequent chapters is the formal proposal which is couched in terms of a logic in which formulas are evaluated with respect to plural information states (sets of assignment functions) that encode dependencies between the values a variable can take on.

Chapter 3 takes up the analysis exceptional scope indefinites. Brasoveanu and Farkas (2011) argues that wide scope indefinites can be conceptualized in terms of independence. An indefinite can take wide scope in the semantics by signalling that the variable introduced by the indefinite cannot vary with respect to the value of another variable.

(2) Every^x student read a^y paper.

A narrow scope reading of the sentence above arises when the witness for the indefinite is allowed to vary from student to student. If the value of the paper is not allowed to vary from student to student a wide scope reading arises. Crucially in a semantics in which indefinites can signal non-variation while appearing in the scope of other quantifiers, wide scope can be achieved without LF movement of the indefinite or any other LF operation (e.g. existentially binding choice/skolem function variables).

In order to implement their analysis Brasoveanu and Farkas (2011) utilize a logic like the one developed in chapter 2. Formulas are evaluated with respect to sets of assignment functions. The contribution of chapter 3 is an argument against the implementation of this idea in Brasoveanu and Farkas (2011). I show that the static logic they provide does not quite capture the truth conditions of sentences containing exceptional scope indefinites. Instead show that a dynamic logic is needed in order to capture the correct truth conditions; the basic conclusion is that universal quantifiers need to be sensitive not only to the fact that an independent witness was chosen for an indefinite in their scope but also the identity of the particular witness. This requires maximization over output assignments.

Chapter 4 turns to Dynamic Plural Logic (van den Berg, 1996), a dynamic logic which contains the resources to manage dependencies between variables while handling plural discourse reference. I discuss some of the maximization operations that have been proposed in the literature, and show that only some of these allow wide scope indefinites to be analysed in terms of independence. I show that the simplest maximization operator, one that returns the largest set of potential output assignments allows for the intuition underlying Brasoveanu and Farkas (2011) to be correctly implemented.

In Dynamic Plural Logic dependencies are managed by means of distributivity operators. Only assignment updates occurring in the scope of distributivity operators can introduce variables that depend on the values of previous variables. Capturing wide scope interpretations thus reduces to managing the scope of distributivity operators within Dynamic Plural Logic.

Chapter 5 provides a final analysis of wide scope indefinites. An analysis of wide scope indefinites in Dynamic Plural logic leaves one with a syntax-semantic interface problem. If dependencies are managed by controlling the scope of distributivity operators, then wide scope indefinites must be equipped with some way of semantically escaping the scope of a distributivity operator. I accomplish this by decomposing distributivity into two operators, a signalling operator, $\downarrow_x$, that indicates that x should be distributed over and an operator $^{\Delta}$ that implements distributivity. This is accomplished by enriching the interpretive resources of the logic to include a store of variables that should be distributed over. The signalling operator adds a variable to the store while the distributivity operator empties the store of variables while distributing over them. By breaking distributivity up into two components a new operator $\uparrow_x$ can be defined that removes variables from the store. This in effect keeps a formula in the scope of $\uparrow_x$ from being interpreted distributively with respect to x. By equipping wide scope indefinites with $\uparrow_x$ operators they are able to control which variables they are allowed to vary with respect to. Thus like in Brasoveanu and Farkas (2011) indefinites are able to choose their scope.

Chapter 2

Bare Cardinal Partitives

2.1 Introduction

This chapter discusses the semantics of BARE CARDINAL PARTITIVES; a celebrated example is given in (3).

(3) That book could belong to one of three people. (Ladusaw, 1982)

I distinguish bare cardinal partitives, which have the form given in (4), from what I will call DEFINITE CARDINAL PARTITIVES by which I mean DPs of the form given in (5), and from ORDINAL PARTITIVES, like those given in (6).

(4) ⌜*n* of *k* NP⌝, e.g. BARE CARDINAL PARTITIVE
  a. one of two truth values
  b. two of four candidates for local office
  c. one of twenty two roses sitting in a vase on the table

(5) ⌜*n* of the *k* NP⌝, e.g. DEFINITE CARDINAL PARTITIVE
  a. one of the two truth values
  b. two of the four candidates for local office
  c. one of the twenty two roses sitting in a vase on the table

(6) ⌜$n/$a *k*th of the NP⌝, e.g. ORDINAL PARTITIVE
  a. A fourth of the roses
  b. Two thirds of the tomatoes
  c. One Sixth of the book

Throughout this chapter I will refer to the syntactically higher but numerically lower numeral as the OUTER CARDINAL (or OUTER NUMERAL). In the case of (4), n is the outer cardinal. I refer to the syntactically lower but numerically higher numeral as the INNER CARDINAL (or INNER NUMERAL). In (4), k is the inner cardinal. I will often need to refer to the NP or the set of entities that satisfy the NP in a partitive; I sometimes refer to the NP in (4) as the restrictor of the inner cardinal and sometimes as the restrictor of the partitive as a whole. When I use the term restrictor set when discussing English partitives, I am referring to the set of entities picked out by the NP; when I use the term restrictor set when referring to an quantifier in a formula of a formal system I refer to the set of entities that satisfy the syntactic restrictor of some quantifier, i.e. to ϕ in a formula like $[Qx : \phi]\psi$.

This chapter makes three contributions to the literature:

i. This chapter offers the first in depth description of bare cardinal partitives. I isolate four uses to which bare cardinal partitives can be put: (a) a partial ignorance use identified by Ladusaw (1982), (b) an exhaustive use identified by Barker (1998), (c) a cumulative use which is available when a bare cardinal partitive appears in the scope of a quantifier, and (c) a fractional use, in which a bare cardinal partitive teams up with an adverbial quantifier to communicate the same content as a ordinal partitive.

ii. This chapter presents new empirical data showing that adjuncts PPs headed by *in/across* have similar effects on the interpretation of sentence internal readings of the adjectives *same* and *different* and on the interpretation of cumulative readings of bare cardinal partitives. This suggests an logical connection between the analysis of adjectives like *same/different* and bare cardinal partitives.

iii. This chapter proposes a novel analysis of bare cardinal partitives in terms of a static plural logic similar to C-FOL developed in Brasoveanu and Farkas (2011). In this logic formulas are interpreted relative to plural information states (sets of assignment functions) that encode not only the values a certain variable can take but also the relationships between the values of different variables. This is broadly the same type of mechanism that can be used to analyse adjectives like *same* and *different* (see e.g. Kuhn (2015) for a recent attempt). The system I propose also utilizes two types of pluralities: informational pluralities (sets of assignment functions) and ontological pluralities (those familiar from Link (1983)).

My formal proposal also bears certain similarities to a proposal by Bumford (2016) in that I treat the determination of the witness of both the inner and outer cardinals as a simultaneous constraint satisfaction problem—the two are 'scopeless' with respect to one another.

The remainder of the chapter is organized as follows. §2.2 provides a broad overview of the previous literature and the empirical landscape. It also develops the central intuitions that guide the development of the analysis. §2.3 delves into the data, isolating the different readings of bare cardinal partitives. §2.4 develops a formal analysis of bare cardinal partitives. In this section I show that FOL does not provide the resources to handle bare cardinal partitives and instead propose an analysis couched in a logic in which formulas are evaluated with respect to multiple assignment functions. §2.5 concludes.

2.2 Overview

2.2.1 Previous Literature

In the previous literature bare cardinal partitives have played a relatively modest role, appearing in lists of sentences counter-exemplifying the partitive constraint (see e.g. Ladusaw 1982; Reed 1991; Abbott 1996; de Hoop 1997; Barker 1998; Zamparelli 1998; Ionin *et al.* 2006; Chen 2011). The partitive constraint has been cashed out in several different ways (see (7)), but roughly the partitive constraint requires that the embedded DP in a partitive construction be (in some sense) definite (Jackendoff, 1977; Selkirk, 1977).

(7) THE PARTITIVE CONSTRAINT:

> In an *of-N"'* construction interpreted as a partitive, the *N"'* must have a demonstrative or a genitive specifier. (Jackendoff, 1977)
>
> The Partitive Constraint can be stated... by requiring that the NP in a partitive phrase always denotes an individual (Ladusaw, 1982)
>
> The Partitive Constraint can be restated such that the embedded NP within a partitive construction must always denote an individual, either entity level or group level. That is, *of NP* yields the components of the generator set if the NP denotes and individual, and is undefined otherwise. (de Hoop, 1997)
>
> Empirically speaking, the Partitive Constraint describes the restrictions on [the embedded NP]. First of all [it] cannot be quantified or bare... Secondly, [it] can be a definite plural, or an indefinite one if it is specific/referential in

> the sense of Fodor and Sag 1982 — the unique entity such that the speaker intends to refer to it... (Ionin et al., 2006)
>
> ⋮

Since the focus of previous literature has been reconciling sentences like (3) with the partitive constraint, there has been no systematic investigation of the semantics of bare cardinal partitives in their own right. In particular, the literature contains very little discussion about possible interpretations or uses of bare cardinal partitives, nor has there been a serious attempt to tease out possible differences between bare cardinal partitives and definite cardinal partitives. The absence of empirical interest is reflected in a paucity of formal treatments: as far as I am aware, only Winter (2000, 2005) formalizes a sentence like (3).

2.2.2 Expanding the Empirical Landscape

The central empirical contribution of this chapter is an in-depth exploration of the interpretation of bare cardinal partitives. I identity four uses of bare cardinal partitives. The first, which I will call the PARTIAL IGNORANCE use, is exemplified in (8). The sentence in (8) could be used naturally by a speaker who is uncertain about which poem Otis recited, but who is not completely ignorant about which poems Otis might have recited. The sense one gets from (8) is that the speaker is partially ignorant about the identity of the poem Otis recited.

(8) Otis recited one of three poems. PARTIAL IGNORANCE
⇒ Otis recited one poem, but I'm not sure which; there are three possibilities.

A second use, which I will call the EXHAUSTIVE use, is exemplified in (9). The sentence in (9) can normally only be used if there are exactly three people Otis admires. If Otis admires more than three people, then (9) is unacceptable.[1]

(9) Sybil is one of three people Otis admires. EXHAUSTIVE
⇒ Sybil is one person Otis admires and there are three people Otis admires.

A third use, which I will call the CUMULATIVE use, is exemplified in (10). Uttering (10) allows a speaker to communicate two facts about which local businesses endorsed which candidates: first, (10) conveys that each local business endorsed one candidate — I will call this the DISTRIBUTIVE INFERENCE—and second, (10) conveys that between every local business three candidates were endorsed overall—I will call this the CUMULATIVE INFERENCE. This interpretation differs from both the partial ignorance and the exhaustive uses. A speaker uttering (10) need not have three specific candidates in mind, nor must she be ignorant about which candidate any business endorsed. Likewise, (10) does not commit the speaker to the belief that there are only three candidates—(10) in fact seems to suggest that there are more than three candidates.

(10) Every local business endorsed one of three candidates for city council. CUMULATIVE
⇒ Every local business endorsed one candidate for city council and three candidates were endorsed overall.

The fourth use, which I will call the FRACTIONAL use, is exemplified in (11). In its fractional use, a bare cardinal partitive is equivalent to an ordinal partitive or some other fractional expression, e.g. 25%, *every third*, etc. Fractional uses occur when a bare cardinal partitive occurs in sentences that are generic in some sense.

(11) Generally, students read one of four assigned papers. FRACTIONAL
⇒ Generally, students read a fourth of the papers assigned.

[1]The only way to salvage (9) in the case in which Otis admires more than three people is to give (9) a partial ignorance interpretation. The only way to set this up given the fact that (9) is an identification sentence is to imagine a scenario in which the speaker knows who Otis admires under names other than Sybil. The speaker could then use (9) to indicate that they were partially ignorant about how to line up the description Sybil with the description of one of the (many) people Otis admires.

I will not have much to say about fractional uses in this paper and will set them aside. Nevertheless, it is important to keep the possibility of a fractional interpretation in mind when deciding how a sentences should be judged, as I will sometimes mark with a # sentences that are acceptable on readings other than those I am interested in.

The most empirically rich descriptions of bare cardinal partitives are found in Ladusaw (1982) and Barker (1998), both of which discuss ways of reconciling bare cardinal partitives with the partitive constraint. Ladusaw, discussing the example given earlier in (3), writes:

> **The [sentence] in [(3) is] appropriately used only when the user has a particular group of individuals in mind.** [(3)] invites a continuation: "namely, John, Mary and Bill"...The indefinite NPs which would normally introduce individuals which later may be referred to by definite NPs are used here to simultaneously introduce and refer to a group of individuals. (Ladusaw, 1982)

The intuition expressed above is most consistent with a partial ignorance interpretation. The speaker using a bare cardinal partitive has some specific set of entities in mind. In my terms: the inner cardinal expresses the size of a speaker's set of epistemic possibilities for the identity of witness for the outer cardinal. This is the intuition followed by Winter (2000, 2005) who tries to formalize this reading by means of choice functions.

Barker, discussing an example from Abbott (1996) given in (12), gives the examples in (13):

(12) He brought back several of twenty of his roses that were sick to get a refund, but had to just throw out the rest, which was about fifteen. (Abbott, 1996)

(13) a. Fortunately, two truth values suffice for most purposes. (Barker, 1998)
b. Because each proposition denotes at most one of two truth values...

Barker suggests that bare cardinal partitives are acceptable if the inner cardinal gives the cardinality of the restrictor set:

> ... in a context in which *true* and *false* are the only (relevant) truth values, the indefinite NP *two truth values* accidentally denotes the same generalized quantifier as *the two truth values*... indefinite examples like those in [(13b)] will be acceptable only in contexts in which the indefinite NPs have denotations that are accidentally (isomorphic to) group individuals. **One prediction is that indefinites will be unacceptable in contexts that entail the existence of additional entities with the relevant properties...** [The context in (12)] renders the indefinite *twenty of his roses that were sick* acceptable as a partitive NP only on the implicit assumption that the set of twenty roses exhausts the set of sick roses. If there were 24 roses in total, and 22 of them were sick, then [(12)] is out. (Barker, 1998)

Here Barker is describing the exhaustive use of bare cardinal partitives. The inner cardinal gives the cardinality of the set of entities that there are; it is thus equivalent to a sentence containing a definite cardinal partitive in its place.

If the uses discussed in Ladusaw (1982) and Barker (1998) exhausted the uses to which bare cardinal partitives can be put, we would expect the condition in (14) to govern all uses of bare cardinal partitives.

(14) A sentence, *S*, containing a bare cardinal partitive, *n-of-k Ps* can be used iff

a. the speaker has *k P*s in mind, and these are the speaker's epistemic possibilities for the identity of the *n P*s involved in the eventuality described by *S* or

b. there are exactly *k P*s.

In this chapter I introduce novel data suggesting that not all uses of bare cardinal partitives are governed by (14). Consider the contrast in (15).

(15) a. # Otis recited one of three poems, but I have no idea which three.
b. Every student recited one of three poems, but I have no idea which three.

The sentence in (15a) is predicted to be out given:

i. *which three P*s requires that there be more than three salient *P*s.

ii. The sluice as a whole denies that the speaker has three candidates in mind.

iii. The bare cardinal partitive requires, pace (14), either that there be exactly three candidates or that the speaker have three candidates in mind.

Since (i-iii) are jointly inconsistent, the acceptability of (15a) is expected. However, since (14) makes no reference to the presence or absence of an additional quantifier, the same reasoning predicts the unacceptability of (15b). Changing the subject from a name to a quantifier seems to call off the requirements of (14).

The empirical contribution of this chapter consists of motivating in detail the replacement of (14) with the condition in (16).[2]

(16) A sentence, *S*, containing a bare cardinal partitive, *n-of-k Ps* can be used iff
a. the speaker has *k P*s in mind, and these are the speaker's epistemic possibilities for the identity of the *n P*s involved in the eventuality described by *S*,
b. there are exactly *k P*s, or
c. if *S* contains a quantified DP *Q*, then for each *Q n P*s participated in the eventuality described by *S* and *k P*s participated overall.

2.2.3 Outline of the Analysis

My analysis will focus on cumulative uses, since these are the most difficult to achieve compositionally. My analysis exploits a proposed similarity between cumulative readings of bare cardinal partitives and the semantics of *same/different*. Imagine that you

[2]Here I am ignoring the fractional use of bare cardinal partitives.

set out to verify a sentence like (17) using a piece of scratch paper. You might first create a list of dogs, then list for each dog which cat(s) they chased, and finally look back on your list to make sure that no cat appears twice.

(17) Every dog chased a different cat.

Fido	Whiskers	} the cat Fido chased
Rex	Socks	
Spot	Evander	
⋮	⋮	
Every dog	no repeats	

Adjectives like *same/different* are special because they involve column-wise reasoning. You have to look at every row at once to when evaluating (17).

I will argue that similar reasoning is involved in the interpretation of bare cardinal partitives. If you set out to verify a sentence like (18) with a piece of scratch paper, you would make a list of dogs, then list the cat each chased, and finally look back to make sure your list includes exactly three cats.

(18) Every dog chased one of three cats.

Fido	Whiskers	} the one cat Fido chased
Rex	Socks	
Spot	Evander	
⋮	⋮	
Every dog	three cats	

My formal analysis builds on a plural logic, First Order Logic with Choice (C-FOL) (Brasoveanu and Farkas, 2011)[3]. This logic implements the 'scratch paper reasoning' laid out above by evaluating expressions with respect to sets of assignment functions. Each assignment function in a set of assignment functions corresponds to a row on our scratch paper. Row-wise reasoning involves looking at single assignment functions and column-wise reasoning involves looking at the set of values a variable takes on across assignment functions. I argue that cumulative uses of bare cardinal partitives occur when the ontological plurality picked out by the inner cardinal is identified with the informational plurality picked out by the outer cardinal.

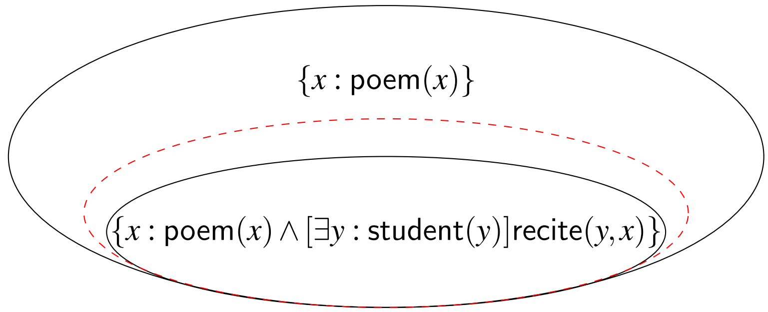
The analysis I propose allows partial ignorance and cumulative uses to be unified. If we accept that utterances are understood as implicitly embedded under an epistemic necessity operator (either as part of the pragmatic reasoning process or covertly at LF as in Meyer (2013)), then we can treat partial ignorance as cumulation across a speaker's epistemically accessible worlds. To say *Otis recited one of three poems* communicates that in each world one regards as epistemically possible Otis recited one poem and summing up the poems across these worlds gives you a set of three poems.

We arrive at the conclusion that the witness for embedded DP cannot be any arbitrary subset of the restrictor set. In the case of definite cardinal partitives, this is trivial since the definite presupposes a familiar or unique referent. In the case of exhaustive interpretations, the witness is just the maximal plural individual that exhausts the restrictor set. In the case of cumulative interpretations, the witness set is the maximal plural individual consisting of those individuals who participate distributively in the eventuality of described by the sentence. Schematically, we have the options in (19):

(19) Every student recited one of three poems.
   a. $|\{x : \mathsf{poem}(x)\}| = 3$ or EXHAUSTIVE

[3] C-FOL is related to several other frameworks including Dynamic Plural Logic van den Berg (1996) and independence friendly logics Hintikka (1973); Väänänen (2007). I choose C-FOL since it utilizes both ontological pluralities (familiar from Link (1983)) and informational pluralities (i.e. the value(s) that a variable may take on), see Brasoveanu (2008); Henderson (2014) for discussion in related dynamic frameworks.

b. $|\{x : \text{poem}(x) \wedge [\exists y : \text{student}(y)]\text{recite}(y,x)\}| = 3$ CUMULATIVE



The inner cardinal can give the cardinality either of its restrictor as in (19a) in which case an exhaustive interpretation arises, or it can give the cardinality of the smallest set of relevant entities that participated in the event as in (19b) in which case a cumulative reading arises. The inner cardinal cannot give the cardinality of some arbitrary intermediate set. This is only possible for definite cardinal partitives which can give the cardinality of any subset of the restrictor that is already familiar/unique in the discourse.

2.3 Three Readings of Bare Cardinal Partitives

This section discusses three readings of bare cardinal partitives, the PARTIAL IGNORANCE, EXHAUSTIVE, and CUMULATIVE readings. I exclude here the FRACTIONAL interpretations. The discussion is divided into several subsections based on the complexity of the sentences in which the bare cardinal partitive occurs. In §2.3.1 I examine bare cardinal partitives in simple sentences with no additional quantifiers. In such sentences only the partial ignorance and exhaustive uses can be detected. Next in §2.3.2 I examine bare cardinal partitives in sentences containing other quantifiers. Here the cumulative interpretations become detectable. The near total absence of examples containing both bare cardinal partitives and other quantifiers in the literature largely explains why these interpretations have not been previously recognized. Finally, in §2.3.3 I note an empirical similarity between the interaction between *in/accross* adjuncts and adjectives like *same/different* and the interaction between *in/across* adjuncts and the cumulative interpretation of bare cardinal partitives. Although I will not provide an analysis of the difference between *in/across*, the empirical similarity provides a hint that a plural logic capable of providing an analysis of *same/different* is an appropriate tool to handle bare cardinal partitives.

2.3.1 Bare Cardinal Partitives in Sentences Without Quantifiers

In this section, I examine bare cardinal partitives in simple sentences (i.e. sentence without any quantificational elements). I will be particularly interested in one question: Under what conditions can a partitive of the form ⌜*n* of *k* NP⌝ be used felicitously in contexts in which there are more than *k* relevant things?

2.3.1.1 Bare Cardinal Partitives Signal Ignorance

Consider the two sentences in (20), which seem at first blush to have nearly identical truth conditions.

(20) a. Otis read one paper.
b. Otis read one of three papers.

Both sentences in (20) require that Otis read one paper. If Otis read didn't read any papers or read more than one paper, then both sentences would be judged false. The sentence in (20b) carries an additional requirement that there be (at least) three papers under discussion. For example, if one was discussing the readings assigned for a particular class on a particular day and it was known to all participants that only two papers were assigned, (20a) could be used felicitously while (20b) would be unacceptable.

A further difference between the examples in (20) can be brought about by considering each sentence relative to different questions under discussion (QUDs) (Ginzburg, 1996; Roberts, 1996; Büring, 2003). Consider each sentence as an answer to a HOW-MANY question.

(21) How many papers did Otis read?
  a. Otis read one paper.
  b. #[4] Otis read one of three papers.

As an answer to (21) the sentence in (21a) containing a bare numeral seems felicitous—it is a direct answer to the question. In the same context, the sentence in (21b) containing a bare cardinal partitive seems odd. Since the question under discussion is how many papers Otis read, the additional information that the one paper Otis read is among three is superfluous resulting in a feeling of infelicity.

Next consider the sentences in (20) as answers to a *which* question.

(22) Which paper did Otis read?
  a. # Otis read one paper.
  b. Otis read one of three papers.

In response to the question in (22) the acceptability of the two answers flips. The sentence in (22a) containing only a bare numeral seems uncooperative. Note that (22a) seems much worse than (21b). Intuitively, this feeling arises due to the fact that the question in (22) contains a presupposition that Otis read a paper (Belnap, 1963) and the response in (22a) merely restates the presupposition of (22). The sentence (22a) thus fails to answer the question or even advance the conversation toward an answer. If all one can truthfully say in response to (22) is (22a), then one should instead admit ignorance: *I'm not sure* is the appropriate response. More interesting is the acceptability of (22b) in this context. Intuitively, (22b) does advance the conversation by signalling that the speaker has some information about which papers Otis read. But, crucially, the answer in (22b) communicates ignorance about the particular paper Otis read.

[4]Here I follow a long tradition of severely overloading the #. In this particular case the feeling is one of getting slightly more information than was asked for.

I will refer to the ignorance inference discussed above as the IGNORANCE CONDITION, which is provided in (23).

(23) IGNORANCE CONDITION: A sentence ⌜*n* of *k* *P*s *Q*ed⌝ conveys that the speaker is not aware of which particular *n* *P*s *Q*ed.

Since we have not yet discussed the interpretation of *n*-of-*k* partitives in sentences with quantifiers, I restrict the ignorance condition to just these cases.

2.3.1.2 Refining the Ignorance Condition

Although (22b) does communicate ignorance, it communicates only partial ignorance. (22b) does not suggest that the speaker does not have any idea which papers Otis read. In fact, it suggests that the speaker can eliminate some possibilities. Consider the exchange in (24).

(24) Sybil: Which paper about partitives did Otis read?
Kashif: I'm not sure, I know he read one of three papers.
Sybil: That's potentially very helpful, which three?
   a. Kashif: It was either Ladusaw '82, Abbott '96, or Barker '98
   b. # Kashif: Oh, I don't know that either.

The dialogue in (24) begins in the same way as (22). However, it contains a follow-up question in which Sybil asks for further clarification about which candidates Kashif has in mind. If Kashif comes back with a list of papers as in (24a), then the conversation seems fine. If instead Kashif responds by claiming further ignorance as in (24b), the conversation seems deviant. The contrast between (24a) and (24b) suggests that by using a bare cardinal partitive to convey ignorance Kashif puts himself on the hook to provide additional information about potential candidates. A partitive of the form ⌜*n* of *k* NP⌝ thus seems to require that the speaker have some specific *k* things in mind.

I will refer to this inference as the SPECIFICITY CONDITION which is defined in (25).

(25) SPECIFICITY CONDITION: A sentence ⌜*n* of *k* *P*s *Q*-ed⌝ communicates that the speaker has *k* particular *Q*'s in mind.

The specificity condition is stated somewhat vaguely here, but it captures Ladusaw's (1982) characterization of the requirements imposed by bare cardinal partitives.

By putting the ignorance and specificity conditions together, we can come to a more precise characterization of PARTIAL IGNORANCE.

(26) PARTIAL IGNORANCE: An utterance, *S*, containing a bare cardinal partitive, like ⌜*n* of *k* *P*s *Q*-ed⌝ communicates partial ignorance iff *S* communicates that the speaker

a. is unaware of which *n* *P*s *Q*-ed and

b. has *k* candidates in mind.

2.3.1.3 Exhaustive Interpretations

We can now ask whether every bare cardinal partitive conveys partial ignorance, i.e. if it is subject to the ignorance and specificity conditions, or if there are uses of bare cardinal partitives which do not convey partial ignorance. To answer this question, we need to engage in a closer examination of the data.

The discussion of (22) and (24) above has left one aspect of the context implicit, viz. how many papers are under consideration when the question is posed. In both (22) and (24) it is natural to assume a context in which there are several possible papers that Otis could have read. In fact, as seen in (27) if we set up the context so that there are only five salient papers, we will discover that the bare cardinal partitive requires that inner cardinal is less than five.

(27) *Sybil has created a list five of semantics papers available to her class and requires that every student read at least one of these papers. Sybil is interested in knowing what a particular student Otis has decided to read and she asks Yngve:*
Which paper did Otis read?
Yngve replies:

a. Otis read one of three papers.
b. # Otis read one of five papers. (cf. # Otis read one paper.)

Yngve's response in (27a) is acceptable. It conveys Yngve's ignorance, but suggests that he is able to provide Sybil with some information. The response in (27b), however, is infelicitous. Intuitively, the feeling of infelicity arises because (27b) does not provide any information that cannot be recovered between the presupposition of the question and the details explicitly mentioned in the context. If Yngve considers all five papers to be live possibilities, then his ignorance is not partial. The requirement on partial ignorance uses examined above can be stated in terms of the relative size of the domain and the inner cardinal. The size of the restrictor set in a context must be larger than the inner numeral for a partial ignorance reading to arise.

The above data might lead one to believe that bare cardinal partitives require that the context include more items in the restrictor set than the inner cardinal. However, it is easy to show that this is not the case. To illustrate the point it is helpful to turn to identification sentences like (28).

(28) Sam is a doctor.

While the indefinite in (28) can be analysed as a predicate, identification sentences can contain also contain quantifiers as seen in (29).

(29) a. Sam and Kari are some doctors Yngve admires.
b. (?) Sam and Kari are all the doctors Yngve admires.

While (29b) seems slightly degraded both sentences are readily interpretable. Both examples in (29) communicate that Sam and Kari are doctors that Yngve admires. The sentence in (29a) implicates, additionally, that there are other doctors Yngve admires, while the example in (29b) entails that Yngve admires no doctors except Sam and Kari.

It is easy to show that the inference associated with (29a) is an implicature. It can easily be called off in a cancellation test as seen in (30).

(30) Sam and Kari are some doctors Yngve admires, and, in fact, they are the only doctors Yngve admires.

Like *some* and *all* bare numerals can also occur in identification sentences. Consider the example in (31).

(31) Warren and Otis are two people Sybil admires.

The example in (31) leads to same inference present in (29a). It suggests that Sybil admires Warren and Otis, but that Sybil admires other people as well. Again, this inference is readily cancellable, as shown in (32).

(32) Warren and Otis are two people Sybil admires, and, in fact, they are the only two people Sybil admires.

Now consider the interpretation of a bare cardinal partitive in an identification sentence. One such example is given in (33).

(33) Warren and Otis are two of five people Sybil admires.

There is no hint of partial ignorance about (33). This sentence like those in (29) and (31) entails that Sybil admires Warren and Otis. However, it conveys some additional information. First, (33), like (29a) and (31) communicates that the speaker admires other people besides Warren and Otis. Unlike (29a) and (31), this inference is an entailment of (33). It cannot be cancelled as seen in (34).

(34) # Warren and Otis are two of five people Yngve admires, and, in fact, they are the only two people Yngve admires.

Second, (33) communicates that there are (exactly) five people Yngve admires. The infelicity of the sentence given in (35) shows that this inference is also an entailment of (33).

(35) # Warren and Otis are two of five people Sybil admires, but, in fact there are $\left\{\begin{array}{l}\text{fewer}\\ \text{more}\end{array}\right\}$ than five people Sybil admires.

In identification sentences bare cardinal partitives do not convey ignorance, but instead convey the size of restrictor set in a context. This is the EXHAUSTIVE use, which I define in (36).

(36) EXHUASTIVE: An utterance, *S*, containing a bare cardinal partitive, like ⌜*n* of *k P*s⌝ communicates exhuastivity iff *S* communicates that the speaker believes there to be only *k P*s.

It is unsurprising that identification sentences provide excellent examples in which bare cardinal partitives communicate exhustivity. It would be very odd to say that Warren is one of the people you admire, but you're not sure which one of the people you admire he is. To the extent that such a scenario makes sense one has to think that you know none of the people you admire by the name 'Warren' but have been informed that one of the people you admire is also someone you know (in a different context) as Warren. You might then have some speculation about which of the people you admire goes by Warren. Note that in the course of coercing a partial ignorance interpretation, one looses the inference that the speaker admires only three people.

It is extremely easy to find naturally occurring examples of bare cardinal partitives that convey exhuastivity. A google search turns up countless examples like those in (37) in which bare cardinal partitives are used in identification sentences.

(37) a. $^{\gamma}$ Fischer's 100-footer Ragamuffin is **one of four supermaxis vying for line honours** along with the defending champion Wild Oats, fellow Australian Perpetual Loyal and the American invader Comanche.

b. $^{\gamma}$ McNamara and Vaughan are **two of five alternates for this year's Titans of Mavericks competition** and set out Thursday morning at first light to practice on the challenging break.

The example in (37a) comes from a discussion of the 2015 Sydney-Hobart race. The *n*-of-*k* partitive is used to indicate that Ragamuffin is a supermaxi and that there are exactly four supermaxis in the race; (37a) would be unequivocally false had a fifth supermaxi entered the race. The example in (37b) comes from the Santa Cruz Sentential discussing a particularly nasty wipeout on Steamer Lane that was recorded by onlookers. This example clearly communicates that the Titans of Mavericks competition has five alternates this year.

It is also easy to find examples of bare cardinal partitives which convey exhuastivity that do not occur in identification sentences. Consider the example in (38).

(38) $^{\gamma}$ A Tennessee couple holding **one of three winning tickets for this week's record $1.6 billion U.S. Powerball lottery jackpot** said on Friday they will keep their jobs because "you just can't sit down and do nothing."

The example in (38) occurs in a newspaper article discussing the recent power-ball lottery. This example is not an identification sentence, yet the context makes clear that the identity of the ticket the couple is holding is not unknown (it's not even clear what it would mean for it to be unknown—the ticket they are holding is the ticket they are holding and that's all that matters). In this sentence it is clear that the inner cardinal, *three* in (38) conveys the total number of winning tickets.

2.3.1.4 Summary

The data discussed in this section lend themselves to the following conclusion: bare cardinal partitives convey either exhaustivity or partial ignorance in sentences without other quantificational expressions. Moreover, it appears that whether the partitive conveys domain exhaustivity or partial ignorance is dependent on the relative size of the restrictor set and the inner cardinal. If the set is larger than than the inner cardinal, then the partitive conveys partial ignorance. If one the other hand, the partitive does not convey partial ignorance, then it communicates that the restrictor set has cardinality of the inner numeral. An bare cardinal partitive like (39) has two paraphrases, given in

(39a) and (39b).

(39) Otis read one of three papers.
  a. There are three papers Otis could have read.
  b. There are three (contextually salient) papers and Otis read one paper.

Both paraphrase in (39) are modelled on Ladusaw's (1982) paraphrase of (3). The first, in (39a), suggests ignorance about the particular paper Otis read and the second, in (39b), gives information about the number of papers in the domain.

In the next section, I turn to more complex sentences that contain quantificational expressions. However, one should keep in mind that partial ignorance and exhaustive uses of bare cardinal partitives are expected to be present in all sentences. The interesting question that I tackle in the next section is whether there are additional readings of bare cardinal partitives that are not covered by these two uses.

2.3.2 Bare Cardinal Partitives in Quantified Sentences

This section examines the behaviour of *n*-of-*k* partitives that occur in the scope of overt universal quantifiers. I will show that when an *n*-of-*k* partitive occurs in the scope of a universal quantifier it can convey the range of variation that potential witnesses take with respect to the quantifier. A sentence like (40) conveys (i) that each business endorsed one candidate and (ii) that overall three candidates were endorsed by local Businesses.

(40) Every local business endorsed one of three candidates for City Council.

This use of *n*-of-*k* partitives allows one to convey the number of candidates endorsed both distributively (one) and cumulatively (three). In this section I will also show that in sentences like (40) there is no specificity requirement on *n*-of-*k* partitives.

Scenarios can easily be constructed in which a sentence like (40) can be used by a speaker who does not have any three candidates in mind.

Imagine that Sybil, a professor, wanting her students to be well read, makes available a list of ten papers relevant to a course, and, not wanting to overburden her students, tells them that they should each read at least one of the papers. To keep track of which papers students have read, Sybil instructs her teaching assistants to make a database that records for each student the papers that they have read. We can now imagine our professor making queries to the data-base and think about the inferences the answers justify.

(41) Sybil wonders which paper a particular student, Yngve, has read. She queries the database and it returns "Shelah & Rudin 1978". She concludes:

a. Yngve read one paper.

b. # Yngve read one of three papers.

(42) Sybil wonders which paper(s) Yngve read. She queries the database in it returns "Shelah & Rudin 1973". Sybil realizes there is a problem, because there is no paper corresponding to that citation on the list. The list does however contain: Shelah & Rudin 1978, Rudin & Erdös 1973, and Shelah & Erdös 1973. She reasons that this entry must have either the wrong date or one wrong name. She concludes:

a. Yngve read one paper.

b. Yngve read one of three papers.

The data in (41-42) confirm the empirical generalizations developed in the previous section. The initial scenario indicates that the context contains ten papers, thus prohibiting a domain exhaustivity use of an *n*-of-*k* partitive. The scenario described in (41) sets up a situation in which Sybil knows exactly how many papers Yngve read, permitting the conclusion in (41a) containing a bare numeral. Scenario (41) does not, however, permit the conclusion in (41b) since Sybil knows which paper Yngve read and thus lacks partial ignorance. The scenario described in (42) sets up a situation in which Sybil knows how many papers Yngve read but does not know its exact identity. Scenario (42) thus licenses both conclusion (42a) containing a bare numeral and conclusion (42b) containing an *n*-of-*k* partitive. Notice that to permit the conclusion (42b) Sybil has to have some idea of which particular papers Yngve might have read. If no paper written by Shelah or by Rudin or in 1973 was present in the list, giving Sybil nothing to go on, (42b) would not be a felicitous conclusion.

Now consider the scenario described in (43).

(43) Sybil wonders how many papers students have been reading. She queries the database and finds that every student has read only one paper. Wondering if, perhaps, as a group, they are well read, she queries the database about how many papers have been read. And finds to her surprise that it returns "3". She concludes:

a. Every student read one paper.

b. Every student read one of three papers.

In this scenario, Sybil has complete ignorance about both (i) which students read which papers and (ii) which papers were read by any students. Instead Sbyil knows (i) how many papers every student read and (ii) how many papers were read by some student or other. In such a scenario both conclusion (43a) containing a bare numeral and (43b) containing an *n*-of-*k* partitive are permitted. This suggests that the *n*-of-*k* partitive in (43b) is compatible with complete ignorance about the identity of the three papers.

The same point can be made by observing sentences with two universal quantifiers like (44).

(44) In every section, every student read one of three papers.

On its most natural interpretation the sentence provided in (44) describes a situation in which (i) every student read one paper and (ii) in every section three papers were read overall. On this reading, the three papers read vary from section to section, but (44) conveys that within each section only three papers were read. On this interpretation (44) does not suggest anything about the speaker's state of knowledge with respect to which student read which papers or which papers were read in which section.

The data above strongly suggest that in the scope of universal quantifiers *n*-of-*k* partitives have readings which convey neither domain exhuastivity nor partial ignorance. In order to better characterize their meanings, I draw attention to the distributive/cumulative ambiguities that arise between bare numerals and universal quantifiers.

(45) Two dogs chased every cat.
  a. Two dogs each chased every cat.
  b. Two dogs chased every cat between them.

Sentences like (45) have two readings, a distributive reading paraphrased in (45a) and a cumulative reading paraphrased in (45b). In contrast, universal quantifiers with bare numerals in their scope allow only distributive readings:

(46) Every dog chased two cats.

The sentence in (46) can only mean that each dog has chased two cats. It cannot be used to describe a situation in which each dog chased some cat and two cats were chased overall. Now consider the sentence in (47) which contains an *n*-of-*k* partitive in the scope of a universal.

(47) Every dog chased two of three cats.

The sentence in (47) seems to get at both a distributive interpretation relative to the outer numeral (*two*) and a cumulative interpretation relative to the inner numeral (*three*). The sentence (47) is true if (i) every dog chased two cats and (ii) three cats were chased overall. On this way of looking at things *n*-of-*k* partitives allow speakers to access normally unavailable cumulative readings. One can also characterize the meaning of (47) in terms of variation. When verifying a sentence like (46) one checks each dog and makes sure that one can find exactly two cats that it chased. The choice of cats can vary without constraint from dog to dog. In (47), on the other hand, one has to select dog after dog two cats from the same set of three.

I refer to the use of *n*-of-*k* partitives under discussion here in terms of partial variation which I define in (48).

(48) PARTIAL VARIATION: An utterance *S*, $\ulcorner$ Every *P Q*-ed *n*-of-*k R*s$\urcorner$, containing an *n*-of-*k* partitive occuring in the scope of a quantificational expression conveys partial varation iff *S* conveys that the speaker believes that $\ulcorner$ Every *P Q*-ed *n R*s$\urcorner$ and that these *n R*s are drawn from the same set of *k R*s.

Before turning to the next section, it is worth considering two additional questions. First, which determiners license the partial variation interpretation of *n*-of-*k* partitives? Second, how does the partial variation interpretation interact with distributivity in general?

Beginning with the first question, we can see that in addition to universal quantifiers like *each, every,* and *all* proportional quantifiers also licence partial variation readings.

(49) $\left\{\begin{array}{c}\text{Most of}\\ \text{The majority of}\\ \text{A fourth of}\end{array}\right\}$ the students read one of three papers.

The expressions in (49) all express proportional quantifiers, and each has an interpretation in which the *n*-of-*k* partitive expresses partial variation. If most of the students read one of three papers, this suggests that if one were to look at all of the students, one would find more than three papers read overall.

Definite descriptions can also license partial variation interpretations, but only if they take distributive interpretations.

(50) a. The students each read one of three papers.
b. # The students read one of three papers altogether.

The sentence in (50a) which contains distributive *each* has a reading in which the students each read one paper and overall three papers were read. The sentence in (50b) which contains the anti-distributive adverbial *altogether* does not have a reading in which each student read one paper and overall three papers were read. Instead (50b) has only the familiar readings from the previous section; it can convey either that there are only three contextually salient papers or that the speaker is unsure of exactly which one paper the students read.

The adverbial *between them* can be used to bleed the inherent distributivity of the universal quantifier *every* as seen in (51). The sentence in (51) does not require that every dog chased every cat, but only that every dog chased some cat and every cat was chased by some dog.

(51) Between them, every dog chased every cat.

The deep connection between distributivity and partial variation readings is also illustrated by example (52).

(52) # There were ten cats around, and every dog chased one of three cats, between them.

The sentence in (52) sounds odd because every use of the *n*-of-*k* partitive is precluded: the exhuastive use of the partitive is inconsistent with the first conjunct, the ignorance interpretation would require that every dog chase one cat, in which case the anti-distributive *between them* is superfluous, and the partial variation reading is bled by the anti-distributive adjunct.

2.3.3 *in/across*, *same/different*, and cumulative readings

In this section, I call attention to a data point that suggests that cumulative readings of bare cardinal partitives are sensitive to the same expressions as sentence internal readings of adjectives like *same* and *different*. In particular I will focus on how adverbial PPs headed by *in* vs. *across* have different effects on the interpretation of *same* and *different* and that these effects are identical to the effects *in* vs. *across* have on cumulative readings of bare cardinal partitives.

Carlson (1987) describes two readings of adjectives like *same* and *different*. These adjectives can have EXTERNAL readings, exemplified in (53). In (53a) the DP *the same cat* refers to the cat Otis saw introduced in the first conjunct. Likewise, in (53b) *a different dog* cannot refer to the dog introduced in the first conjunct. Carlson refers to these uses as sentence external because the adjective compares/contrasts its value with some previously mentioned individual.

(53) a. Otis saw one cat in the morning, and Warren saw the same cat in the afternoon.
b. Otis saw one dog in the morning, and Warren saw a different dog in the afternoon.

In addition to their sentence external use, adjectives like *same* and *different* have INTERNAL readings that are available in the presence of other quantificational elements. These uses are exemplified in (54).

(54) a. Every student recited the same poem.
b. Every student recited a different poem.

These readings are called internal because the sentence itself sets up the context on which *same/different* operate.

Now, imagine a situation in which a class is divided into a number of discussion sections s.t. every student is in exactly one section. Say that each student is tasked with picking a poem to recite during their section. After the recitations, one might utter the sentence in (55).

(55) In every section, every student recited the same poem.

The sentence in (55) would be false if no two students in the same section recited the same poem, but could be true even if two students from different sections recited the same poem. Now, consider what happens if we replace *in* with *across* as in (56).

(56) Across every section, every student recited the same poem.

The sentence in (56) would be false if two students from different sections recited the different poems. Intuitively, (55) involves partitioning the set of students according to which section they attend and then predicating *recited the same poem* of each cell of the partition. The sentence in (56) does not partition the set of students but rather predicates *recited the same poem* of the entire set of students.

The same data can be replicated for *different*:

(57) a. In every section, every student recited a different poem.
b. Across every section, every student recited a different poem.

The sentence in (57a) might be true even if two students from different sections recited the same poem; as long as no two students from the same section both recited a single poem (57a) is true. The sentence in (57b) is stronger—it would be false if any two students from any section both recited the same poem.

Interestingly, the pattern above can be replicated for cumulative readings of bare cardinal partitives. Consider the data in (58).

(58) a. In every section, every student recited one of three poems.
b. Across every section, every student recited one of three poems.

The sentence in (58a) conveys that within each section three poems were recited—(58a) could be true if three different poems were read in each section. The sentence in (58b) would be false if more than three poems were read by students in the class taken as a whole.

Though, I will not offer a full account of the distinction between *in* and *across* as it occurs in (55-58), the pattern suggests that *in* is distributive while *across* is cumulative. Moreover, bare cardinal partitives in their cumulative use and *same* and *different* show exactly the same behaviour with respect to *in* and *across*. This suggests that the semantic account of cumulative readings should make use of similar logical mechanisms.

2.4 Analysis of Bare Cardinal Partitives

In this section, I develop an analysis of *n*-of-*k* partitives in several steps. My approach is novel in several respects. First, I will not worry about the partitive constraint. Second, I will defer the discussion of the scope of the inner (*k*) cardinal as long as possible; the analysis takes us through two logics and not until the second logic will we come to any firm conclusions about where the inner (*k*) cardinal takes scope.

The first section outlines an initial logic that is fairly standard in the semantics literature. It includes the usual expressions and interpretations familiar from first order logic (FOL) enriched with a domain of plural individuals and the standard array of distributive and maximization operators. An important aspect of the logic we develop will be its ability to cleanly, if not compositionally, handle van Bentham's puzzle (van Benthem, 1986); this will allow us to develop a 'scopeless' analysis of *n*-of-*k* partitives allowing us to dodge the scope issue in order to focus on the underlying logic.

The second section shows that with only these standard resources we can easily capture only domain exhuastivity readings: I will show that FOL does not provide the resources necessary to handle partial variation readings.

The third section introduces a logic in which formulas are interpreted relative to sets of assignment functions. This logic allow one to talk about all the values a variable might take while calculating the semantic value of a formula. I show that the additional resources provided by this logic allow for a relatively straightforward account for partial variation readings.

The fourth section provides the final analysis. I show that exhaustive and cumulative readings can be captured in the system and argue that partial ignorance uses can be unified with cumulative interpretations: partial ignorance arises as a result of a cumulative interpretation across epistemically accessible worlds.

2.4.1 Initial Logic: Structured Domains & First Order Interpretation

This section provides a leisurely development of a relatively standard logic capable of translating plurals and bare numerals. The logic handles plurals by means of a structured domain, i.e. a domain consisting of both atomic and non-atomic individuals along with designated symbols $\prec$, $\oplus$, and δ that permit the logic to access the underlying semi-lattice formed by the sum operation. Quantification over this domain is completely standard: constants take individuals in the domain as their semantic value and formulas are interpreted relative to single assignment functions that map variables to individuals in the domain of the model. I show that the logic can provide clean and truth conditionally adequate translations of sentences involving cumulative/distributive ambiguities, one- and two-sided interpretations of numerals, and sentences containing multiple numerals.

We begin with models familiar from Link (1983) that consists of a domain of individuals, $\mathfrak{D}$ that consists of the power-set of a set of atomic entities, Ind, minus the empty set, i.e. $\mathfrak{D} := \wp(\mathsf{Ind}) - \emptyset$ and an interpretation function, $\mathfrak{I}$. Formulas are interpreted relative to a model $\mathfrak{M} := \langle \mathfrak{D}, \mathfrak{I} \rangle$ and an assignment function. I assume the usual definitions of $\neg, \wedge, \vee, =, \exists$, and $\forall$.

To handle quantification we define assignment function difference for single variables and generalization to sequences of variables.

(59) $h[v]g :=$ h differs from g at most with respect to the value assigned to v

(60) $h[v_1, v_2, \ldots, v_n]g :=$ there exists a k s.t. $h[v_1]k$ and $k[v_2, \ldots, v_n]g$

Individual terms are interpreted as elements of $\mathfrak{D}$ in the usual way.

