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CHARACTERIZATION AND SIMPLIFICATION OF NON-SIMPLE MARKED POINT PROCESSES

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Summary. Simple point processes are often characterized by their associated compensators or conditional intensities. For non-simple point processes, however, the conditional intensity and compensator do not uniquely determine the distribution of the process. Various ways of characterizing non-simple multivariate point processes are discussed here, some important classes of separable non-simple processes are investigated, and methods of simplification involving thinning, rescaling, and changing the mark space are presented.

Abbreviated Title: Non-simple marked point processes.

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1 Introduction.

The existence and uniqueness of the compensator, for multivariate or marked point processes with at most one point at any time, has long been known and is the basis for using the compensator for modeling these processes (Jacod, 1975). However, relatively little is known about processes which may have simultaneous points. For such non-simple point processes, the non-uniqueness of the compensator is not only an obstacle to modeling, but also in model evaluation: the horizontal rescaling result of Meyer (1971), which is useful for assessing point process models, requires this simplicity assumption, and the same is true for the extensions and alternate proofs of Meyer's theorem, including those involving vertical rescaling (see e.g. Merzbach and Nualart, 1986; Brown and Nair, 1988; Nair, 1990; Schoenberg, 1999).

The failure of the compensator uniquely to characterize non-simple point processes applies even to the case of unmarked point processes on the line. For an elementary example, given a simple temporal Poisson process of unit rate, consider a non-simple point process constructed to have two points at each time at which the Poisson process has a point. The resulting double-point process has a compensator identical to that of a simple Poisson process of rate two.

The requirement of simplicity for the characterization and rescaling of point processes is unfortunate and does not appear to be met for some applications to data. For instance, point processes have been actively used in modeling the occurrences of earthquakes, and recent seismological research on faulting suggests that earthquakes may begin with multiple ruptures at the same or infinitesimally different times (Kagan, 1994); in such cases even if the underlying earthquake process is simple, the recorded observations of such events are

not.

The aim of the current paper is to explore conditions under which non-simple multivariate point processes may be uniquely characterized. Following some introductory definitions and notation in Section 2, methods for characterizing a general non-simple point process are given in Section 3. Section 4 describes separability criteria for non-simple point processes, defined via conditions on the simplified conditional intensities, and relations between the simplified and ordinary conditional intensities are derived. Finally, Section 5 provides some results on randomly rescaling and thinning non-simple point processes and thus transforming them into simple or separable point processes.

2 Preliminaries.

In this section we review some basic point process constructs; for further details on point processes and conditional intensities see Papangelou (1972), Jacod (1975), Brémaud (1981), Kallenberg (1986), and Daley and Vere-Jones (2003).

A temporal point process N is a σ -finite random measure on the real line \mathbf{R} or a portion thereof, taking values in the non-negative integers or infinity. $N(B)$ represents the number of points in a subset B of \mathbf{R} . For a temporal *marked* point process (hereafter abbreviated t.m.p.p.), to each point there corresponds a random variable from some measurable mark space \mathcal{X} . We consider here the case of temporal marked point processes where the mark space is countable; such a process N may be viewed as a sequence $\{N_i ; i = 1, 2, \dots\}$ of temporal point processes, where the sum $\sum_i N_i(B)$ is σ -finite on \mathbf{R} (see e.g. Brémaud, 1981).

We consider the case where the temporal domain is the real half-line $(0, \infty]$. To the

random measure N_i there corresponds the right-continuous stochastic process

$$N_i(t) := N_i(0, t] \tag{1}$$

for $t > 0$. To avoid ambiguity, we distinguish between $N_i(t)$ as defined in (1) and $N_i(\{t\})$, the number of points at exactly time t and mark x_i . The collection of processes $\{N_i(t)\}$ is considered adapted to some filtered probability space $(\Omega, \mathcal{F}_t, P)$. A *conditional intensity* $\lambda(t)$ of a temporal point process N is a non-negative, \mathcal{F} -predictable process such that $N(t) - \int_0^t \lambda(u) du$ is an \mathcal{F} -martingale; its integral $A(t) = \int_0^t \lambda(u) du$ defines N 's *compensator*, whose general existence and uniqueness are established in Jacod (1975). In the marked setting λ and A are collections $\{\lambda_i\}$ and $\{A_i\}$ of conditional intensities and compensators, respectively, so that each $N_i(t) - A_i(t)$ is an \mathcal{F} -martingale. We assume throughout that the t.m.p.p. N admits a conditional intensity λ .

