Statistics of Certain Automorphic Representations through the Stable Trace Formula

by

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Abstract

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Since Automorphic representations for general groups are very difficult to study individually, they are often studied in families instead. The Arthur-Selberg trace formula lends itself naturally to answering questions about averages of various parameters of the local components of automorphic representations in so-called harmonic families. In their 2016 work, Shin and Templier realized that, in the special case of representations with discrete series at infinity, the trace formula simplified dramatically enough to compute statistics with good error bounds. These bounds were good enough for applications: first, an averaged Sato-Tate law analogous to Sato-Tate for families of elliptic curves and second, computations of the specific random-matrix statistics that low-lying zeros of L-functions in the family follow. Following Shin-Templier's idea, we solve two further problems about discrete-at-infinity families.

First, Shin-Templier's work used the invariant trace formula which disallowed families that distinguish representations with infinite component in the same L-packet. However, which member of this L-packet a representation might correspond to determines some important characteristics—whether the representation is holomorphic or quaternionic for example. Methods related to the stable trace formula can remove this restriction. The key idea is applying a certain "hyperendoscopy" formulation of stabilization used first by Ferrari, though many technical difficulties come up.

Second, while the equidistribution results achieved are interesting in their own right, they also provide a proof-of-concept that the tools developed for proving them are sufficient for studying very general questions about discrete-at-infinity families. As a further demonstration, we also use these methods to solve a very different problem of computing explicit dimensions of spaces of quaternionic forms on the exceptional group G_2.

To my family: Mom, Dad, Neha, and Megha.

Contents

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For the application to quaternionic forms on G_2, I would like to thank Kimball Martin and Aaron Pollack for discussions at conferences that eventually led to the consideration of this problem. The key proposition 5.1.2.1 that made this problem solvable grew out of email conversations with Sam Mundy. Olivier Taïbi provided a great deal of help in pointing out many, many tricks and previous results I could use to keep computations reasonably simple and actually feasible. In particular, I am grateful for the suggestion to use “method 2” described in section 5.2.2 that he passed on from Gaëtan Chenevier. Alexander Bertoloni-Meli again provided much help, this time teaching me things about transfer factors. The root system picture for G_2 was heavily modified from an answer by user Heiko Oberdiek on TeX Stack Exchange.

Much thanks also to the reviewers at Algebra and Number Theory in which material from chapters 2-4 of this thesis are to appear sometime in the future. They provided a very thorough reading, invaluable help with making my writing clearer, and caught a lot of typos I had in earlier drafts.

Chapter 1

Introduction

1.1 Overview

Automorphic representations are spaces of certain very symmetric complex valued functions that mysteriously encode information about much else in mathematics, with applications in fields as diverse as constructing higher-dimensional expanders for computer algorithms and computing scattering amplitudes in string theory. As a particularly interesting example for number theorists, if one believes something called the Langlands conjectures, information about automorphic representations directly produces information about structures called Galois representations that are the main tool used to solve problems in modern algebraic number theory

While useful, general automorphic representations are unfortunately quite difficult to work with. Key problems in the field, such as the Langlands functoriality and generalized Ramanujan conjectures, have been open for decades. Much previous work has been in developing complicated representation-theoretic techniques to solve the overarching abstract problem of functorial transfer, the most important being more and more sophisticated versions of Arthur's trace formula and its comparisons.

Because of their complexity, these more recent trace formulas have so far not been applied much towards more explicit statistical or analytic problems—for example, proving various equidistribution laws or bounds towards the generalized Ramanujan conjecture. However, the formulas simplify dramatically in a particular special case of representations with ”discrete series at infinity”, becoming tractable to compute with. Restriction to discrete-at-infinity is analogous to studying just holomorphic modular forms instead of also Maass forms—in particular, it still includes much interesting and rich behavior.

This thesis attempts to demonstrate that the simplified trace formulas are powerful enough to answer many desired statistical questions about discrete-at-infinity automorphic representations. In chapter 2, we go over current trace formula techniques. Chapter 3 is the technical heart where we then build the extensions of these formulas needed for our specific problems. Chapter 4 applies the results of chapter 3 towards computing equidistribution laws on certain families of automorphic representations on very general groups. Finally, chapter 5 uses the developed techniques to solve a very different problem of getting explicit counts of a particularly interesting class of automorphic representations about which very little is currently known—quaternionic representations on G_2.

We hope that the example uses of the techniques in this write-up are a useful guide for others attempting similar computations.

1.2 Mathematical Background

We start with a "pop-science", general-math-audience introduction to the material covered in this thesis. Any expert in the field should skip ahead to the two technical introductions.

1.2.1 Foundational Notions

Reductive Groups

Automorphic representations are built from certain mathematical objects called reductive groups. These can be thought of as subgroups of $N \times N$ matrices under matrix multiplication. The "reductive" condition is that they have particularly nice representation theory—roughly that their representations all reduce into a direct sum of irreducibles.

The key point is that a reductive group needs to be defined by only polynomial conditions on the coordinates of the matrix, making it completely agnostic as to which exact matrix entries are allowed—$\mathbb{C}$-valued, $\mathbb{Q}$-valued, etc. If G is a reductive group, we denote $G(R)$ to be the group of matrices satisfying the polynomial conditions with entries in R. The language of algebraic geometry lets us abstractly prove results about a given reductive group that stay true for very general choices of R.

Some examples of reductive groups are all the classical matrix groups: linear groups GL_n and SL_n, unitary groups U_n, orthogonal groups SO_n, symplectic groups Sp_{2n}, etc. A non-example is the group of upper triangular $N \times N$ matrices—even though it is a matrix group picked out by polynomial conditions on the coordinates, it has representations that do not decompose into a direct sum of irreducibles.

There is more-or-less a classification of all reductive groups defined by polynomial equations with coefficients in some nice enough field. Over the complex numbers and up to center, the only examples are the families of matrix groups SL_n, SO_n, and Sp_{2n}, together with 5 strange "exceptional groups" denoted G_2, F_4, E_6, E_7, and E_8 of dimensions 14, 52, 78, 133, and 248 respectively. Over a non-algebraically closed field, each of these complex groups has various "forms" that can basically be classified by certain Galois cohomology groups. For example, the unitary groups U_n are specific forms of GL_n and all the various orthogonal groups $\mathrm{SO}(m, n)$ are forms of SO_{m+n}.

The book [56] is a great reference for the full theory.

Adeles

The specific types of matrix entries that we will specialize our reductive matrix groups to are called the adeles.

Recalling first-semester real analysis, the real numbers are produced from the rational numbers together with the the standard metric $d_\infty(x, y) = |x - y|$. This done by a process called completion that fills in all the “holes” in the rational numbers and makes limits with respect to $d(x, y)$ work nicely.

There are many other possible choices of metric on the rational numbers however. For every prime p we can define $d_p(x, y)$ to be the power p^n of p we need to multiply $x - y$ by so that if $p^n(x-y) = a/b$ in lowest common form, a and b are relatively prime to p. For example, $d_2(0, 1/2) = 2$ and $d_7(1/2, 343 + 1/2) = 1/343$. The p-adic numbers $\mathbb{Q}_p$ are analogous to the real numbers: produced by completing with respect to d_p. For example, there is a $\sqrt{2}$ in $\mathbb{Q}_7$ that is the limit of a sequence of certain integers a_n such that $a_n^2 \equiv 2 \pmod{7^n}$, just like there is a $\sqrt{2}$ in $\mathbb{R}$ that is the limit of a sequence of rational a_n such that a_n^2 gets closer and closer to 2 in d_∞.

In number theory, all these notions of distance are important—we not only care directly about how big a number is, we also care about congruences mod powers of primes the number satisfies. Speaking extremely vaguely, the adeles $\mathbb{A}$ are a way to put $\mathbb{R}$ and all the $\mathbb{Q}_p$ together in a way so that we can do analysis with respect to all distances at the same time. Roughly,

$$\mathbb{A} \approx \mathbb{R} \times \prod_p \mathbb{Q}_p,$$

where the “$\approx$” hides a technical fix to make sure $\mathbb{A}$ is locally compact so it has reasonable analytic properties.

For the purposes of automorphic representations, we focus on some key properties of the adeles. First, $\mathbb{A}$ is a locally compact abelian group under addition, so functions on it have a good notion of Fourier transform. In addition, the diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$ realizes $\mathbb{Q}$ as a discrete subgroup such that $\mathbb{Q}\backslash\mathbb{A}$ is compact. The setup is analogous to the case of $\mathbb{Z}$ inside $\mathbb{R}$ and lets us get number theoretic-information out of a lot of powerful techniques from harmonic analysis, such as Poisson summation.

Even better, for non-abelian reductive matrix groups G, there is similarly a good theory of non-abelian Fourier transforms for functions on $G(\mathbb{A})$. In addition, $G(\mathbb{Q})$ diagonally embeds discretely into $G(\mathbb{A})$ and, up to issues with centers, $G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume. This allows us to similarly get number-theoretic information out of many powerful techniques in representation theory/non-abelian harmonic analysis, such as trace formulas.

Basic information about the adeles can be found in any graduate algebraic number theory text, such as [61]. The book [64] is a good reference for facts about the adelic points of reductive groups.

1.2.2 Automorphic Representations

What are they?

If G is a reductive matrix group over a number field K, then an automorphic representation on G is an appropriately-defined notion of irreducible subrepresentation of the square-integrable functions $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ as a representation under right translation by $G(\mathbb{A})$:

$$(g \cdot f)(x) = f(xg).$$

The space $G(\mathbb{Q})\backslash G(\mathbb{A})$ turns out to approximately be a limit of quotients $G(\mathbb{R})$ by all the subgroups in $G(\mathbb{Z})$ defined by congruence conditions on matrix entries. As a more familiar example, in the case $G = \mathrm{GL}_2$, $G(\mathbb{Q})\backslash G(\mathbb{A})$ is related to quotients of the complex upper-half plane (which is the same as $\mathrm{GL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$) by modular subgroups. Automorphic representations for GL_2 then correspond to classical new-eigen-modular and Maass forms. This correspondence takes some work to show.

Automorphic representations roughly factor over the primes and infinity:

$$\pi \approx \pi_\infty \otimes \bigotimes_p \pi_p,$$

where π_∞ is a representation of $G(\mathbb{R})$ and the π_p are representations of the individual $G(\mathbb{Q}_p)$. Not every combination of π_v forms an automorphic representation. Exactly which combinations work is what encodes most of the useful information in applications. In the case of classical modular forms, the π_v have to do with the Fourier coefficients a_p.

The component π_∞ describes the qualitative "type" of the automorphic representation. In the GL_2 case, it determines whether it is a Maass form or modular form and what its weight is. The nicest possible automorphic representations correspond to the nicest possible π_∞: representations that live discretely inside $L^2(G(\mathbb{R}))$, or discrete series. In the GL_2 case, discrete-at-infinity is the case of modular forms of weight $k \geq 2$.

The book [24] discusses automorphic representations on GL_2 and how they connect to classical modular forms and is therefore a great reference for building intuition about the general theory. A good general reference is the book draft [26].

Why do we care?

However strange and unmotivated their definition might be, automorphic representations are important for one big reason: they mysteriously come up in disparate areas across mathematics, thereby providing a common bridge and creating web of unexpected and useful interconnections. As a non-comprehensive list:

- in Number Theory: Galois representations (Langlands conjectures),
- in Computer Science: expander graphs/higher-dimensional expanders,

- in Differential Geometry: spectra of Laplacians on locally symmetric spaces,
- in Combinatorics: identities for the partition function,
- in Finite Group Theory: representation theory of large sporadic simple groups (moonshine),
- in Mathematical Physics: representations of infinite-dimensional Lie algebras,
- in String Theory: black hole partition functions, 4-graviton scattering amplitudes.

The first construction of expander graphs by [52] demonstrates the utility of the interconnections. Expander graphs are graphs without many edges, but where all vertices are still connected by very short paths relative to the number of vertices—a pretty useful property for designing algorithms. This "expansion property" can be restated as bounds on eigenvalues of a certain "Laplacian" operator acting on functions on the set of vertices.

Surprisingly, for certain graphs, the eigenvectors can be interpreted as coming from automorphic representations.The bound on eigenvalues then reduces to proving something called the Ramanujan conjecture for these automorphic representations. This conjecture is wide open in general, but luckily, the specific automorphic representations that come up can be looked at through the number-theoretic perspective.

This is where the power of the bridge comes in—first, the Ramanujan conjecture mysteriously reduces to a problem about counting solutions to certain polynomial equations mod p. The desired result about point counts, called the Weil conjectures, is known. However, it is only known through extremely sophisticated techniques in algebraic geometry. In total, automorphic representations allowed us to apply a deep result in number theory towards constructing combinatorial graphs with surprising and useful properties. For a long time, this was in fact the only known way to construct expander graphs, though there are direct combinatorial methods now.

So as to not oversell, the discrete-at-infinity automorphic representations studied in this work are of course not relevant in all of these applications. In addition, there are few other current examples of connections as striking as the expander graph one, though I am personally hopeful that there might be many more in the future.

1.2.3 Trace Formulas

Idea

We are now left with the question of how to actually study these objects. Automorphic representations are approximately subrepresentations of a larger representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Recalling what happens in any introductory course in representation theory, this means that a good way to study them should be to look at traces of operators on L^2 that relate somehow to the action of $G(\mathbb{A})$

How do we produce these operators? One standard way is to take nice enough functions $f : G(\mathbb{A}) \to \mathbb{C}$ and consider the convolutions:

$$R_f : v \mapsto \int_{G(\mathbb{A})} f(g)(h \cdot v)\, dh,$$

where dg is some invariant measure on $G(\mathbb{A})$. This should be though of as a sort of "averaged" action where the average is weighted by the function f.

How do we compute their traces? The Arthur-Selberg trace formula is roughly a formula for the trace of convolution operators of test functions on $G(\mathbb{A})$ against the space of all automorphic representations:

$$\sum_{\pi \text{ automorphic}} \operatorname{tr}_\pi f \approx \sum_{\gamma \in [G(\mathbb{Q})]} \operatorname{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, dg. \tag{1.1}$$

The right side is approximately sum over rational conjugacy classes of the volume of a particular quotient of the centralizer of γ times an integral over the conjugation orbit of γ in $G(\mathbb{A})$.

The trace formula lets us probe which combinations of local π_v produce an automorphic representation. For example, choose a test place w and pick a function f_w that traces to a desired parameter of π_w. Choose test functions f_r at all other r that trace to 1 if π_r satisfies a desired condition and 0 otherwise. Then plug $\prod_v f_v$ into the trace formula. Applied in this way, the trace formula naturally lends itself to computing statistics of a fixed local component over families of automorphic representations defined by other local conditions—the harmonic families of [70].

Technical issues

If $G(\mathbb{Q}) \backslash G(\mathbb{A})$ is compact, then the "$\approx$" in (1.1) is actually a strict equality. Otherwise, very few of the terms actually converge and we need to use various methods of truncation developed by Arthur. Truncating produces a variety of different and unfortunately extremely complicated formulas that all go under the name "the Arthur-Selberg trace formula".

The different versions of Arthur's trace formula lie on a spectrum. On one end, there are more explicit versions where the individual terms, while still complicated, are not as horrific to compute. However, on this end, the approximation is more brutal and destroys any nice abstract properties the terms might have—conjugation invariance, etc. On the other end, the terms are extremely technical and may even not have explicit formulas, only proofs of existence. However, they keep a lot of nice abstract properties and may even gain better ones then the basic, compact quotient formula.

In certain special cases, better abstract properties make the more advanced trace formulas simplify dramatically, to the point where the terms are even easier than in the explicit trace formulas. The case used in this work is Arthur's invariant trace formula,

$$I_{\text{spec}}(f) = I_{\text{geom}}(f),$$

when the infinite place is restricted to be discrete series as in [4].

The paper [3] is a good introduction to all these trace formulas.

1.3 Summary of Results

1.3.1 Shin-Templier's Result

Shin and Templier's paper [79] realized that the discrete-at-infinity formula of [4] is explicit enough that averaged statistics of local components π_p are computable up to reasonable error bounds. More specifically, let $\mathcal{F}$ be a "family" defined by giving to an automorphic representation π a weight $a_{\mathcal{F}}(\pi)$ that vanishes except for finitely many π and depends only on spectral data of the components π_v. Then, if f_v is an unramified test function at some place v and $\mathcal{F}$ is of a certain form that in particular only includes discrete-at-infinity representations, Shin-Templier's result bounds averages over the family. The bound has shape:

$$\frac{1}{|\mathcal{F}|}\sum_{\pi\in\mathcal{F}} a_{\mathcal{F}}(\pi)\,\mathrm{tr}_{\pi_v}(f_v) = \mu^{\mathrm{pl}}(\widehat{f_v}) + O(q_v^{A+B\kappa}|\mathcal{F}|^{-C}), \tag{1.2}$$

Here, $\mu^{\mathrm{pl}}(\widehat{f_v})$ is a canonical notion of the average of $\mathrm{tr}\, f_v$ on the space of representations of $G(F_v)$ and κ is a measure of the size of the support of f_v. The constants $A, B > 0$ and $C \geq 1$ are inexplicit but basically only depend on G.

The error bound's shape gives useful applications—first, an automorphic Sato-Tate equidistribution law for local components analogous to averaged Sato-Tate for families of elliptic curves, and second, a proof that distributions of low-lying zeros of the L-functions of the families match some expected random matrix laws.

1.3.2 Splitting the L-packet

Discrete series representations of $G(\mathbb{R})$ are partitioned into sets called L-packets. Because [79] used the invariant version of Arthur's trace formula, the coefficients $a_{\mathcal{F}}$ defining $\mathcal{F}$ needed to be constant on automorphic representations with π_∞ in the same L-packet. In other words, it could not distinguish between π with π_∞ in the same packet. While this is irrelevant for some applications, the different elements of an L-packet have differences that are significant in others. For example, only part of the L-packet might correspond to automorphic representations that are representable as holomorphic or quaternionic functions or that have a Whittaker model.

Chapters 2-4 of this thesis apply techniques related to the stable trace formula to prove Shin-Templier's bound (1.2) for families where $a_{\mathcal{F}}$ could depend on the particular π_∞ within an L-packet, thereby distinguishing between representations corresponding to the same packet at infinity. As far as I know, it is currently the only application of the fully-general stable trace formula towards computing statistics.

The work involves three main technical steps: first, a generalization of a "hyperendoscopy" formula of [17] to cases where hyperendoscopic groups do not have simply connected derived subgroup. Second, it used some bounds on endoscopic transfers produced by the full, character-theoretic formulation of the fundamental lemma in [34]. This required some combinatorial formulas from [12] and [32]. Finally, the result needed an extension of [4] to general groups with discrete series at infinity instead of just those satisfying a technical condition on their center. This removed the corresponding technical conditions in [79] as a side effect.

1.3.3 Quaternionic forms on G_2

. There are a few types of automorphic representations on specific groups that have been studied heavily historically—classical modular forms corresponding to discrete series representations on $\mathrm{GL}_2(\mathbb{R})$, Seigel modular forms corresponding to holomorphic discrete series on $\mathrm{Sp}_{2n}(\mathbb{R})$, etc. Because the groups involved were simpler, a lot of progress was made using ad-hoc methods that didn't engaging fully with general representation-theoretic aspects.

More recently, Gross and Wallach in [31] found another special class of automorphic representations—quaternionic forms corresponding to automorphic representations whose infinite component is a so-called quaternionic discrete series. They were quickly found to have many interesting properties. First, [23] showed that the Fourier transforms of those on G_2 encoded interesting arithmetic information, a result [67] extended to all exceptional groups. More bizarrely, they seemed to appear in certain string theory computations about black holes (see conjecture 15.13 in [21] for example).

Focusing just on G_2, the ad-hoc techniques developed for classical groups of course do not apply. Therefore, not very much is currently known about quaternionic G_2-automorphic representations. However, these representations are therefore also a great test application for the general representation-theoretic techniques developed here. Even more interestingly, quaternionic discrete series come in L-packets that also contain non-quaternionic members, so studying quaternionic automorphic representations specifically requires splitting the L-packet with our stable trace formula techniques.

As one technical pitfall, quaternionic discrete series do not satisfy a technical condition of being "regular" necessary for invariant trace formula methods to apply. However, a miracle occurs that being regular is not at all necessary for specifically quaternionic discrete series, even though it is for other members of the L-packet.

Using this miracle, we are able to compute dimensions of spaces of level-1, discrete, quaternionic automorphic representations on G_2. We are also able to give a full listing of all level-1 quaternionic representations in terms of automorphic representations on compact-at-infinity form G_2^c together with pairs of classical modular forms.

1.3.4 Some Selected New Techniques

One main hope of this work is that it can serve as a blueprint and set of guiding examples for other statistics computations using the stable trace formula. Here, we highlight some practical methods to get around common difficulties that may arise:

- Section 3.1 gives a version of the hyperendoscopy formula from [17] that works when groups without simply connected derived subgroup appear in hyperendoscopy. Formula 5.2 gives a telescoped version of this that only has stable terms.
- Section 3.3 gives a generalization of the simple trace formula in [4] to both non-cuspidal groups with fixed central character datum and also to test functions with just a pseudocoefficient at infinity instead of just an Euler-Poincaré function. We hope that this is also useful for people studying the cohomology of Shimura varieties.
- Sections 3.2.4 and 3.2.5 present tractable formulas, computation examples, and bounds for unramified transfers.
- Lemma 5.1.2.1 demonstrates how one might tackle studying certain non-regular discrete series through the trace formula and how one might test when this is feasible for a particular discrete series representation.
- Section 5.4.2 demonstrates a fast and easy way to compute endoscopic character signs in transfer formulas for pseudocoefficients from a given choice of Whittaker datum.
- Finally, the derivation of formula 5.4 demonstrates a trick with stabilization by which computations involving discrete-at-infinity representations can sometimes be reduced to computations on groups that are compact at infinity.

We also attempt to comprehensively summarize the relevant endoscopy and trace formula background in chapter 2, focusing mostly on computational practicalities.

1.4 Technical Introduction to the Equidistribution Problem

1.4.1 Context

Chapters 2-4 of this write-up generalize work in [77] and [79] on equidistribution of local components of families of automorphic representations (see the summary next section). We roughly extend their weight-aspect to the case where the infinite component can be restricted to a single discrete series instead of an entire L-packet.

Slightly more specifically, we consider certain increasing-size sets of automorphic representations $\mathcal{F}_k$ with more and more complicated component at infinity. For appropriate test functions $\widehat{f}$ on the space of possible local components at a finite set of primes S, we estimate

$$\sum_{\pi \in \mathcal{F}_k} \widehat{f}(\pi_S) \text{ as } k \to \infty.$$

These estimates are good enough to show an averaged, automorphic version of Sato-Tate equidistribution of the the components π_v for a fixed v and all $\pi \in \mathcal{F}_k$ as $k, v \to \infty$ jointly in an appropriate way. The additional families that this work addresses, beyond those in [79], are analogous to those corresponding to specifically holomorphic Siegel modular forms or specifically quaternionic modular forms on exceptional groups. The main result appears as theorem 4.3.1.1.

Generally, problems of statistics of families automorphic representations are interesting for a few potential reasons. First, when interpreted classically, such statistics are information on the spectra of lattices in locally symmetric spaces.

Second, they give so-called globalization results such as [6, lem 6.2.2] through probabilistic method-style arguments. These allow the construction of automorphic forms satisfying desired local conditions. This is important since a very standard technique in studying local representations is to find a global representation with the local representation as a component and then use global methods to study the global representation: see for example the classification in [6] or the cohomology formula in [78]. Globalization results were the motivation for [77].

Next, certain bounds on automorphic representations—in particular the generalized Ramanujan conjecture and what it says about the sizes of Fourier coefficients—have various bizarre, unexpected implications. These include some striking ones outside of number theory such as the original construction of expander graphs. See [69] for a review of this subject. As is common in analytic number theory, bounds on averages in families instead of bounds on individual representations are often good enough for these applications. Conveniently enough, average bounds over families are also directly provided by studying statistics. This seems to be the original motivation for studying the problem in [79].

As far as we know, this is the first work to apply the general stable trace formula to computing statistics of automorphic representations. A more common method seems to be using the non-invariant trace formula. This has the advantage of working for very general types of automorphic representations like Maass forms, but the disadvantage of requiring difficult explicit computations that create problems when dealing with general groups (as mentioned later, see [19] and [18] for current progress removing this difficulty). One of the key insights of [79] is that, for certain families, the nicer abstract properties of terms in the invariant trace formula simplify computations to the point where good error bounds can be derived even for very general groups. As a next step, the more powerful stable trace formula allows generalizing the class of more-easily-studied families. Here, we focus on a first example of distinguishing between elements of an L-packet at infinity. Another potential example could be families appearing in cohomologies of locally symmetric spaces—like the type studied in [25] but maybe coming from groups that are not anisotropic. The main trace formula term counting this family comes from endoscopic groups.

While automorphic representations with components in the same L-packet are almost definitionally indistinguishable from the point of view of Galois representations and L-functions, they do differ in other important aspects. For example, a discrete series L-packet can contain both holomorphic and non-holomorphic discrete series as in the case of GSp_4 (see

[72, §3.2]). Breaking up L-packets is therefore useful in studying, for example, holomorphic Siegel modular forms. Breaking up L-packets can be similarly useful for accessing the forms corresponding specifically to the quaternionic discrete series from [31].

We point out some relevant previous work: pseudocoefficients and their simplification of the trace formula were developed by Clozel and Delorme [15] and Arthur [4]. They were used to study statistics of families by Clozel [14]. The exact families studied and the setup to study them are of course a small modification from [77] and [79]. The use of the stable trace formula is through the hyperendoscopy formula in [17], although the results of [63] give a different potential strategy. The paper [38] solves this problem for GSp_4 with far more explicit bounds through different methods. For a fuller history of this field of "limit multiplicity"-type problems, see the introduction to [19].

As for using the theory of endoscopy to count automorphic representations, there are a few articles by Marshall and collaborators, such as [54], [55], and [25], that use endoscopic character identities to bound cohomology dimensions of symmetric spaces for certain unitary groups. In addition, [80] uses similar inductive methods with stabilization to compute literal dimensions of spaces of discrete forms with specified component at infinity, though requiring formulas from [6] that only work for classical groups.

Finally, this work should be compared to [19] and [18] by Finis, Lapid, and Mueller. These use the non-invariant trace formula to develop similar though much more general results. In particular, they show Shin and Templier's level aspect with the Archimedean component restricted to any set of positive measure in the unitary dual. The result is dependent on some technical estimates on intertwining operators that are satisfied for GL_n and SL_n. A future work promises the estimates for most other groups. In addition, their methods do not currently deal with the weight aspect or give error bounds though they could presumably be pushed to do both.

1.4.2 Summary

Shin-Templier's work

Let G be a reductive group satisfying some technical conditions (described in section 4.1.1). In [79] building off [77], Shin and Templier studied certain families of automorphic representations with level and weight restrictions:

$$\mathcal{F}_{U,\xi} = \{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) : \pi_\infty \in \Pi_{\mathrm{disc}}(\xi), \dim(\pi^\infty)^U \geq 1\}$$

where $\mathcal{AR}_{\mathrm{disc}}(G)$ is the set of the discrete automorphic representations of G, U is an open compact subgroup of $G(\mathbb{A}^{\infty,S_0})$ for some finite set of places S_0, ξ is a regular weight of $G_\mathbb{C}$, and $\Pi_{\mathrm{disc}}(\xi)$ is the discrete series L-packet corresponding to ξ. Pick another finite set of places $S \supseteq S_0$ and consider the empirical distribution,

$$\mu_{\mathcal{F},S} = \sum_{\pi \in \mathcal{F}_{U,\xi}} a_\pi \delta_{\pi_S},$$

of S-components of $\pi \in \mathcal{F}$ weighted by

$$a_\pi = m_{\mathrm{disc}}(\pi) \dim(\pi^{S,\infty})^U.$$

Shin and Templier used Arthur's invariant trace formula to study limits of these distributions under either increasing level ($U \to 1$) or increasing weight ($\xi \to \infty$). In both cases, the limits converged to the Plancherel measure μ_S^{pl} on $\widehat{G}_S$. They furthermore provided bounds on how quickly the integrals $\mu_{\mathcal{F},S}(f)$ converge in the case where both the test function f and the elements of $\mathcal{F}$ are unramified on $S \setminus S_0$. The increasing weight aspect required that the center of G was trivial. The shape of the result is:

Theorem 1.4.2.1. *Let $f = f^{\mathrm{ur}} \otimes f^{\mathrm{ram}}$ be a test function on G_S factoring into components with f^{ur} unramified. Then for $\mathcal{F}$ in either of the two limits above,*

$$\frac{1}{|\mathcal{F}|}\mu_{\mathcal{F},S}(\widehat{f}) = \mu_S^{\mathrm{pl}}(\widehat{f}) + O_{f^{\mathrm{ram}}}(q_{f^{\mathrm{ur}}}^{A_G + B_G \kappa_{f^{\mathrm{ur}}}} |\mathcal{F}|^{-C_G}),$$

where κ is a measure of the size of the support of f^{ur},$|\mathcal{F}|$ is a measure of the size of $\mathcal{F}$ depending on $U^{S,\infty}$ and ξ, $q_{f^{\mathrm{ur}}}$ is the product of residue field sizes over the places where f is unramified, and A_G, B_G, C_G are constants determined by G. (Recall that $\widehat{f}$ denotes the Fourier transform $\pi \mapsto \mathrm{tr}_\pi f$).

Their method was in a few broad steps:

1. Realize the empirical distribution $\mu_{\mathcal{F},S}$ as the trace of a function with a special Archimedean component η_ξ against the discrete automorphic spectrum. Here, η_ξ is the Euler-Poincaré function from [15].
2. Since the Archimedean component is an Euler-Poincaré function, Arthur's invariant trance formula reduces to the simple trace formula in [4] giving a reasonably tractable expression for this trace.
3. Bound the appropriate terms and take a limit. This is most of the work.

The form of the error bound allowed the proving of Sato-Tate equidistribution limits of $\mu_{\mathcal{F},v}$ for a single place v as v and ξ jointly go to infinity. They also provided some results on the statistics of low-level zeros of L-functions over the entire family.

The extension

Here, we extend Shin-Templier's weight aspect ($\xi \to \infty$). First, instead of looking at a sequence of entire L-packets $\Pi_{\mathrm{disc}}(\xi_k)$, we fix a single representation $\rho_k \in \Pi_{\mathrm{disc}}(\xi_k)$ for each k. Second, we allow G to have trivial center.

Then we consider the limit as $k \to \infty$ of the empirical distribution,

$$\mu_{\mathcal{F}_k,S} = \sum_{\pi \in \mathcal{F}_{U,\rho_k}} a_\pi \delta_{\pi_S},$$

of representations with $\pi_\infty = \rho_k$ weighted by

$$a_\pi = m_{\mathrm{disc}}(\pi)\dim(\pi^{S,\infty})^U$$

and compute error bounds on its convergence to Plancherel measure as in theorem 1.4.2.1. The precise definition of the family we study is in section 4.1.1 and the final result is theorem 4.3.1.1.

Here are the broad steps of the argument:

1. Realize the empirical distribution $\mu_{\mathcal{F},S}$ as the trace of a function with a special Archimedean component φ_π against the discrete automorphic spectrum. The function φ_π is the pseudocoefficient from [15].

2. Notice that pseudocoefficients have the same stable orbital integrals as Euler-Poincaré functions.

3. Use the stable trace formula to write this trace as a linear combination of traces of functions with Euler-Poincaré components at infinity on the smaller endoscopic groups.

4. Proceed as before to bound each term in the sum. Showing that enough technical conditions are satisfied and that the bounds are uniform enough that you are allowed to do so is most of the new work.

5. Redo the computations showing the versions of Plancherel and Sato-Tate equidistribution that the new main term gives.

It is worth discussing these in more detail. For step (3), the key difficulty is that Arthur's simple trace formula only works when the Archimedean component is Euler-Poincaré instead of a pseudocoefficient. However, the stable trace formula roughly gives the trace of a function as a linear combination of stable traces of transfers of the function on smaller endoscopic groups—we get an expansion of shape:

$$I^G(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} S^H(f^H).$$

Since pseudocoefficients have the same stable orbital integrals as their corresponding Euler-Poincaré functions, the f^H can without loss of generality be chosen to have Euler-Poincaré components at infinity. See section 3.2.1 for details on these transfers.

The most direct way to proceed is to then repeat the work in [4] on the stable distributions S^H instead of the invariant distribution I^G. We choose to instead use the hyperendoscopy formula of [17] (see the remark at the beginning of section 3.1).

It gives an expansion of shape

$$I^G(f) = I^G(f^*) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{\mathcal{H}}((f - f^*)^{\mathcal{H}}).$$

Here f^* is a function with the same stable orbital integrals as f, $\mathcal{HE}_{\mathrm{ell}}(G)$ is roughly the set of groups that can show up in a sequence of iteratively choosing an endoscopic group starting from G, and $\iota(G, \mathcal{H})$ is a non-troublesome constant. See section 3.1 for the full details. The distributions $I^{\mathcal{H}}$ can then be treated exactly as in [79] provided technical conditions still hold.

There are also some complications in step (4). First, the distribution $I^G_{\mathrm{spec}}(f)$ is not obviously the trace of f against the discrete automorphic spectrum like we want it to be. The paper [4] shows this for Euler-Poincaré at infinity and an unpublished lemma of Vogan (appearing here as lemma 3.3.3.1) is needed to extend to the pseudocoefficient case. Next, the groups appearing in $\mathcal{HE}_{\mathrm{ell}}(G)$ do not satisfy the technical simplifying conditions of [4]. We therefore need to slightly generalize the result, in particular to non-cuspidal groups. This is section 3.3. Finally, we need some bounds on endoscopic transfers of test functions so that Shin-Templier's orbital integral bounds apply. This takes some work in the non-Archimedean case and is sections 3.2.4 and 3.2.5.

For step (5), as explained in section 4.2.3, allowing a non-trivial center changes the main term in theorem 4.3.1.1 to something more complicated than originally in [79]. We therefore have to redo the computations for Sato-Tate and Plancherel equidistribution. This produces slightly different limiting measures that can be roughly thought of as Sato-Tate or Plancherel measure conditioned to be on a certain subset of $\widehat{G}_S$: representations with central character contained in a particular discrete set. The computations appear in section 4.4. We do not do the computation for low-lying zeros of L-functions due to complexity.

Finally, we save the level aspect computation for a future write-up. The main difficulty here is that as level gets larger, the test function f becomes more and more ramified adding more and more non-zero terms to the sum over $\mathcal{HE}_{\mathrm{ell}}(G)$. This necessitates proving much stronger uniformity of the bounds in [79, §8] over endoscopic groups.

1.5 Technical Introduction to the Counting Problem

1.5.1 Context

Chapter 5 of this work tries to describe level-1, discrete, quaternionic automorphic representations on G_2. Let $\mathcal{Q}_1(k)$ be the set of such representations of weight k. For each $k > 2$, we give a formula, (5.10), for $|\mathcal{Q}_1(k)|$ in terms of counts of automorphic representations on the compact-at-infinity inner form G_2^c that were calculated by Chenevier and Renard in [13]. We also give a Jacquet-Langlands-style result (corollary 5.6.2.1) describing all elements of $\mathcal{Q}_1(k)$ in terms of certain automorphic representations on G_2^c and certain pairs of classical modular forms.

Quaternionic automorphic representations were developed as a way to generalize to other groups the special place holomorphic modular forms have among automorphic representations of GL_2. Just like holomorphic modular forms, they are characterized by their infinite component being in a particular nice class of discrete series representations: the quaternionic discrete series of [31]. Just like modular forms, they have a nice theory of Fourier expansions with interesting arithmetic content—this was described for G_2 in [23] and generalized to all exceptional groups in [67]. Quaternionic forms have been studied a lot by Pollack: see [65] for an introductory article on them and [66] for good exposition specifically on G_2-quaternionic forms.

We attempt to study discrete, quaternionic representations on G_2 using the trace formula. Since quaternionic discrete series appear in L-packets with non-quaternionic members, this provides a great test case of the efficacy of the techniques in chapter 3 developed to split L-packets with the trace formula. The computation also relies heavily on methods developed in [13] and [80] to get exact counts of level-1 automorphic representations with the invariant/stable trace formulas.

Finally, there is a particular miracle about quaternionic discrete series on G_2 that crucially underpins this result. A priori, chapter 3 cannot be applied: such discrete series are not regular, implying that there may not be a test function at infinity whose trace picks out exactly a quaternionic discrete series without also picking up some unwanted contributions from non-tempered representations. However, it turns out that specifically quaternionic discrete series on G_2 don't get entangled in this way, even though other members of their L-packet do. The proof of this depends on results about Adams-Johnson packets for G_2 that Mundy developed for studying Eisenstein cohomology in [59].

1.5.2 Summary

We summarize the method of computation. Proposition 5.1.2.1 shows that traces against a pseudocoefficient of a quaternionic discrete series with weight $k > 2$ are 0 against all other unitary representations. This allows us to get a formula (5.1.2.2) for traces of finite-place test functions against the space of all quaternionic representations of weight k.

Next, section 5.2 develops a general stabilized formula (5.2) for I_{geom} applied to test functions like ours. We work out what this formula reduces to in section 5.3 using a computation of the endoscopy of G_2 in section 5.3.1. Instead of using formula (5.2) directly, we compare it applied to G_2 to it applied to the compact real form G_2^c to construct a formula for $I_{\text{spec}}^{G_2}$ involving just $I_{\text{spec}}^{G_2^c}$- and I_{spec}^{H}-terms. Here, H is the endoscopic group $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ of G_2.

Section 5.4 then tells us which exact $I_{\text{spec}}^{G_2^c}$- and I_{spec}^{H}-terms appear by computing endoscopic transfers at infinity. The difficult part of this computation is pinning down various signs coming from transfer factors. As a last piece of the puzzle, section 5.5 uses results about level-1 forms from [13] to reduce counts of forms on H to counts of classical modular forms.

Section 5.6 uses all these formulas to characterize representations in $\mathcal{Q}_k(1)$ with $k > 2$ in terms of automorphic representations on G_2^c and certain pairs of classical modular forms. Finally, we substitute in values for the $I_{\text{spec}}^{G_2^c}$-terms from [13] and present a final table of dimensions, table 5.1, in section 5.7.

1.6 Notational Conventions

Here are some notational conventions we will use throughout:
Basics

- F is a fixed number field.
- G is a fixed reductive group over F. In certain sections where we are working locally, G will be the local component instead.
- $\mathbb{A}$ is $\mathbb{A}_F$ for shorthand.
- $\mathbb{A}_\infty$, $\mathbb{A}^\infty$ are the at infinity and away from infinity parts of $\mathbb{A}$ respectively.
- W_E is the Weil group of local or global field E.
- $\mathcal{O}_E$ is the ring of integers of local field E.
- k_E is the residue field of local field E.
- $\mathbf{1}_X$ is the indicator function for set X.
- $\widehat{H}$ is the reductive dual of reductive group H.
- $\widehat{S}$ is the unitary dual of abstract group S.
- $\widehat{S}^{\mathrm{temp}}$ is the tempered part of $\widehat{S}$.
- $\widehat{f}$ is the Fourier transform of function f on an abstract group S that should be clear from context.
- $\bar{f}$ is the Fourier transform of f restricted to some subgroup of the center of S with respect to that subgroup. The exact subgroup should be clear from context.

Reductive Groups

- Z_H is the center of abstract or reductive group H.
- $Z_H(G)$ is the centralizer of H inside G.
- A_H is the maximum split component in the center of reductive group H.
- H_S for group H over F and finite set of places S of F is $H(\mathbb{A}_S)$. Use the standard conventions where an upper index means everything except S.
- H_∞ may be equivalently defined as $(\mathrm{Res}^F_{\mathbb{Q}} H)(\mathbb{R})$ since $(\mathrm{Res}^F_{\mathbb{Q}} H)(\mathbb{R}) = H(F \otimes_{\mathbb{Q}} \mathbb{R}) = H(\mathbb{A}_\infty)$. It is in particular a real reductive group.

- $A_{H,\mathrm{rat}}$ for group H over F is $A_{\mathrm{Res}^F_{\mathbb{Q}} H}(\mathbb{R})^0$ (the connected component is in the real topology).
- $A_{H,\infty} := A_{(\mathrm{Res}^F_{\mathbb{Q}} H)_{\mathbb{R}}}(\mathbb{R})^0$.
- $H(\mathbb{A})^1 := H(\mathbb{A})/A_{H,\mathrm{rat}}$.
- $H^1_\infty := H_\infty/A_{H,\infty}$.
- H_γ is the centralizer of γ in H for H either an algebraic or abstract group.
- I^H_γ is the connected component of the identity in the centralizer of γ in H.
- $\iota^H(\gamma)$ is the set of connected components of H_γ with an F-point.
- $[H]$, $[H]^{\mathrm{ss}}$, $[H]^{\mathrm{ell}}$ are the sets of (semisimple, elliptic) conjugacy classes in H.
- $D^H(\gamma)$ is the Weyl discriminant for H.
- K_S where S is a finite set of places of F is a chosen hyperspecial of $G(\mathbb{A}_S)$.
- M usually represents some Levi subgroup.
- P usually represents some parabolic subgroup.
- $K_{S,H}$ for S some finite set of places usually represents some kind of maximal compact of $H(\mathbb{A}_S)$.

Lie Theory

- $\Phi^*(H), \Phi^+(H), \Phi^*_F(H), \Phi^+_F(H)$ are the sets of (positive, rational) roots of H.
- $\Phi_*(H), \Phi_+(H), \Phi_{*,F}(H), \Phi_{+,F}(H)$ are the sets of (positive, rational) coroots of H.
- $\Delta^*(H), \Delta^*_F(H)$ are the sets of (rational) simple roots of H.
- $\Delta_*(H), \Delta_{*,F}(H)$ are the sets of (rational) simple coroots of H.
- Ω_H is the Weyl group of $H_{\mathbb{C}}$ for H a reductive group.
- $\Omega_{H,E} = \Omega_E$ for H over F and E an extension of F is the subset of Ω_H generated by conjugating by elements of $H(E)$. Note that this depends on the maximal torus chosen to define Ω.

Volumes

- $\mu^{\mathrm{tam}}, \mu^{\mathrm{can}}, \mu^{EP}$ are the Tamagawa, Gross' canonical, or Euler-Poincaré measures on various groups.

- $\bar{\mu}^\star$ is the quotient of measure $\mu^\star$ by something that should be clear from context.
- $\tau(H)$ is the Tamagawa number of H.
- $\tau'(H)$ is the modified Tamagawa number using the canonical measure $\mu^{\mathrm{can},EP}$.

Endoscopy

- $(H, \mathcal{H}, s, \eta)$ is an endoscopic quadruple for G.
- $(\tilde{H}, \tilde{\eta})$ is a z-pair for $(H, \mathcal{H}, s, \eta)$.
- (H_1, η_1) will also sometimes be used to represent a z-pair to keep diacritics from stacking too much.
- $\mathcal{E}_{\mathrm{ell}}(H)$ is the set of elliptic endoscopic quadruples of reductive group H.
- $\mathcal{H}\mathcal{E}_{\mathrm{ell}}(H)$ is the set of elliptic hyperendoscopic paths of reductive group H.
- $(\mathfrak{X}, \chi)$ is a central character datum on some reductive group.
- $\mathcal{H}$ is further overloaded: when context is clear, it can also refer to either a hyperendoscopic path or the last group in the path.

Automorphic Representations and the Trace Formula

- $\mathscr{H}(H, \chi) = \mathscr{H}(H, (\mathfrak{X}, \chi))$ is the space of compactly supported functions on $H(\mathbb{A})$ that transform according to character χ^{-1} on $\mathfrak{X} \subseteq Z_G(\mathbb{A})$.
- $\mathscr{H}(H_S, \chi_S)$ for S a finite set of places of F is compactly supported functions on $H(\mathbb{A}_S)$ similarly transforming according to χ_S^{-1}.
- $\mathscr{H}(H_S, K_S, \chi_S)$ if K_S is a product of hyperspecial subgroups and χ_S is unramified is the Hecke algebra of K_S-bi-invariant elements of $\mathscr{H}(H_S, \chi)$.
- $\mathscr{H}(H_S, K_S, \chi_S)^{\leq \kappa}$ is the truncated Hecke algebra from section 3.2.3.
- $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$ for $(\mathfrak{X}, \chi)$ a central character datum is the unitary $G(\mathbb{A})$-representation of L^2-up-to-$\mathfrak{X}$ functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ transforming according to χ^{-1}.
- $L^2_{\mathrm{disc}}(\cdot)$ is the discrete part of unitary representation $L^2(\cdot)$.
- $\mathcal{AR}_{\mathrm{disc}}(H, \chi)$ is the set of discrete automorphic representations on H with character χ on $A_{H,\infty}$.
- $O^H_\gamma(f)$ is the integral of f on the conjugacy orbit of γ. This can either be local or global; f can be a function on $H(\mathbb{A})$ or some $H(F_v)$.

- $I^{G,\chi}_{\text{spec}}, I^{G,\chi}_{\text{disc}}, I^{G,\chi}_{\text{geom}}$ are the distributions on G defined by Arthur's invariant trace formula depending on central character datum $(\mathfrak{X}, \chi)$.
- $S^{H,\chi}_{\text{spec}}, S^{H,\chi}_{\text{disc}}, S^{H,\chi}_{\text{geom}}$ are the distributions on H defined by Arthur's stable trace formula depending on central character datum $(\mathfrak{X}, \chi)$.
- $\mathscr{L}$ is the set of rational Levi's of G containing a fixed minimal Levi.
- $\mathscr{L}^{\text{cusp}}$ is the $M \in \mathscr{L}$ such that $A_{M,\text{rat}}/A_{G,\text{rat}} = A_{M,\infty}/A_{G,\infty}$. This is a generalization of the definition of cuspidal Levi from [4] to the case where G isn't itself cuspidal.