(61) a. If c is an individual constant (in practice: $a, b, c, j \ldots$), then $[\![c]\!]^{\mathfrak{M},g} = \mathfrak{I}(c)$
b. If v is an individual variable (in practice: $x, y, z, \ldots$), then $[\![v]\!]^{\mathfrak{M},g} = g(v)$

Note that since our domain consists of both atomic individuals, i.e. singleton subsets of Ind, and plural individuals, i.e. non-singleton subsets of Ind, I will sometimes talk about 'one individual' in which case I mean one element (singular or plural) in $\mathfrak{D}$. If I mean to indicate only singular individuals I will use the phrase 'atomic individual' and to pick out pluralities I will use the term 'non-atomic individual' or 'plurality'.

To handle plural individuals we define sum formation, $\oplus$, as set union, and designate $\prec$ as the 'part-of' relation defined as the subset relation.

(62) a. $[\![t_1 \oplus t_2]\!]^{\mathfrak{M},g} = [\![t_1]\!]^{\mathfrak{M},g} \cup [\![t_2]\!]^{\mathfrak{M},g}$
b. $[\![t_1 \prec t_2]\!]^{\mathfrak{M},g} = \mathbb{T}$ iff $[\![t_1]\!]^{\mathfrak{M},g} \subset [\![t_2]\!]^{\mathfrak{M},g}$

n-ary predicates are interpreted as elements of $\mathfrak{D}^n$, and predication is handled standardly.

(63) $[\![P(t_1, \ldots t_n)]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff $\langle [\![t_1]\!]^{g,\mathfrak{M}}, \ldots [\![t_n]\!]^{g,\mathfrak{M}} \rangle \in \mathfrak{I}(P)$

In addition, for every predicate, P, we define a predicate ${}^{\bigstar}P$ that consists of the cumulative closure of P which is defined in (64).

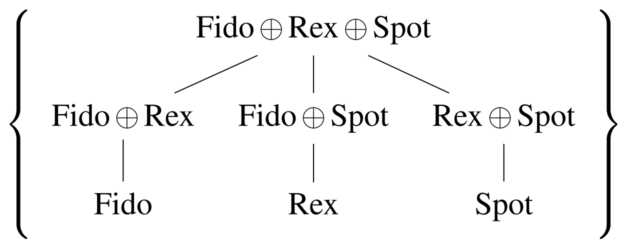
(64) For any predicate P, ${}^{\bigstar}P$ denotes the cumulative closure of P, i.e. $\mathfrak{I}({}^{\bigstar}P)$ is the smallest set s.t. if $\langle a_1, \ldots a_n \rangle, \langle b_1, \ldots b_n \rangle \in \mathfrak{I}(P)$, then $\langle a_1 \oplus b_1, \ldots a_n \oplus b_n \rangle \in \mathfrak{I}({}^{\bigstar}P)$

I assume that some predicates, e.g. dog, will include only atomic individuals while others, e.g. gather, will include only non-atomic individuals. If a predicate like dog includes only atomic individuals, the predicate $^\star\mathsf{dog}$ contains every plural individual consisting of individuals included in the denotation of dog as illustrated in (65).

(65)

$$\mathfrak{I}(\mathsf{dog}) = \{\text{Fido, Rex, Spot}\}$$

$\mathfrak{I}(^\star\mathsf{dog}) =$



Since the distinction between atomic and plural individuals plays a significant role in our discussion, we designate the predicate **atom** to pick out all the atomic individuals in $\mathfrak{D}$.

(66) $[\![\mathbf{atom}(t)]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff $|[\![t]\!]^{g,\mathfrak{M}}| = 1$

We can get away with the definition above because we've chosen to model plural individual as sets of atomic individuals. We can take the cardinality of a plural individual directly, since each plural is just a set of elements of Ind.

In order to 'look inside' non-atomic individuals we define a distributive operator, δ, which distributes a predicate across the atomic subparts of a non-atomic individual.

(67) $\delta_x\phi := [\forall y : y \prec x \wedge \mathbf{atom}(y)]\phi(y/x)$, where $\phi(y/x)$ is the formula obtained by replacing the variable x in ϕ with the fresh variable y.

(68) A variable v is fresh in a formula ϕ iff v does not occur in ϕ.

The distributive operator δ allows us to capture the difference between cumulative and distributive uses of conjunctions. The sentence in (69) has three readings.

(69) Otis and Mary lifted a piano.

a. $[\exists x : x = \mathsf{Otis} \oplus \mathsf{Mary}][\exists y : \mathsf{piano}(y)]\mathsf{lift}(x,y)$

b. $[\exists x : x = \mathsf{Otis} \oplus \mathsf{Mary}][\exists y : \mathsf{piano}(y)]\delta_x \mathsf{lift}(x, y)$
c. $[\exists x : x = \mathsf{Otis} \oplus \mathsf{Mary}]\delta_x[\exists y : \mathsf{piano}(y)]\mathsf{lift}(x, y)$

On its cumulative reading reading, (69) entails that Otis and Mary jointly lifted a piano. The formula in (69a) captures this reading, since it requires that the plural individual $\mathsf{Otis} \oplus \mathsf{Mary}$ lifted some piano. In addition to a cumulative reading, (69) has two distributive readings. One in which Otis and Mary separately lifted the same piano, and one in which Otis and Mary separately lifted two different pianos. The first distributive reading is captured by the formula in (69b), which holds that each atomic individual in $\mathsf{Otis} \oplus \mathsf{Mary}$ lifted some piano, i.e. that there is a piano that Otis lifted and Mary lifted. The second distributive reading is captured by the formula in (69c), which entails that Otis lifted a piano and Mary lifted a piano, but does not require that the two pianos be the same.

I want to flag one important, but easily overlooked entailment of this particular definition: no sentence like that in (70) can be expressed in terms of the δ-operator as defined in (67).

(70) $[\exists x : \phi][\forall y : y \prec x \wedge \mathbf{atom}(y)]R(x, y)$

To see why consider how one would unpack the sentence $[\exists x : \phi]\delta_x R(x, x)$:

(71)
$$\begin{aligned} [\exists x : \phi]\delta_x \psi(x, x) &= [\exists x : \phi][\forall y : y \prec x \wedge \mathbf{atom}(y)]R(y, y) \\ &\neq [\exists x : \phi][\forall y : y \prec x \wedge \mathbf{atom}(y)]R(x, y) \end{aligned}$$

The sentence (70) cannot be expressed in terms of δ, since the δ-operator overrides all instances of a variable x. The upshot is that sentences that can be expressed in terms of δ cannot directly express relationships between wholes and their parts within the scope of δ. We could imagine an alternative distributive operator D that would allow us to express (70). Such an operator could be defined as in (72):

(72) $D^x_y \phi := [\forall y : y \prec x \wedge \mathbf{atom}(y)]\phi$

The definition of D provided above is not defined in terms of variable change and thus allows us to abbreviate (70) as $[\exists x : \phi]D^x_y\psi(x,y)$.

I can think of two reasons to prefer the δ-notation to the D-notation. The first is psychological: the δ-notation allows us to write formulas with fewer variables and more transparent connections between their subparts. The second is logical and less obvious: δ-notation interfaces more cleanly with maximization operators since $\{x : \delta_x\phi\}$ is a join-semilattice, closed under $\oplus$, while $\{x : D^x_y\phi\}$ is not guaranteed to have a greatest upper bound. I first show that $\{x : \delta_x\phi\}$ is always a join-semilattice: From the definition of δ, it follows that $a \in \{x : \delta_x\phi\}$ iff every atomic part of a, a', is s.t. $a' \in \{x : \phi\}$. Now, assume $b, c \in \{x : \delta_x\phi\}$, it follows that every atomic part of b is in $\{x : \delta_x\phi\}$, as is every atomic part of c. But that means every atomic part of $b \oplus c$ is in $\{x : \delta_x\phi\}$, since there is no atomic part of $b \oplus c$ that is not an atomic part of either b or c. So we conclude, $b \oplus c \in \{x : \delta_x\phi\}$. Thus, $\{x : \delta_x\phi\}$ is a join-semilattice. I now show that $\{x : D^x_y\phi\}$ need not be a join-semilattice. Let $\phi = R(x,y)$ and let $\mathfrak{I}(R) = \{\langle a \oplus b, a\rangle, \langle a \oplus b, b\rangle\ \langle c \oplus d, c\rangle, \langle c \oplus d, d\rangle\}$, where a,b,c,d are all atomic. Obviously, $a \oplus b \in \{x : D^x_yR(x,y)\}$ and $c \oplus d \in \{x : D^x_yR(x,y)\}$. However, $a \oplus b \oplus c \oplus d \notin \{x : D^x_yR(x,y)\}$. Hence, $\{x : D^x_y\phi\}$ may not be a join-semilattice.

To enable us to talk about the size of pluralities, we introduce numerical terms of the form **n**.**atoms** which pick out individuals that contain n atomic individuals.

(73) $[\![\mathbf{n}.\mathbf{atoms}(t)]\!]^{\mathfrak{M},g} = |[\![t]\!]^{\mathfrak{M},g}| = n$

Again, we can get away with the definition above because we have chosen to interpret all individuals as non-empty subsets of Ind. This choice privileges the atomic entities allowing us to count the atomic subparts of a non-atomic individual directly.

We are now in a position to start translating sentences containing bare numerals. A bare numeral like *two* can be translated by means of an existential and a cardinality operator.

(74) two $\rightsquigarrow \lambda P_{et}.\lambda Q_{et}.[\exists x : P(x) \wedge \delta_x Q(x)]\mathbf{2.atoms}(x)$

(75) Two dogs barked. $\rightsquigarrow [\exists x : {}^{\star}\mathsf{dog}(x) \wedge \delta_x {}^{\star}\mathsf{bark}(x)]\mathbf{2.atoms}(x)$

The interpretation of the bare numeral in (74) involves first finding a (possibly non-atomic) individual that satisfies the restrictor and nuclear scope and then checking that the individual contains two atomic parts. In (75) one first finds a non-atomic individual *d* consisting of dogs that barked and then checks that *d* consists of two individuals.

The translation of the bare numeral above is ultimately inadequate since it captures only one interpretation of the bare numeral. Bare numerals have both one- (at least) and two- (exact) sided interpretations. These two types of readings are illustrated in (76).

(76) a. **A**: Are you 21?
B: Yes, I'm 28 in fact.

b. **A**: Are you 28?
B: No, I'm 30.

In (76a) **B**'s agreeing response to **A**'s question suggests that **A** took 21 to mean at least 21. This is a plausible interpretation on **B**'s part if we imagine that **A** is a bouncer asking **B** whether they should be allowed into a bar. Since 21 years is the cut-off for being allowed in, a one-sided (at least) reading is plausible. In (76b), **B**'s disagreeing response to **A**'s question suggests that **A** took 28 to mean exactly 28. While the 'classical' analysis (Horn, 1972) of two-sided numerals relies on a scalar implicature from *n* to *not* $n+1$, data like that in (76) has convinced researches that one- and two-sided meanings arise as a matter of semantics (see Kennedy (2013) for a recent overview).

The formula in (74) captures only the one-sided (at least) reading of the bare numeral in (75). To see why, consider a scenario in which three dogs, Fido, Rex, and Spot all barked. In this scenario it will also be true that the plural individual Fido and Rex barked. There will then be a witness for the existential in (75), viz. $\mathsf{Fido} \oplus \mathsf{Rex}$, that consists of barking dogs and has cardinality two. Note that that the cardinality check in (74) is exact, we do check that the witness consists of exactly two atomic individuals—the one-sidedness arises because the witness is allowed to be non-maximal. The translation in (74) does not ensure that the witness includes every dog that barked.

To handle two-sided (exact) interpretations of bare numerals, we introduce a maximization operator, σ, defined below.

(77) $[\![[\sigma_v : \phi](\psi)]\!]^{\mathfrak{M},g}$ = their exists an $h[v]g$ s.t.
  a. $[\![\phi]\!]^{\mathfrak{M},h} = \mathbb{T}$ and there is no $h'[v]g$ s.t. $h(v) \prec h'(v)$ and $[\![\phi]\!]^{\mathfrak{M},h'} = \mathbb{T}$
  b. $[\![\psi]\!]^{\mathfrak{M},h} = \mathbb{T}$

The maximization operator finds an individual d that satisfies its restrictor, ensures that there is no individual d' that properly includes d that also satisfies its restrictor and then passes d to its nuclear scope.

With the help of σ we can capture two-sided interpretations of numerals.

(78) two $\rightsquigarrow \lambda P_{et} \lambda Q_{et}.[\sigma_x : P(x) \wedge \delta_x Q(x)]\mathbf{2.atoms}(x)$
(79) Two dogs barked. $\rightsquigarrow [\sigma_x : {}^{\star}\mathsf{dog}(x) \wedge \delta_x {}^{\star}\mathsf{bark}(x)]\mathbf{2.atoms}(x)$

The formula in (79) requires that the maximal set of dogs, each of whom barked consists of two dogs. This captures the intuitive truth conditions of sentences containing bare numerals on their two-sided readings.

Notice, that the translations for bare numerals which we have so far considered have involved distributive operators over their nuclear scope. This seems right, but will prevent us from captureing various cumulative uses of numerical expressions. Consider the sentence in (80). This sentence has both distributive and cumulative interpretations. Ignoring inverse scope, the distributive interpretation of (80) requires that there be exactly two dogs each of whom chased exactly three cats.

(80) (Exactly) two dogs chased (exactly) three cats

The distributive reading can be captured by the formalism developed so far. We would associate the sentence (80) with the translation in (81). This formula captures the intuitive truth conditions of (80) on its distributive reading. It collects the maximal set of dogs that chased exactly three cats and checks that this set contains just two dogs.

(81) $[\sigma_x : {}^{\star}\mathsf{dog}(x) \wedge \delta_x[\sigma_y : {}^{\star}\mathsf{cat}(y) \wedge \delta_y {}^{\star}\mathsf{chase}(x,y)]\mathbf{3}.\mathbf{atoms}(y)]\mathbf{2}.\mathbf{atoms}(x)$

On its cumulative reading (80) requires (i) that there are two dogs that chased any cats and (ii) there are three cats that were chased by any dogs. To capture this reading, we need to simultaneously maximize over both dogs and cats that stand in the chase relation.

To account for (80) we generalize the σ operator to handle more than one variable at a time.

(82) $[\![[\sigma_{v_1,\ldots,v_n} : \phi](\psi)]\!]^{\mathfrak{M},g}$ = their exists an $h[v_1,\ldots,v_n]g$ s.t.

a. $[\![\phi]\!]^{\mathfrak{M},h} = \mathbb{T}$

b. there is no $h'[v_1,\ldots,v_n]g$ s.t. $h(v_1) \prec h'(v_1)$, ..., or $h(v_n) \prec h'(v_n)$ and $[\![\phi]\!]^{\mathfrak{M},h'} = \mathbb{T}$

c. $[\![\psi]\!]^{\mathfrak{M},h} = \mathbb{T}$

With generalized maximization in hand, we are in a position to formalize the cumulative reading of (80). The formalization, given in (83), first maximizes over the dog-cat pairs that stand in the chase relation and then checks that two dogs and three cats are involved overall. Thus, (83) captures the reading of (80) in which exactly two dogs chased any cats and exactly three cats were chased by any dogs.

(83) $[\sigma_{x,y} : {}^{\star}\mathsf{dog}(x) \wedge {}^{\star}\mathsf{cat}(y) \wedge {}^{\star}\mathsf{chase}(x,y)](\mathbf{2}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$

The crucial aspect of the representation in (83) is the 'scopeless' nature of the two numerals. Maximization is simultaneous and the cardinalities of the two plural individuals are checked only after they are arrived at jointly by the σ operator. I will not at present describe how to arrive at the representation in (83) compositionally. See van Benthem (1986) for further analysis of the problem and Brasoveanu (2013) for a compositional treatment.

The logic laid out in this section is quite expressive. It can handle a variety of cumulative and distributive interpretations and handle both one- and two-sided interpretations of numerals. It even allows the formalization of various ‘scopeless’ readings of numerical expressions. Its hands are tied in only one way: it involves first order interpretation, i.e. variables are assigned single values by single assignment functions. In the next section we turn to *n*-of-*k* partitives and see how far in formalizing their meaning we can get with our initial logic.

2.4.2 Bare Cardinal Partitives in FOL

In this section I develop a preliminary account of bare cardinal partitives. The account developed here has several crucial components. First, the two numerals are ‘scopeless’ with respect to one another. The account utilizes simultaneous maximization over two variables. Second, the account assumes that bare cardinal partitives take low scope. In this way the account differs from previous treatments available in the literature which all assume that the inner numeral takes exceptionally wide scope (see e.g. Winter (2000, 2005)). Third, I show that the first order logic in the previous section cannot capture the full range of readings. In particular a unified account of cumulative readings occurring in the scope of both universal and proportional quantifiers will elude a logic that limits its interpretive resources to single assignment functions.

The account developed in this section has a few downsides. First, it is not strictly compositional. The two numerals do not make independent contributions but contribute to the semantics I assign to the structure as a whole. I have chosen not to pursue a strictly compositional analysis so that I can focus on the relationship between the meaning of the bare cardinal partitive and other quantificational elements in the sentence. Second, my analysis does not make use of current degree-theoretical analyses of the interpretation of numerals (e.g. Kennedy (2015)). I have chosen not to pursue this tack because there has been no worked out solution to van Benthem's puzzle around which I can base my denotations.

We begin with a definition of bare cardinal partitives in (84). Here we treat *n*-of-*k* partitives as quantifiers.

(84) $\ulcorner n\text{-of-}k \urcorner \rightsquigarrow \lambda P_{et} \lambda Q_{et} [\sigma_{x,y} : P(y) \wedge x \preccurlyeq y \wedge Q(x)](\mathbf{n}.\mathbf{atoms}(x) \wedge \mathbf{k}.\mathbf{atoms}(y))$

(85) Otis read one of three papers.
$\rightsquigarrow [\sigma_{x,y} : {}^{\star}\mathsf{paper}(y) \wedge x \preccurlyeq y \wedge {}^{\star}\mathsf{read}(\mathsf{Otis}, x)](\mathbf{1}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$

The formula in (85) captures the exhaustive use of the bare cardinal partitive. To see this clearly it is helpful to pull out the restrictor of $\sigma_{x,y}$.

(86) $\{\langle x, y \rangle : {}^{\star}\mathsf{paper}(y) \wedge x \preccurlyeq y \wedge {}^{\star}\mathsf{read}(\mathsf{Otis}, x)\}$

Notice how the conjuncts work together in the restrictor of the quantifier. Any variable assignment satisfying the first conjunct, ${}^{\star}\mathsf{paper}(y)$, along with the second conjunct, $x \preccurlyeq y$, will also satisfy ${}^{\star}\mathsf{paper}(x)$. Together with the third conjunct, ${}^{\star}\mathsf{read}(\mathsf{Otis}, x)$, we should be satisfied that x must store the largest set of papers that Otis read. Hence, (85) entails that Otis read one paper. Notice that $\prec$ is transitive, i.e. $x \prec y$ and $y \prec z$, then $x \prec z$. This ensures that given ${}^{\star}\mathsf{paper}(x)$ both ${}^{\star}\mathsf{paper}(y)$ and $x \preccurlyeq y$ can be satisfied by an assignment that assigns to y the sum of papers in the model and to x the sum of papers that Otis read. Hence, we should be satisfied that (85) conveys that there are three papers total present in the model.

While the representation in (84-85) are enough to account for the exhaustive interpretation, it will not convey partial ignorance, nor will it scale up to cases cumulative interpretation. Take for instance the translation in (87).

(87) Every student read one of three papers.

$$\rightsquigarrow [\forall z : \mathsf{student}(z)][\sigma_{x,y} : {}^{\bigstar}\mathsf{paper}(y) \wedge x \preccurlyeq y \wedge {}^{\bigstar}\mathsf{read}(z,x)](\mathbf{1.atoms}(x) \wedge \mathbf{3.atoms}(y))$$

The exact same reasoning applied to (85) can be applied to (87). This sentence entails that every student read one paper, and that there are three papers overall. It does not indicate that not all three papers were read. It is initially tempting to blame this state of affairs on my analysis' (unjustified) requirement that the two numerals are both (a) scopeless and (b) low with respect to the universal. If the inner (k) numeral were allowed to take wide scope, the correct reading might be forthcoming.

Implementing such an idea would result in the representation in (88).

(88) $[\sigma_y : {}^{\bigstar}\text{paper}(y) \wedge [\forall z : \text{student}(z)][\sigma_x : x \prec y \wedge {}^{\bigstar}\text{read}(z,x)]\mathbf{1.atoms}(x)]\mathbf{3.atoms}(y)$
In prose: there is a set of three papers s.t. every student read one of them and didn't read the other two and every paper that isn't one of those three was read by some student or other.

The formula in (88) is insidiously wrong. It is simultaneously too weak and too strong. Too weak because it does not entail either (i) that every student read (exactly) one paper nor (ii) that (exactly) three papers were read overall. Too strong because if every student read exactly one paper, it entails that there are exactly three papers. To see this consider that the restrictor in (88) picks out the maximal member of the set given in (89).

(89) $\{y : {}^{\bigstar}\text{paper}(y) \wedge [\forall z : \text{student}(z)][\sigma_x : x \prec y \wedge {}^{\bigstar}\text{read}(z,x)]\mathbf{1.atoms}(x)\}$

The set in (89) contains just those pluralities of papers s.t. every student read exactly one of them. If there were a plural individual p consisting of four papers s.t. some student read two of them, then p would not be included in (89). Hence, (88) does not entail that every student read only one paper, nor does it entail that only three papers were read overall. The closest paraphrase for (88) is "there is a set of three papers s.t. every student read one of them and didn't read the other two and every paper that isn't one of those three was read by some student or other". This is not a reading associated with the partial variation reading, and indeed does not seem to be available for n-of-k partitives at all.

The issue with (88) arises because the maximization operator associated with the outer (n) numeral is inside the restrictor of the maximization operator associated with the inner (k) numeral. The formula in (88) sends one on a quest for the maximal set of papers, s.t. every student read exactly one of them. A scopeless analysis avoids this problem since it simultaneously maximizes over both variables.

One might try to get around this by giving the inner (k) numeral a one-sided interpretation when it takes wide scope. This move would result in the translation in (90).

(90) $[\exists y : {}^{\star}\text{paper}(y) \wedge [\forall z : \text{student}(z)][\sigma_x : x \prec y \wedge {}^{\star}\text{read}(z,x)]\mathbf{1}.\mathbf{atoms}(x)]$
$\mathbf{3}.\mathbf{atoms}(y)$

This translation suffers from similar deficiencies. Like (88), this sentence does not entail either (i) that every student read (exactly) one paper nor (ii) that (at least) three papers were read overall. The formula in (90) will be true just in case, there are three papers s.t. every student read exactly one of them—perhaps every student read the same paper and as for other papers perhaps they were read and perhaps they weren't.

Likewise, one cannot allow both the inner (k) and the outer (n) numerals to take one-sided readings. The result, given in (91), does not improve upon (90).

(91) $[\exists y : {}^{\star}\text{paper}(y) \wedge [\forall z : \text{student}(z)][\exists x : x \prec y \wedge {}^{\star}\text{read}(z,x)]\mathbf{1}.\mathbf{atoms}(x)]$
$\mathbf{3}.\mathbf{atoms}(y)$

The formula in (91) again does not entail either (i) that every student read (exactly) one paper nor (ii) that (exactly) three papers were read overall. Instead it says there there are three papers such that every student read at least one of them—it could be that every student read the same paper of these three or that they read any number of papers beyond these three.

For the sake of completeness we should consider what would happen if we allowed the inner (k) cardinal to take wide scope and a two-sided meaning and the outter (n) numeral to take narrow scope and a one-sided interpretation. The result is given in (92).

(92) $[\sigma_y : {}^\star\text{paper}(y) \wedge [\forall z : \text{student}(z)][\exists x : x \prec y \wedge {}^\star\text{read}(z,x)]\mathbf{1}.\mathbf{atoms}(x)]$
$\mathbf{3}.\mathbf{atoms}(y)$

Unlike (88, 90-91), the formula in (92) does correspond to a reading of associated with n-of-k partitives. However, it is one we have seen before, viz. the exhaustive reading. To sum up this brief discussion, scope alone cannot account for partial variation readings.

The previous discussion should make two points clear. The first and more important takeaway is that cumulative readings cannot be analysed by scoping the inner cardinal above the quantifier. This is an important point since Winter (2000, 2005) which contains the only formally explicit analyses of bare cardinal partitives in the literature does exactly this. In Winter's analysis the inner cardinal contributes a property that is true of pluralities of 3 papers. This property is then bound by a choice function at the matrix level resulting in either of the two readings given below depending on whether the outer cardinal is given a two-sided or one sided interpretation.

(93) a. $\exists f : [\forall z : \text{student}(z)]$
$[\exists x : x \prec f(\{y : {}^\star\text{paper}(y) \wedge \mathbf{3}.\mathbf{atoms}(y)\}) \wedge {}^\star\text{read}(z,x)]$
$\mathbf{1}.\mathbf{atoms}(x)$

b. $\exists f : [\forall z : \text{student}(z)]$
$[\sigma_x : x \prec f(\{y : {}^\star\text{paper}(y) \wedge \mathbf{3}.\mathbf{atoms}(y)\}) \wedge {}^\star\text{read}(z,x)]$
$\mathbf{1}.\mathbf{atoms}(x)$

Notice that these formulas assign one-sided interpretations to the inner numeral; the sentence could just as well be true if $\{y : {}^\star\text{paper}(y) \wedge \mathbf{3}.\mathbf{atoms}(y)\}$ were replaced with the set $\{y : {}^\star\text{paper}(y) \wedge \mathbf{4}.\mathbf{atoms}(y)\}$. Depending on whether the outer cardinal is given a one-sided interpretation (as in (93a)) or a two-sided interpretation (as in (93b)), Winter's analysis derives truth conditions equivolent to (90) or (91), neither of which are adequate.

The second and less important takeaway is that it is not necessary to scope the inner cardinal out in order to arrive at sensible truth conditions. Although the formula in (92) does derive sensible truth conditions, the scopeless analysis achieves equivalent results. This observation is important since a common assumption in the previous literature is that non-definite partitives, to the extent that they can be analysed as bona-fide quantifiers necessarily take scope over the DP that hosts them (see e.g. Abbott (1996); Chen (2011)).

The way forward with the scopeless analysis is to reflect on how the exhaustive reading arises. This reading arises in (85-87) due to the fact that $\prec$ is transitive. If the first conjunct, $^{\bigstar}\mathsf{paper}(y)$, were missing, the resulting sentence would indicate the size of the entire domain since the σ operator could keep finding larger and larger pluralities until the domain of the entire model was exhausted. The first conjunct puts a break on this process of accumulation. The way forward then is to find a way to put the breaks on maximization before reaching the full set of papers and to instead stop maximization once every paper that some student or other read is included in the plurality.

Put this way, our goal is to identify some formula ϕ that added to the restrictor of n-of-k partitives that says that every atom was read by some student or other. The idea is to assign a translation like that in (94) to (87).

(94) $$[\forall z : \text{student}(z)]\\ [\sigma_{x,y} : \phi \wedge {}^{\bigstar}\text{paper}(y) \wedge x \prec y \wedge {}^{\bigstar}\text{read}(z,x)](\mathbf{n}.\mathbf{atoms}(x) \wedge \mathbf{k}.\mathbf{atoms}(y)),\\ \text{where } \phi = \delta_y[\exists u : \text{student}(u)]^{\bigstar}\text{read}(u,y)]$$

In (94) we've added a formula ϕ to put the brakes on the accumulation of papers for the inner (k) cardinal. The additional requirement says that every part of y must have been read by some student or other. So, if a paper wasn't read by any student it will not satisfy ϕ and will not be included in the final count. As an analysis, (94) has one glaring flaw: the extra restriction is cobbled together from other parts of the sentence. The predicate student appears twice: first as the restrictor of the universal and second as the restrictor of an existential quantifier inside the *n*-of-*k* partitive. Likewise, the predicate $^{\bigstar}$read appears twice, once to restrict maximization for the outer (n) cardinal, and again to apply to parts of the inner (k) cardinal.

A further complication arises when we consider sentences involving proportional quantifiers like *most*.

(95) Most students read one of three papers.

To handle (95) we will have to decide how to translate *most*. There seem to me to be two options. We could simply assume that *most Ps Qed* involves showing that $P \cap Q$ is at least half as large as P. A second options would be to treat most as an existential quantifier over plural individuals that picks out some plural individual consisting of at least half the Ps. I will take the second strategy because it seems easiest to implement in this system[5].

(96) $\text{most} \rightsquigarrow \lambda P_{et} \lambda Qet.[Mx : P(x)]\delta_x Q(x)$

(97) $[Mx : P(x)]Q(x) := [\exists x : P(x) \wedge [\sigma_y : P(y)]\frac{|x|}{|y|} \geq \frac{1}{2}]Q(x)$

With the translation of *most* we can see what happens when we try to translate (95).

(98) $[Mz : \text{student}(z)]\delta_z[\sigma_{x,y} : \phi \wedge {}^{\bigstar}\text{paper}(y) \wedge x \prec y \wedge {}^{\bigstar}\text{read}(z,x)](\mathbf{1}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$,

where $\phi = \delta_y[\exists u : \text{student}(u)]^{\bigstar}\text{read}(u,y)]$

The formula in (98) is actually too strong since it entails that only three papers were read by any student. This entailment is fine in the case of (94) since the partitive occurs

[5] By making this decision I am essentially treating *most* as denoting the same thing as *a majority*. Nothing here turns on this.

in the scope of a universal quantifier. In (98), however, only a majority of students are under consideration. The sentence in (95) does not entail that only three papers were read overall. Consider the sentence in (99).

(99) Most of the students read one of three papers, and most of the rest read a fourth.

The sentence in (99) is not consistent with the formula given in (98), which maximizes over the set of papers that any student read. In other words, (98) entails that only three papers were read between all the students. The sentence in (99) however is consistent with there being any number of papers that were read by some student or other. This suggests that ϕ is deeply sensitive to which quantifier appears above it.

2.4.3 Plural Logic with Structured Domains

In plural logic (PL) we retain the same model structure, but enrich the interpretive context. A formula in PL is evaluated with respect to a set G of total variable assignments. A set of variable assignments can be conceptualized as a matrix that encodes dependencies between the values a variable can take on. Consider the set of assignment functions depicted in (100).

(100)

...	x	y	...
...	Fido	Whiskers	...
...	Rex	Evander	...

The first row stores *Fido* in the x slot and *Whiskers* in the y slot, while the second row stores *Rex* in the x slot and *Evander* in the y slot. This assignment could be used to evaluate a formula like $\mathsf{chase}(x,y)$ in which case it would be true iff Fido chased Whiskers and Rex chased Evander.

An atomic formula expressing a lexical relation, ϕ, is true relative to a set of assignment functions$G$ iff ϕ is true relative to each $g \in G$:

(101) $[\![R(x_1,\ldots,x_n)]\!]^G = \mathbb{T}$ iff $\langle g(x_1),\ldots,g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

Normal lexical items are thus evaluated row-wise across the matrix of assignment functions. Notice that according to the definition above the empty set of assignment functions satisfies every lexical relation. This is because the size of G is inversely correlated with the information it contains; to say that a set of assignments G satisfies a formula ϕ is to say that at least the values stored in G satisfy the formula ϕ—the smaller G is the weaker this claim.

To handle quantification we need to generalize assignment update from single assignment functions to sets of assignment functions.

(102) $H[x]G := \forall_{g\in G} : \exists_{h\in H} : g[x]h \ \&\ \forall_{h\in H} : \exists_{g\in G} : h[x]x$

The definition above says that every assignment in G has some H-counterpart that differs from it at most with respect to the values assigned to x and likewise that every assignment in H has some G-counterpart that differs from it at most with respect to the values assigned to x. This ensures that every assignment in G is mapped to some assignment in H and every assignment in H is mapped to from some assignment in G. If we lose the first conjunct we would allow assignment updates in which some assignments in G 'go missing' in the transition to H, while losing the second conjunct would allow new assignments to arrive as we transition to H.

Universal quantification involves maximizing over potential updates:

(103) Universal Quantification:
$[\![[\forall x : \phi]\,\psi]\!]^G = \mathbb{T}$ iff $[\![\psi]\!]^H = \mathbb{T}$, for some H that is maximal relative to x, ϕ and G

(104) H is maximal relative to x, ϕ, and G iff

a. $H[x]G$ and $[\![\phi]\!]^H = \mathbb{T}$

b. there is no $K \supsetneq H$ s.t. $H'[x]G$ and $[\![\phi]\!]^K = \mathbb{T}$

A universal quantifier evaluates its restrictor with as large as set of assignments as could possibly satisfy its scope.

In order to fully implement the analysis we need to define a maximization operator that finds maximal plural individuals in a point-wise manner. The definition that accomplishes this is given in (105).

(105) $[\![[\sigma_x : \phi]\psi]\!]^G = \mathbb{T}$ iff there is some $H[x]G$ s.t.

a. $[\![\phi]\!]^H = \mathbb{T}$

b. there is no $K[x]G$ s.t. $[\![\phi]\!]^K = \mathbb{T}$ and $H \prec_x K$

c. $[\![\psi]\!]^H = \mathbb{T}$

(106) a. $H \blacktriangleleft_v K := H[x]K$ and for every $h \in H, k \in K$ if $h[x]k$, then $h(x) \preccurlyeq k(x)$

b. $H \preccurlyeq_v K := H = K$ or $H \blacktriangleleft_v K$

The definitions of $\blacktriangleleft_v$ and $\preccurlyeq_v$ require some comment: $\preccurlyeq_v$ is defined as the reflexive closure of $\blacktriangleleft_v$ because, $\blacktriangleleft_v$ is (i) transitive, (ii) anti-symmetric, but (iii) only sometimes reflexive[6]. In other words it does not quite define a partial order.

(107) $\blacktriangleleft_v$ is transitive.

Proof: Assume (i) $G \blacktriangleleft_v H$ and (ii) $H \blacktriangleleft_v K$. The first conjunct is trivial: From (i) infer $G[v]H$ and from (ii) $H[v]K$. So we have $G[v]K$. Let g, h, k be assignment functions s.t $g \in G$, $h \in H$ and $h[v]g$ and $k \in K$ and $k[v]h$. It follows immediately that $g[v]k$. From (i) infer $g(v) \preccurlyeq h(v)$ and from (ii) $h(v) \preccurlyeq k(v)$. Hence, $g(v) \preccurlyeq k(v)$. Since g, k are arbitrary we infer that for every $g \in G$, $k \in K$, if $k[v]g$, then $g(v) \preccurlyeq k(v)$. So, $G \blacktriangleleft_v K$.

(108) $\blacktriangleleft_v$ is antisymmetric.

[6]This is why we do not use $\blacktriangleleft$ in place of $\prec$ in (105b)—we might get in trouble when dealing with those assignment functions G s.t. $G \blacktriangleleft_v G$.

Proof: Assume (i) $G \blacktriangleleft_v H$ and (ii) $H \blacktriangleleft_v G$. Let g name an arbitrary assignment function s.t. $g \in G$. From (i) (or (ii)) we infer that $G[v]H$ and therefore that there exists a $h \in H$ s.t. $g[v]h$. From (i) infer $g(v) \preccurlyeq h(v)$ and from (ii) infer $h(v) \preccurlyeq g(v)$. We conclude $g(v) = h(v)$ and therefore $g \in H$. Since g is arbitrary conclude $G \subseteq H$. We can prove $H \subseteq G$ by identical reasoning, therefore $G = H$.

(109) $\blacktriangleleft_v$ is not always reflexive.

Proof: Let $G = \{g : g(x) = a\} \cup \{g : g(x) = a \oplus b\}$, for some a, b s.t. $a \prec a \oplus b$. We have $G \not\blacktriangleleft G$.

In order to understand how the maximization operator above works, it is extremely helpful to go through a few cases. My methodology here will be to display two sets of assignment functions and ask whether (a) one is larger than the other or (b) the two are incomparable[7]. This will allow us to develop an intuitive sense of how σ works and what it does.

Consider first the two sets of assignment functions below:

(110) $A =$

...	x	...
...	Evander	...

$B =$

...	x	...
...	Evander $\oplus$ Whiskers	...

The sets A and B have rows which are identical except that each row in A assigns x just Whiskers while each row in B assigns to x the plurality consisting of Evander and Whiskers. It is then the case that for every assignment in $a \in A$ and $b \in A$ s.t. $a[x]b$ it is the case that $b(x) \succcurlyeq a(x)$. So it follows that $B \blacktriangleright_v A$. It is also clear that $A \not\blacktriangleright_v B$.

[7]The existence of incomparable sets of assignment functions introduces the possibility that σ may not return a unique set of assignment functions for a given restrictor—it may deliver a set of assignment functions with point-wise maximal individuals and not the maximal set. In other words $\preccurlyeq_x$ may define a partial order over sets of assignment functions and any particular restrictor ϕ may or may not be compatible with a set of assignment functions that contains multiple maximal elements.

Next consider the pair of assignments below:

(111) $A =$

...	x	...
...	Evander	...
...	Evander ⊕ Whiskers	...

$B =$

...	x	...
...	Evander ⊕ Whiskers	...

Now for every row in B there are two rows in A which are identical except for the values assigned to x. For each row in $b \in B$ there is a row $a \in A$ s.t. a assigns Evander to x and a second row $a' \in A$ that assigns the plurality consisting of Evander and Whiskers to x. That means that when we consider pairs $a \in A$ and $b \in B$ s.t. $a[x]b$, we can find some in which we can find some in which $b(x) \prec a(x)$, namely when we consider an assignment from A that maps x to Evander and its (lone) counterpart in B that maps x to Evander and Whiskers. However, when we consider these pairs we will find that each of them satisfies $b(x) \preccurlyeq a(x)$ because when we start with an assignment from B, we will find that both of its counterparts in A store individuals that are no greater than the individual stored by b. This pair of assignment functions shows that σ delivers compact sets of assignment functions whenever possible. Note also that A is such that $A \not\prec A$.

Let's go through a few examples to see how pointwise maximization works:

(112) Rex chased (exactly) two cats. $\rightsquigarrow [\sigma_x : {}^{\bigstar}\mathsf{cat}(x) \wedge \mathsf{chase}(\mathsf{Rex}, x)]\mathbf{2.atoms}(x)$

...	...	...
...	...	...

$\xRightarrow{[\sigma_x : {}^{\bigstar}\mathsf{cat}(x) \wedge {}^{\bigstar}\mathsf{chase}(\mathsf{Rex}, x)]}$

...	x	...
...	Evander ⊕ Whiskers	...

$\xRightarrow{\mathbf{2.atoms}(x)} \mathbb{T}$

The σ quantifier delivers the largest plural individual consisting of cats that Rex chased. The updated set of assignment functions is then passed to to the nuclear scope where it is checked to make sure that each row contains a plurality of two. If we assume that Rex chased only Evander and Whiskers, the sentence as a whole will come out true.

Generalized maximization is handled just like maximization over a single variable:

(113) $[\![[\sigma_{v_1,\ldots,v_n} : \phi]\psi]\!]^G = \mathbb{T}$ iff there is some $H[x]G$ s.t.

a. $[\![\phi]\!]^H = \mathbb{T}$

b. there is no $K[x]G$ s.t. $[\![\phi]\!]^H = \mathbb{T}$ and $H \prec_{v_1,\ldots,v_n} K$

c. $[\![\psi]\!]^H = \mathbb{T}$

(114) a. $H \blacktriangleleft_{v_1,\ldots,v_n} K := H[v_1,\ldots,v_n]K$ & $\forall_{h\in H} : \forall_{k\in K} :$ if $h[v_1,\ldots,v_n]k$ then $h(v_1) \preccurlyeq k(v_n)$ & $,\ldots,$ & $h(v_n) \preccurlyeq k(v_n)$

b. $H \preccurlyeq_{v_1,\ldots,v_n} K := H = K$ or $H \blacktriangleleft_{v_1,\ldots,v_n} K$

Proofs that $\blacktriangleleft_{v_1,\ldots,v_n}$ has the same relation properties as $\blacktriangleleft_v$ go through identically, since $h(v_1) \preccurlyeq k(v_n)$ & $,\ldots,$ & $h(v_n) \preccurlyeq k(v_n)$ is a transitive and anti-symmetric relation over single assignment functions.

2.4.4 Bare Cardinal Partitives in Plural Logic

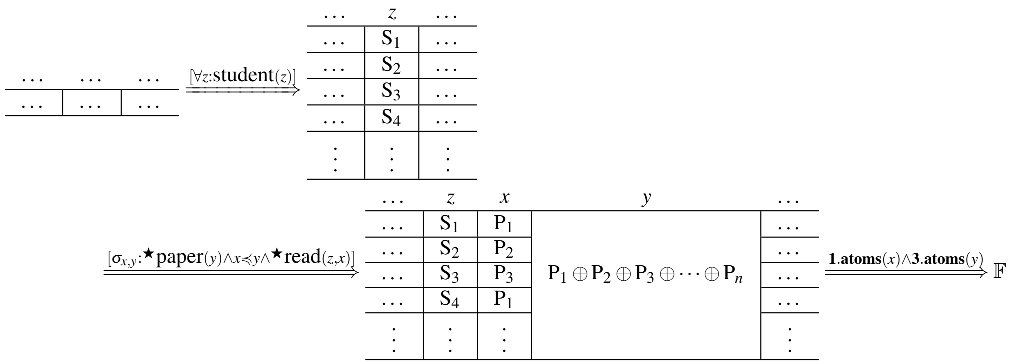
With generalized maximization in our new logic we can provide a full analysis of bare cardinal partitives. I will first show that our previous analysis for exhaustive interpretations still works in this new setting. Next I will show how the interpretation of the partitive relation can be enriched so that it places appropriate breaks on the maximization operation so that the inner cardinal picks out the maximal set of papers that some student or other read. Finally, I will extend the analysis to partial ignorance readings, arguing that these readings are a species of cumulative readings in which cumulation occurs over the set of epistemicly accessible worlds.

2.4.4.1 Exhaustive Readings

Exhaustive readings are captured by our familiar mechanisms. Consider the sentence below, in which we have interpreted the partitive in terms of the simple part-whole relation. If we imagine a situation in which there are more than three papers, the formula below comes out false.

(115) Every student read one of three papers.

$$\rightsquigarrow [\forall z : \text{student}(z)][\sigma_{x,y} : {}^{\star}\text{paper}(y) \wedge x \lhd y \wedge {}^{\star}\text{read}(z,x)] (\mathbf{1}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$$



To see why this result obtains, consider the formula that is being maximized over: $^\bigstar\text{paper}(y) \wedge x \preccurlyeq y \wedge {}^\bigstar\text{read}(z,x)$. This formula will be true of a set of assignment functions just in case the following constraints are jointly satisfied: (i) y stores a plurality of papers, (ii) z stores a plurality that x read, and (iii) x is pointwise smaller than y. Like we noted above, the cumulation of y-papers can go on until the domain of papers is exhausted. Notice also that the plurality of y-papers is the same for every value of x and z. This result arises organically: at each point the cumulation can continue until the domain of papers is reached because y is related to each z-student only indirectly—since y must contain every paper y must contain every paper z read.

2.4.4.2 Cumulative Readings

With our new logical resources it becomes possible to define an enrichment to the part-of relation that allows us to formalize cumulative readings as well.

We do so by defining the relation $\lhd$:

(116) $[\![x \lhd y]\!]^G = \mathbb{T}$ iff

a. $[\![x \preccurlyeq y]\!]^G = \mathbb{T}$

b. $\bigoplus G(x) = g(y)$, for all $g \in G$

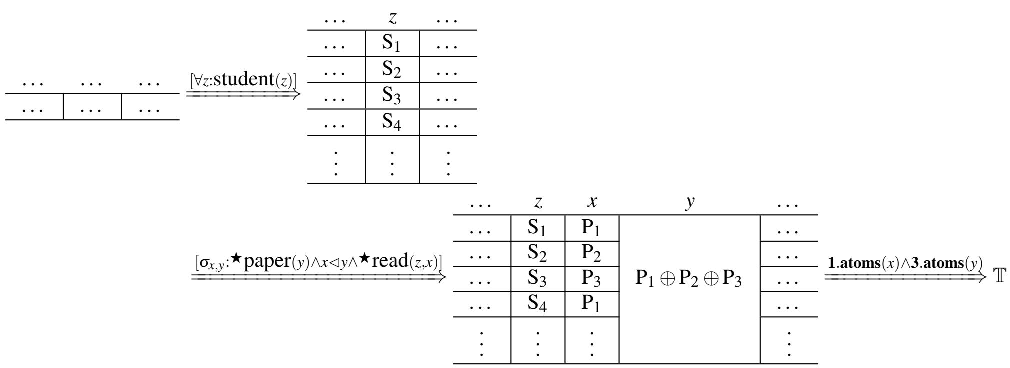
The relation $\lhd$ has two requirements. The first clause contains a condition that can be evaluated point-wise. It says that x stores an atomic part of y. The second clause contains a condition that relates the individuals stored in x and y globally. It says that

y stores the value that would be obtained by summing every individual stored as the value of x anywhere. The function of $\lhd$ is to make available both a single value x and the plural individual consisting of the values of x—y is the ontologically plural entity that corresponds to x's informational plurality.

We can see how this would work below:

(117) Every student read one of three papers.

$$\rightsquigarrow [\forall z : \text{student}(z)][\sigma_{x,y} : {}^{\bigstar}\text{paper}(y) \wedge x \lhd y \wedge {}^{\bigstar}\text{read}(z,x)]$$
$$(\mathbf{1}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$$



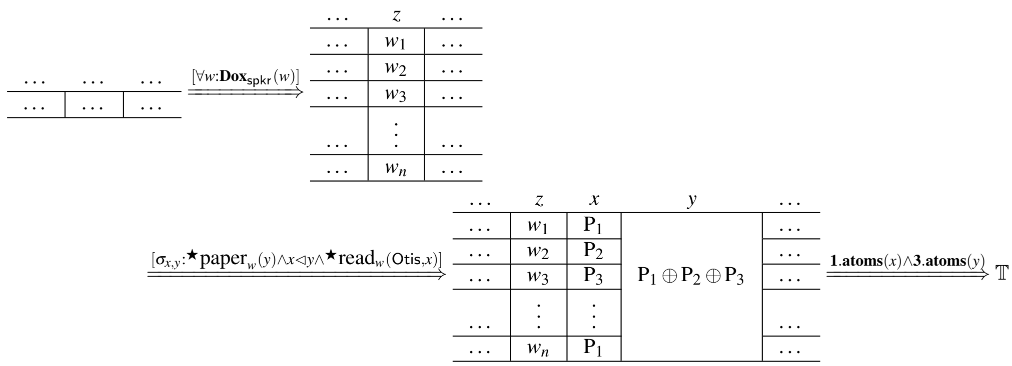
When we maximize over x and y simultaneously we capture the right reading. The restrictor here ensures that both x and y are pluralities of papers since we require that ${}^{\bigstar}\mathsf{paper}(y)$ and since $x \lhd y$ entails that $x \preccurlyeq y$. Likewise, we ensure that x consists of papers that the local student read since we have ${}^{\bigstar}\mathsf{read}(z,x)$. Moreover, we capture the cumulative interpretation since $x \lhd y$ entails that the y stores the value of that is the sum of the values that x can take. As a result, y stores the sum of papers that were read by some student or other. The $\lhd$ relation puts the breaks on cumulation for y—now y can only be a sum of values that appear in x somewhere.