A t.m.p.p. is *simple* if with probability one, all its points are unique, i.e. no two points occur at the same time and mark. We say a t.m.p.p. is *completely simple* (or, in the terminology of Vere-Jones and Schoenberg (2003), has *simple ground process*), if with probability one, all its points occur at distinct times. For a completely simple t.m.p.p. N , the conditional intensity λ completely characterizes the finite-dimensional distributions of N (Daley and Vere-Jones 2003). Hence in modeling N it suffices to prescribe a model for λ . Important examples of point process models include the Poisson process, for which λ is deterministic, and Hawkes processes (Hawkes, 1971), which have the characteristic that a point at (t, \mathbf{x}) increases the conditional intensity thereafter; Hawkes processes are described in more detail below.

3 Direct characterizations.

The most elementary way to characterize uniquely a non-simple marked point process is via a change in the mark space \mathcal{X} , as in the following three definitions; see also page 195 of Daley and Vere-Jones (2003).

Definition 3.1. Let $\mathcal{X}^* = \bigcup_{k=1}^{\infty} \mathcal{X}^k$, with $\mathcal{X}^k = \mathcal{X} \times \dots \times \mathcal{X}$ the collection of all possible combinations of k marks.

Definition 3.2. Define the function $\phi_i(\underline{x})$ as the multiplicity of the mark x_i in the vector \underline{x} , for $x_i \in \mathcal{X}$ and \underline{x} in \mathcal{X}^* , and let $\phi(\underline{x}) = \sum_i \phi_i(\underline{x})$. That is, if $\underline{x} = \{x_1^{a_1}, x_2^{a_2}, \dots\}$, then $\phi_i(\underline{x}) = a_i$ for each i , and $\phi(\underline{x}) = \sum_i a_i$.

Definition 3.3. Given a t.m.p.p. N on the mark space \mathcal{X} , define the simplified t.m.p.p. N^* on the mark space \mathcal{X}^* as the collection of temporal processes $\{N_{\underline{x}}^*\}$, for $\underline{x} \in \mathcal{X}^*$, each defined via

$$N_{\underline{x}}^*(\{t\}) := \prod_i \mathbf{I}_{\{N_i(\{t\})=\phi_i(\underline{x})\}}, \quad (2)$$

for each t , with \mathbf{I} the indicator function.

Thus $N_{\underline{x}}^*$ has a single point at time t iff. N has a_i points at time t and mark x_i for each i , where $\underline{x} = \{x_1^{a_1}, x_2^{a_2}, \dots\}$.

The most elementary characterization of a non-simple t.m.p.p. N is via the conditional intensity λ^* of the simplified process N^* . The fact that λ^* characterizes N is quite trivial, but is an essential starting point for further considerations and representations, and is given in the following result.

Theorem 3.1. Suppose N is a t.m.p.p. on mark space \mathcal{X} , and that there exists a non-negative \mathcal{F} -predictable sequence of processes $\lambda^* := \{\lambda_{\underline{x}}^*(t); \underline{x} \in \mathcal{X}^*\}$ such that $N_{\underline{x}}^*(t) -$

$\int_0^t \lambda_{\underline{x}}^*(u) du$ is an \mathcal{F} -martingale for each $\underline{x} \in \mathcal{X}^*$, with N^* defined as in (2). Then λ^* completely determines the finite-dimensional distributions of N with respect to the filtered probability space $(\Omega, \mathcal{F}_t, P)$.