Representation theory

- $\pi(\lambda, w_0)$, $\pi(w_0(\lambda + \rho))$ are two different parametrizations for discrete series representations for λ a dominant weight.
- $\Pi_{\text{disc}}(\lambda)$ is a discrete series L-packet where λ is a dominant weight.
- Θ_π is the Harish-Chandra character for representation π.
- ω_π is the central character of representation π.
- φ_π is the pseudocoefficient for discrete series representation π.
- η_λ is the Euler-Poincaré function for the L-packet $\Pi_{\text{disc}}(\lambda)$.

Families

- φ^∞ is a specific function defined in section 4.1.1.
- $\mathcal{F}$ is a specific family (as in [79]) of automorphic representations defined in section 4.1.1.
- $a_{\mathcal{F}}(\pi)$ are the coefficients defining $\mathcal{F}$.
- S_0, S_1, $U^{S\cup\infty}$, φ_{S_1}, f_{S_0} are data used to define φ^∞ and $\mathcal{F}$ as explained in section 4.1.1.
- $S_{\text{bad},G}$ is the unknown finite set of bad places depending on reductive group G defined in section 4.3.
- $S_{\text{bad}',G}$ is the version of $S_{\text{bad},G}$ needed for the results from [79].
- L is the lattice $Z_G(F) \cap U^{S,\infty} \subseteq Z_{G_{S,\infty}}/A_{G,\text{rat}}$.
- $E^{\text{pl}}(\widehat{\varphi}|\omega)$ is the expectation defined in section 4.2.3.
- $E^{\text{pl}}(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$ is defined in proposition 4.2.3.5.

Counting

- $\mathcal{S}_k(1)$ is the set of normalized, classical, cuspidal eigenforms on GL_2 of level 1 and weight k.
- $\mathcal{Q}_k(1)$ is the set of discrete, quaternionic automorphic representations on G_2 of level 1 and weight k (see section 5.1.2).

Dimensional Analysis

A lot of the formulas here depend on choices of Haar measure. Since we are explicitly bounding terms, it is sometimes helpful to have notation for how they depend on these choices. For example, if we say that a value has dimension $[G][H]^{-1}$, then it is proportional to a choice of Haar measure on G and inversely proportional to a choice on H.

In any formula, dimensions on both sides need to match. In addition, any quantity with dimension needs to be normalized by a formula expressing it in terms of just dimensionless quantities and Haar measures—for example, the formulas defining traces of Hecke algebra elements, orbital integrals, or pseudocoefficients.

Chapter 2

Background Materials

2.1 Trace Formula Background

2.1.1 Invariant Trace Formula

Let G be a connected reductive group over a number field F. Let $\mathbb{A} = \mathbb{A}_F$. Fix a central character χ on $A_{G,\mathrm{rat}}$. Let $\mathscr{H}(G, \chi)$ be the space of functions on $G(\mathbb{A})$ that are smooth and compactly supported when restricted to $G(\mathbb{A})^1$ and satisfy $f(ax) = \chi^{-1}(a)f(x)$ for all $a \in A_{G,\mathrm{rat}}$.

Over a long series of papers that are summarized in [3] Arthur defines two equal distributions on $\mathscr{H}(G, \chi)$:

$$I^{G,\chi}_{\mathrm{geom}} = I^{G,\chi}_{\mathrm{spec}}.$$

Intuitively, one should think of I_{geom} as a sum of modified orbital integrals of f and I_{spec} as a sum of modified traces of f against components of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$. The exact definitions of these distributions are impractically complicated to use directly. However, enough useful special cases and abstract properties have been worked out—the most relevant being the simple trace formula in [4]. The χ will often be suppressed in notation.

Both sides have dimension $[G(\mathbb{A})^1]$. The individual terms in the expansions for both sides can have more complicated dimensions.

Spectral side

As a very rough description of the spectral side, Arthur defines components

$$I^G_{\mathrm{spec}} = I^G_{\mathrm{cts}} + \sum_{t \geq 0} I^G_{\mathrm{disc},t}.$$

$I_{\mathrm{disc},t}$ is 0 except for countably many t and is much easier to evaluate. Expanding further,

$$I_{\mathrm{disc},t} = \sum_{M \in \mathscr{L}} \frac{|\Omega_M|}{|\Omega_G|} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|_{\mathfrak{a}^G_M}|^{-1} \operatorname{tr}(M_{P,t}(\omega)\mathcal{I}_{P,t}(f)).$$

To describe the most relevant terms, $\mathscr{L}$ is the set of Levi's of G containing a chosen minimal Levi, P is a chosen parabolic for M, $W(M)_{\mathrm{reg}}$ is a particular set of elements of a relative Weyl group (this and the Weyl group factor are a combinatorial term roughly parametrizing parabolics containing the Levi), and $M_{P,t}(\omega, \chi)$ is an intertwining operator between parabolic inductions through different parabolics containing M from the theory of Eisenstein series.

The last term is the most important for us. The χ induces a character on $A_{M,\mathrm{rat}}$ by pullback. Then $\mathcal{I}_P(\chi)$ is the representation of $G(\mathbb{A})$ produced from parabolically inducing $L^2_{\mathrm{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi)$. The term $\mathcal{I}_{P,t}$ is the subrepresentation of this with archimedean infinitesimal character having imaginary part of norm t. By lots of work, all these decompositions makes sense and the convolution operators $\mathcal{I}_{P,t}(f)$ for $f \in \mathscr{H}(G, \chi)$ are trace class. Finally, a much later result in [20] implies that the sum over t converges absolutely.

There are well-known and simple sufficient conditions on f such that $I_{\mathrm{cts}}(f) = 0$:

Definition ([3, paragraph above cor. 23.6])**.** If v is a place of F, $f \in \mathscr{H}(G(F_v))$ is *cuspidal* if for all Levi's M_v of G_v and π_v tempered representations of M_v:

$$\mathrm{tr}_{\pi_v^G}(f) = 0.$$

Here π_v^G is (any) parabolic induction of π_v.

Note that this is an alternate definition to the original one from [8].

Theorem 2.1.1.1 ([8, thm 7.1])**.** *If f factors as $f_v \otimes f^v$ for some place v with f_v cuspidal, then $I_{\mathrm{cts}}(f) = 0$.*

Geometric side

The geometric side can be succinctly written as

$$I_{\mathrm{geom}}(f) = \sum_{M \in \mathscr{L}} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \sum_{\gamma \in [M(\mathbb{Q})]_{M,S}} a^M(S, \gamma) I_M^G(\gamma, f).$$

Here S is a large enough set of places in particular including those at which f is not the characteristic function of a hyperspecial and $[M(\mathbb{Q})]_{M,S}$ is the set of conjugacy classes mod a complicated equivalence relation involving the away-from-S components of the unipotent parts. For γ semisimple,

$$a^M(S, \gamma) = |\iota^M(\gamma)|^{-1} \operatorname{vol}(I_\gamma^M(\mathbb{Q})\backslash I_\gamma^M(\mathbb{A})^1)$$

where $|\iota^M(\gamma)|$ is the number of connected components of M_γ that have an F-point. In general, there is no explicit description of $a^M(S, \gamma)$.

Next, I_M^G is a weighted orbital integral of the S-components of f. If $M = G$, it is simply the orbital integral at γ. If γ is semisimple, there is an explicit formula weighting the integral by a complicated combinatorial factor. Otherwise, it is only defined though some analytic continuations. The term I_M^G satisfies splitting and descent formulas ([3, p. 23.8] and [3, p. 23.9]) that factor it into local components in terms of traces of f against parabolic inductions. When f is cuspidal at some place, these splitting formulas of course then greatly simplify.

If γ is semisimple, the a^M have dimension $[I_\gamma^M(\mathbb{A})^1]$ while the I_M^G have dimensions of $[G(\mathbb{A})^1][I_\gamma^M(\mathbb{A})^1]^{-1}$. Otherwise the dimensions are more complicated.

A technicality

Arthur actually defines two slightly different versions of his local distributions $I_{M,v}^G(\gamma, f)$. Looking at just the place at ∞ for notational ease, the key issue is that the weighting factor v_M in his orbital integrals depends on a choice of the space $A_{M,\star}/A_{G,\star}$ where $\star \in \{\infty, \mathrm{rat}\}$.

The version appearing in his splitting formula [3, p. 23.8] is $\star = \mathrm{rat}$,which we will denote by $I_{M,\infty}^G(\gamma, f)$. The version in his descent formula [3][23.9] is the purely local choice $\star = \infty$, which we will denote by $\tilde{I}_{M,\infty}^G(\gamma, f)$.

Lemma 2.1.1.2. *If cuspidal $f \in \mathscr{H}(G_\infty, \chi)$ (so that $I_{M,\infty}^G(\gamma, f)$ is defined), then*

$$I_{M,\infty}^G(\gamma, f) = \begin{cases} \tilde{I}_{M,\infty}^G(\gamma, f) & A_{M,\mathrm{rat}}/A_{G,\mathrm{rat}} = A_{M,\infty}/A_{G,\infty} \\ 0 & \textit{else} \end{cases}.$$

Proof. If the two spaces are equal, then the weighting factors v_M at the beginning of [3][§18] and the sum over Levi's in [3][thm. 23.2] are equal. Note that while $\widehat{I}_M^L(\gamma, \phi_L(f))$ in [3][thm. 23.2] ostensibly looks like it depends on the choice of $\star$, this is just based on different descriptions of certain spaces of functions to make conditions for containment in the two versions of $\mathcal{I}_{\mathrm{ac}}$ easier to describe. In particular, the distinction does not matter as long as f is in both versions of $\mathscr{H}_{\mathrm{ac}}$. In total, stepping through the definitions of I_M^G and $\tilde{I}_M^G$ shows that they are the same since the above are the only parts that depend on the various A's.

Otherwise, this follows from the generalized descent formula [7][thm. 8.1], setting $\mathfrak{b}$ to be $X_*(A_{M,\mathrm{rat}}A_{G,\infty}) \otimes \mathbb{R}$ inside $\mathfrak{a}_M = X_*(A_{M,\infty}) \otimes \mathbb{R}$. This is the example considered at the bottom of page 361 in [7]. We fill in the details for why the descent formula applies:

To check if $\mathfrak{b}$ is special, we can without loss of generality assume $A_{G,\infty}$ is trivial by modding out by $X_*(A_{G,\infty}) \otimes \mathbb{R}$ everywhere and noting that it is perpendicular to all roots. Then, $\mathfrak{b}$ is the fixed points in $\mathfrak{a}_M$ of a finite group (Galois) action that preserves the inner product on $\mathfrak{a}_M$. The sums on page 355 of [7] testing specialness are invariant under the group action so their evaluation on any v is the same as their evaluation on the average of v over the action. However, averaging over the action is the same as orthogonally projecting onto $\mathfrak{b}$, so the sums need to vanish on the orthogonal complement of $\mathfrak{b}$.

Next, $M_\mathfrak{b} = M$ since $\mathfrak{a}_{M_\mathfrak{b}}$ needs to contain all simple coroots for which the corresponding simple-coroot-coordinate in some element of $\mathfrak{b}$ is non-zero. Therefore, inducing a conjugacy class from M to $M_\mathfrak{b}$ doesn't do anything, so the left side $I_\mathfrak{b}(\gamma^{M_\mathfrak{b}}, f) = I_{M,\infty}^G(\gamma^M, f) = I_{M,\infty}^G(\gamma, f)$.

Finally evaluating the right side of the formula, f being cuspidal implies that the only possibly non-zero f_L is $L = G$. However, then $d_M^G(\mathfrak{b}, G) = 0$ so all terms in the sum vanish. $\square$

2.1.2 The Simple Trace Formula

Whenever G_∞ has discrete series, the trace formula can be simplified by setting the test function to have a special real component.

Parametrizing discrete series

The classification of discrete series is work of Harish-Chandra that can be found summarized in [48, §III.5]. They only exist when G_∞ has an elliptic maximal torus or equivalently if $C\Omega_G$ on any torus contains $-\mathrm{id}$ where C is complex conjugation.

Therefore, for this subsection and the next only, let G be reductive group over $\mathbb{R}$ with fixed elliptic maximal torus T. Let K be a maximal compact of $G(\mathbb{R})$ containing $T(\mathbb{R})$, B_K a Borel of $K_\mathbb{C}$ containing T, and B a Borel of $G_\mathbb{C}$. Let Ω_G be the Weyl group of $(G_\mathbb{C}, T_\mathbb{C})$ and $\Omega_{G,\mathbb{R}}$ be the subgroup given by only conjugating by elements of $G(\mathbb{R})$.

The characters of $T(\mathbb{R})$ are contained in $T(\mathbb{C})$ so the root space of K is contained in G. Let ρ be half the sum of the positive roots of G. Finally, let $\Omega(B_K)$ be a particular set of coset representatives of $\Omega_{G,\mathbb{R}}\backslash\Omega_G$: namely, w such that $w\lambda$ is B_K-dominant for any λ that is B-dominant.

The discrete series representations of G are parametrized by B-dominant weights $\lambda \in X^*(T)_\mathbb{C}$ and elements $w^* \in \Omega(B_K)$. Call the representation parameterized by λ and w_0 either $\pi(\lambda, w_0)$ or $\pi(w_0(\lambda+\rho))$. It is the unique representation with trace character

$$\Theta_{\pi(\lambda,w_0)} = (-1)^{1/2\dim(G(\mathbb{R})/KA_{G,\infty})} \frac{\sum_{w\in\Omega_K} \operatorname{sgn}(ww_0) e^{ww_0(\lambda+\rho)}}{\sum_{w\in\Omega_G} \operatorname{sgn}(w) e^{w\rho}}$$

on T. The infinitesimal character of $\pi(\lambda, w)$ is $\lambda+\rho$: the same as that of V_λ, the finite dimensional representation with highest weight λ. Therefore the $\pi(\lambda, w)$ for a fixed λ are all in the same L-packet $\Pi_{\mathrm{disc}}(\lambda)$. We call $\pi(\lambda, w_0) = \pi(w_0(\lambda+\rho))$ *regular* if λ is. Finally, we call λ the *weight* of $\pi(\lambda, w_0) = \pi(w_0(\lambda+\rho))$.

Pseudocoefficients and Euler-Poincaré functions

Given a discrete series representation π of a real reductive group $G(\mathbb{R})$ with character χ on $A_{G,\infty}$, Clozel and Delorme in [15] define a pseudocoefficient $\varphi_\pi \in C_c^\infty(\chi^{-1})$. The function φ_π is compactly supported and has the property that for irreducible representations ρ with character χ,

$$\operatorname{tr}_\rho(\varphi_\pi) = \begin{cases} 1 & \pi = \rho \\ 0 & \pi \neq \rho, \rho \text{ basic} \\ ? & \text{else} \end{cases}.$$

Here, a basic representation is a parabolic induction of a discrete series or limit of discrete series (up to central character). The non-basic case is much more complicated. Pseudocoefficients have dimension $[G(\mathbb{R})^1]^{-1}$.

If $\Pi_{\mathrm{disc}}(\lambda)$ is the discrete series L-packet for π, it is also useful to consider Euler-Poincaré functions:

$$\eta_\lambda = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|} \sum_{\pi' \in \Pi_{\mathrm{disc}}(\lambda)} \varphi_{\pi'}.$$

Traces against Euler-Poincaré functions can be interepreted as Euler characteristics of certain cohomologies for basic representations and therefore all representations by the Langlands classification. If λ is regular, these Euler characteristics can be shown to be 0 on non-tempered representations. Therefore, if λ is regular we get

$$\mathrm{tr}_\rho(\eta_\lambda) = \begin{cases} |\Pi_{\mathrm{disc}}(\lambda)|^{-1} & \pi \in \Pi_{\mathrm{disc}}(\lambda) \\ 0 & \text{else} \end{cases}$$

for all irreducible representations ρ (see sections 1 and 2 in [4]). Beware that this normalization is different from the one in [79]. It makes endoscopic computations easier.

Note that both pseudocoefficients and Euler-Poincaré functions are cuspidal since they have 0 trace against any non-discrete series basic representation and therefore against all parabolic inductions of tempered representations.

Simple trace formula

The simple trace formula is the main result of [4]. A more textbook exposition is in [3, §24]. We state it here. First, assume

- G is connected,
- G is cuspidal over $\mathbb{Q}$: $\mathrm{Res}^F_{\mathbb{Q}} G/A_{G,\mathrm{rat}}$ has an $\mathbb{R}$-anisotropic maximal torus.

The last condition in particular gives that G_∞ has an elliptic maximal torus and therefore has discrete series mod center. In the case where G_∞ has discrete series mod center, cuspidal is equivalent to $A_{G,\mathrm{rat}} = A_{G,\infty}$: in other words, taking infinite place points of the maximum split torus in the center is the same as base changing to $\mathbb{R}$, looking at the maximal split torus in the center, and taking $\mathbb{R}$-points.

Consider a test function of the form $h = |\Pi_{\mathrm{disc}}(\xi)|\eta_\xi \otimes h^\infty$ for regular weight ξ and $h^\infty \in \mathscr{H}(G(\mathbb{A}^\infty))$. Let χ be the character on $A_{G,\infty}$ determined by ξ. Then

$$I_{\mathrm{spec}}(h) = I_{\mathrm{disc}}(h) = \sum_{\pi : \pi_\infty \in \Pi_{\mathrm{disc}}(\xi)} m_{\mathrm{disc}}(\pi)\, \mathrm{tr}_{\pi^\infty}(h^\infty) \tag{2.1}$$

where $m_{\mathrm{disc}}(\pi)$ is the multiplicity of π in $\mathcal{AR}_{\mathrm{disc}}(G,\chi)$. Let $\mathscr{L}$ be the set of Levi's containing a chosen minimal Levi of G. For each $M \in \mathscr{L}$, choose P_M a parabolic for M. Then

$$I_{\mathrm{geom}}(h) = \sum_{M \in \mathscr{L}^{\mathrm{cusp}}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \times \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I_\gamma^M) |\iota^M(\gamma)|^{-1} \Phi_M(\gamma_\infty, \xi) O_\gamma^M(h_M^\infty). \quad (2.2)$$

Here $\iota^M(\gamma)$ is the set of connected components of the full centralizer M_γ that have an F-point and

$$\chi(H) = (-1)^{q(H)} \operatorname{vol}(H(F)A_{H,\infty}\backslash H(\mathbb{A})) \operatorname{vol}(A_{H,\infty}\backslash \bar{H}_\infty)^{-1} |\Omega(B_{K_{H_\infty}})|,$$

where $\bar{H}_\infty$ is an inner form of H_∞ such that $H_\infty/A_{H,\infty}$ has anisotropic center, $\Omega(B_{K_{H_\infty}})$ is the analog of $\Omega(B_K)$ for H_∞, and $q(H) = 1/2\dim(H_\infty/K_{H,\infty}A_{H,\infty})$ is the Kottwitz sign. Also

$$h_M^\infty(\gamma^\infty) = \delta_{P_M}(\gamma^\infty)^{1/2} \int_{K^\infty} \int_{N_M(\mathbb{A}^\infty)} h(k^{-1}\gamma^\infty n k)\, dn\, dk,$$

where N_M is the unipotent group for P_M and K some chosen maximal compact. To make dimensions work out, the Haar measures choices should satisfy:

- The choices on I_γ^M, M, and in the orbital integral need to coincide,
- The measure on $\bar{I}_\gamma^M$ comes from that on I_γ^M through them both coming from the same top form on $I_{\mathbb{C}}^M$,
- The choices on N_P, K, M, and G need to coincide according to the Iwasawa decomposition.

Finally,

$$\Phi_M(\gamma_\infty, \xi) = \begin{cases} \left| \frac{D^G(\gamma_\infty)}{D^M(\gamma_\infty)} \right|^{1/2} \sum_{\pi \in \Pi^G_{\mathrm{disc}}(\xi)} \Theta_\pi(\gamma^\infty) & \gamma_\infty \text{ in an elliptic torus of } M \\ 0 & \text{else.} \end{cases}$$

As written, this is only defined on regular elements, but Arthur proves it extends to a function that is continuous on every elliptic torus.

As some notes for using this:

- Comparing character formulas computes that $\Phi_G(\gamma_\infty, \xi) = \operatorname{tr} \xi(\gamma_\infty)$ where ξ is overloaded to also denote the finite dimensional representation with highest weight ξ.
- If $M \neq G$, Φ_M cannot be evaluated through the standard Harish-Chandra character formula since it involves Θ_π's evaluated on tori that are not elliptic in G. See [4, §4] for an algorithm to actually do so.

- The only M that contribute to the outer sum are those in L^{cusp}; in this case, those that are cuspidal over $\mathbb{Q}$. Arthur's original paper implicitly showed this for M cuspidal over $\mathbb{R}$. There is a small correction using lemma 2.1.1.2 that the formula in [4][thm. 5.1] is zero for M not cuspidal over $\mathbb{Q}$ (Arthur was surely aware of this but seems to have forgotten to mention it). Alternatively, [27] shows vanishing using different methods. See section 3.3.4 for more details.
- Because of the dimensions on η_ξ, both sides of this formula have dimension $[G^\infty]$. However, explicitly computing the $\chi(I_\gamma^M)$ terms still requires choosing Haar measures at ∞.

2.1.3 Trace Formula with Central Character

Stabilization requires a slightly different version of the trace formula where the fixed character χ is on a larger closed subgroup of $Z(\mathbb{A})$. There is a full theory in [2] that takes quite a bit of work to describe. We summarize the relevant parts here.

Definition. A central character datum on G is $(\mathfrak{X}, \chi)$ where

- $\mathfrak{X} \supseteq \mathbb{A}_{G,\infty}$ is closed inside $Z(\mathbb{A})$ such that $Z(F)\mathfrak{X}$ is also a closed subgroup.
- $\chi : \mathfrak{X} \cap Z(F)\backslash\mathfrak{X} \to \mathbb{C}^\times$ is a continuous character.

Furthermore, $\mathscr{H}(G, (\mathfrak{X}, \chi)) = \mathscr{H}(G, \chi)$ is the set of smooth functions f on $G(\mathbb{A})$ such that $f(gx) = \chi^{-1}(x)f(g)$ and f is compactly supported mod $\mathfrak{X}$.

Note. For our purposes here, it suffices to consider $\mathfrak{X}$ that are the product of the adelic points of some algebraic subtorus of Z multiplied by some abstract subgroup of $Z_{G_\infty}(\mathbb{R})$.

Fix central character data $(\mathfrak{X}, \chi)$. In [6, §3], Arthur defines $I_{\text{disc},t,\chi}$ as a distribution on $\mathscr{H}(G, \chi)$:

$$I_{\text{disc},t,\chi}(f) = \sum_{M \in \mathscr{L}} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \sum_{w \in W(M)_{\text{reg}}} |\det(w-1)|_{\mathfrak{a}_M^G}|^{-1} \operatorname{tr}(M_{P,t}(\omega, \chi)\mathcal{I}_{P,t}(\chi, f)). \tag{2.3}$$

This is a generalization of $I_{\text{disc},t}$ and most of the terms are the same. The relevant part is how $\mathcal{I}_{P,t}$ changes. First, χ induces a character on $A_{M,\text{rat}}\mathfrak{X}$ by pullback and therefore lets us define $L^2_{\text{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi)$ analogous to other L^2 spaces with character: as the discrete part of χ^{-1}-invariant, L^2-up-to-$\mathfrak{X}$ functions on $M(\mathbb{Q})\backslash M(\mathbb{A})$ as an $M(\mathbb{A})$-representation. Then, $\mathcal{I}_{P,t}(\chi, f)$ can be defined analogously to $\mathcal{I}_{P,t}$ from the trace formula without central character. Decompositions and traces making sense in this context requires some extra work summarized on [6, pg 123]. The dimensions change to $[G(\mathbb{A})][\mathfrak{X}]^{-1}$.

For our work here, we only need to worry about the spectral side so we will not mention the geometric version.

2.2 Endoscopy and Stabilization Background

The standard reference for this material, [46], is written for the more general case of twisted endoscopy. It is therefore easier to follow the summary in [36, §1.3]. The simpler summary in [76, §2] for the simply connected derived subgroup case is also helpful. Finally, [48] is a course-notes style writeup of this material and therefore more motivated albeit far less general.

For this section, allow F to be a local or global number field.

2.2.1 Endoscopic Groups

Endoscopic quadruples

Definition ([46, pg 18])**.** An *endoscopic quadruple* for G is a tuple $(H, \mathcal{H}, s, \eta)$ with

- H a quasisplit connected reductive group over F,
- $\mathcal{H}$ is a split extension of $\widehat{H}$ by W_F such that action of W_F on $\widehat{H}$ determined by the splitting is the same as the one coming from H (in $\mathrm{Out}(\widehat{H})$),
- $s \in Z_{\widehat{H}}$ and semisimple in $\widehat{G}$,
- $\eta : \mathcal{H} \to {}^L G$ an L-embedding

such that

1. η restricts to an isomorphism $\widehat{H} \xrightarrow{\sim} \widehat{G}^0_{\eta(s)}$.
2. There is then a W_F-equivariant sequence

$$1 \to Z_{\widehat{G}} \to Z_{\widehat{H}} \to Z_{\widehat{H}}/Z_{\widehat{G}} \to 0$$

   which induces a map $(Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F} \to H^1(F, Z_{\widehat{G}})$. We require that $s \in (Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F}$ and maps to something locally trivial under this.

It is furthermore *elliptic* if

3. $(Z_{\widehat{H}}^{W_F})^0 \subseteq Z_{\widehat{G}}$.

For future reference, we let $\mathfrak{K}(s, \eta)$ be the elements that map to something locally trivial under $(Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F} \to H^1(F, Z_{\widehat{G}})$.

Definition. Two endoscopic quadruples $(H, \mathcal{H}, s, \eta), (H', \mathcal{H}', s', \eta')$ are isomorphic if there is an element $g \in \widehat{G}$ such that

1. $\eta(\mathcal{H})$ and $\eta'(\mathcal{H}')$ are conjugate by g,

2. s and gsg^{-1} are equal in $Z_{\widehat{H}}/Z_{\widehat{G}}$.

Call the set of isomorphism classes of elliptic endoscopic quadruples $\mathcal{E}_{\mathrm{ell}}(G)$.

Note that the definition implicitly uses a fact which we state directly here to cite more easily later:

Lemma 2.2.1.1. *Let G be a reductive group over global or local field K and $(H, \mathcal{H}, \eta, s)$ an elliptic endoscopic quadruple. Then there is a map $Z_G \hookrightarrow Z_H$.*

Proof. See [46] pg. 53. □

Endoscopic pairs

Endoscopic quadruples actually contain a lot of redundant data. A more basic and easier to think about notion is the endoscopic pair defined in [44, §7]:

Definition. An *endoscopic pair* for group G is (s, ρ) where

- s is a semisimple element of $\widehat{G}/Z_{\widehat{G}}$,
- ρ is a map $W_F \to \mathrm{Out}(\widehat{H})$ where $\widehat{H} = \widehat{G}_s^0$

satisfying

1. $\rho(\sigma)$ for $\sigma \in W_F$ is conjugation by an element in the normalizer of $\widehat{H}$ in ${}^L G$ that projects to σ.
2. Then, ρ induces a W_F-action on $Z_{\widehat{G}_s^0}$ which fits into W_F-equivariant sequence

$$1 \to Z_{\widehat{G}} \to Z_{\widehat{H}} \to Z_{\widehat{H}}/Z_{\widehat{G}} \to 0$$

   which induces a map $(Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F} \to H^1(F, Z_{\widehat{G}})$. We require that $s \in (Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F}$ and maps to something locally trivial under this.

It is furthermore *elliptic* if

3. $(Z_{\widehat{H}}^{W_F})^0 \subseteq Z_{\widehat{G}}$.

For future reference, we let $\mathfrak{K}(s, \rho)$ be the elements that map to something locally trivial under $(Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F} \to H^1(F, Z_{\widehat{G}})$.

The ρ action can be further clarified: if $a \rtimes \gamma \in {}^L G$ and $(b, 1) \in \widehat{G} \subset {}^L G$,

$$\begin{aligned}(a \rtimes \gamma)(b \rtimes 1)(a \rtimes \gamma)^{-1} &= (a \rtimes \gamma)(b \rtimes 1)(\gamma^{-1}(a^{-1}) \rtimes \gamma^{-1}) \\ &= (a\gamma(b) \rtimes \gamma)(\gamma^{-1}(a^{-1}) \rtimes \gamma^{-1}) = (a\gamma(b)a^{-1} \rtimes 1)\end{aligned}$$

so if ρ is part of an endoscopic pair, any $\rho(\gamma)$ is of the form $b \mapsto a_\gamma \gamma_{\widehat{G}}(b) a_\gamma^{-1}$ for some $a_\gamma \in \widehat{G}$ where the subscript $\widehat{G}$ denotes that the γ action is as it is on $\widehat{G}$. The choices of a_γ are unique up to

$$a_\gamma \in \operatorname{Int} \widehat{H} \backslash \widehat{G} / Z_{\gamma_{\widehat{G}} \widehat{H}}(\widehat{G}) = \widehat{H}_{\mathrm{ad}} \backslash \widehat{G} / Z_{\gamma_{\widehat{G}} \widehat{H}} = \widehat{H}_{\mathrm{ad}} \backslash \widehat{G} / Z_{\widehat{H}} = \widehat{H} \backslash \widehat{G}$$

since $\gamma_{\widehat{G}} \widehat{H}$ is the centralizer of $\gamma_{\widehat{G}} s$.

Definition. An isomorphism of endoscopic pairs (s, ρ) and (s', ρ') is an element $g \in \widehat{G}$ such that

- $\widehat{G}_s^0, \widehat{G}_{s'}^0$ and ρ, ρ' are g-conjugate,
- s, s' have the same image in $\mathfrak{K}(s, \rho)$.

As explained in [44, pg 630-631], ρ determines a quasisplit group H from $\widehat{H}$ and therefore the (H, s, η) part of an endoscopic quadruple. Given H and G, we can define $\mathcal{H}$ as follows: $\widehat{H}$ embeds into both ${}^L H$ and ${}^L G$. Let $\mathcal{H}$ be the set of $x \in {}^L G$ such that there exists $y \in {}^L H$ such that conjugation by x, y are the same on $\widehat{H}$ and x, y project to the same element of W_F. In terms of the a_γ from above, we can realize

$$\mathcal{H} = \bigcup_{\gamma \in W_F} \widehat{H} a_\gamma \rtimes \gamma$$

where we can choose representatives for a_γ so that conjugation by $a_\gamma \rtimes \gamma$ fixes a pinning of $\widehat{H}$. Isomorphisms are also the same on each side, so in summary:

Lemma 2.2.1.2 ([44, §7])**.** *The set of elliptic endoscopic pairs of G up to isomorphism are in bijection with $\mathcal{E}_{\mathrm{ell}}(G)$ where the bijection is as described above.*

Motivation and the group $\mathfrak{K}$

There are two motivations for this definition, either spectral or geometric. We briefly and very roughly describe the geometric explanation since it is somewhat relevant later. We ignore many, many Galois cohomology details. In increasing generality and detail, more information can be found in [48, §III.3], [45, §9], and [46, §6-7].

Let semisimple $\gamma \in G(F_v)$ be contained in maximal torus T. If γ is strongly regular, then we can write its stable orbit as $(T \backslash G)(F_v)$ and its orbit as $T(F_v) \backslash G(F_v)$. Therefore, the fibers of the map from $(T \backslash G)(F_v)$ onto

$$\mathfrak{D}(F_v, T \backslash G) = \ker(H^1(F_v, T) \to H^1(F_v, G))$$

are exactly the unstable conjugacy classes making up $(T \backslash G)(F_v)$. Let

$$\mathfrak{E}(F_v, T \backslash G) = \ker(H^1(F_v, T) \to H^1_{\mathrm{ab}}(F_v, G))$$

be the abelian group version of this and

$$\mathfrak{K}(F_v, T\backslash G) = \mathfrak{E}(F_v, T\backslash G)^\vee.$$

Elements $\kappa \in \mathfrak{K}$ are called endoscopic characters.

If v is a place of F and $\kappa \in \mathfrak{K}(F_v, T\backslash G)$, this allows the definition of twisted orbital integrals

$$O_\gamma^\kappa(f) = \int_{(T\backslash G)(F_v)} \kappa(g) f(g^{-1}\gamma g)\, dg$$

using the map $(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$.

We can also define adelic versions of these groups $\mathfrak{D}(\mathbb{A}, T\backslash G), \mathfrak{E}(\mathbb{A}, T\backslash G)$, and $\mathfrak{K}(\mathbb{A}, T\backslash G)$ using corresponding cohomology groups $H^1(\mathbb{A}, \cdot)$. If $\gamma \in G(\mathbb{A})$ is strongly regular, $\mathfrak{D}(\mathbb{A}, T\backslash G)$ parametrizes the γ' that have every component stably conjugate to γ. It is a restricted direct product of the $\mathfrak{D}(F_v, T\backslash G)$ by $\mathfrak{D}(\mathcal{O}_v, T\backslash G)$ which happens to be trivial. Define a measure on it by taking the product of the counting measures on $\mathfrak{D}(F_v, T\backslash G)$. Then for $\kappa \in \mathfrak{K}(\mathbb{A}, T\backslash G)$ we can define global twisted orbital integral

$$O_\gamma^\kappa(f) = \sum_{e \in \mathfrak{D}(\mathbb{A}, T\backslash G)} \kappa(\mathrm{obs}(\gamma_e)) O_{\gamma_e}(f)$$

where γ_e is the conjugacy class corresponding to e with base point γ and obs is the obstruction defined in [45] and [46].

Stabilization of the trace formula first produces sums of $O_\gamma^\kappa(f)$'s over triples of these (T, γ, κ) over F. The result ([46, lem 7.2.A]) shows that such triples are in bijection with quintuples $(H, \mathcal{H}, s, \eta, \gamma_H)$: endoscopic quadruples with a choice of strongly regular element $\gamma_H \in H$ up to appropriately defined equivalence. Through this equivalence, the group $\mathfrak{K}$ for T ends up being the same as the group $\mathfrak{K}$ defined above for (s, η) (see [46, pg 105-106]).

2.2.2 z-Extensions

Our next goal is to define transfers of functions. This naïvely needs an embedding ${}^L H \hookrightarrow {}^L G$, but in general ${}^L H \ncong \mathcal{H}$ so we do not have one. There are two possible strategies for dealing with this: the original in [50] is to take a nice enough central extension of G. This works for the standard endoscopy described here but not for the more general twisted endoscopy, so more modern sources prefer to take central extensions of H as described in [46]. As we will remark after proposition 2.2.2.2, these methods are more or less interchangeable in the standard endoscopy case.

We describe the second method in detail:

Definition. A *z-pair* $(\tilde{H}, \tilde{\eta})$ for endoscopic quadruple $(H, \mathcal{H}, s, \eta)$ is an extension $\tilde{H}$ by a central induced torus such that

1. $\tilde{H}_{\mathrm{der}}$ is simply connected (we call such an $\tilde{H}$ a *z-extension*).

2. $\tilde{\eta} : \mathcal{H} \to {}^L\tilde{H}$ is an L-embedding that restricts to the map $\widehat{H} \to \widehat{\tilde{H}}$ dual to the projection $\tilde{H} \to H$.

By Lemma 2.2.A in [46], as long as (1) is satisfied, a valid η satisfying (2) always exists.

Lemma 2.2.2.1. *Let H be a reductive group that splits over K'. Then there exists a z-extension of H splitting over K'. Furthermore, the dimension of the extending torus is bounded by $[K' : \mathbb{Q}](\operatorname{rank}_{\mathrm{ss}} H)$.*

Proof. We just go through the construction in [49] or [57, pg 299] explicitly seeing how big things get at each step. Let T^{sc} be the maximal torus in the simply connected cover of H^{der}. Let $P = X_*(T)/X_*(T^{\mathrm{sc}})$ as a Galois module. A z-extension would correspond to an extension of $X_*(T)$ making this quotient have no torsion. The torsion part has less than $\operatorname{rank}_{\mathrm{ss}} G$ generators.

The argument starts with a lemma writing P as a quotient of Galois modules

$$0 \to M \to Q \to P \to 0$$

with M free over $Z[G]$ and Q free over P. The construction is [57, prop 3.1] and bounds $\operatorname{rank}_{\mathbb{Z}} M$ by $\dim K'$ times the number of generators of the torsion of $P/\mathbb{Z}$ which we can further bound by $(\dim K')(\operatorname{rank}_{\mathrm{ss}} H)$.

Some work with reductive groups shows that M can be chosen to be the cocharacter space of the extending torus, thereby finishing the argument. □

In the case where G has simply connected derived subgroup, the Z-extension can be chosen to be trivial and $\mathcal{H} \simeq {}^LH$. In this case, an endoscopic triple (H, s, η) contains all the needed data.

z-extensions and central character datum

If $(\mathfrak{X}, \chi)$ is a central character datum for G, any $(H, \mathcal{H}, s, \eta)$ and $(\tilde{H}, \tilde{\eta})$ quadruple and extension determine a central character datum $(\mathfrak{X}_{\tilde{H}}, \chi_{\tilde{H}})$ on $\tilde{H}$. The central subgroup $\mathfrak{X}_{\tilde{H}}$ is produced from $\mathfrak{X}$ by first taking the image under the map $Z_G \hookrightarrow Z_H$ and then taking the preimage under $\tilde{H} \to H$.

To get $\chi_{\tilde{H}}$, pick a section c for $\mathcal{H} \to W_F$. Then if T is the extending torus defining $\tilde{H}$, the composition

$$W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\tilde{\eta}} {}^L\tilde{H} \to {}^LT$$

is an L-parameter for T. This determines a character $\lambda_{\tilde{\eta}}^{-1}$ on $T(F)$ if F is local or $T(F)\backslash T(\mathbb{A})$ if F is global through the Langlands correspondence for Tori. The inverse is to match our convention for defining Hecke algebras.

Through considerations of transfer factors (see section 2.2.3), $\lambda_{\tilde{\eta}}$ can be extended to the preimage of Z_G in $Z_{\tilde{H}}$. Therefore, we can set $\chi_{\tilde{H}}$ to be $\chi\lambda_{\tilde{\eta}}$ (where χ is defined on $\mathfrak{X}_H$ by pullback). We will discuss this and more properties of $\lambda_{\tilde{\eta}}$ when we discuss transfer. In particular, we will show that in the relevant cases, $\lambda_{\tilde{\eta},v}$ at a place v can be extended to a character on $\tilde{H}_v$.

z-extensions do not change much

There is a vague intuition that taking a z extension should not change a groups endoscopy:

Proposition 2.2.2.2. *Let G be a group over F.*

(a) If G_1 is a central extension of G by induced torus T, then the (elliptic) endoscopic tuples for G are in bijection with those of G_1. This bijection takes a group H to a central extension H_1 by T.

(b) If H is an (elliptic) endoscopic group of G and H_1 is a central extension of H by induced torus T, then there is a central extension G_1 of G by T such that H_1 is an (elliptic) endoscopic group of G_1. Furthermore, the endoscopic tuples determining H and H_1 correspond under the bijection from (a).

Proof. Part (a):
The s: The map $\widehat{G} \to \widehat{G}_1$ gives a canonical W_F-equivariant isomorphism $\widehat{G}/Z_{\widehat{G}} \to \widehat{G}_1/Z_{\widehat{G}_1}$ so choices for s are the same. Given such an s, set $\widehat{H}_1 = (\widehat{G}_1)^0_s$. Then we have the diagram

$$\begin{tikzcd}
\widehat{H} \arrow[r, hook] \arrow[d, hook] & \widehat{G} \arrow[d, hook] \\
\widehat{H}_1 \arrow[r, hook] \arrow[d, two heads] & \widehat{G}_1 \arrow[d, two heads] \\
\widehat{T} \arrow[r, "\sim"] & \widehat{T}
\end{tikzcd}$$

The ρ and H: This gives a canonical isomorphism $\widehat{H}_1\backslash\widehat{G}_1 \to \widehat{H}\backslash\widehat{G}$ so assignments $\gamma \to a_\gamma$ as in the comment after the definition of endoscopic pair are the same for G and G_1. There are two conditions for this assignment to give a valid ρ: The first is that $\gamma \mapsto \operatorname{Int} a_\gamma \circ \gamma$ is a homomorphism up to $\operatorname{Int} \widehat{H} = \operatorname{Int} \widehat{H}_1$. This condition is clearly the same with respect to either $\widehat{H}$ or $\widehat{H}_1$.

The second condition is that $\operatorname{Int} a_\gamma \circ \gamma$ needs to fix the appropriate group: $\widehat{H}$ or $\widehat{H}_1$. By construction, $\widehat{H} = \widehat{G} \cap \widehat{H}_1$. Therefore, since $\widehat{G}$ is W_F and $\operatorname{Int} \widehat{G}_1$-invariant, if such a map fixes $\widehat{H}_1$, it fixes $\widehat{H}$. For the other direction, since these are all complex groups and $\widehat{G} \supseteq (\widehat{G}_1)^{\mathrm{der}}$, all elements of $\widehat{G}_1$ can be written as zg for $z \in Z^0_{\widehat{G}_1}$ and $g \in \widehat{G}$. This is an element of $\widehat{H}_1$ if and only if $g \in \widehat{H}$. In total, $\widehat{H}_1 = Z^0_{\widehat{G}_1}\widehat{H}$ so we are done since $Z^0_{\widehat{G}_1}$ is fixed by W_F and Int_G. Therefore, this second condition is true for $\widehat{G}$ if and only if it is true for $\widehat{G}_1$.

Note that for any such ρ, the columns of the above diagram and the isomorphism between $\widehat{T}$'s are Γ-equivariant. Undoing this dual, this will give that H_1 is an extension of H by T.
The cohomology condition: In total, the possible pairs (s, ρ) ignoring the cohomology condition are the same for G and G_1. It remains to show that the cohomology condition holds

with respect to G if and only if it does for G_1. We have W_F-equivariant diagram where the first two rows are exact sequences (note that the actions on $Z_{\widehat{G}}$ from ρ and $\widehat{G}$ coincide so the action here is according to ρ):

$$\begin{tikzcd}
1 \arrow[r] & Z_{\widehat{G}} \arrow[r] \arrow[d] & Z_{\widehat{H}} \arrow[r] \arrow[d] & Z_{\widehat{H}}/Z_{\widehat{G}} \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & Z_{\widehat{G}_1} \arrow[r] \arrow[d] & Z_{\widehat{H}_1} \arrow[r] \arrow[d] & Z_{\widehat{H}_1}/Z_{\widehat{G}_1} \arrow[r] & 1 \\
& \widehat{T} \arrow[r, "\sim"] & \widehat{T} & &
\end{tikzcd}$$

This gives a corresponding diagram in cohomology:

$$\begin{tikzcd}
(Z_{\widehat{H}}/Z_{\widehat{G}})^\Gamma \arrow[r, "\varphi_1"] \arrow[d, "\sim"] & H^1(\Gamma, Z_{\widehat{G}}) \arrow[d, "\psi"] \\
(Z_{\widehat{H}_1}/Z_{\widehat{G}_1})^\Gamma \arrow[r, "\varphi_2"] & H^1(\Gamma, Z_{\widehat{G}_1})
\end{tikzcd}$$

Here $\Gamma \subseteq W_F$ is some local Galois group. The cohomology conditions for H and H_1 matching at Γ is equivalent to $\ker \varphi_1 = \ker \varphi_2$. To show this, consider the sequence

$$\pi_0(\widehat{T}^\Gamma) \to H^1(\Gamma, Z_{\widehat{G}}) \xrightarrow{\psi} H^1(\Gamma, Z_{\widehat{G}_1}) \to H^1(\Gamma, \widehat{T}).$$

Since T is an induced torus, $\widehat{T}$ is a power of $\mathbb{G}_m$ with a Γ action by permuting coordinates. This gives first, that $\widehat{T}^\Gamma$ is connected and second, that $\widehat{T}$ is induced, so $H^1(\Gamma, \widehat{T}) = 0$. Therefore, ψ is an isomorphism and the cohomology conditions are equivalent at every place.

Ellipticity: The elliptic condition is that $(Z_{\widehat{H}_*}^{W_F})^0 \subseteq Z_{\widehat{G}_*}^{W_F}$. As before, $Z_{\widehat{H}_1} = Z_{\widehat{H}} Z_{\widehat{G}_1}$ and $Z_{\widehat{H}} \cap Z_{\widehat{G}_1} = Z_{\widehat{G}}$. Then we get the sequence

$$1 \to Z_{\widehat{G}} \to Z_{\widehat{H}} \times Z_{\widehat{G}_1} \to Z_{\widehat{H}_1} \to 1$$

where the first map is the antidiagonal. This gives a map in cohomology:

$$Z_{\widehat{H}}^{W_F} \times Z_{\widehat{G}_1}^{W_F} \to Z_{\widehat{H}_1}^{W_F} \to H^1(W_F, Z_{\widehat{G}}) \to H^1(W_F, Z_{\widehat{G}_1}) \oplus H^1(W_F, Z_{\widehat{H}}).$$

From previous arguments, T being induced gives that the last map in injective into the first coordinate. Therefore the middle is 0 and the first is surjective. Therefore $Z_{\widehat{H}_1}^{W_F} = Z_{\widehat{H}}^{W_F} \times Z_{\widehat{G}_1}^{W_F} / Z_{\widehat{G}}^{W_F}$ and $(Z_{\widehat{H}_1}^{W_F})^0 \subseteq (Z_{\widehat{G}_1}^{W_F})^0 (Z_{\widehat{H}}^{W_F})^0 Z_{\widehat{G}}^{W_F}$. This gives that the elliptic condition on H implies that on H_1.

For the other direction, $Z_{\widehat{H}} = Z_{\widehat{H}_1} \cap \widehat{G}$ gives that $Z_{\widehat{H}}^{W_F} = Z_{\widehat{H}_1}^{W_F} \cap \widehat{G}^{W_F}$ which gives $(Z_{\widehat{H}}^{W_F})^0 \subseteq (Z_{\widehat{H}_1}^{W_F})^0 \cap \widehat{G}^{W_F}$. Assuming $(Z_{\widehat{H}_1}^{W_F})^0 \subseteq Z_{\widehat{G}_1}^{W_F}$ and further using that $Z_{\widehat{G}_1} \cap \widehat{G} = Z_{\widehat{G}}$ implies $Z_{\widehat{G}_1}^{W_F} \cap \widehat{G}^{W_F} = Z_{\widehat{G}}^{W_F}$ finally giving that $(Z_{\widehat{H}}^{W_F})^0 \subseteq Z_{\widehat{G}}^{W_F}$.