2.4.4.3 Partial Ignorance Readings

To handle partial ignorance readings I assume that sentences are interpreted relative to a speakers epistemically accessible states. An utterance of ϕ is interpreted as $[\forall w : \mathbf{Dox}_{\mathsf{spkr}}(w)]\phi(w)$. Cashed out this way, partial ignorance readings are simply cumulative readings interacting not with an overt quantificational expression but with an implicit and perhaps pragmatically inferred epistemic universal:

(118) Otis read one of three papers.

$$\rightsquigarrow [\forall w : \mathbf{Dox}_{\mathsf{spkr}}(w)][\sigma_{x,y} : {}^{\bigstar}\text{paper}_w(y) \wedge x \lhd y \wedge {}^{\bigstar}\text{read}_w(\mathsf{Otis}, x)] (\mathbf{1}.\mathbf{atoms}(x) \wedge \mathbf{3}.\mathbf{atoms}(y))$$



As we can see the particular paper that Otis read varies from world to world as we examine the worlds epistemically accessible to the speaker. However, when we look across all these worlds we see the same three papers arise over and over. Thus the speaker communicates not only that Otis read one paper in every world consistent with their beliefs but also that they only consider three papers to be papers that Otis might have read.

2.5 Conclusions

This chapter provided new data examining bare cardinal partitives. I identified and distinguished several readings. I also argued that standard first order logic could not capture cumulative interpretations of bare cardinal partitives, and provided a formal analysis of bare cardinal partitives in a simple plural logic that included both atomic and non-atomic entities. I argued that cumulative readings arise when the part-of relation is enriched so that it forces identity between the ontological plurality provided by the inner cardinal and the informational plurality associated with the outer cardinal.

One interesting aspect of my analysis that I have not commented upon is the 'scopeless' nature of the inner and outer cardinal. While this may at first seem exotic, my analysis bares a certain similarity to other definite descriptions.

Imagine a situation in which there are two hats and two rabbits: a rabbit in a hat (and thus a hat with a rabbit), and a rabbit not in a hat (and thus a hat without a rabbit). In such a situation it seems felicitous to utter either of the following sentences:

(119) a. The rabbit in the hat is eating grass.
b. The hat with the rabbit in it is big.

The felicity of these sentences is somewhat unexpected because the scenario as described contains neither a unique or salient hat nor a unique or salient rabbit. Instead there is a unique rabbit that is in a hat and a unique hat that has a rabbit in it. Only when the entire description is taken into account can the participants be jointly and uniquely identified.

Bumford (2016) provides a split-scope analysis of sentences like these by decomposing definiteness into (i) an existential component and (ii) a maximization component. The existential component delivers a meaning like that below:

(120) $\{\langle x, y\rangle : \mathsf{rabbit}(x) \wedge \mathsf{hat}(y) \wedge \mathsf{in}(x, y)\}$

The maximization component then indicates that the x and y satisfying the description are both unique. The two definite thus act as if both had wide scope over the entire description. Scoplessness thus may not be exotic at all.

Appendix A

Technical Appendix

A.1 Notational Conventions

(121) ASSIGNMENT UPDATE:
$H[x]G := \forall_{g \in G} : \exists_{h \in H} : g[x]h \;\&\; \forall_{h \in H} : \exists_{g \in G} : h[x]x$

(122) MEASURING ONTOLOGICAL PLURALITIES:

a. $H \blacktriangleleft_{v_1,\ldots,v_n} K := H[v_1,\ldots,v_n]K \;\&\; \forall_{h \in H} : \forall_{k \in K} :$ if $h[v_1,\ldots,v_n]k$ then $h(v_1) \preccurlyeq k(v_n) \;\&\; ,\ldots,\; \&\; h(v_n) \preccurlyeq k(v_n)$

b. $H \preccurlyeq_{v_1,\ldots,v_n} K := H = K$ or $H \blacktriangleleft_{v_1,\ldots,v_n} K$

A.2 Plural Logic with Structured Domains

(123) LEXICAL RELATIONS:
$[\![R(x_1,\ldots,x_n)]\!]^G = \mathbb{T}$ iff $\langle g(x_1),\ldots,g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

(124) UNIVERSAL QUANTIFICATION:
$[\![[\forall x : \phi]\,\psi]\!]^G = \mathbb{T}$ iff $[\![\psi]\!]^H = \mathbb{T}$, for some H that is maximal relative to x, ϕ and G

(125) H is maximal relative to x, ϕ, and G iff

a. $H[x]G$ and $[\![\phi]\!]^H = \mathbb{T}$

b. there is no $K \supsetneq H$ s.t. $H'[x]G$ and $[\![\phi]\!]^K = \mathbb{T}$

(126) POINT-WISE MAXIMIZATION:

$[\![[\sigma_x : \phi]\psi]\!]^G = \mathbb{T}$ iff there is some $H[x]G$ s.t.

a. $[\![\phi]\!]^H = \mathbb{T}$

b. there is no $K[x]G$ s.t. $[\![\phi]\!]^K = \mathbb{T}$ and $H \prec_x K$

c. $[\![\psi]\!]^H = \mathbb{T}$

Chapter 3

First Order Logic With Choice

3.1 Introduction

The previous chapter offered a novel empirical argument for interpreting formulas relative to sets of assignment functions. This chapter takes a more in depth look at this idea focusing on another domain, the interpretation of indefinites, in which these tools have been put to use. I will be focusing in particular on the interpretation of wide scope indefinites. There are two basic claims from the literature that I will take as read in this chapter: first, indefinites can take arbitrarily high scope even out of scope islands, and, second, the scope of indefinites is constrained by the appearance of a bound pronoun inside their restrictors—an indefinite cannot take scope beyond a quantifier that binds into its restrictor.

The first claim is illustrated by the sentences given below in (127). In this sentence the indefinite *a (certain) dock* can take wide scope over the universal *every boat*.

(127) Every boat that was launched from a (certain) dock did well in the race.
(128) A boat that was launched from every dock did well in the race.

Notice that if the indefinite were replaced with the universal *every dock* a wide scope reading is impossible despite the fact it is implausible that any boat be launched from every dock and world knowledge thus militates against a narrow scope reading of the universal.

The second claim is illustrated by the sentence given below in (129). In this sentence the indefinite *a problem* contains a pronoun that is bound by the quantifier *every student*. Notice that this sentence does not have an interpretation in which the indefinite takes scope over the universal.

(129) Every^x student that picked a problem that interested her$_x$ wrote a good paper.

Brasoveanu and Farkas (2011) dub the constraint illustrated by (129) the binder roof constraint (sometimes called the integrity condition (see e.g. Schwarz (2001); Chierchia (2001))) because the binder *every student* limits the upward scope of the indefinite.

In this chapter I will primarily be interested in the claim advanced in Brasoveanu and Farkas (2011) that exceptional scope can be reduced to independence between the values different variables can take on. Their claim, in brief, is that wide scope indefinites occur when an indefinite is unable to vary with respect to the value of variable introduced by another quantifier. Non-variation amounts to wide scope. Since this mechanism is available without recourse to syntactic movement the ability of indefinites to take scope outside of scope islands is thereby explained.[1]

In this chapter I offer a critique, not of the intuition underling Brasoveanu and Farkas (2011), but of their formal implementation. I will show that their logic delivers sentences that are systematically stronger than their natural language counterparts. The sentences that give them trouble are those in (130) in which a wide scope indefinite occurs inside the restrictor of a universal quantifier.

[1]Note that this claim is different from claims advanced in the late 70s that wide scope interpretations of indefinites could be reduced to narrow scope interpretations + vagueness about whether there was accidental variation between the entities involved (see e.g. Reinhart (1976, 1979); Cooper (1979)).

(130) Every dog that chased a certain cat barked.

The semantics Brasoveanu and Farkas assign to sentences like (130) works by first finding a maximal group of dogs for which an independent choice of cat can be made and saying that each of the dogs barked. The trouble for this account arises when we consider a situation in which two dogs, Fido and Rex, that chased a cat, Whiskers, and barked and three dogs, Fido, Rex, and Dudley chased a cat, Socks, and did not (all) bark. In such a scenario the largest set of dogs for which an independent choice of cat can be made is Fido, Rex, and Dudley because they all chased Socks. I will show that in their semantics Whiskers cannot act as a witness for the indefinite in (130).

The problem presented by the example in (130) is more formal than conceptual. In the static logic in which Brasoveanu and Farkas present their idea, universal quantifiers do not have access to information about how discourse referents will be assigned by indefinites inside their scope. I will show that a dynamic implementation of their logic does capture the correct truth conditions. The surprising lesson is that a full implementation of Brasoveanu and Farkas (2011) requires a dynamic logic.

I end the chapter on a critical note: the straightforward re-implementation of their logic in a dynamic system gets the truth conditions of single sentences right but is ill suited to handle discourse dynamics. It does not admit of a coherent theory of structured singular and plural discourse reference.

In the following chapter I will explore the range of systems that can handle plural discourse reference and settle on a new system Dynamic Plural Logic with Unselective Maximization that can handle a broad range of phenomena while offering the raw ingredients needed re-implement the basic ideas present in Brasoveanu and Farkas (2011).

3.2 Wide Scope Indefinites & The Binder Roof Constraint

It has been long noted that indefinites are not subject to the same constraints on upward scope as other quantifiers (see e.g. Farkas (1981), Fodor & Sag (1982), Abusch (1994), a.o.). This contrast is exemplified by the sentences in (131).

(131) a. John read a^x paper that every^y professor recommended. Brasoveanu and Farkas (2011)
b. John read every^x paper that a^y professor recommended.

In (131a) the quantifier *every* occurs in a relative clause attached to *a paper*. Moreover, this sentence does not have a reading in which papers can co-vary with professors, i.e. (131a) does not have an interpretation in which for every professor there is some paper that they recommended which John read. This leads to the conclusion that the usual mechanisms responsible for inverse scope will not deliver an inverse scope reading for (131). However, in (131b) has an interpretation in which the indefinite, *a professor* takes scope outside of the relative clause over *every paper*, i.e. the sentence in (131b) has an interpretation in which there is one professor s.t. John read every paper that professor recommended—he may or may not have read any papers that were only recommended by other professors.

In response to data like that in (131), Fodor and Sag (1982) analyse indefinites as being potentially referential. The reading of (131a) in which the indefinite takes wide scope is analysed in terms of a referential interpretation of the indefinite.

Others, notably Farkas (1981), have argued that indefinites are always quantificational and thus that wide scope readings of sentences like (131) are not the result of referential interpretations. The strongest argument for this position comes form the existence of intermediate readings.

(132) Every student^x read every paper^y that a^z professor recommended. Brasoveanu and Farkas (2011)

a. Narrow Scope (NS):
for every student x,
for every paper y s.t.
there is a professor z that recommended y,
x read y.

b. Intermediate Scope (IS):
for every student x,
there is a professor z
for every paper y s.t. that z recommended y,
x read y.

c. Widest Scope (WS):
there is a professor z
for every student x,
for every paper y s.t. that z recommended y,
x read y.

The sentence in (132) has three readings, a narrow scope reading provided in (132a) in which the indefinite takes narrow scope with respect to both universal quantifiers, an intermediate scope reading provided in (132b) in which the indefinite takes scope over *every paper* but occurs in the scope of *every student*, and a wide scope reading provided in (132c) in which the indefinite takes scope over both quantifiers. The existence of the intermediate scope reading suggests that the indefinite can take scope beyond the relative clause without simply taking on a referential interpretation. The contrast initially given between (131a-b) thus supports broad theoretical architecture in which there are at least two scope taking mechanisms. One that handles *bona fide* quantifiers like *every* and another that handles indefinites like *a*. Notice we can be quite agnostic about what the 'usual' mechanisms responsible for inverse scope are. They could be relatively standard mechanisms like QR or alternative mechanisms involving continuations, type-shifting, cooper storage, etc. The important point here is that something prevents *every* from taking scope over the indefinite in (131a) but no such constraints are placed on the indefinite in (131b).

At the same time indefinites are not completely unconstrained in their ability to take wide scope (see Ruys (1992), Abusch (1994), Chierchia (2001), Schwarz (2001), a.o.). An indefinite cannot take wide scope over an quantifier that binds into its restrictor as the pair in (133) shows.

(133) a. Every^x student read every^y paper that one^z professor recommended.
b. Every^x student read every^y paper that one^z of his$_x$ favourite authors recommended.

The sentence in (133) has the familiar NS, IS, and WS readings; *one professor* can take scope over both universals, only *every paper* or neither universal. However, the sentence in (133b) has only the IS and NS readings; *one of his favourite authors* cannot take scope over *every student*. The only difference between these two is the presence of the bound pronoun in the restrictor of the indefinite. The fact that *every student* binds into the restrictor of *one of his favourite authors* places an upper bound on how high the indefinite can scope. Brasoveanu and Farkas dub this the binder roof constraint.

One goal of this chapter is to lay out how Brasoveanu and Farkas account for (i) the ability of indefinites to take upward scope in a way that is not usually available to quantificational expressions and (ii) the binder roof constraint. I will advance the idea that their intuitions are fundamentally sound, but that a fully adequate formal implementation of their analysis requires a dynamic logic.

3.3 From FOL to C-FOL

This section lays out the background necessary for the remainder of the chapter. First, I introduce first order models and partial assignment functions and quickly review first order interpretation relative to single assignment functions. Next, I introduce plural information states (sets of partial assignment functions in which every element is defined for the same set of variables) and show that they can be viewed as a matrix or database like that depicted below:

...	x	y	...
...	Henrik	Sybil	...
...	Albert	Kashif	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$

I then define the notions of PROJECTION (a way of extracting a vertical slice from a database) and SUB-STATE (a way of shrinking a database horizontally). Using these notions I show that notions of DEPENDENCE and INDEPENDENCE can be defined.

I close this section with a discussion of the conceptual relationship between plural information states and run-of-the-mill first order interpretation. In first order logic the interpretation of a universally quantified formula involves 'visiting' many different assignment functions. Each row in a plural information state can be viewed as one of these assignments. Seen this way plural information states represent a 'God's eye of view' of first order interpretation.

3.3.1 First order models

The models we work with in this section are familiar from FOL. A model is a pair $\langle\mathfrak{D},\mathfrak{I}\rangle$ consisting of a domain, $\mathfrak{D}$ of individuals and an interpretation function $\mathfrak{I}$ that determines the meaning of lexical expressions in the language; each n-ary predicate P is associated with a subset of $\mathfrak{D}^n$:

(134) $\mathfrak{I}(P^n) \subseteq \mathfrak{D}^n$, e.g.

a. $\mathfrak{I}(\mathsf{dog}) = \{\mathsf{fido},\mathsf{rex},\mathsf{spot}\}$

b. $\mathfrak{I}(\mathsf{chase}) = \{\langle\mathsf{fido},\mathsf{rex}\rangle,\langle\mathsf{rex},\mathsf{spot}\rangle\}$

3.3.2 Partial assignment functions

A partial assignment function is a function between a subset of the variables in a language $\mathfrak{L}$ and and the domain of the model. We can think of the variables in the domain of a partial assignment function as a set of discourse referents. As a formula is processed discourse referents can be changed in controlled ways. To manage assignment updates we utilize the notation $h[x]g$ defined below:

(135) $h[x]g :=$ h differs from g at most with respect to the value (if any) associated with x

As defined in (135) assignment difference does not care about whether x is defined in g or h or both: x may or may not be in the domain of g, x may or may not be in the domain of h, and if x is in the domain of g and h it may or may not be assigned the same value. The condition $h[x]g$ is really about other variables; it holds just in case (i) every variable besides x is either defined in both g and h or defined in neither g nor h and (ii) every variable besides x that is defined in both g and h is assigned the same value by both g and h. The relation $h[x]g$ is transitive, reflexive, and symmetric.

When working with partial assignment functions a useful notion is assignment function extension, which is defined in (136).

(136) $h \geq g := \forall_v : (v \notin \mathbf{Dom}(g) \text{ or } h(v) = g(v))$

The definition in (136) says that an assignment h is an extension of g just in case, for every variable in the language either g is undefined for v or g and h assign v the same value.

With assignment extension defined we can define guarded non-destructive update of an assignment function with respect to a single variable.

(137) $h \overset{x}{\Leftarrow} g := x \notin \mathbf{Dom}(g)\ \&\ h \geq g\ \&\ h[x]g$

An assignment function h is an x-extension of g just in case (i) x is undefined in g, (ii) h is an non-trivial extension of g and (ii) h and g differ at most with respect to the variable x. The $\Leftarrow_x$ relation represents the growth of the discourse context—x is not tracked in g but is in h.

3.3.3 First order interpretation

In static first order logic all expressions are interpreted with the help of an assignment function; an assignment function is (i) used to interpret variables, (ii) incremented by quantifiers, and (iii) passed down unchanged to conjuncts.

(138) Some expressions of FOL:

a. $[\![R(v_1,\ldots,v_n)]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff $\langle g(v_1),\ldots,g(v_n)\rangle \in \mathfrak{I}(R)$

b. i. $[\![[\forall v : \phi]\,\psi]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff for every $h \Leftarrow_v g$ if $[\![\phi]\!]^{h,\mathfrak{M}} = \mathbb{T}$, then $[\![\psi]\!]^{h,\mathfrak{M}} = \mathbb{T}$

ii. $[\![[\exists v : \phi]\,\psi]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff there is some $h \Leftarrow_v g$, s.t. $[\![\phi]\!]^{h,\mathfrak{M}} = \mathbb{T}$ and $[\![\psi]\!]^{h,\mathfrak{M}} = \mathbb{T}$

c. $[\![\phi \wedge \psi]\!]^{g,\mathfrak{M}} = \mathbb{T}$ iff $[\![\phi]\!]^{g,\mathfrak{M}} = \mathbb{T}$ and $[\![\psi]\!]^{g,\mathfrak{M}} = \mathbb{T}$

In the logic, truth for a formula is defined relative to the empty assignment function:

(139) $[\![\phi]\!]^{\mathfrak{M}} = \mathbb{T}$ iff $[\![\phi]\!]^{\emptyset,\mathfrak{M}} = \mathbb{T}$

When we consider the interpretation of natural language we will occasionally want to consider the interpretation of a formula relative to a partial assignment function in which the values of some variables are already defined. These values are a partial representation of the context of conversation: they indicate which entities may be referred to by means of pronouns.

3.3.4 Sets of assignment functions

Formulas in C-FOL are interpreted relative to sets of partial assignment functions which I will refer to as a plural information state.

A set of partial assignment functions might take the following form:

(140) $G = \{g_1, g_2\} = \{\{\langle x, \mathsf{a}\rangle, \langle y, \mathsf{d}\rangle\}, \{\langle x, \mathsf{b}\rangle, \langle y, \mathsf{e}\rangle\}\}$

This set of assignment functions can be perspicuously depicted as a matrix. The set in (140) can be unpacked as follows:

(141) $G =$

	x	y
g_1	a	d
g_2	b	e

The rows of the matrix in (141) correspond to the assignment functions in G, while the columns correspond to the values associated with each variable. For instance x is associated with $\{\mathsf{a}, \mathsf{b}\}$, while y is associated with $\{\mathsf{d}, \mathsf{e}\}$. In addition to representing the set associated with a variable, the matrix representation in (141) reveals that plural information states represent dependencies between variables; the value associated with y in any given row varies depending on the value of x in the same row.

When working with sets of partial assignment functions we need to confront the possibility that different variable assignments in a set might be defined for different variables. Since allowing this possibility would only complicate matters, I assume that all sets of variable assignments we work with contain no elements which are defined over different sets of variables. The condition in (142) holds throughout my discussion.

(142) $\forall_{g,g' \in G} : \mathbf{Dom}(g) = \mathbf{Dom}(g')$

Working with partial assignment functions it is also helpful to designate the assignment function for which no values are defined. This assignment function is the empty-set when we consider single assignment functions. When we consider sets of assignment functions the intial state must be the singleton set containing the empty-set. Since this is cumbersome to write I designate the initial state as **0**:

(143) $\mathbf{0} := \{\emptyset\}$, i.e. the set containing the assignment function with an empty domain.

3.3.5 Projections and Sub-states

Since we only allow sets of assignment functions that have the same domain, we can identify the domain of the entire state as the domain of any particular component.

(144) DOMAIN of G:
$\mathbf{Dom}(G) := \{v : v \in \mathbf{Dom}(g)\}$, where g is any member of G.
'The names of the columns'

To recover the plurality associated with an plural information state, we collect all the values associated with a variable x in any row. The set of values associated with a variable x in a plural information state G is called the PROJECTION of x in G. This is defined below:

(145) PROJECTION (for single variables):
$G(x) := \{g(x) : g \in G\}$, if $x \in \mathbf{Dom}(G)$ and $\emptyset$ otherwise.
'The values found in the x column'

The projection of a variable x recovers a horizontal slice of the matrix. Notice that if x is not in the domain of G, the projection of x in G is the empty set. Projections can be generalized to include multiple variables:

(146) PROJECTION (for sequences variables):

$$G(v_1, ..., v_n) := \begin{cases} \{\langle g(v_1), \ldots, g(v_n) \rangle : g \in G\}, \\ \qquad\qquad \text{if } v_1 \in \mathbf{Dom}(G),\ \&\ \ldots,\ \&\ v_n \in \mathbf{Dom}(G) \\ \\ \emptyset \text{ otherwise.} \end{cases}$$

The projection of two variables, say x and y, is a relation between the values of x and the values of y, and provides us with a way of talking about the dependencies between x and y. If $G(x,y) = G(x) \times G(y)$, then there is no special association between the values of x and the values of y; every y value is related to every x value.

Another concept that is useful is that of a SUB-STATE. A sub-state of a plural information state G is just a subset of G that has a fixed (possibly singleton) set of values for some variable x. To pick out such sub-states we adopt the following notation:

(147) SUB-STATES:

$$\begin{array}{lcl} G|_{x=\mathsf{d}} & := & \{g : g \in G \;\&\; g(x) = \mathsf{d}\} \\ G|_{x\in D} & := & \{g : g \in G \;\&\; g(x) \in D\} \\ G|_{x\notin D} & := & \{g : g \in G \;\&\; g(x) \notin D\} \end{array}$$

We have three ways of picking out sub-states; the first picks out the sub-state of G in which x has a single value, the second picks out a sub-state where x has any one of a set of values, and the third picks out a sub-state in which x takes a value outside of some domain. Notice that these sub-states will return the empty set if x is not defined in G. The notation in (147) will play a role in the definitions of dependence and the distributive operator.

Several sub-states are illustrated below:

(148)
$$G = \begin{array}{cc} x & y \\ \hline \mathsf{a} & \mathsf{d} \\ \mathsf{b} & \mathsf{e} \\ \mathsf{c} & \mathsf{f} \end{array} \quad G|_{x=a} = \begin{array}{cc} x & y \\ \hline \mathsf{a} & \mathsf{d} \end{array} \quad G|_{x\in\{a,b\}} = \begin{array}{cc} x & y \\ \hline \mathsf{a} & \mathsf{d} \\ \mathsf{b} & \mathsf{e} \end{array} \quad G|_{x\notin\{a,b\}} = \begin{array}{cc} x & y \\ \hline \mathsf{c} & \mathsf{f} \end{array}$$

3.3.6 Dependence and Independence

A notion of DEPENDENCE can be defined in either in terms of sub-states or in terms of the projection of multiple variables: In terms of sub-states variable y is said to depend on a variable x in a state G iff there are sub-states $G|_{x=\mathsf{d}}$, $G|_{x=\mathsf{d}'}$ that disagree with respect to the values assigned to y. This is formalised below:

(149) DEPENDENCE (sub-state version):
y depends on x in G iff $\exists_{\mathsf{d},\mathsf{d}'\in G(x)} : G|_{x=\mathsf{d}}(y) \neq G|_{x=\mathsf{d}'}(y)$

In terms of the projection of two values. A variable x is said to depend on a variable x in a state G iff the projection of the two variables is a strict subset of the cross-product of the projections of the variables individually.

(150) DEPENDENCE (projection version):
y depends on x in G iff $G(x,y) \subsetneq G(x) \times G(y)$

Notice that these two definitions are equivalent. If $G(x,y) = G(x) \times G(y)$, then for every $d \in G(x)$ the sub-state $G|_{x=d}$ will contain every value associated with y in G. Likewise, if x is associated with the same values of y in every sub-state in which x takes on a particular value, then $G(x,y) = G(x) \times G(y)$.

Let's take two examples. The first, given in (151), shows a plural state H in which y depends on x. This can be verified by noting that $H|_{x=\mathsf{a}}(y) = \{\mathsf{c}, \mathsf{d}\} \neq H|_{x=\mathsf{b}}(y) = \{\mathsf{c}, \mathsf{e}\}$.

(151)

$$H = \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{a} & \mathsf{c} \\ & \mathsf{d} \\ \hline \mathsf{b} & \mathsf{c} \\ & \mathsf{e} \\ \hline \end{array} \qquad H|_{x=\mathsf{a}} = \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{a} & \mathsf{c} \\ & \mathsf{d} \\ \hline \end{array} \qquad H|_{x=\mathsf{b}} = \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{b} & \mathsf{c} \\ & \mathsf{e} \\ \hline \end{array}$$

$$H|_{x=\mathsf{a}}(y) = \{\mathsf{c}, \mathsf{d}\} \qquad H|_{x=\mathsf{b}}(y) = \{\mathsf{c}, \mathsf{e}\}$$

$$H(x,y) = \{\langle \mathsf{a}, \mathsf{c} \rangle, \langle \mathsf{a}, \mathsf{d} \rangle, \langle \mathsf{b}, \mathsf{c} \rangle, \langle \mathsf{bd} \rangle\} \subset \{\mathsf{a}, \mathsf{b}\} \times \{\mathsf{c}, \mathsf{d}\} = H(x) \times H(y)$$

The second example, given in (152), shows a plural state in which y does not depend on x.

(152)

$$\begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{a} & \mathsf{c} \\ & \mathsf{d} \\ \hline \mathsf{b} & \mathsf{c} \\ & \mathsf{d} \\ \hline \end{array}$$

Notice that here we can find no sub-states in which different values of x are associated with different values of y.

3.4 Plural information states provide a God's eye view of first order interpretation

It is not uncommon when teaching a introductory logic or semantics course to help students get a feel for the operation of assignment functions by depicting the evaluation of a formula by means of a tree:

$$[\forall x:\phi]\psi \quad\quad g$$
$$\psi \quad\quad h_0 \;\ldots\; h_n \quad \text{where } \{h_0,\ldots h_n\} = \{h : h \overset{x}{\Leftarrow} g \;\&\; [\![\phi]\!]^h = \mathbb{T}\}$$

The top formula is true relative to the assignment function on its tier iff the formula below it is satisfied by every assignment function on its tier. In the above example the formula $[\forall x:\phi]\psi$ is satisfied by g iff ψ is satisfied by each of $h_0,\ldots,h_n$.

Quantifiers also impose requirements on the general shape of the tree. An existential quantifier would simply generate a single daughter:

(153)
$$[\exists x:\phi]\psi \quad\quad g$$
$$\psi \quad\quad h \quad \text{where } h[x]g \;\&\; [\![\phi]\!]^h = \mathbb{T}$$

The effects of an existential in the scope of a universal quantifier generate variation:

(154)
$$[\forall x:\phi][\exists y:\psi]\theta \quad\quad g$$
$$[\exists y:\psi]\theta \quad\quad h_0 \;\ldots\; h_n \quad \text{where } \{h_0,\ldots h_n\} = \{h : h \overset{x}{\Leftarrow} g \;\&\; [\![\phi]\!]^h = \mathbb{T}\}$$
$$\theta \quad\quad k_0 \;\ldots\; k_n \quad \text{where } k_i \overset{x}{\Leftarrow} h_i \;\&\; [\![\psi]\!]^{k_i} = \mathbb{T}$$

There is no guarantee that $k_0,\ldots k_n$ assign the same value to y. The choice is entirely free and can change depending on the values of x assigned to the assignment functions $h_0,\ldots,h_n$. Of course there is likewise no requirement that $k_0,\ldots,k_n$ assign different values to y—they may all happen to assign the same value to y.

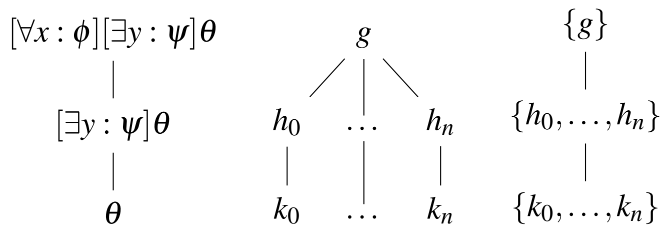
If an existential takes scope outside a universal, there is no chance for variation since the existential requires only one changed variable assignment.

(155)
$$[\exists y:\psi][\forall x:\phi]\theta \quad\quad g$$
$$[\forall x:\phi]\theta \quad\quad h \quad \text{where } h[y]g \;\&\; [\![\psi]\!]^h = \mathbb{T}$$
$$\theta \quad\quad k_0 \;\ldots\; k_n \quad \text{where } \{k_0,\ldots k_n\} = \{k : k[x]h \;\&\; [\![\phi]\!]^k = \mathbb{T}\}$$

The intuition underlying Brasoveanu and Farkas (2011) is that wide scope and dependent indefinites are able to signal how the value of their witness depends or does not depend on the values that are taken by other variables, i.e. that indefinites can talk about the relationship between $k_0, \ldots, k_n$. This idea cannot be expressed in FOL, since formulas are evaluated only with respect to single assignment functions—there is no way to talk about the other assignment functions that might be looked at when calculating the truth conditions of a sentence. Instead Brasoveanu and Farkas need a logic that can access global aspects of the computation.

The key insight leading toward C-FOL is that each tier of the trees above can be represented as a set of assignment functions.

(156)



By enriching the context of evaluation so that it includes sets of assignment functions instead of single assignment functions, formulas can be made aware of the entire computation of the truth conditions.

3.5 Exceptional Scope in C-FOL

3.5.1 Preliminaries

In the logic presented in Brasoveanu and Farkas (2011), formulas are evaluated with respect to pairs $\langle G, \mathscr{V} \rangle$ consisting of a set of total assignment functions G and a set of 'live variables' $\mathscr{V}$. Tracking the set of live variables is necessary because Brasoveanu and Farkas work with sets of total variable assignments. When working with sets of partial variable assignments, the set of live variables can be read directly off the domain of G. In our logic $\mathsf{Dom}(G)$ will serve as a stand in for Brasoveanu and Farkas's $\mathscr{V}$.

Lexical relations are interpreted in a distributive fashion relative to a set of assignment functions:

(157) LEXICAL RELATIONS: $[\![R(x_1,\ldots,x_n)]\!]^G = \mathbb{T}$ iff
  a. $G \neq \emptyset$
  b. $\{x_1,\ldots,x_n\} \subseteq \mathsf{Dom}(G)$
  c. $\langle g(x_1),\ldots,g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

The interpretation of a lexical relation has three components:

i. The first conjunct requires that the set of assignment functions be non-empty.

ii. The second conjunct requires that every variable appearing in the atomic formula be in the domain of G. This ensures that every variable is assigned some value or other by every assignment function in G.

iii. The third conjunct takes every assignment function $g \in G$ and evaluates the atomic formula with respect to g in the way familiar from first order logic. The evaluation here is essentially distributive. Atomic formula are checked row by row.

The distributive interpretation of lexical relations sheds light on the connection between FOL and C-FOL. In FOL lexical relations are interpreted with respect to single assignment functions. The goal of C-FOL is to provide existential quantifiers with a way of restricting the dependencies in which a variable x can be involved. Lexical relations in C-FOL essentially drop back into a FOL-like interpretation.

Conjunction in C-FOL is defined exactly as it is in FOL:

(158) CONJUNCTION:
$[\![\phi \wedge \psi]\!]^G = \mathbb{T}$ iff $[\![\phi]\!]^G = \mathbb{T}$ and $[\![\psi]\!]^G = \mathbb{T}$

Since we work with partial assignment functions we define bottom line truth in terms of the set consisting of the empty assignment function:

(159) TRUTH:
$[\![\phi]\!] = \mathbb{T}$ iff $[\![\phi]\!]^{\mathbf{0}} = \mathbb{T}$, where ϕ contains no free variables.

Interpretation starts with an empty context and slowly builds up discourse referents.

3.5.2 Existential Quantification

To define existential quantification we require two auxiliary components. First, extension is generalized to from single variable assignments to sets of partial variable assignments:

(160) $H \overset{x}{\Leftarrow} G := \forall_{h \in H} : \exists_{g \in G} : h \overset{x}{\Leftarrow} g \;\&\; \forall_{g \in G} : \exists_{h \in H} : h \overset{x}{\Leftarrow} g$

The definition above says that every assignment in H is an x-extension of some variable assignment in G. And that every variable assignment in G is extended by some variable assignment in H—i.e. the domain of the $\overset{x}{\Leftarrow}$ relation is G and its range is H. Note that H may have multiple assignment functions that are extensions of the same variable assignment in G. This means that the cardinality of H may be greater than or equal to G.

Second, we define the partialization of a set of variable assignments with respect to a set of variables. Let $\mathscr{V}$ be a set of variables and G a set of partial variable assignments. We define the restriction of G to $\mathscr{V}$, written $G_{\mathscr{V}}$ as follows:

(161) RESTRICTION OF G TO $\mathscr{V}$:

$$G_{\mathscr{V}} := \{h : h \subseteq g \;\&\; \mathsf{Dom}(h) = \mathscr{V} \;\&\; g \in G\}$$

We obtain $G_{\mathscr{V}}$ by taking each element of g and removing from g's domain any variables that are not in the set $\mathscr{V}$. We can think of $G_{\mathscr{V}}$ as representing a stage of the derivation before the variables in $\mathscr{V}$ were added to G:

(162) $G =$ <table><tr><th>x</th><th>y</th></tr><tr><td rowspan="2">Rex</td><td>Whiskers</td></tr><tr><td>Evander</td></tr><tr><td>Spot</td><td>Socks</td></tr></table>

(163) $G_{\{x\}} =$

x
Rex
Spot

Notice that in the examples above, that by restricting G to just the variable x we actually eliminate one of the rows. In general, restriction can only reduce the size of G.

Existential quantification in C-FOL comes parametrized with a set $\mathscr{U}$ of variables upon which the introduced variable's value may depend:

(164) EXISTENTIAL QUANTIFICATION:
$[\![[\exists^{\mathscr{U}} x : \phi]\psi]\!]^G = \mathbb{T}$ iff $\mathscr{U} \subseteq \mathsf{Dom}(G)$ & $[\![\psi]\!]^H = \mathbb{T}$ for some H s.t.
a. $H \overset{x}{\Leftarrow} G$
b. $[\![\phi]\!]^{H_{\mathscr{U} \cup \{x\}}} = \mathbb{T}$
c. $h(x) = h'(x)$, for all $h, h' \in H$ that are $\mathscr{U}$-identical

(165) h, h' are $\mathscr{U}$-identical iff $\forall_{u \in \mathscr{U}} : h(u) = h'(u)$

The first conjunct, $\mathscr{U} \subseteq \mathsf{Dom}(G)$, requires that the existential depend only on the values that have been previously introduced[2]. The second conjunct checks that there is some assignment H that makes the nuclear scope true and has the following three properties:

i. The first clause ensures that H is an x-extension of G.

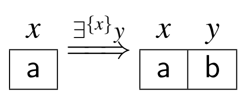
ii. The second clause requires that the restrictor be true relative to $H_{\mathscr{U} \cup \{x\}}$, i.e. we roll back the derivation to a point in which only the variables in $\mathscr{U}$ were defined and show that this point can be extended with x in a way that makes the restrictor true.[3]

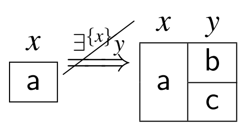
[2]The may-depend-upon relation is actually transitive. If y depends on x and z is introduced in such a way that in the resulting set of assignments H $h(z) = h'(z)$, for all $h, h' \in H$ that are $\{x\}$-identical, then z may still depend on y insofar as x depends on y.

iii. The final clause is the most complicated. It says that if any two assignment functions $h, h' \in H$ assign the same values to every variable in $\mathscr{U}$, then h, h' also assign the same value to x. In other words, the value assigned to x can vary with the value assigned to another variable iff there is corresponding variation in the value assigned to some other variable in $\mathscr{U}$. I will refer to this condition as the FIXED VALUE CONDITION.

Clauses (ii) and (iii) both account for different sets of empirical phenomena. Clause (ii) together with the interpretation of lexical relations entails the binder roof constraint. Clause (iii) on the other hand is the component of the definition that allows existential quantifiers in C-FOL to behave (in some instances) as if they took scope beyond higher quantifiers.

The first thing to note is that the restrictor is evaluated with respect to a (potentially) smaller set of assignment functions than the nuclear scope, since only the variables x can depend on are present. Second, notice that the existential never splits rows:

(166) a. 

b. 

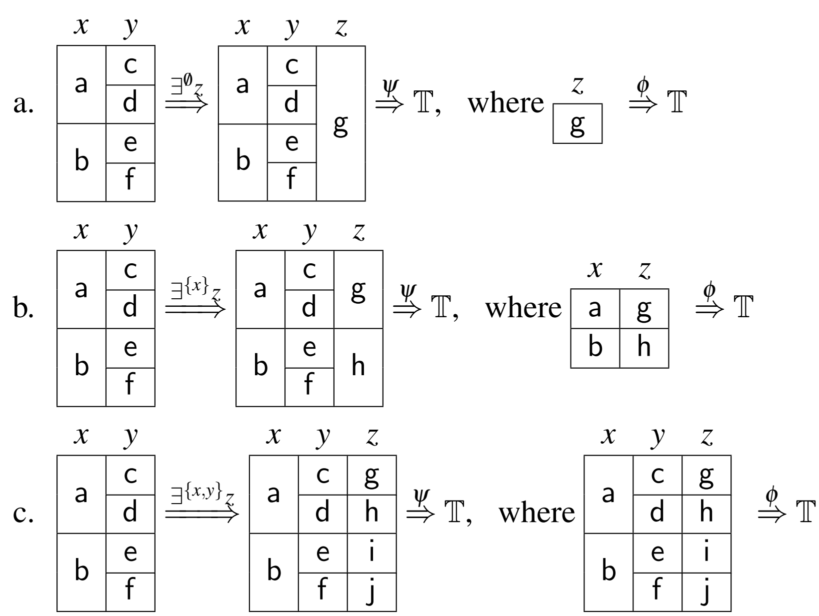
The transition in (166b) is illicit because it violates the fixed value condition. Notice that the two rows are $\{x\}$-identical (since both assign x the value a), yet they differ with respect to the value assigned to the variable y. The problem here is that the output contains assignment functions h, h' are non-identical x-extensions of the same input row.

[3]The corresponding clause in Brasoveanu and Farkas (2011) makes use of the set of 'live variables' and as a result looks highly stipulative. In contrast when working with partial variable assignments this move can be seen to represent a concrete intuition about the mechanics wide scope indefinites. An indefinite takes exceptional wide scope by unwinding the interpretation to a previous state. This idea aligns my presentation of C-FOL with the proposal in Farkas (1997).

This point is general. To see why, consider the fact that $H \overset{x}{\Leftarrow} G$ is defined so that for every $g \in G$, there is at least one $h \in H$ s.t. for every variable for which they are both defined. At the same time the existential requires that every $h, h' \in H$ are $\mathscr{U}$-identical for some $\mathscr{U}$. Now, let g be some arbitrary assignment function in G and let $H_{\overset{x}{\Leftarrow} g} := \{h : h \overset{x}{\Leftarrow} g \;\&\; h \in H\}$, i.e. the set of h's that are x-extensions of g. Now, let h, h' name arbitrary elements of $H_{\overset{x}{\Leftarrow} g}$. Now, for any set of variables $\mathscr{U}$ that does not include x, every element of $H_{\overset{x}{\Leftarrow} g}$ will be $\mathscr{U}$-identical. This lets us derive $h(x) = h'(x)$, but we also know that h, h' have the same values of every other variable as well. Hence, $h = h'$. There is thus a one-to-one correspondence between the initial value provided to the existential and the manipulated set of variable assignments that is passed to its nuclear scope.

The easiest way to see how existential quantification works by working through an example in which an existentially quantified formula is evaluated with respect to some set of assignment functions in which some variables are already live:

(167) $[\exists^{\emptyset/\{x\}/\{x,y\}} z : \phi]\psi$



Note that in the diagrams above, the update sequence terminates in the symbol $\mathbb{T}$. This is to emphasize that C-FOL is static. Passing a set of assignment functions across a quantifier increments them but updates are lost once the interpretation encounters a lexical relation. It also means that C-FOL does not pass variables introduced inside the restrictor of a quantifier into the nuclear scope of a quantifier. These are like in FOL evaluated on 'separate tracks' this is why I split the evaluation of the restrictor and nuclear scope in the above diagram[4].

3.5.3 Universal Quantification (Preliminary)

We turn now to universal quantification. Brasoveanu and Farkas give a preliminary (flawed) definition of universal quantification in terms of maximization:

(168) UNIVERSAL QUANTIFICATION (preliminary):
$[\![[\forall x : \phi]\, \psi]\!]^G = \mathbb{T}$ iff $[\![\psi]\!]^H = \mathbb{T}$, where H is the maximal set of assignments that satisfies ϕ relative to x and G

(169) H is the maximal set of assignments that satisfies ϕ relative to the variable x, the set of assignments G and the set of variables iff

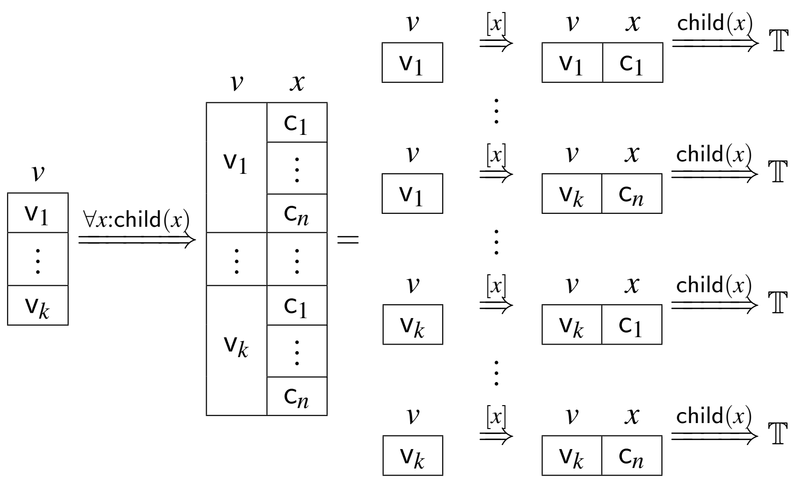
$$H = \bigcup_{g \in G} \left\{ h : h \overset{x}{\Leftarrow} g \;\&\; [\![\phi]\!]^{\{h\}} = \mathbb{T} \right\}$$

The universal quantifier finds the maximal set of assignment functions satisfying its restrictor and evaluates its nuclear scope with respect to this set. The maximal set is found by going through each assignment function $g \in G$, then collecting the set of singleton sets $\{h\}$ s.t. $h \overset{x}{\Leftarrow} g$ and $[\![\phi]\!]^{\{h\}} = \mathbb{T}$. Taking all these together gives the maximal set.

[4] I hope that these diagrams also serve to illustrate that all logics are dynamic in some sense or other. So called 'static logics' simply restrict all dynamics to the syntactic scope. Contexts are passed down not across. In fact I can think of no *a priori* reason one could not define a logic that utilized both 'static' context passing and 'dynamic' context passing for different aspects of the context of evaluation. Whether this would ever be a good idea is another question.

Let us assume that the model contains some children, $\mathsf{c}_1, \ldots, \mathsf{c}_n$. We can calculate the max set relative to the sentence $\mathsf{child}(x)$ by taking each row of the input and including every way of adding to that row one of the children. Below I have depicted the calculation of the maximal set of assignments[5]:

(170)



There are two key observations here. First, the universal quantifier does not introduce any dependencies because the set of children will be the same for each set of values assigned to variables in the initial set. Second, the universal quantifier can (and usually will) split rows. The intuition here is exactly that underlying the tree representation of FOL quantification. A universal induces branching while the existential creates a single daughter.

3.5.4 Truth Conditions I: *some* in the nuclear scope of *every*

The first case we tackle is the interpretation of wide scope existentials that appear in inside the nuclear scope of a universal. In this section I will show that in C-FOL existential quantifiers can take unbounded upward scope from the restrictors of any number

[5]The output assignment does not terminate in a truth value in this diagram because it represents the set of assignments that would be passed to the restrictor of the universal which I have chosen not to picture.

of universal quantifiers. I will illustrate this by showing how narrow, intermediate, and wide scope readings can be derived for the sentence in (171)

One sentence that illustrates this configuration is given in (171).

(171) Every^x student noticed that ever^y professor recommended a^z paper about scope.

a. NS: every$^x \gg$ every$^y \gg$ a^z

On this reading the sentence is consistent with a situation in which different students noticed different professors recommending different papers.

b. IS: every$^x \gg$ a$^z \gg$ every^y

On this reading every student noticed that there is a paper that every professor recommended. It may have been that multiple papers about scope were recommended by every professor and that different students noticed different that different books were recommended.

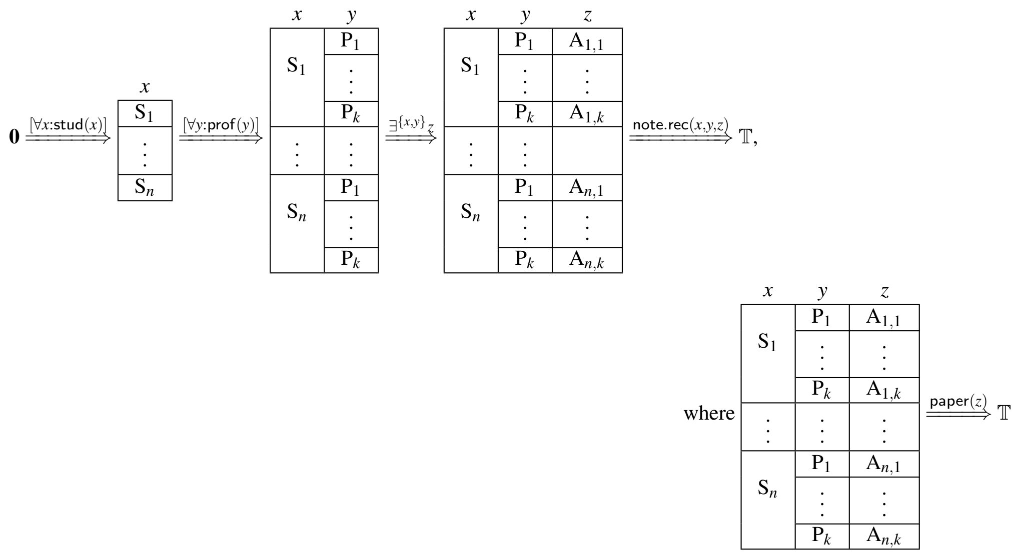
c. WS: a$^z \gg$ every$^x \gg$ every^y

On this interpretation there is at least one paper s.t. every student noticed that every professor recommended that paper.