Proof. It is evident from the definition of N^* that its finite-dimensional distributions determine those of N . So from the characterization theorem for the conditional intensities of completely simple point processes (see e.g. Jacod, 1975; Brown et al., 1986; or Daley and Vere-Jones, 2003) one need merely verify that N^* is indeed such a process. That N^* is non-negative and integer-valued is immediate from (2). Since for any Borel $B \subset (0, \infty)$, $\sum_{\underline{x} \in \mathcal{X}^*} N_{\underline{x}}^*(B) \leq \sum_i N_i(B)$ and $N_{\underline{x}}^*(B) \leq N_i(B)$ if $\phi_i(\underline{x}) > 0$, N^* inherits its σ -finiteness from N . Therefore N^* is a t.m.p.p., and its complete simplicity follows directly from the construction in (2). \square

The next result relates the two conditional intensities, λ and λ^* .

Theorem 3.2. Suppose the t.m.p.p. N has conditional intensity λ and simplified conditional intensity λ^* . For each i , for almost all t ,

$$\lambda_i(t) = \sum_{\underline{x} \in \mathcal{X}^*} \phi_i(\underline{x}) \lambda_{\underline{x}}^*(t). \quad (3)$$

Proof. Fix i . Observe from (2) that $N_i(t) = \sum_{\underline{x} \in \mathcal{X}^*} \phi_i(\underline{x}) N_{\underline{x}}^*(t)$. So

$$N_i(t) - \int_0^t \sum_{\underline{x} \in \mathcal{X}^*} \phi_i(\underline{x}) \lambda_{\underline{x}}^*(u) du = \sum_{\underline{x} \in \mathcal{X}^*} \phi_i(\underline{x}) \left[N_{\underline{x}}^*(t) - \int_0^t \lambda_{\underline{x}}^*(u) du \right],$$

which is a linear combination of \mathcal{F} -martingales and is therefore itself an \mathcal{F} -martingale. Hence the sum in (3) is an \mathcal{F} -conditional intensity of N_i , and thus coincides almost everywhere with λ_i by the uniqueness theorem for point process compensators (Jacod, 1975). \square

Alternatively, one may consider describing a non-simple t.m.p.p. N in terms of a sequence of point processes whose points are identical to those of N but whose multiplicities are the powers of those of N , as defined below.

Definition 3.4. For $i, j = 1, 2, \dots$, let $N_i^{(j)}(B) = \sum_{\underline{x} \in B} \int [\phi_i(\underline{x})]^j dN_{\underline{x}}^*$.

Each $N^{(j)}$ is an \mathcal{F} -adapted t.m.p.p. provided the same is true of N , and one may consider whether the collection of their conditional intensities $\{\lambda_i^{(j)}; i, j, = 1, 2, \dots\}$ uniquely determines the distribution of N . The answer in general is no, since N may have simultaneous points at different marks, and $\{\lambda_i^{(j)}\}$ do not uniquely determine the likelihood of such occurrences for each combination of marks. However, a separability condition under which the conditional intensities of $N^{(j)}$ do uniquely characterize N is given in the following Section.

4 Separable conditional intensity.

Theorem 3.1 showed that a non-simple t.m.p.p. N may be characterized via its simplified conditional intensity λ^* , assuming it exists. Certain special cases are worth considering. One natural case to consider is the following.

Definition 4.1. N has *separable* simplified intensity λ^* if, for all $\underline{x} \in \mathcal{X}^*$ and almost all $t \in (0, \infty)$,

$$\lambda_{\underline{x}}^*(t) = (\lambda^*(t))^{(1-\psi(\underline{x}))} \prod_{\substack{\underline{x}' = \{x_i^a\}; \\ a = \phi_i(\underline{x}) > 0}} \lambda_{\underline{x}'}^*(t), \quad (4)$$

where $\lambda^*(t) := \sum_{\underline{x} \in \mathcal{X}^*} \lambda_{\underline{x}}^*(t)$, and $\psi(\underline{x}) := \sum_i \mathbf{1}_{\phi_i(\underline{x}) > 0}$ is the number of distinct marks in \underline{x} .