Part (b):
We are given G, endoscopic group H, and extension H_1 by T. There is a map $Z_G \hookrightarrow Z_H$ (see [46] pg. 53) so we can pullback the extension Z_{H_1} to an extension Z_{G_1} of Z_G by T.

Set $G_1 = Z_{G_1} \times G_{\mathrm{der}}/Z_{G^{\mathrm{der}}}$ as an algebraic group where the $Z_{G^{\mathrm{der}}}$ is embedded antidiagonally. Then, since $G = Z_G \times G_{\mathrm{der}}/Z_{G^{\mathrm{der}}}$, G_1 is an extension of G by T. If H comes from data (s, ρ), then through the construction of the bijection in (a), (s, ρ) gives data for H_1 and is elliptic if and only if (s, ρ) is. □

Consider H an endoscopic group of G and H_1 a z-extension (so it has simply connected derived subgroup). Let $(H_1, \mathcal{H}_1, s, \eta)$ be the quadruple for G_1 produced by part (b). Then the map ${}^LH_1 \to \mathcal{H}_1$ is an isomorphism, so we actually do have an embedding ${}^LH_1 \hookrightarrow {}^LG_1$. This is the z-extension construction described in [50].

2.2.3 Transfer

Consider quadruple $(H, \mathcal{H}, s, \eta)$ for G over local or global K and associated z-extension (H_1, η_1). There is a transfer map

$$\begin{aligned}\mathcal{T} : &\{\text{strongly } G\text{-regular semisimple conjugacy classes in } H(K)\} \\ &\to \{\text{strongly regular stable conjugacy classes } G(K)\} \cup \{*\}\end{aligned}$$

where the $*$ is a dummy variable to allow maps that are not necessarily defined everywhere. We say that $\gamma_H \in H(K)$ is a norm of $\gamma_G \in G(K)$ if $\mathcal{T}$ takes the conjugacy class of γ_H to that of γ_G. Respectively, $\gamma_{H_1} \in H_1(K)$ is a norm of something if its projection to $H(F)$ is.

Local Transfer

Now, consider local F_v. If strongly G-regular γ_{H_1} is a norm of strongly regular γ_G, a transfer factor $\Delta(\gamma_{H_1}, \gamma_G) = \Delta_G^{H_1}(\gamma_{H_1}, \gamma_G)$ can be defined (this is the content of sections $4.1 - 5.1$ in [46]). The factor is non-canonical up to a uniform constant. We recall some useful properties from [46, §5.1]:

- $\Delta(\gamma_{H_1}, \gamma_G)$ is 0 unless γ_{H_1} is a norm of γ_G.
- $\Delta(\gamma_{H_1}, \gamma_G)$ is constant over the stable conjugacy class of γ_{H_1}.
- Let $Z_{G_1} = Z_{H_1} \times_{Z_H} Z_G$. There exists a character λ_{η_1} on $Z_{G_1}(F_v)$ such that if $(z_1, z) \in Z_{G_1}(F_v)$,
$$\Delta_G^{H_1}(z_1\gamma_{H_1}, z\gamma_G) = \lambda_{\eta_1}^{-1}(z_1, z)\Delta_G^{H_1}(\gamma_{H_1}, \gamma_G).$$
In fact, λ_{η_1} even extends to a character on $G_1(F_v)$ (see the construction on pg. 53 in [46] or pg. 55 in [50]).

- Let the quadruple $(H, \mathcal{H}, s, \eta, \gamma_{H_1})$ correspond to the triple (T, γ_G, κ). Then γ_{H_1} is a norm of γ_G. If γ'_G is a stable conjugate of γ_G,

$$\kappa(\gamma'_G)\Delta(\gamma_{H_1}, \gamma_G) = \Delta(\gamma_{H_1}, \gamma'_G).$$

Fix central character datum $(\mathfrak{X}, \chi)$ for G. Let $f \in \mathscr{H}(G(F_v), \chi_v)$. We say that $f^{H_1} \in \mathscr{H}(H_1(F_v), \chi_{H_1,v})$ matches f if

$$SO_{\gamma_{H_1}}(f^{H_1}) = \sum_{\gamma_G} \Delta(\gamma_{H_1}, \gamma_G) O_{\gamma_G}(f)$$

for all strongly G-regular $\gamma_{H_1} \in H_1(F_v)$ where γ_G ranges over representatives of unstable conjugacy classes such that γ_{H_1} is a norm of γ_G. Note that the right-hand side is a twisted orbital integral multiplied by an appropriate constant.

Since γ_{H_1} and γ_G are strongly regular, if T is a maximal torus for G and Z is the extending torus defining H_1 from H, the orbital integrals have dimension $[G(F_v)][T(F_v)]^{-1}$ and $[H_1(F_v)][T(F_v)]^{-1}[Z(F_v)]^{-1} = [H(F_v)][T(F_v)]^{-1}$. Therefore, f^{H_1} needs to have dimensions $[G(F_v)][H(F_v)]^{-1}$.

A big theorem is that such an f^H always exists. The Archimedean case is from Shelstad in [73] while the non-Archimedean case was reduced to the fundamental lemma (which will be discussed later) by Waldspurger in [84]. Call such an f^H a transfer of f.

Global Transfer

If F is global, then the endoscopic datum determine local endoscopic datum at each place v. This lets us define a global transfer factor $\Delta_{\mathbb{A}}(\gamma_{H_1}, \delta_G)$ as the product of all the local transfer factors. [46, cor 7.3.B] gives that all the choices defining the local factors can be made consistently giving a canonical choice of global factor.

If $f \in \mathscr{H}(G, \chi)$ factors into local factors at each place, then transferring each of the local factors gives a transfer f^H satisfying a similar identity. By the fundamental lemma, this is unramified almost everywhere and is therefore an element of $\mathscr{H}(H_1, \chi_{H_1})$.

After lots of cohomology work, f^H can be shown to satisfy a global identity

$$SO_{\gamma_{H_1}}(f^H) = O^{\kappa}_{\gamma_G}(f)$$

where $(H, \mathcal{H}, s, \eta, \gamma_H)$ corresponds to (T, γ_G, κ). This is [46, lem 7.3.C].

Characters from Transfer

By the above, endoscopy always defines a character on $Z_{H_1}(F_v)$. However, for v non-Archimedean, this actually extends to a character on $H_1(F_v)$. We will need this to state some bounds on non-Archimedean transfers later.

Fix such a v and assume without loss of generality that G has simply connected derived subgroup (possibly by taking a z-extension and using proposition 2.2.2.2). Take the extension

G_1 of G as in proposition 2.2.2.2(b). Then G_1^{der} is an isogenous cover of G^{der}, so the two are equal. The map η determines a character λ_{η_1} on $Z_{G_1}(F_v) = Z_{H_1}(F_v) \times_{Z_H(F_v)} Z_G(F_v)$. Since this lifts to a character on $G_1(F_v)$, it is actually a character on $G_1(F_v)/G_1^{\mathrm{der}}(F_v)$. If F is local then $H^1(F_v, G_1^{\mathrm{der}}) = 0$ since G_1^{der} is semisimple and simply connected. Therefore this is a character on $(G_1)_{\mathrm{ab}}(F_v)$ so let it correspond to the L-parameter $\alpha : W_{F_v} \hookrightarrow {}^L(G_1)_{\mathrm{ab}}$.

Next

Lemma 2.2.3.1. *Let G be a reductive group over F_v. Then $Z^0_{\widehat{G}} = \widehat{G_{\mathrm{ab}}}$ as groups with W_{F_v}-action.*

Proof. Let G have maximal torus T. As W_F-modules, $X_*(\widehat{G_{\mathrm{ab}}}) = X^*(G_{\mathrm{ab}}) = X^*(T)^\Omega$ and $X_*(Z^0_{\widehat{G}}) = X_*(\widehat{T})^\Omega = X^*(T)^\Omega$. This equality of cocharacters induces an equality of torii. $\square$

Since $\widehat{H}_1$ is a connected centralizer in $\widehat{G}_1$, we get a map $Z^0_{\widehat{G}_1} \hookrightarrow Z^0_{\widehat{H}_1}$. Since H_1 is endoscopic, the map is Galois-equivariant so it extends to a map ${}^L(G_{1,\mathrm{ab}}) \to {}^L(H_{1,\mathrm{ab}})$. Therefore α can be pushed forward and determines a character λ'_{H_1} on H_1.

Note that λ_{H_1} and λ'_{H_1} are equal on $Z_{G_1}(F_v)$ since they correspond to the same parameter of $Z_{G_1}(F_v)$. This common value is the character λ_{η_1} from before that determined which Hecke algebra transfers landed in. The discussion here simply shows that it extends to a character on H_v.

A trick for computing transfers with z-extensions

Most formulas for transfers in the literature only apply in the case when ${}^LH \cong \mathcal{H}$. To use these in the general case, consider the same quadruple and z-extension as before with $T \hookrightarrow H_1$ as the extending torus. Proposition 2.2.2.2(b) lets us find G_1 such that $(H_1, \mathcal{H}_1, s, \eta)$ is an endoscopic quadruple for G_1 with ${}^LH_1 \cong \mathcal{H}_1$. Let $\pi : G_1 \to G$ be the projection.

The key property we use is that

$$\Delta^{H_1}_{G_1}(\gamma_1, \delta_1) = \Delta^{H_1}_{G}(\gamma_1, \delta)$$

whenever $\delta_1 \in G_1(F)$ projects to $\delta \in G(F)$ and γ_1 is a norm of δ_1 (see [50] pg. 55). Therefore, given $f \in \mathscr{H}(G(F), \chi)$, let

$$f_1(g) = f \circ \pi(g)$$

for $g \in G_1(F)$. If f_1 and ${f_1}^{H_1}$ match, then for all appropriate γ_1, δ,

$$SO_{\gamma_1}(f_1^{H_1}) = \sum_{\delta_1} \Delta^{H_1}_{G_1}(\gamma_1, \delta_1) O_{\delta_1}(f_1) = \sum_{\delta_1} \Delta^{H_1}_{G}(\gamma_1, \pi(\delta_1)) O_{\pi(\delta_1)}(f),$$

which is the condition for f and ${f_1}^{H_1}$ matching. Therefore we can compute f^{H_1} by transferring f_1.

As a sanity check, note that γ_1 being a norm of δ_1 is true if and only if $z\gamma_1$ is a norm of $z\delta_1$ for all $z \in Z_{G_1}$. In particular, if $x = (z_1, z) \in Z_{G_1}$ then

$$\begin{aligned}\Delta_{G_1}^{H_1}(x\gamma_1, x\delta_1) = \Delta_{G_1}^{H_1}(z_1\gamma_1, x\delta_1) = \Delta_G^{H_1}(z_1\gamma_1, z\pi(\delta_1)) \\ = \lambda_{\eta_1}(x)^{-1}\Delta_G^{H_1}(\gamma_1, \pi(\delta_1)) = \lambda_{\eta_1}(x)^{-1}\Delta_{G_1}^{H_1}(\gamma_1, \delta_1).\end{aligned}$$

Therefore, the transfer factor transforms appropriately so that this transfer will be in the Hecke algebra $\mathscr{H}(H_1(F), \chi_H)$.

Beware that there is a small technical issue here. Theorems in the literature only give the existence of transfers of compactly supported functions. We get around this by finding a compactly supported function f' that averages to $f \circ \phi$ along the central character datum (see lemma 3.3.1.1 for example) and then transferring f'. We then average $(f')^{H_1}$ against the central character datum.

2.2.4 Stabilization

Using all the above and with much work, $I_{\mathrm{disc},t}^{G,\chi}(f)$ can be stabilized. In other words, it can be expanded as

$$I_{\mathrm{disc},t}^{G}(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H)\widehat{S}_{\mathrm{disc},t}^{\tilde{H},\chi_{\tilde{H}}}(f^{\tilde{H}})$$

for some choice of z-extensions. Here $\widehat{S}_{\mathrm{disc},t}^{\tilde{H},\chi_{\tilde{H}}}$ is a stable distribution on $\mathscr{H}(\tilde{H}, \chi_{\tilde{H}})$ depending only on $t, \tilde{H}$. We will not use any properties of S except that it is stable. There is no explicit construction of f^H in general, so its known properties will be cited as needed.

The constant ι has an explicit formula. Recall the definition in section 2.2.1 of automorphisms of quadruples $(H, \mathcal{H}, s, \eta)$ by elements $g \in \widehat{G}$. Let $\Lambda(H, \mathcal{H}, s, \eta)$ be the image of $\mathrm{Aut}(H, \mathcal{H}, s, \eta) \to \mathrm{Out}(\widehat{H})$. Then

$$\iota(G, H) = |\Lambda(H, \mathcal{H}, s, \eta)|^{-1}\tau(G)\tau(H)^{-1}$$

where τ is the Tamagawa number.

2.2.5 Some Properties

Endoscopy and root data

The following is a summary of the relation between roots data of endoscopic groups and the original group:

Lemma 2.2.5.1. *Let G be a reductive group over global or local field K, $(H, \mathcal{H}, \eta, s)$ an elliptic endoscopic quadruple and $(\tilde{H}, \tilde{\eta})$ a z-extension. Let T_H be a maximal torus for $H_{\overline{K}}$. Then there is a maximal torus T of $G_{\overline{K}}$ and an isomorphism $T_H \to T$. The choice of T and the map are unique up to $G_{\overline{K}}$-conjugacy. Let $T_{\tilde{H}}$ be the pullback of T_H to $\tilde{H}$.*

Then the following also hold:

1. *The positive (co)roots of (H, T_H) can be chosen to be a subset of those of (G, T) through $T_H \to T$.*

2. *For any root of α of (H, T_H), $s_\alpha \in \Omega_H$ is the same as $s_\alpha \in \Omega_G$ through the isomorphism $T_H \to T$.*

3. *The positive roots of $(\tilde{H}, T_{\tilde{H}})$ can be chosen to be a subset of those of (G, T) through $T_{\tilde{H}} \to T_H \to T$.*

4. *The Weyl action on the roots of $(\tilde{H}, T_{\tilde{H}})$ restricts to that on (H, T_H) through $X^*(T_H) \hookrightarrow X^*(T_{\tilde{H}})$.*

Proof. The construction of T_H and (1),(2) are done in [45, §3.1] and [50, §1.3].

To deal with $\tilde{H}$, let the extension be $1 \to Z \to \tilde{H} \to H \to 1$. Every maximal torus of $\tilde{H}$ is the preimage of one of H so $X^*(T_H)$ maps into the corresponding $X^*(T_{\tilde{H}})$. Since in the sequence

$$0 \to \operatorname{Lie} Z \to \operatorname{Lie} \tilde{H} \to \operatorname{Lie} H \to 0,$$

$\operatorname{Lie} Z$ maps into the center, the roots of $\tilde{H}$ have to be the images of those of H. Choose a Borel $\tilde{B}$ containing B_H to get containment of positive roots. The last statement on Weyl groups comes from $\Omega_{(H,T_H)} \cong N_H(T)/Z_H(T)$. $\square$

Be careful that this lemma says nothing about the Galois actions on the roots. We will not need that information and getting it requires G to be quasisplit. Also beware that this does not give that the simple roots of H are a subset of the simple roots of G or that the coroots of $\tilde{H}$ are a subset of the coroots of G.

Real endoscopic characters

As another computational tool, the character $\kappa = \kappa_{G,H}$ for elliptic elements has a nice form in the real case. If G is a real group and T is elliptic, there is an isomorphism

$$\Omega_{\mathbb{C},G}/\Omega_{\mathbb{R},G} \to \mathfrak{D}(\mathbb{R}, T\backslash G).$$

An endoscopic character κ can therefore be extended to $\Omega_{\mathbb{C}}(G)$. [48, §IV.1] gives that the extension is left-$\Omega_{\mathbb{C},H}$ invariant.

In addition, the composition

$$\Omega(B_K) \to \Omega_{\mathbb{C},G} \to \Omega_{\mathbb{C},G}/\Omega_{\mathbb{R},G}$$

is a bijection. This gives a bijection between any regular $\Pi_{\mathrm{disc}}(\xi)$ and $\mathfrak{D}(\mathbb{R}, T\backslash G)$ that depends on the choice of B_K.

This interpretation of κ will be used when computing transfers of pseudocoefficients.

Chapter 3

New Formulas

3.1 The Hyperendoscopy Formula

Here we will describe Ferrari's hyperendoscopy formula with some modifications in the case where groups without simply connected derived subgroup appear in the hyperendoscopic paths. Using this formula may appear a little bizarre since it may seem more reasonable to try to directly mimic the work of [4] on the stable distributions $SO^H(f^H)$ like the main result of [63].

The advantage of using hyperendoscopy is that we can directly apply the work already done in [79] instead of proving slightly different bounds for the slightly different terms appearing in the stable trace formula. There are two disadvantages: first, it gives worse constants in bounds, but the constants were already not explicit due to the model theory bounds that go into them. Second, hyperendoscopy requires extending Shin-Templier's results to groups with fixed central character datum, but this is interesting in its own right. In addition, the hyperendsocopic formula itself may be a useful tool for studying future forms of the invariant trace formula that, unlike [63], do not have a reasonable stabilization.

3.1.1 Raw Formula

Recalling the key trick from [17], rearrange the stabilized trace formula:

$$\widehat{S}^{G^{\mathrm{qs}}}_{\mathrm{disc},t}(f^{G^{\mathrm{qs}}}) = I^G_{\mathrm{disc},t}(f) + \sum_{\substack{H \in \mathcal{E}_{\mathrm{ell}}(G) \\ H \neq G^{\mathrm{qs}}}} (-\iota(G,H))\widehat{S}^{\tilde{H}}_{\mathrm{disc},t}(f^{\tilde{H}})$$

where G^{qs} is the quasisplit form of G. We want to continue this expansion inductively to get a formula in terms of I_{disc} for the various groups. The result in [17] uses endoscopic triples, seemingly assuming that if a group has simply connected derived subgroup, then so do all its endoscopic groups. This is not true as there can be SO_{2k} factors in endoscopic groups of Sp_{2n} (see [85, §1.8]). Nevertheless, with a little more work, a formula more-or-less equivalent to Ferrari's can be derived.

Inductively substituting in the expansions for $\widehat{S}^{\tilde{H}}_{\mathrm{disc},t}(f^{\tilde{H}})$ since the $\tilde{H}$ are all quasisplit gives something like

$$\widehat{S}^{G^{\mathrm{qs}}}_{\mathrm{disc},t}(f^{G^{\mathrm{qs}}}) = I^{G}_{\mathrm{disc},t}(f) + \sum_{\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_{\mathcal{H}}}}_{\mathrm{disc},t}(f^{\mathcal{H}}).$$

Because of the non-canonical z-extensions, the notation defining the indexing set becomes somewhat painful. We will later find a nicer set to index over.

Definition. A consistent choice of length-1 raw endoscopic paths for G is a set $\mathcal{HE}^0_{\mathrm{ell}}(G)_1$ consisting of pairs (H,z) where H ranges over the proper isomorphism classes in $\mathcal{E}_{\mathrm{ell}}(G)$ and z is a choice of z-pair for H.

Given a consistent choice of length-$(n-1)$ raw hyperendoscopic paths $\mathcal{HE}^0_{\mathrm{ell}}(G)_{n-1}$, a consistent choice of length-n hyperendoscopic paths is a set $\mathcal{HE}^0_{\mathrm{ell}}(G)_n$ consisting of tuples $(\mathcal{H},H,z)$ where $\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)_{n-1}$, H ranges over proper isomorphism classes in $\mathcal{H}_{\mathrm{ell}}(\mathcal{H})$ (overloading notation so that $\mathcal{H}$ also refers to the group in the last z-pair of $\mathcal{H}$), and z is a choice of z-pair for H.

A consistent choice of raw hyperendoscopic paths $\mathcal{HE}^0_{\mathrm{ell}}(G)$ is the union of an (inductively-chosen) consistent choice of $\mathcal{HE}^0_{\mathrm{ell}}(G)_n$ for all $n>0$.

The sum is over a choice of $\mathcal{HE}^0_{\mathrm{ell}}(G)$. If $\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)$, let $n_{\mathcal{H}}$ be its length. As shorthand, we will sometimes write

$$\mathcal{H} = (H_1, H_2, \cdots, H_{n_{\mathcal{H}}})$$

where H_n is the group in the z-pair for the nth step in the path. As further shorthand, $\mathcal{H}$ will sometimes be overloaded to refer to $H_{n_{\mathcal{H}}}$. For indexing purposes, $H_0 = G$. Similarly define:

$$\iota(G,\mathcal{H}) = (-1)^{n_{\mathcal{H}}} \prod_{i=1}^{n_{\mathcal{H}}} \iota(H_{i-1},H_i) \qquad f^{\mathcal{H}} = (\cdots(f^{H_1})^{H_2\cdots})^{H_{n_{\mathcal{H}}}}.$$

Note that $f^{\mathcal{H}}$ is not canonical and the choice of $f^{\mathcal{H}}$ needs to be consistent with the choice of $f^{\mathcal{H}'}$ where $\mathcal{H}'$ is $\mathcal{H}$ truncated by removing the last step. Finally, a hyperendoscopic path $\mathcal{H}$ determines central character datum $(\mathfrak{X}_n,\chi_n)$ for each H_n.

This expansion of course only works if the paths are all finite. This holds:

Lemma 3.1.1.1. *Every element of* $\mathcal{HE}^0_{\mathrm{ell}}(G)$ *has* $n_{\mathcal{H}} \le |\Phi^+(G)|+1$.

Proof. Consider the quadruple $(H_i,\mathcal{H},s_i,\eta_i)$ of H_{i-1}. If H_i is quasisplit, the group $\widehat{H}_i$ is a centralizer of $s_i\in\widehat{H}_{i-1}$ that is not $\widehat{H}_{i-1}$ since $H_{i-1}\neq H_i$ and H_{i-1} is necessarily quasisplit. Therefore H_{i+1} either has fewer positive roots than $\widehat{H}_{i-1}$ or changes from non-quasisplit to quasisplit. The result follows. □

The key point then is that

$$I^{G}_{\mathrm{disc},t}(f) + \sum_{\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_{\mathcal{H}}}}_{\mathrm{disc},t}(f^{\mathcal{H}})$$

is a stable distribution in $f^{G^{\text{qs}}}$. Finally, since G^{qs} corresponds to the trivial endoscopic character, if $f^{G^{\text{qs}}} = f_1^{G^{\text{qs}}}$, then f, f_1 have the same stable orbital integrals. Setting this equal for two such functions:

Proposition 3.1.1.2 ([17, prop 3.4.3] corrected)**.** *Let f and f_1 be functions on $G(\mathbb{A})$ that have the same stable orbital integrals. Then*

$$I^G_{\text{disc},t}(f) = I^G_{\text{disc},t}(f_1) + \sum_{\mathcal{H}\in\mathcal{HE}^0_{\text{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_{\mathcal{H}}}}_{\text{disc},t}((f_1 - f)^{\mathcal{H}}).$$

3.1.2 Simplifying Hyperendoscopic Paths

To control which groups appear, it is nice to have an easier definition of hyperendoscopic path.

Definition. An endoscopic path for G is a sequence $(Q_1, \dots, Q_n)$ where $Q_1 \in \mathcal{E}_{\text{ell}}(G)$ and $Q_i \in \mathcal{E}_{\text{ell}}(H_{i-1})$ for $i > 1$ where H_{i-1} is the group in Q_{i-1}. Note that if two endoscopic quadruples are isomorphic, then so are their groups.

We use the same notation for endoscopic paths as for raw endoscopic paths. The set of endoscopic paths for G will be called $\mathcal{HE}_{\text{ell}}(G)$.

Definition. A z-pair path for an endoscopic path $(Q_1, \dots, Q_n)$ is a sequence of z-pairs $(\tilde{Q}_1, \dots, \tilde{Q}_n)$ where

- $\tilde{Q}_1 = (\tilde{H}_1, \tilde{\eta}_1)$ is a choice of z-pair for Q_1.
- For $i > 1$ assume we have already chosen $Q_1, \dots, Q_{i-1}$. We get a quadruple Q'_i for H_{i-1} through repeated applications of the bijection from lemma 2.2.2.2(a) down through the Q_i (it will be clear that H_{i-1} can be produced from the group in Q_{i-1} by a sequence of central extensions by induced torii). Then $\tilde{Q}_i = (\tilde{H}_i, \tilde{\eta}_i)$ should be a z-pair for Q'_i.

If $\mathcal{H} \in \mathcal{HE}_{\text{ell}}(G)$ with z-pair path $\tilde{\mathcal{H}}$, we will sometimes overload notation and use $\tilde{\mathcal{H}}$ to denote that last group $\tilde{H}_n$ in the path. If $(\mathfrak{X}, \chi)$ is a central character datum for G, we will also let $(\mathfrak{X}_{\tilde{\mathcal{H}}}, \chi_{\tilde{\mathcal{H}}})$ be the induced datum on $\tilde{\mathcal{H}}$. We can also define $\iota(G, \tilde{\mathcal{H}})$ and transfers $f^{\tilde{\mathcal{H}}}$ similarly.

As in the definition of raw hyperendoscopic paths, we can similarly inductively define a consistent choice of z-pair paths for all elements of $\mathcal{HE}_{\text{ell}}(G)$.

Lemma 3.1.2.1. *Choose a consistent set of z-pair paths $\tilde{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{HE}_{\text{ell}}(G)$. Then the set of combined data $\{[\mathcal{H}, \tilde{\mathcal{H}}] : \mathcal{H} \in \mathcal{HE}_{\text{ell}}(G)\}$ concatenated properly form a consistent set of raw hyperendoscopic paths for G.*

Proof. We show this inductively on length. For length 1, this works by definition. For longer length, we use lemma 2.2.2.2(a): if we know this for length i and H_i is the ith group in $\mathcal{H}$, the corresponding H'_i in the corresponding raw endoscopic path has the same possible "next steps"—the elliptic quadruples of the two are in bijection. $\square$

Finally,

Lemma 3.1.2.2. *Let $\tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}'$ be two different z-extensions for the hyperendoscopic path H. Let $f \in \mathscr{H}(G, \chi)$ for some central character datum $(\mathfrak{X}, \chi)$. Then the two terms $S^{\tilde{\mathcal{H}}}_{\chi_{\tilde{\mathcal{H}}}}(f^{\tilde{\mathcal{H}}})$ and $S^{\tilde{\mathcal{H}}'}_{\chi_{\tilde{\mathcal{H}}'}}(f^{\tilde{\mathcal{H}}'})$ are equal. In addition, $\iota(G, \tilde{\mathcal{H}}) = \iota(G, \tilde{\mathcal{H}}')$.*

Proof. First, let G be a group, H an endoscopic group, and $f \in \mathscr{H}(G, \chi)$ for some χ. Let $(\tilde{H}, \tilde{\eta})$ and $(\tilde{H}', \tilde{\eta}')$ be two z-pairs. Then part of the formalism of the stable trace formula gives that $S^{\tilde{H}}_{\chi_{\tilde{H}}}(f^{\tilde{H}}) = S^{\tilde{H}'}_{\chi_{\tilde{H}'}}(f^{\tilde{H}'})$. By definition, $\iota(G, \tilde{H}) = \iota(G, H) = \iota(G, \tilde{H}')$.

Second, if G_1 is a z-extension of G and f_1 the pullback of f to some $\mathscr{H}(G_1, \chi_1)$ where χ_1 is the pullback of χ, it induces extension H_0 of H according lemma 2.2.2.2(a). We can find a z-pair (H_1, η_1) of H such that H_1 is a z-extension of H_0. By a similar argument to section 2.2.3, $f^{H_1} = f_1^{H_1}$ and $\chi_{H_1} = (\chi_1)_{H_1}$. Therefore $S^{H_1}_{(\chi_1)_{H_1}}(f_1^{H_1}) = S^{H_1}_{\chi_{H_1}}(f^{H_1})$. Since Tamagawa measures are products of Tamagawa measures of factors, $\iota(G, H_1) = \iota(G, H) = \iota(G_1, H_1)$ by the explicit formula.

The result follows from an induction alternating on these two steps. $\square$

Define $\iota(G, \mathcal{H})$ to be the common value of all the $\iota(G, \tilde{\mathcal{H}})$. In total, we can choose whichever z-extensions we want and ignore the consistency condition:

Theorem 3.1.2.3 (The Hyperendoscopy Formula)**.** *Let f and f_1 be functions on $G(\mathbb{A})$ that have the same stable orbital integrals. Then*

$$I^G_{\mathrm{disc},t}(f) = I^G_{\mathrm{disc},t}(f_1) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G, \mathcal{H}) I^{\tilde{\mathcal{H}}}_{\mathrm{disc},t}((f_1 - f)^{\tilde{\mathcal{H}}})$$

where $\tilde{\mathcal{H}}$ is a choice of z-extension path for $\mathcal{H}$ and where we suppress the central character datum.

3.1.3 Central Characters from Hyperendoscopy

Let $\mathcal{H}$ be a hyperendoscopic path for G with z-extension $\tilde{\mathcal{H}}$ corresponding to the sequence of groups and embeddings $(\tilde{H}_i, \eta_i)$. We can, without loss of generality, assume that $H_0 = G$ has simply connected derived subgroup by taking further extensions. Then we can inductively define character on each $(H_i)_v$:

- χ_1 is the character λ_{η_1} on $(\tilde{H}_1)_v$ defined by η_1 as in section 2.2.3.

- Let χ_i' be the character on $(\tilde{H}_{i+1})_v$ coming from character χ_i on $(\tilde{H}_i)_v$ as in section 2.2.3. Let λ_{i+1} be the character on $(H_i)_v$ determined by η_{i+1}. Then set $\chi_{i+1} = \chi_i' \lambda_{i+1}$.

From all the previous discussion, we know χ_i are the characters such that given central character datum $(\mathfrak{X}, \chi)$ and $f \in \mathscr{H}(G, \chi)$, the successive transfers $f^{\tilde{H}_i}$ lie in $\mathscr{H}(G, (\mathfrak{X}_{\tilde{H}_i}, \chi\chi_i))$.

3.1.4 Remarks on Usage

Some notes for using this:

- Beware that the transfers $(f_1 - f)^{\mathcal{H}}$ must be chosen explicitly, since the stable orbital integrals of $(f^{H_1})^{H_2}$ depend on the standard orbital integrals of f^{H_1}. Care should be taken in these choices since the ease of evaluating I_{disc} depends much on properties of f^{H_1} that are not determined by stable orbital integrals.

- As a sum of distributions, the sum over $\mathcal{E}_{\text{ell}}(H_i)$ can be infinite. However, for any particular f only finitely many terms are non-zero. Nevertheless, the number of such terms depends on the choices of $f^{\mathcal{H}}$ and can be arbitrarily large. Thankfully, if we choose the $f^{\mathcal{H}}$ so that they stay unramified outside of a finite set of places S, then there is a finite set of terms depending only on S that are non-zero. See lemma 3.2.6.1.

- If we can choose the f^H to be cuspidal, we do not need to worry that this formula is only in terms of I_{disc} instead of I_{spec}.

- If each of the H_i in path $\mathcal{H}$ are unramified, we can choose $\tilde{\mathcal{H}}$ to only have unramified groups since z-extensions can be chosen to have the same splitting field as the original group.

3.2 Lemmas on transfers

3.2.1 Formulas for Archimedean Transfer

This section will compute transfers of pseudocoeffcients. We take the Whittaker normalization of transfer factors as in [74] and [48]. Because pseudocoefficients already have the correct dimensions, we do not need to fix Haar measures.

Recall the parametrization of discrete series in section 2.1.2. We first make a basic remark:

Lemma 3.2.1.1. *Let $\pi \in \Pi_{\text{disc}}(\xi)$ be a discrete series representation. Then for any $\gamma \in G_\infty$,*

$$SO_\gamma(\varphi_\pi) = SO_\gamma(\eta_\xi).$$

Proof. Transfers from G to G^{qs} are determined by the identities

$$SO^G_\gamma(\varphi_\pi) = SO^{G^{\text{qs}}}_\gamma(\varphi^{G^{\text{qs}}}_\pi), \qquad SO^G_\gamma(\eta_\xi) = SO^{G^{\text{qs}}}_\gamma(\eta^{G^{\text{qs}}}_\xi).$$

By [74], transfers of pseudocoefficients can be chosen to be linear combinations of Euler-Poincaré functions. Such linear combinations are determined by evaluations on an elliptic torus so both of the transfers will be equal if we can show the lemma statement for just elliptic elements . The transfers being equal will suffice to prove the lemma.

We therefore just need the computation from [48, §IV.3] with κ trivial. The key point is that the $\Omega(B_K)$ parametrizing $\pi \in \Pi_{\mathrm{disc}}(\xi)$ also parametrizes conjugacy classes in an elliptic stable class by section 2.2.5. □

Now let $(H_\infty, \mathcal{H}, \eta, s)$ be an endoscopic quadruple of G_∞. Fix an elliptic maximal torus T and let and κ be the corresponding endoscopic character on Ω_G.

Trivial z-Extension case

We will first work out the formula for transfers in the case where $\mathcal{H} \cong {}^L H$ where we do not need a z-extension. To start,

Lemma 3.2.1.2. *Unless all elliptic tori G_∞ are transfers of elliptic torii of H_∞, transfers of pseudocoefficients can be taken to be* 0.

Proof. See lemma 3.2 in [74] or the computation of κ-orbital integrals on page 186 of [43]. □

Therefore, we can choose isomorphic maximal torii T_H and T of $H_\mathbb{C}$ and $G_\mathbb{C}$ respectively that are both elliptic over $\mathbb{R}$. The Weyl chambers of (H, T_H) are a coarser partition than those of (G, T) by lemma 2.2.5.1. Therefore, we can choose a positive Weyl chambers for H that contains a chosen one for G. Let B_H and B_G be the corresponding Borel subgorups. Let $\rho' = \rho_G - \rho_H$ be the half-sum of positive roots of G that are not roots of H.

The transfer of pseudocoefficients is worked out in [43, §7]. Special cases are worked out in terms of roots in [48, §IV.3]. For full generality when ρ' is not a character of T, we have to use a corrected transfer factor from [73, pg 396] as worked out in [17]. This involves an Ω_H-invariant $\mu^* = \mu^*_{G,H}$ such that $\mu^* - \rho'$ is a character of T. The μ^* is determined by the exact chosen isomorphism ${}^L H \to \mathcal{H}$. Finally, recall the endoscopic character $\kappa := \kappa_{G,H}$ on Ω_G defined in section 2.2.5.

Proposition 3.2.1.3 ([17, prop. 4.3.1])**.** *We can take*

$$(\varphi_{\pi_G(\lambda)})^H = \sum_{\omega_* \in \Omega_*} \kappa(\omega_*) \operatorname{sgn}(\omega^* \omega_0) \varphi_{\pi_H(\omega_* \lambda - \mu^*)}$$

where $\omega_0^{-1}\lambda$ is B-dominant and Ω_* is the set of representatives w of $\Omega_H \backslash \Omega_G$ such that $w\lambda$ is B_H-dominant.

As a sanity check, note that if $A_{G,\infty} \in \mathfrak{X}$, then ellipticity forces $A_{H,\infty} \in \mathfrak{X}_H$.

[17] explicitly computes the extension to hyperendoscopy: let $\Omega(G,H)$ be a set of representatives w of $\Omega_H\backslash\Omega_G$ such that $w\mu$ is B_H dominant for any μ that is B_G dominant. Reindexing $\omega_* = \omega_1\omega_0^{-1}$

$$(\varphi_{\pi_G(\lambda)})^H = \sum_{\omega_1\in\Omega(G,H)} \kappa(\omega_1\omega_0^{-1})\,\mathrm{sgn}(\omega_1)\varphi_{\pi_H(\omega_1\omega_0^{-1}\lambda-\mu^*)}.$$

Next, note that the Euler-Poincaré function φ_λ has the same stable orbital integrals as the pseudocoefficient $f_{\pi_H(\lambda+\rho_H)}$. Let $\mu = \omega_0^{-1}\lambda - \rho_G$ so that $\pi_G(\lambda)$ becomes $\pi_G(\mu,\omega_0)$. Then

Corollary 3.2.1.4. *We can take*

$$(\varphi_{\pi_G(\mu,\omega_0)})^H = \sum_{\omega_1\in\Omega(G,H)} \kappa(\omega_1\omega_0^{-1})\,\mathrm{sgn}(\omega_1)\eta^H_{\omega_1(\mu+\rho_G)-\rho_H-\mu^*}.$$

Next, since κ is $\Omega_\mathbb{R}$-right invariant,

$$\sum_{\omega_0\in\Omega(B_K)} \kappa(\omega_1\omega_0^{-1}) = \sum_{[\omega]\in\Omega_\mathbb{R}\backslash\Omega_\mathbb{C}} \kappa(\omega_1\omega^{-1}) = \sum_{[\omega]\in\Omega_\mathbb{C}/\Omega_\mathbb{R}} \kappa(\omega_1\omega) = \sum_{[\omega]\in\Omega_\mathbb{C}/\Omega_\mathbb{R}} \kappa(\omega),$$

where it does not matter which representatives ω we choose. Therefore, averaging over $\omega_0 \in \Omega(B_K)$,

Corollary 3.2.1.5 (see [17, prop. 4.3.2])**.** *We can take*

$$(\eta_\mu)^H = \bar\kappa \sum_{\omega_1\in\Omega(G,H)} \mathrm{sgn}(\omega_1)\eta_{\omega_1(\mu+\rho_G)-\rho_H-\mu^*}$$

where $\bar\kappa = \bar\kappa_{G,H}$ is the average value of κ over $\Omega_\mathbb{C}/\Omega_\mathbb{R}$.

General case

For $\mathcal{H} \not\cong {}^LH$, we use the trick in section 2.2.3. Let $\varphi : (G_1)_\infty \to G_\infty$ be the surjective map coming from the z-extension $G_1 \to G$: if f is a function on G_∞, we choose $f^{H_1} = (f\circ\phi)^{H_1}$.

Given elliptic torii T_{G_1} and T_{H_1} as before, we can also get elliptic torus T_G by taking images under the z-extensions. The function $\varphi : (G_1)_\infty \to G_\infty$ gives a map $\phi^* : X^*(G_\infty, T_G) \hookrightarrow X^*((G_1)_\infty, T_{G_1})$. Then $f_\pi(\lambda)\circ\phi = f_{\pi(\phi^*\lambda)}$ so we can still use the above formulas in the general case as long as we treat λ as an element of $X^*(G_1, T_{G_1})$.

Note that the character λ_{H_1} shows up through the weight μ^*—each may be used to compute the other (not that we've explicitly described either here).

Hyperendoscopic Transfers

To simplify notation, for any weight μ of a group G, endoscopic group H, and $\omega \in \Omega(G,H)$ as before, let

$$T_{G,H}(\mu,\omega) = \omega(\mu+\rho_G) - \rho_H - \mu^*_{G,H}.$$

As in the previous section, we interpret μ as an character of G_1 corresponding to the chosen z-extension H_1.

For any hyperendoscopic path $\mathcal{H} = (H_i)_{0\leq i\leq n}$, let

$$\Omega(\mathcal{H}) = \prod_{i=1}^{n} \Omega(H_{i-1}, H_i) \qquad \bar{\kappa}_{\mathcal{H}} = \prod_{i=2}^{n} \bar{\kappa}_{H_{i-1},H_i}.$$

For $\omega = (\omega_i)_{i\leq i\leq n} \in \Omega(\mathcal{H})$ let

$$\epsilon(\omega) = \prod_{i=1}^{n} \epsilon(\omega_i)$$

and let

$$T_{\mathcal{H}}(\mu, \omega) = T_{H_{n-1},H_n}(\cdots T_{G,H_1}(\mu_1, \omega_1)\cdots, \omega_n)$$

be the composition of all the T_{H_{i-1},H_i}. Inductively applying propositions 3.2.1.3 and 3.2.1.5 while keeping in mind section 3.2.1 then gives:

Proposition 3.2.1.6 (see [17, prop. 4.4.2])**.** *We can take*

$$(\varphi_{\pi_G(\mu,\omega_0)})^{\mathcal{H}} = \bar{\kappa}_{\mathcal{H}} \sum_{\omega\in\Omega_{\mathcal{H}}} \kappa_{G,H_1}(\omega_1\omega_0^{-1}) \operatorname{sgn}(\omega)\eta_{T_{\mathcal{H}}(\mu,\omega)}$$

with the terms defined as in the above paragraph.

Note that all the coefficients in the sum have norm 1 and define $\Xi_{\mu,\mathcal{H}}$ to be the set of $T_{\mathcal{H}}(\mu, \omega)$ for $\omega \in \Omega(\mathcal{H})$.

3.2.2 Bounds on Archimedean Transfers

Here are few lemmas on the terms that appear in proposition 3.2.1.6. For μ a weight of G define:

- $m(\mu) = m_G(\mu) = \min_{\alpha\in\Phi^+(G)}\langle\alpha, \mu + \rho_G\rangle$,
- $n(\mu) = n_G(\mu) = \min_{\alpha\in\Phi^+(G)}\langle\alpha, \mu\rangle$,
- $\dim \mu = \dim_G(\mu)$ is the dimension of the finite dimensional representation with highest weight μ.

Lemma 3.2.2.1. *If μ is a weight of G and $\mathcal{H}$ as before, then for all $\mu' \in \Xi_{\mu,\mathcal{H}}$, $n_G(\mu') \geq n_{\mathcal{H}}(\mu)$. In particular, μ' is regular if μ is.*

Proof. In the situation where H is just an endoscopic group, consider $\omega \in \Omega_G$ such that $\mu' = \omega(\mu + \rho_G) - \rho_H - \mu^* \in \Xi_{\mu,H}$. Consider $\alpha \in \Phi^+(H)$. Since μ^* is invariant under Ω_H, $\langle\mu^*, \alpha\rangle = 0$ so

$$\langle\mu', \alpha\rangle = \langle\omega\mu, \alpha\rangle + \langle\omega\rho_G - \rho_H, \alpha\rangle.$$

Next, ρ_G is the sum of the fundamental weights so it is a regular weight. This implies that $\omega\rho_G$ is too. Therefore, for all $\beta \in \Phi^+(G)$, $\beta^\vee(\omega\rho_G) \in \mathbb{Z} \setminus \{0\}$. In particular, since $\omega\rho_G$ is B_H-dominant, for $\alpha \in \Phi^+(H)$, $\alpha^\vee(\omega\rho_G) \geq 1$. If α is in addition simple, we can compute

$$\alpha^\vee(\omega\rho_G - \rho_H) \geq 1 - \alpha^\vee(\rho_H) = 0,$$

so $\omega\rho_G - \rho_H$ is B_H-dominant. This gives

$$\langle \mu', \alpha \rangle \geq \langle \omega\mu, \alpha \rangle.$$

To finish this one-step case,

$$n_H(\mu') = \min_{\alpha \in \Phi^+(H)} \langle \mu', \alpha \rangle \geq \min_{\alpha \in \Phi^+(H)} \langle \omega\mu, \alpha \rangle = \min_{\alpha \in \Phi^+(H)} \langle \mu, \omega^{-1}\alpha \rangle.$$

All the terms in the last two minimums have to be positive. However, μ is B_G-dominant so this means the $\omega^{-1}\alpha$ are all in $\Phi^+(G)$ giving

$$n_H(\mu') \geq \min_{\alpha \in \Phi^+(G)} \langle \mu, \alpha \rangle = n_G(\mu).$$

Finally, for an arbitrary endoscopic path, inductively continue this argument through each step. □

Lemma 3.2.2.2. *If μ is a weight of G and $\mathcal{H}$ as before, then for all $\mu' \in \Xi_{\mu,\mathcal{H}}$*

$$\frac{\dim_{\mathcal{H}}(\mu')}{\dim_G(\mu)} = O(m_G(\mu)^{-1})$$

with the implied constant only depending on G and $\mathcal{H}$.

Proof. This follows from the Weyl character formula. If H is just an endoscopic group, let $\mu' = \omega(\mu + \rho_G) - \rho_H - \mu^*$ for appropriate $\omega \in \Omega_G$. Using that μ^* pairs to zero with any root of H,

$$\frac{\dim_H(\mu')}{\dim_G(\mu)} = \frac{\prod_{\alpha \in \Phi^+(G)} \langle \alpha, \rho_H \rangle}{\prod_{\alpha \in \Phi^+(H)} \langle \alpha, \rho_G \rangle} \frac{\prod_{\alpha \in \Phi^+(H)} (\langle \alpha, \omega\mu \rangle + \langle \alpha, \omega\rho_G \rangle)}{\prod_{\alpha \in \Phi^+(G)} (\langle \alpha, \mu \rangle + \langle \alpha, \rho_G \rangle)}.$$

The first fraction is a constant depending only on G and H. The second terms in the products in the second fraction are also. A priori, the $\langle \alpha, \omega\mu \rangle = \langle \omega^{-1}\alpha, \mu \rangle$ are a subset of the $\langle \pm\beta, \mu \rangle$ for $\beta \in \Phi^+(G)$. However, since they all have to be positive since $\omega\mu$ is B_H-dominant, they are actually a subset of the $\langle \beta, \mu \rangle$. Denote by A the subset of such β. Then

$$\frac{\dim_H(\mu')}{\dim_G(\mu)} = C \frac{\prod_{\alpha \in A} (\langle \alpha, \mu \rangle + O(1))}{\prod_{\alpha \in \Phi^+(G)} (\langle \alpha, \mu \rangle + O(1))} = O\left(\prod_{\alpha \in \Phi^+(G) \setminus A} \langle \alpha, \mu \rangle^{-1} \right)$$

using that the pairings are bounded below by a constant. Bounding the pairings again by $m_G(\mu)$, this is $O(m_G(\mu)^{|\Phi^+(H)| - |\Phi^+(G)|})$. Finally, since endoscopic groups have smaller rank, they do not have the same root data as the original group so this difference has to be negative.