Since I am not interested in the lexical semantics of noticing or the compositional semantics of the sentence in (171) , I will interpret it with the help of a three place relation $\mathsf{note.rec}(x,y,z)$ which should be read 'x noticed that y recommended z'. The semantic representations associated with these readings is given below:

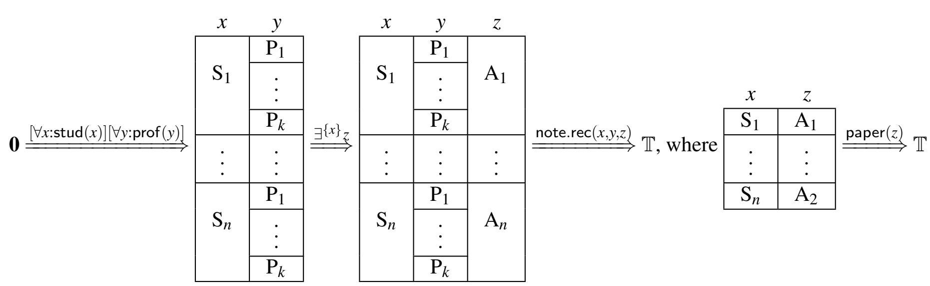
(172) a. NS: $[\forall x : \mathsf{stud}(x)][\forall y : \mathsf{prof}(y)][\exists^{\{x,y\}} z : \mathsf{paper}(z)]\mathsf{note.rec}(x,y,z)$
b. IS: $[\forall x : \mathsf{stud}(x)][\forall y : \mathsf{prof}(y)][\exists^{\{x\}} z : \mathsf{paper}(z)]\mathsf{note.rec}(x,y,z)$
c. WS: $[\forall x : \mathsf{stud}(x)][\forall y : \mathsf{prof}(y)][\exists^{\emptyset} z : \mathsf{paper}(z)]\mathsf{note.rec}(x,y,z)$

The narrow scope reading is the simplest and is depicted below:

(173) 

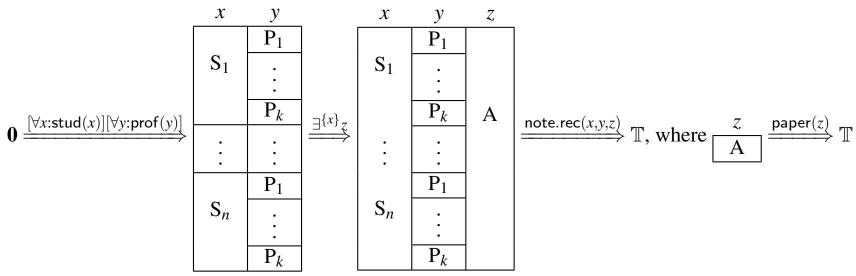
The two universal quantifiers first introduce maximal sets of students and professors. The existential can then generate one article per row as long as the output is s.t. (i) the same article shows up in every row that is $\{x,y\}$-identical and (ii) the restrictor is true relative to the entire initial assignment function.

The intermediate scope reading is depicted below:

(174) 

Notice that here the fixed value condition requires that the articles vary only with the students and not with the professors. This means that for every student x there is some paper z s.t. for every professor y is s.t. x noticed that y recommended z.

For completeness the wide scope reading is depicted below:

(175) 

Here articles are not allowed to vary with either papers or students. There must be a single article s.t. every student noticed that every professor recommended it.

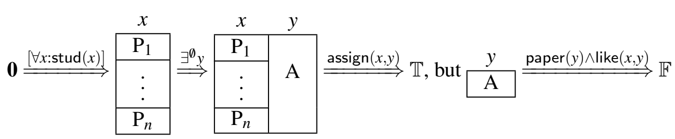
3.5.5 The Binder Roof Constraint

The logic set out so far manages to account for the binder roof constraint. Consider the sentence below:

(176) Every^x professor assigned a^y paper she$_x$ liked.

This sentence lacks a wide scope reading. The indefinite cannot take scope over the universal. To see that the theory so far presented accounts for this fact consider the derivation of a hypothetical wide scope reading:

(177) $[\forall x : \mathsf{professor}(x)][\exists^{\emptyset} y : \mathsf{paper}(y) \wedge \mathsf{like}(x,y)]\mathsf{assign}(x,y)$

(178) 

The restrictor of the existential comes out false, because x is not defined in the restricted state over which the restrictor of the existential is defined.

3.5.6 Truth Conditions II: *some* in the restrictor *every*

The definition of universal quantification in (168) breaks down when we consider indefinites in the scope of a the restrictor of the universal. Lets take the sentence like (179) which has a reading in which every dog that chased, say, Whiskers is a good dog, but dogs that chased Evander are not.

(179) Every dog that chased a certain cat is a good dog.

$$\rightsquigarrow [\forall x : \mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)]\mathsf{goodDog}(x)$$

The desired interpretation for the formula in (179) is one in which there is a certain cat, c, s.t. every dog who chased c is a good dog. However, the formula in (179) given the definition of maximality in (169) will be true just in case every dog who chased any cat is a good dog.

We begin by setting up a scenario:

(180) $\mathfrak{I}(\mathsf{Dog}) = \{\mathsf{f},\mathsf{r},\mathsf{d},\mathsf{s}\}$
$\mathfrak{I}(\mathsf{cat}) = \{\mathsf{w},\mathsf{e}\}$
$\mathfrak{I}(\mathsf{chase}) = \{\langle \mathsf{f},\mathsf{w}\rangle, \langle \mathsf{r},\mathsf{w}\rangle, \langle \mathsf{d},\mathsf{e}\rangle, \langle \mathsf{s},\mathsf{e}\rangle\}$
$\mathfrak{I}(\mathsf{good}) = \{\mathsf{f},\mathsf{r}\}$

This model makes the English sentence (179) true on the reading in which the indefinite scopes above *every dog*, since every dog that chased w is good, but false on the reading where the indefinite scopes below *every dog* because some dogs that chased cats are not good, viz. d,s chased e but are not good dogs.

Consider which set assignment functions H is maximal with respect to x and the formula $\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$, i.e. the set of assignment functions H s.t. for all h if $[\![\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)]\!]^{\{h\}} = \mathbb{T}$, then $h \in H$. Looking at the four possible singleton sets of assignment functions we see that every singleton assignment function in which x maps to a dog satisfies the formula:

(181) a. $\overset{x}{\boxed{\mathsf{f}}} \xRightarrow{\mathsf{dog}(x)} = \mathbb{T}$ & $\overset{x}{\boxed{\mathsf{f}}} \xRightarrow{\exists^{\emptyset} y} \overset{x \quad y}{\boxed{\mathsf{f} \mid \mathsf{w}}} \xRightarrow{\mathsf{chase}(x,y)} \mathbb{T}$ where $\overset{y}{\boxed{\mathsf{w}}} \xRightarrow{\mathsf{cat}(y)} \mathbb{T}$

b. $\overset{x}{\boxed{\mathsf{r}}} \xRightarrow{\mathsf{dog}(x)} = \mathbb{T}$ & $\overset{x}{\boxed{\mathsf{r}}} \xRightarrow{\exists^{\emptyset} y} \overset{x \quad y}{\boxed{\mathsf{r} \mid \mathsf{w}}} \xRightarrow{\mathsf{chase}(x,y)} \mathbb{T}$ where $\overset{y}{\boxed{\mathsf{w}}} \xRightarrow{\mathsf{cat}(y)} \mathbb{T}$

c. $\overset{x}{\boxed{\mathsf{d}}} \xRightarrow{\mathsf{dog}(x)} \mathbb{T}$ & $\overset{x}{\boxed{\mathsf{d}}} \xRightarrow{\exists^{\emptyset} y} \overset{x \quad y}{\boxed{\mathsf{d} \mid \mathsf{e}}} \xRightarrow{\mathsf{chase}(x,y)} \mathbb{T}$ where $\overset{y}{\boxed{\mathsf{e}}} \xRightarrow{\mathsf{cat}(y)} \mathbb{T}$

d. $\overset{x}{\boxed{\mathsf{s}}} \xRightarrow{\mathsf{dog}(x)} \mathbb{T}$ & $\overset{x}{\boxed{\mathsf{s}}} \xRightarrow{\exists^{\emptyset} y} \overset{x \quad y}{\boxed{\mathsf{s} \mid \mathsf{e}}} \xRightarrow{\mathsf{chase}(x,y)} \mathbb{T}$ where $\overset{y}{\boxed{\mathsf{e}}} \xRightarrow{\mathsf{cat}(y)} \mathbb{T}$

Each assignment given above satisfies the formula $\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$ by itself. Since each stores only a single value for x, the values for y after updating will show non-variation trivially. Because of this, $H = \{$(181a), (181b), (181c), (181d)$\}$ is the maximal assignment assignment relative to the restrictor:

(182) $$H = \begin{array}{c} x \\ \hline \multicolumn{1}{|c|}{\mathsf{f}} \\ \hline \multicolumn{1}{|c|}{\mathsf{r}} \\ \hline \multicolumn{1}{|c|}{\mathsf{s}} \\ \hline \multicolumn{1}{|c|}{\mathsf{d}} \\ \hline \end{array} \xRightarrow{\mathsf{goodDog}(x)} \mathbb{F}$$

The problem with H is that it includes dogs that did not chase the same cat, which is part of our desired interpretation.

This problem is general: definition (169) examines only one assignment function h in a potential H at a time. This guarantees the fixed-value condition contributed by the existential quantifier will be satisfied trivially. Since, the set $\{h\}$ of assignment functions contains only h it trivially satisfies the final clause of (169) which has the form: $\forall h, h' \in \{h\}$: if h, h' are $\{x\}$-identical, then $h(y) = h'(y)$. This is can be satisfied simply by finding a cat that the dog in question chased.

The definition in (169) does not succeed in ensuring that the same y-cat is present for each x-dog in H. The problem is with the definition of maximization. This definition evaluates its restrictor with respect to each row considered by itself. When evaluating the fixed value condition, we want to evaluate it with respect to the entire set of assignment functions.

The lesson is general: for the fixed value condition to do its work we can never break up the set of assignment functions into sub-states and evaluate formulas relative to them. If we do, we evaluated the fixed value condition only against each sub-state—once we add the pieces back together we have no guarantee that the fixed value condition will hold globally.

Brasoveanu and Farkas are aware of this problem and propose an amended definition for universal quantification:

(183) UNIVERSAL QUANTIFICATION (revised):

$[\![[\forall x : \phi]\psi]\!]^G = \mathbb{T}$ iff $[\![\psi]\!]^H = \mathbb{T}$, for some H that is maximal relative to x, ϕ, and G

(184) H is maximal relative to x, ϕ, and G iff

a. $H \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^H = \mathbb{T}$

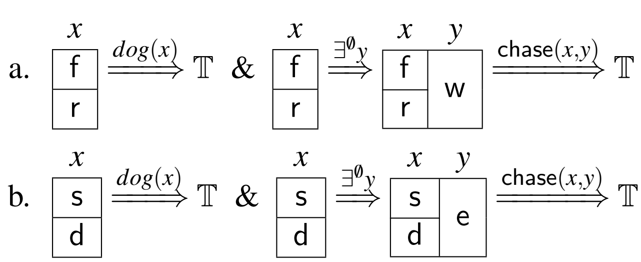
b. there is no $H' \neq H$ s.t. $H \subseteq H'$ and $H' \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^{H'} = \mathbb{T}$

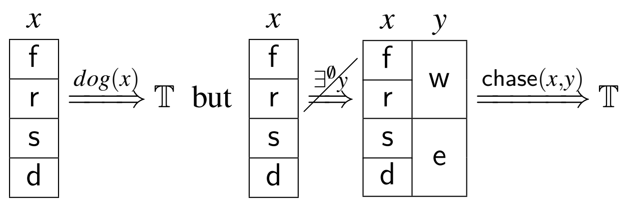
While definition (169) finds the maximal set of assignment functions H, the revised definition (184) finds a maximal set of assignment functions. Definition (184) also makes available to the formula in the restrictor of a universal the full set of assignment functions that could satisfy it.

Since definition (184) finds a maximal set of assignments, it can be thought of in two steps. Step one (corresponding to clause (184a)) finds a witness set, H, that satisfies the restrictor. Step two (corresponding to clause (184b)) ensures that there is no larger set of assignments, H', that also satisfies the restrictor. Importantly the search for a larger H' is conditioned on the original witness set, because clause (184b) requires that $H \subseteq H'$.

Consider the scenario from (180) again and the sets of assignment functions in (185):

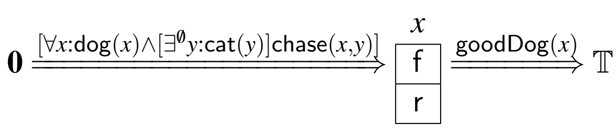
(185)



c. 

Both assignments in (185a-b) satisfy the formula $[\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$. Notice that the set of assignments given in (185c) cannot. The only way to satisfy $\mathsf{chase}(x,y)$ runs afoul of the fixed value condition contributed by the existential. If instead, one were to select just one cat, one would fail to satisfy $\mathsf{chase}(x,y)$ since there is no cat that every dog chased. Thus both (185a) and (185b) are maximal with respect the formula $[\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$ and the initial empty set of assignments.

With the revised definition the logic delivers the right result for the formula (179) when evaluating it in the scenario given in (180):

(186) 

3.6 Revisiting *some* in the restrictor of *every*

In this section, I will show that static C-FOL cannot account for exceptional wide scope indefinites. I will show that when a widest scope existential occurs inside the restrictor of a universal, the truth conditions of the resulting formulas are systematically stronger than their natural language counterparts.

Consider the following model:

(187) $\mathfrak{I}(\mathsf{dog}) = \{\mathsf{f},\mathsf{r},\mathsf{d}\}$
$\mathfrak{I}(\mathsf{cat}) = \{\mathsf{w},\mathsf{e}\}$
$\mathfrak{I}(\mathsf{chase}) = \{\langle\mathsf{f},\mathsf{w}\rangle,\langle\mathsf{r},\mathsf{w}\rangle,\langle\mathsf{f},\mathsf{e}\rangle,\langle\mathsf{r},\mathsf{e}\rangle,\langle\mathsf{d},\mathsf{e}\rangle\}$
$\mathfrak{I}(\mathsf{bark}) = \{f,r\}$

The model above makes the English sentence given in (188) true on the reading in which the existential takes wide scope out of the relative clause. On this reading (188) is true because there is a cat, viz. w, such that every dog that chased it barked.

(188) Every dog that chased a certain cat barked.

This model we are considering differs minimally from the model I provided in the previous section. In the previous model the set of w-chasers and the set of e-chasers was disjoint. In the model above, the set of w-chasers is a strict subset of the set of e-chasers. In order to capture the widest scope reading of (188) both the set of w-chasers and the set of e-chasers will have to count as maximal with respect to the formula in the restrictor of the universal in (189):

(189) $[\forall x : \mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)]\mathsf{bark}(x)$

I will show that the above formula comes out false relative to the scenario under consideration. The issue arises because, as defined, maximization attempts to find a maximal set of dogs that are all chasing the same cat. It doesn't look for the maximal set of dogs that are chasing e or the maximal set of dogs that are chasing w.

Recall the definition of maximization given above:

(190) H is maximal relative to x, ϕ, and G iff
a. $H \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^H = \mathbb{T}$
b. there is no $H' \neq H$ s.t. $H \subseteq H'$ and $H' \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^{H'} = \mathbb{T}$

To show that (189) is false, we need to show that the set of assignment functions given in (191) is not maximal with respect to the formula $\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$.

(191) $H =$

x
f
r

To show that (191) is not maximal, we need to find some H' and verify that (i) $H' \supset H$ and (ii) $[\![\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)]\!]^{H'} = \mathbb{T}$.

It is in fact easy to find such an H'. Consider the set of assignment functions given in (192):

(192) $$H' = \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \mathsf{d} \\ \hline \end{array}$$

It is clear that $H' \supset H$ meeting criteria (i). It is also the case that H' satisfies the restrictor of the universal in (189).

(193) $$\begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \mathsf{d} \\ \hline \end{array} \xrightarrow{\mathsf{dog}(x)} \mathbb{T} \;\&\; \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \mathsf{d} \\ \hline \end{array} \xrightarrow{\exists^{\emptyset} y} \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{f} & \\ \cline{1-1} \mathsf{r} & \mathsf{e} \\ \cline{1-1} \mathsf{d} & \\ \hline \end{array} \xrightarrow{\mathsf{chase}(x,y)} \mathbb{T} \text{ where } \begin{array}{|c|} \multicolumn{1}{c}{y} \\ \hline \mathsf{e} \\ \hline \end{array} \xrightarrow{\mathsf{cat}(y)} \mathbb{T}$$

Since H' also satisfies the restrictor it meets criteria (ii) above. Hence H does not count as a maximal assignment function relative to the formula $\mathsf{dog}(x) \wedge [\exists^{\emptyset} y : \mathsf{cat}(y)]\mathsf{chase}(x,y)$. Instead the only maximal assignment function that does satisfy the restrictor in our scenario is H'. The problem is that H' does not satisfy the nuclear scope of the formula in (189).

(194) $$\begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \mathsf{d} \\ \hline \end{array} \xrightarrow{\mathsf{bark}(x)} \mathbb{F}$$

We are led to the conclusion that (189) is false in our scenario. Thus C-FOL does not deliver the correct truth conditions for the English sentence given in (188).

To see what the problem is, we should briefly show that H itself does satisfy the restrictor of (189):

(195) $$\begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \end{array} \xrightarrow{\mathsf{dog}(x)} \mathbb{T} \;\&\; \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{f} \\ \hline \mathsf{r} \\ \hline \end{array} \xrightarrow{\exists^{\emptyset} y} \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \mathsf{f} & \\ \cline{1-1} \mathsf{r} & \mathsf{w} \\ \hline \end{array} \xrightarrow{\mathsf{chase}(x,y)} \mathbb{T} \text{ where } \begin{array}{|c|} \multicolumn{1}{c}{y} \\ \hline \mathsf{w} \\ \hline \end{array} \xrightarrow{\mathsf{cat}(y)}$$

Comparing the derivation in (193) to the derivation in (195) we see that they rely on picking different different cats as the witness for the existential. This is allowed by the definition of maximization given in (190), since the only comparisons are between H and H': it is important only that (i) one is larger than the other and (ii) both satisfy the restrictor. It does not matter that they satisfy the restrictor by picking different witnesses for the existential. The scenario is set up in such a way that every dog that chased e also chased w. The set of e-chasers and the set of w-chasers will both satisfy the restrictor of (189). However, since the set of w-chasers is larger, only this set will be maximal.

The problem with the definition of maximisation in (190) comes down to the fact that it is not dynamic. Maximization needs to compare not only the input to the restrictor but also the output; it must be sensitive to the fact that assignment functions H and H' are completed in different ways—they require picking different witnesses for the existential and should not compete with each other.

3.7 Dynamic Predicate Logic with Choice

In this section I will develop a dynamic version of C-FOL based largely on DPL. I call this logic Dynamic Predicate Logic with Choice (C-DPL). While developing the logic, my strategy will be to divide the work performed by the complex definitions of C-FOL quantifiers into as many pieces as possible:

- Random assignment, $[x]$, captures the common features of universal and existential quantification, viz. that the set of assignment functions is changed.
- To capture the fixed value condition, I define a formula, **fix**(_), that requires that its argument vary only with the variables already available in the context.
- An operator I call 'jump', $\curvearrowleft_{\mathscr{U}}(\phi)$, causes a formula ϕ to be evaluated with respect to a prior context in which only the variables in $\mathscr{U}$ are present. This operator does the work of set-indexing on the existential in C-FOL.

- Universal quantification is handled in terms of a maximization operator, $\mathbf{M}_x(\phi)$, that maximizes its output in a way that avoids the objection to C-FOL given in the previous section.

I will show that while C-DPL avoids the major issue confronting C-FOL it is ill suited for analysis of the dynamics of natural language. In particular is suffers from two major flaws that I will spell out in the final section:

- C-DPL does not have a way of keeping universal quantification dynamically closed. Universal quantifiers take semantic scope over everything conjoined to their left.
- C-DPL does not admit of a coherent analysis of plural and singular discourse reference.

Though these objections are unrelated to the interpretation of indefinites, they are important for understanding the broader landscape of plural logics.

3.7.1 Preliminaries

Lexical relations in C-DPL are tests, i.e. they do not change their input. Like in C-FOL, lexical relations are evaluated row-by-row in C-DPL.

(196) $[\![R(x_1,\ldots,x_n)]\!]^{G,H} = \mathbb{T}$ iff
  a. $G = H$
  b. $G \neq \emptyset$
  c. $\{x_1,\ldots,x_n\} \subseteq \mathsf{Dom}(G)$
  d. $\langle g(x_1),\ldots,g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

The only change we make is the addition of the first conjunct that requires that $G = H$, i.e. that the input and the output be identical.

All quantifiers in C-DPL will invoke random assignment, $[v]$, in some way or another. We define random assignment in (197). The first conjunct, (197a), ensures that x is fresh in the previous context. The second conjunct, (197b), requires the output state H be an x-extension of the input assignment G.

(197) $[\![[x]]\!]^{G,H} = \mathbb{T}$ iff
  a. $x \notin \mathsf{Dom}(G)$
  b. $H \stackrel{x}{\Leftarrow} G$

Random assignment will play a role in both the definitions of existential and universal quantification, as such we do not require that the output satisfy the fixed value condition or be maximal in any respect, but instead provide separate operators to handle these properties.

Dynamic conjunction is defined in the standard way.

(198) $[\![\phi \wedge \psi]\!]^{G,H} = \mathbb{T}$ iff there exists a K s.t. $[\![\phi]\!]^{G,K} = \mathbb{T}$ and $[\![\psi]\!]^{K,H} = \mathbb{T}$

Dynamic conjunction simply evaluates the first conjunct and then passes its updates, if any, to the second conjunct.

3.7.2 The fixed value condition

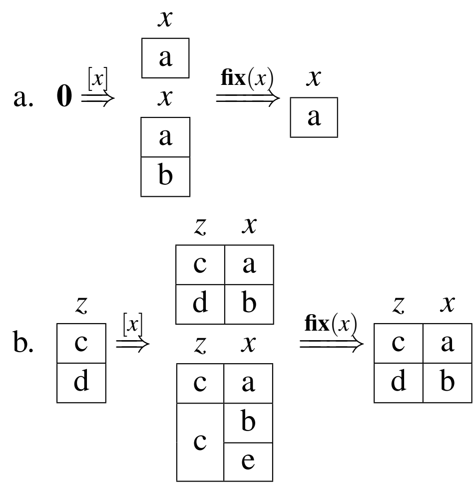
To handle the fixed value condition we define a predicate **fix**:

(199) $[\![\textbf{fix}(x)]\!]^{G,H} = \mathbb{T}$ iff
  a. $G = H$
  b. for all $g, g' \in G$ if $g[x]g'$, then $g(x) = g'(x)$

The first clause ensures that **fix** is a test. The second clause ensures that values of x do not vary from assignment function to assignment function unless the value of some other variable also varies, i.e. there are no assignment functions that differ with respect to x and only x.

The predicate **fix** provides an injunction against splitting rows:

(200) $[x] \wedge \mathbf{fix}(x)$



In (200a-b) we see random updates to the empty assignment function. The first update, (200a), satisfies the fixed value condition because it contains only one value for the variable *x*. The second, (200b), does not satisfy the fixed value condition because the update contains multiple values for *x* despite there being no other values for *x* to vary with respect to. The third update, in (200b), updates an assignment function that already has two values for the variable *z*. Here multiple updates to *x* can satisfy the fixed value condition because the values of *x* can vary with the values of *z*.

3.7.3 Interpolating Assignment Updates: Merge

To help define exceptional wide-scope existentials we must first define a function **merge** that combines two sets of assignment functions. The goal is a function that takes sets G, H of assignment functions and combines them into a new set K in the following way: we will find pairs g, h from G and H that agree with respect to all variables in the domain of both g and h and combine them into a single assignment function k. Taking all such assignment functions k together will give us the new set K.

(201) MERGE:

$$\mathbf{merge}(G,H) = \{g \cup h : g \in G \,\&\, h \in H \,\&\, \forall_{v \in \mathsf{Dom}(g) \cap \mathsf{Dom}(h)} : g(v) = h(v)\}$$

The formula above takes pairs of assignment functions from G and H that agree with respect to all the variables they both assign values to and puts them together.

I've provided some examples below:

(202) $G =$

x
a
b

$H =$

y
c
d

$\mathbf{merge}(G,H) =$

x	y
a	c
	d
b	c
	d

In the example above we see that the domains of G and H are completely disjoint. Putting them together involves taking the union of the two elements from the cross product.

(203) $G =$

x	y
a	c
b	d

$H =$

x	z
a	e
b	f

$\mathbf{merge}(G,H) =$

x	y	z
a	c	e
b	d	f

In the example above we see that the domains of G and H overlap with respect to the variable x. So for every element of g we combine it with the corresponding elements of H that agree with g with respect to the variable assigned to x.

One interesting case concerns the merger of a set of assignments G, H where $G \subseteq H$. In this case $\mathbf{merge}(G,H) = G$. To see why consider that for arbitrary sets $g \in G$ and $h \in H$, if $\forall_v \in \mathbf{Dom}(G) \cup \mathbf{Dom}(H)$ it is the case that $g(v) = h(v)$, then it is the case that $g = h$. Since these are the only pairs meeting the description in the definition of **merge** it follows that the only assignments in the merged sets will be those in G. Note that this does not hold if one of the merged sets is the emptyset. In this case the merger of the two sets will be the non-empty set of assignments.

3.7.4 Existential Quantification

Existential quantification can be defined in terms of dynamic conjunction, random assignment, and the **fix** predicate:

(204) $[\exists x : \phi]\psi := ([x] \wedge \mathbf{fix}(x) \wedge \phi) \wedge \psi$

Going through an example will require cheating a bit and assuming that we have a working definition of universal quantification. Waving our hands for a moment, let us assume that against the empty input a universal quantifier can give us the output in (205).

(205) $$\emptyset \xrightarrow{\text{every}^x \text{ student}} \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \text{Student}_1 \\ \hline \text{Student}_2 \\ \hline \text{Student}_3 \\ \hline \vdots \\ \hline \end{array} \xrightarrow{[y]\wedge\mathbf{fix}(y),\checkmark\mathsf{paper}(y),\checkmark\mathsf{read}(x,y)} \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \text{Student}_1 & \text{Paper}_1 \\ \hline \text{Student}_2 & \text{Paper}_2 \\ \hline \text{Student}_3 & \text{Paper}_3 \\ \hline \vdots & \vdots \\ \hline \end{array}$$

Against this backdrop we can consider the update provided by an existential like $[\exists y : \mathsf{paper}(y)]\mathsf{read}(x,y)$. We first feed the output of (205) to the formula $([y]; \mathbf{fix}(y) \wedge \mathsf{paper}(y)) \wedge \mathsf{read}(x,y)$ and expect an output like the final output above. Above we see that random assignment $[y]$ initiates an update, $\mathbf{fix}(y)$ ensures that we can vary papers with the students assigned to x, the restrictor $\mathsf{paper}(y)$ ensures that each row stores a paper, and the nuclear scope $\mathsf{read}(x,y)$ checks that each x-student read each y-paper.

Turning to wide scope existentials, we would like to implement the following procedure:

i. Pick a subset, $\mathscr{U}$, of the variables defined in the incoming assignment function G.

ii. Feed the restriction of G to $\mathscr{U}$, i.e. $G_{\mathscr{U}}$, into the restrictor of the existential: $([x] \wedge \mathbf{fix}(x) \wedge \phi)$ and obtain an output K. Given the formula $\mathbf{fix}(x)$ present in the existential, this will ensure that x varies only with the variables in $\mathscr{U}$.

iii. Make sure that $\mathsf{Dom}(K) \cap \mathsf{Dom}(G) = \mathscr{U}$, i.e. that evaluating $([x] \wedge \mathbf{fix}(x) \wedge \phi)$ does not involve adding variables already defined in G. If $\mathsf{Dom}(G) = \{z, y\}$ and $\mathscr{U} = \{y\}$, we would not want a K that included values of z, since this would amount to overwriting the values already in the input G.

iv. Combine K with the original input G to obtain the output H. For this step we will use the **merge** function.

The formula to accomplish this task is given in (206).

(206) $[\![\underset{\mathscr{U}}{\curvearrowleft} \phi]\!]^{G,H} = \mathbb{T}$ iff

a. $\mathscr{U} \subseteq \mathsf{Dom}(G)$

b. there exists a K s.t. $[\![\phi]\!]^{G_{\mathscr{U}},K} = \mathbb{T}$

i. $\mathsf{Dom}(K) \cap \mathsf{Dom}(G) = \mathscr{U}$

ii. $H = \mathbf{merge}(G, K)$

The first clause of (206) ensures that the variables in $\mathscr{U}$ exist in the previous context. The second clause then gets an update from ϕ against the input $G_{\mathscr{U}}$, which consists of G restricted to only those variables in U. The output K is then interpolated back into G to get the global output H.

We can now define exceptional wide scope indefinites:

(207) $[\exists^z x : \phi]\psi := \underset{z}{\curvearrowleft}([x] \wedge \mathbf{fix}(x) \wedge \phi) \wedge \psi$

Lets return to our example in (205) to see how this works:

(208) $\mathbf{0} \xRightarrow{\text{every}^x \text{ student}}$

x
Student_1
Student_2
Student_3
$\vdots$

$\xRightarrow{\underset{\emptyset}{\curvearrowleft}([y] \wedge \mathbf{fix}(y) \wedge \mathsf{paper}(y))}$???

To get a handle on this, lets explore the complex update that occurs after we update with *every student*. The first thing that happens is that the formula in the scope of $\underset{x}{\curvearrowleft}$ is evaluated with respect to a rolled back state, in this case the empty state. This part of the computation is depicted in (209).

(209) $\mathbf{0} \xRightarrow{[y]\mathbf{fix}(y)\wedge\mathsf{paper}(y)} \begin{array}{|c|} \multicolumn{1}{c}{y} \\ \hline \text{Paper}_1 \\ \hline \end{array}$

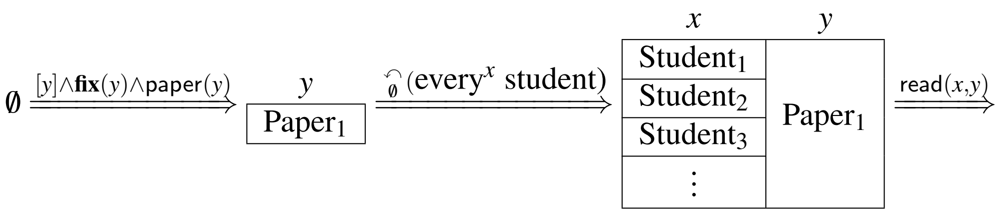
The last part of the computation involves putting together the output of (209) and the output of *every student* from (208). This occurs first by checking every pair of rows from the two outputs and seeing if they agree with respect to the values assigned to variables in their common domain. If they do, then we combine that pair of rows and add it to the output. In this particular case, the task is easy: since the domains are disjoint, all the rows agree vacuously and so we should just combine every pair of rows from the two outputs:

(210) $\mathbf{0} \xRightarrow{\text{every}^x \text{ student}} \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \text{Student}_1 \\ \hline \text{Student}_2 \\ \hline \text{Student}_3 \\ \hline \vdots \\ \hline \end{array} \xRightarrow{\overset{\curvearrowleft}{\emptyset}([y]\wedge\mathbf{fix}(y)\wedge\mathsf{paper}(y))} \begin{array}{|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{y} \\ \hline \text{Student}_1 & \\ \cline{1-1} \text{Student}_2 & \text{Paper}_1 \\ \cline{1-1} \text{Student}_3 & \\ \cline{1-1} \vdots & \\ \hline \end{array} \xRightarrow{\mathsf{read}(x,y)}$

Our new operator $\overset{\curvearrowleft}{\mathscr{U}}$ will let existential quantifiers take scope over universals, but it will not allow universal quantifiers to take scope over existential quantifiers. We can satisfy ourselves that this is true before we even have a satisfactory translation of universal quantification, since we can use our hypothesised output from (205). Consider the following derivation:

(211) $\mathbf{0} \xRightarrow{[y]\wedge\mathbf{fix}(y)\wedge\mathsf{paper}(y)} \begin{array}{|c|} \multicolumn{1}{c}{y} \\ \hline \text{Paper}_1 \\ \hline \end{array} \xRightarrow{\overset{\curvearrowleft}{\emptyset}(\text{every}^x \text{ student})} ???$

Again, we evaluate *every student* against the rolled back assignment function, giving us the output familiar from (205) and (208). When we get to the combination stage we realize what the problem is, there is still only one paper available when it comes time to re-combine the assignment functions. This means that we end up with the same set of assignment functions:

(212) 

I will assume, that since there is no truth conditional affect of $\frown_v$ when taking scope over universal quantifiers, that this option is unavailable for building non-existential quantifiers.

I now turn to the question of defining universal quantification. In normal DPlL, universal quantification is defined in terms of a maximization operator **M** and dynamic conjunction. The maximization operator finds some maximal set of assignment functions that satisfies the restrictor and then passes this set to the nuclear scope of the universal. Since we already have a dynamic conjunction, our task is to define the maximization operator **M**. I will argue at length for the definition below in the subsequent chapter, but here I simply note that maximization defined in terms of the subset operator will work for our purposes here.

(213) $[\![\mathbf{M}_x\phi]\!]^{G,H} = \mathbb{T}$ iff
  a. $[\![[x] \wedge \phi]\!]^{G,H} = \mathbb{T}$
  b. there is no $K \supsetneq H$ s.t. $[\![[x] \wedge \phi]\!]^{G,K} = \mathbb{T}$,

This definition says that $\mathbf{M}_x\phi$ is true of an output H relative to an input G and a formula ϕ iff H is an allowable output given the input G and formula $[x] \wedge \phi$ and there is no larger output that also satisfies this formula relative to the input G.

(214) Every student left. $\rightsquigarrow \mathbf{M}_x\mathsf{student}(x);\mathsf{left}(x)$

(215) $\emptyset \xrightarrow{\mathbf{M}_x\mathsf{student}(x)}$???

The desired output should be the largest of every possible update available to $[x] \wedge \mathsf{student}(x)$. There are three such updates shown in (216).

(216) a. $\mathbf{0} \xRightarrow{[x]\wedge\mathsf{student}(x)} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Otis}\\ \hline \text{Sybil}\\ \hline\end{array}$

b. $\mathbf{0} \xRightarrow{[x];\mathsf{student}(x)} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Sybil}\\ \hline\end{array}$

c. $\mathbf{0} \xRightarrow{[x];\mathsf{student}(x)} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Otis}\\ \hline\end{array}$

If we take the largest of these possible updates, then we get the update in (216a). Hence, the output for $\mathbf{M}_x\mathsf{student}(x)$ should be the output of (216a). The end result is the update in (217).

(217) $\mathbf{0} \xRightarrow{\mathbf{M}_x\mathsf{student}(x)} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Otis}\\ \hline \text{Sybil}\\ \hline\end{array}$

Now, let us consider a case in which we have a wide scope indefinite in the scope of the $\mathbf{M}_x$-operator.

(218) Every student that read a (certain) paper succeeded.
$$\rightsquigarrow \mathbf{M}_x(\mathsf{student}(x) \wedge \overset{\frown}{\{x\}}([y] \wedge \mathsf{paper}(y)) \wedge \mathsf{read}(x,y)) \wedge \mathsf{succeed}(x)$$

(219) $\emptyset \xRightarrow{\mathbf{M}_x(\mathsf{student}(x)\wedge \overset{\frown}{\{x\}}([y]\wedge\mathsf{paper}(y))\wedge\mathsf{read}(x,y))} ???$

Let's assume that Henrik, Otis, and Sybil are students, that Henrik and Otis both read paper_1, and that Henrik, Otis, and Sybil read paper_2. Let's also assume for simplicity that no other students read any other papers. Recall that this is the configuration that C-FOL has trouble with because the readers of paper_1 are a proper subset of the readers of paper_2.

(220) a. $\mathbf{0} \xRightarrow{[x\wedge\mathsf{student}(x)]} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Otis}\\ \hline\end{array} \xRightarrow{\overset{\frown}{\{x\}}([y]\wedge\mathsf{paper}(y))\wedge\mathsf{read}(x,y)} \begin{array}{|c|c|}\multicolumn{1}{c}{y} & \multicolumn{1}{c}{x}\\ \hline \text{Paper}_1 & \text{Otis}\\ \hline\end{array}$

b. $\mathbf{0} \xRightarrow{[x\wedge\mathsf{student}(x)]} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Henrik}\\ \hline\end{array} \xRightarrow{\overset{\frown}{\{x\}}([y]\wedge\mathsf{paper}(y))\wedge\mathsf{read}(x,y)} \begin{array}{|c|c|}\multicolumn{1}{c}{y} & \multicolumn{1}{c}{x}\\ \hline \text{Paper}_1 & \text{Henrik}\\ \hline\end{array}$

c. $\mathbf{0} \xRightarrow{[x\wedge\mathsf{student}(x)]} \begin{array}{|c|}\multicolumn{1}{c}{x}\\ \hline \text{Otis}\\ \hline \text{Henrik}\\ \hline\end{array} \xRightarrow{\overset{\frown}{\{x\}}([y]\wedge\mathsf{paper}(y))\wedge\mathsf{read}(x,y)} \begin{array}{|c|c|}\multicolumn{1}{c}{y} & \multicolumn{1}{c}{x}\\ \hline \text{Paper}_1 & \text{Otis}\\ \cline{2-2} & \text{Henrik}\\ \hline\end{array}$

The question is, which of these are maximal outputs? They all satisfy $[\![[x] \wedge \mathsf{student}(x) \wedge \underset{x}{\curvearrowleft}([x] \wedge \mathsf{paper}(y))]\!]^{G_\emptyset, H_{y,x}} = \mathbb{T}$. Notice that (220c) contains (220a-b). So, unless there is some other set of assignment functions H satisfying $[\![[x] \wedge \mathsf{student}(x) \wedge \underset{x}{\curvearrowleft}([x] \wedge \mathsf{paper}(y))]\!]^{G_\emptyset, H_{y,x}} = \mathbb{T}$ that properly includes the output of (220c), then we conclude that (220c) is maximal. Now, since the model does not contain any more students who read paper$_1$, there is no output that contains more students but contains paper$_1$ in the y slot. Consider the output in (221) that gave us trouble for C-FOL:

(221) $\mathbf{0} \xRightarrow{[x \wedge \mathsf{student}(x)]}$

x
Otis
Henrik
Sybil

$\xRightarrow{\underset{\{x\}}{\curvearrowleft}([y] \wedge \mathsf{paper}(y)) \wedge \mathsf{read}(x,y)}$

y	x
	Otis
Paper$_2$	Henrik
	Sybil

So, the question is: does (221) properly include (220c)? The answer is no, since the output of (220c) and the output of (221) differ with respect to the paper introduced at y.

The conclusion we reach is that by maximizing over output assignments we can generate a system that behaves appropriately with respect to wide scope indefinites in the scope of maximization operators.

3.8 Problems (and basic solutions) for C-DPL

There are two objections to utilizing C-DPL for analysing natural language expressions. The first objection stems from the fact that the universal quantifier in C-DPL is dynamically open. It takes scope over everything conjoined to its right:

(222) $([\forall x : \phi]\psi) \wedge \chi := (\mathbf{M}_x(\phi) \wedge \psi) \wedge \chi = \mathbf{M}_x(\phi) \wedge \psi \wedge \chi$

(223) $[\forall x : \phi](\psi \wedge \chi) := \mathbf{M}_x(\phi) \wedge (\psi \wedge \chi) = \mathbf{M}_x(\phi) \wedge \psi \wedge \chi$

In similar systems this does not end up being a problem because existential are by default outfitted with some means of enforcing non-variation with previously introduced plural discourse referents. For instance in van den Berg's Dynamic Plural Logic an indefinite can only vary with a quantifier if it occurs inside the scope of a distributivity operator. Here however we allow indefinites to vary with the values of any variables previously introduced. This will end up being a problem for discourses like the following:

(224) Every student left the bar. A local entered.

On our current account expect that this discourse to have a reading in which *a local* takes scope under *every student* and thus have a reading in which there are potentially as many locals entering the bar as there are students leaving the bar. However on any sensible interpretation the second sentence in this discourse introduces only a single individual.

One way to manage this problem is to collapse informational pluralities down into ontological pluralities after we are done processing the nuclear scope of the universal. A definition like the one below would accomplish this:

(225) $[\![[\forall x : \phi]\,\psi]\!]^{G,H} = \mathbb{T}$ there is some K s.t.
  a. $[\![\mathbf{M}_x\phi \wedge \psi]\!]^{G,K} = \mathbb{T}$
  b. $H = \{\{\langle v, \bigoplus K(v)\rangle : v \in \mathbf{Dom}(K)\}\}$

The first conjunct of the definition packs together the familiar analysis of universal from the main body of the chapter. It takes the input and finds some output K that consists of a plural information state of the type we have been working with. The second conjunct indicates that the output of the entire quantifier is built by taking each variable v in the domain of the output assignment and associating it with the ontological plurality that is formed by summing up all the values that v takes on in K. Notice that at this stage there will be only a single assignment function in H and so indefinites in subsequent sentences will not be able to be able to vary with the values introduced by the universal.

Several basic facts concerning plural discourse reference to sets will also be captured. A plural pronoun like *they* will be able to pick up a variable x that had a non-singleton projection in a previous sentence.

(226) Every^x student left her$_x$ bike at home. They^x had to take the bus.

(227) Every tourist caught a^x fish. They$_y$ were very big.

Cases like those above can be handled because the singular pronoun *her* appears in the scope of the universal quantifier when each value is (presumably) atomic. After the sentence is interpreted the students are packed together into an ontological plurality which can be refered to with the plural pronoun *they*. Similar considerations apply to the second sentence.

The downside of such an analysis is that it annihilates the structure that obtains between variables introduced by sentences like (227). Dependencies between variables are destroyed when they are packed wholesale into ontological pluralities. Thus phenomena that show that these dependencies can be picked back up on in subsequent discourse will be impossible to handle.

3.9 Conclusion

This chapter advanced the argument that C-FOL did not deliver the correct truth conditions for sentences in which an existential occurs in the restrictor of a universal quantifier. I showed that a dynamic variant could provided the correct truth conditions but would not work as a general analysis. In particular, I showed that a dynamic variant of C-FOL faced problems acting as a theory of discourse reference. In order to keep a universal quantifier from taking unbounded rightward scope it is necessary to collapse the structural information generated while it is processed. In the next two chapters I will take up the task of embedding the intuition underlying Brasoveanu & Farkas (2011) in a system that is already designed to handle plural discourse reference.

Appendix B

Technical Appendix

B.1 Notational Conventions

(228) $\mathbf{Dom}(G) := \{v : v \in \mathbf{Dom}(g)\}$, where g is any member of G.

(229) $\mathbf{0} := \{\emptyset\}$, i.e. the set containing the assignment function with an empty domain.

(230) $G(x) := \{g(x) : g \in G\}$, if $x \in \mathbf{Dom}(G)$ and $\emptyset$ otherwise.

(231) $$G(v_1,...,v_n) := \begin{cases} \{\langle g(v_1),\ldots,g(v_n)\rangle : g \in G\}, \\ \qquad \text{if } v_1 \in \mathbf{Dom}(G),\ \&\ \ldots,\ \&\ v_n \in \mathbf{Dom}(G) \\ \emptyset \text{ otherwise.} \end{cases}$$

(232) $H \overset{x}{\Leftarrow} G := \forall_{h \in H} : \exists_{g \in G} : h \overset{x}{\Leftarrow} g\ \&\ \forall_{g \in G} : \exists_{h \in H} : h \overset{x}{\Leftarrow} g$

(233) $G_{\mathscr{V}} := \bigcup_{g \in G} \{h : h \subseteq g\ \&\ \mathsf{Dom}(h) = \mathscr{V}\}$

(234) $\mathbf{merge}(G,H) = \{g \cup h : g \in G\ \&\ h \in H\ \&\ \forall_{v \in \mathsf{Dom}(g) \cap \mathsf{Dom}(h)} : g(v) = h(v)\}$

B.2 First Order Logic with Choice

(235) Lexical Relations: $[\![R(x_1,\ldots,x_n)]\!]^G = \mathbb{T}$ iff

a. $G \neq \emptyset$

b. $\{x_1,\ldots,x_n\} \subseteq \mathsf{Dom}(G)$

c. $\langle g(x_1),\ldots,g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

(236) CONJUNCTION:
$[\![\phi \wedge \psi]\!]^G = \mathbb{T}$ iff $[\![\phi]\!]^G = \mathbb{T}$ and $[\![\psi]\!]^G = \mathbb{T}$

(237) EXISTENTIAL QUANTIFICATION:
$[\![[\exists^{\mathscr{U}} x : \phi]\psi]\!]^G = \mathbb{T}$ iff $\mathscr{U} \subseteq \mathsf{Dom}(G)$ & $[\![\psi]\!]^H = \mathbb{T}$ for some H s.t.

a. $H \overset{x}{\Leftarrow} G$
b. $[\![\phi]\!]^{H_{\mathscr{U}\cup\{x\}}} = \mathbb{T}$
c. $h(x) = h'(x)$, for all $h, h' \in H$ that are $\mathscr{U}$-identical

(238) h, h' are $\mathscr{U}$-identical iff $\forall_{u\in\mathscr{U}} : h(u) = h'(u)$

(239) UNIVERSAL QUANTIFICATION:
$[\![[\forall x : \phi]\psi]\!]^G = \mathbb{T}$ iff $[\![\psi]\!]^H = \mathbb{T}$, for some H that is maximal relative to x, ϕ, and G

(240) H is maximal relative to x, ϕ, and G iff

a. $H \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^H = \mathbb{T}$
b. there is no $H' \neq H$ s.t. $H \subseteq H'$ and $H' \overset{x}{\Leftarrow} G$ and $[\![\phi]\!]^{H'} = \mathbb{T}$

B.3 Dynamic Predicate Logic with Choice

(241) LEXICAL RELATIONS:
$[\![R(x_1, \ldots, x_n)]\!]^{G,H} = \mathbb{T}$ iff

a. $G = H$
b. $G \neq \emptyset$
c. $\{x_1, \ldots, x_n\} \subseteq \mathsf{Dom}(G)$
d. $\langle g(x_1), \ldots, g(x_n)\rangle \in \mathfrak{I}(R)$, for all $g \in G$

(242) CONJUNCTION:
$[\![\phi \wedge \psi]\!]^{G,H} = \mathbb{T}$ iff there exists a K s.t. $[\![\phi]\!]^{G,K} = \mathbb{T}$ and $[\![\psi]\!]^{K,H} = \mathbb{T}$

(243) ASSIGNMENT UPDATE:
$[\![[x]]\!]^{G,H} = \mathbb{T}$ iff

a. $x \notin \mathsf{Dom}(G)$
b. $H \overset{x}{\Leftarrow} G$

(244) FIXED VALUE CONDITION:
$[\![\mathbf{fix}(x)]\!]^{G,H} = \mathbb{T}$ iff

a. $G = H$
b. for all $g, g' \in G$ if $g[x]g'$, then $g(x) = g'(x)$

(245) JUMP:
$[\![\curvearrowright_{\mathscr{U}} \phi]\!]^{G,H} = \mathbb{T}$ iff

a. $\mathscr{U} \subseteq \mathsf{Dom}(G)$

b. there exists a K s.t. $[\![\phi]\!]^{G_{\mathscr{U}},K} = \mathbb{T}$

i. $\mathsf{Dom}(K) \cap \mathsf{Dom}(G) = \mathscr{U}$

ii. $H = \mathbf{merge}(G,K)$

(246) MAXIMIZATION:
$[\![\mathbf{M}_x\phi]\!]^{G,H} = \mathbb{T}$ iff

a. $[\![[x] \wedge \phi]\!]^{G,H} = \mathbb{T}$

b. there is no $K \supsetneq H$ s.t. $[\![[x] \wedge \phi]\!]^{G,K} = \mathbb{T}$,

(247) UNIVERSAL QUANTIFICATION:
$[\![[\forall x : \phi]\psi]\!]^{G,H} = \mathbb{T}$ there is some K s.t.

a. $[\![\mathbf{M}_x\phi \wedge \psi]\!]^{G,K} = \mathbb{T}$

b. $H = \{\{\langle v, \bigoplus K(v)\rangle : v \in \mathbf{Dom}(K)\}\}$

Chapter 4

Dynamic Plural Logic with Unselective Maximization

4.1 Introduction

The previous chapter showed that while C-DPL could adequately handle exceptional scope indefinites within a single sentence, it did not scale up to handle multiple sentences. The logic needed to be outfitted either with a mechanism that compressed informational pluralities into ontological pluralities at various points. Either route would result in a serious decrease in empirical coverage when compared to theories that utilize systems like Dynamic Plural Logic (DPlL) (van den Berg, 1994, 1996; Nouwen, 2003), or Plural Compositional Discourse Representation Theory (PCDRT) (Brasoveanu, 2007). In each of these theories universals are dynamically open and information about the relationships between the values of different variables stays available and so can be elaborated upon in subsequent discourse. These theories aim to cover not only plural discourse reference but also a variety of donkey sentences, telescoping, and quantificational and modal subordination.