After a quick check that the $m_{\mathcal{H}_i}(\mu') = O(m_G(\mu))$, inducting on this argument for each step of the hyperendoscopic path $\mathcal{H}$ finishes the proof . □

3.2.3 Truncated Hecke algebras

We now move on to the unramified finite places. Fix a place v at which G_v is quasisplit. Since we are only working at v, for this subsection G will always mean G_v to simplify notation.

Choose (B,T) to be a Borel and maximal torus defined over F_v. By G being quasisplit, all such choices are conjugate and T automatically contains a maximal split torus A. Furthermore, Ω_F can be identified with the fixed points Ω^{W_F} and therefore the Weyl group of the relative root system of rational roots in $X^*(A)$. Let K be a hyperspecial subgroup from a hyperspecial point in the apartment corresponding to A.

Eventually, we will evaluate $I_{\mathrm{geom}}(f)$ up to some error bounds which depend on how big the support of the finite part of f is. To precisely measure this size, we slightly modify the notion of truncated Hecke algebras as in [79, §2].

Recall then that the elements $\tau^G_\lambda = \mathbf{1}_{K\lambda(\varpi)K}$ for a chosen uniformizer ϖ and $\lambda \in X_*(A)^+$ generate $\mathscr{H}(G,K)$. Pick a basis $\mathcal{B}$ for the $X_*(A)$ and define norm

$$\|\lambda\|_{\mathcal{B}} = \max_{\omega\in\Omega}(\text{biggest } |\mathcal{B}\text{-coordinate of } \omega\lambda|)$$

for $\lambda \in X_*(A)$. Define truncated Hecke algebra

$$\mathscr{H}(G,K)^{\leq\kappa,\mathcal{B}} = \langle \tau^G_\lambda : \|\lambda\|_{\mathcal{B}} \leq \kappa\rangle.$$

It turns out (see [79, §2]) that for any two $\mathcal{B},\mathcal{B}'$, $\|\lambda\|_{\mathcal{B}} = \Theta(\|\lambda\|_{\mathcal{B}'})$. All the bounds we use will depend on κ only up to an unspecified constant. Therefore we can suppress the $\mathcal{B}$.

There is also a truncated Hecke algebra with central character data: choose an $(\mathfrak{X},\chi)$ such that χ is unramified. In the case we care about, $\mathfrak{X}$ is a subtorus of Z_G. Let $A_{\mathfrak{X}}$ be its split part. Define

$$\mathscr{H}(G,K,\chi)^{\leq\kappa,\mathcal{B}} = \langle \tau^G_\lambda : \|\lambda+\zeta\|_{\mathcal{B}} \leq \kappa \text{ for some } \zeta \in X_*(A_{\mathfrak{X}})\rangle \cap \mathscr{H}(G,K,\chi).$$

Note that for $x \in K\lambda(\varpi)K$ and $z \in \mathfrak{X}$, then there is $k \in K$ and $\zeta \in X_*(A_{\mathfrak{X}})$ such that $z = \zeta(\varpi)k$, implying $zx \in K(\lambda+\zeta)(\varpi)K$. Therefore, this is a reasonable, non-empty intersection.

A useful projection

Working with the basis of τ^G_λ, it is sometimes useful to consider the following maps. First, there is a map $Q : \chi \mapsto \sum_{\omega\in\Omega_G} \omega\chi$ on $X_*(T)$. This sends every coroot of G to 0. Normalizing Q by $|\Omega_G|^{-1}$ gives a projection P on $X_*(T)\otimes\mathbb{Q}$. Note that this projection is onto $X_*(Z_G)\otimes\mathbb{Q}$ since Weyl-invariant cocharacters are the same as central cocharacters (they pair to zero with every root).

Recall $X_*(A)$ embeds into $X_*(T)$ as the W_F invariants.

Lemma 3.2.3.1. *Let $\lambda \in X_*(A)$. Then, $Q\lambda \in X_*(A)$.*

Proof. It suffices to show this for $P\lambda$. The map P is an orthogonal projection onto W_F-invariant $X_*(Z_G) \otimes \mathbb{Q}$ with respect to a W_F-invariant inner product. Therefore it commutes with W_F and sends W_F invariants to W_F invariants. $\square$

Therefore, we can consider Q and P as maps of $X_*(A)$ and $X_*(A) \otimes \mathbb{Q}$ respectively. The kernel of P is the span of the roots of G so the kernel in $X_*(A) \otimes \mathbb{Q}$ is V_F where V_F is the span of $\{\alpha^\vee | \alpha \in \Phi_F^*\}$ inside $X_*(A) \otimes \mathbb{Q}$.

3.2.4 Formulas for Unramified Non-Archimedean Transfers

Fix a place v at which G_v is quasisplit. Since we are only working at v, for this subsection G will always mean G_v to simplify notation.

The Fundamental Lemma

The fundamental lemma allows for computation of unramified non-Archimedean transfers (the lemma is actually enough to show the existence of all non-Archimedean transfers). We will eventually use this to control which $\mathscr{H}(H_v, K_{H,v}, \chi_{H,v})^{\leq \kappa}$ transfers end up being in. Use the notation T, A, and K analogous to the last section.

As explained in [79, §2.2], the Satake transform gives two isomorphisms

$$\varphi_G : \mathscr{H}(G, K) \to \mathscr{H}(A, A \cap K)^{\Omega_F} \to \mathbb{C}[X_*(A)]^{\Omega_F}.$$

We mention that this implies:

Lemma 3.2.4.1. *The space $\widehat{G}^{\mathrm{ur}}$ can be identified with $\Omega_F \backslash \widehat{A}$. The tempered part is $\Omega_F \backslash \widehat{A}_c$ where $\widehat{A}_c$ is the maximum compact torus in $\widehat{A}$.*

Proof. A result in representation theory of p-adic groups says that unramified representations of G are the same as characters of $\mathscr{H}(G, K)$ and therefore characters on $\mathbb{C}[X_*(A)]^{\Omega_F}$ (see [10, §10]). These are the same as elements of $\Omega_F \backslash \widehat{A}$. Tempered representations need to correspond to tempered characters of $\mathscr{H}(G, K)$ which forces the element to be in $\widehat{A}_c$. $\square$

There are more implications: let ${}^L G^{\mathrm{ur}} := {}^L G^{\mathrm{ur}}$ be defined like ${}^L G$ except that the semidirect product is only with $W_{F_v}^{\mathrm{ur}}$. Define $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$ to be the algebra of trace characters of representations of ${}^L G^{\mathrm{ur}}$ restricted to $(\widehat{G} \rtimes \mathrm{Frob})_{\mathrm{ss}}$. There is a third isomorphism

$$\mathcal{T} : \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \to \mathbb{C}[X_*(A)]^{\Omega_F}$$

that takes a representation π to a function on $\widehat{T}$ given by $a \mapsto \mathrm{tr}_\pi(a \rtimes \mathrm{Frob})$. This function can be shown to factor through $\widehat{A}$ (see [10, prop 6.7]).

If we have a map $\eta : {}^L H^{\mathrm{ur}} \hookrightarrow {}^L G^{\mathrm{ur}}$, we get a pullback map $b_\eta : \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \to \mathbb{C}[\mathrm{ch}({}^L H^{\mathrm{ur}})]$. We pick the Whittaker normalization for transfer factors and choose the measures μ^{can} on H and G that give K and K_H volume 1.

Theorem 3.2.4.2 (Full Fundamental Lemma)**.** *Let G be an unramified reductive group over the local field F_v. Let $(H, \mathcal{H}, \eta, s)$ be an elliptic endoscopic quadruple for G such that $\mathcal{H} \cong {}^L H$. Then, for $f \in \mathscr{H}(G, K)$ we can take*

$$f^H = \begin{cases} \varphi_H^{-1} \circ b_\eta \circ \varphi_G(f) & H \text{ unramified} \\ 0 & H \text{ ramified} \end{cases}.$$

Here we recall that if H and G are unramified, then the embedding $\mathcal{H} \hookrightarrow {}^L G$ descends to one $\mathcal{H}^{\mathrm{ur}} \hookrightarrow {}^L G^{\mathrm{ur}}$. In addition, H being unramified allows us to pick an $\eta : {}^L H \xrightarrow{\sim} \mathcal{H}$ that also descends to unramfied L-groups. The pullback b_η is defined through such an η.

Proof. The statements defining η come from the construction of $\mathcal{H}$ and the proof of 7.2A in [46].

The ramified H case is by [45, §7.5]. Otherwise, it is reduced in [34] to proving the result for just $\mathbf{1}_K$. This was further reduced to a fundamental lemma for Lie algebras in [84] which was finally proven in [62]. [34] removes a restriction on the size of the residue field of F_v. □

Representations of ${}^L G^{\mathrm{ur}}$

To compute with the fundamental lemma, we need to describe representations of ${}^L G^{\mathrm{ur}}$. As a start:

Lemma 3.2.4.3. *Let π be a representation of ${}^L T^{\mathrm{ur}}$. Then there exists λ a character of $\widehat{T}$ up to W_F^{ur}-action and $\alpha \in \mathbb{C}^\times$ such that $\pi = \chi_{\lambda,\alpha}$ where*

$$\chi_{\lambda,\alpha} = \bigoplus_{\gamma \in W_F / \operatorname{Stab} \lambda} V_{\gamma\lambda}$$

and each V_μ is a 1-dimensional space with a chosen generator v_μ on which $\widehat{T}$ acts through μ. Let $\operatorname{Stab}\lambda$ be generated by $\operatorname{Frob}^{i(\lambda)}$. Then $\operatorname{Frob}^{i(\lambda)}$ acts by $v_\lambda \mapsto \alpha v_\lambda$. Finally, $\operatorname{Frob}(v_\lambda) = \beta_\lambda v_{\operatorname{Frob}(\lambda)}$ for some constants β_λ. (Note that by scaling v_μ, without loss of generality all the β_λ are 1 except one that is α).

Proof. Decompose π into eigenspaces V_μ for $\widehat{T}$. We can compute that, $\gamma V_\mu \subseteq V_{\gamma\mu}$ for $\gamma \in W_F^{\mathrm{ur}}$. Let γ_0 generate $\operatorname{Stab}\lambda$ for some non-empty V_λ. Then γ_0 acts as an element of $\mathrm{GL}(V_\lambda)$. Let v_λ be a chosen eigenvector of γ_0 with eigenvalue α. The vectors v_λ generates a $\chi_{\lambda,\alpha}$ inside π. □

Beware that this parametrization depends on the splitting $W_F \hookrightarrow {}^L T$. Next

Proposition 3.2.4.4. *Representations of ${}^L G^{\mathrm{ur}}$ are parametrized by $\chi_{\lambda,\alpha}$ of ${}^L T^{\mathrm{ur}}$ for α dominant. Call the one corresponding to $\chi_{\lambda,\alpha}$ by $\pi_{\lambda,\alpha} := \pi_{\lambda,\alpha}^{{}^L G}$.*

Proof. This is by [42, pg 375-376] . We have that ${}^L T^{\mathrm{ur}}$ is the same as H^+ in the reference because the action of W_F fixes the Borel B used to define ${}^L G$. The construction is similar to that for connected complex Lie groups: $\pi_{\lambda,\alpha}$ forms a highest weight space on which the actions of the root subgroups of $\widehat{G}$ are determined. Together $\widehat{G}$ and ${}^L T^{\mathrm{ur}}$ generate ${}^L G^{\mathrm{ur}}$. □

In fact, if $\pi_\lambda^{\widehat{G}}$ is the representation corresponding to highest weight λ of $\widehat{G}$, then each of the $V_{\gamma\lambda} \subseteq V_{\lambda,\alpha}$ generates a copy of $\pi_{\gamma\lambda}^{\widehat{G}}$ under the action of $\widehat{G}$. The representation $\pi_{\lambda,\alpha}|_{\widehat{G}}$ therefore decomposes as a direct sum of the $\pi_{\gamma\lambda}^{\widehat{G}}$ and any $\gamma \in W_F$ sends $\pi_\mu^{\widehat{G}}$ to $\pi_{\gamma\mu}^{\widehat{G}}$. The exact description of this map in complicated but can be computed by the following trick: For any $\gamma \in \Gamma$, the μ coefficient of tr_π restricted to $\widehat{T} \rtimes \gamma$ is the trace of $1 \rtimes \gamma$ acting on the μ-weight space $V_\mu^{\lambda,\alpha}$ of $\pi_{\lambda,\alpha}$. This trace can be computed by Kostant's character formula [42, thm 7.5].

As an easier way to think about this parametrization, let F_n be the splitting field for G. The groups $\mathrm{Gal}(F_n/F_v)$ and $\Omega_\mathbb{C}$ together generate a group C in automorphisms of the set of roots. Inside this, $\mathrm{Gal}(F_n/F_v)$ is the stabilizer of the positive Weyl chamber and $\Omega_\mathbb{C}$ acts simply on the Weyl chambers so $\mathrm{Gal}(F_n/F_v) \cap \Omega_\mathbb{C} = 1$. In addition, $\Omega_\mathbb{C}$ is normal since T is fixed by Galois. Therefore, $C = \Omega_\mathbb{C} \rtimes \mathrm{Gal}(F_n/F_v)$. The λ parametrizing $\pi_{\lambda,\alpha}$ can be thought of as a C-orbit. This decomposes into $\Omega_\mathbb{C}$ orbits representing the constituent $\pi_{\gamma\lambda}^{\widehat{G}}$.

Some Bases

We also need to describe some bases of the various spaces.

If ϖ is a chosen uniformizer for $\mathcal{O}_F$ and $X_*(A)^+$ a chosen Weyl chamber, then the functions

$$\tau_\lambda^G = \mathbf{1}_{K\lambda(\varpi)K} \text{ for all } \lambda \in X_*(A)^+$$

form a basis for $\mathscr{H}(G,K)$ (the corresponding double cosets partition G by the Cartan decomposition).

$\mathbb{C}[X_*(A)]^{\Omega_F}$ contains functions

$$\chi_\lambda = \frac{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_F(\sigma)\sigma(\lambda\cdot\rho)}{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_F(\sigma)\sigma(\rho)} \in \mathbb{C}[X_*(A)]^{\Omega_F}$$

for $\lambda \in X_*(A)^+$. We write the addition in $X_*(A)$ multiplicatively for clarity. Here, $\rho = \rho_F$ is the half-sum of the positive roots of $\widehat{G}$ over F_v, which is the same as the half-sum of all positive roots since rational roots are sums over orbits of roots. We recall that Ω_F is the same as the Weyl group for the relative root system of rational roots of G_v by quasiplitness (See [10, §6.1]). The sgn_F here are -1 to the power of the number of positive rational roots sent to negative roots. If the rational roots form a reduced root system, this is just the standard sgn on Ω_F.

If the relative root system is reduced, these are the standard characters from Weyl's character formula and are studied in [37]. In the non-reduced case, these are the twisted characters from [12, thm 1.4.1] or [32, thm 7.9]. Either way, χ_λ for dominant weighs λ form a basis for $\mathbb{C}[X_*(A)]^{\Omega_F}$.

Finally,

Lemma 3.2.4.5.

$$\mathcal{T}(\pi_{\lambda,\alpha}) = \begin{cases} \alpha\chi_\lambda & \lambda \in X_*(A) \\ 0 & \textit{else} \end{cases}.$$

Proof. This is just stated in the proof of [79] lemma 2.1. We give details here since there seems to be a minor mistake (that is irrelevant to all the work there and here) when λ is not in $X_*(A)$. This is also proven as [12, thm 1.4.1] and as [32, thm 7.9] in a slightly different form.

We use Kostant's character formula [42, thm 7.5]. Using the notation there, $a = t \rtimes \mathrm{Frob}$ for some $t \in \widehat{T}$ and W_a is the W_F^{ur} invariants in $\Omega_{\mathbb{C}}$ which is Ω_F. Also, let $\Phi_\sigma = \Phi_{\mathbb{C}}^+ \cap \sigma(-\Phi_{\mathbb{C}}^+)$ for $\sigma \in \Omega_{\mathbb{C}}$ where $\Phi_{\mathbb{C}}^+$ is the set of positive roots. Since Frob preserves a pinning, it acts by a permutation on some diagonal basis of $\bigoplus_{\phi \in \Phi_\sigma} \mathfrak{g}_{-\phi}$. Therefore, the determinant of the action of a is

$$\chi_1^\sigma(a) = \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \prod_{\varphi \in \Phi_\sigma} \varphi^{-1}(t).$$

In addition $\chi_1^\delta(a)$ for δ the representation of ${}^L T$ parametrized by (λ, α) is $\alpha\lambda(t)$ if λ is fixed by Frob and 0 otherwise (the 0 otherwise case is what is missing in [79]). By a [50, pg 15], we can find representations of $\sigma \in W_a$ fixed by Frob so we get that $\chi_\sigma^\delta(a) = \alpha\sigma\lambda(t)$.

In total, the trace in the non-zero case is

$$\alpha \frac{\sum_{\sigma \in \Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\, \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \sigma\lambda(t) \prod_{\varphi \in \Phi_\sigma} \varphi^{-1}(t)}{\sum_{\sigma \in \Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\, \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \prod_{\varphi \in \Phi_\sigma} \varphi^{-1}(t)} = \alpha \frac{\rho(t)^{-1} \sum_{\sigma \in \Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\, \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \sigma\lambda(t) \sigma\rho(t)}{\rho(t)^{-1} \sum_{\sigma \in \Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\, \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \sigma\rho(t)}.$$

The $\mathrm{sgn}_{\mathbb{C}}$ here is the sign character for $\Omega_{\mathbb{C}}$: the number of all positive roots sent to negative roots. This differs from the sgn_F in the formula for χ_λ by a factor of $\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma})$ through an argument breaking up Φ_σ into Frob-orbits and noting that each rational root is a sum over an orbit. Therefore, we are done.

Note that the 0 case can be done more easily by thinking about the action in block matrix form with respect to the subspaces $\pi_{\gamma\lambda}^{\widehat{G}}$ and noticing that all diagonal blocks are 0. ☐

The key consequence of this is that the $\mathcal{T}(\pi_{\lambda,1})$ for $\lambda \in X_*(A)$ form a basis for $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$.

3.2.5 Bounds on Unramified Transfers

Trivial z-extension case

As in the Archimedean case, we consider the trivial z-extension case first.

Recall the notation for various bases of spaces related to the Satake isomorphism. From

[29] and [37] (again, see [32, §7] or [12, §1] for the non-split case), we can write

$$\varphi_G(\tau_\lambda^G) = \chi_\lambda + \sum_{\substack{\mu \in X^*(\widehat{A})^+ \\ 0 \le \mu < \lambda}} b_\lambda^G(\mu)\chi_\mu^G,$$

$$\varphi_H^{-1}(\chi_\nu^H) = q^{-\langle \nu, \rho_H \rangle}\tau_\nu^H + \sum_{\substack{\xi \in X^*(\widehat{A}_H)^+ \\ 0 \le \xi < \nu}} q^{-\langle \xi, \rho_H \rangle} d_\nu^H(\xi)\tau_\xi^H.$$

for some constants b and d. Here $\mu \le \lambda$ means that there is some non-negative integer linear combination of roots $\alpha^\vee$ for $\alpha \in \Phi^+$ equal to $\lambda - \mu$.

Lemma 3.2.5.1. *The $d_\lambda^G(\mu)$ and $q^{-\langle \lambda, \rho_H \rangle} b_\lambda^G(\mu)$ are bounded by a polynomial in the norm $\|\mu\|$ that is independent of q and λ.*

Proof. First, let's show this for $d_\lambda^G(\mu)$. By the above, we can ignore the $\lambda = \mu$ case. Otherwise, we apply [79, lem 2.2]. There is a small issue here: this lemma depends on the main result of [37] which only works when the root system is reduced. Nevertheless, [32, thm 7.10] and [12, thm 1.9.1] provide an appropriate substitute in the non-reduced case.

[79, lem 2.2] bounds $d_\lambda^G(\mu)$ by $|\Omega_{G,F_v}|$ times the size of the set of tuples $(c_{\alpha^\vee})$ for α a positive root such that $\sum_{\alpha^\vee} c_{\alpha^\vee}\alpha^\vee = \mu - \lambda$ (since both μ and λ are in the positive Weyl chamber, the max in the lemma is achieved for the trivial element of the Weyl group). Looking at the coordinate of μ in the direction used to define positivity, every $\alpha^\vee$ is positive in this coordinate, so some weighted sum of the $c_{\alpha^\vee}$ is bounded. This implies that the number of tuples is only polynomial in this coordinate of μ. The result follows.

For $b_\lambda^G(\mu)$, note that the $q^{-\langle \beta, \rho_H \rangle} d_\alpha^G(\beta)$ for $\alpha, \beta \le \lambda$ form an upper-triangular matrix with dimension polynomial in the size of λ. Then, $b_\beta^G(\alpha)$ are coordinates of the inverse of this matrix. Making a change of variables, the $q^{-\langle \beta, \rho_H \rangle} b_\beta^G(\alpha)$ are the coordinates of the inverse of the matrix with coordinates $d_\alpha^G(\beta)$ so these are bounded by a polynomial in $\|\mu\|$ by solving through back substitution. □

It remains to understand the map b_η. This is computed exactly in terms of certain partition functions in [12, §2.3], but we only need bounds so we do something slightly different and much simpler. For $\mu \in X_*(A)$, define coefficients $c_\mu(\nu)$ by

$$\pi_\mu^{\widehat{G}}|_{\widehat{H}} = \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \le \nu \le \mu}} c_\mu(\nu)\pi_\nu^{\widehat{H}}.$$

The $c_\nu(\mu)$ are in particular bounded by the dimension of $\pi_\mu^{\widehat{G}}$ so they are polynomial in the size of μ by the Weyl character formula.

Proposition 3.2.5.2. *As elements of $\mathbb{C}[\mathrm{ch}({}^L H^{\mathrm{ur}})]$,*

$$b_\eta(\pi_{\mu,1}^{{}^L G}) = \bigoplus_{\substack{\nu \in X_*(A_H)^+ \\ 0 \le \nu \le \mu}} \alpha_\mu(\nu)c_\mu(\nu)\pi_{\nu,1}^{{}^L H},$$

where A_H is the maximal split torus of H contained in some maximal T_H contained in a rational Borel B_H and we consider $\mu \in X_*(T_H) = X_*(T)$ as dominant element by taking its Weyl-translate in the positive Weyl chamber.

For notational convenience, let $\Gamma = W_{F_v}^{\mathrm{ur}}$. There exists $t_\eta \in (Z_{\widehat{G}}^{\Gamma})^0$ depending only on η such that the constants $\alpha_\mu(\nu)$ satisfy two properties:

- $|\alpha_\mu(\nu)| \le |\nu(t_\eta)|$.
- *Let Y_G be the maximal split torus in Z_G^0. If $\zeta \in X_*(Y_G)$, then $\alpha_{\mu+\zeta}(\nu+\zeta) = \zeta(t_\eta)\alpha_\mu(\nu)$.*

Before starting the proof, note that all such T_H are isomorphic and that the map $X_*(T_H) \to X_*(T)$ is unique up to Weyl element. Therefore, this is well defined.

Proof. Decomposition: To avoid confusion, $\Gamma_{\widehat{G}}$ is Γ acting on $\widehat{G}$ and visa versa for $\widehat{H}$ when it is not clear from context. First,

$$b_\eta(\pi^{{}^LG}_{\mu,1})|_{\widehat{H}} = (\pi^{{}^LG}_{\mu,1}|_{\widehat{G}})|_{\widehat{H}} = \bigoplus_{\gamma} \pi^{\widehat{G}}_{\gamma\mu}|_{\widehat{H}} = \bigoplus_{\gamma} \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \le \nu \le \mu}} c_\mu(\nu)\pi^{\widehat{H}}_{\gamma_G \nu}$$

where the $\gamma\mu$ index the $\Gamma_{\widehat{G}}$-orbit of μ in $X_*(T)$. Note that $c_\mu(\nu)$ is constant on $\Gamma_{\widehat{G}}$ orbits and $\Omega_{\mathbb{C}}(\widehat{G})$ orbits.

The $\Gamma_{\widehat{H}}$-action is the composition of the action of $\Gamma_{\widehat{G}}$ with conjugation by elements of $N_{\widehat{G}}(T)$, so since G is quasisplit, $\Gamma_{\widehat{H}}$ acts on $\widehat{T}_H$ through a subgroup W' with $\mathrm{Gal}(F_n/F) \subseteq W' \subseteq C_H \subseteq C_G$ (recall notation $C_G = \Gamma \rtimes \Omega_{\widehat{G},\mathbb{C}}$). This implies that $c_\mu(\nu)$ is constant on $\Gamma_{\widehat{H}}$-orbits.

Therefore, the sum over such an orbit of the $c_\mu(\nu)\pi^{\widehat{H}}_\nu$ decomposes into $c_\mu(\nu)$ different $\pi^{{}^LH}_{\nu,\alpha_{i,\mu}}$ for possibly different $\alpha_{i,\mu}$. In total

$$b_\eta(\pi^{{}^LG}_{\mu,1}) = \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \le \nu \le \mu}} \bigoplus_{i=1}^{c_\mu(\nu)} \pi^{{}^LH}_{\nu,\alpha_{i,\mu}(\nu)} = \bigoplus_{\substack{\nu \in X_*(A_H)^+ \\ 0 \le \nu \le \mu}} \left(\sum_{i=1}^{c_\mu(\nu)} \alpha_{i,\mu}(\nu)\right) \pi^{{}^LH}_{\nu,1}$$

as elements of $\mathbb{C}[\mathrm{ch}({}^LH^{\mathrm{ur}})]$ and for some $\alpha_{i,\mu}(\nu) \in \mathbb{C}^\times$. Let $\alpha_\mu(\nu)$ be the average of the $\alpha_{i,\mu}(\nu)$.

Properties of $\alpha_\mu(\nu)$: It remains to show the two properties of $\alpha_\mu(\nu)$. Since all (B,T)-pairs in $\widehat{G}$ are conjugate, without loss of generality take an inner automorphism of ${}^LG$ so that $(\widehat{B}_H, \widehat{T}_H)$ is the pullback of $(\widehat{B}, \widehat{T})$. The map η determines a cocycle $c_\gamma \in C^1(\Gamma_{\widehat{G}}, \widehat{G})$ by $\eta(1 \rtimes \gamma) = c_\gamma \rtimes \gamma$. We then have that $\alpha_i(\nu)$ is the factor by which $c_{\mathrm{Frob}} \rtimes \mathrm{Frob}$ acts on the highest weight space V of the ith $\pi^{\widehat{H}}_\nu$.

There exists n such that the conjugation action of $(c_{\mathrm{Frob}} \rtimes \mathrm{Frob})^n$ on $X^*(\widehat{T})$ is trivial. Since this action also fixes a pinning of H, we must have

$$(c_{\mathrm{Frob}} \rtimes \mathrm{Frob})^n = z_0 \rtimes \mathrm{Frob}^n$$

for some $z_0 \in Z_{\widehat{H}}$. By the lemma below, we know $1 \rtimes \mathrm{Frob}$ acts trivially on V. Therefore, $\alpha_{i,\mu}(\nu)^n = \nu(z_0)$.

Next, note that the $\Gamma_{\widehat{H}}$-action is generated by conjugation by $c_{\mathrm{Frob}} \rtimes \mathrm{Frob}$. This fixes z_0 so $z_0 \in Z_{\widehat{H}}^{\Gamma}$. We can without loss of generality make n bigger so that z_0 is trivial in the finite group $\pi_0(Z_{\widehat{H}}^{\Gamma})$—in other words, we may without loss of generality assume $z_0 \in (Z_{\widehat{H}}^{\Gamma})^0$. Then by ellipticity of H, $z_0 \in (Z_{\widehat{G}}^{\Gamma})^0$. Since this a complex torus, there then exists $t_\eta \in Z_{\widehat{G}}^0$ such that $t_\eta^n = z_0$, so taking nth roots, $|\alpha_{i,\mu}(\nu)| = |\nu(t_\eta)|$. Summing over i then produces the bound on the $\alpha_\mu(\nu)$.

To get the central character transformation, $\zeta \in X_*(Y_G)$ if and only if it is a Γ_G and Ω_G-invariant element of $X_*(T) = X^*(\widehat{T})$. Such characters lift to Γ-invariant characters of $\widehat{G}$ and therefore characters on ${}^L G$. For such ζ, $\pi_{\mu+\zeta,1} = \zeta \otimes \pi_{\mu,1}$ so

$$b_\eta(\pi_{\mu+\zeta,1}) = b_\eta(\zeta) \otimes b_\eta(\pi_{\mu,1}) = \zeta(c_{\mathrm{Frob}})\zeta|_{\widehat{H}} b_\eta(\pi_{\mu,1}).$$

Since $c_\mu(\nu)$ is 0 unless μ and ν have the same central character and since $c_{\mu+\zeta}(\nu+\zeta) = c_\mu(\nu)$, this implies that $\alpha_{\mu+\zeta}(\nu + \zeta) = \zeta(c_{\mathrm{Frob}})\alpha_\mu(\nu)$. Therefore, we are done if all the choices defining t_η above are such that t_η has the same image in $\widehat{G}_{\mathrm{ab}}$ as c_{Frob}. □

The lemma used in this proof follows:

Lemma 3.2.5.3. *Let V_ν for $\nu \in X_*(A)$ be a weight space for $\pi_{\mu,\alpha}^{{}^L G}$ for $\mu \in X_*(A)$. Then* $1 \rtimes \mathrm{Frob}$ *acts as multiplication by α on V_ν.*

Proof. For any $\gamma \in W_F^{\mathrm{ur}}$, the trace of γ acting on V_ν is the coefficient of ν in $\operatorname{tr} \pi_{\mu,\alpha}^{{}^L G}$ restricted to $\widehat{T} \rtimes \gamma$. Let n be the splitting degree of G. The same computation as lemma 3.2.4.5 gives that this is $\alpha^{ni+1} \dim V_\nu$ for any $\gamma = \mathrm{Frob}^{ni+1}$. The only representation of $W_F^{\mathrm{ur}} \cong \mathbb{Z}$ with these traces sends 1 to scaling by α. □

The element t_η defines a function χ_η^{-1} on G by $K\lambda(\varpi)K \mapsto \lambda(t_\eta)$ for $\lambda \in X_*(A)$. Since t_η is central, if Q is the map on $X_*(A)$ summing over Ω_G-orbits, this is constant on fibers of Q. In particular, since products of basis elements $\tau_\lambda^G \in \mathscr{H}(G,K)$ are a linear combination of $\tau_{\lambda'}^G$ for λ' in a single fiber, χ is a character of G. This is the character that corresponds to t_η considered as a Weyl-orbit in $\widehat{A}$ through the Satake isomorphism.

Furthermore, the relation $\alpha_{\mu+\zeta}(\nu+\zeta) = \zeta(t_\eta)\alpha_\mu(\nu)$ forces χ_η to be the character associated to η through transfer factors as in section 2.2.3. This all finally gives that the character on H determined by $K_H\lambda(\varpi)K_H \mapsto \lambda(t_\eta)$ for $\lambda \in X_*(A_H)$ is the same as the one from transfer factors.

In summary, we get

$$(\tau_\lambda^G)^H = \delta_H^G(\lambda)\tau_\lambda^H + \sum_{\substack{\xi \in X^*(\widehat{A}_H) \\ 0 \le \xi < \lambda}} a_\lambda(\xi)\tau_\xi^H,$$

where

$$a_\lambda(\xi) = \sum_{\substack{\mu \in X^*(\widehat{A}) \\ \nu \in X^*(\widehat{A}_H) \\ \xi \le_H \nu \le_H \mu \le_G \lambda}} \alpha_\mu(\nu) b^G_\lambda(\mu) c_\mu(\nu) q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi),$$

setting terms of the form $*_\mu(\mu) = 1$ here for ease of indexing. We also know that the $\alpha_\mu(\nu)$ can be bounded in terms of the character on H determined by η.

Going back to the global context, this finally allows us to compute:

Proposition 3.2.5.4. *Let G be a reductive group over a global field and $(H, \mathcal{H}, \eta, s)$ an endoscopic quadruple that has a trivial z-extension. Let S be a finite set of places v such that:*

- *G_v, H_v are unramified.*
- *$|k_v|$ does not divide $|\Omega_G|$.*

Let $\chi_{\eta,S}$ be the product of the characters $\chi_{\eta,v}$ on H_v for $v \in S$ determined by η.

If $f \in \mathscr{H}(G(F_S), K_S)^{\le \kappa}$ with $\|f\|_\infty \le 1$, we can take $f^H \in \mathscr{H}(H(F_S), K_S)^{\le \kappa}$ such that $\|\chi_{\eta,S} f^H_S\|_\infty = O(q_{S_1}^{E\kappa} \kappa^{C|S|})$ for constants C, E independent of f_S and q_S. In addition, E can be chosen uniformly over all G in endoscopic paths from a fixed G'.

Proof. Use the notation from the previous discussion. For $s \in S$, f_s is then a linear combination of some of τ^G_λ. If τ^G_λ has a τ^H_ξ component then $\lambda - \xi$ is in particular a non-negative sum of roots of G. The number of such λ is polynomial in κ. Therefore, if f^H_s is written as a linear combination of τ^H_ξ, the coefficient for τ^H_ξ is bounded by a sum of polynomially many $a_\lambda(\xi)$. Furthermore, all these ξ are smaller than λ.

Moving to what we are actually bounding, if t_η is as in the previous discussion, the corresponding coefficient in $\chi^{-1}_{\eta,s} f^H_s$ is bounded by a sum of polynomially many $\xi(t_\eta)^{-1} a_\lambda(\xi)$. For all $\alpha_\mu(\nu)$ appearing in the sum defining $a_\lambda(\xi)$,

$$|\xi(t_\eta)^{-1} \alpha_\mu(\nu)| \le |\xi(t_\eta)^{-1} \nu(t_\eta)| = 1$$

since ξ and ν have the same Ω_G-orbit sum. In particular, if we define

$$a'_\lambda(\xi) = \sum_{\substack{\mu \in X^*(\widehat{A}) \\ \nu \in X^*(\widehat{A}_H) \\ \xi \le_H \nu \le_H \mu \le_G \lambda}} b^G_\lambda(\mu) c_\mu(\nu) q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi),$$

then $|\xi(t_\eta)^{-1} a_\lambda(\xi)| \le |a'_\lambda(\xi)|$.

It remains to bound the polynomially many summands in $a'_\lambda(\xi)$. Bounding each of these terms, the $c_\mu(\nu)$ are polynomial in how big μ is. By lemma 3.2.5.1, the term

$$b^G_\lambda(\mu) q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi)$$

is a polynomial in the size of λ times a factor of $q^{-\langle \xi, \rho_H\rangle + \langle \lambda, \rho_G\rangle}$. Therefore, we roughly bound the entire product, $a_\lambda(\xi)$, by a polynomial in κ times a factor of $q^{-\langle \lambda, \rho_G\rangle}$

Finally, note that $\langle \lambda, \rho_G\rangle \leq \mathrm{rank}_{\mathrm{ss}}(G)\kappa$. Taking the product of f_s^H over $s \in S$ and setting $E = \mathrm{rank}_{\mathrm{ss}}(G)$ gives the result. □

Note that this lemma can be inductively applied through a hyperendoscopic path by letting χ at each step be the character defined from the hyperendoscopic path as in section 3.1.3.

General case

Starting as in the Archimedean case argument in section 3.2.1, consider z-pair (H_1, η_1) for H. The extension H_1 induces an extension G_1 such that H_1 is an endoscopic group for G_1 by proposition 2.2.2.2. If $\varphi : (G_1)_v \to G_v$ is the projection, we have that $f^{H_1} = (f \circ \varphi)^{H_1}$ for any H_1 on G (interpreted as before).

If H is ramified, then all κ-orbital integrals are still 0 so this transfer is 0.

If H is unramified, T can be pulled back to a maximal torus T_1 of G_1 and A can be pulled back to A_1. By lemma 2.2.2.1 the extending torus Z is without loss of generality unramified so G_1 is too. As explained in [45, §7], the reductive model of G corresponding to the chosen hyperspecial $K_{G,v}$ gives a reductive model of G_1 so we can find a hyperspecial $K_{G_1,v}$ that surjects onto $K_{G,v}$. The map φ induces $\varphi_* : X_*(A_1) \to X_*(A)$ so

$$\varphi(K_{G_1,v}\lambda(\varpi)K_{G_1,v}) = K_{G,v}\varphi_*\lambda(v)K_{G,v}.$$

Therefore,

$$\tau_\lambda^G \circ \varphi = \sum_{\lambda' \in \varphi_*^{-1}(\lambda)} \tau_{\lambda'}^{G_1}$$

and the transfer can be computed by the fundamental lemma.

We describe the transfer of τ_0^G as an example computation:

Lemma 3.2.5.5. *Use the notation above. Then we can take*

$$(\tau_0^G)^{H_1} = \sum_{\lambda \in X_*(A_Z)} \chi_{\eta_1}(\lambda(\varpi))\tau_\lambda^{H_1}.$$

Here A_Z is the split part of the extending torus Z and χ_{η_1} is the character on Z_{G_1} determined by η_1.

Finally, we get an extension of proposition 3.2.5.4: that transfers from $\mathscr{H}(G_v, K_v, \chi)$ land in $\mathscr{H}(H_v^1, K_{H,v}, \chi\chi_{\eta_1})$ with the same bound.

3.2.6 Controlling Endoscopic Groups Appearing

Lemma 3.2.6.1. *Let G be a reductive group over global field F that is cuspidal at infinity together with central character datum $(\mathfrak{X}, \chi)$ such that $\mathfrak{X}$ contains $A_{G,\infty}$. Let $f = \eta_\xi \otimes f^\infty$ be a function on $G(\mathbb{A})$ where η_ξ is some EP-function with central character matching χ. Let R be a finite set of places containing those on which f^∞ or G are ramified. Then there are a finite number of elliptic endoscopic quadruples $(H, \mathcal{H}, \eta, s)$ up to equivalence for which $I_{\mathrm{disc}}(f^{H_1}) \neq 0$ for (all) z-extensions H_1. For each such H_1:*

- *H_1 is cuspidal at infinity and $\mathfrak{X}_{H_1}$ contains $A_{H_1,\infty}$.*
- *f^{H_1} is unramified outside of R and H_1 can be chosen to be.*
- *χ_{H_1} is unramified outisde of R.*

Proof. If H_1 is not cuspidal at infinity, then $I_{\mathrm{disc}}(g) = 0$ for any g with infinite part that is a EP function by the previous section. By corollary 3.2.1.5 and lemma 3.2.1.2, f^H is either a linear combination of such functions or 0. As before, we remark that $\mathfrak{X}_{H_1} \supseteq A_{H_1,\infty}$ due to ellipticity.

If H is ramified outside of R, then by the full fundamental lemma together with the trick to compute transfers on z-extensions, $f^{H_1} = 0$. Otherwise, by lemma 2.2.2.1, H_1 can be chosen to be unramified outside R so f^{H_1} is unramified outside of R by the full fundamental lemma again. The group H_1 being unramified outside of R further implies that χ_{H_1} is too.

Finiteness of the sum is implicit in the stabilization of the trace formula. Repeating the argument here, note that the roots of $H_{\overline{K}}$ are a subset of those of $G_{\overline{K}}$. Therefore, there are a finite number of possibilities for $H_{\overline{K}}$ and the splitting field of H has degree $\leq \Omega_G$. Since the splitting field is also unramified outside of R, there are a finite number of choices for it. This leaves only a finite number of choices for H.

To get finitely many quadruples it then suffices to show there are finitely many choices for $s \in (Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F}$. For this, $Z_H^{W_F}/Z_G^{W_F}$ is finite by ellipticity and $Z_H^{W_F}$ having finitely many connected components. Therefore $(Z_{\widehat{H}}/Z_{\widehat{G}})^{W_F}$ is finite by finiteness of a cohomology group. □

Note that this lemma can be inductively applied through a hyperendoscopic path.

3.3 Simple Trace Formula with Central Character

3.3.1 Set-up

To apply the hyperendoscopy formula, we will need two generalizations of the simple trace formula: first, allowing central characters and second, allowing pseudocoefficients at infinite places on the spectral side. We use a slightly convoluted and indirect argument to avoid having to go into too many technicalities of Arthur's distributions $I(f, \gamma)$ and $I(f, \pi)$:

Fix central character datum $(\mathfrak{X}, \chi)$ and let χ_0 be the restriction of χ to $A_{G,\mathrm{rat}}$. We first define a variant of $I_{\mathrm{disc},\chi}$ that can be more easily related to I_{geom,χ_0}. Let $\mathfrak{X}_F = \mathfrak{X} \cap Z(F)$. There is a map

$$\mathscr{H}(G, \chi_0) \to \mathscr{H}(G, \chi) : f(g) \mapsto \bar{f}_\chi(g) := \int_{\mathfrak{X}/A_{G,\mathrm{rat}}} f(gz)\chi(z)\, dz.$$

Lemma 3.3.1.1. *$f \mapsto \bar{f}_\chi$ is surjective.*

Proof. Let $h \in \mathscr{H}(G, \chi)$. There exists compact $U \subseteq G(\mathbb{A})/A_{G,\mathrm{rat}}$ such that $U\mathfrak{X}$ contains the support of h. Let c be a cutoff function: compactly supported, continuous, non-negative real valued, and positive on U. Then the function

$$m(g) = \int_{\mathfrak{X}/A_{G,\mathrm{rat}}} c(gz)\, dz$$

is continuous and non-zero on the support of h. If we take $f = m^{-1}ch$, then $\bar{f}_\chi = h$. $\square$

We follow a strategy from [40]. For any $\star \in \{\mathrm{geom}, \mathrm{disc}, \mathrm{spec}\}$, also define distributions on $\mathscr{H}(G, \chi_0)$:

$$I'_{\star,\chi}(f) = \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\star,\chi_0}(f_z)\, dz$$

where $f_z : g \mapsto f(gz)$. We of course have that

$$I'_{\mathrm{geom},\chi} = I'_{\mathrm{spec},\chi}.$$

In addition, if f is cuspidal, then so is f_z for any central z so

$$I'_{\mathrm{spec},\chi}(f) = I'_{\mathrm{disc},\chi}(f).$$

For our case, we can only consider central character datum where $A_{G,\infty} \subseteq \mathfrak{X}$. Fix $(\mathfrak{X}, \chi)$ for the rest of this section and let χ_0 be the restriction of χ to $A_{G,\mathrm{rat}}$. The generalized simple trace formula can then be developed in three steps:

1. Find a generalized pseudocoefficient φ so that $\bar{\varphi}_\chi$ is the pseudocoefficient φ_π and traces against φ can be computed easily

2. Compute $I'_{\mathrm{spec},\chi}(\varphi \otimes f^\infty)$ and show this equals $I_{\mathrm{spec},\chi}(\varphi_\pi \otimes \overline{(f^\infty)}_\chi)$. Both these are small modifications of Arthur's original spectral side argument together with an extra lemma of Vogan.

3. Sum over φ to get a generalized Euler-Poincaré function η. Evaluate $I_{\mathrm{geom},\chi_0}(\eta \otimes f^\infty)$ and average to get a formula for $I'_{\mathrm{geom},\chi}(\eta \otimes f^\infty)$.

To see how everything depends on Haar measures, φ will have dimension $[G_\infty/A_{G,\mathrm{rat}}]^{-1}$ and f^∞ will have dimension $[\mathfrak{X}^\infty]^{-1}$ so that both sides of our final formula will have dimension $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

3.3.2 Generalized Pseudocoefficients

We first need to define a version of truncated/generalized pseudocoefficients from [35, §1.9] in the real case. This actually can be done slightly more explicitly than the p-adic case. A lot of this section is probably implicit somewhere in [15].

For this section only, let $G = G(\mathbb{R})$ be a group over $\mathbb{R}$ with discrete series mod center. All other variables ($\mathfrak{a}$, A_G, etc.) will refer to real versions. There is a map

$$H_G : G(\mathbb{R}) \to \mathfrak{a}_*^G : \lambda(H_G(\gamma)) = \log|\lambda(\gamma)| \text{ for all } \lambda \in \mathfrak{a}_G^*.$$

It is well known that this maps $A^0 = A_G(\mathbb{R})^0$ isomorphically to a_*^G so since A^0 is central, we get a splitting $G(\mathbb{R}) = G(\mathbb{R})^1 \times A^0$, where $G(\mathbb{R}^1)$ is the kernel of H_G.

Any character $\lambda \in (\mathfrak{a}_G^*)_{\mathbb{C}}$ of $\mathfrak{a}_*^G$ corresponds to the character $e^{\lambda(H_G(\gamma))}$ on A^0 and therefore G through this isomorphism. The unitary characters correspond to $\lambda \in i\mathfrak{a}_G^*$. Finally, if π is a representation of $G(\mathbb{R})$, let $\pi_\lambda = \pi \otimes e^{\lambda(H_G(\gamma))}$.

Let f be any smooth, compactly supported function on $\mathfrak{a}_*^G$ and π a discrete series representation. The main theorem [15] also allows us to construct a (again not-necessarily unique) compactly supported $\varphi_{\pi,f}$ such that for any unitary ρ

$$\operatorname{tr}_\rho(\varphi_{\pi,f}) = \begin{cases} \widehat{f}(\lambda) & \rho = \pi_\lambda \text{ for some } \lambda \in (\mathfrak{a}_G^*)_{\mathbb{C}} \\ 0 & \rho \text{ basic}, \rho \neq \pi_\lambda \text{ for all } \lambda \in (\mathfrak{a}_G^*)_{\mathbb{C}} \\ ? & \text{else} \end{cases}.$$

Call such a $\varphi_{\pi,f}$ a generalized pseudocoefficient. For any character ω on A^0, we can define

$$\varphi_{\pi,f,\omega}(g) = \int_{A^0} \omega(a)\varphi_{\pi,f}(ag)\, da.$$

This is compactly supported mod center and transforms according to ω^{-1} on A^0. Therefore, if ρ has character ω on A^0, we can define

$$\begin{aligned} \operatorname{tr}_\rho(\varphi_{\pi,f,\omega}) &= \int_{G/A^0} \varphi_{f,\pi,\omega}(g)\Theta_\rho(g)\, dg = \int_{G/A^0}\int_{A^0} \varphi_{\pi,f}(ag)\omega(a)\Theta_\rho(g)\, da\, dg \\ &= \int_{G/A^0}\int_{A^0} \varphi_{\pi,f}(ag)\Theta_\rho(ag)\, da\, dg = \int_G \varphi_{f,\pi}(g)\Theta_\rho(g)\, dg = \operatorname{tr}_\rho(\varphi_{\pi,f}). \quad (3.1) \end{aligned}$$

where Θ is the Harish-Chandra trace character. In particular, $\varphi_{\pi,f,\omega}$ appropriately scaled is a pseudocoefficient.

Averaging $\varphi_{\pi,f}$ over an L-packet $\Pi_{\mathrm{disc}}(\tau)$ for fixed f produces a generalized Euler-Poincaré function $\eta_{\tau,f}$. Since the $\eta_{\tau,f,\omega}$ are averages of pseudocoeffecients over L-packets, they are actually standard Euler-Poincaré functions. Therefore, computation (3.1) gives that whenever τ is regular:

$$\operatorname{tr}_\rho(\eta_{\tau,f}) = \begin{cases} \widehat{f}(\lambda)|\Pi_{\mathrm{disc}}(\tau)|^{-1} & \rho = \pi_\lambda \text{ for some } \pi \in \Pi_{\mathrm{disc}}(\tau), \lambda \in (\mathfrak{a}_G^*)_{\mathbb{C}} \\ 0 & \text{else} \end{cases}.$$

Generalized pseudocoefficients and Euler-Poincaré functions are cuspidal for the same reason as the normal versions.

Finally, as a useful lemma relating our notion to the one in [35],

Lemma 3.3.2.1. *Let π be a discrete series representation with character $e^{\lambda(H_G(a))}$ on A^0 for $\lambda \in (\mathfrak{a}_G^*)_\mathbb{C}$. Let f on $\mathfrak{a}_*^G$ be smooth and compactly supported. Then we can make choices for φ_π and $\varphi_{\pi,f}$ such that $\varphi_{\pi,f} = f\varphi_\pi$.*

Proof. Make a preliminary choice for $\varphi_{\pi,f}$. Then $\widehat{f}(0)^{-1}\varphi_{\pi,f,\lambda}$ is a valid choice of φ_π We evaluate

$$\begin{aligned}
\mathrm{tr}_\rho(f\varphi_{\pi,f,\lambda}) &= \int_G f(g)\varphi_{\pi,f,\lambda}(g)\Theta_\rho(g)\,dg \\
&= \int_{A^0}\int_{G/A_0} f(ag)\varphi_{\pi,f,\lambda}(ag)\Theta_\rho(ag)\,dg\,da \\
&= \int_{A^0} f(a)e^{(\mu-\lambda)(H_G(a))}\int_{G/A_0}\varphi_{\pi,f,\lambda}(g)\Theta_{\rho_{\lambda-\mu}}(g)\,dg\,da
\end{aligned}$$

where we choose $\mu \in (\mathfrak{a}_G^*)_\mathbb{C}$ so that $e^{\mu(H_G(g))}$ is the central character of ρ on A^0. By previous properties, the inner integral therefore becomes $\mathrm{tr}_{\rho_{\lambda-\mu}}(\varphi_{\pi,f})$ and we get

$$\mathrm{tr}_\rho(f\varphi_{\pi,f,\lambda}) = \widehat{f}(\mu-\lambda)\,\mathrm{tr}_{\rho_{\lambda-\mu}}(\varphi_{\pi,f}).$$

Checking each of the three cases in its definition, $f\widehat{f}(0)^{-1}\varphi_{\pi,f,\lambda}$ is then a valid alternative choice for $\varphi_{\pi,f}$. □

A similar property also therefore holds for Euler-Poincaré functions.

A small modification

Generalized pseudocoefficients are in $C_c^\infty(G_\infty)$. We instead want functions in some $C_c^\infty(G_\infty, \chi_0)$ so we make a small modification.

Return to the previous notation where G is a group over F. Let χ_0 be a character on $A_{G,\mathrm{rat}}$ and π_0 a representation of G_∞ consistent with χ_0. Let $\varphi_{\pi_0,f} = f\varphi_{\pi_0}$ be a generalized pseudocoefficient for π_0 and consider the partial average

$$\begin{aligned}
\bar{\varphi}(g) &= \int_{A_{G,\mathrm{rat}}} \chi_0(a)f(ag)\varphi_{\pi_0}(ag)\,da \\
&= \int_{A_{G,\mathrm{rat}}} \chi_0(a)f(ag)\chi_0^{-1}(a)\varphi_{\pi_0}(g)\,da = \varphi_{\pi_0}(g)\int_{A_{G,\mathrm{rat}}} f(ag)\,da.
\end{aligned}$$

This is an element of $C_c^\infty(G_\infty, \chi_0)$ and every function $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$ arises as an integral this way. Finally, by a similar computation to (3.1), this has the same traces against representations π consistent with χ_0 as $\varphi_{\pi_0,f}$.

Therefore, for any function $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$, we can construct analogues of generalized pseudocoefficients $\varphi_{\pi_0,f} = f\varphi_{\pi_0} \in C_c^\infty(G_\infty, \chi_0)$. For computations later, note that such f have Fourier transforms defined on any character of $A_{G,\infty}$ trivial on $A_{G,\mathrm{rat}}$. The same discussion carries over to Euler-Poincaré functions. These are the functions we will actually be using.

We fix f to be dimensionless so these generalized pseudocoefficients have dimension $[G_\infty/A_{G,\mathrm{rat}}]^{-1}[A_{G,\infty}/A_{G,\mathrm{rat}}] = [G_\infty]^{-1}[A_{G,\infty}]$.

3.3.3 Spectral Side with Central Character

To get a simple trace formula with central character, we need two spectral side computations: one for I'_{spec} and one for I_{spec}. Start with a lemma:

Lemma 3.3.3.1. *Let π_0 be a regular discrete series representation of G_∞ with weight ξ_0 and character χ_0 on $A_{G,\infty}$. Then for any real irreducible representation ρ of G_∞ with character χ_0 on $A_{G,\infty}$, $\mathrm{tr}_\rho(\varphi_{\pi_0}) = \delta_{\pi_0}(\rho)$.*

Proof. We thank David Vogan for this argument and note that all mistakes in this writeup are our own.

The case $\rho = \pi_0$ follows immediately. Consider $\rho \neq \pi_0$. In the Grothendieck group, ρ is a linear combination of basic representations with infinitesimal character matching π_0:

$$\rho = \sum_{\rho' \text{ basic}} m_\rho(\rho')\rho'.$$

Taking traces of both sides, $\mathrm{tr}_\rho(\varphi_{\pi_0}) = m_\rho(\pi_0)$. Now, taking the trace against an EP-function η_{ξ_0}:

$$0 = \mathrm{tr}_\rho(\eta_{\xi_0}) = \frac{1}{|\Pi_{\mathrm{disc}}(\xi_0)|} \sum_{\rho' \in \Pi_{\mathrm{disc}}(\xi_0)} m_\rho(\rho')$$

where $\Pi_{\mathrm{disc}}(\xi_0)$ is the L-packet for ξ_0. It therefore suffices to show that the $m_\rho(\rho')$ for $\rho' \in \Pi_{\mathrm{disc}}(\xi_0)$ all have the same sign. This would force them all to be 0.

The most direct way is to use the classification of all unitary representations with infinitesimal character of a discrete series from [68]. These are of the form of certain $A_\mathfrak{q}(\lambda)$ described in terms of Zuckerman functors. These have an explicit decomposition in the Grothendieck group through a version of Zuckerman's character formula proposition 9.4.16 in [82]: λ is a character on Levi L_∞ so first get a character formula λ by twisting both sides of 9.4.16 for L_∞ by λ. Then cohomologically induce to get a character formula on G_∞. Alternatively, by Kazhdan-Lusztig theory, the $m_\rho(\rho')$ are Euler characteristics of stalks of certain perverse sheaves. By theorem 1.12 in [53] their cohomologies are either concentrated in even degree or odd degree. See the comments in the proof to corollary 4.6 in [81], for example, for why this applies to $\mathbb{C}$ in addition to $\overline{\mathbb{F}}_p$. □

Combining with computation (3.1) (note that twisting by a character does not change the regularity of the discrete series) then gives:

Corollary 3.3.3.2. *Let π_0 be a regular discrete series representation of G_∞ with weight ξ_0. Let $f \in C_c^\infty(A_{G,\infty})$. Then for any real representation ρ of G_∞, $\operatorname{tr}_\rho(\varphi_{\pi_0,f}) = f(\rho, \pi_0)$ where*

$$f(\pi, \pi_0) = \begin{cases} \widehat{f}(\lambda) & \pi = \pi_\lambda \text{ for some } \lambda \in (a^*_{G_\infty})_{\mathbb{C}} \\ 0 & \text{else} \end{cases}.$$

A similar result holds for $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$.

This allows us to prove:

Proposition 3.3.3.3. *Let π_0 be a regular discrete series representation of G_∞ with weight ξ_0 and character χ_0 on $A_{G,\mathrm{rat}}$. Let $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Then for all $\varphi^\infty \in \mathscr{H}(G^\infty)$:*

$$I^G_{\mathrm{spec}}(\varphi_{\pi_0,f} \otimes \varphi^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) f(\pi_\infty, \pi_0) \operatorname{tr}_{\pi^\infty}(\varphi^\infty)$$

where

$$f(\pi_\infty, \pi_0) = \begin{cases} \widehat{f}(\lambda) & \pi_\infty = (\pi_0)_\lambda \text{ for some } \lambda \in (a^*_{G_\infty})_{\mathbb{C}} \\ 0 & \text{else} \end{cases}.$$

Proof. This is simply a due-diligence check that none of the steps in the derivation of formula 3.5 in [4] break. First, $\varphi_{\pi_0,f}$ being cuspidal gives

$$\begin{aligned} I^G_{\mathrm{spec}}(\varphi) &= \sum_{t \geq 0} I^G_{\mathrm{disc},t}(\varphi) \\ &= \sum_{t \geq 0} \sum_{L \in \mathscr{L}(G)} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \\ &\quad \times \sum_{s \in W^G(\mathfrak{a}_L)_{\mathrm{reg}}} |\det(s-1)|_{\mathfrak{a}_L/\mathfrak{a}_G}|^{-1} \operatorname{tr}(M_{Q|Q}(s,0)\rho_{Q,t}(0, (\varphi_{\pi_k}\varphi^\infty)^1)), \end{aligned}$$

using that G is connected. This uses a lot of the notation from [4]. In particular, $\mathscr{L}(G)$ is the set of Levi subgroups of G, Q is a parabolic for L, $M_{Q|Q}(s,0)$ is some intertwining operator, $\rho_{Q,t}$ is a sum of parabolically-induced representations from Q with Archimedean infinitesimal character having imaginary part of norm t, and $(\varphi_\pi\varphi^\infty)^1$ is the restriction of the function to $G(\mathbb{A})^1$.

The full definition of the rest of the terms in the inner sum is unnecessary: the only detail Arthur uses is that when $Q \neq G$ it is a sum

$$\sum_{\pi \in \mathcal{AR}(G)} c_\pi \operatorname{tr}_\pi((\varphi_{\pi_0,f}\varphi^\infty)^1)$$

where the c_π vanish whenever the Archimedean infintesimal character of π is regular. However, a property of the pseudocoefficient $\varphi_{\pi_0,f}$ is that it is only supported on representations which have the same infinitesimal character as π_0 (similar to the the proof of [14] lemma 1). This character minus ρ has to be regular. Therefore the sum is 0.

For the leftover term, $Q = G$ so $L = G$ and $M_{Q|Q}(s, 0)$ is trivial. This gives

$$I^G_{\mathrm{disc}}(\varphi_{\pi_0,f} \otimes \varphi^\infty) = \sum_{t\geq 0} \operatorname{tr} \rho_{G,t}(0, (\varphi_{\pi_0,f}\varphi^\infty)^1).$$

By its definition, $\rho_{G,t}(0)$ is the sum of all irreducible, discrete subrepresentations of the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ with Archimedean infinitesimal character having imaginary part with norm t. Arthur's original argument for the sum over discrete representations converging absolutely does not work since there are now potentially infinitely many t on which this trace is supported. However, absolute convergence is now known in general by [20].

Finally, $(\varphi_{\pi_0,f}\varphi^\infty)^1$ acting on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ is the same operator as $\varphi_{\pi_0,f}\varphi^\infty$ acting on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi_0)$. Therefore, summing over the representations that are actually subrepresentations of L^2,

$$\begin{aligned} I^G_{\mathrm{disc}}(\varphi_{\pi_0,f} \otimes \varphi^\infty) &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) \operatorname{tr}_\pi(\varphi_{\pi_0,f}\varphi^\infty) \\ &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) \operatorname{tr}_{\pi_\infty}(\varphi_{\pi_0,f}) \operatorname{tr}_{\pi^\infty}(\varphi^\infty). \end{aligned}$$

Corollary 3.3.3.2 gives that $\operatorname{tr}_{\pi_\infty}(\varphi_{\pi_0,f}) = f(\pi_\infty, \pi_0)$ finishing the argument. □

Next, let $\varphi = \varphi_{\pi_0,f} \otimes \varphi^\infty$. Then

$$I'_{\mathrm{spec},\chi}(\varphi) = \frac{1}{\operatorname{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\mathrm{spec},\chi_0}(\varphi_z)\, dz.$$

Computing

$$\begin{aligned} I_{\mathrm{spec},\chi_0}(\varphi_z) &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) \operatorname{tr}_\pi(\varphi_z) \\ &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi)\omega_\pi^{-1}(z) \operatorname{tr}_\pi(\varphi) \end{aligned}$$

where ω_π is the central character of π. Substituting this in and factoring out the sum and constants from the integral gives

$$\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z)\omega_\pi^{-1}(z)\, dz = \begin{cases} \operatorname{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}) & \chi = \omega_\pi|_{\mathfrak{X}} \\ 0 & \text{else} \end{cases}.$$

Therefore, a lot of terms in the sum go to 0. Finally, since π^∞ has central character χ^∞, it can be traced against functions in $\mathscr{H}(G^\infty, \chi^\infty)$. By definition

$$\mathrm{tr}_{\pi^\infty}(\varphi^\infty) = \mathrm{tr}_{\pi^\infty}(\overline{(\varphi^\infty)}_{\chi^\infty}).$$

Putting it all together,

Corollary 3.3.3.4. *Let π_0 be a regular discrete series representation of G_∞ with weight ξ_0 and character χ_0 on $A_{G,\mathrm{rat}}$. Let $f \in C_c^\infty(A_{G,\infty}, \chi_0)$. Then for all $\varphi^\infty \in \mathscr{H}(G^\infty)$:*

$$I'_{\mathrm{spec},\chi}(\varphi_{\pi_0,f} \otimes \varphi^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) f(\pi_\infty, \pi_0)\, \mathrm{tr}_{\pi^\infty}(\overline{(\varphi^\infty)}_{\chi^\infty})$$

(where we only sum over automorphic representations with the correct central character on all of $\mathfrak{X}$ instead of just $A_{G,\mathrm{rat}}$).

Finally, the same arguments as in 3.3.3.3 again work for the terms in equation (2.3) giving that for $\varphi^\infty \in \mathscr{H}(G^\infty, \chi^\infty)$,

$$I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \delta_{\pi_0,\pi_\infty}\, \mathrm{tr}_{\pi^\infty}(\varphi^\infty)$$

where we factor $\mathfrak{X}_\infty = \mathfrak{X}^1_\infty \times A_{G,\infty}$. Sanity checking dimensions here, we need

$$[G(\mathbb{A})][\mathfrak{X}]^{-1}[G_\infty]^{-1}[A_{G,\infty}] = [\mathfrak{X}_\infty / A_{G,\infty}]^{-1}[G^\infty][\mathfrak{X}^\infty]$$

which holds.

Putting everything together:

Proposition 3.3.3.5. *Let π_0 be a regular discrete series representation of G_∞ with weight ξ_0 and that matches character χ on $\mathfrak{X}$. Let $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$ and $\varphi^{\infty_1} \in \mathscr{H}(G^\infty, \chi^\infty)$ such that $\overline{(\varphi^{\infty_1})}_\chi = \varphi^\infty$. Then:*

$$\begin{aligned}\mathrm{vol}(\mathfrak{X}^1_\infty) I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = &\\ \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \delta_{\pi_0,\pi_\infty}\, \mathrm{tr}_{\pi^\infty}(\varphi^\infty) &\\ = \frac{1}{\widehat{f}(0)} I'_{\mathrm{spec},\chi}(\varphi_{\pi_0,f} \otimes \varphi^{\infty_1}).\end{aligned}$$

The second equality uses that for any $\pi_\lambda \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)$, $\lambda = 0$. We fix φ^∞ to be dimensionless and normalize φ^{∞_1} by it. Therefore, the dimensions are all $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

3.3.4 Geometric Side with Central Character

Vanishing of $I^G_{M,\infty}(\gamma,\psi)$

We explicitly describe all the implicit vanishing arguments in [4] for the ease of the reader. Assume ψ is some cuspidal function. First, by lemma 2.1.1.2, I^G_M vanishes unless $M \in \mathscr{L}^{\mathrm{cusp}}$; i.e., unless $A_{M,\mathrm{rat}}/A_{G,\mathrm{rat}} = A_{M,\infty}/A_{G,\infty}$. Furthermore, in this case, $I^G_{M,\infty}(\gamma,\psi) = \tilde{I}^G_{M,\infty}(\gamma,\psi)$.

Furthermore, as explained in the summary [3, §24], unless γ is elliptic in M over ∞, it is contained in a smaller Levi at ∞, so the descent formula to the smaller Levi shows that $\tilde{I}^G_{M,\infty}(\gamma,\psi)$ vanishes. The main result of [5] also gives this.

Computation of I_{geom,χ_0}

Next, we compute the geometric side. Let $\Pi_{\mathrm{disc}}(\lambda)$ be a regular discrete series L-packet for G_∞ consistent with χ and $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. We again try to mimic Arthur's arguments. Cuspidality of $\eta_{\lambda,f}$ and the splitting formulas reduce the geometric side to

$$I_{\mathrm{geom},\chi_0}(\eta_{\lambda,f}\otimes\varphi^\infty) = \sum_{M\in\mathscr{L}} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \sum_{\gamma\in[M(\mathbb{Q})]_{M,S}} a^M(S,\gamma) I^G_M(\gamma_{\mathbb{R}},\eta_{\lambda,f}) O^M_\gamma(\varphi^\infty_M).$$

Define for $\psi \in C_c^\infty(G_\infty,\chi)$:

$$\Phi_M(\gamma_{\mathbb{R}},\psi) = |D^M(\gamma)|^{-1/2}\tilde{I}^G_M(\gamma,\psi),$$

By the previous subsubsection, we can without loss of generality set $\Phi_M(\gamma,\psi)=0$ if M is not cuspidal over $\mathbb{R}$

For L-packet $\Pi_{\mathrm{disc}}(\lambda)$ and elliptic regular $\gamma \in M_\infty$,

$$\Phi_M(\gamma,\lambda) = (-1)^{q(G)}|D^G_M|^{1/2}\sum_{\pi\in\Pi_{\mathrm{disc}}(\lambda)}\Theta_\pi(\gamma).$$

Arthur shows that $\Phi_M(\gamma,\lambda)$ can be extended by continuity to all elements in elliptic maximal tori. Define it to be 0 for other elements to extend it to all of M_∞; in particular, to non-semisimple elements.

Next, we need a defintion:

Definition. Let χ be a character on $A_{G,\infty}$. A cuspidal function $\psi \in C_c^\infty(G_\infty,\chi)$ is *stable cuspidal* if its trace is supported on discrete series and constant on L-packets.

Note that Euler-Poincaré functions are stable cuspidal. Part of the main result of [15] gives that Euler-Poincaré functions are also K-finite.

As some notation for the next step, if H is a reductive group over $\mathbb{R}$, let $\overline{H}$ be the compact form of H. Any Haar measure on H comes from a differential form on $H_{\mathbb{C}}$ and therefore induces a Haar measure on $\overline{H}$. Then:

Theorem 3.3.4.1 ([4, thm 5.1] slightly rephrased)**.** *Let χ be a character on $A_{G,\infty}$ and $\varphi \in C_c^\infty(G_\infty, \chi)$ be stable cuspidal and K-finite. Then for any $\gamma \in M_\infty$*

$$\Phi_M(\gamma, \varphi) = (-1)^{\dim(A_M/A_G)} \nu(I_\gamma^M)^{-1} \sum_{\substack{\lambda\chi^{-1} \in X^*_{\mathbb{C}}(T) \\ \lambda \text{ matches } \chi}} \Phi_M(\gamma, \lambda) \operatorname{tr}_{\lambda^\vee}(\varphi) :$$

where $\nu(M_\gamma) = (-1)^{q(G)} \operatorname{vol}(\bar{I}^M_{\gamma,\infty}/A_{I^M_\gamma,\infty}) |\Omega(B_{K_{I^M_{\gamma,\infty}}})|^{-1}$.

Note that there is a correction here changing $A_{I^M_\gamma,\mathrm{rat}}$ to $A_{I^M_\gamma,\infty}$ and using $\tilde{I}^G_M$ instead of I^G_M. (see the end of [27, §7]).

Since lemma 3.1 gives that without loss of generality, $\eta_{\lambda,f} = f\eta_\lambda$, we recall the following rephrasing of a fact used in deriving the invariant trace formula:

Lemma 3.3.4.2. *Let $f = f_1 \circ H_{G_\infty}$ be a function on $G_\infty/A_{G,\mathrm{rat}}$ where f_1 is a function on $C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Let φ be any function on G_∞ compactly supported mod center. Then for any $\gamma \in G_\infty$ and Levi M*

$$\tilde{I}^G_M(\gamma, f\varphi) = f(\gamma)\tilde{I}^G_M(\varphi).$$

Proof. Remark 4 after theorems 23.2 and 23.3 in [3] gives that $\tilde{I}^G_M(\gamma, f\varphi)$ only depends on the values of $f\varphi$ on $g \in G_\infty$ with the same image as γ under H_{G_∞}. On this set f is constant so the result follows. □

In particular, keeping in mind our normalization for EP-functions, for any $\gamma \in G_\infty$:

$$|\Pi_{\mathrm{disc}}(\lambda)|\Phi_M(\gamma, \eta_{\lambda,f}) = f(\gamma)\Phi_M(\gamma, \eta_\lambda) = (-1)^{\dim(A_M/A_G)} f(\gamma)\nu(M_\gamma)^{-1}\Phi_M(\gamma, \lambda),$$

so following the computation in [4] section 6 gives:

Corollary 3.3.4.3. *Let λ_0 be weight consistent with χ_0 and $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Then*

$$|\Pi_{\mathrm{disc}}(\lambda_0)| I_{\mathrm{geom},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \sum_{M \in \mathscr{L}^{\mathrm{cusp}}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \times \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I^M_\gamma) |\iota^M(\gamma)|^{-1} f(\gamma) \Phi_M(\gamma, \lambda_0) O^M_\gamma(\varphi^\infty_M)$$

where

$$\chi(I^M_\gamma) = \frac{\operatorname{vol}(I^M_\gamma(F)\backslash I^M_\gamma(\mathbb{A})/A_{I^M_\gamma,\mathrm{rat}})}{\operatorname{vol}(\bar{I}^M_{\gamma,\infty}/A_{I^M_\gamma,\infty})} |\Omega(B_{K_{I^M_{\gamma,\infty}}})|$$

and $\iota^M(\gamma)$ is the set of connected components of M_γ that have an F-point.

As explained on the top of page 19 in [77], we can actually set

$$\chi(I^M_\gamma) = \bar{\mu}^{\mathrm{can},EP}(I^M_\gamma(F)\backslash I^M_\gamma(\mathbb{A})/A_{I^M_\gamma,\mathrm{rat}})$$

by picking measures appropriately on I^M_γ. Note a key change from Arthur's formula: the Levi's that appear are those for which $A_{M,\mathrm{rat}}/A_{G,\mathrm{rat}} = A_{M,\infty}/A_{G,\infty}$ instead of just those satisfying Arthur's notion of cuspidal.

Computation of $I'_{\mathrm{geom},\chi}$

It remains to compute $I'_{\mathrm{geom},\chi}(\eta_{\lambda,f} \otimes \varphi^\infty)$ by averaging. To make the final formula more elegant, without loss of generality assume λ_0 is consistent with χ. We have

$$I'_{\mathrm{geom},\chi}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\mathrm{geom},\chi_0}((\varphi_{\pi_0,f} \otimes \varphi^\infty)_z)\, dz.$$

Without loss of generality, taking $\eta_{\lambda_0,f} = f\eta_{\lambda_0}$ by lemma 3.1:

$$(\eta_{\lambda_0,f} \otimes \varphi^\infty)_z = (\eta_{\lambda_0,f})_{z_\infty} \otimes \varphi^\infty_{z^\infty} = \omega_{\lambda_0}^{-1}(z_\infty)\eta_{\lambda_0,f_{z_\infty}} \otimes \varphi^\infty_{z^\infty}$$

where ω_{λ_0} is the central character associated to λ_0. Here, $\varphi_{\lambda_0,f_{z_a}}$ is still a generalized Euler-Poincaré function so we substitute in corollary 3.3.4.3. The terms that change are

$$f(\gamma)\Phi_M(\gamma,\lambda_0) \mapsto \omega_{\lambda_0}^{-1}(z_\infty) f_{z_\infty}(\gamma)\Phi_M(\gamma,\lambda_0)$$

and

$$O_\gamma^M(\varphi_M^\infty) \mapsto O_\gamma^M((\varphi^\infty_{z^\infty})_M).$$

By our simplifying assumptions, the $\omega_{\lambda_0}^{-1}(z_\infty)$ can be pulled out and partially cancelled against the $\chi(z)$. Finally, we use proposition 3.3.3.5:

$$\mathrm{vol}(\mathfrak{X}^1_\infty) I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\widehat{f}(0)} I'_{\mathrm{spec},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \frac{1}{\widehat{f}(0)} I'_{\mathrm{geom},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty),$$

thereby getting the full formula we will use later:

Proposition 3.3.4.4. *Let $\Pi_{\mathrm{disc}}(\lambda_0)$ be a regular discrete series L-packet of G_∞ with weight ξ_0 and central character χ on $\mathfrak{X}$, f a function pulled back through H_{G_∞} from $C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$, and $\varphi^{\infty_1} \in \mathscr{H}(G^\infty, \chi_0)$ such that $\overline{(\varphi^{\infty_1})}_{\chi^\infty} = \varphi^\infty$. Then we have geometric expansion*

$$\begin{aligned}
&\mathrm{vol}(\mathfrak{X}^1_\infty)|\Pi_{\mathrm{disc}}(\lambda_0)| I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \\
&\quad \frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) \sum_{M \in \mathscr{L}^{\mathrm{cusp}}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \\
&\qquad \times \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I_\gamma^M)|\iota^M(\gamma)|^{-1} f(z_\infty\gamma)\Phi_M(\gamma,\lambda_0) O_\gamma^M((\varphi^{\infty_1}_{z^\infty})_M)\, dz
\end{aligned}$$

where

$$\chi(I_\gamma^M) = \frac{\mathrm{vol}(I_\gamma^M(F)\backslash I_\gamma^M(\mathbb{A})/A_{I_\gamma^M,\mathrm{rat}})}{\mathrm{vol}(\bar{I}^M_{\gamma,\infty}/A_{I_\gamma^M,\infty})} |\Omega(B_{K_{I^M_{\gamma,\infty}}})|$$

and $\iota^M(\gamma)$ is the set of connected components of M_γ that have an F-point.

Further Simplification

Mimicking some simplifications from [40], the integral can be evaluated to remove f and φ^1-dependence. This version of the formula and the method of its derivation are useful for some bounds later.

$\mathfrak{X}_F$ acts on $[M(F)]^{\mathrm{ss}}$ by multiplication. Let the set of orbits be $[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}$. For any γ, let $\operatorname{Stab}_{\mathfrak{X}}(\gamma)$ be the stabilizer of γ under this action. This is finite by using a faithful representation (which always induces a finite-to-one map on semisimple conjugacy classes) to reduce to the case $G = \mathrm{GL}_n$. Here conjugacy classes are just sets of eigenvalues and the $\mathfrak{X}$-action just scales each eigenvalue. Note also that since $\mathfrak{X}$ is central, ι and ν are constant on $\mathfrak{X}$-orbits.

We can therefore move the integral into the inner sum over γ and break it up as

$$\sum_{\gamma \in [M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}$$
$$\times \sum_{x \in \mathfrak{X}_F} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) f_{z_\infty}(x\gamma)\Phi_M(x\gamma, \lambda_0) O^M_\gamma((\varphi^{\infty_1}_{xz^\infty})_M)\, dz.$$

Since χ is defined to be trivial on rational points, the innermost sum simplifies to

$$\sum_{x \in \mathfrak{X}_F} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty x) f(z_\infty x\gamma)\omega^{-1}_{\lambda_0}(x)\Phi_M(\gamma, \lambda_0) O^M_\gamma((\varphi^{\infty_1}_{xz^\infty})_M)\, dz$$
$$= \Phi_M(\gamma, \lambda_0)\left(\int_{\mathfrak{X}_\infty/A_{G,\mathrm{rat}}} f(z\gamma)\, dz\right)\left(\int_{\mathfrak{X}^\infty} \chi(z) O^M_\gamma((\varphi^{\infty_1}_z)_M)\, dz\right).$$

Recalling

$$(\varphi^{\infty_1}_z)_M = \delta_{P_M}(\gamma^\infty)^{1/2}\int_{K^\infty}\int_{N_M(\mathbb{A}^\infty)} \varphi^{\infty_1}(k^{-1}\gamma^\infty z n k)\, dn\, dk,$$

a bunch of Fubini's steps gives that the non-Archimedean integral is $O^M_\gamma((\overline{(\varphi^{\infty_1})_\chi})_M) = O^M_\gamma((\varphi^\infty)_M)$ where we recall

$$\overline{\varphi_\chi}(g) = \int_{\mathfrak{X}^\infty} \varphi(gz)\chi(z)\, dz$$

for any φ.

For the Archimedean integral, let the $G_\infty = G^1_\infty \times A_{G,\infty}$ components of any g be $g_1 \times g_a$. Then $f(z\gamma) = f(z_a\gamma_a)$. This factorization gives a corresponding one $\mathfrak{X}^\infty/A_{G,\mathrm{rat}} = \mathfrak{X}^1_\infty \times A_{G,\infty}/A_{G,\mathrm{rat}}$. Then the integral becomes

$$\int_{\mathfrak{X}^1_\infty}\int_{A_{G,\infty}/A_{G,\mathrm{rat}}} f(z_a\gamma_a)\, dz_a\, dz_1 = \mathrm{vol}(\mathfrak{X}^1_\infty)\widehat{f}(0).$$

Putting it all together:

$$|\Pi_{\mathrm{disc}}(\lambda_0)|I'_{\mathrm{geom},\chi}(\eta_{\lambda_0,f}\otimes\varphi^\infty)=\frac{\mathrm{vol}(\mathfrak{X}^1_\infty)\widehat{f}(0)}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\sum_{M\in\mathscr{L}^{\mathrm{cusp}}}(-1)^{\dim(A_M/A_G)}\frac{|\Omega_{M,F}|}{|\Omega_{G,F}|}$$
$$\times\sum_{\gamma\in[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}}\chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\,\mathrm{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\Phi_M(\gamma,\lambda_0)O^M_\gamma((\varphi^\infty)_M).$$

Using proposition 3.3.3.5 as before finally gives:

Proposition 3.3.4.5. *Let $\Pi_{\mathrm{disc}}(\lambda_0)$ be a regular discrete series L-packet of G_∞ with weight ξ_0 and central character χ on $\mathfrak{X}$. Then, for any $\varphi^\infty\in\mathscr{H}(G^\infty,\chi^\infty)$, we have geometric expansion:*

$$I_{\mathrm{spec},\chi}(\eta_{\lambda_0}\otimes\varphi^\infty)=\frac{1}{|\Pi_{\mathrm{disc}}(\lambda_0)|}\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\sum_{M\in\mathscr{L}^{\mathrm{cusp}}}(-1)^{\dim(A_M/A_G)}\frac{|\Omega_{M,F}|}{|\Omega_{G,F}|}$$
$$\sum_{\gamma\in[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}}\chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\,\mathrm{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\Phi_M(\gamma,\lambda_0)O^M_\gamma((\varphi^\infty)_M).$$

The dimensions on both sides are $[G^\infty][\mathfrak{X}^\infty]^{-1}[\mathfrak{X}^1_\infty]^{-1}=[G^\infty][\mathfrak{X}/A_{G,\infty}]^{-1}$. We state again that the Levi's that appear are those for which $A_{M,\mathrm{rat}}/A_{G,\mathrm{rat}}=A_{M,\infty}/A_{G,\infty}$ instead of just those satisfying Arthur's notion of cuspidal.

3.3.5 Irregular Discrete Series

When λ_0 is not regular, $\mathrm{tr}_{\pi_\infty}\eta_{\lambda_0}$ does not simply test if π_∞ is in a given L-packet. However, it can be interpreted as a cohomology as in [4, §2]. While we will not use this more general result, we state it here in case it is useful in other applications.

Even with irregular λ_0, we still have

$$|\Pi_{\mathrm{disc}}(\lambda_0)|I_{\mathrm{spec},\chi}(\eta_{\lambda_0}\otimes\varphi^\infty)=\frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)}\sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)}m_{\mathrm{disc}}(\pi)\,\mathrm{tr}_{\pi_\infty}(\eta_{\lambda_0})\,\mathrm{tr}_{\pi^\infty}(\varphi^\infty).$$

The Euler-Poincaré function η_{λ_0} always satisfies $\mathrm{tr}_{\pi_\infty}(\eta_{\lambda_0})=\chi_{\lambda_0}(\pi_\infty)$ where χ_{λ_0} is the Euler characteristic

$$\chi_{\lambda_0}(\pi_\infty)=\sum_q(-1)^q\dim H^q(\mathfrak{g}(\mathbb{R}),K_\infty,\pi_\infty\otimes\pi_{\lambda_0}).$$

Here, H^q is the $(\mathfrak{g},K)$-cohomology: K_∞ is a maximal compact of G_∞ and π_{λ_0} is the finite dimensional complex representation with highest weight λ_0. The equality holds in general because it holds on basic representations which generate the Grothendieck group.

In particular, if we define the L^2-Lefschetz number

$$\mathscr{L}_{\lambda_0}(\varphi^\infty)=\sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)}m_{\mathrm{disc}}(\pi)\chi_{\lambda_0}(\pi_\infty)\,\mathrm{tr}_{\pi^\infty}(\varphi^\infty),$$

we get

$$|\Pi_{\mathrm{disc}}(\lambda_0)| I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \mathscr{L}_{\lambda_0}(\varphi^\infty).$$

Combining with the calculations before proposition 3.3.4.5 gives the formula:

Corollary 3.3.5.1. *Let π_0 be a possibly irregular discrete series representation of G_∞ with weight ξ_0 matching character χ on $\mathfrak{X}$. Then, for any $\varphi^\infty \in \mathscr{H}(G^\infty, \chi^\infty)$:*

$$\begin{aligned}\mathscr{L}_{\lambda_0}(\varphi^\infty) = \frac{\mathrm{vol}(\mathfrak{X}^1_\infty)}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \sum_{M \in \mathscr{L}^{\mathrm{cusp}}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \\ \times \sum_{\gamma \in [M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma) |\iota^M(\gamma)|^{-1} |\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \Phi_M(\gamma, \lambda_0) O^M_\gamma((\varphi^\infty)_M).\end{aligned}$$

The dimensions on both sides are $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

Chapter 4

Application to Equidistribution

4.1 Trace Formula Computation Set-Up

Now we can finally set up our main computation.

4.1.1 Conditions on G and Defining Families

Let G be a reductive group over a number field F with discrete series at ∞. By instead looking at $\mathrm{Res}^F_{\mathbb{Q}} G$, we could without loss of generality take $F = \mathbb{Q}$ since $\mathrm{Res}^F_{\mathbb{Q}} G(\mathbb{Q}) = G(F)$ and $\mathrm{Res}^F_{\mathbb{Q}} G(\mathbb{A}) = G(\mathbb{A}_F)$ as topological groups. Fix central character datum $(\mathfrak{X}, \chi)$. Assume G is connected.

Let:

- π_0 be a regular real discrete series representation for G with weight ξ_0 and character χ on $A_{G,\infty}$.
- φ_{π_0} be its pseudocoefficient.
- S_0 be a finite set of finite places and choose $\varphi_{S_0} \in \mathscr{H}(G_{S_0}, \chi_{S_0})$.
- S_1 be another finite set of finite places disjoint from S_0 such that χ_{S_1} is unramfied.
- $S = S_0 \sqcup S_1$.
- $U^{S,\infty} \subset G(\mathbb{A}^{S,\infty})$ an open compact subset on which $\chi^{S,\infty}$ is trivial.
- S_{bad} is a set of places that S_1 needs to be disjoint from that will be defined in section 4.3.

Define a family of automorphic representations $\mathcal{F}$ in $\mathcal{AR}_{\mathrm{disc}}(G, \chi)$ through discrete multiplicities

$$a_{\mathcal{F}}(\pi) = m_{\mathrm{disc}}(\pi)\delta_{\pi_0,\pi_\infty} \dim(\pi^{S,\infty})^{U^{S,\infty}} \frac{\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1})}{\mathrm{vol}(K_{S_1})}.$$

Note that the second-to-last term is just checking if π_{S_1} is unramified. The coefficient $a_{\mathcal{F}}(\pi)$ is dimensionless.

Define function

$$\mathbf{1}_{U^{S,\infty},\chi} = \mathrm{vol}(U^{S,\infty} \cap \mathfrak{X}^{S,\infty})^{-1} \overline{(\mathbf{1}_{U^{S,\infty}})}_{\chi}.$$

This is normalized so that $\mathbf{1}_{U^{S,\infty},\chi}(1) = 1$. For any test function $\varphi_{S_1} \in \mathcal{H}^{\mathrm{ur}}(G_{S_1}, \chi_{S_1})$ let

$$\varphi = \varphi_{\pi_0, f, \varphi_{S_0}} = \varphi_{\pi_0} \otimes \varphi^{\infty} = \varphi_{\pi_0} \otimes \mathbf{1}_{U^{S,\infty},\chi} \otimes \varphi_{S_0} \otimes \varphi_{S_1}$$

where as before, φ_{π} is the pseudocoefficient for π. Test function φ will momentarily be shown to pick out the family $a_{\mathcal{F}}$.

Intuitively, the test function is

- putting weight restrictions on the infinite place,
- putting level restrictions on finite places away from S,
- forcing S_1 parts to be unramified,
- counting possible components at S according to test function φ_S with φ_{S_1} unramified.

To make all the traces well-defined, we fix Haar measures on factors of $G(\mathbb{A}_F)$:

- Use the normalization from [79, §6.6] of Gross' canonical measure from [28] on G_S and the $\mathfrak{X}_S$.
- Use Euler-Poincaré measure on G_∞, $A_{G,\infty}$, $A_{G,\mathrm{rat}}$, and $\mathfrak{X}^1_\infty$.

This determines all appropriate Plancherel measures. We call the product measure $\mu^{\mathrm{can},EP}$ and the volume of the adelic quotient under it the modified Tamagawa number $\tau'(G)$.

4.1.2 Spectral Side

We can now directly compute the spectral expansion of $I_{\mathrm{spec},\chi}(\varphi)$:

Corollary 4.1.2.1. *Let π_0 be a regular discrete series representation of G with weight ξ_0. Then:*

$$I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^{\infty}) = \bar{\mu}^{\mathrm{can}}(U^{S,\infty}_{\mathfrak{X}}) \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}}(\pi) \widehat{\varphi}_S(\pi)$$

where $U^{S,\infty}_{\mathfrak{X}} = U^{S,\infty}/\mathfrak{X}^{S,\infty} \cap U^{S,\infty}$.

Proof. By proposition 3.3.3.5 and using that $\mathrm{vol}(\mathfrak{X}^1_\infty) = 1$,

$$I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^{\infty}) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \delta_{\pi_0, \pi_\infty} \mathrm{tr}_{\pi^\infty}(\varphi^{\infty}).$$

Factoring the finite trace into its S_0, S_1 and other components gives that

$$\operatorname{tr}_{\pi^\infty}(\varphi^\infty) = \widehat{\varphi}_{S_0}(\pi_{S_0}) \frac{\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1})}{\operatorname{vol}(K_{S_1})} \widehat{\varphi}_{S_1}(\pi) \mu^{\mathrm{can}}(U_{\mathfrak{X}}^{S,\infty}) \dim(\pi^{S,\infty})^{U^{S,\infty}},$$

so we are done. □

4.1.3 Geomteric Side Outline

We get a geometric expansion $I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty)$ by using the hyperendoscopy formula (proposition 3.1.2.3). Since Euler-Poincaré functions and pseudocoefficients have the same stable orbital integrals:

$$\begin{aligned} &I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) \\ &\qquad = I^G_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G, \mathcal{H}) I^{\mathcal{H}}_{\mathrm{spec},\chi_{\mathcal{H}}}((\eta_{\xi_0} - \varphi_{\pi_\infty})^{\mathcal{H}} \otimes (\varphi^\infty)^{\mathcal{H}}). \end{aligned}$$

Simplifying and bounding this takes a few steps:

1. Notice that transfers $(\eta_{\xi_0} - \varphi_{\pi_\infty})^{\mathcal{H}}$ through hyperendoscopic paths can be chosen to be linear combinations of regular Euler-Poincaré functions.

2. Substitute in proposition 3.3.4.4 for each hyperendoscopic group.

3. The result will have a main term consisting of central elements of G and an error term consisting of non-central elements, Levi terms, and terms from the hyperendoscopic groups.

4. Use a Poisson summation argument to compute the main term.

5. Bound the error term using bounds on non-Archimedean transfers and small generalizations of the results of [79].

For sanity checks later, note that both sides of our computation have dimension $[G^\infty][\mathfrak{X}/A_{G,\infty}]^{-1}$.

4.2 Geometric Side Details

We are eventually going to use the hyperendoscopic formula with f_1 of the form

$$f_1 = \eta_\xi \otimes \varphi^\infty.$$

All transfers appearing will have linear combinations of Euler-Poincaré functions as infinite parts so we only need to analyze the geometric side with test functions of the form $\eta_\xi \otimes \varphi^\infty$. This is similar to what was done in [79].

4.2.1 Original Bounds

Recall the notation and conditions from 4.1.1. We state the main bounds from [79] for reference. G determines a finite set of places $S_{\mathrm{bad}',G}$ in a complicated, uncontrolled manner. We assume three conditions:

- S does not intersect $S_{\mathrm{bad}',G}$.
- G is cuspidal.
- $\mathfrak{X}$ is trivial.

Then we get the following bounds (changing to our normalization of EP-functions):

Theorem 4.2.1.1 (Weight-aspect bound [79, thm 9.19])**.** *Consider the case where $Z_G = 1$. Let $f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leq \kappa}$ such that $\|f_{S_1}\|_\infty \leq 1$. Let ξ be a dominant weight. Then*

$$\frac{|\Pi_{\mathrm{disc}}(\lambda_0)|}{\tau'(G)\dim(\xi)\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{\varphi}_{S_0})} I_{\mathrm{spec}}(\eta_\xi \otimes \varphi^\infty) = \widehat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{f}_{S_1}) + O_{G,\varphi_{S_0}}(q_{S_1}^{A_{\mathrm{wt}}+B_{\mathrm{wt}}\kappa} m(\xi)^{-C_{\mathrm{wt}}})$$

for some constants $A_{\mathrm{wt}}, B_{\mathrm{wt}}, C_{\mathrm{wt}}$ depending only on G.

Theorem 4.2.1.2 (Level-aspect bound [79, thm 9.16])**.** *Consider the case where $U^{S,\infty}$ is a level subgroup $K^{S,\infty}(\mathfrak{n})$ for some ideal $\mathfrak{n}$ relatively prime to $S_{\mathrm{bad}',G}$. Let $f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leq \kappa}$ such that $\|f_{S_1}\|_\infty \leq 1$. Let ξ be a dominant weight. Then, if $\mathbb{N}(\mathfrak{n})$ is large enough,*

$$\frac{|\Pi_{\mathrm{disc}}(\lambda_0)|}{\tau'(G)\dim(\xi)\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{\varphi}_{S_0})} I_{\mathrm{spec}}(\eta_\xi \otimes \varphi^\infty) = \widehat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{f}_{S_1}) + O_{G,\varphi_{S_0}}(q_{S_1}^{A_{\mathrm{lv}}+B_{\mathrm{lv}}\kappa} \mathbb{N}(\mathfrak{n})^{-C_{\mathrm{lv}}})$$

for some constants $A_{\mathrm{lv}}, B_{\mathrm{lv}}, C_{\mathrm{lv}}$ depending only on G.

For clarity later, we emphasize that the implied constants in the big O depend on G and φ_{S_0}. As noted in errata on the authors' websites, there is a mistake in [79, §7] so the alternate argument in [79, B] must be used for the orbital integral bounds that go into the results. This alternate argument does not provide any control on the constants or $S_{\mathrm{bad}'}$.