In this chapter I introduce the logic that will form the basis of the remainder of the dissertation. The basic system needs two basic properties:

i. The logic must support structured plural and singular discourse reference—universals should be dynamically open passing not only information about the set of entities quantified over but also information about the relationships between these entities and other discourse referents.

ii. The logic should have some natural way of controlling when dependencies between variables can be introduced—the **fix** test from the previous section was ultimately ad hoc. We want a system in which there are more principled ways to control dependencies between the values of different variables.

I will argue that van den Berg's (1996) DPlL offers the best jumping off point as it meets both criteria outlined above:

i. DPlL keeps every expression dynamically open. Like in C-FOL/C-DPL, expressions in DPlL are interpreted relative to sets of assignment functions. Unlike these logics however the sets of assignment functions are used both to keep track of the relations between the values of different variables and also to stand in for pluralities wherever they are needed (there are no ontological pluralities in DPlL).

ii. DPlL utilizes an assignment update operation ε_x that updates a set of assignments in a way that does not introduce any dependencies between x and any previously introduced variables. The ε_x update offers an even more general way than the fixed value condition for managing dependencies. Where the fixed value condition disallowed row-splitting the ε_x update allows row splitting while still preventing new dependencies from arising. Dependencies in DPlL can only arise when an ε_x update occurs in the scope of a distributivity operator δ_v giving DPlL a natural way of constraining how and where dependencies can enter into the interpretation of a sentence.

I develop the logic in several steps. First I contrast two broad families of dynamic plural systems, one based on work by van den Berg (1996) and Nouwen (2003) which features (i) collective (as opposed to distributive) interpretation of lexical items and (ii) assignment updates that do not introduce dependencies and a second based heavily on work by Brasoveanu (and followed by Henderson (2014); Kuhn (2015)) which features (i) distributive interpretation of lexical items and (ii) assignment updates that do introduce dependencies. I show that the basic system laid out by van den Berg and Nouwen offers finder grained control over dependencies and so forms the better basis for development of the final logic.[1]

In the system we will arrive at the problem of wide scope indefinites will be reduced to the problem of the scope of distributivity operators. As the formulas in (248) below show, the difference between a wide and narrow scope reading comes down to whether the restrictor of the indefinite is inside the scope of the distributivity operator associated with the quantifier:

(248) Every student who read a (certain) paper succeeded.

a. NS target:

$$\mathbf{M}^{\mathfrak{R}}(\varepsilon_x \wedge \delta_x(\mathsf{student}(x) \wedge \underbrace{\varepsilon_y \wedge \mathsf{paper}(y)}_{a\ paper} \wedge \delta_y \mathsf{read}(x,y))) \wedge \delta_x \mathsf{succeed}(x)$$

b. WS target:

[1]It is important that my argument not be misconstrued. Brasoveanu (2007) makes a convincing case that at least some assignment updates need to allow uncontrolled dependencies to be introduced. Interestingly, van den Berg (1996) also includes a number of assignment updates that allow different levels of freedom with respect to what sorts of dependencies can be generated by default. Since the only most constrained systems are most conducive to my goals in this dissertation, I will not spend much time going through the various arguments.

$$\mathbf{M}^{\mathfrak{R}}(\varepsilon_x \wedge \underbrace{\varepsilon_y \wedge \mathsf{paper}(y)}_{a\ paper} \wedge \delta_x(\mathsf{student}(x) \wedge \delta_y \mathsf{read}(x,y))) \wedge \delta_x \mathsf{succeed}(x)$$

In the formula above $\mathbf{M}^{\mathfrak{R}}$ is a maximization operator that finds the $\mathfrak{R}$-greatest output of the formula in its scope. In the narrow scope target formula the ε_y update occurs in the scope of the distributivity operator δ_x. This means that the formula in the scope of the maximization operator allows updates in which y varies with x. In the wide scope target formula the ε_y update is not inside the restrictor of the distributivity operator δ_x and so the value of y cannot vary with the value of x. The analysis in this chapter thus retains the same basic semantic intuitions developed in the previous chapter; all that is new is the logic in which these intuitions are embedded.

The second part of the chapter is concerned with examining exactly which relations can do the work of $\mathfrak{R}$ in the example above. I show that not every maximality operator that has been proposed in the literature generates the right predictions when slotted into the formulas in (248). In particular I will show that the maximization operator utilized by Brasoveanu (2007), Henderson (2014), and Kuhn (2015) is subject to the same class of counter-examples that afflicted C-FOL.

The third part of the chapter generalizes the account so that it can handle a wide variety of quantifiers. Here I stay very close to van den Berg's account. Generalized quantifiers have their semantics broken up into lexical and a dynamic components and are expressed by means of formulas like that below:

(249) $$\overbrace{\underbrace{\mathbf{M}^{\mathfrak{R}}(\varepsilon_x \wedge \delta_x \phi)}_{restrictor} \wedge \underbrace{\mathbf{M}^{\mathfrak{R}}(\varepsilon_{x' \subseteq x \wedge \delta_{x'} \psi})}_{scope}}^{Dynamic} \wedge \overbrace{\mathbf{every}(x,x')}^{Lexical}$$

The dynamic component first finds the maximal set of x's consistent with the formula in the restrictor of the quantifier. The scope then finds the maximal subset of the x's that satisfy the nuclear scope of the quantifier (this is done utilizing a copy-by-value assignment update: $\varepsilon_{x' \subseteq x}$). Finally the lexical aspect takes over indicating in the case above that the sentence is true just in case x and x' contain the same plurality of individuals.[2]

Even after the logic is updated to include generalized quantifiers, the basic analysis of wide scope indefinites in terms of the scope of distributivity operators remains. The chapter thus leaves off with a syntax semantics interface problem: if distributivity operators are provided by the quantifiers as suggested by the formula in (249) suggests, how do wide scope indefinites escape the scope of distributivity operators? While a full answer to this question will have to wait until the next chapter, a preview of the final analysis will help situate some of the decisions made in this chapter. In the final analysis I will suggest that distributivity operators should be decomposed into two parts: one part is responsible for signalling distributivity and the second part is responsible for contributing the quantificational force. This idea is sketched below:

(250) $\delta_x(\phi \wedge \psi) = \downarrow_x ({}^{\Delta}\phi \wedge {}^{\Delta}\psi)$

In this formula the $\downarrow_x$ signals to its scope that x is among the variables that are being distributed over (this information is passed down the tree in the recursive definition of truth). The operators ${}^{\Delta}$ then contribute quantificational force; ϕ and ψ separately look at the atomic elements of x. This creates space for an operator $\uparrow_x$ that calls off distributivity over the variable x to intervene before a ${}^{\Delta}$-operator has a chance to force distributive interpretation. In the final analysis indefinites will be able to remove themselves from the scope of distributivity operators.

Since the final analysis involves decomposing distributivity operators, one of the

[2]An alternative route for accounting for generalized determiners in this framework is perused by Brasoveanu (2007). In his system an input state G is fed first through the formula ϕ associated with the restrictor and a maximal output H is collected. Then G is fed through the formula $\phi \wedge \psi$ and a maximal output H' is collected. The values assigned to a variable in H can be compared to those assigned to the same variable in H'. I have not yet found anything that might turn on this distinction, but it is not obvious to me that the two formulations are equivalent.

concerns in this chapter will be the extent to which distributivity operators can be pushed around without effecting the truth conditions of a sentence. We will see that, minimally, distributivity operators distribute over conjunction and that for some values of $\mathfrak{R}$ they can even be pushed past maximization operators without effecting the truth conditions of a sentence.

Lastly, its worth pointing out the extent to which various aspects of this chapter are severable from the analysis proposed in the subsequent chapter. I think here the main contribution is in spelling out the typology of maximization. In this chapter I show that not all ways of defining the maximization operator will lead to intuitively correct truth conditions for formulas of the form in (248). These results will stand even if one wanted to find another route to enforce independence between a variable introduced by an indefinite and a variable introduced by a syntactically higher quantifier. If one thinks of the previous chapter as an extended argument showing that a dynamic semantics is necessary to capture wide scope indefinites, then the current chapter is an extended argument showing that dynamic semantics is not by itself sufficient; only certain maximization operators will do.

The rest of this chapter is structured as follows: I first describe two flavours of dynamic plural logic that one could imagine. I will ultimately adopt a flavour in which random assignment updates by default do not introduce new dependencies. I then cover the typology of maximization operators. I show that the simplest maximization operator can meet both of the desideratum outlined above. Then I turn to generalized quantifiers showing how the system can scale up to handle arbitrary generalized quantifiers. This is important since our analysis of wide scope indefinites should not be limited to cases in which they appear in the restrictor or nuclear scope of a universal.

4.2 Two Flavours of Dynamic Plural Logic

In this section I will outline two flavours of DPlL that have been proposed in the literature. The first is a simplified variant of the logic proposed by van den Berg (van den Berg, 1994, 1996) and Nouwen (2003). Its overarching goal is to account for plural discourse reference. In this logic, lexical relations are interpreted collectively with respect to the sets of assignment functions that serve as the input-output contexts, and assignment update operators generate no dependencies between introduced and previously available discourse referents. The dependency structure is managed entirely by distributivity operators: a distributivity operator generates local contexts that allow for (i) distributive interpretations of lexical relations and (ii) new (global) dependencies that arise outside the scope of the distributivity operator. The second logic is based around PCDRT[3](Brasoveanu (2007)) and following work especially Henderson (2014); Kuhn (2015). In this logic, lexical relations are interpreted distributively (as in C-FOL or C-DPL) and assignment updates can introduce new dependencies (again similar to C-FOL and C-DPL). In such a logic distributivity operators have a smaller role to play as they are required neither to (i) achieve distributive interpretations of lexical items nor (ii) allow new dependencies to arise. This system has an easier time accounting for cumulative interpretations.

I will examine the relative merits of these systems through the lens of sentences like (251).

(251) Two dogs chased two cats.

The sentence above has three interpretations of interest: (i) a distributive interpretation in which two dogs chased two cats each, (ii) a collective interpretation in which two

[3]These systems are different in that PCDRT offers a compositional system in a dynamic type logic that builds on CDRT (Muskens, 1996) while van den Berg's DPlL simply offers a dynamic logic.

dogs as a group chased two cats as a group, and (ii) a cumulative interpretation in which two dogs chased two cats between them, i.e. dog A chased cat A while dog B chased cat B. I will show that the van den Berg-Nouwen system has an easy time capturing readings (i) and (ii), but struggles to capture the cumulative reading. On the other hand the Brasoveanu-Henderson-Kuhn system easily captures readings (ii) and (iii), but struggles with the first reading (i) without introducing additional interpretive resources or plural individuals.

One of the key takeaways from this discussion is that collective lexical relations (van den Berg-Nouwen style) are useful for capturing collective readings, while dependency introducing assignment updates (Brasoveanu-Henderson-Kuhn style) are needed to capture the cumulative readings. However, Kuhn (2015) provides an argument against combining these two features into a single logic.

Ultimately, for the current purposes, I come down in favour of the system set up by van den Berg. In order to handle wide scope indefinites in the way outlined in the introduction, it will be necessary to have assignment update operations that are dependency free. Such updates will form the basic building blocks for indefinites capable of taking unbounded upward scope.

The remainder of this section is organized as follows. First, I will discuss the common features of the two systems: both conjunction and distributivity are defined identically in both logics. Second, I will discuss a simplified version of van den Berg's system and show that it is capable of accounting for collective and distributive interpretations of simple sentences like (251). Next, I discus Brasoveanu's system and show how it handles both cumulative and distributive interpretations of these sentences. I close the section by considering Kuhn's (2015) argument against systems that combine collective interpretation and dependency introducing assignment update.

4.2.0.1 Conjunction & Distributivity

Conjunction in all forms of DPlL is defined like its counterpart in DPL. Dynamic conjunction simply takes the output of the first formula and passes it to the second formula. A conjunction $\phi \wedge \psi$ is true relative to an input G and an output H iff there is some K s.t. ϕ is true relative to input G and output K and ψ is true relative to input K and output H.

(252) DYNAMIC CONJUNCTION :
$[\![\phi \wedge \psi]\!]^{G,H} = \mathbb{T}$ iff $\exists K : [\![\phi]\!]^{G,K} = \mathbb{T}$ & $[\![\psi]\!]^{K,H} = \mathbb{T}$

All variants of DPlL utilize a distributivity operator, δ. The distributivity operator δ_x, takes an input assignment function, breaks it up into sub-states corresponding to each value x takes in the output, passes each state on to the input, and knits together each of these outputs and returns them as a single a global output.

(253) DISTRIBUTIVITY:
$[\![\delta_x \phi]\!]^{G,H} = \mathbb{T}$ iff
a. $x \in \textbf{Dom}(G)$ and $G(x) = H(x)$
b. $\forall_{d \in G(x)} : [\![\phi]\!]^{G|_{x=d}, H|_{x=d}} = \mathbb{T}$

The first conjunct ensures that no new values for x are added to H that are not already present in the input. Since $G(x) = H(x)$, there is a one-to-one function between elements of $\{G|_{x=d} : d \in G(x)\}$ and $\{H|_{x=d} : d \in H(x)\}$. The second conjunct conjunct breaks the input assignment one sub-state for each value x can take and passes it on to ϕ.

One point bares emphasis: the semantic effect of a distributivity operator varies between various systems. I will show that in van den Berg's original logic a distributivity operator performs two roles (i) it allows for non-collective interpretations of predicates and (ii) provides a semantic context in which dependencies can be introduced. In alternative systems distributivity operators are not needed to handle non-collective interpretations or for the introduction of dependencies. Instead distributivity operators serve to block cumulative interpretations.[4]

4.2.1 van den Berg-Nouwen Style DPlL

In this section, I will outline a variant of DPlL based heavily on work by van den Berg (1996) and Nouwen (2003), though it simplifies their systems in several respects: first, I present a bivalent logic that conflates the definedness and truth conditions given by van den Berg and Nouwen.[5]Second, I ignore in this section the dummy value, $\star$, that indicates that a value is undefined in a particular assignment function in the plural state. While this value has its uses and will play a role in the final substantive portion of this chapter, it introduces several complications that are not pertinent at this juncture.

4.2.1.1 Model

The models van den Berg works with are almost entirely standard. A model $\mathfrak{M}$ is an ordered pair $\langle \mathfrak{D}, \mathfrak{I} \rangle$, consisting of a domain, $\mathfrak{D}$, of (atomic) individuals and an interpretation function, $\mathfrak{I}$. The interesting difference between the model of DPlL and models of FOL comes with the interpretation of predicates.

A one-place predicate is interpreted as a subset of $\wp^{+}(\mathfrak{D}) = \wp(\mathfrak{D}) - \emptyset$, i.e. a set of sets of entities. A simple predicate like dog, could have as its interpretation the set of singleton sets containing dogs,[6]while a collective predicate like gather would have only non-singleton sets in its denotation.

[4]The point here is this: the semantic effect of a distributivity operator can't just be read off the recursive definitions of truth for the operator. It is the interaction between distributivity operators and other elements of the logic that determines what the semantic effects of a distributivity operator are.

[5]Conflating falsity and definedness seems to be a standard move in the literature (see e.g. Brasoveanu (2007); Kuhn (2015)). If nothing else this move eases the exposition and facilitates the comparison between various systems.

[6]This is meant for expository purposes only, to model natural language the denotation of *dog* should

(254) $\mathfrak{I}(\mathsf{dog}) = \{\{\mathsf{f}\},\{\mathsf{r}\},\{\mathsf{s}\},\dots\}$
(255) $\mathfrak{I}(\mathsf{gather}) = \{\{\mathsf{a},\mathsf{b},\mathsf{c}\},\{\mathsf{d},\mathsf{e},\mathsf{f}\},\dots\}$

A two-place predicate is interpreted as a subset of $\wp^+(\mathfrak{D}) \times \wp^+(\mathfrak{D})$, i.e. as a relation between sets of entities. Consider the predicate chase below. It sets up a relationship between the singleton containing f and the singleton containing w indicating that f chased w. It also sets up a relationship between the non-singleton $\{\mathsf{a},\mathsf{b}\}$ and the singleton set containing c indicating that a and b chased c together, but not necessarily separately.

(256) $\mathfrak{I}(\mathsf{chase}) = \{\langle\{\mathsf{f}\},\{\mathsf{w}\}\rangle,\langle\{\mathsf{a},\mathsf{b}\},\{\mathsf{c}\}\rangle,\dots\}$

Notice that the model I have provided does not close the predicates under sum-formation/set-union by default. Instead we build these basic predicates so that they encode collective readings. If we deem a predicate to be cumulatively closed it must be modeled as a set of sets that is closed under set-union.

In this set-up there are no plural individuals in the domain, but there may be pluralities in the denotations of predicates. Pluralities play a role only in the interpretation of predicates. Van den Berg's goal is to replace ontological pluralities entirely with plural information states. In this way van den Berg's logic differs in a fundamental way from C-FOL and C-DPL despite utilizing similar data structures. Where C-FOL grants one a gods-eye-view of first order quantification, van den Berg's DPlL gives one a way to talk about pluralities without structuring the domain.

4.2.1.2 Lexical Relations

In van den Berg's system, lexical relations are interpreted cumulatively with respect to their input/output state. This is necessary if one is to represent plural individuals only

be modelled as cumulatively closed. Here I mean only to indicate that one place relations are properties of sets of entities—any property of sets of entities will do.

by means of plural information states. The truth conditions of a lexical relation feed the sets associated with each variable into the interpretation of the predicate. These sets thus play the role of plural individuals. Dependencies between variables are lost at this stage of interpretation since lexical relations are interpreted as relations between sets and do not make any reference to any relations between the values of the variables that might be encoded in the plural information states.

(257) LEXICAL RELATIONS:
$[\![R(v_1,\ldots,v_2)]\!]^{G,H} = \mathbb{T}$ iff
a. $G = H$
b. $\langle G(v_1),\ldots,G(v_2)\rangle \in \mathfrak{I}(R)$

The first conjunct is familiar: it simply says that lexical relations are tests. The second conjunct differs from the definitions of lexical relations that we have seen so far. Notice that lexical relations between two or more variables are not evaluated with respect to the projection of those variables. Instead they are evaluated with respect to the unstructured sets of values that the variables take on. In other words if a lexical relation $R(x,y)$ is true of the input-output state G, it is because $\langle G(x),G(y)\rangle \in \mathfrak{I}(R)$. This in itself places no constraints at all on $G(x,y)$—any relation at all between the two variables will be consistent with the lexical relation.

To handle number morphology in this system we introduce two special predicates **sg** and **pl**. These predicates are evaluated exactly like lexical relations. The formula $\mathbf{sg}(x)$ is true in a context iff x is assigned only one value, while $\mathbf{pl}(x)$ is true whenever x is assigned more than one value.

(258) SINGULAR and PLURAL:
a. $[\![\mathbf{sg}(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| = 1$
b. $[\![\mathbf{pl}(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| > 1$

Numerals can be handled in a similar fashion. We define a series of one place predicates $1,2,\ldots$ that are true only of pluralities of a certain cardinality:

(259) NUMERALS:
$[\![n(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| = n$

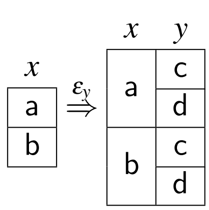
4.2.1.3 Dependency Free Assignment Update & Collective Readings

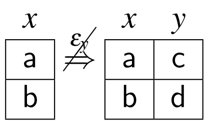
Assignment updates in van den Berg's system are number neutral and dependency-free, i.e. an assignment update is define in such a way that it is compatible with both singular and plural updates but it will never introduce a dependency between the new variable and any previously introduced variable.

(Dependency-Free) assignment update is defined below. The first conjunct ensures that updates are non-destructive. Random assignment never over-writes variables that are already in use. The second conjunct says that for every $g \in G$ and $d \in D$, H contains the assignment $g^{[x \to d]}$. Note that $|H| = |G| \times |D|$ and that ε_x does not introduce any new dependencies between x and any other variable.

(260) (DEPENDENCY-FREE) RANDOM ASSIGNMENT:
$[\![\varepsilon_x]\!]^{G,H} = \mathbb{T}$ iff
a. $x \notin \mathbf{Dom}(G)$
b. $\exists D \subseteq \mathfrak{D} : H = \{g^{[x \to d]} : g \in G \,\&\, d \in D\}$

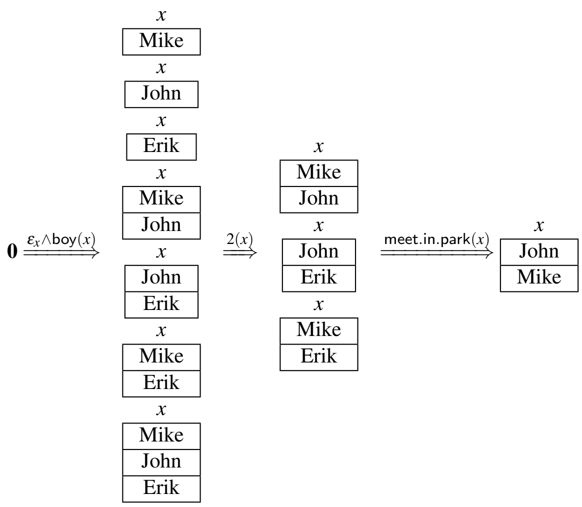
The workings of ε are illustrated below for an update of the variable y with the values in $D = \{\mathsf{c}, \mathsf{d}\}$. The update in (261) is well formed since every input assignment is updated with every value in D. Notice that the output in (262) does not have any dependencies between the variables x and y. The update in (262) does not satisfy ε_x. Notice that it does induce dependencies between x and y. However, there are input assignments, e.g. $g = \{\langle x, \mathsf{a}\rangle\}$, that are not updated with every value in D.

(261)


(262) 

With the material already at hand we can generate reasonable representations of sentences with collective predicates.

(263) Two boys met in the park. $\rightsquigarrow \varepsilon_x \wedge \mathsf{boy}(x) \wedge 2(x) \wedge \mathsf{meet.in.park}(x)$

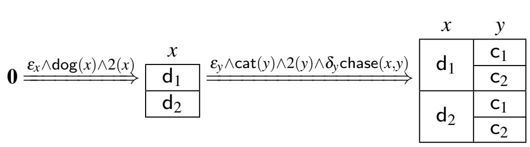


The first update, $\varepsilon_x \wedge \mathsf{boy}(x)$, is compatible with any update in which x contains a set of boys.[7]The next update $2(x)$ is compatible only with those sets of assignments in which x contains two elements. The final update $\mathsf{meet.in.park}$ then removes any sets of assignments in which the x-boys did not collectively meet in the park.

Collective interpretations of other predicates like *chase* or *lift* will also be the default case in the logic we have developed.

(264) Two dogs chased two cats.

$$\rightsquigarrow \varepsilon_x \wedge \mathsf{dog}(x) \wedge 2(x) \wedge \varepsilon_y \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \mathsf{chase}(x,y))$$

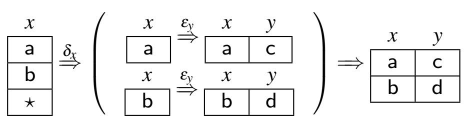
(265) 

[7]I am assuming here that the predicate boy is cumulatively closed. If however we wanted to treat the predicate boy as picking out only single boys, then we would have to modify the formula here to contain $\delta_x \mathsf{boy}(x)$

First we update with a set of two dogs, next we update with a set of two cats that the two dogs collectively chased in a dependency free way. Notice that when we interpret any predicates we gather up the set of dogs and the set of cats.

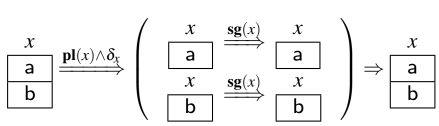
4.2.2 Distributivity & Distributive Readings

The dependency free assignment update ε_x by itself will never introduce any dependencies. New dependencies come into the state by means of the distributive operator, δ_x. In the scope of a distributivity operator assignment updates are able to introduce new dependencies.

(266)


The distributivity operator breaks up the set of assignments into sub-states defined in terms of the value taken on by x. The assignment update ε_y then operates on each of these sub-states separately. In none of these local updates are any dependencies introduced. However, when taken together a global dependency can arise.

Distributivity operators can also change the number associated with a variable. Under a distributivity operator a global context that assigns a plurality to a variable x will by definition contain only a single value in each of the sub-states that would be accessed by distributing over x.

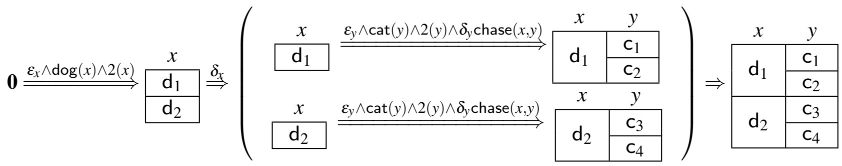
(267)


In the global state, x is associated with the plurality $\{\mathsf{a}, \mathsf{b}\}$, but when being distributed over x contains only a single value. In this way a distributivity operator manipulates the number associated with a variable.

With distributive operators we can provide interpretations for distributive interpretations for simple sentences like that in (251).

(268) Two dogs chased two cats.

$$\rightsquigarrow \varepsilon_x \wedge \mathsf{dog}(x) \wedge 2(x) \wedge \delta_x(\varepsilon_y \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \delta_y \mathsf{chase}(x,y))$$

(269) 

Because we break up the set of assignments up into sub-states for each dog we ultimately end up relating two (possibly) different cats to each dog. Likewise, because the predicate $\mathsf{chase}(x,y)$ is evaluated within the scope of two distributivity operators it will be evaluated relative to sub-states corresponding to each row. That is, in each row the x-dog must have chased the y-cat.

4.2.2.1 Interim Summary

In summary van den Berg-Nouwen style DPlL has the following three properties:

i. Collective interpretation of lexical relations by default.

ii. Dependency free assignment updates.

iii. Distributivity operators that open up the possibility of dependencies and alter the interpretation of all lexical items including cardinality predicates.

4.2.3 Brasoveanu-Henderson-Kuhn Style

This section describes a variety of DPlL that can be extracted from work by Brasoveanu (2007); Henderson (2014); Kuhn (2015).

4.2.3.1 Models

The models here entirely standard. Crucially the interpretation function assigns n-ary predicates denotations that are subsets of $\mathfrak{D}^n$. Unless the domain itself contains plural individuals, lexical relations denote relations only between atomic entities, not relations between sets of entities as in van den Berg's logic.

4.2.3.2 Lexical Relations

Lexical relations in Brasoveanu-Henderson-Kuhn style DPlL are interpreted distributively as they are in C-FOL and C-DPL.

(270) LEXICAL RELATIONS:
$[\![R(v_1,\ldots,v_2)]\!]^{G,H} = \mathbb{T}$ iff
a. $G = H$
b. $\forall_{g\in G} : \langle g(v_1),\ldots,g(v_2)\rangle \in \mathfrak{I}(R)$

The first conjunct ensures that the input and output assignments are identical. The second checks each assignment function in the plural state against the lexical relation. Notice that unlike the definition in (257) the definition above is not neutral with respect to the dependency structure in the input state. The relationships between variables matters when evaluating lexical relations.

Interestingly, number cannot be handled in terms of lexical relations. This is because a lexical relation is always evaluated with respect to each assignment function in the plural state. In order to determine whether a plural state actually contains any particular number of entities as a whole requires looking at more than one assignment function at a time. Although this is the default case in van den Berg style DPlL, a variant that utilizes distributive interpretation for lexical relations needs special predicates to handle number morphology.

(271) NUMBER MORPHOLOGY:
a. $[\![\mathbf{sg}(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| = 1$
b. $[\![\mathbf{pl}(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| > 1$
c. $[\![n(x)]\!]^{G,H} = \mathbb{T}$ iff $G = H$ & $|G(x)| = n$

Notice that the definitions above are exactly the same as those provided for in van den Berg style DPlL. However, they have a different status since they are not simple lexical relations.

4.2.3.3 Random Assignment

Assignment update is defined entirely in terms of variable difference.

(272) (DEPENDENCY-INTRODUCING) RANDOM ASSIGNMENT:
$[\![[x]]\!]^{G,H} = \mathbb{T}$ iff

a. $x \notin \mathbf{Dom}(G)$

b. i. $\forall_{g \in G} : \exists_{h \in H} : h \Leftarrow_x g$

ii. $\forall_{h \in H} : \exists_{g \in G} : h \Leftarrow_x g$

The first conjunct here is familiar; it guards against overwriting a variable that is already introduced in the input state. The second conjunct ensures that every element of g is extended by some element of h and that every element of h is an extension of some element of g. The update created by the formula $[x]$ is compatible with outputs that generate dependencies between x and previously introduced variables.

Consider the examples below which contrast updates that assign to y the set $\{\mathsf{a}, \mathsf{b}\}$ that are possible with ε_y and $[y]$:

(273)

x
a
b

$\overset{\varepsilon_y}{\Rightarrow}$

x	y
a	c
	d
b	c
	d

(274)

x
a
b

$\overset{[y]}{\Rightarrow}$

x	y
a	c
	d
b	c
	d

x	y
a	d
b	c
	d

x	y
a	c
	d
b	c

⋮

x	y
a	c
b	d

These examples show that $[x]$ is more permissive than ε_x. There is only one way to update y that is permitted by ε_y because it always generates the cross product of the original set of of assignment functions and the set it is updating with. In contrast the $[y]$ update allows any relation to obtain between any variable in the input and the variable introduced by the update.

4.2.3.4 Distributivity

Its worth pausing to reflect on the semantic effect of a distributivity operator in the system we are now considering. First, since lexical relations are interpreted distributively by default, a distributivity operator will not figure in an account of distributive vs collective interpretations of simple predicates. Second, since random assignment updates can introduce dependencies between the introduced variable and previously introduced discourse referents, distributivity operators will not be needed to license dependencies between variables.

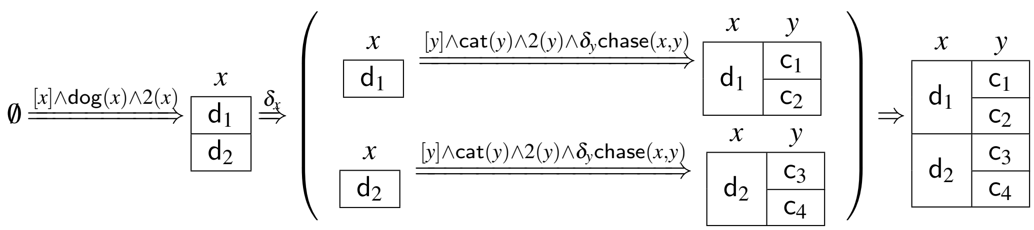
Distributivity operators are thus limited in their effect: they only provide local contexts which effect the evaluation of expressions used to encode number morphology. Since these expressions retain the same collective interpretation that they had in van den Berg style DPlL, they retain the same behaviour with respect to distributivity operators.

4.2.3.5 Distributive vs Cumulative Interpretation

The system developed so far does a nice job of capturing both cumulative and distributive interpretations. Consider our previous cats and dogs example. The distributive interpretation looks exactly like the one provided in the van den Berg system modulo the change in the assignment updates.

(275) Two dogs chased two cats.

$$\rightsquigarrow [x] \wedge \mathsf{dog}(x) \wedge 2(x) \wedge \delta_x([y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \delta_y \mathsf{chase}(x,y))$$

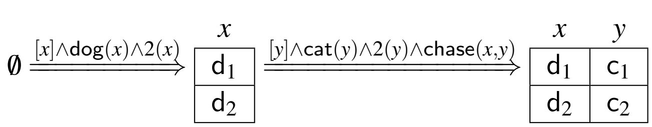
(276) 

The first set of updates introduces the variable x with and assigns it two dogs. The sub-states corresponding to each dog are then updated distributively with the complex update $[y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \mathsf{chase}(x,y)$. The action occurs when we evaluated the expression $2(y)$ in the scope of δ_x. The lexical relation checks that two y-cats are assigned in the sub-state associated with each dog. This leads ensures that d_1 chased two cats and d_2 chased two cats.

In order to achieve the cumulative interpretation we simply remove the distributivity operator associated with x. The resulting interpretation is shown below.

(277) Two dogs chased two cats.

$$\rightsquigarrow [x] \wedge \mathsf{dog}(x) \wedge 2(x) \wedge [y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \mathsf{chase}(x,y)$$

(278) 

The first set of updates again introduces a set of two dogs. The second update introduces two cats, since the assignment update can generate new dependencies, it can match d_1 with c_1 and d_2 with c_2. Furthermore, since $2(y)$ is not evaluated within the scope of a distributivity operator, it is not necessary that there be two cats associated with each dog. Finally, since the interpretation of lexical relations is distributive the predicate $\mathsf{chase}(x,y)$ requires that the chase relation holds in each row. It does not, as in van den Berg's system, require that the plurality $\{\mathsf{d}_1,\mathsf{d}_2\}$ have chased the plurality $\{\mathsf{c}_1,\mathsf{c}_2\}$.

Notice that this analysis is not open to us in the van den Berg-Nouwen style system because it crucially depends on the possibility of an assignment update introducing new dependencies.

4.2.3.6 Interim Summary

In summary the Brasoveanu-Henderson-Kuhn style DPlL has the following three features:

i. Lexical relations are interpreted distributively always.

ii. Assignment updates are fully random and can generate new dependencies.

iii. Distributivity operators only effect the interpretation of cardinality checks, which are interpreted collectively.

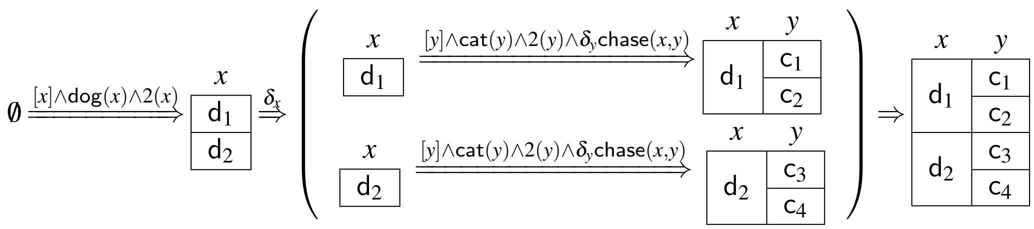
4.2.4 Issues with Hybrid Systems

In the discussion above we saw that a system that featured collective interpretation of predicates made it easy to handle collective readings, while a system that had dependency introducing random assignments made it easy to handle cumulative readings. This raises the question of whether these two features should be combined into a single system that features both (a) collective interpretation of predicates and (b) dependency introducing random assignments.

Kuhn (2015) considers this possibility but notices the following problem: a system that combines collective interpretation with dependency introducing assignment can account for all three readings, but over-generates dependencies with collective readings.

Distributive readings are accomplished in the familiar way. A distributivity operator takes scope over the material in the nuclear scope of the both numerals:

(279) Two dogs chased two cats. $\rightsquigarrow [x] \wedge \mathsf{dog} \wedge 2(x) \wedge \delta_x([y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \delta_y \mathsf{chase}(x,y))$

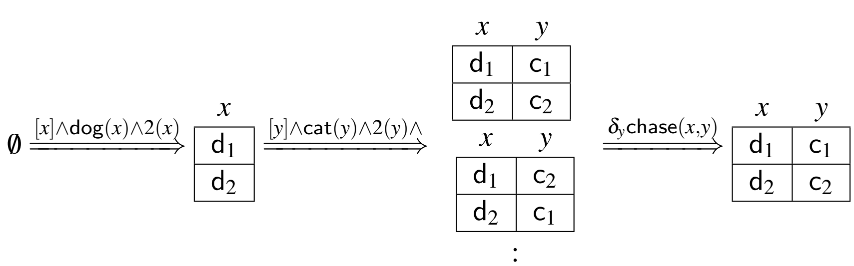
(280) 

This derivation looks exactly like the derivation used to derive distributive readings in van den Berg's system.

To handle cumulative readings we leave only the distributivity operator associated with the syntactically lower numeral. Imagine a situation in which dog_1 chased cat_1 and dog_2 chased cat_2. We would then get an update like that below:

(281) Two dogs chased two cats.

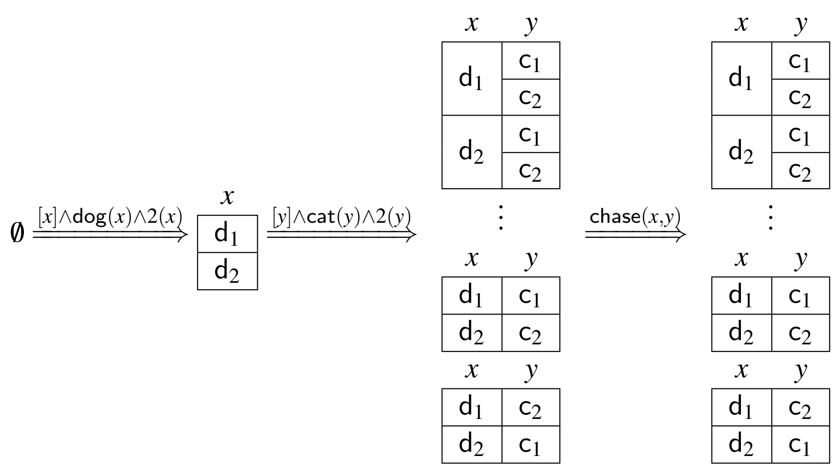
$\rightsquigarrow [x] \wedge \mathsf{dog} \wedge 2(x) \wedge [y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \delta_y \mathsf{chase}(x,y)$

(282) 

The update again starts with a set of two dogs followed by a set of cats which can depend on the dogs. Finally since the distributivity operator associated with y takes scope over chase(x, y) we look at each y-cat and make sure it was chased by the corresponding set of x-dogs. This filters out output sets of assignments which don't get the dependencies between cats and dogs right.

Collective readings can also be captured by translations in which distributivity operators are entirely absent. The problem with these readings is that they are agnostic about the dependency structure that holds between the two variables. Consider the calculation below:

(283) Two dogs chased two cats. $\rightsquigarrow [x] \wedge \text{dog} \wedge 2(x) \wedge [y] \wedge \text{cat}(y) \wedge 2(y) \wedge \text{chase}(x, y)$

(284) 

The first set of updates delivers a plurality of dogs. The second group of updates delivers plurality of cats. Because assignment update is dependency introducing the full range of relations between the cats and the dogs is generated. Finally, the predicate chase(x, y) is interpreted collectively. Since each set of assignment functions assigns the same set to the projections of both x and y none of these cases are filtered out. The set of available outputs can accommodate any relation between the dogs and the cats.

Ideally we would like only top output in which there is no dependency between cats and dogs. Indeed discourses like the one below suggest that the output of collective interpretations does not provide an articulated relationship between the dogs and the cats that can be accessed in subsequent discourse.[8]

(285) Two dogs collectively chased two cats. # They each caught it. / They caught them.

One cannot distribute over the dogs and find a single cat associated with each. Instead, one can only reference the plurality of dogs and the plurality of cats. This would only be predicted by a system that allowed only a dependency free output for collective interpretations. If collective interpretations provided outputs that allowed for dependencies one would expect to be able to elaborate on the structure in subsequent discourse.

4.2.5 Summary

In this section I have described two variants of DPlL. The first is based heavily on van den Berg's original formulation while the second utilizes alternative choices made in the systems used by Brasoveanu (2007); Henderson (2014); Kuhn (2015). Because the overall goal of this project is to interpret wide scope indefinites in the spirit of Brasoveanu and Farkas (2011), I will utilize a van den Berg style logic. This is because we want the underlying system to limit dependencies to just those indefinites that occur in the scope of distributivity operators.

At the same time this section has highlighted a shortcoming of the resulting system: cumulative interpretations. These seem to require dependency introducing assignment (as argued in Brasoveanu (2012)). Brasoveanu (2007) also points to mixed weak/strong readings of donkey sentences as a reason to countenance dependency introducing random assignment. Additionally Kuhn (2015) points out a system with both collective interpretation and dependency introducing random assignment seems untenable.

[8] Though see Brasoveanu (2007) for an argument against this perspective based on mixed weak/strong readings of donkey anaphora. A sentence like *Everyone who bought a book on amazon.com and has a credit card used it to pay for it* seems to make reference not only to the books people bought on amazon.com and the credit cards they have but also the relationship between books and credit cards. This suggests that when the nuclear scope of the universal is processed it allows outputs in which there are dependencies between books and credit cards despite the absence of linguistic material that would force them to arise at that point in the interpretation.

4.3 The Typology of Maximization

This section argues for the inclusion into the logic of an unselective maximization operator, i.e. a maximization operator that maximizes the output set of assignments in a way that does not privilege any particular variable over another. The maximization operator I propose is defined simply in terms of the subset relation—an output H is maximal relative to an input G and a sentence ϕ iff (i) $[\![\phi]\!]^{G,H} = \mathbb{T}$ and (ii) there is no $K \supset H$ s.t. $[\![\phi]\!]^{G,K} = \mathbb{T}$. This maximization operator offers two advantages over its competitors: (i) non-deterministic updates in ϕ are correctly percolated to maximal output assignments and (ii) this maximization operator supports distributive normal form, i.e. a distributivity operator can be freely permuted with a maximization operator defined this way.

I will also show in this section that other maximization operators proposed in the literature lack one or both of these properties. A maximization operator proposed by Brasoveanu (2007) and utilized widely in the literature, lacks the first property—it does not correctly percolate non-determinism generated by a formula in its scope. The maximization operator proposed by van den Berg (1996) and utilized in closely related work, e.g. Nouwen (2003), does correctly percolate non-deterministic updates but does not support distributive normal form. This property while not strictly necessary for the analysis sketched in the introduction will become important in the next chapter where I present a system in which wide scope indefinites can be derived compositionally.

Moreover, I show that neither of these maximization operators suppress dependencies. That is both the Brasoveanu (2007) maximization and van den Berg (1996) maximization will non-deterministically allow outputs that have more structural information than is supported by the material in the scope of the maximization operators.

This remainder of this section is split into several parts. The first section discusses two abstract schema that can be used to define a number of maximization operators. The first schema, which I attribute to van den Berg, is simplest: it parametrises maximization with respect to a relation $\mathfrak{R}$ that is used to determine which among a set of potential outputs are maximal. One perversity of this schema is that a maximization operator can take scope over a test in which case the operator is necessarily vacuous. In all natural language applications, maximization operators take scope over dynamic formulas. This has lead subsequent authors to define maximization in terms of both a relation $\mathfrak{R}$ and a variable x. The maximization operator itself updates the discourse with values for x. This change is subtle and has not been remarked on in the literature as far as I am aware. I will argue for a system in which maximization is parametrized only with respect to a relation $\mathfrak{R}$ and does not include assignment update in its own definition. I will show that in the system we are working with, in which assignment updates do not introduce any new dependencies outside the scope of a maximization operator, maximization operators that themselves update the assignment function do not support distributive normal form.

The second section forms the logical core of this chapter. It develops a typology of maximization operators and defines a number of properties possessed (or not) by different maximization operators proposed in the literature. The most important are (i) whether the maximization operator handles non-deterministic updates in a way that will support the analysis of wide scope indefinites sketched in the introduction and (ii) whether the maximization operator supports distributive normal form.

4.3.1 Maximization Schema

This section sets out the general notation I will use for defining maximization operators. There are two recipes for defining maximization operators that can be extracted from the literature. The first comes from van den Berg's early work on DPlL, while the second I have extracted from much of the subsequent literature. The difference between these two recipes is almost always functionally non-existent. However, for my current project there are logical reasons to prefer definitions of maximization more in line with van den Berg's: with van den Berg's definition it becomes possible to define maximization operators that support distributive normal form; more recent definitions of maximality operators prevent this possibility.

The first recipe for maximization, I attribute to van den Berg. Note that this is not a definition he explicitly provides, but his initial output maximization operator can be fit into this schema:

(286) Van den Berg Recipe:
$[\![\mathbf{M}^{\mathfrak{R}}(\phi)]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H} = \mathbb{T}$
b. $\neg\exists_{K\mathfrak{R}H} : [\![\phi]\!]^{G,K} = \mathbb{T}$

The definition in (286) will provide a maximization operator as long as $\mathfrak{R}$ is a transitive, asymmetric, and antireflexive relation (i.e. a partial order without the reflexive links). This maximization operator finds and delivers an $\mathfrak{R}$-largest output consistent with (i) the formula in its scope and (ii) its input assignment.

An astute reader may notice that if the sentence ϕ is a test, the restrictor will only be true for outputs identical to the input and so maximization will be trivial; in order for a maximization to have a non-trivial effect it must take scope over a externally dynamic formula. The practising semanticist will always find herself utilising formulas like those in (287) in which a maximization operator takes scope over a conjunction that contains at least (i) an assignment update operation and (ii) a test that restricts possible values assigned to the variable.

(287) $\mathbf{M}^{\mathfrak{R}}(\varepsilon_x \wedge P(x))$

This observation has lead subsequent authors to amend van den Berg's recipe along the lines shown in (288).

(288) Brasoveanu Recipe:
$[\![\mathbf{M}^{\mathfrak{R}}_{x}(\phi)]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\varepsilon_x \wedge \phi]\!]^{G,H} = \mathbb{T}$
b. $\neg\exists_{K\mathfrak{R}H} : [\![\varepsilon_x \wedge \phi]\!]^{G,K} = \mathbb{T}$

The definition in (288) adds an assignment update to van den Berg's definition. This ensures that the maximization operator is never vacuous and provides a generally more compact notation to the working semanticist. For instance, the expression in (287) can be more perspicuously expressed using the notation in (288).

(289) $\mathbf{M}^{\mathfrak{R}}_{x}P(x)$

While it bares emphasis that there is little substantive difference between the notation given in (287) and that in (288), there are trade-offs between the two notations. The Brasoveanu recipe has several concrete advantages for natural language semantics: it is not clear maximization in natural language would ever fail to involve an assignment update and the definition in (288) builds this property into the notation. However, there are two disadvantages to the Brasoveanu Recipe in the current context:

i. This recipe obscures the relationship between the variable x and the relation $\mathfrak{R}$. Consider the maximization operator obtained by setting $\mathfrak{R}$ to the strict subset relation. The variable x plays no special role in determining which potential outputs are maximal. The Brasoveanu recipe notationally privileges the variable x despite the fact that it has no special role to play in determining maximal outputs. In other candidates for maximization particular variables do play important roles in determining the set of maximal outputs to the exclusion of other variables. In these cases we will see that $\mathfrak{R}$ is parametrized with these variables. The van den Berg recipe thus wears its (un)selectivity on its sleeve.

ii. One logical issue that will take special importance in the next chapter is the issue of distributive normal form. A maximisation operator **M** supports DNF iff $\delta_v \mathbf{M}\phi = \mathbf{M}\delta_v\phi$. No maximization operator put into the Brasoveanu notation supports DNF unless it also utilizes dependency introducing random assignment: since the assignment update is baked into the maximization operator, pushing the distributivity operator past the maximization operator also pushes it past the implicit assignment update. This is important when we are working with assignment updates that can only introduce new dependencies when they occur in the scope of distributivity operators. Pushing the distributivity operator past the maximization operator necessarily changes the set of allowable outputs because it restricts possible dependency relations in the set of possible outputs.

Note that the second criticism of the Brasoveanu recipe is only applicable to the current context. Recall that Brasoveanu's system utilizes assignment updates that can introduce new dependencies. The assignment update $[x]$ is such that $\delta_v([x] \wedge \phi) = \delta_v[x] \wedge \delta_v\phi = [x] \wedge \delta_v\phi$ so criticism (ii) above does not apply; it is only when we utilize dependency free assignment update that trouble arises since $\delta_v\varepsilon_x \neq \varepsilon_x$.