Clarifying a minor detail

As another note, there is a small detail assumed in the bound for $a_{\gamma,M}$ used in proving the weight aspect bound: corollary 6.16 used to bound the L function in the formula for $\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\mathrm{rat}})$ only applies to groups with anisotropic center. However 6.17 uses it for centralizers of elements and these can have arbitrary center. We can use the following lemma to get an alternate bound for $\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\mathrm{rat}})$ in general in terms of the bound for groups with anisotropic center:

Lemma 4.2.1.3. *Let G be a connected reductive group over F and $G' = G/A_G$. Then*

$$\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\mathrm{rat}}) = \bar{\mu}^{\mathrm{can},EP}(G'(F)\backslash G'(\mathbb{A}))\bar{\mu}^{\mathrm{can},EP}(A_G(F)\backslash A_G(\mathbb{A})/A_{A_G,\mathrm{rat}}).$$

Note that the factor $\mu^{\mathrm{can},EP}(A_G(F)\backslash A_G(\mathbb{A})/A_{A_G,\mathrm{rat}})$ is a constant depending only on the field F and the dimension of A_G.

Proof. If G is quasisplit at finite v, there is a special model $\underline{G}$ over F_v. Then $\underline{G}(\mathcal{O}_v) \cap A_G(F_v)$ is a maximal (a bigger subgroup times $G(\mathcal{O}_v)$ is otherwise a bigger compact) connected compact subgroup and therefore corresponds to a model $\underline{A_G}$ consistent with the inclusion. Consider the quotient model $\underline{G/A_G}$. By Lang's theorem, $\underline{G'}(k_v) = \underline{G}(k_v)/\underline{A_G}(k_v)$, so by Hensel's lemma and smoothness of quotient maps by smooth subgroups, $\underline{G/A_G}(\mathcal{O}_v) = \underline{G}(\mathcal{O}_v)/\underline{A_G}(\mathcal{O}_v)$. By Hilbert 90, $G'(F_v) = G(F_v)/A_G(F_v)$ for any local F_v. This gives that $G'(\mathbb{A}) = G(\mathbb{A})/A_G(\mathbb{A})$ implying $G'(\mathbb{A})^1 = G'(\mathbb{A}) = G(\mathbb{A})^1/A_G(\mathbb{A})^1$.

Using $G'(F) = G(F)/A_G(F)$, we then get an isomorphism of topological spaces

$$G(F)\backslash G(\mathbb{A})^1 \cong G'(F)\backslash G'(\mathbb{A}) \times A_G(F)\backslash A_G(\mathbb{A})^1.$$

Next, $\mu^{\mathrm{can},EP}$ on $G'(\mathbb{A})$ and $G(\mathbb{A})$ induces a measure μ_A on $A_G(\mathbb{A})$. By the above factorization, it suffices to show that this equals $\mu_A^{\mathrm{can},EP}$ place by place. At the infinite place, they are the same by definition (see [79, §6.5]).

If G is quasisplit at finite v, then μ^{can} is characterized by giving any special subgroup volume 1. As before, $\underline{G/A_G}(\mathcal{O}_v) = \underline{G}(\mathcal{O}_v)/\underline{A_G}(\mathcal{O}_v)$. In particular, $\underline{G/A_G}(\mathcal{O}_v)$ also needs to be maximal connected so it is special. Since these are all special subgroups, this forces $\mu_A = \mu_A^{\mathrm{can}}$ at v.

If G is not quasisplit at v, then μ^{can} is determined by the transfer of a top-form $\omega_{G^{\mathrm{qs}}}$ from G^{qs} (since the normalization factor Λ in [79] depends only on the motive for G which depends only on the quasisplit form of G). The isomorphism $G_{\overline{k}} \xrightarrow{\sim} G^{\mathrm{qs}}_{\overline{k}}$ carries $(A_G)_{\overline{k}}$ to $(A_{G^{\mathrm{qs}}})_{\overline{k}}$ since centers are identified between inner forms. This means that $G'^{\mathrm{qs}} = G^{\mathrm{qs}}/A_{G^{\mathrm{qs}}}$ through the isomorphism over $\overline{k}$. By the previous paragraph, the defining top-forms for G'_{qs} and $A_{G^{\mathrm{qs}}}$ wedge together to that of G^{qs}. Therefore, this same property holds for G and A_G, which is what we want. $\square$

The previous lemma is implicit in later sections of [79] but not explained in detail.

4.2.2 New Bounds Set-up

For our use, we will need a generalization of these bounds that works when $Z_G \neq 1$ and when G is not necessarily cuspidal. We will also need the big O, choices of $S_{\mathrm{bad},H}$, and the constants A, B, C to be uniform over all groups H appearing in hyperendoscopic paths of G. The final statement requires some notation and will be in Theorem 4.3.1.1.

Let ξ be a dominant weight and choose central character datum $(\mathfrak{X}, \chi)$ where $A_{G,\infty} \subseteq \mathfrak{X}$ and χ is consistent with ξ. Let χ_0 be its restriction to $A_{G,\mathrm{rat}}$. We start similar to [77, thm 4.11] and [79, thm 9.19], instead trying to apply proposition 3.3.4.4. This requires making some choices:

- a cutoff function $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$,
- a $\varphi^{\infty_1} \in \mathscr{H}(G^\infty, \chi_0)$ such that $\overline{(\varphi^{\infty_1})}_\chi = \varphi^\infty$,
- lots of Haar measures: fix them to be $\mu^{\mathrm{can} \times EP}$ whenever necessary.

We need to bound the term for all endoscopic groups. Considering all the previous lemmas on transfers, we are interested in the case where:

- φ and χ are unramified outside of S_0 and ∞.
- χ extends to a character on G_v.
- $(\varphi^{S,\infty})^1$ can be chosen to be $\mathrm{vol}(\mathfrak{X}^{S,\infty} \cap U^{S,\infty})^{-1}\mathbf{1}_{U^{S,\infty}}$. For endoscopic groups we will without loss of generality expand S_0 so that $U^{S,\infty} = K^{S,\infty}$. Then this follows from the computation of transfers in section 3.2.5.
- $\varphi_s \in \mathscr{H}(G_s, K_s, \chi_s)^{\leq \kappa}$ and $\|\chi_s \varphi_s\|_\infty \leq 1$ for all $s \in S_1$.

We choose a specific φ_s^1 for $s \in S_1$ according to the following lemma.

Lemma 4.2.2.1. *Pick unramified character datum $(\mathfrak{X}_v, \chi_v)$ such that χ_v extends to a character on G. Let $\varphi_v \in \mathscr{H}(G_v, K_v, \chi_v)^{\leq \kappa}$ such that $\|\chi_v \varphi_v\|_\infty \leq 1$. Fix the canonical measure on $\mathfrak{X}_v$ so that $\mathrm{vol}(K \cap \mathfrak{X}_v) = 1$. Then there exists $\varphi_v^1 \in \mathscr{H}(G_v, K_v)^{\leq \kappa}$ such that $\overline{(\varphi_v^1)}_{\chi_v} = \varphi_v$ and $\|\chi_v \varphi_v^1\|_\infty \leq 1$.*

Proof. Let

$$\varphi_v = \sum_{\lambda \in X_*(A)} a_\lambda \tau_\lambda.$$

Let $A_{\mathfrak{X}_v}$ be the split part of $\mathfrak{X}_v$. Then for any $\zeta \in X_*(A_{\mathfrak{X}_v})$, $a_{\lambda+\zeta} = \chi(\zeta(\varpi))^{-1} a_\lambda$. For each λ such that $a_\lambda \neq 0$, there is a representative λ' of its class $[\lambda] \in X_*(A)/X_*(A_{\mathfrak{X}_v})$ such that $\|\lambda'\| \leq \kappa$. Let Λ be the set of all these chosen representatives. Then

$$\varphi_v^1 = \varphi_v = \sum_{\lambda \in \Lambda} a_\lambda \tau_\lambda$$

satisfies $\overline{(\varphi_v^1)}_{\chi_v} = \varphi_v$ The L^∞ bound on φ_v gives that $|\chi_v(\lambda(\varpi)) a_\lambda| = 1$ implying the needed bound on φ_v^1. $\square$

Note. There is a small technicality here. The original χ_v chosen on the subgroup of G_v may not necessarily extend to G_v. However, section 3.1.3 still gives that $\chi_{\mathcal{H},v}$ on any $\mathcal{H}_v$ is a character λ that extends to $\mathcal{H}_v$ times χ_v. Since $Z_{G^{\mathrm{der}}}$ is finite, χ_v can be factored as a unitary character times a character on G_v. Since the bounds here are only up to absolute value, this does not matter.

Beginning the computation:

$$\frac{|\Pi_{\mathrm{disc}}(\lambda_0)|}{\tau'(G)\dim(\xi)} I_{\mathrm{spec},\chi}(\eta_\xi \otimes \varphi^\infty) = \frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty)$$
$$\times \sum_{M \in \mathscr{L}^{\mathrm{cusp}}} \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} a_{M,\gamma} |\iota^M(\gamma)|^{-1} f(z_\infty \gamma) \frac{\Phi_M(\gamma,\xi)}{\dim \xi} O^M_\gamma((\varphi^{\infty_1}_{z^\infty})_M)\, dz.$$

Here

$$a_{M,\gamma} = \tau'(G)^{-1} \frac{|\Omega_{M,F}|}{|\Omega_{G,F}|} \frac{\overline{\mu}^{\mathrm{can},EP}(I^M_\gamma(F) \backslash I^M_\gamma(\mathbb{A}_F)/A_{I^M_\gamma,\mathbb{Q}})}{\overline{\mu}^{EP}(\bar{I}^M_{\gamma,\infty}/A_{I^M_\gamma,\infty})}$$

(see the top of page 19 in [77]).

This double sum breaks into three pieces: $M = G$ and $\gamma \in Z_G$, $M = G$ otherwise, and $M \neq G$. For $M = G$, $\Phi_M(\gamma,\xi) = \mathrm{tr}\,\xi(\gamma_\infty)$. For central γ, the centralizer is everything so $|\iota^G(\gamma)| = 1$. In addition, the measure on the quotient is just counting measure on a point so $O^M_\gamma(\varphi^{\infty_1}_{z^\infty}) = \varphi^{\infty_1}(z^\infty\gamma)$. Finally,

$$a_{G,\gamma} = \tau'(G)^{-1} \frac{\overline{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\mathrm{rat}})}{\overline{\mu}^{EP}(\bar{G}_\infty/A_{G,\infty})} = \overline{\mu}^{EP}(\bar{G}_\infty/A_{G,\infty})^{-1} = 1$$

since existence of a discrete series requires that the last group is compact and therefore has EP-measure 1.This leaves us with

$$\frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) \sum_{\gamma \in Z_G(F)} \varphi^{\infty_1}(\gamma) f(z\gamma) \frac{\mathrm{tr}\,\xi(z_\infty\gamma)}{\dim \xi}\, dz.$$

Next, note that by a Fourier inversion formula

$$\frac{\mathrm{tr}\,\xi(\gamma)}{\dim \xi} = \omega_\xi^{-1}(\gamma) = \omega_\xi(z_\infty)\omega_\xi^{-1}(z_\infty\gamma) = \omega_\xi(z_\infty)\eta_\xi(z_\infty\gamma)\eta_\xi(1)^{-1}$$

where ω_ξ is the central character for ξ. Therefore, the term inside the sum is simply $\omega_\xi(z_\infty) f(z\gamma)\varphi^1(z\gamma)$ where $\varphi^1 = \eta_\xi \varphi^{\infty_1}$.

Combining the $\omega_\xi(z_\infty)$ factor with the χ, we get a main term

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) \sum_{\gamma \in Z_G(F)} f(z_\infty\gamma)\varphi^1(z\gamma)\, dz \tag{4.1}$$

The leftovers form an error term

$$\frac{1}{\widehat{f}(0)}\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z^\infty)$$
$$\times\left(\sum_{\substack{\gamma\in[G(F)]^{\mathrm{ss}}\\ \gamma\notin Z_G}} a_{G,\gamma}|\iota^G(\gamma)|^{-1}f(z_\infty\gamma)\frac{\mathrm{tr}\,\xi(\gamma_\infty)}{\dim\xi}O^G_{z^\infty\gamma}(\varphi^{\infty_1})\right.$$
$$\left.+\sum_{\substack{M\in\mathscr{L}^{\mathrm{cusp}}\\ M\neq G}}\sum_{\gamma\in[M(F)]^{\mathrm{ss}}} a_{M,\gamma}|\iota^M(\gamma)|^{-1}f(z_\infty\gamma)\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_{z^\infty\gamma}(\varphi^{\infty_1}_M)\right)\,dz \quad (4.2)$$

We compute these separately since they require pretty different ideas to understand.

4.2.3 The Main Term

Central Fourier transforms

This section uses material on Fourier analysis on non-abelian groups. See [22] chapter 7 for a good reference. That p-adic reductive groups are type I is a classic result from [9].

The main term initially simplifies in terms of the Fourier transform $\bar{f}_S$ of f_S with respect to $(Z_G)_S$. To actually get a reasonable interpretation, we need to relate $\bar{f}_S$ to $\widehat{f}_S$. Therefore, for this subsection only, redefine $G = G_S$, $Z = (Z_G)_S$ and consider arbitrary $f \in \mathcal{H}(G_S)$. Note that the following results probably hold for general type I unimodular groups with an appropriate modification of $\mathcal{H}(G)$ to a more complicated function space; the case of p-adic groups just makes the analytic issues a lot nicer.

There is a map from $P : \widehat{G} \to \widehat{Z}_G$ taking π to its central character ω_π.

Lemma 4.2.3.1. *P is measurable with respect to the usual sigma algebras on $\widehat{G}$ and $\widehat{Z}$.*

Proof. Fix a Hilbert space H_i of dimension i for $i \in \mathbb{N}$ or countable infinity. Let Π be the set of irreducible unitary representations of G on some H_i. Consider the functions on Π defined by $\pi \mapsto \langle \pi(g)v, w\rangle$ for $g \in G$ and v, w in the appropriate Hilbert space. Since G is type I, the σ-algebra on $\widehat{G}$ is the quotient of the smallest one on Π that makes these functions continuous. An analogous statement holds for $\widehat{Z}$.

Then, since central elements act by central characters, the functions defined by $z \in Z$ on $\widehat{G}$ are exactly the pullbacks by P of the analogous functions on $\widehat{Z}$. □

Denote the Fourier transform of $f|_{Z_G}$ by $\bar{f}$.

Lemma 4.2.3.2. *For any functions $\varphi \in \mathcal{H}(\widehat{Z})$ and $f \in \mathcal{H}(G)$*

$$\int_{\widehat{Z}}\varphi\bar{f}\,d\mu^{\mathrm{pl}} = \int_{\widehat{G}}(\varphi\circ P)\widehat{f}\,d\mu^{\mathrm{pl}}.$$

Proof. Using both Fourier inversion theorems, for any $z \in Z$

$$\int_{\widehat{Z}} \omega(z)\bar{f}(\omega)\,d\omega = f(z) = \int_{\widehat{G}} \omega_\pi(z)\widehat{f}(\pi)\,d\pi.$$

For a general φ

$$\begin{aligned}\int_{\widehat{G}} \varphi(\omega_\pi)\widehat{f}(\pi)\,d\pi &= \int_{\widehat{G}}\int_Z \bar{\varphi}(z)\omega_\pi^{-1}(z)\widehat{f}(\pi)\,dz\,d\pi \\ &= \int_Z \bar{\varphi}(z)\int_{\widehat{G}} \omega_\pi^{-1}(z)\widehat{f}(\pi)\,d\pi\,dz \\ &= \int_Z \bar{\varphi}(z)\int_{\widehat{Z}} \omega^{-1}(z)\bar{f}(\omega)\,d\omega\,dz \\ &= \int_{\widehat{Z}}\int_Z \bar{\varphi}(z)\omega^{-1}(z)\bar{f}(\omega)\,dz\,d\omega = \int_{\widehat{Z}} \varphi(\omega)\bar{f}(\omega)\,d\omega\end{aligned}$$

so we are done. □

Intuitively, we can therefore think of $\bar{f}(\omega)$ as an average of $\widehat{f}$ over representations with central character ω. To make this notion precise, push $\widehat{f}\,d\mu^{\mathrm{pl}}$ forward to a measure $\mu_{\widehat{f}}$ on $\widehat{Z}_G$.

Lemma 4.2.3.3. *$\mu_{\widehat{f}}$ is absolutely continuous with respect to Haar measure on $\widehat{Z}_G$.*

Proof. Let $X \subset \widehat{Z}$ have measure 0. By σ-finiteness, outer regularity, and continuity of $\bar{f}$, for any $\epsilon > 0$, X is contained in a union X_ϵ of countably many compact open sets such that $\int_{X_\epsilon} \bar{f} d\mu^{\mathrm{pl}} < \epsilon$. Then

$$\mu_{\widehat{f}}(X) \le \mu_{\widehat{f}}(X_\epsilon) = \int_{\widehat{G}} \mathbf{1}_{P^{-1}(X_\epsilon)}\widehat{f}\,d\mu^{\mathrm{pl}} = \int_{\widehat{Z}} \mathbf{1}_{X_\epsilon}\bar{f}\,d\mu^{\mathrm{pl}} < \epsilon.$$

Since this is true for every $\epsilon > 0$, $\mu_{\widehat{f}}(X) = 0$, so we are done. □

Therefore we can define:

Definition. The *conditional Plancherel expectation* is the Radon-Nikodym derivative

$$E^{\mathrm{pl}}(\widehat{f}|\omega) := \frac{d\mu_{\widehat{f}}}{d\mu^{\mathrm{pl}}_{Z_G}}(\omega).$$

This is defined up to a set of measure 0. However, note that the measures $E^{\mathrm{pl}}(\widehat{f}|\omega)\,d\mu^{\mathrm{pl}}$ and $\bar{f}d\mu^{\mathrm{pl}}$ are the same on $\widehat{Z}$ so:

Corollary 4.2.3.4. *$E^{\mathrm{pl}}(\widehat{f}|\omega)$ can be taken to be continuous. If so $E^{\mathrm{pl}}(\widehat{f}|\omega) = \bar{f}(\omega)$.*

We borrow the notation of conditional expectation from probability theory to emphasize first, the same definition in terms of Radon-Nikodym derivatives and second, the analogous intuition as an average over the measure-zero set of representations with central character ω. Beware that under this analogy, E^{pl} is an unnormalized expectation since $E^{\mathrm{pl}}(\widehat{f}|\omega) = \bar{f}$ and the operation $f \mapsto \bar{f}$ multiplies in a factor of $[Z]$ to the dimensions of f.

Main term computation

Proposition 4.2.3.5. *The main term* (4.1) *simplifies to*

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)}\sum_{\omega_S\in\widehat{Z}_{S,L,\xi,\chi}} E^{\mathrm{pl}}(\widehat{\varphi}_S|\omega_S),$$

where $Z'_{S,\infty}=Z_{G_{S,\infty}}/A_{G,\mathrm{rat}}$, $L=Z_G(F)\cap U^{S,\infty}$, *and* $\widehat{Z}_{S,L,\xi,\chi}$ *is the set of* $\omega_S\in\widehat{Z}_S$ *such that* $\omega_S|_L=\omega_\xi|_L$ *and* $\omega_S|_{\mathfrak{X}_S}=\chi_S$. *The normalizing factors are:*

- $\mu=\mu_{Z'_\infty}/\mu^{EP}_{Z'_\infty}$ *where* $\mu_{Z'_\infty}$ *is the measure chosen on* Z'_∞ *to compute the other terms.*
- X *is the finite group* $\mathfrak{X}^{S,\infty}/\mathfrak{X}^{S,\infty}\cap\overline{Z_G(F)}Z_{U^{S,\infty}}$ *where the closure is taken in* $Z^{S,\infty}$.

For shorthand, we denote this sum $E(\widehat{\varphi}_S|\omega_\xi,L,\chi_S)$.

Proof. Start with (4.1):

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)}\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z)\sum_{\gamma\in Z_G(F)}f(z_\infty\gamma)\varphi^1(z\gamma)\,dz.$$

$Z_G(F)$ is cocompact and discrete inside $Z^1=Z_G(\mathbb{A})/A_{G,\mathrm{rat}}$. Then by Poisson summation, the inner sum becomes

$$\frac{1}{\mathrm{vol}(Z/Z_G(F))}\sum_{\substack{\omega\in\widehat{Z^1}\\ \omega(Z_G(F))=1}}\omega^{-1}(z)\overline{f\varphi^1}(\omega)$$

since if $\varphi_z:x\mapsto\varphi(zx)$, then $\bar{\varphi}_z(\omega)=\omega^{-1}(z)\bar{\varphi}(\omega)$. Integrating over z, all terms with $\omega\neq\chi$ vanish so (4.1) becomes

$$\frac{1}{\widehat{f}(0)}\frac{1}{\mathrm{vol}(Z^1/Z_G(F))}\sum_{\substack{\omega\in\widehat{Z^1}\\ \omega(Z_G(F))=1\\ \omega|_{\mathfrak{X}}=\chi}}\overline{f\varphi}(\omega).$$

Here we use that φ^∞ has Fourier transforms on any ω^∞ in the sum and $\overline{\varphi^\infty}=\overline{\varphi^{\infty 1}}$ on these characters. We next break this up into local components to make it more interpretable. First,

$$\bar{\varphi}(\omega)=\overline{f\eta_\xi}(\omega_\infty)\bar{\varphi}_S(\omega_S)\bar{\varphi}^{S,\infty}(\omega^{S,\infty})$$

after choosing Haar measures on the components of Z^1. Let ω_ξ be the central character associated to ξ. For any test function ψ compactly supported on $Z'_\infty=Z_{G,\infty}/A_{G,\mathrm{rat}}$, by lemma 4.2.3.2 applied to $G_\infty/A_{G,\mathrm{rat}}$,

$$\int_{\widehat{Z'_\infty}}\psi(\omega)\overline{f\eta_\xi}(\omega)\,d\omega^{\mathrm{pl}}=\int_{(G_\infty/A_{G,\mathrm{rat}})^\vee}\psi(\omega_\pi)\widehat{f\eta_\xi}(\pi)\,d\pi^{\mathrm{pl}}=$$

$$\int_{\widehat{A}}\int_{\widehat{G^1_\infty}}\psi(\omega\omega_\pi)\widehat{f\eta_\xi}(\pi\otimes\omega)\,d\pi^{\mathrm{pl}}\,d\omega^{\mathrm{pl}}=\mathrm{vol}_{\widehat{G^1_\infty}}(\Pi_{\mathrm{disc}}(\xi))\int_{\widehat{A}}\psi(\omega_\xi\omega)\widehat{f}(\omega)\,d\omega^{\mathrm{pl}}$$

where $A = A_{G,\infty}/A_{G,\mathrm{rat}}$. We want to change the integral to be over $\widehat{Z'_\infty}$. The measure chosen on on Z'_∞ induces Plancherel measure on $\widehat{Z'_\infty}$ which restricts to a measure on $\widehat{A}$ lying co-discretely inside. This corresponds to the quotient measure on A coming from setting $\mathrm{vol}(Z^1_\infty) = 1$. Therefore, if we had EP-measure on Z'_∞, our fixed EP-measure on $\widehat{A}$ would have matched that on $\widehat{Z'_\infty}$. The choices of measures we made also fix EP-measure on G^1_∞ so the volume factor becomes 1.

In general, let $\mu = \mu_{Z'_\infty}/\mu^{EP}_{Z'_\infty}$. Then the identity finally simplifies to

$$\int_{\widehat{Z'_\infty}} \psi(\omega)\overline{f\eta_\xi}(\omega)\,d\omega^{\mathrm{pl}} = \mu \int_{\widehat{Z'_\infty}} \psi(\omega_\xi\omega)\mathbf{1}_{\widehat{A}}(\omega)\widehat{f}(\omega)\,d\omega^{\mathrm{pl}}$$

for any test function ψ. Therefore we get

$$\overline{f\varphi_\infty}(\omega) = \mu\delta_{\omega|_{Z^1_\infty}=\omega_\xi|_{Z^1_\infty}}\widehat{f}(\omega\omega_\xi^{-1}).$$

In our case $A_{G,\infty} \subseteq \mathfrak{X}_\infty$ so for $\omega|_{\mathfrak{X}_\infty} = \omega_\xi|_{\mathfrak{X}_\infty}$, this simplifies to

$$\overline{f\varphi_\infty}(\omega) = \mu\delta_{\omega_\infty=\omega_\xi}\widehat{f}(0).$$

Next, let $Z_{U^{S,\infty}} = U^{S,\infty} \cap Z^1$. Since it is an integral over a subgroup

$$\bar{\varphi}^{S,\infty}(\omega^{S,\infty}) = \begin{cases} \mathrm{vol}(Z_{U^{S,\infty}}) & \omega^{S,\infty}|_{Z_{U^{S,\infty}}} = 1 \\ 0 & \text{else} \end{cases}.$$

In total, cancelling the $\widehat{f}(0)$ factors, the terms that do not vanish are

$$\frac{\mu\,\mathrm{vol}(Z_{U^{S,\infty}})}{\mathrm{vol}(Z^1/Z_G(F))}\bar{\varphi}_S(\omega_S)$$

for every character ω satisfying

1. $\omega(Z_G(F)) = 1$,

2. $\omega|_{\mathfrak{X}} = \chi$,

3. $\omega_\infty = \omega_\xi$,

4. $\omega^{S,\infty}(Z_{U^{S,\infty}}) = 1$.

We try to characterize such ω. Consider $\omega = \omega_\infty\omega_S\omega^{S,\infty}$. Let $L = Z_G(F) \cap U^{S,\infty}$. These conditions require that $\omega_S\omega_\infty = 1$ on L and that $\omega_S\chi_S^{-1} = 1$ on $\mathfrak{X}_S$. Given ω_S satisfying this, the conditions determine $\omega^{S,\infty} = \omega_S^{-1}\omega_\infty^{-1}$ on $Z_G(F)$. Since the determined $\omega^{S,\infty}$ is trivial on $Z_G(F) \cap U^{S,\infty}$, it extends to a continuous character on $\overline{Z_G(F)} \subseteq Z^{S,\infty}$. The character $\omega^{S,\infty}$ is also determined on $U^{S,\infty}$ and $\mathfrak{X}^{S,\infty}$ so in total, the possible choices of $\omega^{S,\infty}$ are those that restrict to a particular value on $E^{S,\infty} = \overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}$.

Since quotient maps of groups are open, $Z_{U^{S,\infty}}$ is open mod $\overline{Z_G(F)}$. Therefore, since $Z^{S,\infty}/\overline{Z_G(F)}$ is compact, $Z^{S,\infty}/E^{S,\infty}$ is a finite group. Therefore the choices are in bijection with $Z^{S,\infty}/E^{S,\infty}$.

By comparing $U^{S,\infty}$ times a fundamental domain for $Z^1/Z_G(F)$ to a fundamental domain for $Z^1_{S,\infty}/L$, we get

$$\frac{\mathrm{vol}(Z_{U^{S,\infty}})}{\mathrm{vol}(Z^1/Z_G(F))} = \frac{1}{\mathrm{vol}(Z'_{S,\infty}/L)|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|}.$$

Therefore, pulling out just the non-zero terms in the sum gives

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)} \sum_{\substack{\omega_S \in \widehat{Z}_S \\ \omega_S \omega_\xi^{-1}(L)=1 \\ \omega_S \chi_S^{-1}(\mathfrak{X}_S)=1}} \bar{\varphi}_S(\omega_S)$$

where

$$|X|^{-1} = \frac{|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}|}{|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|} = |\overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|^{-1}.$$

An application of corollary 4.2.3.4 to $G_S/\mathfrak{X}_S$ then finishes the argument. $\square$

The formula here is complicated and requires some discussion. First, ω_ξ determines a character on L consistent with χ_S. Therefore, ω_ξ and χ_S together determine a character λ on $L\mathfrak{X}_S$. The term $E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$ can be thought of as some sort of normalized average of $\hat{\varphi}_S$ along representations with central character extending λ.

Note that if Z_G is compact and $\mathfrak{X} = A_{G,\mathrm{rat}} = 1$, we can choose a measure so that $\mu(Z_v) = 1$ for all v. This gives $\mu = 1$ so

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z_{S,\infty}/L)} = \frac{1}{\mu(Z_{S,\infty}/L)} = |L| = |Z_G(F) \cap U^{S,\infty}|$$

and $\widehat{Z}_S$ has the counting measure. Therefore, $E^{\mathrm{pl}}(\widehat{f}|\omega)$ is the literal integral of $f\, d\mu^{\mathrm{pl}}$ over representations with character ω. This is in line with the result in [39].

This computation can be compared to the very short argument at the beginning of [18, §2]. Reconciling notation, Θ in that paper is the same as L here and S there is $S \cup \infty$ here. Our argument is much longer since we are factoring out the infinite part of $\mu_{\Theta,S}$ requiring a sum over a complicated set of ω_S instead of just a term for $E^{\mathrm{pl}}(\varphi_{S,\infty}|1)$. In addition, issues involving $\mathfrak{X}$ appear.

Main term bound

It will also be useful to have a very rough bound on the magnitude of this main term.

Proposition 4.2.3.6. *Let $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq \kappa}$ such that $|\chi_{S_1}(x)\varphi_{S_1}(x)| \leq 1$ for all x. Then for some constant C depending only on G, the main term (4.1) is $O_{\varphi_{S_0}}(q_{S_1}^{C \log \kappa})$ where the implied constant is independent of φ_{S_1} and the L-packet weight ξ.*

Proof. Start with the expression (4.1):

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X} / A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X} / A_{G,\mathrm{rat}}} \chi(z) \sum_{\gamma \in Z_G(F)} f(z_\infty \gamma) \varphi^1(z\gamma)\, dz.$$

Here, it is actually convenient to evaluate the integral, giving the central terms in 3.3.4.5:

$$\frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X} / A_{G,\mathrm{rat}})} \sum_{\gamma \in [Z_G(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \omega_\xi^{-1}(\gamma)\varphi(\gamma).$$

The sum becomes

$$\sum_{\gamma \in [Z_G(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi_{S_1}(\gamma)\varphi_{S_1}(\gamma)\chi_{S_0}(\gamma)\varphi_{S_0}(\gamma)\varphi^{S,\infty}.$$

By construction, $\varphi^1_{S_1}$ and $(\varphi^1)^{S,\infty}$ intersect every $\mathfrak{X}$-class in $Z_G(F)$ that φ_{S_1} does. Pick a $\varphi^1_{S_0}$ with the same property. Finally let $U_\infty \subset Z_\infty$ be such that every point with non-zero summand can be translated into it. We will choose specific U_∞ later.

We may then instead bound

$$\sum_{\gamma \in [L]^{\mathrm{ss}}_{\mathfrak{X}}} \mathbf{1}_{U_\infty}(\gamma)\chi_{S_1}(\gamma)\varphi^1_{S_1}(\gamma)\chi_{S_0}(\gamma)\varphi^1_{S_0}(\gamma)$$

where $L = Z_G(F) \cap U^{S,\infty}$. We will do this by first bounding the number of terms in this sum by the size of $L \cap U_\infty \operatorname{Supp} \varphi_S$.

If K_s are the chosen maximal compacts, for each $s \in S_1$, $\varphi^1_s \in \mathscr{H}(G_s, K_s)^{\leq \kappa}$ so φ^1_s is a linear combination of indicator functions $\mathbf{1}_{K_s \lambda(\omega) K_s}$ for a number of possible ω that is polynomial in κ. Therefore, for some constant C, φ_{S_1} is supported on a union of $O(\kappa^{C|S_1|})$ double cosets of K_{S_1}. Since $\varphi^1_{S_0}$ is compactly supported, this gives that φ^1_S is supported on a union of $O_{\varphi^1_{S_0}}(\kappa^{C|S_1|})$ double cosets of K_S. Note that $\kappa^{C|S_1|} \leq \kappa^{C \log q_{S_1}} = q_{S_1}^{C \log \kappa}$.

Let $Z_{K_S} = Z_S \cap K_S$ be the maximal compact for abelian Z_S. Consider the double coset $D = K_S \alpha K_S$. If $D \cap Z_S \neq \emptyset$, without loss of generality let α be in the intersection. Then $D = \alpha K_S$ and $D \cap Z_S$ is a union of cosets of Z_{K_S} in Z_S. Consider two of these cosets xZ_{K_S} and yZ_{K_S}. Then there exists $k \in K_S$ such that $x = ky \implies k = xy^{-1} \implies k \in Z_S$. Therefore, $x \in Z_{K_S}$ and the two cosets are equal. In total, $D \cap Z_S$ is either empty or a coset of K_S. This finally implies that $\operatorname{Supp} \varphi \cap Z_S$ is contained in a union of $O_{\varphi^1_{S_0}}(q_{S_1}^{C \log \kappa})$ cosets of Z_{K_S}.

To continue, we need to choose a particular U_∞. First, Z_∞ factors as $A_{G,\infty}/A_{G,\mathrm{rat}}$ times a compact real torus Z_c. Let U'_∞ be some subset of $A_{G,\infty}/A_{G,\mathrm{rat}}$ and choose f to be the pullback of the characteristic function of U'_∞ through H_{G_∞} (we are not technically allowed to do this due to the smoothness restriction but we can take a close enough approximation in L^1). Then f has support on $U_\infty = U'_\infty \times Z_c$.

Let $c_1 = |L \cap Z_{K_S} U_\infty|$ and assume for now that this is finite. If coset $C = \alpha_S Z_{K_S} U_\infty$ contains an element of L, without loss of generality let this element be α_S. Multiplying by α_S^{-1} bijects $L \cap C$ to $L \cap Z_{K_S} U_\infty$ so $|L \cap C| = c_1$. Counting all possible cosets, $|L \cap \operatorname{Supp}(f\varphi_S)| = O_{\varphi_{S_0}}(c_1 q_{S_1}^{C \log \kappa})$. By a similar argument, $|L \cap \operatorname{Supp}(f\varphi_S)_z| = O_{\varphi_{S_0}}(c_z q_{S_1}^{C \log \kappa})$ where $c_z = |L \cap Z_{K_S} z_\infty^{-1} U_\infty|$.

It remains to bound

$$c_z = |Z_G(F) \cap Z_{K_S} Z_{U^{S,\infty}} z_\infty^{-1} U_\infty| \leq |Z_G(F) \cap Z_{K_S} Z_{K^{S,\infty}} z_\infty^{-1} U_\infty|$$

where $K^{S,\infty}$ is the maximal compact (since $Z^{S,\infty}$ is abelian). This is finite since $Z_G(F)$ is discrete inside $Z/A_{G,\mathrm{rat}}$. Then, $Z_G(F) \cap Z_{K_S} Z_{K^{S,\infty}}$ is a co-compact lattice inside Z_∞. It is still so when projecting down to $A_{G,\infty}/A_{G,\mathbb{Q}}$. Choose U'_∞ to be a fundamental domain for this lattice. Then $c_z = 1$ for all z and $\widehat{f}(0) = \operatorname{vol}(U'_\infty)$ which depends only on G.

Finally, the terms in the sum all have norm 1 up to the factor $\chi_{S_0} \varphi_{S_0}^1$ that depends on φ_{S_0}. Therefore the sum is $O_{\varphi_{S_0}}(q_{S_1}^{C \log \kappa})$ for all z. The factor in front depends only on $(G, \mathfrak{X})$ so the entire term is $O_{\varphi_{S_0}, G}(q_{S_1}^{C \log \kappa})$. $\square$

4.2.4 The Error Term

We need to do a few things to bound the error term. First, the orbital integral bounds used only apply to elements in $\mathscr{H}(H_v, K_{H,v})^{\leq \kappa}$ so we need to extend them to spaces like $\mathscr{H}(H_v, K_{H,v}, \chi)^{\leq \kappa}$.

Second, a given group has infinitely many endoscopic groups. Unfortunately, the alternate proof of orbital integral bounds in [79, §B] gives no control over constants and $S_{\mathrm{bad}'}$. Therefore, it is useful to have some result that allows the use of the same constants and a choice of uniform S_{bad}.

Finally, we need to do another due-diligence check that one, all the lemmas used in the proofs of theorems 4.2.1.1 and 4.2.1.2 still hold over to the non-trivial center case, and two, all the constants from those lemmas can also be uniformly bounded over all hyperendoscopic groups that contribute a non-zero term. This in particular uses the correction to [79, cor 6.17].

Uniform bounds for orbital integrals

The model-theoretic method for bounding orbital integrals gives the following

Theorem 4.2.4.1 ([79, thm B.2]). *Let Ξ be the root datum for an unramified group over some non-Archimedean local field (so the Galois action is determined by the Frobenius action). Choose a norm of the form $\|\cdot\|_{\mathcal{B}}$ on $X_*(A)$. Then there exist T_Ξ, a_Ξ, b_Ξ depending only on $(\Xi, \|\cdot\|_{\mathcal{B}})$ such that for all non-Archimedean local fields F (including ones of positive characteristic) with residue field degree $q \geq T_\Xi$ the following holds:*

Let G^F be the unramified group over F with root datum Ξ, K a hyperspecial of G^F, A a maximal split torus, and ϖ a uniformizer for F. Then for all $\lambda \in X_*(A)$ with $\|\lambda\| \leq \kappa$ and semisimple $\gamma \in G^F(F)$:

$$|O_\gamma(\tau_\lambda^{G^F})| \leq q^{a_\Xi \kappa + b_\Xi} D^{G^F}(\gamma)^{-1/2}$$

where as before, $\tau_\lambda^{G^F} = \mathbf{1}_{K\lambda(\varpi)K}$.

Note. Elements of $\mathscr{H}(G_{S_1}, K_{S_1})^{\leq\kappa}$ are linear combinations of a number of $\mathbf{1}_{K\lambda(\varpi)K}$ that is bounded by a polynomial in κ. Therefore, this can be used to get a bound of shape

$$|O_\gamma(\varphi_{S_1})| = O(\|\varphi_{S_1}\|_\infty q_{S_1}^{a_\Xi\kappa+b_\Xi} D^G_{S_1}(\gamma)^{-1/2})$$

for $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1})^{\leq\kappa}$ and where we slightly increase a_Ξ to absorb the $\kappa^{d|S_1|}$ factor from the polynomials.

Note. We actually need a bound on $\varphi \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq\kappa}$. By shifting double cosets by central elements, we can extend it at the cost of a factor of $|\chi_{S_1}^{-1}(\gamma_{S_1})|$. (Recall that this is well defined by the note in section 4.2.2).

By the following lemma, we can choose a_Ξ, b_Ξ, and T uniformly over all H appearing in an endoscopic path of G and all places v where H is unramified:

Lemma 4.2.4.2. *Let H be a group appearing in a hyperendoscopic path for G, M_H a Levi of H, v a place where H is unramified, and Ξ the unramified root data for $(M_H)_v$. Then Ξ is an element of a finite set depending only on G.*

Proof. The (co)root spaces of M_H are isomorphic to those of G and the (co)roots of M_H are a subset of those of G so there are only finitely many possibilities for the root system of M_H (without Galois action) since its rank is bounded by a finite number through iteratively applying lemma 2.2.2.1. Then, there are only finitely many ways for Frobenius to map into the automorphisms of this root system. □

This bound is absurdly inefficient—in particular, it involves factorials nested to the degree of $|\Phi^+(G)|$. In any application, one should use properties of the exact group being studied to describe the set more explicitly.

Error term bound for weight aspect

We can now show

Proposition 4.2.4.3. *Assume that $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq\kappa}$ with $\|\chi_{S_1}\varphi_{S_1}\|_\infty \leq 1$. Consider error term (4.2) for any group H unramified on S_1 and appearing in an endoscopic path of G with induced central character datum $(\mathfrak{X}, \chi)$ such that $A_{H,\infty} \subseteq \mathfrak{X}$ and χ is unramified on S_1. It is $O_{\varphi_0,H}(q_{S_1}^{A_{\mathrm{wt},H}+B_{\mathrm{wt},H}\kappa} m(\xi)^{-C_{\mathrm{wt},H}})$ for some constants $A_{\mathrm{wt},H}, B_{\mathrm{wt},H}$, and $C_{\mathrm{wt},H}$ as long as S_1 contains no fields with residue degree less than some M_G that is uniform over all H.*

Proof. Let M_G be the maximum of the T_Ξ from 4.2.4.1 over all root data Ξ from lemma 4.2.4.2. This is then a due-diligence check that all the steps in [79, thm 9.19] still hold. We start by evaluating the integral in (4.2) getting term

$$\frac{\mathrm{vol}(\mathfrak{X}^1_\infty)}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{H,\mathrm{rat}})}\left(\sum_{\substack{\gamma\in[H(F)]^{\mathrm{ss}}\\ \gamma\notin Z_H}} a_{H,\gamma}|\iota^H(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\frac{\operatorname{tr}\xi(\gamma_\infty)}{\dim\xi}O^M_\gamma(\varphi^\infty_M)\right.$$
$$\left.+\sum_{\substack{M\in\mathscr{L}^{\mathrm{cusp}}_H\\ M\neq H}}\sum_{\gamma\in[M(F)]^{\mathrm{ss}}} a_{M,\gamma}|\iota^H(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_\gamma((\varphi^\infty)_M)\right).$$

Without loss of generality, expand S_0 so that φ is the characteristic function of a hyperspecial $K^{S,\infty}$ away from $S\cup\infty$ and that the places less than M_G are contained in S_0. If a conjugacy class intersects the support of φ_{S_1}, then we can scale it by an element of $\mathfrak{X}_S$ so that it intersects the support of $\varphi^1_{S_1}$. The same holds for $\varphi^{S,\infty}$ which has support $K^{S,\infty}$. Choose φ_{S_0} and φ_S similarly and let their supports after taking constant terms to M be $U_{S_0,M}$ and $U_{\infty,M}$. We can then replace terms in the sum through the rule

$$\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_\gamma((\varphi^\infty)_M)\mapsto \mathbf{1}_{U_{\infty,M}}\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_\gamma((\varphi^\infty)^1_M).$$

Let $U_{S_1,M}=\operatorname{Supp}\mathscr{H}^{\mathrm{ur}}(M_{S_1})^{\le\kappa}$. Let Y_M be the set of semisimple rational conjugacy classes intersecting the set $U_{S_1,M}U_{S_0,M}U_{\infty,M}K^{S,\infty}_M$. The number of terms in the sum is less than or equal to $|Y_M|$.

We check that each of factors can be bounded as in the proof of [79, thm 9.19]. The finite set of places $S_{M,\gamma}$ disjoint from S can be defined in the same way. Then:

- [79, cor 6.17] still applies to $a_{M,\gamma}$, (see the missing step lemma 4.2.1.3 for why this works for general center).
- The bound in [79, lem 6.11] still applies to the $\Phi_M(\gamma,\xi)$ terms. There is an extra factor of $\chi^{-1}_\infty(\gamma_\infty)$.
- A version of [79, thm A.1] modified to work on functions with central character still applies to bound $O^M_\gamma(\varphi_{S_0,M})$. There is an extra factor of $|\chi^{-1}_{S_0}(\gamma_{S_0})|$.
- Proposition 4.2.4.1 still bounds $O^M_\gamma(\varphi_{S_1,M})$. There is an extra factor of $|\chi^{-1}_{S_1}(\gamma_{S_1})|$.
- Proposition 4.2.4.1 still gives the same bound for $O^M_\gamma(\varphi_{v,M})$ for $v\in S_{M,\gamma}$ since $M_H\le M_G$.There is again an extra factor of $|\chi|$.
- [79, lem 2.18] and [79, lem 2.21] still provide a bound on the D^M terms since we can still construct the embedding from [79, prop 8.1].

- $|Y_M|$ can still be bounded by [79, cor 8.10] (this also applies to groups with general center).
- $|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \le 1$.

Since χ is trivial at rational elements, all the χ_v terms cancel. Therefore, the entire term can similarly be bounded by

$$O(q_{S_1}^{A_{\mathrm{wt},H}+B_{\mathrm{wt},H}\kappa} m(\xi)^{-c_{\mathrm{wt},H}}),$$

folding in the constant that only depends on H and $\mathfrak{X}$. $\square$

This very weak uniformity in just M_G is all we will need for the weight aspect.

4.3 Final Computation

4.3.1 Weight Aspect

Assume the previous conditions on $(G, \mathfrak{X}, \chi)$ from section 4.1.1. Let π_k be a sequence of discrete series representations of G_∞ such that their corresponding finite-dimensional representations ξ_k have regular weights $m(\xi_k) \to \infty$. Let S_1 be disjoint from $S_{\mathrm{bad},G}$: the set of places with residue degree less than the uniform M_G from proposition 4.2.4.3. Choose constant S_0, φ_{S_0} and $U^{S,\infty}$ to define a sequence of families $\mathcal{F}_k$ for each ξ_k.

Theorem 4.3.1.1. *There are constants $A'_{G,\mathrm{wt}}$ and $B'_{G,\mathrm{wt}}$ such that for any φ_{S_0} and $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\le\kappa}$,*

$$\frac{\bar{\mu}^{\mathrm{can}}(U_{\mathfrak{X}}^{S,\infty})|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_S(\pi) = E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S) + O(q_{S_1}^{A'_{\mathrm{wt}}+B'_{\mathrm{wt}}\kappa} m(\xi_k)^{-1})$$

(using notation from corollary 4.1.2.1 and theorem 4.2.3.5). The constants in the error depend on $(G, \mathfrak{X}, \chi)$, φ_{S_0}, and $U^{S,\infty}$.

Proof. For $\|\varphi_{S_1}\chi_{S_1}\|_\infty \le 1$, let

$$\varphi_k = \varphi_{\pi_k} \otimes \mathbf{1}_{U^{S,\infty},\chi} \otimes \varphi_{S_1} \otimes \varphi_{S_0}$$

as in section 4.1.1. Let $\varphi_k^1 = \eta_{\xi_k} \otimes \varphi_k^\infty$. Then φ_k and φ_k^1 are unramified outside of S. Let $\mathcal{A}$ be the set of hyperendoscopic tuples that contribute a non-zero value to the hyperendoscopy formula as in lemma 3.2.6.1.

Then using the hyperendoscopy formula:

$$\frac{|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k) = \frac{|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} \left(I^G_{\mathrm{disc}}(\varphi^1_k) + \sum_{\mathcal{H}\in\mathcal{A}} \iota(G,\mathcal{H}) I^{H_{n_\mathcal{H}}}_{\mathrm{disc}}((\varphi^1_k - \varphi_k)^{\mathcal{H}}) \right).$$

We choose arbitrary transfers of φ_0. Choose $(\mathbf{1}_{K_G^{S,\infty}})^{\mathcal{H}}$ according to lemma 3.2.5.5 since by lemma 3.2.6.1, $\mathcal{H}$ stays unramified away from S,∞. Let $\Pi_{\mathrm{disc}}(\xi_k)$ be the L-packet containing π_k and let its size be X_k. Then

$$(\varphi^1_k - \varphi_k)^\infty = \varphi^\infty_k \qquad (\varphi^1_k - \varphi_k)_\infty = \frac{1}{X_k} \sum_{\pi_k \neq \pi \in \Pi_{\mathrm{disc}}(\xi_k)} \varphi_\pi - \frac{X_k - 1}{X_k} \varphi_{\pi_k}.$$

By proposition 3.2.1.6, we can choose the infinite part transfer to be a linear combination of EP-functions

$$\sum_{\xi \in \Xi_{\xi_k,\mathcal{H}}} c_\xi \eta_\xi$$

for some constants

$$|c_\xi| \le (X_k - 1)\frac{1}{X_k} + \frac{X_k - 1}{X_k} \le 2.$$

Now, checking some conditions:

- All groups in the hyperndoscopic paths are unramified on S_1 and cuspidal at infinity. In addition, each $\mathfrak{X}_\mathcal{H} \supseteq A_{\mathcal{H},\infty}$ by lemma 3.2.6.1.
- Let $\chi_\mathcal{H}$ be the character determined by $\mathcal{H}$ as in section 3.1.3. The transfer $\chi_{\mathcal{H},S_1}\varphi^{\mathcal{H}}_{S_1}$ can be chosen to be in $\mathscr{H}(G_{S_1}, K_{S_1}, \chi_{\mathcal{H},S_1})^{\le\kappa}$ and have L^∞-norm bounded by some $q_{S_1}^{E_\mathcal{H}\kappa}\kappa^{|S_1|}$ by repeated application of proposition 3.2.5.4. We can apply this due to the above.
- The ξ are regular by lemma 3.2.2.1.
- Without loss of generality, enlarge S_0 so that $U^{S,\infty} = K^{S,\infty}$. Then $\mathbf{1}^{\mathcal{H}}_{U^{S,\infty}}$ is still the indicator function of an open compact subgroup averaged over $\chi^{S,\infty}_\mathcal{H}$.

We can therefore apply the main term bound in proposition 4.2.3.6 and the error term bound in propostion 4.2.4.3 to each term in the sum and get

$$I^{\mathcal{H}}_{\mathrm{disc}}((\varphi^1_k - \varphi_k)^{\mathcal{H}}) = I^{\mathcal{H}}_{\mathrm{spec}}((\varphi^1_k - \varphi_k)^{\mathcal{H}}) = \sum_{\xi \in \Xi_{\xi_k,\mathcal{H}}} \iota(G,\mathcal{H}) \frac{\tau'(\mathcal{H})\dim(\xi)}{|\Pi^{\mathcal{H}}_{\mathrm{disc}}(\xi)|} O_{\varphi_0^{\mathcal{H}}, U^{S,\infty},\mathcal{H}}(q_{S_1}^{(A_{\mathrm{wt},\mathcal{H}} + E_\mathcal{H} + \epsilon)\kappa + B_{\mathrm{wt},\mathcal{H}}})$$

for some constant $U_{\xi,\mathcal{H}}$. We use here that $O(\kappa^{C|S_1|})O(q_{S_1}^{(A+E)\kappa+B}) = O(q_{S_1}^{(A+E+\epsilon)\kappa+B})$.

By the computation in 4.2.3.5 and the error term bound 4.2.4.3,

$$\frac{|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} I^G_{\mathrm{disc}}(\varphi^1_k) = E + O(q_{S_1}^{A_{\mathrm{wt},G}+B_{\mathrm{wt},G}\kappa_1} m(\xi_k)^{-C_{\mathrm{wt},G}})$$

where we shorthand $E = E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$. Multiplying through,

$$\begin{aligned}\frac{|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k) = \\ E + \sum_{\mathcal{H}\in\mathcal{A}} \sum_{\xi\in\Xi_{\xi_k,\mathcal{H}}} W_{\xi,\mathcal{H}} \frac{\dim(\xi)}{\dim(\xi_k)} O_{\mathcal{H},\varphi_0^{\mathcal{H}}}(q_{S_1}^{(A_{\mathrm{wt},\mathcal{H}}+E_{\mathcal{H}}+\epsilon)\kappa+B_{\mathrm{wt},\mathcal{H}}}) \\ + O(q_{S_1}^{A_{\mathrm{wt},G}\kappa+B_{\mathrm{wt},G}} m(\xi_k)^{-C_{\mathrm{wt}}})\end{aligned}$$

where

$$\left| W_{\xi,\mathcal{H}} = \iota(G,\mathcal{H}) \frac{\tau'(\mathcal{H})}{\tau'(G)} \frac{|\Pi^G_{\mathrm{disc}}(\xi_k)|}{|\Pi^{\mathcal{H}}_{\mathrm{disc}}(\xi)|} \right| \leq W$$

for some constant W independent of ξ_k, k, and q_{S_1}. Finally, by lemma 3.2.2.2 the ratio of dimensions is $O(m(\xi_k)^{-1})$.

In total, the inner sum has $|\Omega_{\mathcal{H}}|$ elements so the entire double sum has finite size independent of S_1 and ξ. Therefore, it can be bounded to

$$\frac{|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k) = E + O(q_{S_1}^{A'_{\mathrm{wt}}+B'_{\mathrm{wt}}\kappa} m(\xi_k)^{-1}).$$

where $A'_{\mathrm{wt}}, B'_{\mathrm{wt}}$ are anything bigger than the maxima over all groups appearing in $\mathcal{A}$ (Note that C_{wt} can be chosen to be ≥ 1). Finally, plug in corollary 4.1.2.1. □

4.4 Corollaries

Theorem 4.3.1.1 can be substituted in for [79]'s 9.19 to most of the same corollaries. We leave the result on zeros of L-functions for the future because the computations are complicated—the term β_v^{pl} gets replaced by something far more complicated in the case with central character.

Recall the notation from last section and for brevity define

$$\mu_{\mathcal{F}_k}(\widehat{\varphi}_S) = \frac{\bar{\mu}^{\mathrm{can}}(U_{\mathfrak{X}}^{S,\infty})|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_S(\pi)$$

for any $\widehat{\varphi}_S$ on $\widehat{G}_S$. Theorem 4.3.1.1 computes this when $\varphi_S \in \mathscr{H}(G_S,\chi_S)$ and φ_{S_1} is unramified.

4.4.1 Plancherel Equidistribution

First, we get a version of [79, cor 9.22] using a similar Sauvageot density argument. We thank the reviewers for pointing out that [60, pg. 111] discusses a possible gap in [71, pg. 181]. Here, Bernstein components are conflated with "$\mathfrak{l}$-components" (defined in [60, §23.7]) when arguing that a certain algebra separates points. Most of this section is dependent on that gap being fixed.

We phrase things as in [18]. Restrict to the case where all the ξ_k have the same central character ω_ξ and S_1 is trivial. Let $\Theta = L\mathfrak{X}_S$ and let ψ be the character on Θ induced by ω_ξ and χ_S. Let $\widehat{G}_{S,\psi} \subseteq \widehat{G}_S$ be all representations with central character extending ψ. We can define a measure μ^{pl}_ψ on $\widehat{G}_{S,\psi}$ by $\mu^{\mathrm{pl}}_\psi(\widehat{f}) = E(\widehat{f}^*|\omega_\xi, L, \chi_S)$ where $\widehat{f}^*$ is a continuous extension of $\widehat{f}$ to $\widehat{G}_S$.

Here is a summary of the unconditional results for reference:

Corollary 4.4.1.1 (Unconditional Plancherel equidistribution up to central character)**.** *In the notation above:*

1. *For any $f \in \mathscr{H}(G_S, \chi_S)$,*
$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\widehat{f}) = \mu^{\mathrm{pl}}_\psi(\widehat{f}).$$

2. *For any Riemann integrable $\widehat{f}$ supported on $\widehat{G}^{\mathrm{ur}}_{S,\psi}$,*
$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\widehat{f}) = \mu^{\mathrm{pl}}_\psi(\widehat{f}).$$

3. *For any bounded $A \subseteq \widehat{G}_{S,\psi} \setminus \widehat{G}^{\mathrm{temp}}_{S,\psi}$,*
$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\mathbf{1}_A) = 0.$$

Proof. The first statement is a quick consequence of 4.3.1.1 restated in the terminology of this section.

We then note which parts of the following arguments in gray hold unconditionally. For the second statement, remark 9.5 in [79] replaces Sauvageot's result for functions on the unramified spectrum $\widehat{G}^{\mathrm{ur}}_S$ so (2) of 4.4.1.2 holds unconditionally for f on $\widehat{G}^{\mathrm{ur}}_{S,\psi}$. This implies the corresponding piece of 4.4.1.3.

The third statement depends on using [71, cor. 6.2] to show (1) in 4.4.1.2. Corollary 6.2 only depends the interaction between Bernstein components and $\mathfrak{l}$-components through the use of Sauvageot's lemma 5.1 and corollary 3.3. The issue [60] raises is about distinguishing representations in an $\mathfrak{l}$-component that might have the same infinitesimal character. However, the application of 3.3 in 5.1 only cares about representations up to infinitesimal character. □

Note that there is no uniformity in this result—the rate of convergence depends heavily on the exact f.

The following is all conditional on [71]: When ψ is trivial, $\mu_\psi^{\mathrm{pl}} = \mu_{\Theta,\mathrm{pl}}$ from [18] up to some constant. The lemma in the middle of the proof of [18, thm 2.1] extends to our case of non-trivial ψ and Θ a general subgroup of Z_{G_S}.

Lemma 4.4.1.2. *Let $\epsilon > 0$:*

1. *For any bounded $A \subseteq \widehat{G}_{S,\psi} \setminus \widehat{G}_{S,\psi}^{\mathrm{temp}}$, there exists $h \in \mathscr{H}(G_S, \chi_S)$ such that $\widehat{h} \geq 0$ on $\widehat{G}_{S,\psi}$, $\widehat{h} \geq 1$ on A, and $\mu_\psi^{\mathrm{pl}}(\widehat{h}) \leq \epsilon$.*

2. *For any Riemann integrable function $\widehat{f}$ on $\widehat{G}_{S,\psi}^{\mathrm{temp}}$, there exist $h_1, h_2 \in \mathscr{H}(G_S, \chi_S)$ such that $|\widehat{f} - \widehat{h}_1| \leq \widehat{h}_2$ on $\widehat{G}_{S,\psi}$ and $\mu_\psi^{\mathrm{pl}}(\widehat{h}_2) \leq \epsilon$.*

Proof. We try to mimic the argument in [18, thm 2.1]. Let $\Theta_f = \Theta \cap Z_{G_{\mathrm{der}}}(F_S)$ and $\overline{\Theta} = \Theta/\Theta_f$. Then Θ_f is finite. In addition, if we denote by $X(\cdot)$ taking complex-valued characters, the map $X(G_S) \to X(Z_{G,S}/Z_{G_{\mathrm{der}}}(F_S)) \to X(\overline{\Theta})$ is surjective. Choose a set-theoretic section s of this map.

We can ignore normalization constants by, without loss of generality, changing ϵ. Then this result for Θ trivial follows from the main result of [71]. If $\overline{\Theta}$ is trivial, then the various $\widehat{G}_{S,\psi}$ are positive-measure clopen subsets of $\widehat{G}_S$ so we can use the h_i for either A or the extension of f by 0 on $\widehat{G}_S$.

For the general case, given f on $\widehat{G}_{S,\psi}$ define F on $\widehat{G}_{S,\psi|_{\Theta_f}}$ by

$$F(\pi) = f(\pi \otimes s(\omega_\pi^{-1}\psi))\mathbf{1}_C(\omega_\pi^{-1}\psi),$$

where $C \subseteq X(\overline{\Theta})$ is compact. Choose H_1 and H_2 satisfying $|\widehat{F} - \widehat{H}_1| \leq \widehat{H}_2$ on $\widehat{G}_{S,\psi|_{\Theta_f}}$ and $\mu^{\mathrm{pl}}_{\psi|_{\Theta_f}}(\widehat{H}_2) \leq \epsilon/(2\operatorname{vol}(C))$. For any finite subset $T_0 \subseteq C$, the averages

$$h_i = \frac{1}{|T_0|}\sum_{\lambda \in T_0} s(\lambda) H_i$$

satisfy $|\widehat{f} - \widehat{h}_i| \leq \widehat{h}_2$ (each individual term in the sum does), so we simply need to find a T_0 that allows us to prove $\mu_\psi^{\mathrm{pl}}(\widehat{h}_2) \leq \epsilon$.

Up to some constants:

$$\mu_\psi^{\mathrm{pl}}(\widehat{h}_2) = \int_{\overline{\Theta}} \psi(z) h_2(z)\, dz = \frac{1}{|T_0|}\sum_{\lambda \in T_0} \overline{H}_2(\lambda\psi)$$

by variations of the arguments in section 4.2.3. Taking Riemann sums, we can find finite $T_0 \subseteq C$ such that such that this sum is within $\epsilon/2$ of

$$\operatorname{vol}(\psi C)\int_{\psi C} \widehat{H}_2(\chi)\, d\chi = \operatorname{vol}(C)\int_{p^{-1}(\psi C)} \widehat{H}_2(\pi)\, d\mu^{\mathrm{pl}}_{\psi|_{\Theta_f}}(\pi).$$

where the equality is again by results in section 4.2.3. Since $\widehat{H}_2$ is positive, this is further bounded by $\mathrm{vol}(C)\mu^{\mathrm{pl}}_{\psi|_{\Theta_f}}(\widehat{H}_2) \le \epsilon/2$, finally showing that $\mu^{\mathrm{pl}}_{\psi}(\widehat{h}_2) \le \epsilon$.

The argument for subsets A is the same averaging trick—in place of the function F, we use set $A' = \{\pi \otimes \lambda : \pi \in A, \lambda \in C \subseteq X(\overline{\Theta})\}$.

Finally, this only produces functions on $\mathscr{H}(G_S,)$, so we simply average against χ_S to get functions in $\mathscr{H}(G_S, \chi_S)$ that have all the same trace properties on $G_{S,\psi}$. □

The same "3ϵ"-argument as [79, cor 9.22] then gives:

Corollary 4.4.1.3 (Plancherel equidistribution up to central character)**.** *Recall the conditions and notation from the above discussion. Then*

1. *For any bounded $A \subseteq \widehat{G}_{S,\psi} \setminus \widehat{G}^{\mathrm{temp}}_{S,\psi}$,*

$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\mathbf{1}_A) = 0.$$

2. *For any Riemann integrable $\widehat{f}$ on $\widehat{G}^{\mathrm{temp}}_{S,\psi}$,*

$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\widehat{f}) = \mu^{\mathrm{pl}}_{\psi}(\widehat{f}).$$

Beware that part (1) does not give a Ramanujan conjecture at S on average; it cannot count that the total number of π in $\mathcal{F}$ with non-tempered π_S is $O(m(\xi_k))^{-1}$ since A needs to be bounded. It is nevertheless somewhat close.

4.4.2 Sato-Tate Equidistribution

For this section we need to slightly modify our notation. Allow S_1 to be infinite and define modified measure

$$\mu^{\natural}_{\mathcal{F}_k,v}(\widehat{\varphi}_v) = \frac{\bar{\mu}^{\mathrm{can}}(U^{S,\infty}_{\mathfrak{X}})|\Pi_{\mathrm{disc}}(\xi_k)|}{\tau'(G)\dim(\xi_k)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_{S_0}(\pi_{S_0})\widehat{\varphi}_v(\pi_v)$$

for any $v \in S_1$. Then $\mu^{\natural}_{k,v}(\widehat{\varphi}_v)$ can still be picked out by a test function φ of the form we have been considering by setting $\varphi_w = \mathbf{1}_{K_w}$ for all $w \in S_1 \setminus v$.

Sato-Tate measures

We recall the definition of the Sato-Tate measure from [79, §3, §5]. Recall the Satake isomorphism $\mathscr{H}(G_v, K_v) \to \mathbb{C}[X_*(A)]^{\Omega_F}$ in the notation of section 3.2.4 and how it identifies $\widehat{G}^{\mathrm{ur,temp}}_v$ with $\Omega_{F_v}\backslash\widehat{A}_c$.

We can find a maximal compact $\widehat{K}$ of $\widehat{G}$ invariant under Frob_v. Then since G_v is unramified, $\Omega_{F_v}\backslash\widehat{A}_c$ can be identified with the $\widehat{G}$ classes in $\widehat{K} \rtimes \mathrm{Frob}_v \subseteq {}^L G$ and also $\widehat{T}_{c,v} = \Omega_{F_v}\backslash\widehat{T}_c/(\mathrm{Frob}_v - \mathrm{id})\widehat{T}_c$ (see [79, lem 3.2]).

In general, let G split over F_1 and let $\Gamma_1 = \mathrm{Gal}(F_1/F)$. Given $\Theta \in \Gamma_1$, define

$$\widehat{T}_{c,\Theta} = \Omega_G^\Theta \backslash \widehat{T}_c / (\Theta - \mathrm{id})\widehat{T}_c.$$

Given $\tau \in \Gamma_1$, $t \mapsto \tau t$ canonically identifies $T_{c,\Theta}$ with $T_{c,\tau\Theta\tau^{-1}}$. All these identifications are consistent with each other so $T_{c,\Theta}$ depends only on the Γ_1-conjugacy class of Θ. Note then that $\widehat{T}_{c,\mathrm{Frob}_v} = \widehat{T}_{c,v}$ since G_v is quasisplit.

Choose the Haar measure on $\widehat{K}$ with total volume 1. This induces a quotient measure on the set of conjugacy classes in $\widehat{K} \rtimes \Theta$ and therefore on $\widehat{T}_{c,\Theta}$. Call this μ_Θ^{ST}. Finally, let $\mathcal{V}_F(\Theta)$ be the set of places v such that F_1 is unramified at v and Frob_v is in the conjugacy class of Θ. For such a v, we get a measure $\mu_v^{\mathrm{pl,ur}}$ from the identification $T_{c,\Theta}$ with $\widehat{G}_v^{\mathrm{ur,temp}}$. Normalize this to also have total volume 1.

Proposition 4.4.2.1 ([79, prop 5.3])**.** *For any $\Theta \in [\Gamma_1]$, let $v \to \infty$ in $\mathcal{V}_F(\Theta)$. Then there is weak convergence $\mu_v^{\mathrm{pl,ur}} \to \mu_\Theta^{\mathrm{ST}}$.*

Proof. by the explicit formulas [79, prop 3.3] and [79, lem 5.2] □

Central character issues

Recall all the notation from proposition 4.2.3.5. Our result is in terms of $E(\widehat{\varphi}|\omega_\xi, L, \chi_S)$ instead of $\mu_v^{\mathrm{pl,ur}}$ so we need to define an alternate Sato-Tate measure in terms of this. First, we need to understand $E_v^{\mathrm{pl,ur}}$ better.

There is a central character map $T_{c,\Theta} \to \widehat{Z}_{G_v}$. This lets us define $E^{\mathrm{ST},\Theta}(\widehat{\varphi}|\omega)$ for any $\widehat{\varphi}$ on $T_{c,\Theta}$ similar to $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega)$ from section 4.2.3. Now Langlands for tori gives that $\widehat{Z}_{G_v}$ is the set of L-parameters $\varphi : W_{F_v}^{\mathrm{ur}} \hookrightarrow {}^L(Z_G)_{F_v}^{\mathrm{ur}}$. If Frob_v and Frob_w are conjugate in Γ_1, we can identify the set of these parameters and therefore $\widehat{Z}_{G_v}$ and $\widehat{Z}_{G_w}$. For $v \in \mathcal{V}_F(\Theta)$, call this common set $\widehat{Z}_\Theta$. Note that these identifications commute with the identifications of $\widehat{T}_{c,v}$ and the map taking central characters.

Lemma 4.4.2.2. *Fix a common measure on $\widehat{Z}_\Theta$. Choose $\widehat{\varphi}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega) \to E^{\mathrm{ST},\Theta}(\widehat{\varphi}|\omega)$ pointwise for $\omega \in \widehat{Z}_\Theta$.*

Proof. The previous result gives weak convergence $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}_\Theta|\omega) \to E^{\mathrm{ST},\Theta}(\widehat{\varphi}_\Theta|\omega)$ in $L^2(\widehat{Z}_\Theta)$. By the formula [79, prop 3.3], the $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega)$ are equicontinuous so this implies pointwise convergence. □

To understand the more complicated $E(\widehat{\varphi}|\omega_\xi, L, \chi_S)$, we now have to parametrize $Z_{S,\xi,L,\chi}$ in terms of local components. Assume $\omega_S = \omega_{S_1}\omega_{S_0} \in Z_{S,\xi,L,\chi}$: i.e. $\omega_S\omega_\xi = 1$ on L and $\omega_S|_{\mathfrak{X}_S} = \chi_S$. Assume also that ω_{S_1} is unramified. Let $L_0 = L \cap K_{S_1}$. It is a cocompact lattice in Z_{S_0}. Then we always have that $\omega_{S_0}\omega_\xi = 1$ on L_0 and that $\omega_{S_0}|_{\mathfrak{X}_{S_0}} = \chi_{S_0}$.

Given such ω_{S_0}, it forces $\omega_{S_1} = \omega_{S_0}^{-1}\omega_\xi^{-1}$ on L. The determined ω_{S_1} is trivial on $L \cap K_{S_1}$ and therefore extends to a continuous character on $\overline{L} \subseteq Z_{S_1}$. Therefore, the possible choices for ω_{S_1} are those that restrict to $\omega_{S_0}^{-1}\omega_\xi^{-1}$ on L, restrict to χ_{S_1} on $\mathfrak{X}_{S_1}$, and are unramified.

Let E_{S_1} be the group $\overline{L}K_{S_1}\mathfrak{X}_{S_1}$. Since Z_{S_1}/E_{S_1} is finite, there are finitely many choices for ω_{S_1} and we can factor

$$\sum_{\substack{\omega_S \in \widehat{Z}_S \\ \omega_S\omega_\xi(L)=1 \\ \omega_S|_{\mathfrak{X}_S}=\chi_S}} E^{\mathrm{pl}}(\widehat{\varphi}_S|\omega_S) = \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}}=\chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\substack{\omega_S \in \widehat{Z}^{\mathrm{ur}}_{S_1} \\ \omega_{S_1}\omega_{S_0}\omega_\xi(L)=1 \\ \omega_{S_1}|_{\mathfrak{X}_{S_1}}=\chi_{S_1}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_1}|\omega_{S_1}).$$

To compute $\mu^{\natural}_{k,v}$, we consider $\varphi_{S_1} = \mathbf{1}_{K_{S_1\setminus v}}\varphi_v$ so

$$E^{\mathrm{pl}}(\widehat{\varphi}_{S_1}|\omega_{S_1}) = E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v)\prod_{w\in S_1\setminus v}\mathrm{vol}(Z_w\cap K_w) = \frac{\mathrm{vol}(Z_{S_1}\cap K_{S_1})}{\mathrm{vol}(Z_v\cap K_v)}E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v).$$

Let the set of summands for the second sum be $\widehat{Z}_{v,\omega_{S_0},\chi_v} \subseteq \widehat{Z}_v$ and let $\omega_S \in \widehat{Z}_{v,\omega_{S_0},\chi_v}$. The possible ω_v components are those satisfying two conditions: $\omega_v\omega_{S_0}\omega_\xi$ extends continuously to $\overline{L} \subseteq Z_{S\setminus v}$, and $\omega_v|_{\mathfrak{X}_v} = \chi_v$. The first condition is equivalent to ω_v being the F_v-component of a character ω on $Z_G(\mathbb{A})/Z_G(F)$ trivial on $U^{S,\infty}$ that also has $F_{S_0,\infty}$-component $\omega_{S_0}\omega_\xi$.

Next, by global Langlands for tori, this is equivalent to its parameter $\psi_{\omega_v} : W_{F_v} \to {}^L(Z_G)_{F_v}$ being a restriction of a global parameter $\psi_\omega : W_F \to {}^LZ_G$ satisfying certain conditions. However, if Frob_w is conjugate to Frob_w, then $\psi_\omega|_{W_{F_w}}$ is the transport of $\psi_\omega|_{W_{F_v}}$ through the identification before. In particular, if we identify all the $\widehat{Z}_v$ for $v \in \mathcal{V}_F(\Theta)\cap S_1$, $\widehat{Z}_{v,\omega_{S_0},\chi_v}$ depends on v only through Θ. Call the common value $\widehat{Z}_{\Theta,\omega_{S_0},\chi_v} \subseteq \widehat{Z}_\Theta$.

In total, if we set $\varphi_{S_1} = \mathbf{1}_{K_{S_1\setminus v}}\varphi_v$ for some $v \in \mathcal{V}_F(\Theta)\cap S_1$,

$$E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S) = \frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)}\frac{\mathrm{vol}(Z_{S_1}\cap K_{S_1})}{\mathrm{vol}(Z_v\cap K_v)} \times \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}}=\chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\omega_v\in\widehat{Z}_{\Theta,\omega_{S_0},\chi_v}} E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v).$$

This allows us to define an $E_{\mathrm{ST},\Theta}(\widehat{\varphi}_v|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0})$ analogously:

$$E_{\mathrm{ST},\Theta}(\widehat{\varphi}_S|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0}) = \frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)}\frac{\mathrm{vol}(Z_{S_1}\cap K_{S_1})}{\mathrm{vol}(Z_v\cap K_v)} \times \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}}=\chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\omega_v\in\widehat{Z}_{\Theta,\omega_{S_0},\chi_v}} E^{\mathrm{ST},\Theta}(\widehat{\varphi}_v|\omega_v).$$

Then we get:

Proposition 4.4.2.3. *Choose a sequence $v \to \infty$ in $\mathcal{V}_F(\Theta) \cap S_1$ such that the characters χ_v all correspond in $\widehat{\mathfrak{X}}_\Theta$. Choose $\widehat{\varphi}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then*

$$E(\widehat{\mathbf{1}}_{K_{S_1\setminus v}}\widehat{\varphi}_\Theta\widehat{\varphi}_{S_0}|\omega_\xi, L, \chi_S,) \to E_{\mathrm{ST},\Theta}(\widehat{\varphi}_\Theta|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0}).$$

Proof. Use the above formula for E_{ST} and E together with the previous lemma. We can compute both sides by fixing a common measure on $\widehat{Z}_\Theta$ which makes $\mathrm{vol}(Z_v \cap K_v)$ constant on $v \in \mathcal{V}_F(\Theta)$. □

This is a replacement for [79, prop 5.3] in our case.

Final Statment

Arguing as in [79, thm 9.26], we get the full corollary. Note that remark 9.5 in [79] removes the dependence on Sauvageot's result.

Corollary 4.4.2.4 (Sato-Tate equidistribution up to central character)**.** *Choose a sequence $v_j \to \infty$ in $\mathcal{V}_F(\Theta) \cap S_1$ such that the characters χ_v all correspond in $\widehat{\mathfrak{X}}_\Theta$. Choose a Riemann integrable function $\widehat{f}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then*

$$\lim_{(j,k)\to\infty} \mu^\natural_{\mathcal{F}_k,v_j}(\widehat{f}_\Theta) = E_{\mathrm{ST},\Theta}(\widehat{f}_\Theta|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0})$$

where the limit is over any sequence of pairs (j,k) such that $q_{v_j}^N m(\xi_k)^{-1} \to 0$ for all integers N.

This can be thought of as sort of a "diagonal" equidistribution as opposed to the "vertical" Plancherel equidistribution involving $\lim_{k\to\infty} \mu^\natural_{\mathcal{F}_k,v_j}(\widehat{f}_\Theta)$ or the conjectural "pure horizontal" Sato-Tate equidistribution involving $\lim_{j\to\infty} \mu^\natural_{\mathcal{F}_k,v_j}(\widehat{f}_\Theta)$.

Chapter 5

Application to Quaternionic Forms on G_2

We use some extra notation in this chapter:

- G_2 is as the split, simple, and simply-connected exceptional group. It can be defined over $\mathbb{Z}$.
- G_2^c is the sole inner form of G_2 over $\mathbb{Q}$. It is compact at infinity, equal to split G_2 at all finite places, and can also be defined over $\mathbb{Z}$.
- H will generally refer to the specific endoscopic group $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ (split SO_4) of G_2.
- $\alpha_i, \lambda_i, \epsilon_i, \delta_i, s_{\alpha_i}$ are various pieces of the root data for G_2 defined in section 5.1.1.

Finally, as shorthand, any variable requiring a general reductive group subscript will be for G_2 if the group isn't specified.

5.1 G_2 and Quaternionic Discrete Series

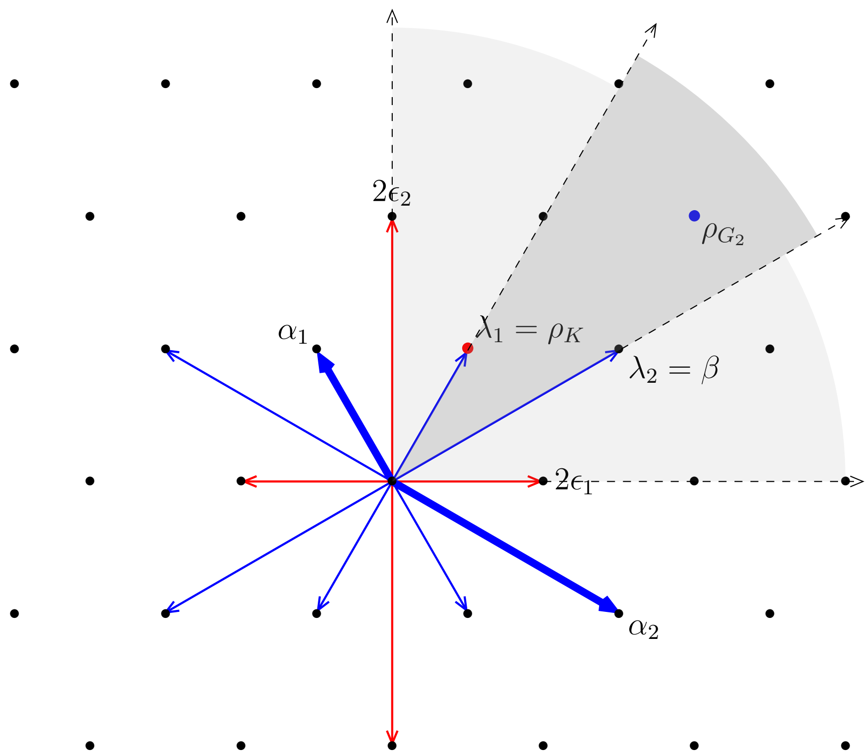
5.1.1 Root System of G_2

Roots

We use notation from [51] to specify the root system of G_2. Let K be the maximal compact $\mathrm{SU}(2) \times \mathrm{SU}(2)/\pm 1$ of $G_2(\mathbb{R})$. Choose a dominant chamber for K and the choice of simple roots of G_2 consistent with this. Let β be the highest root with respect to this and note that it is long .

We now give explicit coordinates. As a mnemonic convention, roots indexed 1 will be short and roots indexed 2 will be long. Figure 5.1 displays all the roots and shades our choices of dominant Weyl chambers with respect to both G_2 and K. Compact roots at infinity are in red and non-compact in blue.

Figure 5.1: Character lattice, roots, and choices of dominant chamber for G_2



If the roots of the short and long SU_2 are $2\epsilon_1$ and $2\epsilon_2$ respectively, then the simple roots of G_2 are:

$$\text{(short) } \alpha_1 = -\epsilon_1 + \epsilon_2, \qquad \text{(long) } \alpha_2 = 3\epsilon_1 - \epsilon_2.$$

The other positive roots are:

$$\begin{aligned} \text{(short) } 2\epsilon_1 &= \alpha_1 + \alpha_2, & \epsilon_1 + \epsilon_2 &= 2\alpha_1 + \alpha_2, \\ \text{(long) } 2\epsilon_2 &= 3\alpha_1 + \alpha_2, & 3\epsilon_1 + \epsilon_2 &= 3\alpha_2 + 2\alpha_2. \end{aligned}$$

The fundamental weights are:

$$\lambda_1 = 2\alpha_1 + \alpha_2, \qquad \lambda_2 = 3\alpha_2 + 2\alpha_2.$$

Of course $\beta = \lambda_2$.

The Weyl group is generated by simple reflections:

$$s_{\alpha_1}\begin{pmatrix} 2\epsilon_1 \\ 2\epsilon_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 + \epsilon_2 \\ 3\epsilon_1 - \epsilon_2 \end{pmatrix}, \qquad s_{\alpha_2}\begin{pmatrix} 2\epsilon_1 \\ 2\epsilon_2 \end{pmatrix} = \begin{pmatrix} -\epsilon_1 + \epsilon_2 \\ 3\epsilon_1 + \epsilon_2 \end{pmatrix}.$$

Finally:

$$\begin{aligned} \rho_K &= \epsilon_1 + \epsilon_2 = 2\alpha_1 + \alpha_2, \\ \rho_G &= 4\epsilon_1 + 2\epsilon_2 = 5\alpha_1 + 3\alpha_2. \end{aligned}$$

Coroots

Coroots will follow the opposite mnemonic: coroots indexed 1 are long and coroots indexed 2 are short.

Let T be a split maximal torus. Since G_2 has trivial center, $X^*(T)$ is the root lattice:

$$X^*(T) = \{a\epsilon_1 + b\epsilon_2 : a, b \in \mathbb{Z}, a + b \in 2\mathbb{Z}\}.$$

Let (δ_1, δ_2) be the dual basis to $(2\epsilon_1, 2\epsilon_2)$: i.e. $(\delta_i, \epsilon_j) = 1/2\mathbf{1}_{i=j}$. Then:

$$X_*(T) = \{a\delta_1 + b\delta_2 : a, b \in \mathbb{Z}, a + b \in 2\mathbb{Z}\}.$$

Since ϵ_1 and ϵ_2 are perpendicular:

$$(2\epsilon_1)^\vee = 2\delta_1,$$
$$(2\epsilon_2)^\vee = 2\delta_2.$$

More generally, the Weyl action gives:

$$(\alpha_1^\vee, 2\epsilon_1) = -1, \qquad (\alpha_1^\vee, 2\epsilon_2) = 3,$$
$$(\alpha_2^\vee, 2\epsilon_1) = 1, \qquad (\alpha_2^\vee, 2\epsilon_2) = -1,$$

so we get simple coroots:

$$\alpha_1^\vee = -\delta_1 + 3\delta_2,$$
$$\alpha_2^\vee = \delta_1 - \delta_2.$$

This reproduces that the coroot lattice is $X_*(T)$, implying that G_2 is simply connected. For completeness:

$$\lambda_1^\vee = \delta_1 + 3\delta_2,$$
$$\lambda_2^\vee = \delta_1 + \delta_2.$$

5.1.2 Quaternionic Discrete Series

Description

Recall the notation from 2.1.2 to discuss discrete series. In particular, recall the two parametrizations of discrete series on $G_2(\mathbb{R})$:

$$\pi^{G_2}_{\lambda,\omega} = \pi^{G_2}_{\omega(\lambda+\rho_{G_2})}$$

for λ a dominant (but possibly irregular) weight of G_2 and ω a Weyl-element that takes the Ω_{G_2} dominant chamber to something Ω_K-dominant—in other words, $1, s_{\alpha_1}$, or s_{α_2}. Note that

$\pi^{G_2}_{\lambda,\omega}$ has infinitesimal character $\lambda + \rho_{G_2}$. Recall that $\omega(\lambda + \rho_{G_2})$ is called the Harish-Chandra parameter of this discrete series.

The quaternionic discrete series of weight k for $k \geq 2$ lies in the L-packet

$$\Pi_{\mathrm{disc}}((k-2)\beta).$$

The members of this L-packet have Harish-Chandra parameters:

$$(k-2)\beta + \rho_G, \qquad s_{\alpha_1}((k-2)\beta + \rho_G), \qquad s_{\alpha_2}((k-2)\beta + \rho_G).$$

As in [23], the quaternionic member is the one with minimal K-type $\lambda_B = 2k\epsilon_2$. We know that the discrete series $\pi(\omega, \lambda)$ has minimal K-type

$$\lambda_B = \omega(\lambda + 2\rho_G) - 2\rho_K$$

by [41, Thm. 9.20]. Therefore the weight-k quaternionic discrete series π_k is specifically $\pi(s_{\alpha_2}, (k-2)\beta)$—computing, s_{α_2} fixes ρ_K so

$$s_{\alpha_2}(\lambda + 2\rho_G) - 2\rho_K = s_{\alpha_2}(\lambda + 2\rho_G - 2\rho_K) = s_{\alpha_2}(\lambda + 2\beta) = s_{\alpha_2}(k\beta) = 2k\epsilon_2.$$

This is the discrete series with Harish-Chandra parameter

$$\lambda_{k,H} := s_{\alpha_2}((k-2)\beta + \rho_G).$$

Their pseudocoefficients

Let φ_k be a pseudocoefficient for π_k. A priori, π_k is not a regular discrete series, so the trace against φ_k may be non-zero for certain non-tempered representations in addition to just π_k. This could make it not work as a test function to pick out just automorphic representations π with $\pi_\infty = \pi_k$. However, this is miraculously not a problem for specifically quaternionic discrete series.

Proposition 5.1.2.1. *Let $k > 2$. Then for any unitary representation ρ of $G_2(\mathbb{R})$:*

$$\mathrm{tr}_\rho(\varphi_k) = \begin{cases} 1 & \rho = \pi_k \\ 0 & \textit{else} \end{cases}.$$

Proof. By the same argument of Vogan described in lemma 3.3.3.1, the trace is 0 unless $\mathrm{tr}_\rho(\eta_{(k-2)\beta}) \neq 0$ for $\eta_{(k-2)\beta}$ the Euler-Poincaré function. This is only possible if ρ appears in an appropriate (g, k)-cohomology. By the main result of [83], the only representations that do so are the packet $\Pi_\lambda((k-2)\beta)$ and something denoted $A_{\mathfrak{q}}((k-2)\beta)$ for $\mathfrak{q}$ corresponding to the Levi subgroup with roots $\pm\alpha_1$ and Weyl group $\Omega_L = \{s_{\alpha_1}, 1\}$. This is because $k > 2$ implies that this is the only Levi such that $(k-2)\beta$ is fixed by Ω_L. See also [51][lem. 2.2].

If $\rho \in \Pi_\lambda((k-2)\beta)$ and $\mathrm{tr}_\rho(\varphi_k) = 0$, then $\rho = \pi_k$ by definition. It therefore suffices to exclude the case of $A_\mathfrak{q}((k-2)\beta)$. We use that $\mathrm{tr}_{A_\mathfrak{q}((k-2)\beta)}(\varphi_k)$ is the coefficient of π_k in the sum expansion

$$A_\mathfrak{q}((k-2)\beta) = \sum_{\rho' \text{ basic}} m(\rho')\rho'$$

in the Grothendieck group and will show that this coefficient is 0.

By [59][thm. 6.4.4], $A_\mathfrak{q}$ is a Langlands quotient of a discrete series on a GL_2 Levi. Let the corresponding parabolic induction be I. Since GL_2 is a maximal proper Levi and the infinitesimal character is regular, the other terms in the expansion need to be discrete series. There have to be two of these since $\mathrm{tr}_{A_\mathfrak{q}((k-2)\beta)}(\eta_{(k-2)\beta}) = -2$ (see [51][lem. 2.2] again). Call these ρ_1 and ρ_2 to make the expansion into basics

$$A_\mathfrak{q}((k-2)\beta) = I - \rho_1 - \rho_2.$$

The computation of the A-packet in [59][thm. 6.4.4] shows that the character

$$A_\mathfrak{q}((k-2)\beta) - \pi_k = I - \rho_1 - \rho_2 - \pi_k$$

is stable. By [75][lem. 5.2], I is stable so $\rho_1 + \rho_2 + \pi_k$ also has to be. Then, [75][lem. 5.1] further implies that $\rho_1 + \rho_2 + \pi_k$ is fully Weyl-invariant on an elliptic maximal torus. Examining Harish-Chandra's character formula for discrete series, this is only possible if the three discrete series are exactly the three members $\Pi_{\mathrm{disc}}((k-2)\beta)$. In particular, $\pi_k \neq \rho_1, \rho_2$ so we are done. □

Corollary 5.1.2.2. *Let f^∞ be a compactly supported locally constant function on $G_2(\mathbb{A}^\infty)$ and $k > 2$. Then*

$$\begin{aligned} I_{\mathrm{spec}}(\varphi_k \otimes f^\infty) &= \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G_2)} m_{\mathrm{disc}}(\pi)\delta_{\pi_\infty = \pi_k} \mathrm{tr}_{\pi^\infty}(f^\infty) \\ &= \sum_{\pi \in \mathcal{AR}_{\mathrm{cusp}}(G_2)} m_{\mathrm{cusp}}(\pi)\delta_{\pi_\infty = \pi_k} \mathrm{tr}_{\pi^\infty}(f^\infty). \end{aligned}$$

Proof. The statement for discrete representations is the same argument as 3.3.3.3 after we know proposition 5.1.2.1. Since $\pi_\infty = \pi_k$ is necessarily discrete series for the non-zero terms, the main result of [86] shows that $m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi)$. □

We therefore have

$$|\mathcal{Q}_k(1)| = I_{\mathrm{spec}}(\varphi_k \otimes \mathbf{1}_K) \tag{5.1}$$

if we choose Gross' canonical measure from [28] at finite places. Note again that this heavily depends on the miracle of proposition 5.1.2.1 and a similar result does *not* hold either for the other members of $\Pi_{\mathrm{disc}}((k-2)\beta)$ or for the Euler-Poincaré function.

5.2 Geometric Side/Application of Endoscopy

5.2.1 Notation

We will need to recall some extra notation related to general reductive group H over F:

- Ω_H^c is the Weyl group generated by compact roots at infinity.
- $d(H_\infty)$ is the size of the discrete series L-packets of H_∞. Alternatively, $d(H_\infty) = |\Omega_H|/|\Omega_H^c|$.
- $k(H_\infty)$ is the size of the group $\mathfrak{K} = \ker(H^1(\mathbb{R}, T_{\mathrm{ell}}) \to H^1(\mathbb{R}, G_\infty))$ that appears in the theory of endoscopy for G_∞.
- $q(H_\infty) = \dim(H_\infty/K_\infty Z_{H_\infty})$ where K_∞ is a maximal compact subgroup of H_∞.
- H_∞^* is the quasisplit inner form of H_∞.
- $\bar{H}_\infty$ is the compact form. If H_∞ has an elliptic maximal torus, this is inner.
- $e(H_\infty)$ is the Kottwitz sign $(-1)^{q(H_\infty^*)-q(H_\infty)}$.
- $[H : M] = [H : M]_F = \dim(A_M/A_G)$, where $A_\star$ is the maximal F-split torus in the center of $\star$. We call this the index of M in H.
- $\tau(H)$ is the Tamagawa number of H.
- Mot_H is the Gross motive for H.
- $L(\mathrm{Mot}_H)$ is the value of the corresponding L-function at 0 (or residue of the pole).
- $\iota^H(\gamma) = \iota_F^H(\gamma)$ for $\gamma \in H(F)$ is the number of connected components of H_γ that have an F-point.

5.2.2 The Hyperendoscopy Formula

Preliminaries

We will use the hyperendoscopy formula of [17] to compute $I_{\mathrm{geom}}(\varphi_k \otimes f^\infty)$. A priori, we need to apply the general case of Theorem 3.1.2.3 since G_2 has endoscopy without simply connected derived subgroup.

Let η_k be the Euler-Poincaré function for $\Pi_{\mathrm{disc}}((k-2)\beta)$. Let $\mathcal{HE}_{\mathrm{ell}}(G_2)$ be the set of non-trivial hyperendoscopic paths for G_2. Then, in the notation of section 3.1,

$$I_{\mathrm{geom}}^{G_2}(\varphi_k \otimes f^\infty) = I_{\mathrm{geom}}^{G_2}(\eta_k \otimes f^\infty) + \sum_{H \in \mathcal{HE}_{\mathrm{ell}}(G_2)} \iota(G, \mathcal{H}) I_{\mathrm{geom}}^{\tilde{\mathcal{H}}}(((\eta_k - \varphi_k) \otimes f^\infty)^{\tilde{\mathcal{H}}}),$$

where the $\tilde{H}$ are choices of z-pair paths when they are needed.

Telescoping

Next, an unpublished work of Kottwitz summarized in [58, §5.4] and proved by other methods in [63] stabilizes $I_{\mathrm{geom}}(\varphi \otimes f^\infty)$ when φ is stable-cuspidal (as all terms on the right side above can be taken to be).

Theorem 5.2.2.1. *Let φ be stable cuspidal on $G_2(\mathbb{R})$ and f^∞ a test function on $G(\mathbb{A}^\infty)$. Then*

$$I^{G_2}_{\mathrm{geom}}(\varphi \otimes f^\infty) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G_2)} \iota(G,H) S^{\tilde{H}}_{\mathrm{geom}}((\varphi \otimes f^\infty)^{\tilde{H}}),$$

where $\mathcal{E}_{\mathrm{ell}}(G_2)$ is the set of elliptic endoscopic groups for G_2 and the $\tilde{H}$ are z-extensions if necessary. The transfers $(\varphi \otimes f^\infty)^{\tilde{H}}$ depend on choices of measures for G and H.