In the rest of the chapter I will adopt the van den Berg recipe for maximization. I will assume that as a matter of fact natural language does not involve maximization over tests and that the translations of quantifiers with involve not only maximization operators but also assignment updates in their scope.

4.3.2 Two Selective Maximization Operators

The two maximization operators I discuss in this section are both taken from the literature. The first, which comes from Brasoveanu (2007); Henderson (2014); Kuhn (2015) a.o., is altered in two ways: first, I have altered it to fit the maximization scheme discussed above, and, second, I have defined it in terms of dependency free assignment update. The second comes directly from van den Berg (1996). Both of these maximization operators are selective, i.e. there is a specific variable that plays a privileged role in the relation over which the output is maximized. I will show that Brasoveanu's maximization operator is not suited for a logic that attempts to account for wide scope indefinites in terms of the scope of distributivity operators and included dependency free assignment update. In fact, using this maximization operator falls prey to exactly the same counter-example as the original C-FOL was subject to. Van den Berg's maximization operator, however, is immune to this particular criticism.

The first selective maximization operator I will discuss is defined in terms of the relation $\supset_x$ defined below:

(290) $K \supset_x H := K(x) \supset H(x)$

The operator defined in terms of the relation in (290) finds outputs that associate as many values with the variable x as possible:

(291) $G[\![\mathbf{M}^{\supset_x}(\phi)]\!]H$ iff $G[\![\phi]\!]H$ & $\neg\exists_{K \supset_x H} : G[\![\phi]\!]K$

This definition is well suited to Brasoveanu's purposes; it plays a crucial role in his account of cumulative readings of numerals and quantifiers like *every* and works well in systems in which variable assignment has the potential to introduce new variables.[9]

However, the maximization operator given in (291) will not allow us to handle wide scope indefinites in the manner outlined in the introduction to this chapter. Recall that our target analysis is one in which the wide scope of an indefinite can be reduced to its scope relative, not to the maximization operator, but to the scope of a distributivity

[9]Brasoveanu's maximization operator also gives one a sort of dynamic λ-abstraction. See Brasoveanu (2007) §3.4 when combined correctly with other elements of his system.

operator. The problem with the definitions in (290-291) can be seen by considering the scenario which proved impossible to handle in C-FOL.

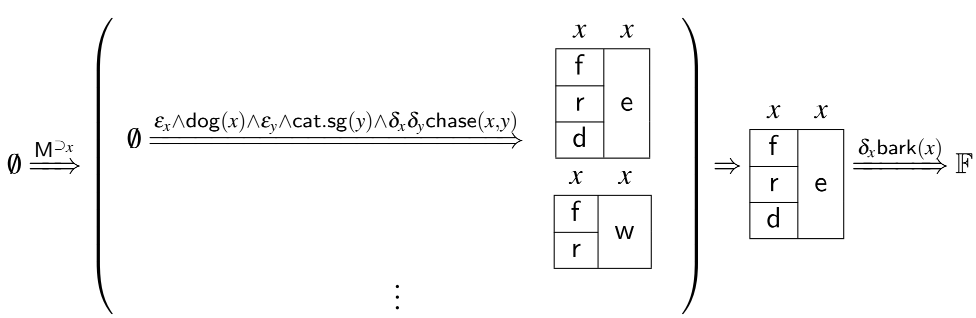
Recall that we have a situation in which every dog that chased Whiskers barked, not every dog that chased Evander barked, and every dog that chased Whiskers also chased Evander. In this scenario the sentence in (292) is intuitively true on its wide scope interpretation.

(292) Every dog that chased a certain cat barked.

The target translation of the sentence in (292) is given below:

(293) $\mathsf{M}^{\supset x}(\varepsilon_x \wedge \mathsf{dog}(x) \wedge \varepsilon_y \wedge \mathsf{cat.sg}(y) \wedge \delta_x \delta_y \mathsf{chase}(x,y)) \wedge \delta_x \mathsf{bark}(x)$

The first update will deliver a maximal output that (i) contains a set of dogs as the value of x, (ii) a set of cats as the value of y, (iii) no dependencies between the cats and the dogs (since ε_y is not in the scope of a δ_x), s.t. every dog x chased the cat(s) y associated with it. If we assume that Fido, Rex, and Dudley chased Evander, while only Fido and Rex chased Whiskers, we can depict the dynamics of the sentence in (293) in the following way.

(294) 

The formula in the scope of the maximization operator is itself compatible with outputs in which all the dogs chased Evander and outputs in which all the dogs chased Whiskers. However, since the Whiskers-chasers are a proper subset of the Evander-chasers, the relation $\supset_x$ selects only the output consisting of all the Evander-chasers.

This results in the entire sentence being evaluated as false since not all the Evander chasing dogs barked. This problem is exactly the problem that confronted C-FOL.

The trouble with the maximization operator defined in (291) is that it is too selective. It selects maximal output assignments in a way that is sensitive only to the values assigned to the variable x. If we wish to define a maximization operation that allows both outputs depicted in (294), then the maximization operator will have to be sensitive to the values of other variables as well. This is not to say that it is must attempt to maximize over other variables, but that it must not compare outputs that differ with respect to the values taken on by other variables.

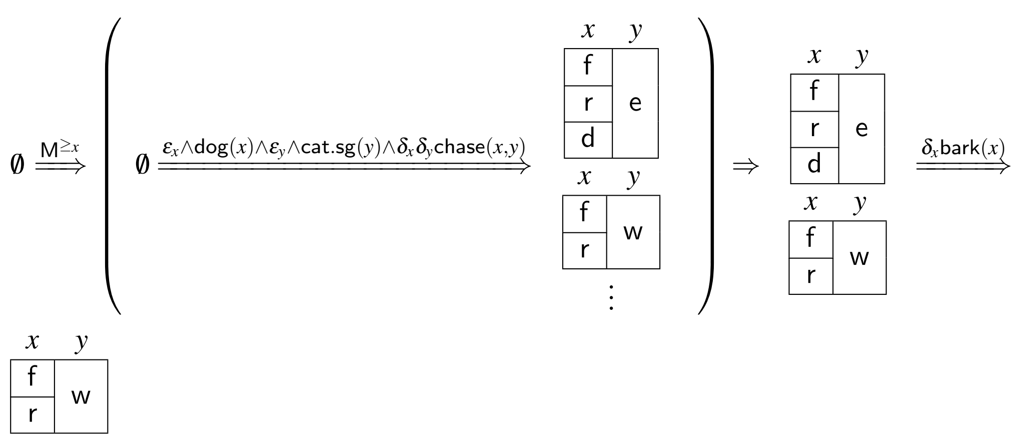
Interestingly van den Berg was aware of this behaviour when he defined his maximization operator in terms of the more complicated relation $>_x$ defined in (295):

(295) $K >_x H := K(x) \supset H(x) \;\&\; K \supseteq H$

Van den Berg's definition simply adds a condition to Brasoveanu's. His definition requires that comparable information states must stand in a subset/superset relation. This relation allows us to define the following maximization operator:

(296) $G[\![\mathbf{M}^{>_x}(\phi)]\!]H$ iff $G[\![\phi]\!]H \;\&\; \neg\exists_{K>_xH} : G[\![\phi]\!]K$

If we use this definition of maximization in our target translation, we see that the correct results are achieved:

(297) 

Notice that both desired outputs from the maximization operator are now allowed. They are both candidates because they do not stand in a subset/superset relation and thus are not comparable in terms of the $>_x$ relation.

Interestingly, van den Berg made this argument on purely logical grounds. His justification is worth quoting in full:

> ...consider why there might be different output states for a given input state. The only reason for different outputs are random assignments inside ϕ adding variables to the input with different values. In the case of x... the extra variable y introduced need not be dependent on x, and in that case we need to be careful. Suppose that y can take the values $\{a\}$. if x is $\{c\}$ and y is $\{b\}$ if $\{c,d\}$ Then we do not want to lose the value $\{a\}$ just because x is maximized. It stays a matter of further research whether such cases every really occur.

This justification fits exactly the scenario described above. Van den Berg thus defined his maximization operator so that it correctly percolates indeterminacy from its nuclear scope. Moreover, as the final sentence in the quotation above makes clear, van den Berg did not have wide scope indefinites in mind as a justification for his maximality operator.

4.3.3 Dependency Suppressing Maximization

In this section, I will discuss a maximization operator that is of only logical interest. This is the dependency suppressing maximization operator. A maximization operator that is dependency suppressing should select outputs that have as few dependencies as possible given the formula in its scope. It should not attempt to increase the values assigned to variables in its scope but instead rule out output states that have more dependencies than are justified given (i) the values assigned to the variables in the output and (ii) the formula in the scope of the maximization operator.

The dependency suppressing maximization operator holds logical interest for two reasons. First, it helps us isolate the property of the relation $\mathfrak{R}$ that will make any maximality operator dependency suppressing. Second, recall Kunh's (2015) criticism against the inclusion of fully dependency introducing assignment updates, $[x]$, into a system that utilized collective interpretations of lexical relations: dependency introducing assignment updates created arbitrary and unjustified dependencies in potential outputs. If such assignment updates occur inside the scope of a dependency suppressing maximality operator, this criticism would be avoided. The maximization operator would ensure that no dependencies that were not justified by the lexical material could occur in the output.

In order to accomplish this task we will need a way to measure two output assignments H and K and determine (i) if H and K are comparable with respect to one another and (ii) if so, whether H has more are less dependencies between its variables than K. This requires being more fine grained in our discussion of dependence and independence between two or more variables than we have up till now. We have only said that in an assignment H the variables $v_1, \ldots, v_n$ are independent of one another iff $H(v_1, \ldots, v_n) = H(v_1) \times \cdots \times H(v_n)$, i.e. we take the n-place relation H assigns to the variables $v_1, \ldots, v_n$ and determine if it is simply the cross-product of the values that H assigns to each of the variables independently.

For any assignment function K, we can generate a set of assignment functions K' s.t. $K(v) = K'(v)$ for every variable v and K' that also contains no dependencies between any variables defined in the info state K. To pick out such a state, I will use the notation in (298).

(298) DEPENDENCY SCRAMBLING:
$K^{+} := \{k : \forall_{v} : \exists_{k' \in K} : k'(v) = k(v)\}$

The definition of K^{+} scrambles the variable assignments in the state K: K^{+} consists of those assignment functions k s.t. for every variable, there is some $k' \in K$ s.t. k and k' agree on the value assigned to that variable. This ensures first that for every variable v $K(v) = K^{+}(v)$ and second that if $\langle \mathsf{a}, \mathsf{b} \rangle \in K(x) \times K(y)$, then $\langle \mathsf{a}, \mathsf{b} \rangle \in K^{+}(x, y)$, i.e. that there are no dependencies between any variables in K^{+}.

The $^{+}$ operation places an upper bound on which output states should be considered to have fewer dependencies between the variables present in a particular output. If an output $H \supset K+$, then there must be some variable v s.t. $H(v) \supset K(v)$, i.e. H assigns more values to some variable than K assigns to that variable, and thus the two should not be compared by the dependency suppressing maximization operator, since the operator aims only to prevent dependencies that are not justified by the material in its scope from entering into the computation.

A dependency suppressing maximization operator can be defined in terms of the relation $\supset^{+}$ given below:

(299) $K \supset^{+} H := H^{+} \supseteq K \supset H$

If $K \supset^{+} H$ holds then two requirements must be met: (i) K must be a (improper) subset of H^{+}, i.e. it must not include any more values for any variables that are already present in H, and (ii) K must be a proper superset of H, i.e. it must include strictly more rows than H. Since these additional rows cannot include new values for any variables they must hold between values already present in H.

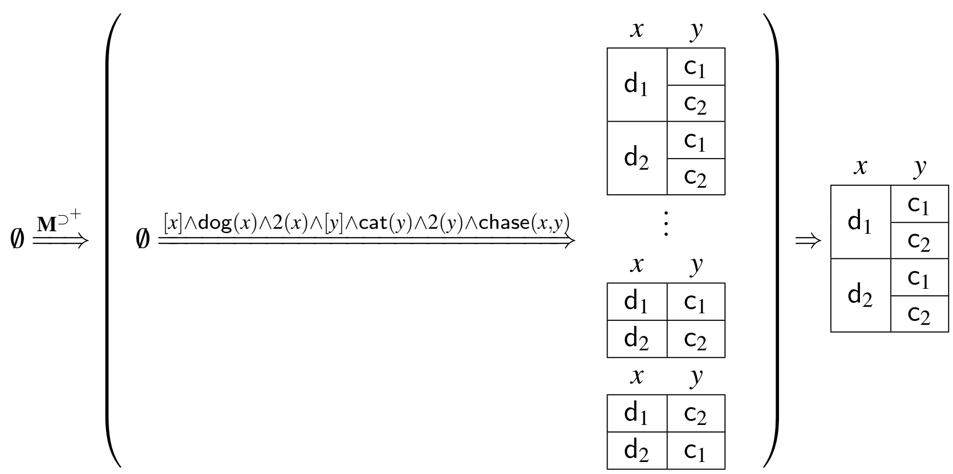
The maximization operator defined in terms of $\supset^{+}$ is given below:

(300) DEPENDENCY SUPPRESSING MAXIMIZATION:
$[\![\mathbf{M}^{\supset^{+}}(\phi)]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H} = \mathbb{T}$

b. $\neg\exists_{K \supset^+ H} : [\![\phi]\!]^{G,K}$

Interestingly, the dependency suppressing maximization operator gives us a way of re-introducing dependency introducing assignment updates into a system with collective interpretation of lexical items. Recall that collective interpretations proved to be a problem in this system because the final output states could encode arbitrary relationships between pluralities. If instead we wrap collective interpretations inside a dependency suppressing maximization operator, collective interpretations will only deliver outputs which include no dependencies between the pluralities in question:

(301) Two dogs chased two cats. $\rightsquigarrow \mathbf{M}^{\supset^+}([x] \wedge \mathsf{dog} \wedge 2(x) \wedge [y] \wedge \mathsf{cat}(y) \wedge 2(y) \wedge \mathsf{chase}(x,y))$

(302) 

The formula in the scope of the maximality operator is consistent with any relationship between dogs and cats since the predicate $\mathsf{chase}(x,y)$ is interpreted collectively, i.e. without reference to the relationship encoded in the plural information state. From the among potential outputs, the dependency suppressing maximality operator selects only the output in which the x-dogs and y-cats do not depend upon one another.

The maximality operator is defined in terms of the smallest relation that will suppress arbitrary dependencies in the output information state. We can ask under what circumstances a maximality relation defined in terms of a relation $\mathfrak{R}$ will suppress arbitrary dependencies. In fact, any relation that meets the criteria below will have the effect of suppressing dependencies.

(303) A relation $\mathfrak{R}$ defines a dependency suppressing maximization operator iff $K \supset^{+} H \Rightarrow K\mathfrak{R}H$

The criteria above states that any relation $\mathfrak{R}$ that is entailed by the $\supset^{+}$ relation will suppress dependencies. The $\supset^{+}$ relation is the smallest relation in terms of which a dependency suppressing maximization operator can be defined.

It turns out that neither of the selective maximality operators defined above suppress arbitrary dependencies. Both $K >_x H$ and $K \supset_x H$ entail that $K(x) \supset H(x)$ which in turn entails that $H^{+} \not\supseteq K$ as required by the $\supset^{+}$ relation. Neither the Brasoveanu nor the van den Berg maximality operators suppress dependencies.

4.3.4 Unselective Maximization

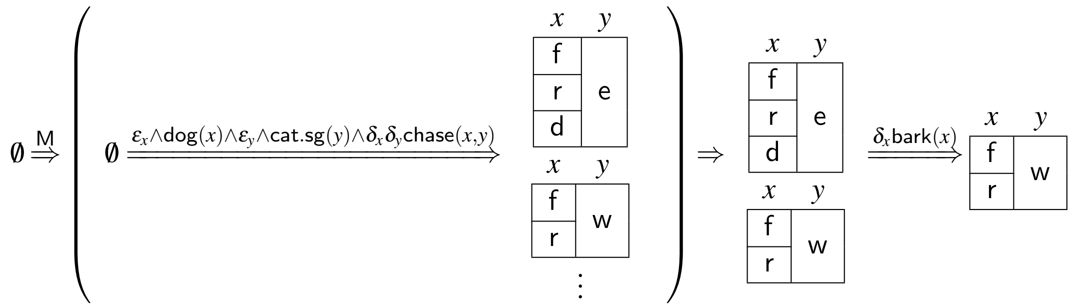
In this section I describe the simplest possible maximization operator, one defined in terms of the subset relation directly. This maximization operator (i) correctly handles non-determinism in the scope of maximization, (ii) eliminates arbitrary dependencies generated inside its scope, and (iii) supports distributive normal form. This is the maximization operator I will use for the remainder of the dissertation.

Unselective maximization is defined below:

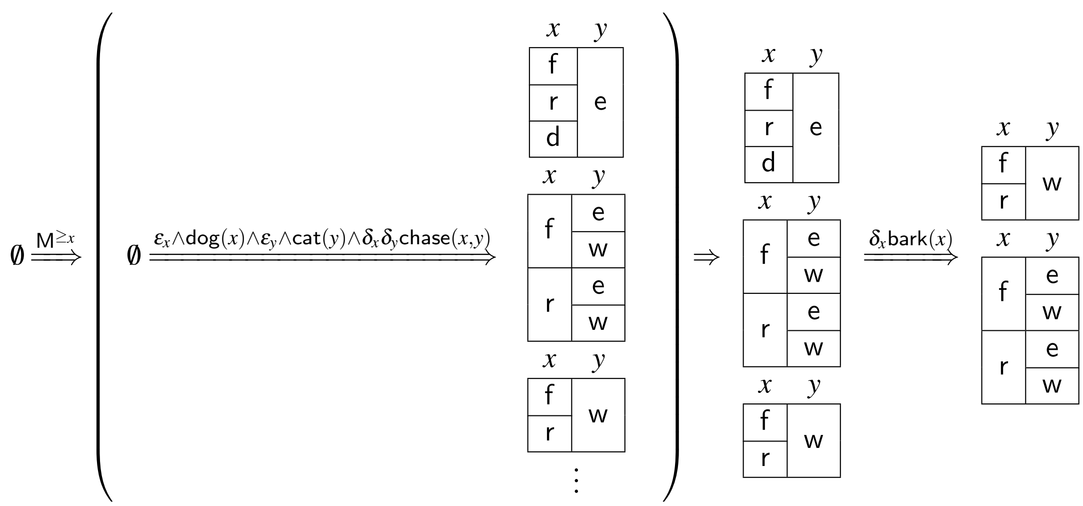
(304) UNSELECTIVE MAXIMIZATION:
$[\![\mathbf{M}(\phi)]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H} = \mathbb{T}$
b. $\neg\exists_{K \supset H} : [\![\phi]\!]^{G,K} = \mathbb{T}$

Since, this maximization operator is the primary maximization operator for the remainder of the dissertation, I simply use **M**, leaving off the superset relation.

Non-determinism is correctly handled by this maximization operator. The dynamics of our target translation are given below:

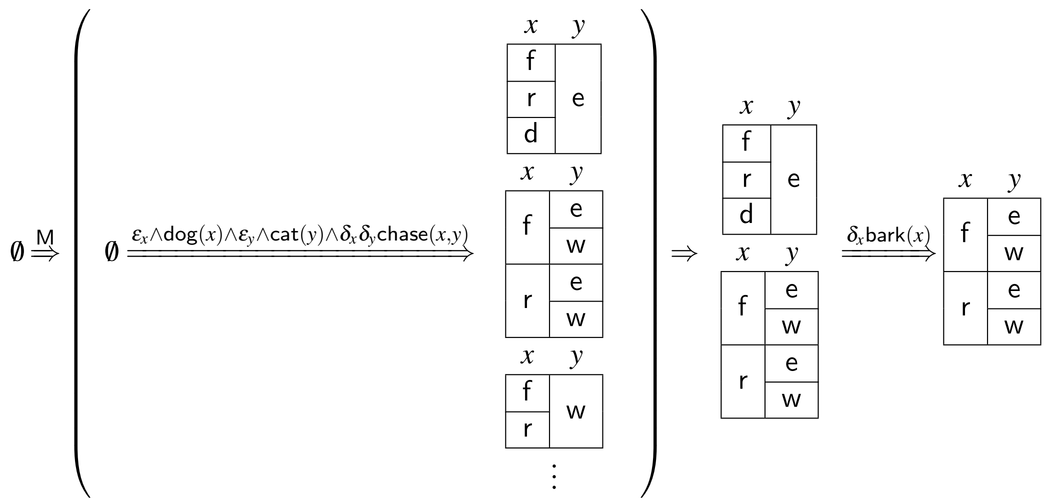
(305) 

The dynamics are exactly like those of the maximization operator defined in terms of the $>_x$ relation in this case. There are however minor differences between the two relations. Consider what happens if we alter the formula in (305) so that we do not require that the y to denote only a singleton cat, i.e. we replace cat.sg with the predicate cat In such a case the maximality operator defined in terms of $>_x$ delivers three possible outputs:

(306) 

Since the predicate cat is true of any plurality of cats, and since both Fido and Rex in our scenario chased both Evander and Whiskers the output of the scope of the maximality operator can generate more outputs than the we saw in the formula above. All three outputs are maximal as determined by the $>_x$ relation. The first output is not comparable to the two below it because it does not stand in a subset-superset relation with either. The middle output is not comparable with the bottom output because they assign the same set of values to the variable x. Notice that both of the bottom two outputs survive the update by the formula $\delta_x\mathsf{bark}(x)$ because they assign to x the same set of dogs.

If instead we utilize the unselective maximality operator, we see that we have a restricted range of possible outputs.

(307) 

In the case depicted above, we again see that the formula in the scope of the maximization operator delivers the same set of potential outputs. However, of the bottom two outputs only the middle output is selected because it is a strict subset of the bottom output. This is what makes the maximization operator unselective—it tries to find as many values as can be associated with any variable not just those associated with a specific variable. Note, however, that the formula in the scope of the maximization operator is responsible for determining the candidates. Predicates sensitive to the cardinality of the set like $\mathbf{sg}(y)$ or $n(y)$ and updates that restrict possible relations between two variables, like ε_y will constrain which sets are even candidates for maximization.

Unslective maximization is also dependency suppressing. This is easy to show. Notice that $K \supset^{+} H$ contains as part of its definition that $K \supset H$. So, if K outranks H

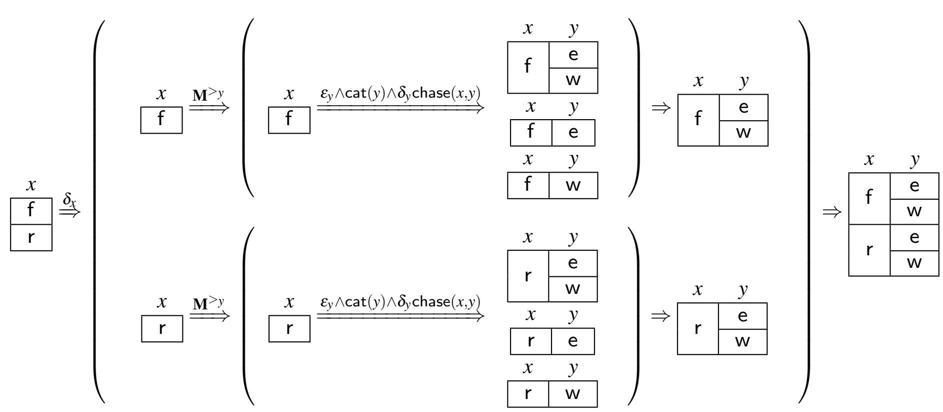
with respect to the $\supset^+$ relation, K will also outrank H with respect to the $\supset$ relation. Unselective maximization thus guarantees the prevention of arbitrary dependencies.

The final point I would like to make is the most complicated. The unselective maximization operator defined above, unlike the selective maximization operators utilized by van den Berg and Brasoveanu, supports distributive normal form. A distributivity operator can be pushed past an unselective maximization operator without any affect on the available output assignments.

(308) $\delta_x \mathbf{M}^{\supset x}(\phi) \neq \mathbf{M}^{\supset x}(\delta_x \phi)$
(309) $\delta_x \mathbf{M}^{>x}(\phi) \neq \mathbf{M}^{>x}(\delta_x \phi)$
(310) $\delta_x \mathbf{M}(\phi) = \mathbf{M}(\delta_x \phi)$

It is easy to create a counter example showing that the selective maximization operators do not support distributive normal form. To do so we need (i) an input assignment in which the variable x has multiple values (so that the distributivity operator is not vacuous), (ii) a concrete formula to fill in for ϕ that introduces at least one variable and relates it to x, and (iii) a model specification.

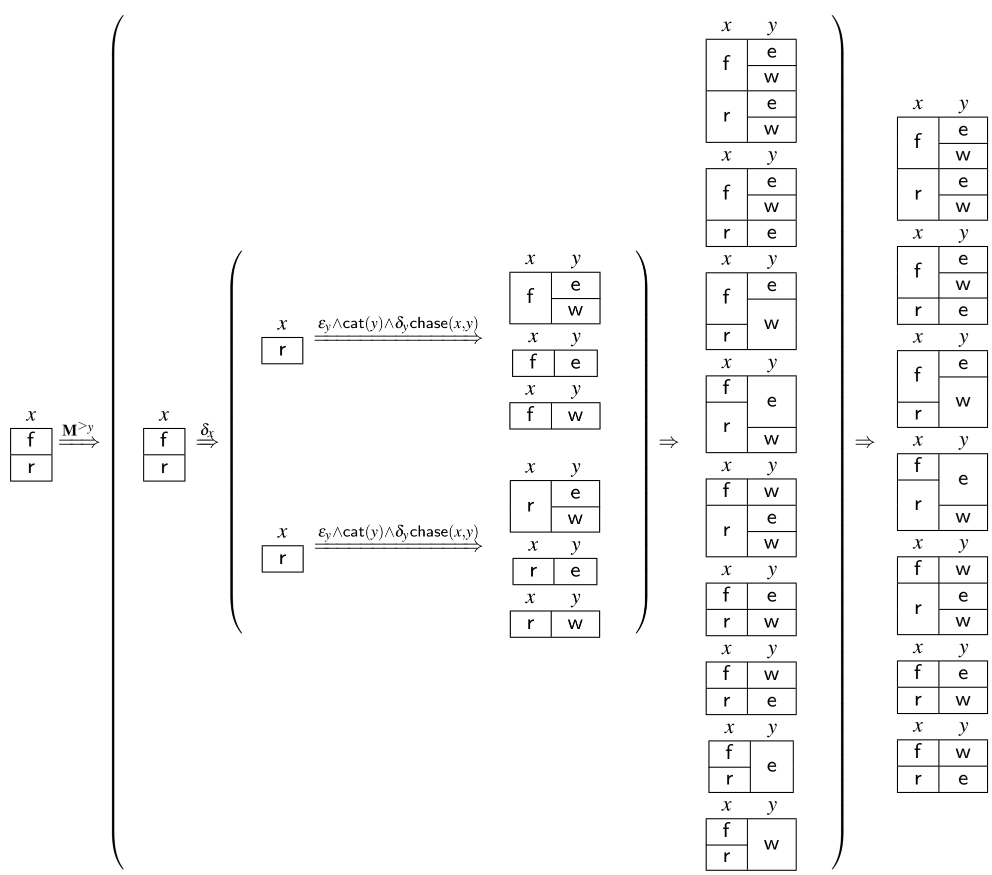
Let our input assignment be one that assigns to x the dogs Fido and Rex. Let $\phi = \varepsilon_y \wedge \mathsf{cat}(y) \wedge \delta_y \mathsf{chase}(x,y)$. And let the model be our previous model, i.e. one in which Evander and Whiskers are cats and both Fido and Rex chased both Evander and Whiskers. Let's first consider van den Berg's maximality operator:

(311)


The input state contains two values for the variable x which the distributivity operator immediately breaks up into sub-states corresponding to each value that x takes. The maximality operators are then applied pointwise. The scope of each maximality operator is consistent with three possible output states: one containing just Whiskers, one containing just Evander, and one containing both cats (n.b. the fact that the cats are the same for both dogs is the result of the model under consideration). The maximization operator selects the outputs with the most cats. At each point these happen to be unique so there is only one way to re-combine the pointwise outputs to generate the final output state: one in which relates both Fido and Rex to both Evander and Whiskers.

Now consider the update associated with the formula in which the distributivity operator is pushed past the maximization operator:

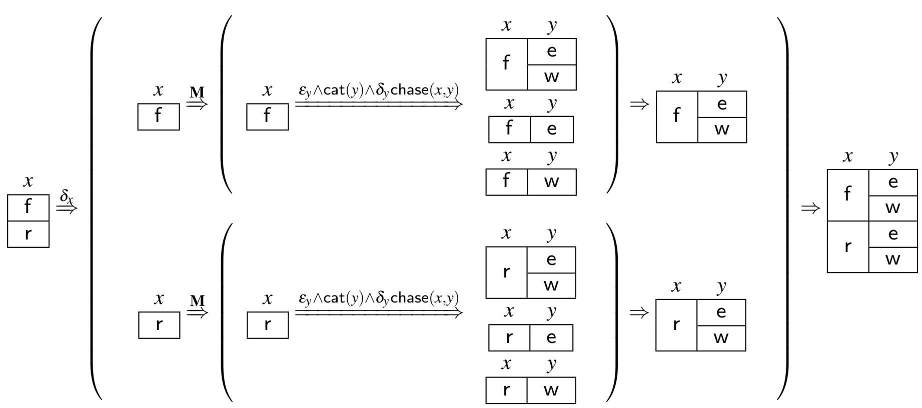
(312)



Here maximization takes scope over a formula that is evaluated pointwise instead of maximizing pointwise over a formula. Inside the scope of the maximality operator the distributivity operator breaks the state into substates corresponding to each value taken on by x. Again, the scope of the distributivity operator generates three potential outputs for each state. This generates nine possible outputs for the entire formula. The maximization operator eliminates only two of these outputs (the bottom two). The remaining outputs all count as maximal since each assign the same set of values to y—only the dependencies differ. We conclude that van den Berg's maximality operator does not support distributive normal form: pushing a distributivity operator past a maximality operator generates new possible outputs for a given input assignment.

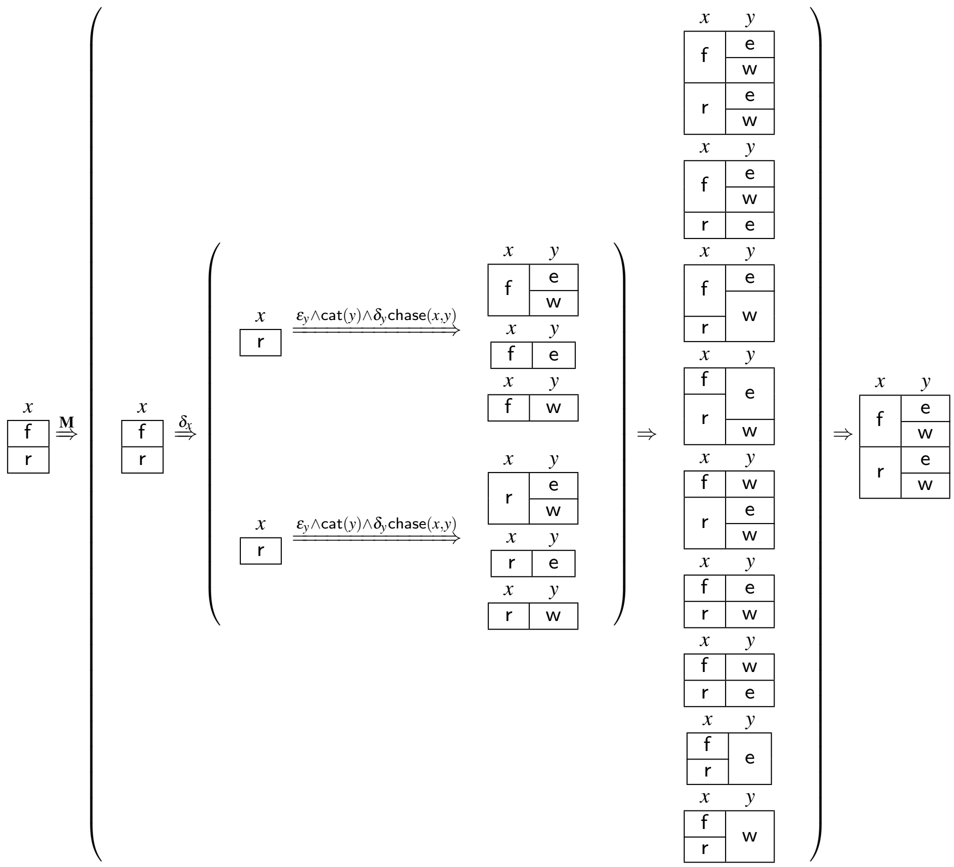
If we run the same example using an unselective maximality operator we will see that we get the same set of outputs regardless of which side of the maximality operator the distributivity operator occurs on. If the maximality operator occurs in the scope of a distributivity operator we get the same result we saw for van den Berg's selective maximality operator:

(313)



The dynamics are exactly the same for this maximization operator as we saw for van den Berg's maximization operator and the results are thus the same.

When we permute the distributivity and maximization operators we get the same result for the final output:

(314) 

Here things work exactly as they did with van den Berg's selective maximality operator until the maximization operator kicks in. Only the top candidate is chosen because each other output candidate is a subset of it. Because unselective maximization compares the values associated with every variable and the dependencies between variables, it selects from the outputs licensed by the distributive update in its scope only those outputs that are also pointwise maximal.

4.3.5 Summary

In this section, I reviewed two maximization operators from the literature and provided arguments against utilizing either. Brasoveanu's max operator, which is defined by the $\supset_x$ relation, will not allow us to correctly handle wide scope indefinites that occur inside the restrictor of universal quantifiers. Van den Berg's max operator, defined by the $>_x$ relation, does correctly percolate non-deterministic updates generated inside the scope of a max operator. However, I showed that van den Berg's max operator does not freely permute with distributivity operators. Any logic that utilized this max operator will not support distributive normal form.

Instead I argued for utilizing an unselective max operator, which is defined by the $\supset$ relation. I have showed that this operator has three properties:

i. It correctly percolates non-deterministic updates generated by formulas inside its scope to its output. This will allow an analysis of exceptional scope indefinites in terms of the scope that distributivity operators take.

ii. It can be freely permuted with distributivity operators. A logic utilising unselective maximization may (depending on other operators) support distributive normal form.

iii. Unselective maximization also prevents arbitrary dependencies from arising from the formula in its scope. Because it selects the largest outputs full stop, it naturally favours outputs that encode fewer dependencies.

This third point is important since the encoding of arbitrary dependencies in output states was one of the arguments advanced against the inclusion of both dependency introducing variable assignment updates and cumulatively interpreted lexical relations.

The full logic that we adopt is given below:

(315) $[\![R(v_1,\ldots,v_n)]\!]^{G,H} = \mathbb{T}$ iff
   a. $G = H$
   b. $\langle G(v_1),\ldots G(v_n)\rangle \in \mathfrak{I}(R)$

(316) $[\![\varepsilon_v]\!]^{G,H} = \mathbb{T}$ iff
   a. $G(v) = \emptyset$
   b. $\exists_{D\subseteq\mathfrak{D}} : H = \{g^{[v\to d]} : d \in D \;\&\; g \in G\}$

(317) $[\![\phi \wedge \psi]\!]^{G,H} = \mathbb{T}$ iff $\exists_K : [\![\phi]\!]^{G,K} = \mathbb{T} \;\&\; [\![\psi]\!]^{K,H} = \mathbb{T}$

(318) $[\![\mathbf{M}\phi]\!]^{G,H} = \mathbb{T}$ iff

a. $[\![\phi]\!]^{G,H} = \mathbb{T}$

b. $\neg\exists_{K\supset H} : [\![\phi]\!]^{G,K} = \mathbb{T}$

(319) $[\![\delta_v\phi]\!]^{G,H} = \mathbb{T}$ iff

a. $G(v) = H(v)$

b. $\forall_{d\in G(v)} : G|_{v=d}[\![\phi]\!]H|_{v=d}$

In the next section we will see how to extend this logic to include arbitrary generalized quantifiers.

4.4 Adding Generalized Quantifiers

In this section I describe how the system can be extended to handle not only universal quantifiers but the full range of quantifiers found in natural language. This is done by means of introducing a designated value $\star$ that stands in for the row being unfilled. That is we can distinguish between a variable being outside the domain of our assignment functions and a row having no value (from the model). We work with partial assignment functions like those below:

(320) For $\star \notin \mathfrak{D}$, a partial assignment function g is a function from variables v to elements of $\mathfrak{D} \cup \{\star\}$.

We can now express sets of partial assignment functions like those below:

(321) $G =$

	x	y
g_1	a	d
g_2	b	e
g_3	c	$\star$

The interpretation of this should be as follows: the variables x and y are both present, but in g_3 y is not mapped to anything in the model. In other words, all the assignment functions in G have the same domain, but not all of them map every variable to something in the domain of the model: $y \in \mathbf{Dom}(g_3)$ but $g_3(y) \notin \mathfrak{D}$.

This means that when we collect the projection of a variable x in G we need to exclude cases in which $g(x) = \star$ because no set containing the $\star$-value is in any lexical relation which are defined as elements of $\wp^+(\mathfrak{D})^n$. To recover the plurality associated with an assignment function, we must collect all the values associated with a variable x in any row, while discarding rows which map x to $\star$:

(322) PROJECTION (for single variables):
$G(x) := \{g(x) : g \in G \; \& \; g(x) \neq \star\}$

When we generalize projection to multiple variables we take care to exclude any rows in which any of the variables in question are undefined:

(323) PROJECTION (for multiple variables):
$G(v_1, ..., v_n) := \{\langle g(v_1), \ldots, g(v_n)\rangle : g \in G \; \& \; g(v_1) \neq \star \; \& \; \ldots \; \& \; g(v_n) \neq \star\}$

One final thing we will need to be careful of: we can no longer assume that if $G(x) = \emptyset$ that $x \notin \mathbf{Dom}(G)$. If every element of G maps x to $\star$, then $G(x) = \star$, but $x \in \mathbf{Dom}(G)$. We are in a sense juggling two ways for an assignment function to be partial: it can fail to assign a variable to anything, or it could assign that variable to the dummy object.

Our sub-state notation can be modified to pick out just those rows in which a variable is undefined:

(324) NEW SUB-STATE:

$$G|_{x=\star} \quad := \quad \{g : g \in G \; \& \; g(x) = \star\}$$

4.4.1 Lexical Relations & Conjunction

Lexical relations retain their previous definitions. Notice that since we exclude $\star$ values from the projection of any variable, the presence or absence of such values will be invisible to lexical relations.

(325) LEXICAL RELATIONS:
$[\![R(v_1,\ldots,v_2)]\!]^{G,H} = \mathbb{T}$ iff
a. $G = H$
b. $\langle G(v_1),\ldots,G(v_2)\rangle \in \mathfrak{I}(R)$

Conjunction also retains its familiar definition. A conjunction $\phi \wedge \psi$ is true relative to an input G and an output H iff there is some K s.t. ϕ is true relative to input G and output K and ψ is true relative to input K and output H.

(326) CONJUNCTION :
$[\![\phi \wedge \psi]\!]$ iff $\exists K : [\![\phi]\!]^{G,K} = \mathbb{T}$ & $[\![\psi]\!]^{K,H} = \mathbb{T}$

Dynamic conjunction simply takes the output of the first formula and passes it to the second formula.

4.4.2 Assignment Update

(Dependency-Free) random assignment has to be slightly modified. Instead of updating with any subset of $\mathfrak{D}$ we update with an element of $\mathfrak{D}^{\star} := \wp^{+}(\mathfrak{D}) \cup \{\{\star\}\}$. That is we select a non-empty subset of $\mathfrak{D}$ updating each row in the input with each element of our selection or we pick the set contain $\star$ and assign $\star$ to every row. This allows us to register the fact that we have updated a variable assignment with a new variable without populating any of the rows with elements of the model. Think of this as adding a column name to a database without populating any the rows with values.

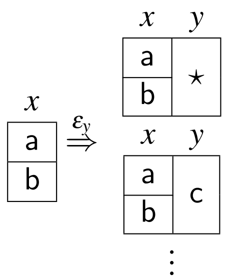
The first conjunct ensures that updates are non-destructive. Random assignment never over-writes variables that are already in use. The second conjunct says that for every $g \in G$ and $d \in D$, H contains the assignment $g^{[x \to d]}$. Note that $|H| = |G| \times |D|$ and that ε_x does not introduce any new dependencies between x and any other variable.

(327) (DEPENDENCY-FREE) RANDOM ASSIGNMENT:
$[\![\varepsilon_x]\!]^{G,H}$ iff
a. $x \notin \mathbf{Dom}(G)$

b. $\exists D \in \mathfrak{D}^{\star} : H = \{g^{[x \to d]} : g \in G \,\&\, d \in D\}$

One key property of ε_x in this new context is that it does not necessarily add any additional values. We allow empty updates as well:

(328)



In addition to ε, van den Berg provides a number of variants of random assignment. The most important is the subset assignment, $\varepsilon_{x \subseteq y}$. This operation is a copy-by-value update of the variable x. It takes some subset of the values that y stores on and copies them over to x. The remainder of the xs are assigned the dummy value $\star$.

(329) (DEPENDENCY-PRESERVING) SUBSET ASSIGNMENT:
$[\![\varepsilon_{x \subseteq y}]\!]^{G,H}$ iff

a. $x \notin \mathbf{Dom}(G)$

b. $\exists_{D \in G(y)} : H = \{g^{[x \to g(y)]} : g \in G|_{y \in D}\} \cup \{g^{[x \to \star]} : g \in G|_{y \notin D}\}$

The first clause again serves to prevent destructive updates. The second clause requires some unpacking. For some $D \subseteq G(y)$ we generate an output consisting of the union of two sets: the first set is constructed by taking the every $g \in G|_{y \in D}$ and duplicating g's y value into the value g assigns to x, i.e. $g^{[x \to g(y)]}$—this covers those rows where y takes some value in D; the second set is generated by assigning x to $\star$ in the sub-state $G|_{y \notin D}$. Subset assignment thus extends a state G by copying some values from y to x and giving x a place-holder value everywhere else.

The operation of subset assignment is illustrated in below. Notice that an update $\varepsilon_{z \subseteq x}$ to an input G has $|\wp(G(x))|$ possible output assignments, one corresponding to every subset of $G(x)$. For each of these subsets the output is unique.

(330)

x	y		x	y	z
a	c	$\overset{\varepsilon_{z\subseteq x}}{\Longrightarrow}$	a	c	a
b	d		b	d	b

(331)

x	y		x	y	z
a	c	$\overset{\varepsilon_{z\subseteq x}}{\Longrightarrow}$	a	c	a
b	d		b	d	$\star$

(332)

x	y		x	y	z
a	c	$\overset{\varepsilon_{z\subseteq x}}{\Longrightarrow}$	a	c	$\star$
b	d		b	d	b

(333)

x	y		x	y	z
a	c	$\overset{\varepsilon_{z\subseteq x}}{\Longrightarrow}$	a	c	$\star$
b	d		b	d	$\star$

Notice that subset assignment does not introduce any new dependencies but simply copies over values. This allows the new variable to depend on other variables but only in the same way as the source variable. Notice also that the last update above is trivial since it maps z to the empty subset of x.

4.4.3 Distributivity

The definition of distributivity has to be modified to deal with rows in which a variable is mapped to $\star$:

(334) DISTRIBUTIVITY:
$[\![\delta_x(\phi)]\!]^{G,H} = \mathbb{T}$ iff

a. $G(x) = H(x)$

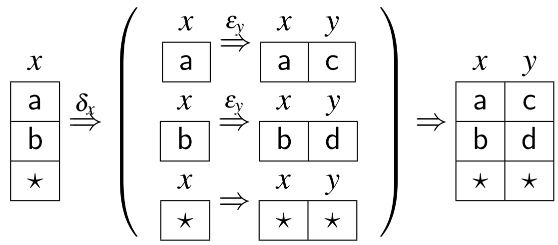
b. $H|_{x=\star} = \{g^{[v\to\star]} : v \in \mathbf{Dom}(H) - \mathbf{Dom}(G)\ \&\ g \in G|_{x=\star}\}$

c. $\forall_{d\in G(x)} : [\![\phi]\!]^{G|_{x=d},H|_{x=d}} = \mathbb{T}$

d. $\forall_{d\in H(x)} : [\![\phi]\!]^{G|_{x=d},H|_{x=d}} = \mathbb{T}$

The first conjunct ensures that no new values for x are added to H that are not already present in the input. The remaining conjuncts break the input into sub-states defined so that each takes on a single value for x: the second conjunct says that in any row in which G assigns x the value $\star$, H assigns not only assigns x to $\star$ but also assigns $\star$ to all the new values introduced in the course of evaluating ϕ, and the third conjunct breaks the input assignment one sub-state for each value x can take and passes them on to ϕ.

In the scope of a distributivity operator random assignment is able to introduce new dependencies:

(335) 

The distributivity operator breaks up the set of assignments into sub-states defined in terms of the value taken on by x. The assignment update ε_y then operates on each of these sub-states separately. Notice that the update does not apply to the sub-state in which x is undefined. This substate is passed on directly without modification.

One case is worth remarking on. If $G(x) = \emptyset$, then $[\![\delta_x(\phi)]\!]^{G,G} = \mathbb{T}$. Notice that the first two conjuncts are satisfied trivially, since $G = G$, hence $G(x) = G(x)$ and $G|_{x=\star} = G|_{x=\star}$. The third conjunct is satisfied vacuously since there is no $d \in G(x)$ since $G(x) = \emptyset$. The conclusion to draw is that distributing over a variable which has only $\star$ values results in a trivially true formula.

4.4.4 Unselective Maximization

Unselective maximization is more difficult to define now that we work with sets of partial assignment functions with dummy values. The issue is that we would like maximization to militate against output assignments that assign dummy values to variables whenever possible. If we define maximization in terms of the subset relation, however, we will end up in trouble.

Let's consider a few cases. First, consider the two assignment functions in (336). It seams clear that a suitable relation should make (336a) $<$ (336b), since (336b) assigns more values to both x and y than (336a).

(336) a.

x	y
a	b

b.

x	y
a	b
c	d

One simple way to get the right results for the cases above would be to follow our previous definition to the letter and identify $<$ with $\subset$. We would say that one set of assignments A was greater than B iff $A \supset B$.

Next, consider the two sets of assignment functions in (337). Here it seems that (337a) $<$ (337b), since (337b) assigns as many values to x and more values to y than (337a). At the same time it is not the case that (337a) $\subset$ (337b), so we should not identify $<$ with the subset relation.

(337) a.

x	y
a	c
b	$\star$

b.

x	y
a	c
b	d

In the case of (337) we would like to say that (337b) is strictly greater than (337a) because the second row in (337b) assigns a value (from the model) to y while the second row in (337a) does not (recall that our aim here is unselective maximization—so if a formula in the scope of a maximization operator is consistent with both these updates we need to select only the output that maximizes the values assigned to the x and the values assigned to the y while minimizing the dependencies between them). In this case we need to select (337b).

If we want a relation that agrees with our intuitions regarding the pairs in (336) and (337), we can adopt the definitions in (338):

(338) a. $g \preceq h := \mathbf{Dom}(g) = \mathbf{Dom}(h)\ \&\ \forall_{v \in \mathbf{Dom}(g)} : (g(v) = \star \text{ or } g(v) = h(v))$

b. $G \precsim H := \forall_{g \in G} : \exists_{h \in H} : g \preceq h$

The definition in (338a) compares single assignment functions. It says that if two assignment functions g and h are such that for every variable either (i) g fails to assign it some value from the model or (ii) h agrees with g with respect to v, then $g \preceq h$. The definition in (338b) generalizes to sets of assignment functions. It says that $G \preceq H$ just in case for every assignment function in G, there is some assignment function in H that is larger in the sense of (338a).[10]

It is easy to verify that the $\preceq$ relation determines a partial order over single assignment functions. However, over sets of assignment functions the $\precsim$ relation defines only a preorder (i.e. it is not anti-symmetric). Consider the two sets of assignments depicted below:

(339) a.

x	y
a	c
b	d

b.