The S_{geom} terms are defined by their values on Euler-Poincaré functions:

$$S^H_{\mathrm{geom}}(\eta_\lambda \otimes f^\infty) = \sum_{M \in \mathcal{L}^{\mathrm{cusp}}(H)} (-1)^{[H:M]} \frac{|\Omega_{M,F}|}{|\Omega_{H,F}|} \tau(M)$$
$$\times \sum_{\gamma \in [M(\mathbb{Q})]_{\mathrm{st,ell}^\infty}} |\iota^M(\gamma)|^{-1} \frac{e(\bar{M}_{\gamma,\infty})}{\mathrm{vol}(\bar{M}_{\gamma,\infty}/A_{\bar{M}_{\gamma,\infty}})} \frac{k(M_\infty)}{k(H_\infty)} \Phi^H_M(\gamma,\lambda) SO^\infty_\gamma((f^\infty)_M),$$

choosing Tamagawa globally measure on all centralizers. The volume on $\bar{M}_{\gamma,\infty}$ is transferred from that on $M_{\gamma,\infty}$ in the standard way for inner forms so that the entire term doesn't depend on a choice of measure at infinity.

There's an alternating sign in the hyperendoscopy formula: if $\mathcal{H}$ is a hyperendoscopic path, then $-\iota(G,\mathcal{H})\iota(\mathcal{H},H) = \iota(G,(\mathcal{H},H))$ for H any endoscopic group of $\mathcal{H}$. Here, $(\mathcal{H},H)$ represents the concatenation and $\mathcal{H}$ is overloaded to also refer to the last group in $\mathcal{H}$.

In particular, substituting in the stabilization telescopes the hyperendoscopy formula.

Final Geometric Formulas and Method of Computation

The final telescoped formula is:

$$I^{G_2}_{\mathrm{geom}}(\varphi_k \otimes f^\infty) = S^{G_2}_{\mathrm{geom}}(\eta_k \otimes f^\infty) + \sum_{\substack{H \in \mathcal{E}_{\mathrm{ell}}(G_2) \\ H \neq G_2}} \iota(G,H) S^{\tilde{H}}_{\mathrm{geom}}((\varphi_k \otimes f^\infty)^{\tilde{H}}). \tag{5.2}$$

We recall:

$$\iota(G,H) = |\Lambda(H,\mathcal{H},s,\eta)|^{-1} \frac{\tau(G)}{\tau(H)},$$

where $\Lambda(H,\mathcal{H},s,\eta)$ is the image in $\mathrm{Out}(\widehat{H})$ of the automorphisms of the endoscopic quadruple.

There are two possible methods to proceed here. We will be using method 2 and mention method 1 in case it is useful for anyone attempting a similar computation on another group.

Method 1:

We can try calculate the S_{geom} terms directly from their formula. We will need to choose Euler-Poincaré measure at $\bar{M}_\gamma$ times canonical measure for the orbital integrals (canonical measure is the same for all inner forms). This adds an extra factor of

$$d(\bar{M}_{\gamma,\infty})\frac{L(\text{Mot}_{M_\gamma})}{e(\bar{M}_{\gamma,\infty})2^{\text{rank}(M_{\gamma,\infty})}}$$

by [79, lem. 6.2]. Since $d(H_\infty) = 1$ and $\text{vol}_{EP}(H_\infty/A_{H_\infty}) = 1$ for H compact, this expands the terms in (5.2) as:

$$S^H_{\text{geom}}(\eta_\lambda \otimes f^\infty) = \sum_{M \in \mathcal{L}^{\text{cusp}}(H)} \left((-1)^{[H:M]}\frac{|\Omega_{M,F}|}{|\Omega_{H,F}|}\right)\left(\tau(M)\frac{k(M_\infty)}{k(H_\infty)}\right)$$
$$\times \sum_{\gamma \in [M(\mathbb{Q})]_{\text{st,ell}^\infty}} 2^{-\text{rank}(M_{\gamma,\infty})}\Phi^H_M(\gamma,\lambda)\left(L(\text{Mot}_{M_\gamma})|\iota^M(\gamma)|^{-1}SO^\infty_\gamma((f^\infty)_M)\right),$$

where the stable orbital integrals are now computed using canonical measure on centralizers.

The hardest terms here are the stable orbital integrals, the L-values, and the characters Φ. Note that since we are using $f^\infty = \mathbf{1}_{K^\infty}$, the constant terms $(f^\infty)_M$ are also indicator functions of hyperspecials.

The L-values may be computed as products of values of Artin L-functions by explicitly describing the motives from [28]. The terms Φ can be reduced to linear combinations traces of γ against finite dimensional representations of G_2 by the algorithm on [4, pg. 273]. These can of be computed by the Weyl character formula and it's extension to irregular elements stated in, for example, [13, prop. 2.3].

The stable orbital integrals unfortunately cause far more difficulty. They are computed and listed in a table on [30, pg. 159]. First, they are interpreted as orbital integrals on compact-at-infinity G_2^c. The spectral side of the trace formula on G_2^c is then possible to compute, allowing the orbital integrals to be solved for once the coefficients in terms of L-values are known.

Even using the previous work of [30], this method is horrendously complicated.

Method 2:

Fortunately, there is a much simpler way to compute our desired count. Recalling that $I^{G_2^c}$ is known from [13], we can compare the expansions (5.2) for G_2 and G_2^c. The term for S^{G_2} a appears in the expansion for $I^{G_2^c}$ and can therefore be solved for and substituted in the expansion for I^{G_2}. In total we get a formula

$$I^{G_2} = I^{G_2^c} + \text{corrections},$$

where the corrections are in terms of S^H for smaller endoscopic H.

In the next section we will see that there aren't actually that many H appearing. Finally, section 5.5 will show that the terms for these H are easily computed through another trick. Method 2 also gives in section 5.6 a Jacquet-Langlands-style result comparing quaternionic representations on G_2 to representations on G_2^c.

5.3 Groups Contributing and Related Constants

5.3.1 Elliptic Endoscopy of G_2

The elliptic endoscopic groups of G_2 are G_2, PGL_3, SO_4, and potentially some tori. This is stated in a thesis [1] but not fully explained, so we work out the computation here for reader convenience. We again use our previous notational conventions for endoscopy as in section 2.2.

By inspecting the root data, the conjugacy classes of centralizers $\widehat{H}$ of a semisimple element in $\widehat{G}_2 \simeq G_2(\mathbb{C})$ are:

- G_2 itself,
- SL_3 from the long roots,
- $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$ from a short and long root that are orthogonal,
- GL_2 from a short root,
- GL_2 from a long root,
- $\mathbb{G}_m^2$.

We compute the possible endoscopic pairs (s, ρ) for each possibility. Recall that, since G_2 is split, ρ is a map from a Galois group to $\mathrm{Out}(\widehat{H}) \cap \Omega_{G_2}$.

Since G_2 has trivial center, the cohomology condition on s is always satisfied so we don't bother checking it. Trivial center further gives that the isomorphism class of the pair cannot change with s. Therefore the only thing that depends on s is whether we can exhibit one that is Galois invariant when ρ is non-trivial.

For each pair we will also compute the automorphism group Λ that comes up in the formula for $\iota(G, H)$.

$\underline{G_2}$:

Then ρ is trivial. This gives the trivial endoscopic group G_2. Since only the trivial element of $\mathrm{Out}(\widehat{H})$ can be realized in Ω_{G_2}, $\Lambda = 1$.

$\underline{\mathrm{SL}_3}$:

There are two possibilities for ρ: trivial or sending a quadratic Galois element to the outer automorphism of SL_3: $(\delta_1, \delta_2) \mapsto (\delta_1, -\delta_2)$ (fixing a long root). We are forced to choose s so that without loss of generality $(\delta_1 + \delta_2)(s) = (\delta_1 - \delta_2)(s) = \zeta_3$ (two short roots of $\widehat{G}_2$ at $120°$). This is preserved by only the trivial ρ.

This gives endoscopic group PGL_3. Here, $\mathrm{Out}(\widehat{H})$ is realized in Ω_{G_2} and commutes with ρ so $|\Lambda| = 2$.

$\underline{\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}}$:

No outer automorphisms can be realized through conjugation in $\widehat{G}_2$ so ρ is trivial. This gives endoscopic group $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$. Since $\mathrm{Out}(\widehat{H}) \cap \Omega_{G_2}$ is trivial, $\Lambda = 1$.

Short GL_2:

Without loss of generality assume the short root is $2\delta_2$. To be elliptic ρ needs to send a quadratic Galois element to the outer automorphism $(\delta_1, \delta_2) \mapsto (-\delta_1, \delta_2)$ of GL_2. To have the right centralizer, $2\delta_1(s) = \alpha$ for some $\alpha \neq 1$ and $2\delta_2(s) = 1$. However, then $(\delta_1+\delta_2)(s) = \pm\sqrt{\alpha}$ which can't be equal to it's inverse. Therefore s can't invariant under ρ. In total, there are no such elliptic endoscopic groups.

Long GL_2:

This is the same as the previous case and gives no elliptic endoscopy.

$\mathbb{G}_m^2$:

Here ρ send Galois elements to any element of Ω_{G_2}. Elliptic means the action can have no invariants except zero in $X^*(T)$. To find which ρ have an invariant, regular s, we look through the possible images of Galois: conjugacy classes of subgroups of D_{12} that don't fix any line.

- C_2: generated by $\alpha \mapsto -\alpha$: Then an invariant s needs to evaluate on all roots to -1 which is impossible
- C_3: Then three short roots in an orbit need to evaluate to the same value on s. The other three short roots evaluate to the square, so we need $\alpha_1(s)^{-1} = \alpha_1(s)^2 \implies \alpha_1(s) = \zeta_3$, which means that long roots evaluate to 1, which is impossible.
- D_4: impossible since C_2 is.
- D_6: impossible since C_3 is
- C_6: impossible since C_2, C_3 are.
- D_{12}: impossible since C_2, C_3 are.

Therefore none are elliptic endoscopic groups.

If a group contributes to the stabilization applied to our test function, then by the fundamental lemma, it needs to be unramified away from infinity. By formulas for transfers of pseudocoefficients, it needs to be elliptic at infinity. The only groups contributing are therefore the G_2 and the $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$.

5.3.2 Endoscopic Constants and Normalizations

The ι

Let $H = \mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ and let G_2^c be the unique non-split inner form of G_2 over $\mathbb{Q}$ which is compact at infinity. Then,

$$\iota(G_2^c, H) = \iota(G_2, H) = |\Lambda(H, \mathcal{H}, s, \eta)|^{-1}\frac{\tau(G)}{\tau(H)} = 1 \cdot \frac{1}{2},$$
$$\iota(G_2^c, G_2) = 1,$$

by Kottwitz's formula for Tamagawa numbers (note that $\ker^1(\mathbb{Q}, Z_H) = \ker^1(\mathbb{Q}, \{\pm 1\}) = 1$).

The transfer factors

We also need to fix transfer factors at all places to compute transfers. The computations in [80] demonstrate how to do so explicitly. First, they can be chosen consistently by fixing a global Whittaker datum. The corresponding local Whittaker datum determine the local transfer factors as in [46]. Since G_2 is defined over $\mathbb{Z}$, we can choose a global datum that is unramified/admissible at all finite places with respect to the $G_2(\mathbb{Z}_p)$ as in [33, §7], so we can use the fundamental lemma at all finite places.

All we will need to know about the Archimedean Whittaker datum for G_2 is which element of $\Pi_{\text{disc}}((k-2)\beta)$ it makes Whittaker-generic. This will have to be $\pi_{(k-2)\beta,1}$ since our choice of dominant Weyl chamber has all simple roots non-compact and is the only possible such choice up to Ω_K (see the discussion before lemma 4.2.1 in [80]. In fact, there is only one possible conjugacy class of Whittaker datum at infinity by considerations explained there).

The stabilizations

We fix canonical measure at finite places so that the fundamental lemma directly gives $\mathbf{1}^H_{K^\infty_{G_2}} = \mathbf{1}_{K^\infty_H}$. Recall that EP-functions and pseudocoefficients are defined depending on measure so we don't need to fix measure at infinity.

Then, (5.2) gives

$$\begin{aligned} I^{G_2}(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)} \otimes \mathbf{1}_{K^\infty_{G_2}}) \\ = S^{G_2}(\eta^{G_2}_{(k-2)\beta} \otimes \mathbf{1}_{K^\infty_{G_2}}) + \frac{1}{2} S^H((\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H \otimes \mathbf{1}_{K^\infty_H}). \end{aligned} \tag{5.3}$$

A simple case of the discrete transfer formula in [48, §IV.3] computes that $(\eta^{G_2^c}_{(k-2)\beta})^{G_2} = \eta^{G_2}_{(k-2)\beta}$ (note that $\Omega_\mathbb{R}(G_2^c)\backslash\Omega_\mathbb{C}(G_2^c)$ is trivial so κ is too), so

$$I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta} \otimes \mathbf{1}_{K^\infty_{G_2^c}}) = S^{G_2}(\eta^{G_2}_{(k-2)\beta} \otimes \mathbf{1}_{K^\infty_{G_2}}) + \frac{1}{2} S^H((\eta^{G_2^c}_{(k-2)\beta})^H \otimes \mathbf{1}_{K^\infty_H}).$$

Since type $A_1 \times A_1$ has no non-trivial centralizer of full semisimple rank, all elliptic endoscopy of $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ is non-split. Therefore, it is ramified at some prime, so the transfers of $\mathbf{1}_{K^\infty_H}$ vanish, implying that, $S^H = I^H$ on our test functions. Substituting one stabilization into another finally gives:

$$\begin{aligned} I^{G_2}(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)} \otimes \mathbf{1}_{K^\infty_{G_2}}) &= I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta} \otimes \mathbf{1}_{K^\infty_{G_2^c}}) \\ &- \frac{1}{2} I^H((\eta^{G_2^c}_{(k-2)\beta})^H \otimes \mathbf{1}_{K^\infty_H}) + \frac{1}{2} I^H((\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H \otimes \mathbf{1}_{K^\infty_H}) \end{aligned} \tag{5.4}$$

under canonical measure at finite places.

This is our realization of method 2. There are three steps remaining to get counts:

1. Compute the transfers of EP-functions to H.

2. Write the resulting $I^H(\eta_\lambda \otimes \mathbf{1}_{K_H})$ terms in terms of counts of level-1, classical modular forms.

3. Look up values for the G_2^c-term from [13].

5.4 Real Endoscopic Transfers

Let H again be the one endoscopic group we care about: $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$. We want to compute $(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H$ and $(\eta^{G_2^c}_{(k-2)\beta})^H$. By our choice of transfer factors, we may do so by the formulas in [48, §IV.3].

As a choice for computation that doesn't affect the final result, we realize the roots of H as $2\epsilon_1$ and $2\epsilon_2$. Orient $X^*(T)$ by setting the 1st quadrant in ϵ_1 and ϵ_2 to be dominant. The Weyl elements $\Omega(G,H)$ that send the G-dominant chamber to an H-dominant one are $\{1, s_{\alpha_1}, s_{\alpha_2}\}$.

5.4.1 Root Combinatorics

Since $\rho_G - \rho_H \in X^*(T)$, [48, §IV.3] gives the transfer of the pseudocoefficient of the quaternionic discrete series to H:

$$(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H = \kappa^H(s_{\alpha_2}^{-1})\eta^H_{(k-2)\beta+\rho_G-\rho_H} - \kappa^H(s_{\alpha_1}s_{\alpha_2}^{-1})\eta^H_{s_{\alpha_1}((k-2)\beta+\rho_G)-\rho_H} - \eta^H_{s_{\alpha_2}((k-2)\beta+\rho_G)-\rho_H} \tag{5.5}$$

for some signs κ.

We compute that $\rho_H = \epsilon_1 + \epsilon_2$. Then

$$(k-2)\beta + \rho_G - \rho_H = (k-2)(3\epsilon_1+\epsilon_2) + (3\epsilon_1+\epsilon) = 3(k-1)\epsilon_1 + (k-1)\epsilon_2.$$

In addition,

$$\begin{aligned} s_{\alpha_1}\rho_G &= 5\epsilon_1+\epsilon_2, & s_{\alpha_1}\beta &= \beta,\\ s_{\alpha_2}\rho_G &= \epsilon_1+3\epsilon_2, & s_{\alpha_2}\beta &= 2\epsilon_2, \end{aligned}$$

so

$$s_{\alpha_1}((k-2)\beta+\rho_G) - \rho_H = (k-2)(3\epsilon_1+\epsilon_2) + (4\epsilon_1) = (3k-2)\epsilon_1 + (k-2)\epsilon_2$$

and

$$s_{\alpha_2}((k-2)\beta+\rho_G) - \rho_H = (k-2)(2\epsilon_2) + (2\epsilon_2) = 2(k-1)\epsilon_2.$$

5.4.2 Endoscopic Characters

It remains to compute the κ terms in 5.5. These signs depend in a very complicated way on the realization of H and the exact transfer factors chosen. They can be pinned down most easily by looking at endoscopic character identities.

Let ψ_H be a (discrete in our case) L-parameter for $H(\mathbb{R})$ and ψ_G the composition with ${}^LH \hookrightarrow {}^LG_2$. The we have an identity of traces over L-packets:

$$S\Theta_{\psi_H}(f^H) = \sum_{\pi \in \Pi_{\psi_G}} \langle \varphi_H, \pi \rangle \Theta_\pi(f),$$

where f^H is a transfer of f, Θ_π is the Harish-Chandra character, $S\Theta_{\psi_H}$ is the stable character corresponding to the L-packet, Π_{φ_G} is the L-packet corresponding to the L-parameter, and $\langle \varphi_H, \pi \rangle$ is a particular pairing depending on transfer factors. See [36, §1] for an exposition of how this works in general.

If π on G_2 is discrete series, Labesse's formula tells us:

$$(\varphi_\pi^{G_2})^H = \sum_\lambda \epsilon(\lambda, \pi) \eta_\lambda^H$$

for some signs ϵ and weights λ. Let ψ_λ be the L-parameter corresponding to weight-λ discrete series on H. Plugging this formula into the character identity for ψ_λ gives that ψ_λ is required to push forward to the parameter for π and that $\epsilon(\lambda, \pi) = \langle \psi_\lambda, \pi \rangle$.

The only fact we need now is that $\epsilon(\lambda, \pi) = \langle \psi_\lambda, \pi \rangle = 1$ whenever π is the Whittaker-generic member of its L-packet. Therefore, in Labesse's formula for the *generic* member $\pi_{1,(k-2)\beta}$,

$$\begin{aligned}(\varphi_{\pi_{G_2}(1,(k-2)\beta)})^H = \eta^H_{(k-2)\beta+\rho_G-\rho_H} &+ \kappa^H(s_{\alpha_1}) \operatorname{sgn}(s_{\alpha_1}) \eta^H_{s_{\alpha_1}((k-2)\beta+\rho_G)-\rho_H} \\ &+ \kappa^H(s_{\alpha_2}) \operatorname{sgn}(s_{\alpha_2}) \eta^H_{s_{\alpha_2}((k-2)\beta+\rho_G)-\rho_H},\end{aligned}$$

all the coefficients need to be 1. The allows to solve

$$\kappa^H(s_{\alpha_1}) = \kappa^H(s_{\alpha_2}) = -1$$

for our choice of transfer factors. Right-$\Omega_\mathbb{R}$-invariance of κ then also gives that

$$\kappa^H(s_{\alpha_1} s_{\alpha_2}) = -1.$$

5.4.3 Final Formulas for Transfers

Therefore, our final transfer is

$$(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H = -\eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} + \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}. \tag{5.6}$$

Transfers from G_2^c are easier. Here, $\Omega_\mathbb{R}(G_2^c) \backslash \Omega_\mathbb{C}(G_2^c)$ is trivial so the average value of κ is 1. Averaging Labesse's formula as in corollary 3.2.1.5 therefore gives:

$$(\eta^{G_2^c}_{(k-2)\beta})^H = \eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} - \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}. \tag{5.7}$$

5.5 The $H = \mathrm{SL}_2 \times \mathrm{SL}_2 / \pm 1$ term

Here we compute the terms $I^H(\eta_\lambda \otimes \mathbf{1}_{K_H})$ for Euler-Poincaré functions η_λ. Any $\lambda = a\epsilon_1 + b\epsilon_2$ is a weight of H if $a+b$ is even. Note first that

$$I^H(\eta_\lambda^H \otimes \mathbf{1}_{K_H}) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(H)} \mathrm{tr}_{\pi_\infty}(\eta_\lambda^H)\,\mathrm{tr}_{\pi^\infty}(\mathbf{1}_{K_H}) = \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(H) \\ \pi \text{ unram.}}} \mathrm{tr}_{\pi_\infty}(\eta_\lambda^H),$$

by Arthur's simple trace formula and using our choice of canonical measure at finite places.

To move forward, we need to understand automorphic reps on H by relating them to other groups. Consider the sequence

$$1 \to \pm 1 \to \mathrm{SL}_2 \times \mathrm{SL}_2 \to H \to 1.$$

It induces on local or global F:

$$1 \to \pm 1 \to \mathrm{SL}_2 \times \mathrm{SL}_2(F) \to H(F) \to F^\times/(F^\times)^2 \to 1,$$

using that $H^1(F, \pm 1) = F^\times/(F^\times)^2$ and $H^1(F, \mathrm{SL}_2) = 1$ for the F we care about (the $\mathbb{R}$ case of the second equality comes from the determinant exact sequence on GL_2). Let H'_F be the image of $\mathrm{SL}_2 \times \mathrm{SL}_2(F)$.

As noted in a similar analysis for SL_2 in [47], unitary irreducibles for H'_F induce to semisimple representations of $H(F)$.

5.5.1 Cohomological Representations of $H(\mathbb{R})$

Next, we recall that the infinite trace measures an Euler characteristic against $(\mathfrak{h}, K_{H,\infty})$-cohomology:

$$\mathrm{tr}_{\pi_\infty}(\eta_\lambda^H) = \chi(H^*(\mathfrak{h}, K_{H,\infty}, \pi_\infty \otimes V_\lambda)),$$

where $\mathfrak{h}$ is the Lie algebra of $H(\mathbb{R})$ and V_λ is the finite dimensional representation of weight λ. Using the definition from [11, §5.1],

$$H^*(\mathfrak{h}, K_{H,\infty}, \pi_\infty \otimes V_\lambda) = H^*(\mathfrak{h}, K^0_{H,\infty}, \pi_\infty \otimes V_\lambda)^{K_{H,\infty}/K^0_{H,\infty}},$$

it suffices to consider the π_∞ whose restrictions to H'_∞ contain a component that is cohomological when pulled back to $[\mathrm{SL}_2 \times \mathrm{SL}_2](\mathbb{R})$. By Frobenius reciprocity and semisimplicity of inductions, these are exactly the irreducible constituents of $\mathrm{Ind}_{H'_\infty}^{H_\infty} \pi'$ for π' cohomological of H'_∞.

Next, H'_∞ is index 2. Pick $h \in H_\infty - H'_\infty$ and let $\pi'^{(h)}$ be the representation $\gamma \mapsto \pi'(h^{-1}\gamma h)$. Define character

$$\chi : H_\infty \mapsto H_\infty / H'_\infty = \pm 1.$$

There are two cases for H'_∞-cohomological π':

1. $\pi' \neq \pi'^{(h)}$: then $\mathrm{Ind}_{H'_\infty}^{H_\infty} \pi'$ is irreducible, and $\mathrm{Res}_{H'_\infty}^{H_\infty} \mathrm{Ind}_{H'_\infty}^{H_\infty} \pi' = \pi' \oplus \pi'^{(h)}$.
2. $\pi' = \pi'^{(h)}$: then $\mathrm{Ind}_{H'_\infty}^{H_\infty} \pi' = V \oplus (V \otimes \chi)$ for some irreducible V. Only one of these factors will have a subspace fixed by $K_{H,\infty}$ so only one of them will have fixed chains and therefore be cohomological. Also, $\mathrm{Res}_{H'_\infty}^{H_\infty} \mathrm{Ind}_{H'_\infty}^{H_\infty} \pi' = \pi' \oplus \pi'$.

Recalling a standard result, the cohomological representations of $\mathrm{SL}_2(\mathbb{R})$ with respect to λ are:

- A discrete series L-packet $\pi_{\lambda,1}$, $\pi_{\lambda,s}$ (where $\Omega_{\mathrm{SL}_2} = \{1, s\}$),
- The trivial representation $\mathbf{1}_{\mathrm{SL}_2}$ if $\lambda = 0$.

By the Künneth rule, cohomological representations of $\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{R})$ are exactly products of those on $\mathrm{SL}_2(\mathbb{R})$. Those of H'_∞ are exactly the $\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{R})$ ones that are trivial on ± 1—in other words, with $\lambda = a\epsilon_1 + b\epsilon_2$ and $a + b$ even.

Consider such λ. There are three cases of inductions to consider to compute the cohomological representations of H. Note that conjugation by $h \in H_\infty - H'_\infty$ switches both factors to the other member of their SL_2-L-packet if they are discrete series and otherwise fixes the trivial representation.

- $a, b \neq 0$: We look at the inductions of products of discrete series. This is case (1) so the 4 products pair up in sums that are 2 members of an L-packet. These are of course $\pi^H_{\lambda,1}$ and $\pi^H_{\lambda,s}$ where s is a length-1 element of Ω_H:

$$\pi^H_{\lambda,1}|_{H'_\infty} = (\pi_{a\epsilon_1,1} \boxtimes \pi_{b\epsilon_2,1}) \oplus (\pi_{a\epsilon_1,s} \boxtimes \pi_{b\epsilon_2,s}),$$
$$\pi^H_{\lambda,s}|_{H'_\infty} = (\pi_{a\epsilon_1,1} \boxtimes \pi_{b\epsilon_2,s}) \oplus (\pi_{a\epsilon_1,s} \boxtimes \pi_{b\epsilon_2,1}).$$

- Without loss of generality, $a = 0, b \neq 0$: We also need to consider inductions of $\mathbf{1} \boxtimes \pi_{b\epsilon_2,\star}$. This is case (1) and both induce to a single irreducible σ^H_λ:

$$\sigma^H_\lambda|_{H'_\infty} = (\mathbf{1} \boxtimes \pi_{b\epsilon_2,1}) \oplus (\mathbf{1} \boxtimes \pi_{b\epsilon_2,s}).$$

- $a = b = 0$: In addition to both the above, we need to consider the induction of $\mathbf{1}_{\mathrm{SL}_2} \boxtimes \mathbf{1}_{\mathrm{SL}_2}$. This is case (2). This trivial representation induces to $\mathbf{1}_{H_\infty} \oplus \chi$ on H_∞. The cohomological piece is $\mathbf{1}_{H_\infty}$.

Grothendieck group relations stay true restricted to H'_∞ so we can compute traces against η_λ. Recall that in $\mathrm{SL}_2(\mathbb{R})$:

$$\mathbf{1} = I - \pi_{0,1} - \pi_{0,s},$$

where I is some non-cohomological parabolically induced representation.

First, by our normalization

$$\mathrm{tr}_{\pi^H_{\lambda,1}}(\eta^H_\lambda) = \mathrm{tr}_{\pi^H_{\lambda,s}}(\eta^H_\lambda) = 1/2.$$

Next, working in H'_∞:

$$\mathbf{1} \boxtimes \pi_{\lambda,\star} = (I - \pi_{0,1} - \pi_{0,s}) \boxtimes \pi_{\lambda,\star} = I \boxtimes \pi_{\lambda,\star} - \pi_{0,1} \boxtimes \pi_{\lambda,\star} - \pi_{0,s} \boxtimes \pi_{\lambda,\star},$$

so

$$\sigma^H_\lambda = \mathbf{1} \boxtimes \pi_{\lambda,1} + \mathbf{1} \boxtimes \pi_{\lambda,s} = I \boxtimes (\pi_{\lambda,1} + \pi_{\lambda,s}) - \pi^H_{0+\lambda,1} - \pi^H_{0+\lambda,s},$$

implying

$$\operatorname{tr}_{\sigma^H_\lambda}(\eta^H_\lambda) = -1.$$

Finally

$$\begin{aligned} \mathbf{1} \boxtimes \mathbf{1} &= (I - \pi_{0,1} - \pi_{0,s}) \boxtimes (I - \pi_{0,1} - \pi_{0,s}) \\ &= I \boxtimes I - I \boxtimes (\pi_{0,1} + \pi_{0,s}) - (\pi_{0,1} + \pi_{0,s}) \boxtimes I + \pi^H_{0+0,1} + \pi^H_{0+0,s}, \end{aligned}$$

so

$$\operatorname{tr}_{\mathbf{1}}(\eta^H_\lambda) = 1.$$

In total, our H-term becomes a count

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(H)} w(\pi_\infty),$$

where w is a weight

$$w^H(\pi_\infty) = \begin{cases} 0 & \pi_\infty \text{ not cohomological} \\ 1/2 & \pi_\infty \text{ one of the } \pi^H_{\lambda,*} \\ -1 & \pi_\infty \text{ one of the } \sigma^H_{\lambda,*} \\ 1 & \pi_\infty \text{ trivial} \end{cases}.$$

Call the cohomological cases type I, II, and III in order.

5.5.2 Reduction to Modular Form Counts

We now recall a result from [13]. Consider central isogeny $G \to G'$ of algebraic groups over $\mathbb{Z}$. If $\pi' = \pi'_\infty \otimes \pi'^\infty$ is an unramified, discrete automorphic representation of G', let $R(\pi')$ be the set of unitary, admissible representations $\pi = \pi_\infty \otimes \pi^\infty$ of $G(\mathbb{A})$ that satisfy:

- π^∞ is unramified with Satake parameters induced from those of π'^∞ through $\widehat{G}' \to \widehat{G}$.
- π_∞ is a constituent of the restriction of π'_∞ through $G(\mathbb{R}) \to G'(\mathbb{R})$.

Note that the size of $R(\pi')$ is the number of constituents of the restriction of π'_∞.

Theorem 5.5.2.1 ([13, cor. 4.10])**.** *Assume that all $\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(G)$ have multiplicity one. Then the same holds for G' and the $R(\pi')$ with $\pi' \in \mathcal{AR}_{\mathrm{disc,ur}}(G')$ partition $\mathcal{AR}_{\mathrm{disc,ur}}(G)$.*

Make similar definitions of type I, II, and III for representations of $[\mathrm{SL}_2 \times \mathrm{SL}_2](\mathbb{R})$ and $[\mathrm{GL}_2 \times \mathrm{GL}_2](\mathbb{R})$. Since type I and II on H decompose into two constituents in $\mathrm{SL}_2 \times \mathrm{SL}_2$ and type III decomposes into 1, our count becomes $\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ weighted by

$$w^{\mathrm{SL}_2 \times \mathrm{SL}_2}(\pi_\infty) = \begin{cases} 1/4 & \pi_\infty \text{ type I} \\ -1/2 & \pi_\infty \text{ type II} \\ 1 & \pi_\infty \text{ type III} \end{cases}.$$

Each $\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ lifts to a rep of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbb{G}_m^2$. This group is further isogenous to $\mathrm{GL}_2 \times \mathrm{GL}_2$ so we apply the theorem again. Type I on $\mathrm{GL}_2 \times \mathrm{GL}_2$ decomposes into 4 constituents on $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbb{G}_m^2$, type II into 2, and type III into 1. Therefore, we get a count of $\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ weighted by

$$w^{\mathrm{GL}_2 \times \mathrm{GL}_2}(\pi_\infty) = \begin{cases} 1 & \pi_\infty \text{ type I} \\ -1 & \pi_\infty \text{ type II} \\ 1 & \pi_\infty \text{ type III} \end{cases}.$$

Let $\mathcal{S}_k(1)$ be the set of normalized, level-1, weight-k cuspidal eigenforms. If $\lambda = a\epsilon_1 + b\epsilon_2$, then type I representations on $\mathrm{GL}_2 \times \mathrm{GL}_2$ correspond to pairs in $S_{a+2}(1) \times S_{b+2}(1)$. Type II is a single form times the trivial representation and Type I is only the trivial representation.

5.5.3 Final Formula for S^H

Therefore, if

$$S_k = |\mathcal{S}_k(1)|,$$

we get:

$$I^H(\eta^H_{a\epsilon_1 + b\epsilon_2} \otimes \mathbf{1}_{K_H}) = (S_{a+2} - \mathbf{1}_{a=0})(S_{b+2} - \mathbf{1}_{b=0}), \tag{5.8}$$

using canonical measure at finite places. By a classical formula ([16, Thm. 3.5.2] for example),

$$S_{a+2} = \begin{cases} 0 & a+2 = 2 \text{ or } a+2 \text{ odd} \\ \lfloor \frac{a+2}{12} \rfloor - 1 & a+2 \equiv 2 \pmod{12} \\ \lfloor \frac{a+2}{12} \rfloor & \text{else} \end{cases}.$$

5.6 A Jacquet-Langlands-style result

5.6.1 First Form

Generalizing (5.4) slightly and substituting in (5.6) and (5.7) gives:

$$I^{G_2}(\varphi_{\pi_k} \otimes f^\infty) = I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta} \otimes f^\infty) - I^H(\eta^H_{(3k-3)\epsilon_1 + (k-1)\epsilon_2} \otimes (f^\infty)^H) \\ + I^H(\eta^H_{(3k-2)\epsilon_1 + (k-2)\epsilon_2} \otimes (f^\infty)^H). \tag{5.9}$$

for any unramified function f^∞ (we use here that $(G_2^c)^\infty = (G_2)^\infty$). This will let us describe the set $\mathcal{Q}_k(1)$ for $k > 2$ in terms of certain representations of G_2^c and H.

Choose $\pi = \pi_k \otimes \pi^\infty \in \mathcal{Q}_k(1)$. Since π^∞ is unramified, it can be described by a sequence of Satake parameters: for each prime p, a semisimple conjugacy class $c_p(\pi^\infty) \in [\widehat{G_2}]_{\mathrm{ss}}$ (note that G_2 is split so we don't need to worry about the full Langlands dual).

The endoscopic datum for H also gives an embedding $\widehat{H} \hookrightarrow \widehat{G_2}$ (noting again that everything is split) whose image contains a chosen maximal torus and therefore induces a map

$$T_H^{G_2} : [\widehat{H}]_{\mathrm{ss}} \twoheadrightarrow [\widehat{G_2}]_{\mathrm{ss}}.$$

The fibers of this map are Ω_{G_2}-orbits of conjugacy classes in H and have size 3 at G_2-regular elements.

Proposition 5.6.1.1. *Let $k > 2$ and π^∞ an unramified representation of $(G_2)^\infty$. Then*

$$m_{\mathrm{disc}}^{G_2}(\pi_k \otimes \pi^\infty) = m_{\mathrm{disc}}^{G_2^c}(V_{(k-2)\beta} \otimes \pi^\infty) - \frac{1}{2}|S^H(\pi^\infty, (3k-3)\epsilon_1 + (k-1)\epsilon_2)| + \frac{1}{2}|S^H(\pi^\infty, (3k-2)\epsilon_1 + (k-2)\epsilon_2)|.$$

Recall here that V_λ is the finite dimensional representation of G_2^c with highest weight λ. Also, $S^H(\pi^\infty, \lambda)$ is the set of $\pi_\infty \otimes \pi_1^\infty \in \mathcal{AR}_{\mathrm{disc}}(H)$ such that:

- $\pi_\infty \in \Pi_{\mathrm{disc}}^H(\lambda)$,
- *For all p, $c_p(\pi_1^\infty) \in (T_H^{G_2})^{-1}(c_p(\pi^\infty))$.*

Proof. This is a standard Jacquet-Langlands-style argument. Through the Satake isomorphism, each f_p can be thought of as a function $[\widehat{G_2}]_{\mathrm{ss}} \to \mathbb{C}$ through $f_p(c_p(\pi)) = \mathrm{tr}_{\pi_p}(f_p)$. It is in fact a Weyl-invariant regular function on a maximal torus in $\widehat{G_2}$. The full version of the fundamental lemma (see the introduction to [34] for example) shows that

$$f_p^H(c_p) = f_p(T_H^{G_2}(c_p))$$

for all $c_p \in \widehat{H}$.

There are only finitely many sequences $c_p(\pi_1^\infty)$ and $T_H^{G_2}(c_p(\pi_1^\infty))$ for π_1^∞ the unramified finite component of an automorphic representation either:

- of G_2 with infinite part π_k,
- of G_2^c with infinite part $V_{(k-2)\beta}$,
- or of H with infinite part in $\Pi_{\mathrm{disc}}((3k-3)\epsilon_1 + (k-1)\epsilon_2)$ or $\Pi_{\mathrm{disc}}((3k-2)\epsilon_1 + (k-2)\epsilon_2)$.

Therefore we can choose an f^∞ that is 0 on all of these sequences $c_p(\pi_1^\infty)$ except 1 on exactly the sequence $c_p(\pi^\infty)$ (this reduces to finding Weyl-invariant polynomials on $(\mathbb{C}^\times)^2$ that take specified values on certain Weyl orbits). The result follows from plugging this f^∞ into (5.9) and noting that $m_{\mathrm{disc}}^H(\pi)$ is always 0 or 1. $\square$

5.6.2 In terms of Modular Forms

We can use the argument from section 5.5.2 to reduce the H-multiplicity terms to GL_2-multiplicty ones. Since we already got multiplicity 1 from comparing H to $\mathrm{SL}_2 \times \mathrm{SL}_2$ there, it will end up being more convenient here to compare H to $\mathrm{PGL}_2 \times \mathrm{PGL}_2$.

First, we have a map on conjugacy classes

$$T^H_{\mathrm{PGL}_2\times\mathrm{PGL}_2} : [\widehat{\mathrm{PGL}_2 \times \mathrm{PGL}_2}]_{\mathrm{ss}} \twoheadrightarrow [\widehat{H}]_{\mathrm{ss}}.$$

Since the first group is $\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})$, the fibers of this map are of the form $\{c, -c\}$ for some $c \in [\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}}$. Composing then gives map

$$T^{G_2}_{\mathrm{PGL}_2\times\mathrm{PGL}_2} : [\widehat{\mathrm{PGL}_2 \times \mathrm{PGL}_2}]_{\mathrm{ss}} \twoheadrightarrow [\widehat{G_2}]_{\mathrm{ss}}.$$

This allows us to define $S^{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi^\infty, \lambda)$ analogous to $S^H(\pi^\infty, \lambda)$ for all $\lambda = a\epsilon_1 + b\epsilon_2$ with both a and b even. For indexing purposes, set it to be empty when a and b aren't even.

Formula (5.8) gives us that $S^H(\pi^\infty, a\epsilon_1 + b\epsilon_2) = \emptyset$ also when a and b aren't both even. In addition, the restriction of discrete series $\pi_\lambda^{\mathrm{PGL}_2\times\mathrm{PGL}_2}$ to $H(\mathbb{R})$ has as components the entire L-packet $\Pi^H_{\mathrm{disc}}(\lambda)$. Therefore, theorem 5.5.2.1 shows that the $R^H_{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi')$ for $\pi' \in S^{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi^\infty, \lambda)$ partition $S^H(\pi^\infty, \lambda)$. Since $R^H_{\mathrm{PGL}_2\times\mathrm{PGL}_2}$ is two-to-one, this gives

$$|S^H(\pi^\infty, \lambda)| = 2|S^{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi^\infty, \lambda)|.$$

Finally, PGL_2 is a quotient of GL_2 by a central torus with trivial Galois cohomology, so automorphic representations on PGL_2 are just those on GL_2 with all components having trivial central character. Recalling injection

$$\iota : [\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}} \hookrightarrow [\mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{C})]_{\mathrm{ss}},$$

this gives:

Corollary 5.6.2.1. *Let $k > 2$ and π^∞ an unramified representation of $(G_2)^\infty$. Then*

$$m^{G_2}_{\mathrm{disc}}(\pi_k \otimes \pi^\infty) = m^{G_2^c}_{\mathrm{disc}}(V_{(k-2)\beta} \otimes \pi^\infty) - |S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty, (3k-3)\epsilon_1 + (k-1)\epsilon_2)| \\ + |S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty, (3k-2)\epsilon_1 + (k-2)\epsilon_2)|.$$

Recall here that V_λ is the finite dimensional representation of G_2^c with highest weight λ. Also, $S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty, \lambda)$ is the set of $\pi_\infty \otimes \pi_1^\infty \in \mathcal{AR}_{\mathrm{disc}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ such that:

- *π_∞ is the discrete series $\pi_\lambda^{\mathrm{GL}_2\times\mathrm{GL}_2}$,*
- *For all p, $c_p(\pi_1^\infty) = \iota(c'_p)$ for some $c'_p \in (T^{G_2}_{\mathrm{PGL}_2\times\mathrm{PGL}_2})^{-1}(c_p(\pi^\infty))$. Here ι is the map $[\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}} \hookrightarrow [\mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{C})]_{\mathrm{ss}}$.*

Of course, since all infinite terms in sight are discrete series, we may again replace the m_{disc} by m_{cusp} using [86].

Note of course that $S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty, a\epsilon_1 + b\epsilon_2) = \emptyset$ unless both a and b are even. Therefore, we can interpret this as, for $k > 2$:

- If k is even: $\mathcal{Q}_k(1)$ is the corresponding set of representations transferred from G_2^c *in addition to* representations transferred from pairs of cuspidal eigenforms in $\mathcal{S}_{3k-2}(1) \times \mathcal{S}_{k-2}(1)$.
- If k is odd: $\mathcal{Q}_k(1)$ is the corresponding set of representations transferred from G_2^c *except for* representations that are also transferred from pairs of cuspidal eigenforms in $\mathcal{S}_{3k-3}(1) \times \mathcal{S}_{k-1}(1)$.

Results for level > 1 would be a lot more complicated since formula (5.4) would have many further hyperendoscopic terms and the comparison to $\mathrm{GL}_2 \times \mathrm{GL}_2$ would not work as nicely.

5.7 Counts of forms

5.7.1 Formula in terms of $I^{G_2^c}$

. To get counts instead of a list, combining formulas (5.1),(5.9), and (5.8) gives that

$$
\begin{aligned}
|\mathcal{Q}_k(1)| = I^{G_2^c}(\eta_\lambda^{G_2^c} \otimes \mathbf{1}_{K^\infty_{G_2^c}}) - (S_{3k-1} - \mathbf{1}_{3k-3=0})(S_{k+1} - \mathbf{1}_{k-1=0}) \\
+ (S_{3k} - \mathbf{1}_{3k-2=0})(S_k - \mathbf{1}_{k-2=0}).
\end{aligned}
$$

This finally becomes, for $k > 2$:

$$
\begin{aligned}
|\mathcal{Q}_k(1)| = {} & I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}}) \\
& + \begin{cases}
\lfloor \frac{k}{4} \rfloor \left(\lfloor \frac{k}{12} \rfloor - 1\right) & k \equiv 2 \pmod{12} \\
\lfloor \frac{k}{4} \rfloor \lfloor \frac{k}{12} \rfloor & k \equiv 0,4,6,8,10 \pmod{12} \\
-\left(\lfloor \frac{3k-1}{12} \rfloor - 1\right)\left(\lfloor \frac{k+1}{12} \rfloor - 1\right) & k \equiv 1 \pmod{12} \\
-\left(\lfloor \frac{3k-1}{12} \rfloor - 1\right)\lfloor \frac{k+1}{12} \rfloor & k \equiv 5,9 \pmod{12} \\
-\lfloor \frac{3k-1}{12} \rfloor \lfloor \frac{k+1}{12} \rfloor & k \equiv 3,7,11 \pmod{12}
\end{cases} . \qquad (5.10)
\end{aligned}
$$

5.7.2 Computing $I^{G_2^c}$

The group $G_2^c(\mathbb{R})$ is compact so the $I^{G_2^c}$ term takes a very simple form: $L^2(G_2^c(\mathbb{Q})\backslash G_2^c(\mathbb{A}))$ decomposes as a direct sum of automorphic representations and the EP-functions η_λ are just scaled matrix coefficients of the finite-dimensional representations V_λ with highest weight λ on $G_2^c(\mathbb{R})$. Therefore

$$I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}}) = \sum_{\pi \in \mathcal{AR}(G_2^c)} \mathbf{1}_{\pi_\infty = V_\lambda} \operatorname{tr}_{\pi^\infty}(\mathbf{1}_{K^\infty_{G_2^c}}),$$

which is just counting the number of unramifed automorphic reps of G_2^c that have infinite component V_λ.

By standard results on unramified representations, taking $K^\infty_{G_2^c}$ invariants sends each such π to a linearly independent copy of V_λ that together span the V_λ-isotypic component of

$$L^2(G_2^c(\mathbb{Q})\backslash G_2^c(\mathbb{A})/K^\infty_{G_2^c}) = L^2(G_2^c(\mathbb{Z})\backslash G_2^c(\mathbb{R})) \subseteq L^2(G_2^c(\mathbb{R})).$$

By Peter-Weyl, $L^2(G_2^c(\mathbb{R}))$ has V_λ-isotypic component $V_\lambda^{\oplus \dim V_\lambda}$. In fact, this component for both the left- and right-actions is the same subspace. Therefore the number of copies of $V_\lambda \subseteq L^2(G_2^c(\mathbb{Z})\backslash G_2^c(\mathbb{R}))$ is $\dim\left(V_\lambda^{G_2^c(\mathbb{Z})}\right)$ by a dimension count.

Summarizing:

$$I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}}) = \dim\left(V_\lambda^{G_2^c(\mathbb{Z})}\right). \tag{5.11}$$

A PARI/GP 2.5.0 program in the online appendix to [13] computes this for all λ by pairing the trace character of $V_\lambda|_{G_2(\mathbb{Z})}$ with the trivial character.

5.7.3 Table of Counts

Table 5.1 gives values of $|\mathcal{Q}_k(1)|$ for $k = 3$ to 52 produced by formula (5.10) and [13]'s table for formula (5.11). The lowest-weight example is bolded, although this work does not rule out the existence of an example with weight 2 or weight 1 (as defined by [67, §1.1]).

Table 5.1: Counts of discrete, quaternionic automorphic representations of level 1 on G_2.

k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$
3	0	13	5	23	76	33	478	43	1792
4	0	14	13	24	126	34	610	44	2112
5	0	15	8	25	121	35	637	45	2250
6	**1**	16	23	26	175	36	807	46	2619
7	0	17	17	27	173	37	849	47	2790
8	2	18	37	28	248	38	1037	48	3233
9	1	19	30	29	250	39	1097	49	3447
10	4	20	56	30	341	40	1332	50	3938
11	1	21	50	31	349	41	1412	51	4201
12	9	22	83	32	460	42	1686	52	4780

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