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>a</td><td>c</td></tr>
<tr><td rowspan="2">b</td><td>d</td></tr>
<tr><td>$\star$</td></tr>
</table>

We can first show that (339a) $\precsim$ (339b). Since the two first rows are identical, each row in (339a) $\preceq$ than the corresponding row in (339b). Hence, (339a) $\precsim$ (339b). However, we can also show that (339b) $\precsim$ (339a). We embrace identical reasoning for the first two rows and note that the third row of (339b) $\preceq$ the second row of (339a) since (i) they agree about the value of x and (ii) the second row of (339a) is defined for more values than the third row of (339b). Hence, (339b) $\precsim$ (339a).

[10] Notice that we can define the subset relation in very similar form: $G \subseteq H := \forall_{g \in G} \exists_{h \in H} : g = h$. The definition of $\precsim$ establishes the same type of relation between G and H as the subset relation. Every thing in g must have a corresponding member in h but not necessarily the other way around—this allows for H to have more rows than G and gives maximization its tendency to expand the set of assignments vertically.

Worse yet are the following two sets of assignment functions:

(340) a.

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td rowspan="2">a</td><td>⋆</td></tr>
<tr><td>c</td></tr>
<tr><td>b</td><td>d</td></tr>
</table>

b.

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>a</td><td>c</td></tr>
<tr><td rowspan="2">b</td><td>d</td></tr>
<tr><td>⋆</td></tr>
</table>

Here were compare the last two rows of (340a) to the first two rows of (340b), the first row of (340a) to the first row of (340b), and the last row of (340b) to the last row (340a).

At this point I would like to make one observational comment: in both the cases I have constructed above, at least one of the sets of assignment functions contains two assignment functions g, g' s.t. $g \prec g'$. Such rows seem redundant: why would we care to store the same value for a variable twice—one with some other variable defined and again with the same variable undefined.

In fact we can show that for any G,H s.t. $G \precsim H$ and $H \precsim G$ and $G \neq H$ it is the case that either in G or in H there are two assignment functions s.t. $a \prec b$. To prove this let g name some arbitrary element of G. From $G \precsim H$ we know that there is some $h \in H$ s.t. $g \preceq h$, and from $H \precsim G$ we know there is some $g' \preceq h$ in G. We conclude that $g' \preceq h \preceq g$. If $g' \prec g$, we have the result we want. If on the other hand $g = g'$ we have $g = h$ and therefore $G \subseteq H$. We reason in the same manner from H concluding the either that H has two assignment functions that are non-identical and stand in the $\prec$ relation to one another or $H \subseteq G$. If we conclude the later we contradict the assumption that $G \neq H$.

The observation above suggests that the problems with (339-340) are surmountable. There are in fact two ways of surmounting the problem:

i. One could show that $\precsim$ determines a partial over over the set of output contexts generated by sentences of the logic.

ii. One could define a suitable $<$ relation in terms of $\preceq$ and some other tie-breaking relation.

Each strategy has its advantages. The first strategy will tell us something about the types of output contexts that can be generated by DPlL and so lead us to consider constraints on potential additions to the logic, so it is the strategy I will pursue.

We note, first, that the designated start state is such that $\neg\exists_{g,g'\in\mathbf{0}} : g \prec g'$. This is trivial since there is only one element in $\mathbf{0}$. Next we show that if $\neg\exists_{g,g'\in G} : g \prec g'$, then no formula ϕ in the language is such that there exists an H s.t. $[\![\phi]\!]^{G,H} = \mathbb{T}$ and $\exists_{h,h'\in H} : h \prec h'$.

First note that if ϕ is a test then $[\![\phi]\!]^{I,O}$ requires that $I = O$, so the result follows. Second, note that neither of our assignment updates will generate such results; if ε_v assigns $\star$ as the value of v in any, row it assigns $\star$ as the value of v in every row. The update $\varepsilon_{v\subseteq v'}$ partitions the set into two parts. To one it assigns the value v' to v in a row and in the other it assigns v the value $\star$.

Conjunction is easy: if ϕ and ψ don't generate outputs which contain rows $g, g' \in G$ are s.t. $g \prec g'$, then neither will $\phi \wedge \psi$.

Distributivity is somewhat harder, if $\neg\exists_{g,g'\in G} : g \prec g'$, then none of the subsets of G that are passed to the scope of δ_v will host such assignments either. Notice that cells across the partition of G into sub-states will never stand in the $\prec$ relation either (because they assign different non-$\star$ values to v) so when they are joined at the end no such pairs will generate. We have to worry about states in which a variable v is given the value $\star$. Notice that from the assumption that the input state had no pairs g, g' s.t. $g \prec g'$ that none of the assignments in the sub-state where $v = \star$ will stand in the $\prec$ relation with anything else in the input state. This cannot be because of the value they assign to x so it must be that for each of these rows there is some other value of some other variable that prevents them from standing in the $\prec$ relation with any other row. This will persist even after the rows are updated with the new $\star$ values.

Finally, maximization, however defined, delivers only as subset of the output states that are consistent with its scope, viz. the maximal ones. So if the formula ϕ does not generate sub-states with problematic pairs of rows, then neither will $\mathbf{M}^{\mathfrak{R}}(\phi)$ for any relation $\mathfrak{R}$.

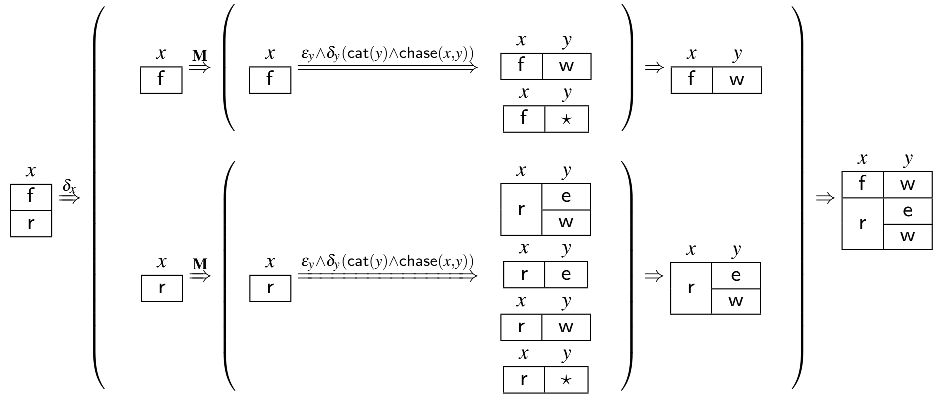
The upshot is that we do not need to worry about the fact that $\precsim$ defines a pre-order over sets of assignment functions because the pairs of information states G, G' that are equal with respect to the relation are either identical or at least one of them can never be generated by the logic.

Maximization is then defined in the usual way:

(341) UNSELECTIVE MAXIMIZATION:
$[\![\mathbf{M}\phi]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H} = \mathbb{T}$
b. $\neg\exists_{K \succ H} : [\![\phi]\!]^{G,K} = \mathbb{T}$

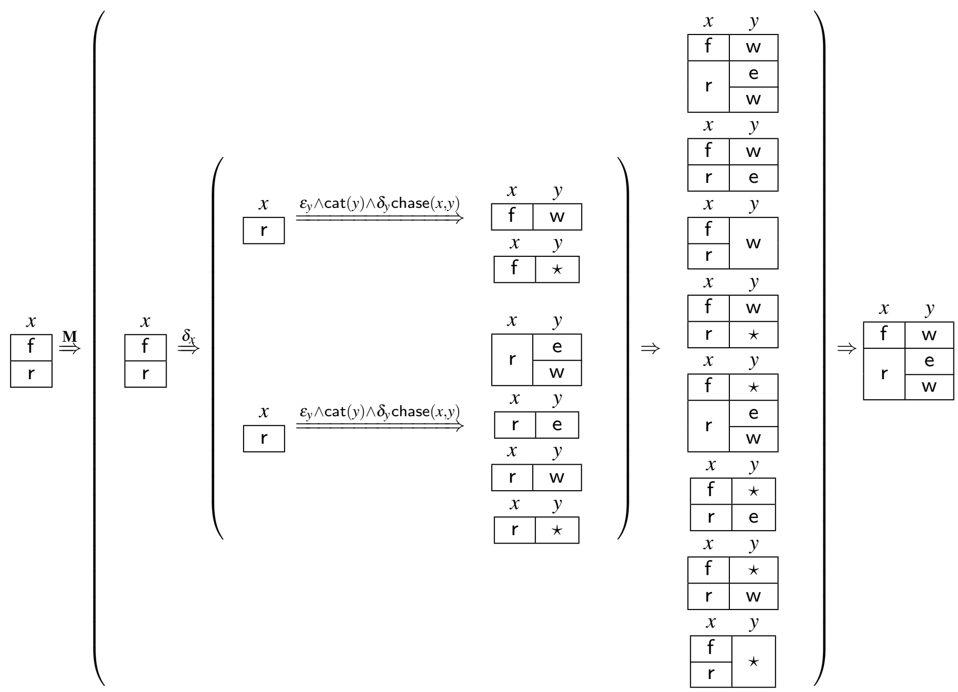
Note that maximization still permutes with distributivity operators. To simplify things for ourselves let's consider a situation in which Fido chased Whiskers and Rex chased Evander and Whiskers.[11] Our story looks exactly like before except that now we have to consider updates in which no cat is present.

[11] Notice that I have scoped the δ_y operator over the predicate cat. This is not part of our official translations of indefinites. I do this because scoping out the δ_y like this allows us to get $\star$ updates for y that come out true—i.e. we actually bleed the existential import. I do this for expository purposes, we want to see that $\mathbf{M}$ and δ still permute when there are $\star$'s in the mix.

(342) 

The dynamics are exactly the same for this maximization operator as we saw for van den Berg's maximization operator and the results are thus the same.

When we permute the distributivity and maximization operators we get the same result for the final output:

(343) 

Notice again the formula in the scope of the maximization operator generates a large number of potential outputs. However, the max operator picks up only the output that is also point-wise maximal.

4.4.5 Definitions For Generalized Quantifiers

Since generalized quantifier can be seen as relations between sets (Barwise and Cooper, 1981) the truth conditional aspects of their meanings can be encapsulated by lexical relations in DPlL.

(344) $\mathfrak{I}(\mathbf{every}) = \{\langle A,B\rangle : A = A\cap B\}$
(345) $\mathfrak{I}(\mathbf{most}) = \left\{\langle A,B\rangle : \frac{A\cap B}{A} > \frac{1}{2}\right\}$
(346) $\mathfrak{I}(\mathbf{no}) = \{\langle A,B\rangle : A\cap B = \emptyset\}$

Having parcelled up the truth conditional components of their meanings, the dynamic aspects of quantifiers can be dealt with:

(347) $\mathbf{Q}\phi\text{s}\psi\text{ed} \rightsquigarrow \mathbf{M}(\varepsilon_x \wedge \delta_x\phi) \wedge \mathbf{M}(\varepsilon_{x'\subseteq x} \wedge \delta_{x'}\psi) \wedge \mathbf{Q}(x,x')$

The formula above first collects the maximal set of x's that satisfy ϕ, then it collects the maximal subset of x's that satisfy ψ by copying over only those values of x to x' that satisfy ψ. Finally the lexical relation determined by the quantifier ensures that the sentences a whole has the right truth conditions.

Indefinites can be translated in terms of an ε operator:

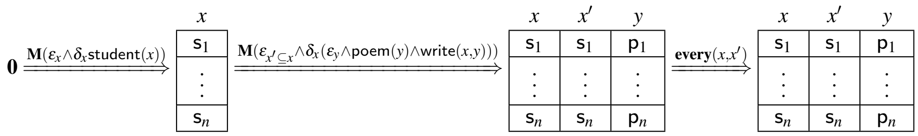
(348) a poem $\rightsquigarrow \varepsilon_x \wedge \mathsf{poem}(x)$

We are now in a position to see the system in action. Consider the sentence and its translation into DPlL below. Here I have provided an interpretation in which the indefinite takes scope below the universal.

(349) Every student wrote a poem.

$$\rightsquigarrow \mathbf{M}(\varepsilon_x \wedge \delta_x\mathsf{student}(x))\wedge$$
$$\mathbf{M}(\varepsilon_{x'\subseteq x} \wedge \delta_x(\varepsilon_y \wedge \mathsf{poem}(y) \wedge \mathsf{write}(x,y))) \wedge \mathbf{every}(x,x')$$

Let us assume that every student s_i read only one poem p_i. In that case we expect the sentence above to be true and below we can see the computation of it's truth conditions:

(350) 

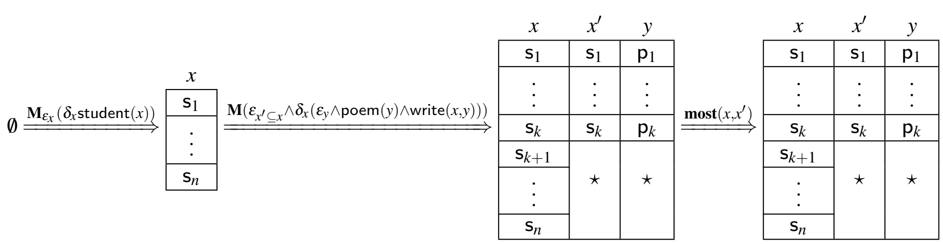
The first conjunct maximizes over students and assigns to x the set of students. The next conjunct finds the maximal subset of x that wrote a poem. Here were are able to a choose a poem or poems for each student because ε_y is inside the scope of the distributivity operator. In the final output assignment given above we see that y is associated with multiple values. Reference back to y should now require a plural pronoun.

Proportional Quantifiers like *most* can also be handled by the system we have developed.

(351) Most students wrote a poem.

$$\rightsquigarrow \mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_x(\varepsilon_y \wedge \mathsf{poem}(y) \wedge \mathsf{write}(x,y))) \wedge \mathbf{most}(x,x')$$

Again The first conjunct maximizes over students and assigns to x the set of students. The next conjunct finds the maximal subset of x that wrote a poem. Below I have assumed that only a subset of students wrote poems. Notice that poems are filled in only for students who wrote poems.

(352) 

4.4.6 Summary

This section introduced plural information states with designated dummy values. This allowed us to encode the lexical information associated with generalized quantifiers while keeping the dynamics identical. I showed that unselective maximization can be retained in this framework and that it can still permute with distributivity.

4.5 Conclusion

This chapter sets the stage for embedding the intuition behind Brasoveanu and Farkas (2011) into a plural logic that allows discourse reference to both discourse pluralities and relations between variables. I showed how in van den Berg's DPlL dependence and independence are managed by means of distributivity operators. In addition I showed that the exact definition of maximization plays a large role in whether the intuition underlying Brasoveanu & Farkas (2011) can be adequately implemented. It would be fair to state that the broad conclusion of this chapter is that a dynamic semantics is not sufficient to capture wide scope indefinites in terms of independence. The right definition of maximization is also needed.

I also showed that unselective maximization has several desirable formal properties. First it provides a logic that can be easily syntactically manipulated because it allows distributivity operators to be pushed past maximization without affecting the underlying truth conditions of the sentence. Unselective maximization was also seen to be dependency suppressing. This might pave the way for the reintroduction dependency introducing random assignments.

Appendix C

Technical Appendix

C.1 Notational Conventions

(353) For $\star \notin \mathfrak{D}$, a partial assignment function g is a function from variables v to elements of $\mathfrak{D} \cup \{\{\star\}\}$.

(354) $G(x) := \{g(x) : g \in G \;\&\; g(x) \neq \star\}$

(355) $G(v_1, ..., v_n) := \{\langle g(v_1), \ldots, g(v_n)\rangle : g \in G \;\&\; g(v_1) \neq \star \;\&\; \ldots \;\&\; g(v_n) \neq \star\}$

(356) $\mathfrak{D}^\star := \wp^+(\mathfrak{D}) \cup \{\{\star\}\}$

C.2 Relations Defined Over Sets of Assignments

(357) $K \supset_x H := K(x) \supset H(x)$

(358) $K >_x H := K(x) \supset H(x) \;\&\; K \supseteq H$

(359) $K \supset^+ H := H^+ \supseteq K \supset H$

(360) a. $g \preceq h := \mathbf{Dom}(g) = \mathbf{Dom}(h) \;\&\; \forall_{v \in \mathbf{Dom}(g)} : (g(v) = \star \text{ or } g(v) = h(v))$

b. $G \precsim H := \forall_{g \in G} : \exists_{h \in H} : g \preceq h$

C.3 Dynamic Plural Logic With Unselective Maximization

(361) LEXICAL RELATIONS:
$[\![R(v_1,\ldots,v_2)]\!]^{G,H} = \mathbb{T}$ iff
a. $G = H$
b. $\langle G(v_1),\ldots,G(v_2)\rangle \in \mathfrak{I}(R)$

(362) CONJUNCTION:
$[\![\phi \wedge \psi]\!]$ iff $\exists K : [\![\phi]\!]^{G,K} = \mathbb{T}$ & $[\![\psi]\!]^{K,H} = \mathbb{T}$

(363) (DEPENDENCY-FREE) RANDOM ASSIGNMENT:
$[\![\varepsilon_x]\!]^{G,H}$ iff
a. $x \notin \mathbf{Dom}(G)$
b. $\exists D \in \mathfrak{D}^\star : H = \{g^{[x\to d]} : g \in G \,\&\, d \in D\}$

(364) (DEPENDENCY-PRESERVING) SUBSET ASSIGNMENT:
$[\![\varepsilon_{x\subseteq y}]\!]^{G,H}$ iff
a. $x \notin \mathbf{Dom}(G)$
b. $\exists_{D\in G(y)} : H = \{g^{[x\to g(y)]} : g \in G|_{y\in D}\} \cup \{g^{[x\to\star]} : g \in G|_{y\notin D}$

(365) DISTRIBUTIVITY:
$[\![\delta_x(\phi)]\!]^{G,H} = \mathbb{T}$ iff
a. $G(x) = H(x)$
b. $H|_{x=\star} = \{g^{[v\to\star]} : v \in \mathbf{Dom}(H) - \mathbf{Dom}(G) \,\&\, g \in G|_{x=\star}\}$
c. $\forall_{d\in G(x)} : [\![\phi]\!]^{G|_{x=d},H|_{x=d}} = \mathbb{T}$
d. $\forall_{d\in H(x)} : [\![\phi]\!]^{G|_{x=d},H|_{x=d}} = \mathbb{T}$

(366) UNSELECTIVE MAXIMIZATION:
$[\![\mathbf{M}\phi]\!]^{G,H} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H} = \mathbb{T}$
b. $\neg\exists_{K\succsim H} : [\![\phi]\!]^{G,K} = \mathbb{T}$

Chapter 5

Wide scope indefinites and Decomposed Distributivity

5.1 Introduction

This chapter has three goals. First, it expands on the data surrounding the binder roof constraint. I will come to several novel empirical generalizations:

i. The binder roof constraint applies not only to bound pronouns but also to donkey pronouns, i.e. an indefinite with a donkey pronoun in its restrictor cannot take wide scope with respect to the universal that hosts the donkey indefinite.

ii. The binder roof constraint applies only to singular pronouns, i.e. an indefinite can take wide scope with respect to a universal that a plural pronoun in its restrictor is anaphoric too. This holds regardless of whether the pronoun is anaphoric to a quantifier or a donkey indefinite.

Second, I will sketch an account of wide scope indefinites in which the scope of the indefinite is reduced to the scope of various distributivity operators provided by syntactically higher quantifiers. The purpose of this exercise is to show that in a system like the one developed at the end of the previous chapter in which new dependencies and cardinality checks are similarly effected by distributivity operators predicts different effects singular and plural anaphora on the possibility of wide scope interpretations of indefinites.

Third, this chapter provides a final analysis of wide scope indefinites in terms of decomposed distributivity operators. A distributivity operator, δ_x, consists of two components:

i. A variable v which is be distributed over.

ii. Universal quantification over sub-states defined in terms of v.

I will present a logic which will split these two components apart. Distributivity will consist of two operations[1]:

i. $\downarrow_v$ which enters v into a cache of variables $\mathscr{V}$.

ii. $^{\Delta}$ which empties the cache of variables and contributes universal quantification over the sub-states definable in terms of the variables in $\mathscr{V}$.

In this logic a quantifier does not itself contribute distributivity but rather indicates which variables should be distributed over. Operators like $^{\Delta}$ can then make use of these variables to generate both (i) distributive readings of lexical items and (ii) assignment updates which can introduce new dependencies. With $\downarrow$ and $^{\Delta}$ defined, it becomes

[1]This proposal joins a long line of proposals that decompose quantification in certain ways. The dynamic logics we have already encountered offer one example; quantification is decomposed into assignment updates and operators that generate quantificational force. Hybrid logics offer another example in these logics binding and quantificational force are severed (Blackburn, 2000). The proposal also bears some similarity to the system put forward by Steedman (2007) in which inverse scope is handled in part by adding a parameter that stores the variables being quantified over to the interpretation of sentences.

possible to define a third operation $\uparrow_v$ that removes v from the cache of stored variables without distributing over it. This prevents formulas in the scope of $\uparrow_v$ from being interpreted distributively with respect to the values that v stores. In this way the logic provides a way for the scope of distributivity operators to be controlled by lexical items themselves.

Since the resulting logic is somewhat cumbersome, I make use of a wide range of truth preserving inference rules that allow formulas containing these new operators to be simplified. For instance we will see equivalences like the one below:

(367) Every student who takes a class ... $\rightsquigarrow$

a. every $\rhd$ a:
$\mathbf{M}(\varepsilon_x \wedge \downarrow_x (^{\Delta}\mathsf{student}(x) \wedge {}^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathsf{class}(y) \wedge {}^{\Delta}\mathsf{take}(x,y)))$
$\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \downarrow_x {}^{\Delta}(\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{class}(y) \wedge \mathsf{take}(x,y)))$
$\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \delta_x(\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{class}(y) \wedge \mathsf{take}(x,y)))$

b. a $\rhd$ every:
$\mathbf{M}(\varepsilon_x \wedge \downarrow_x (^{\Delta}\mathsf{student}(x) \wedge \uparrow_x (^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathsf{class}(y)) \wedge {}^{\Delta}\mathsf{take}(x,y)))$
$\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}(x) \wedge \downarrow_x \uparrow_x {}^{\Delta}(\varepsilon_y \wedge \mathsf{class}(y)) \wedge \downarrow_x {}^{\Delta}\mathsf{take}(x,y)))$
$\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{class}(y) \wedge \downarrow_x {}^{\Delta}\mathsf{take}(x,y)))$
$\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \delta_x\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{class}(y) \wedge \delta_x\mathsf{take}(x,y)))$

$\uparrow$'s provide a way for a formula to escape the universal force of a distributivity operator. This is because the universal force is not contributed by the $\downarrow$-operator itself, but is contributed many times very locally by the $^{\Delta}$-operators. We can get away with this move only because the underlying logic allows us to (i) distribute δ_x over conjunction and (ii) permute $\mathbf{M}$ and δ_x. The nice thing about the resulting logic is that the new symbols (and thus the new interpretive resources) can always be eliminated.

The remainder of the chapter is structured as follows. In §5.1 I provide a broad overview of the chapter that informally walks through a few data points and their associated formal representations. In §5.2 I re-examine the binder roof constraint. Here I argue that the constraint applies not only to binding but also to various anaphoric processes, e.g. donkey anaphora. I also show in this section that the binder roof constraint is limited to singular binding/anaphora and that plural discourse reference and donkey anaphora do not place any constraints on the upward scope of indefinites. In §5.3 I show how the formal tools developed in chapter 4 can be applied to handle both wide scope indefinites and the new data surrounding the binder roof constraint. In §5.4 I put in place the final system that splits the scope of distributivity operators between $\downarrow$ and $^{\Delta}$ operators. §5.5 concludes.

5.2 Overview

This section provides a broad overview of the analysis presented in the following three sections. This chapter is concerned with contrasts like the one provided in (368). Both sentences in (368) contain indefinites that have pronouns in their scope that are co-indexed with a variable introduced by a previous quantifier.

(368) a. Every^{x} student presented on a^{y} topic that interested her$_{x}$.
b. Every^{x} student presented on a^{y} topic that interested them$_{x}$ (all).

The only difference between the sentences in (368) is the number of the bound/anaphoric pronoun. In (368a) the pronoun is singular while in (368) the pronoun is plural. Moreover this distinction in number tracks a distinction in the scope which the indefinite can take. In (368a) in which the pronoun is singular, the indefinite cannot take wide scope with respect to the universal. However, in (368b) in which the pronoun is plural a wide scope interpretation of the indefinite is possible. §5.3 will provide additional data arguing in favour of this empirical generalization.

The logic developed in chapter 4 allows wide scope of the indefinite to be reduced to the scope of a distributivity operator. A sentence like (369) can be translated either by means of the formula in (370a) which corresponds to a narrow scope interpretation or by means of (370b) which corresponds to a narrow scope reading.

(369) Every student noticed that John recommended a (certain) paper about semantics.

(370) a. $\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'} \underbrace{(\varepsilon_y \wedge \mathsf{sem.paper}(y) \wedge \mathsf{note.recommend}(x', j, y))}_{\text{scope of } \delta_{x'}}) \wedge \mathbf{every}(x, x')$$

b. $\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \varepsilon_y \wedge \mathsf{sem.paper}(y) \wedge \delta_{x'} \underbrace{\mathsf{note.recommend}(x', j, y))}_{\text{scope of } \delta_{x'}} \wedge \mathbf{every}(x, x')$$

The formula in (370) is interpreted as follows. The first conjunct finds the maximal set of students and stores it as the value of x. The second conjunct finds the maximal subset of x (which will be stored in x') and maximal y s.t. y is a semantics paper that x' noticed that John recommended. Since the assignment update associated with y occurs inside the scope of the distributivity operator the papers are allowed to vary with the x'-students. The final conjunct ensures that every value in x is also in x'. The formula in (370) is interpreted as follows. The first conjunct again finds the maximal set of students and stores it as the value of x. The second conjunct is consistent with any maximal output in which x' stores a subset of x and y stores a semantics paper that each x noticed that John recommended. Since the assignment update associated with y does not occur in the scope of the distributivity operator $\delta_{x'}$ the variable assigned to y will not vary with respect to the value of x' (which tracks the value of x). Since the scope of a distributivity operator also determines whether a variable will be picked up by a singular or plural pronoun this explanation will automatically capture the correlation between the availability of wide scope readings and the number morphology of an anaphoric pronoun in the scope of an indefinite. This argument is developed in more detail in §5.3.

The account outlined above replaces one syntax-semantics interface issue with another. Left unexplained is how indefinites manage to control the scope of distributivity operators introduced by other elements. To account for this I argue for a decomposition of distributivity into two components. First a component introduced by the quantifier $\downarrow_x$ (the mnemonic is: we look down into the atomic elements of x) that enters x into a cache of variables that are being quantified over. This cache can then be made use of by subsequent $^{\Delta}$ operators to actually implement distributivity using the variables in the cache. This allows an operator $\uparrow_x$ to intervene between (i) the point at which x is entered into the cache of variables to be distributed over and (ii) the implementation of distributivity. The $\uparrow_x$ operator has the effect of removing the variable x from the cache of variables and calls off distributivity for x.

The interaction between these operators is schematized in (371). $\downarrow$-operators generate contexts in which formulas are interpreted distributively while $\uparrow$-operators can create sub-contexts that are immune to distributive interpretation.

(371) $$\downarrow_x \underbrace{(\ldots \uparrow_x (\overbrace{\ldots \; {}^{\Delta}\phi \; \ldots}^{\text{not in scope of } \delta_x}) \ldots)}_{\text{scope of } \delta_x}$$

A compositional treatment of indefinites would then equip them with the ability to chose to have their restrictors interpreted in a non-distributive context.

In order to implement this analysis formulas will be interpreted relative not only to input and output states but also a set of variables $\mathscr{V}$ that indicate which variables are to be distributed over. This interpretive parameter is passed statically from a conjunction to each conjunct identically:

(372) $$[\![\phi \wedge \psi]\!]^{G,H,\mathscr{V}} = \mathbb{T} \text{ iff } \exists_K : [\![\phi]\!]^{G,K,\mathscr{V}} = \mathbb{T} \text{ and } [\![\psi]\!]^{K,H,\mathscr{V}} = \mathbb{T}$$

Notice that G and H are treated as an input-output pair. We find an output for the first conjunct and feed it in as the input for the second conjunct. The cache of variables $\mathscr{V}$ is treated statically. It is passed unchanged to each individual conjunct much like the assignment functions in FOL. §5.4 spells out this analysis in detail.

5.3 Revisiting the Binder Roof Constraint

Many authors have noticed that the presence of a bound pronoun in the scope of an indefinite places a constraint on its otherwise unbounded upward scope:

(373) Every^x professor will rejoice if a^z student of his$_x$ cheats on the exam. (Ruys, 1992)

(374) Every^x professor rewarded every^y student who read a^z book he$_x$ had recommended. (Abusch, 1994)

(375) Every student^x read a^y book I had recommended to him$_x$. (Schwarz, 2001)

Notice that in each of these cases the pronoun inside the restrictor of the indefinite is (i) bound by a syntactically c-commanding element that (ii) matches the number of the element that binds it.

We can test if both these conditions are necessary by examining first cases in which the pronoun is not bound by a c-commanding element but instead is anaphoric to an indefinite in the scope of a c-commanding universal, i.e. donkey sentences. Examples like those below show that donkey anaphora also triggers binder-roof effects:

(376) Every^x department that hired a^y faculty member asked a^z (certain) professor that knew her$_y$ for a letter of recommendation.

(377) Everyone^x who owns a^y credit card was turned down by a^z (certain) company that refused to accept it$_y$.

The sentence in (376) does not have an interpretation in which every department asked the same professor for a letter for every candidate. Even with the presence of *certain* only a functional reading—not a wide scope reading—seems to arise. Likewise, the sentence in (377) does not have a reading in which the same company turned down every credit card owner. The binder roof constraint thus seems to apply to cases of donkey anaphora. Notice that in these cases even the presence of the adjective *certain* is not able to facilitate a wide scope interpretation.

Examples like those below show that the binder roof constraint can be called off if the pronoun in the restrictor of the indefinite is plural:

(378) Every^x professor will rejoice if a^y student of theirs$_x$ cheats on the exam.

(379) Every^x professor rewarded every^y student who read a^z book they$_x$ (all) had recommended.

As soon as the pronoun in the restrictor of the indefinite is plural, that wide scope readings become available. There are good reasons not to accept data like that above as a reason to reject the binder roof constraint for plural pronouns. First, we might worry about the actual existence of wide scope readings for sentences like (378-379). It is unclear whether these are cases of genuine exceptional scope or cases of accidental non-variation that can be confused with wide scope interpretation.

Second, it is not clear that the pronouns above are bound as opposed to anaphoric. In fact good evidence that they are anaphoric comes from the ability of plural epithets to replace the plural pronouns in these sentences.

(380) Every^x corrupt inspector rewarded every business that used a (certain) contractor that [the bastards]$_x$ had recommended.

Singular epithets in the same sentence seem distinctly odd.

(381) ?? Every^x corrupt inspector rewarded every business that used a (certain) contractor that [the bastard]$_x$ had recommended.

Note that with a singular epithet the sentence does not seem to have an interpretation in which the indefinite takes widest scope.

A third issue with data like (378-379) arises when we consider that plausible contexts for uttering these sentences will be ones in which there is already salient a set of professors. In this case it is not clear that the plural pronoun depends on the quantifier in any way for its interpretation.

Starting with the first objection, we need to turn to quantifiers like *no* in which accidental non-variation cannot provide an explanation for apparent wide scope effects.

(382) No^x professor will rejoice if a^y certain student of theirs$_x$ cheats on the exam.

(383) No^x professor rewarded every^y student who read a^z (certain) book they^z (all) had recommended.

While the judgement for (382) is unclear[2]it does seem to me that (383) has a reading in which there is some book that every professor recommended and no professor rewarded every student who read that book. This reading is especially facilitated by the presence of *all* after the pronoun which I believe suppresses a singular-*they* interpretation of the pronoun and explains the difficulty with determining the judgements for (382).

The second and third objections can be met by considering donkey sentences like those in (376-377) in contrast to the sentence in (384). In the donkey sentences given earlier the donkey pronoun was singular and the binder roof constraint was seen to hold. This suggests that the binder roof constraint should be described in broad terms so that it includes not only binding but also other anaphoric interpretations. The example in (384) shows that plural pronouns also call off the binder roof constraint as it applies to donkey anaphora.

(384) Every^x manager that had a^y new employee bought [him or her]$_y$ a^z t-shirt in a^v colour that they$_y$ (all) found acceptable.

The sentence in (384) seems to have a reading in which there is one colour that every new employee found acceptable s.t. every manager who had a new employee bought that employee a t-shirt in that colour. This reading suggests that *a colour* can take

[2]I believe the judgement is unclear due to the fact that *their* can be interpreted as singular and gender neutral. The wide scope interpretation of (383) is facilitated since the *all* can suppress singular interpretation of *they*. Some speakers in fact report a near total inability to interpret *they* as plural in sentences like these without either (i) including a modifier like *all* or (ii) ensuring that the antecedent is non-human.

widest scope despite the presence of the plural pronoun anaphoric to the donkey indefinite. Moreover it is not clear that the set of new employees that were hired by managers would be previously available in a context that makes uttering (384) plausible. Thus the set referred to by *they* only becomes available after the quantifier is processed.

There do however appear to be counter-examples to the claim that this wide scope reading requires a plural pronoun. Consider the example below which is a minimal pair with the example in (384).[3]

(385) Every^x manager that had a^y new employee bought him$_y$ a^z t-shirt in a^v colour that he$_y$ and all the others found acceptable.

Despite the singular donkey pronoun in the restrictor of *a colour* the example in (385) seems to have a reading in which there is a single colour shirt that is bought for every employee. This example thus seems to show that even singular donkey pronouns do not prevent indefinites from taking wide scope.

Before deciding that these examples truly show that the binder roof constraint does not apply to donkey pronouns even when singular, we should consider the possibility that the wide scope interpretation could be the result of accidental non-variation. For instance it makes sense to think that all the t-shirts a bunch of managers would buy would all be the same colour. Once again to test if these are genuinely wide scope interpretations, we need to use a quantifier like *no*.

(386) a. No^x manager that had a^y new employee bought [him or her]$_y$ a^z t-shirt in a^v colour that they$_y$ (all) found acceptable.
b. No^x manager that had a^y new employee bought him$_y$ a^z t-shirt in a^v colour that he$_y$ and all the others found acceptable.[4]

[3]I thank Pranav Anand for bringing examples like these to my attention.

[4]Adrian Brasoveanu points out that a natural reaction this sentence is to pronounce the relative clause *that he and all the others found acceptable* with comma intonation suggesting it's natural interpretation

Consider a situation in which all the new employees find both red and yellow acceptable colours for a t-shirt and all the managers decided to buy their employees red t-shirts. It seems to me that there is a reading on which (386a) is true in such a situation: there is a colour, viz. yellow, s.t. no manager that had a new employee bought him or her a shirt that was yellow (i.e. in a colour that they all found acceptable). On the other hand, it is much less clear that (386b) has a reading in which it is true in this scenario. If this judgement stands, then these example provide evidence that the binder roof constraint does in fact apply to donkey pronouns while differentiating between singular and plural anaphora.

The data presented in this section suggest first that the binder roof constraint should be expanded to include donkey anaphora and second that the binder roof constraint applies to singular but not plural anaphora. The central data that I will try to explain in the next section is the correlation between the number of the pronoun and the scope of the indefinite.

5.4 Distributivity and the Scope of Indefinites

In this section I will reprise the analysis of exceptional scope indefinites sketched in the introduction of the preceding chapter. I will show how wide and narrow scope for indefinites in both the restrictor and the nuclear scope of universal quantifiers can be reduced to the relative scope of the distributivity operator associated with the universal quantifier. My goal is to demonstrate two things:

i. The logic developed in the previous chapter has the expressive capacity to handle

is as a non-restrictive relative clause. In this case the contribution of the relative clause would be not-at-issue (Potts, 2005). It is unclear how non-restrictive relative clauses could be worked into the current account but see AnderBois et al. (2015) for an account of non-restrictive relative clauses in a very similar framework to the one proposed here.

a variety of wide scope and dependent indefinites. One can write formulas that have the same truth conditions as sentences containing such indefinites.

ii. The difference between a formula translating a sentence containing a wide scope indefinite vs one containing a narrow scope indefinite comes down to the extent to which material contributed by the indefinite is inside or outside the scope of a distributivity operator associated with some other quantifier.

It is this second point that will ultimately renders the analyses sketched in this section non-compositional. Since distributivity operators are generally associated with the translations of universal quantifiers (see (387)), it is impossible to take scope beyond the distributivity operator without taking syntactic scope beyond the quantifier itself. Moreover even if maximality and distributivity were split up in the syntax, one would not expect an indefinite to be able to take scope over two or more universals—only intermediate readings would be predicted.

(387) every P Qed $\rightsquigarrow \mathbf{M}(\varepsilon_x \wedge \delta_x P(x)) \wedge \mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'} Q(x)) \wedge \mathbf{every}(x, x')$

We will translate indefinites in terms of ε-updates and number morphology:

(388) A P Qed $\rightsquigarrow \varepsilon_x \wedge \mathbf{sg}(x) \wedge P(x) \wedge Q(x)$

These simple rules of thumb will result in translations that capture a variety of narrow scope readings, but we will need to depart from them in order to capture exceptional scope interpretations.

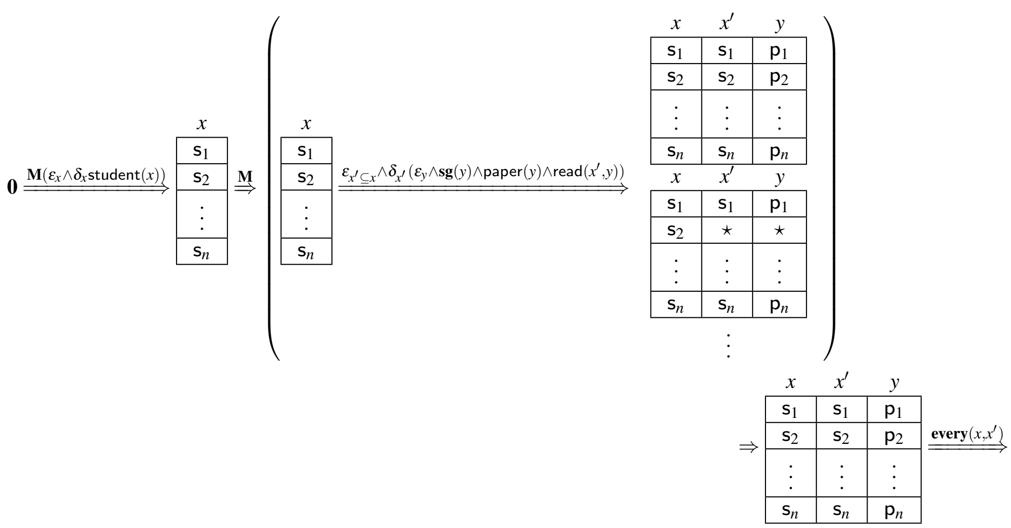
(389) Every student read a paper.

The sentence in (389) is ambiguous. On its narrow scope interpretation, it expressed that every student read some paper or other, leaving open the possibility that different students read different papers. This reading is easily captured using our heuristic interpretive procedure outlined above. We would assign the sentence in (389) the translation given below in (390).

(390)
$$\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'} \underbrace{(\overbrace{\varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y)}^{\text{NS of } \textit{a paper}} \wedge \mathsf{read}(x', y))}_{\text{scope of } \delta_{x'}}) \wedge \mathbf{every}(x, x')$$

Notice that the formula translating *a paper* in (390) occurs in the scope of the distributivity operator associated with the universal quantifier. The calculation of the truth conditions, depicted below in (391) shows how the narrow scope reading of (389) is captured.

(391)



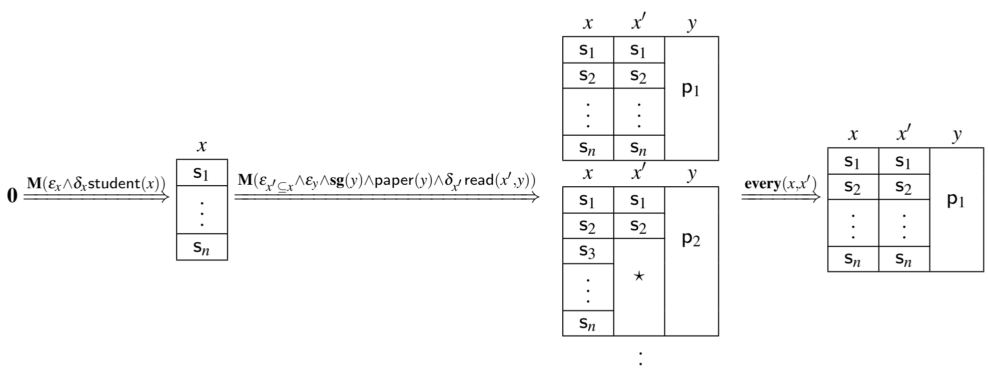
The first step depicted in (391) delivers the maximal set of students. Set of students that distributively each read a book are non-deterministically copied over by the formula in the scope of the maximality operator. Of these only the largest is selected and passed on to the final test. The final output assignment function assigns to x a plurality of students and to y a plurality of papers while also encoding a non-trivial relation between students and papers. The y-papers depend on the x-students. Assuming that every student read a book, the set of x'-students who read books is identical to the set of x-students and the sentence is true.

By shrinking the scope of the distributivity operator so that it doesn't encompass linguistic material associated with the restrictor of *a paper* we arrive at the representation in (392).

(392) $\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$

$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \overbrace{\varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y)}^{\text{NS of } a\ paper} \wedge \delta_{x'} \underbrace{\mathsf{read}(x', y)}_{\text{scope of } \delta_{x'}}) \wedge \mathbf{every}(x, x')$$

Notice that the formula translating *a paper* in (392) occurs outside the scope of the distributivity operator associated with the universal quantifier. Let us assume that every student read p_1 and that p_1 was the only paper every student read. The calculation of the truth conditions, depicted below in (393) show how the wide scope reading of (389) is captured.

(393) 

Again the first step delivers the maximal set of students. The next conjunct allows for non-deterministic updates. There are as many ways to update x' and y by copying over as many x''s as read a single y-paper as there are papers that at least one student read, so a number of updates is possible. Notice that y cannot depend on x' because the ε_y update does not occur inside the scope of a distributivity operator $\delta_{x'}$. This is also why y is defined even in rows in which x' is not—these rows are not eliminated when ε_y is contributed as they would be if ε_y were to occur in the scope of $\delta_{x'}$. Additionally y can only contain a single paper due to the number requirement on the predicate paper.

The final update eliminates all but one of the possible outputs since only p_1 was read by each x-student. Also consider what would happen if there were no papers. In this case there would be no output that satisfied the restrictor of the universal and the whole sentence would be false—existential import would be achieved.

5.4.1 The Binder Roof Constraint & Singular Pronouns

The sentence in (394), unlike the sentence in (389) is unambiguous. It only has a reading in which *a paper that interested him* takes narrow scope with respect to the quantifier *every student*.

(394) Every^x student read a paper that interested him$_x$.

This is a result of the fact that the pronoun bound by *every student* is singular. Consider the translation of (394) given in (395) below:

(395) $\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$

$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'} \underbrace{(\overbrace{\varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y) \wedge \mathbf{sg}(x') \wedge \mathsf{interest}(y,x')}^{\text{NS of } \textit{a paper}} \wedge \mathsf{read}(x',y))}_{\text{scope of } \delta_{x'}}) \wedge \mathbf{every}(x,x')$$

In the formula in (395), the relative clause is translated as $\mathsf{interest}(x,y) \wedge \mathbf{sg}(x)$—the number morphology of the pronoun contributes a test indicating that x contains only one value. Moreover, like the narrow scope translation of (389) given in (390) above, the material introduced by the restrictor of the indefinite appears inside the scope of the distributivity operator provided by the universal quantifier.

If we assume that every student, s_i, read a paper, p_i, that interests him, we expect the formula in (395) to come out true and deliver an output contexts which contains every student and their associated paper. The calculation of the truth conditions depicted in (396) shows that this result obtains.

(396) $$\mathbf{0} \xRightarrow{\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x))} \begin{array}{|c|} \multicolumn{1}{c}{x} \\ \hline \mathsf{s}_1 \\ \hline \mathsf{s}_2 \\ \hline \vdots \\ \hline \mathsf{s}_n \\ \hline \end{array} \xRightarrow{\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'}\, \varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y) \wedge \mathbf{sg}(x') \wedge \mathsf{interest}(y,x') \wedge \mathsf{read}(x',y))) \wedge \mathbf{every}(x,x')} \begin{array}{|c|c|c|} \multicolumn{1}{c}{x} & \multicolumn{1}{c}{x'} & \multicolumn{1}{c}{y} \\ \hline \mathsf{s}_1 & \mathsf{s}_1 & \mathsf{p}_1 \\ \hline \mathsf{s}_2 & \mathsf{s}_2 & \mathsf{p}_2 \\ \hline \vdots & \vdots & \vdots \\ \hline \mathsf{s}_n & \mathsf{s}_n & \mathsf{p}_n \\ \hline \end{array}$$

The only new component of the calculation depicted above is the evaluation of the text $\mathbf{sg}(x')$ in the scope of the distributivity operator. Notice that since the distributivity operator has broken apart its input information state into sub-states in which x stores only one value. The test thus applies to each of these sub-states individually. Since these sub-states are defined so that x' is atomic in each, the test will always be satisfied.

The argument for treating number morphology as part of the indefinite comes from Brasoveanu (2007) who works with pairs like those in (397) from Karttunen (1976).

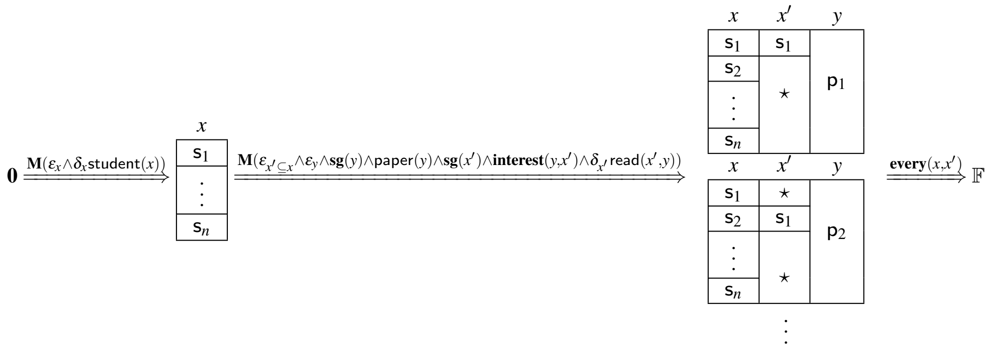
(397) a. Harvey courts a woman at every convention. She is very beautiful.
b. Harvey courts a woman at every convention. She is always very beautiful.

In (397a) the use of the singular pronoun to refer back to *a woman* forces a unique / wide scope interpretation of the indefinite *a woman* in the previous sentence. If however a quantificational adverb like *always* appears in the sentence this inference is called off. These facts fall out of a system in which singular pronouns require singular antecedents: in (397a) if there are multiple women at multiple conventions there will not be a single woman to refer back to unless the pronoun appears in the scope of an operator that distributes over conventions—and even here there must be one woman per convention.[5]

If instead we attempt to translate (394) along the lines of the wide scope translation of (389) given in (392) we end up with a formula that is necessarily false unless there

[5]In order to capture this we would have to translate *a woman* without checking the number on the indefinite *a woman* in the initial sentence. This would cause us to treat indefinites as ambiguous between readings in which their number morphology contributes a test and readings in which the number morphology is semantically uninterpreted. This is probably a good thing since we uniqueness condition appears must less robust than the uniqueness condition associated with (397a) and is subject to a number of pragmatic and discourse factors (Heim, 1982; Kadmon, 1990; Roberts, 2003). Moreover treating indefinites as ambiguous in some way plays a large role in some theories that try to capture the distinction between weak and strong donkey readings (van den Berg, 1996; Brasoveanu, 2007).

is exactly one student.

(398) 

The translation in (398) takes the restrictor of the indefinite out of the scope of the distributivity operator associated with the universal. Crucially this also removes the test associated with the singular bound pronoun from the scope of the distributivity operator. Since the test $\mathbf{sg}(x')$ does not occur in the scope of the distributivity operator, it is evaluated relative to an plural information state in which x contains many values (again assuming that there are several students). Thus any formula satisfying the restrictor of the second max formula will only copy a single value for x' so it will never satisfy the formula $\mathbf{every}(x,x')$ This ensures that the sentence cannot be true as long as there is more than one student. Note that this result obtains merely due to the number morphology of the pronoun.

5.4.2 The Binder Roof Constraint & Plural Pronouns

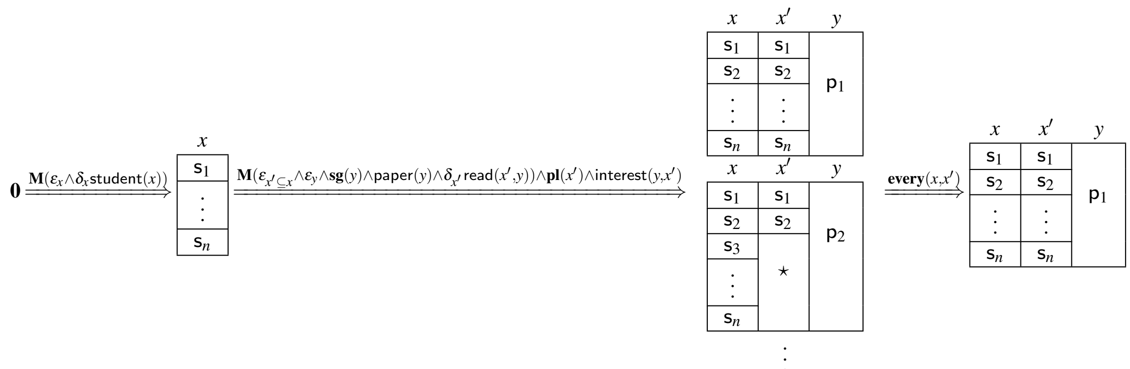
Now consider the sentence in (399) which forms a minimal pair with the sentence in (395)—the singular bound pronoun has been replaced with a plural pronoun anaphoric to the same quantifier.

(399) Every^x student read a paper that interested them$_x$ (all).

Notice that the sentence in (399) does have a reading in which the indefinite takes scope over the universal. This reading can be represented by means of the translation given below in (400).

(400)
$$\mathbf{M}(\varepsilon_x \wedge \delta_x \mathsf{student}(x)) \wedge$$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \delta_{x'} \underbrace{(\overbrace{\varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y) \wedge \mathbf{pl}(x') \wedge \mathsf{interest}(y,x')}^{\text{NS of } a\ paper} \wedge \mathsf{read}(x',y))}_{\text{scope of } \delta_{x'}}) \wedge$$
$$\mathbf{every}(x,x')$$

(401)



The formula in (400) differs from the formula in (398) only with respect to the number assigned to the pronoun. The derivation begins in exactly the same way as the derivation in (393). The first step delivers the maximal set of the students, and the next few updates introduce non-deterministically single papers in a way that ensures papers do not depend upon students. Notice here that the derivation does not terminate because the plurality condition imposed on the x variable is met since the test $\mathbf{pl}(x')$ does not occur in the scope of a distributivity operator, δ'_x. This leaves only two potential updates, which are shaved down to one by the final test. Notice that the universal still binds into the restrictor of the indefinite, but it is still able to take wide scope with respect to it. Because wide scope is achieved by manipulating the scope of a distributivity operator, it will have ramifications for tests in the restrictor of indefinites that make reference to the cardinality of other variables.

5.4.3 The Binder Roof Constrain & Donkey Anaphora

This analysis extends naturally to donkey pronouns as well. Consider the sentence given below in (402), in which a singular donkey pronoun occurs in the restrictor of the indefinite in the scope of the universal.

(402) Every student who picked a problem talked to a professor who know about it.

Like the sentence in (395), the sentence given above in (402) does not have a reading in which the indefinite, *a professor* takes semantic scope outside the universal *every student*.

The absence of a wide scope interpretation follows exactly in exactly the same way for donkey pronouns as it does for bound pronouns. A putative wide scope translation involves removing the indefinite from the scope of the distributivity operator associated with the universal resulting in a translation like that given below in (403).

(403)
$$\mathbf{M}(\varepsilon_x \wedge \delta_x(\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{problem}(y) \wedge \mathbf{sg}(y) \wedge \mathsf{pick}(x,y))) \wedge$$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \overbrace{\varepsilon_z \wedge \mathsf{prof}(z) \wedge \mathbf{sg}(z) \wedge \mathsf{k.about}(z,y) \wedge \mathbf{sg}(y)}^{\text{NS of } \textit{a prof...}} \wedge \delta_{x'} \underbrace{\mathsf{talk.to}(x,z)}_{\text{scope of } \delta'_x} \wedge$$
$$\mathbf{every}(x,x')$$

Now consider a scenario in which students $\mathsf{s}_1,\ldots,\mathsf{s}_n$ picked problems $\mathsf{p}_1,\ldots,\mathsf{p}_n$, there are two professors, $\mathsf{t}_1,\mathsf{t}_2$, who both know about every problem, and students s_{1-n} talked to professor t_1 and no other professor was talked to by every student who picked a problem. In this scenario the sentence in (402) is true on its wide scope interpretation because there is a professor t_1 who knows about all the problems picked by the students and every student who picked a problem talked talked to t_1.

Even in this scenario the translation in (403) comes out false due to the cardinality check associated with the pronoun; the calculation of the truth conditions comes to a halt when the cardinality test is met. Because there are multiple y-problems (one per student in fact) y is not associated with an atomic individual when the statement $\mathbf{sg}(y)$ is encountered. Notice that when y is first introduced it is inside the scope of δ_x this means that y can vary with x and that because of the test $\mathbf{sg}(y)$ that there is only one value of y per value of x but multiple values can be associated with y overall.

Notice that if we alter the sentence in (402) so that the donkey pronoun is plural, a wide scope reading for the indefinite is available:

(404) Every student who picked a problem talked to a professor who know about them (all).

The translation of (404) is given below in (405). Notice that this translation is identical to the formula in (403) except that the test $\mathbf{sg}(y)$ has been replaced by the test $\mathbf{pl}(y)$.

(405)
$$\mathbf{M}(\varepsilon_x \wedge \delta_x(\mathsf{student}(x) \wedge \varepsilon_y \wedge \mathsf{problem}(y) \wedge \mathbf{sg}(y) \wedge \mathsf{pick}(x,y))) \wedge$$
$$\mathbf{M}(\varepsilon_{x' \subseteq x} \wedge \overbrace{\varepsilon_z \wedge \mathsf{prof}(z) \wedge \mathbf{sg}(z) \wedge \mathsf{k.about}(z,y) \wedge \mathbf{pl}(y)}^{\text{NS of } \textit{a prof...}} \wedge \delta_{x'} \underbrace{\mathsf{talk.to}(x,z)}_{\text{scope of } \delta'_x} \wedge$$
$$\mathbf{every}(x,x')$$

The calculation of the truth conditions for the formula in (405) begins exactly like the calculation of (403) except that when we encounter the test associated with the pronoun we do see a plurality of y-problems. Here we don't expect to encounter any difficulties because although y is introduced as singular it is introduced in the scope of a distributivity operator. When it is later picked back up and said to be plural in the formula in (405) the test $\mathbf{pl}(y)$ is outside the scope of any distributivity operator so there will be a plurality of values stored in the y column.

Note that since distributivity operators in our system tightly couple (i) whether a variable can be referenced via a singular or plural pronoun and (ii) whether an indefinite can vary with that pronoun or not. This predicts a more fine grained version of the binder roof constraint which is sensitive not only to binding but also discourse anaphora and in addition is sensitive to the morphological number associated with the anaphoric pronoun.

5.5 Decomposing Distributivity

In this section I outline a logic in which distributivity operators are split into two component pieces: a component that determines which variables are being distributed over and a component that distributes over them. This involves enriching the evaluation function to include three parameters: an input state, G, an output state, H, and a stock of stored variables $\mathscr{V}$. The input-output states will be passed dynamically from one conjunct to the other while the store of variables will be passed statically to each conjunct separately.

Pursuing this analysis requires us to think about how multiple variables could be distributed over at the same time. In our definitions so far we have only ever distributed over a single variable at a time. If instead we have have input-output sets G and H and a stock of variables over which we will be distributing, then we will need a way to define distributivity with respect to all of the variables in the stock $\mathscr{V}$ simultaneously. We can get at an appropriate definition by defining the set $G^{\mathscr{V}}$ to be the set of local contexts that would be available in DPlL if the variables in $\mathscr{V}$ were all distributed over:

(406) $$G^{\mathscr{V}} := \begin{array}{l} \{G\}, \text{ if } \mathscr{V} = \emptyset \\ \{G|_{v_1=d_1,\ldots,v_n=d_n} : \{v_1,\ldots,v_n\} = \mathscr{V} \;\&\; \langle d_1,\ldots,d_n\rangle \in G(v_1,\ldots,v_n)\}, \text{otherwise} \end{array}$$

The definition in (406) defines a set of subsets of G in terms of the values of the variables in $\mathscr{V}$. Each cell consists of a sub-state $G|_{v_1=d_1,\ldots,v_n=d_n}$, where $v_1,\ldots,v_n$ are the variables in $\mathscr{V}$, and $\langle d_1,\ldots,d_n\rangle$ is an element in $G(v_1,\ldots,v_n)$. Recall that the set $G(v_1,\ldots,v_n)$ consists of just those n-tuples containing values of of the variables

$v_1, \ldots, v_n$ that co-occur in a single assignment function. This ensures that every unique n-tuple receives its own sub-state in $G^{\mathscr{V}}$. Recall also that $G(v_1, \ldots, v_n)$ excludes any rows in which any of the variables are mapped to the value $\star$. This ensures that any assignment in G that sets the value of any $v \in \mathscr{V}$ to the $\star$ value is not in any sub-state contained in $G^{\mathscr{V}}$. This is the desired behaviour since distributivity operators in a sequence would each shave off all the assignment functions in which their particular values were assigned the $\star$ value.

Below I give some examples of the sets that can be formed from an underlying set of assignments G and a set of variables $\mathscr{V}$:

(407)

$G^{\emptyset} =$

x	y
a	c
	d
b	c
	d

$G^{\{x\}} =$

x	y
a	c
	d
b	c
	d

$G^{\{x,y\}} =$

x	y
a	c
a	d
b	c
b	d

Notice that in each of the above examples each cell contains only one value for the sequence of relevant variables. The set $G^{\mathscr{V}}$ thus contains all those sub-states that would be obtained by distributing over the variables in $\mathscr{V}$.

5.5.1 Conjunction

Conjunction is defined so that the input-output states are treated dynamically while the cache of variables is treated statically.

(408) $[\![\phi \wedge \psi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff there is some K s.t. $[\![\phi]\!]^{G,K,\mathscr{V}} = \mathbb{T}$ and $[\![\psi]\!]^{K,H,\mathscr{V}} = \mathbb{T}$

Notice that this definition looks exactly like the definition of conjunction for DPlL except that in addition to finding an intermediate set of assignment functions it also passes the stock of stored variables $\mathscr{V}$ down to each conjunct. This is because distributivity is itself non-dynamic. Distributing over a variable has the same effects on both the input and output contexts in DPlL: both are split up and later re-assembled. This fact is mirrored in the definition of conjunction—the stock of variables is treated statically while the input-output states are handled dynamically.

Notice also that the way that the set $\mathscr{V}$ is passed between conjuncts mirrors the manner in which distributivity operators in DPlL distribute over conjunction.

(409) $\delta_x(\phi \wedge \psi) = \delta_x\phi \wedge \delta_x\psi$

5.5.2 Distributivity

We define the distributivity operator $\downarrow_x$ to simply modify the input context so that and additional variable is being distributed over:

(410) $\downarrow$-DISTRIBUTIVITY:
$[\![\downarrow_x (\phi)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff $[\![\phi]\!]^{G,H,\mathscr{V}\cup\{x\}} = \mathbb{T}$

Notice that the $\downarrow$-operator has several of the same properties as the distributivity operator with respect to (i) sequences of distributivity operators, (ii) permutation of distributivity operators, and (iii) distributing over conjunction:

(411) INFERENCES WITH $\downarrow$:
a. $\downarrow_x\downarrow_x \phi = \downarrow_x \phi$
b. $\downarrow_x\downarrow_y \phi = \downarrow_y\downarrow_x \phi$
c. $\downarrow_x (\phi \wedge \psi) = \downarrow_x \phi \wedge \downarrow_x \psi$

All of these inferences are easy enough to prove. The first follows since $\mathscr{V} \cup \{x\} \cup \{x\} = \mathscr{V} \cup \{x\}$. The second follows from the fact that$\mathscr{V} \cup \{x\} \cup \{y\} = \mathscr{V} \cup \{y\} \cup \{x\}$. The third follows from the fact that conjunction simply passes down the store of variables to each conjunct. It doesn't matter if the variables are added then passed down or added to the store for each conjunct separately.

The key difference between $\downarrow_x$ and δ_x is that $\downarrow_x$ does not itself break apart the input or the output states. They remain whole and so operations in the scope of $\downarrow_x$ can make use of all the information available in the information state instead of just the subcomponents available to formula's in the scope of δ_x. This is because $\downarrow_x$ is in a real sense not a distributivity operator but an operator that signals that a variable should be distributed over.

In order to actually distribute over the variables in the store, we define a distributivity operator $^{\Delta}$ that breaks apart the incoming set of assignment functions and passes them on to its nuclear scope piece by piece.

(412) Δ-DISTRIBUTIVITY:
$[\![^{\Delta}\phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $G(\mathscr{V}) = H(\mathscr{V})$

b. $\forall_{G' \in G^{\mathscr{V}}} : [\![\phi]\!]^{G',H|_{\forall_{v\in\mathscr{V}}:v=G'(v)},\emptyset} = \mathbb{T}$

c. $\forall_{H' \in H^{\mathscr{V}}} : [\![\phi]\!]^{G|_{\forall_{v\in\mathscr{V}}:v=H'(v)},H',\emptyset} = \mathbb{T}$

d. $H|_{\exists_{v\in\mathscr{V}}:v=\star} = \{h : \exists_{g\in G|_{\exists_{v\in\mathscr{V}}:v=\star}} : \forall_{v\in\mathbf{Dom}(G)} : h(v) = g(v)$
$\& \forall_{v\in\mathbf{Dom}(H)-\mathbf{Dom}(G)} : h(v) = \star\}$

The truth conditions are relatively complex. It is worth going through each conjunct:

a. $G(\mathscr{V}) = H(\mathscr{V})$

This condition says that G and H have the same projections for the set of variables in $\mathscr{V}$ that is they contain the same values and the same relations between these values.

b. $\forall_{G' \in G^{\mathscr{V}}} : [\![\phi]\!]^{G',H|_{\forall_{v\in\mathscr{V}}:v=G'(v)},\emptyset} = \mathbb{T}$

This condition starts by breaking apart G into the sub-states defined by $G^{\mathscr{V}}$. It then says that for each such sub-state in G' has as its output. relative to the scope of $^{\Delta}$ the sub-state $H|_{\forall_{v\in\mathscr{V}}:v=G'(v)}$. This is the sub-state of H that assigns the same values to the variables in $\mathscr{V}$ as are assigned by G'. This is the sub-state in $H^{\mathscr{V}}$ that corresponds to G'.

c. $\forall_{H' \in H^{\mathscr{V}}} : [\![\phi]\!]^{G|_{\forall_{v\in\mathscr{V}}:v=H'(v)},H',\emptyset} = \mathbb{T}$

This condition works exactly like the first conjunct except that we make sure that for every sub-state in $H^{\mathscr{V}}$ we can find an origin sub-state in $G^{\mathscr{V}}$. Together with the first conjunct this makes sure that every sub-state in $G^{\mathscr{V}}$ has a destination in $H^{\mathscr{V}}$ and every destination in $H^{\mathscr{V}}$ has some origin in $G^{\mathscr{V}}$.

d. $H|_{\exists_{v\in\mathscr{V}}:v=\star} = \{h : \exists_{g\in G|_{\exists_{v\in\mathscr{V}}:v=\star}} : \forall_{v\in\mathbf{Dom}(G)} : h(v) = g(v)$

$\& \; \forall_{v\in\mathbf{Dom}(H)-\mathbf{Dom}(G)} : h(v) = \star\}$

This condition relates the sets $H|_{\exists_{v\in\mathscr{V}}:v=\star}$ and $G|_{\exists_{v\in\mathscr{V}}:v=\star}$. These are the subsets of H and G in which one or more of the values in $\mathscr{V}$ is given the $\star$ value. That is the set of rows that are not in any sub-state in $H^{\mathscr{V}}$ or $G^{\mathscr{V}}$ respectively. The condition itself states that every $h \in H|_{\exists_{v\in\mathscr{V}}:v=\star}$ is an extension of some element of $G|_{\exists_{v\in\mathscr{V}}:v=\star}$ that (i) agrees with g with respect to all values in their common domain and (ii) assigns $\star$ values to all the variables which are in the domain of H but not G.

A final point with emphasizing about this definition is that $^{\Delta}$-operators zero-out the values being distributed over. They are removed from the set of values being distributed over. By emptying the stock of variable this operator signals to the formula in its scope that there is nothing that needs to be distributed over.

Before providing the rest of the logic, I want to address a few logical matters. First, if no variables are being distributed over, then Δ can freely added or deleted:

(413) $[\![\phi]\!]^{G,H,\emptyset} = [\![^{\Delta}\phi]\!]^{G,H,\emptyset}$

To see why we only need to recall that $G^{\emptyset} = \{G\}$ (likewise $H^{\emptyset} = \{H\}$). In other words the ${}^{\Delta}$-operator in these circumstances creates only the trivial partition over the assignment function and passes the entire assignment function to its scope.

It then becomes obvious that a sequence of ${}^{\Delta}$-operators is equivalent to a single ${}^{\Delta}$-operator:

(414) $\Delta \ldots \Delta\phi = \Delta\phi$

This follows since the first operator empties the cache of variables and the semantics of the remaining operators is vacuous.

Additionally, a ${}^{\Delta}$-operator can be prefixed to a formula $\downarrow_x {}^{\Delta}\phi$ harmlessly.

(415) $\downarrow_x ({}^{\Delta}(\phi)) = {}^{\Delta}(\downarrow_x ({}^{\Delta}(\phi)))$

This inference holds because the order in which variables are distributed over does not matter. We can either add x to a set $\mathscr{V}$ and distribute over $\mathscr{V} \cup \{x\}$ or we can distribute first over the variables in $\mathscr{V}$ and then distribute over the variables in $\{x\}$.

Finally, ${}^{\Delta}$, like δ_v, distributes over conjunction:

(416) ${}^{\Delta}(\phi \wedge \psi) = {}^{\Delta}\phi \wedge {}^{\Delta}\psi$

One last operation that is useful to define for peace of mind is the distributivity operator δ_v:

(417) δ-DISTRIBUTIVITY:
$[\![\delta_x \phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $\mathscr{V} = \emptyset$
b. $G(x) = H(x)$
c. $\forall_{G' \in G^{\{x\}}} : [\![\phi]\!]^{G',H||_{x=G'(x)},\emptyset} = \mathbb{T}$
d. $\forall_{H' \in H^{\{x\}}} : [\![\phi]\!]^{G|_{x=H'(x)},H',\emptyset} = \mathbb{T}$
e. $H|_{x=\star} = \{h : \exists_{g \in G|_{x=\star}} : \forall_{v \in \mathbf{Dom}(G)} : h(v) = g(v)$
$\& \forall_{v \in \mathbf{Dom}(H) - \mathbf{Dom}(G)} : h(v) = \star\}$

Notice that the definition above (i) is exactly like the definition of $^{\Delta}$ except that we (a) utilize a singleton set to find the sets we distribute over and (b) require that $\mathscr{V} = \emptyset$ and (ii) that this definition is substantively identical to the definition of distributivity in DPlL modulo these two changes.

Turning back to logical matters, it becomes clear that we can use δ-operators cancel out $\downarrow$ and $^{\Delta}$ operators.

(418) $\downarrow_x {}^{\Delta}\phi = {}^{\Delta}\delta_x\phi$

We reason first from $\downarrow_x {}^{\Delta}\phi$ to ${}^{\Delta} \downarrow_x {}^{\Delta}\phi$, then note that when we evaluate $\downarrow_x {}^{\Delta}\phi$ the store of variables will be empty (due to the initial $^{\Delta}$). When we next evaluate ${}^{\Delta}\phi$ it will contain only x. This leads to an interpretation that is identical to the interpretation of $\delta_x\phi$.

5.5.3 Lexical Relations & Assignment Updates

Lexical relations are defined almost identically to their DPlL counterparts. The truth conditions for a lexical relation is given (419).

(419) LEXICAL RELATIONS: $[\![R(v_1,\ldots,v_n)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff
a. $\mathscr{V} = \emptyset$
b. $G = H$
c. $\langle G(v_1),\ldots,G(v_n)\rangle \in \mathfrak{I}(R)$

The truth conditions are familiar; the lexical relation is defined only for identical input-output pairs and lexical relations are interpreted collectively. The only new addition is the first conjunct which requires that the store of variables be empty. That is to say either (i) no variables were entered into the set at all in which case the lexical relation should be interpreted collectively or (ii) the store has been emptied of variables by a $^{\Delta}$ operator that distributes over these variables.

The work here is done when we translate expressions from English into the logic. We always prefix predicates with $^{\Delta}$-operators to ensure that appropriate variables are being distributed over and the store is zeroed out.

(420) a. dog $\rightsquigarrow {}^{\Delta}\mathsf{dog}(x)$
b. chase $\rightsquigarrow {}^{\Delta}\mathsf{chase}(x,y)$

In essence formulas like ${}^{\Delta}\mathsf{dog}(x)$ say, "when distributing over the variables over which I have been told to distribute, I find that x stores a value in the interpretation of dog".

The next alteration we need to make is to the definition of assignment update. Again we minimally alter the definition to include the requirement that the store of variables being distributed over is empty.

(421) ASSIGNMENT UPDATE:
$[\![\varepsilon_x]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff
a. $\mathscr{V} = \emptyset$
b. $x \notin \mathbf{Dom}(G)$
c. $\exists_{D \in \mathfrak{D}^\star} : H = \{g^{[x \to d]} : d \in D \,\&\, g \in G\}$

(422) SUBSET ASSIGNMENT UPDATE:
$[\![\varepsilon_{x' \subseteq x}]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff
a. $\mathscr{V} = \emptyset$
b. $x' \notin \mathbf{Dom}(G)$
c. $\exists_{D \subseteq G(x)} : H = \{g^{[x' \to g(x)]} : g \in G|_{x \in D}\} \cup \{g^{[x' \to \star]} : g \in G|_{x \notin D}\}$

Notice again that these definitions are identical to their DPlL counterparts except that we require that the store of variables be empty.

Again we ensure that no variables are being distributed over in the intput-output by prefixing random assignment operators with a $^{\Delta}$-operator whenever we would use random assignment when translating something into DPlL:

(423) an egg $\rightsquigarrow {}^{\Delta}\varepsilon_x \wedge {}^{\Delta}\mathsf{egg}(x) = {}^{\Delta}(\varepsilon_x \wedge \mathsf{egg}(x))$

5.5.4 Maximization

We keep unselective maximization exactly as is. Since δ_v can normally permute with **M**, we simply pass the store of variables to be distributed over to the scope of the maximization operator.

(424) MAXIMIZATION:
$[\![\mathbf{M}(\phi)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff
a. $[\![\phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$
b. $\neg\exists_{K\supset H} : [\![\phi]\!]^{G,K,\mathscr{V}} = \mathbb{T}$

Sentences with quantifiers are translated exactly as before modulo two changes: (i) we now use $\downarrow$-operators lieu on δ-operators and (ii) we attach $^{\Delta}$-operators locally to all lexical relations and assignment updates.

(425) Every student read a paper.
$\rightsquigarrow \mathbf{M}(^{\Delta}\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}) \wedge$
$\mathbf{M}(^{\Delta}\varepsilon_{x'\subseteq x} \wedge \downarrow_{x'} (^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathsf{paper.sg}(y) \wedge {}^{\Delta}\mathsf{read}(x,y)) \wedge {}^{\Delta}\mathbf{every}(x,x')$

5.5.5 Truth

Truth for a sentence is defined in terms of the initial state **0** and the empty cache of variables:

(426) TRUTH:
$[\![\phi]\!] = \mathbb{T}$ iff $\exists_H : [\![\phi]\!]^{\mathbf{0},H,\emptyset} = \mathbb{T}$

Because our definition of truth always starts with the empty store, if we write out formulas that are string identical to the formulas in DPlL, then we end up with the same truth conditions, as their DPlL counterparts. This is because the store of variables will never be incremented and so the only conjuncts that differ in the recursive definitions of truth for the two languages (i.e. the ones referring to the store) will be trivially satisfied. The empty store will just be passed down through conjunction and maximization until it hits a lexical relation, an assignment update, or one of the δ_x operators (which itself will not change the store).

Additionally, if there are no occurances of $\downarrow_x$ in a formula we can immediately remove any occurrences of $^\Delta$. If the store of variables starts empty and stays that way, then any $^\Delta$-operators that occur in the formula will have only a semantically vacuous effect.

One of the other benefits of the current system is that we can manipulate these representations syntactically. Consider the example from (425). This can be simplified just by manipulating the $^\Delta$ and $\downarrow$-operators:

(427) $$\begin{aligned}&\mathbf{M}({}^\Delta\varepsilon_x \wedge \downarrow_x {}^\Delta\mathsf{student}) \wedge \mathbf{M}({}^\Delta\varepsilon_{x'\subseteq x} \wedge \downarrow_{x'} ({}^\Delta\varepsilon_y \wedge {}^\Delta\mathsf{paper}(y) \wedge {}^\Delta\mathsf{read}(x,y)) \wedge {}^\Delta\mathbf{every}(x,x')\\ &\Leftrightarrow \mathbf{M}({}^\Delta\varepsilon_x \wedge \downarrow_x {}^\Delta\mathsf{student}) \wedge \mathbf{M}({}^\Delta\varepsilon_{x'\subseteq x} \wedge \downarrow_{x'} {}^\Delta(\varepsilon_y \wedge \mathsf{paper}(y) \wedge \mathsf{read}(x,y)) \wedge {}^\Delta\mathbf{every}(x,x')\\ &\Leftrightarrow \mathbf{M}({}^\Delta\varepsilon_x \wedge {}^\Delta\delta_x\mathsf{student}) \wedge \mathbf{M}({}^\Delta\varepsilon_{x'\subseteq x} \wedge {}^\Delta\delta_{x'}(\varepsilon_y \wedge \mathsf{paper}(y) \wedge \mathsf{read}(x,y)) \wedge {}^\Delta\mathbf{every}(x,x')\\ &\Leftrightarrow \mathbf{M}(\varepsilon_x \wedge \delta_x\mathsf{student}) \wedge \mathbf{M}(\varepsilon_{x'\subseteq x} \wedge \delta_{x'}(\varepsilon_y \wedge \mathsf{paper}(y) \wedge \mathsf{read}(x,y)) \wedge \mathbf{every}(x,x')\end{aligned}$$

In the first step we bring $^\Delta$ out from single conjuncts to take scope over whole conjunctions. In the next step we simplify the sequence $\downarrow_v {}^\Delta$ to ${}^\Delta\delta_v$. In the final step we remove any $^\Delta$ operators that are not inside the scope of any $\downarrow$-operators since these will be semantically vacuous. Since this formula no longer contains any of the new symbols ($\downarrow_x$, $^\Delta$) its truth conditions are exactly the same as the identical sentence of DPlL. Thus this sentence captures a reading in which the indefinite takes scope below the universal quantifier.

5.6 Scope Control

The only operator we need to implement wide scope indefinites is the $\uparrow_v$-operator. This operator un-distributes over v by removing it from the stock of variables in $\mathscr{V}$:

(428) $\uparrow$-SCOPE CONTROL:
$$[\![\uparrow_v \phi]\!]^{G,H,\mathscr{V}} = \mathbb{T} \text{ iff } [\![\phi]\!]^{G,H,\mathscr{V}-\{v\}} = \mathbb{T}$$

The formula above shares much in common with the distributivity operator $\downarrow_v$. The only difference is that instead of adding a variable to the store of those to be distributed over, it removes a variable from the store without doing anything with it.

If we think of the $\downarrow_x$-operator as sending a signal to its scope that the variable x must be distributed over, then the $\uparrow_x$-operator contravenes this instruction ensuring that its scope is not evaluated distributively with respect to x.

This gives u a new set of inference rules that handle the $\uparrow$-operator:

(429) INFERENCE RULES FOR $\uparrow$:
a. $\downarrow_x\uparrow_x \phi = \uparrow_x \phi$
b. $\uparrow_x\downarrow_x \phi = \uparrow_x\uparrow_y \phi$

Together these two inferences tell us that sequence of $\uparrow$- and $\downarrow$-operators (that work with the same variable) reduce to the final operator in the sequence. This means that $\downarrow$-operators can be cancelled and distributivity can be called off.

Wide scope indefinites can now be translated with the help of these operators. Consider just the interpretation of the restrictor of a universal hosting a wide scope indefinite.

(430) Every student who read a certain paper ...
$\rightsquigarrow \mathbf{M}({}^{\Delta}\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}(x)) \wedge \downarrow_x (\uparrow_x ({}^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathbf{sg}(y) \wedge {}^{\Delta}\mathsf{paper}(y)) \wedge {}^{\Delta}\mathsf{read}(x,y))$

Notice that the indefinite *a certain paper* has been translated with an $\uparrow_x$ wrapped around its restrictor. In other words this indefinite comes with an instruction to prevent its restrictor from being interpreted distributively with respect to the variable x.

The representation above can be simplified in several steps. First, we can distribute $\downarrow_x$ over the conjunction:

(431) $\mathbf{M}({}^{\Delta}\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}(x)) \wedge \downarrow_x\uparrow_x ({}^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathbf{sg}(y) \wedge {}^{\Delta}\mathsf{paper}(y)) \wedge \downarrow_x {}^{\Delta}\mathsf{read}(x,y))$

Next we simplify the sequence $\downarrow_x\uparrow_x (\dots)$ by cancelling the outer $\downarrow_x$-operator.

(432) $\mathbf{M}({}^{\Delta}\varepsilon_x \wedge \downarrow_x {}^{\Delta}\mathsf{student}(x)) \wedge {}^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathbf{sg}(y) \wedge {}^{\Delta}\mathsf{paper}(y) \wedge \downarrow_x {}^{\Delta}\mathsf{read}(x,y))$

Next we reduce sequences of $\downarrow_x {}^{\Delta}$ to ${}^{\Delta}\delta_x$.

(433) $\mathbf{M}({}^{\Delta}\varepsilon_x \wedge {}^{\Delta}\delta_x\mathsf{student}(x)) \wedge {}^{\Delta}\varepsilon_y \wedge {}^{\Delta}\mathbf{sg}(y) \wedge {}^{\Delta}\mathsf{paper}(y) \wedge {}^{\Delta}\delta_x\mathsf{read}(x,y))$

Finally we remove any Δ's that are not in the scope of any $\downarrow$ operator since their contribution is semantically vacuous:

(434) $\mathbf{M}(\varepsilon_x \wedge \delta_x\mathsf{student}(x)) \wedge \varepsilon_y \wedge \mathbf{sg}(y) \wedge \mathsf{paper}(y) \wedge \delta_x\mathsf{read}(x,y))$

Notice that this gives us exactly the representation outlined in the previous section. Exceptional scope indefinites can thus choose not to be in the scope of any given distributivity operator. This will have predictable effects as far as (i) whether they can vary with a particular variable—no unless they are in the scope of distributivity—and (ii) whether singular or plural pronouns in their restrictors are required to pick up a particular variable—again this will depend on whether or not they come with an $\uparrow$-operator that shelters them from the scope of other distributivity operators.

5.7 Conclusions

In this chapter I outlined a logic that differed minimally from DPlL with unselective maximization. I broke distributivity up into two components. A signalling operator $\downarrow_x$ that indicates that x is to be distributed over and an distributivity operator ${}^{\Delta}$ that implements distributivity. For this to work formulas of the logic have to be interpreted relative to a store indicating which variables were to be distributed over.

This allowed the logic to be enriched with an operator $\uparrow_x$ that removed x from the store of variables, thus sheltering a formula in its scope from being interpreted distributively with respect to x. In this system wide scope indefinites achieve their interpretation by signalling non-variation with the values of a variable introduced by a syntactically higher quantifier. They do this by manipulating the scope of distributivity operators.

Appendix D

DPlL with Decomposed Distributivity

NOTATIONAL CONVENTIONS:

(435) $G^{\mathscr{V}} := \begin{array}{l} \{G\}, \text{ if } \mathscr{V} = \emptyset \\ \{G|_{v_1=d_1,\ldots,v_n=d_n} : \{v_1,\ldots,v_n\} = \mathscr{V} \;\&\; \langle d_1,\ldots,d_n\rangle \in G(v_1,\ldots,v_n)\}, \text{otherwise} \end{array}$

RECURSIVE DEFINITION OF TRUTH:

(436) LEXICAL RELATIONS: $[\![R(v_1,\ldots,v_n)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $\mathscr{V} = \emptyset$

b. $G = H$

c. $\langle G(v_1),\ldots,G(v_n)\rangle \in \mathfrak{I}(R)$

(437) ASSIGNMENT UPDATE:
$[\![\varepsilon_x]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $\mathscr{V} = \emptyset$

b. $x \notin \mathbf{Dom}(G)$

c. $\exists_{D\in\mathfrak{D}^\star} : H = \{g^{[x\to d]} : d \in D \;\&\; g \in G\}$

(438) SUBSET ASSIGNMENT UPDATE:
$[\![\varepsilon_{x'\subseteq x}]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $\mathscr{V} = \emptyset$

b. $x' \notin \mathbf{Dom}(G)$

c. $\exists_{D\subseteq G(x)} : H = \{g^{[x'\to g(x)]} : g \in G|_{x\in D}\} \cup \{g^{[x'\to\star]} : g \in G|_{x\notin D}\}$

(439) MAXIMIZATION:
$[\![\mathbf{M}(\phi)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $[\![\phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$

b. $\neg\exists_{K\supset H} : [\![\phi]\!]^{G,K,\mathscr{V}} = \mathbb{T}$

(440) δ-DISTRIBUTIVITY:
$[\![\delta_x\phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $\mathscr{V} = \emptyset$

b. $G(x) = H(x)$

c. $\forall_{G'\in G^{\{x\}}} : [\![\phi]\!]^{G',H||_{x=G'(x)},\emptyset} = \mathbb{T}$

d. $\forall_{H'\in H^{\{x\}}} : [\![\phi]\!]^{G|_{x=H'(x)},H',\emptyset} = \mathbb{T}$

e. $H|_{x=\star} = \{h : \exists_{g\in G|_{x=\star}} : \forall_{v\in\mathbf{Dom}(G)} : h(v) = g(v)$
$\& \forall_{v\in\mathbf{Dom}(H)-\mathbf{Dom}(G)} : h(v) = \star\}$

(441) $\downarrow$-DISTRIBUTIVITY:
$[\![\downarrow_x (\phi)]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff $[\![\phi]\!]^{G,H,\mathscr{V}\cup\{x\}} = \mathbb{T}$

(442) $\uparrow$-SCOPE CONTROL:
$[\![\uparrow_v \phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff $[\![\phi]\!]^{G,H,\mathscr{V}-\{v\}} = \mathbb{T}$

(443) Δ-DISTRIBUTIVITY:
$[\![^{\Delta}\phi]\!]^{G,H,\mathscr{V}} = \mathbb{T}$ iff

a. $G(\mathscr{V}) = H(\mathscr{V})$

b. $\forall_{G'\in G^{\mathscr{V}}} : [\![\phi]\!]^{G',H|_{\forall_{v\in\mathscr{V}}:v=G'(v)},\emptyset} = \mathbb{T}$

c. $\forall_{H'\in H^{\mathscr{V}}} : [\![\phi]\!]^{G|_{\forall_{v\in\mathscr{V}}:v=H'(v)},H',\emptyset} = \mathbb{T}$

d. $H|_{\exists_{v\in\mathscr{V}}:v=\star} = \{h : \exists_{g\in G|_{\exists_{v\in\mathscr{V}}:v=\star}} : \forall_{v\in\mathbf{Dom}(G)} : h(v) = g(v)$
$\& \forall_{v\in\mathbf{Dom}(H)-\mathbf{Dom}(G)} : h(v) = \star\}$

Chapter 6

Conclusions

The bulk of this dissertation has been devoted to providing a solution to the problem of wide scope indefinites that follows the central intuitions guiding the account in Brasoveanu and Farkas (2011). Indefinites can take wide semantic scope outside of scope islands by signalling non-variation with respect to the values of variables introduced by other expressions. In my system variation and non-variation are controlled by distributivity operators and so indefinites signal non-variation by exercising control over the scope of distributivity operators introduced by other elements. Formally this is implemented by splitting distributivity operators into two parts: a signalling operator, $\downarrow_v$, that a variable needs to be distributed over and an operator that contributes quantificational force, $^{\Delta}$.

One of the interesting formal elements of the system is that it blends static and dynamic style reasoning. Input-output plural information states are passed from one conjunct to another dynamically left to right, while the store of variables to be distributed over is passed top down, from the conjunction to each conjunct identically.

At this point in the dissertation, it is worth stepping back from the particular formal implementation to discuss broader claims that will apply to other potential implementations of the same basic intuition.[1]

i. In chapter 3 I argued that the intuition that indefinites signal independence has to be embedded inside a dynamic logic. This way a universal quantifier can be made aware of the particular way in which the updates associated with a wide scope existential in its scope will be completed. It is not enough to look for a maximal set of entities *x* for which an independent choice of *y* can be made. One needs to look for the maximal set of entities *x* for which a particular independence choice of *y* can be made, and this necessitates looking at an output set of assignments.

ii. In chapter 4 we learn that an dynamic logic by itself does not guarantee that independence amounts to wide scope. Instead only a logic with certain maximization operators will achieve correct interpretations for sentences containing wide scope indefinites.

iii. In chapter 5 I showed that utilizing DPlL indefinites could signal independence by means of controlling the scope of a distributivity operators associated with higher quantifiers. It bears emphasis that the particular implementation in this chapter is one many potential implementations of this idea.

6.1 Extensions

Below I outline two extensions to the basic system that that may extend its empirical coverage.

[1] It seems to me that there may be alternative routes to allow indefinites to signal non-variation while still sitting in the syntactic scope of a quantifier (the analysis at the end of chapter 3 is a case in point). This would be an overly parochial contribution if nothing could be learned from it about alternative routes.

6.1.1 Disjunction

Evaluating formulas with respect to sets of assignment functions allows us to analyse disjunction as a sort of conjunction and account for its wide scope behaviour. In essence we have a disjunction split up the incoming set of assignment functions into two pieces sending one to each disjunct.

(444) $[\![\phi_1 \vee_{\mathscr{T}} \phi_2]\!]^{G,H} = \mathbb{T}$ iff $\mathscr{T} \subseteq \mathbf{Dom}(G)$ and $\exists_{D_1, D_2 \subseteq G(\mathscr{V})}$ s.t. there are sets G_1, G_2, H_1, and H_2 s.t.

a. $G_1(\mathscr{T}) = D_1$ and $G_2(\mathscr{T}) = D_2$ and $H_1(\mathscr{T}) = D_1$ and $H_2(\mathscr{T}) = D_2$

b. $G_1 \cup G_2 = G$ & $H_1 \cup H_2 = H$

c. $[\![\phi]\!]^{G_1,H_2} = \mathbb{T}$

d. $[\![\psi]\!]^{G_1,H_2} = \mathbb{T}$

Going through the definition step-by-step. Six sets are quantified over existentially:

- D_1, D_2 which are subsets of $G(\mathscr{T})$ that jointly exhaust $G(\mathscr{T})$. Recall that $G(\mathscr{T})$ is the set of tuples of values that the alphabetized variables in $\mathscr{T}$ in G. So, D_1, D_2 are each sets of tuples of values that these variables can take on.

- G_1, G_2 are subsets of G that have as their projections of the variables in $\mathscr{T}$ the sets of values in D_1, D_2 respectively.

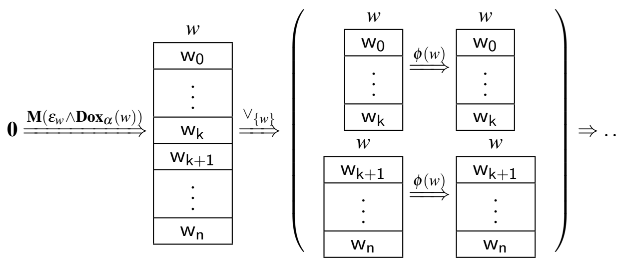
- H_1, H_2 which are subsets of the output assignment H that jointly exhaust H.

The final clause links D_1 with H_1 and D_2 with H_2. For $D_{1/2}$, the state $G_{1/2}$ is used as the input for the conjunct, $\phi_{1/2}$ obtaining the output $H_{1/2}$.

If we assume that all sentences are taken as being implicitly embedded under an epistemic necessity operator, this definition will help derive ignorance inferences for disjunctions.

Let us say that $\phi(w)$ holds for $w_1, \ldots, w_k$ and $\psi(w)$ holds for $w_{k+1}, \ldots, w_n$. Let's also say that $w_1, \ldots w_n$ represent a speaker α's belief worlds. We want to get the fact that a utterance "ϕ or ψ" should indicate that a speaker believes both ϕ and ψ to be possible. Let's see how this works:

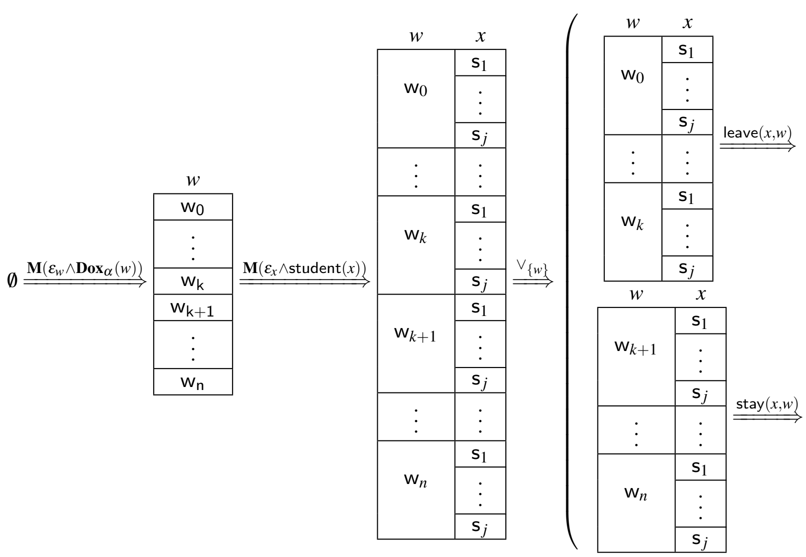
(445) $\mathbf{M}_{\varepsilon_w}(\mathbf{Dox}_\alpha(w)) \wedge (\phi(w) \vee_w \psi(w))$

(446) 

The first conjunct populates the variable assignment with the set of epistemically accessible worlds. Then the disjunction partitions G based on the values assigned to w and feeds them into one or the other disjunct. This indicates that the speaker thinks that worlds in which both ϕ and ψ hold are possible.

We can now see how exceptional scope disjunction would work. For simplicity's sake, let's assume that the domains of every world are the same and that the same individuals are students in every world.

(447) $\mathbf{M}(\varepsilon_w \wedge \mathbf{Dox}_\alpha(w)) \wedge \mathbf{M}(\varepsilon_x \wedge \mathsf{student}(x)) \wedge (\mathsf{leave}(x,w) \vee_w \mathsf{stay}(x,w))$

(448) 

Notice that because the disjunction is only indexed with world variables it can only break up the set of assignments by worlds, not students. This results in an interpretation of the disjunction in which it takes wide scope over the universal.

If instead we index the disjunction with both world and student variables, then we would allow the set of assignment functions to be partitioned along lines of arbitrary world-student pairs.

(449) $\mathbf{M}(\varepsilon_w \wedge \mathbf{Dox}_\alpha(w)) \wedge \mathbf{M}(\varepsilon_x \wedge \mathsf{student}(x)) \wedge (\mathsf{leave}(x,w) \vee_{w,x} \mathsf{stay}(x,w))$

The sentence above would be true as long as every world-student pair satisfied the predicate leave or the predicate student. Once we allow $\vee$ to depend on two variables the set of models that can be described really opens up. It might be that the speaker has belief worlds in which every student left and others in which every student stayed or that their belief worlds are all such that some students left and other students stayed but different students left or stayed in different worlds, etc. All that is required is that there every world-student pair is s.t. the student left in that world or the student stayed in that world.

One benefit of this account is that it ties the ignorance inference to the scope taken by the disjunction. It predicts that an ignorance inference associated with disjunction should be attenuated when it appears in the scope of another quantifier.

This account would hopefully scale up to handle cases of donkey anaphora like the sentence below:

(450) Every tourist that saw a certain pigeon or a pelican took a picture of it.

On its most natural interpretation the indefinite *a pigeon* takes scope above the universal, while both the disjunction and the indefinite *a pelican* scope below the universal. The pronoun *it* is then anaphoric to either the pelican or the pigeon as the case may be. By splitting up the set of assignment functions one hopes an account of disjunction developed along the lines about would allow the indefinite *a certain pigeon* to signal its independence from the variable introduce by *every* in the part of the global state that it was fed. The narrow scope indefinite, *a pelican*, could signal dependence within its part of the global state. Taken together a single variable could then store tourist-pigeon/pelican pairs.

6.1.2 Extraposed Relative Clauses

Restrictive relative clauses can appear well after the DP that hosts them. Consider the examples below:

(451) a. A man left who had been looking angry for a while.
b. Every student did well who took their studies seriously.

The semantic difficulty associated with extraposed relative clauses arises form the need to interpret them as if they occurred in the restrictor of a DP. For instance (451b) does not require that every student do well; the sentence is true as long as the students who took their studies seriously did well.

With the semantic resources available in DPlL we can define an operator **S**—pronounced 'shave'—that can interpret extraposed restrictive relative clauses *in situ*, i.e. outside the restrictor of the DP they modify.

(452) SHAVE:
$[\![\mathbf{S}(\phi)]\!]^{G,H} = \mathbb{T}$ iff $\exists_{G' \subseteq G} : [\![\phi]\!]^{G',H}$ and $\neg\exists_{K \supset H} : \exists_{G \subseteq G} : [\![\phi]\!]^{G'',K} = \mathbb{T}$

$\mathbf{S}(\phi)$ works by finding the a subset of the input context that can satisfy its scope s.t. there is no other subset of the input that would lead to a greater output set. In essence $\mathbf{S}(\phi)$ shaves off rows from the input assignment that render it incompatible with the formula ϕ. This means that we will be able to begin the interpretation of the sentence in (451b) by first collecting the full set of students. Then we can shrink down the set to include only those students who take their studies seriously. Finally we can ensure that every student in this modified set took their studies seriously.

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