Note that relation (4) implies that for all $\underline{x} = \{x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}\}$, for almost all t ,

$$\lambda_{\{x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}\}}^*(t) = \lambda_{\{x_1^{a_1}, \dots, x_{k-1}^{a_{k-1}}\}}^*(t) \lambda_{\{x_k^{a_k}\}}^*(t) / \lambda^*(t).$$

Heuristically, separability means that the likelihood of N simultaneously having a_1 overlapping points at mark x_1 , and a_2 overlapping points at mark x_2 , etc., is proportional to the products of the likelihoods of each of these phenomena occurring individually. Note also that if λ^* is separable then the finite-dimensional distributions of N are completely determined by the collection of processes $\{\lambda_{\{x^a\}}^*(t)\}$ alone.

Definition 4.2. λ^* is *completely separable* if it is separable and also, for all $x \in \mathcal{X}$ and integers $a > 0$, for almost all t , $\lambda_{\{x^a\}}^*(t) = (\lambda^*(t))^{(1-a)} \left(\lambda_{\{x\}}^*(t)\right)^a$.

Note that complete separability of λ^* implies that if $\underline{x} = \{x_1^{a_1}, \dots, x_k^{a_k}\}$ with $\sum a_k = a$, then $\lambda_{\underline{x}}^*(t) = (\lambda^*(t))^{(1-a)} \prod_{i=1}^k \left(\lambda_{\{x_i\}}^*(t)\right)^{a_i}$, a.e. With complete separability, the addition into \underline{x} of a new term x_i (or equivalently the addition of one to a_i for some i) results in the multiplication of $\lambda_{\underline{x}}^*(t)$ by $\lambda_{\{x_i\}}^*(t)/\lambda^*(t)$, a.e. If λ^* is completely separable, then λ^* is governed (almost everywhere) by the collection of processes $\{\lambda_{\{x\}}^*(t)\}$ alone.

We now relate the separability condition to the processes defined at the end of Section 3.

Theorem 4.1. Suppose N is a t.m.p.p. with separable simplified intensity λ^* , and suppose that an intensity $\lambda_i^{(j)}(t)$ as defined following Definition 3.4 exists for any positive integers i and j . Then the collection of processes $\{\lambda_i^{(j)}(t) ; i, j = 1, 2, \dots\}$ uniquely determine the finite-dimensional distributions of N .

Proof. For any $j = 1, 2, \dots$ and any \mathcal{F} -predictable process $Y(t, x)$ on $(0, \infty) \times \mathcal{X}$,

$$\begin{aligned} E \sum_i \int Y(t, x_i) \lambda_i^{(j)}(t) dt &= E \sum_i \int Y(t, x_i) dN_i^{(j)}(t) \\ &= E \sum_i \sum_{\underline{x}} \int (\phi_i(\underline{x}))^j Y(t, x_i) dN_{\underline{x}}^*(t) \\ &= E \sum_i \sum_{\underline{x}} \int (\phi_i(\underline{x}))^j Y(t, x_i) \lambda_{\underline{x}}^*(t) dt. \end{aligned}$$

So for $i, j = 1, 2, \dots$, for almost all t ,

$$\begin{aligned} \lambda_i^{(j)}(t) &= \sum_{\underline{x}} (\phi_i(\underline{x}))^j \lambda_{\underline{x}}^*(t) \\ &= \sum_{\underline{x}} (\phi_i(\underline{x}))^j (\lambda^*(t))^{(1-\psi(\underline{x}))} \prod_{\substack{\underline{x}'=\{x_k^a\}; \\ a=\phi_k(\underline{x})>0}} \lambda_{\underline{x}'}^*(t) \\ &= c_i \sum_{a=1}^{\infty} a^j \lambda_{\{x_i^a\}}^*(t), \end{aligned}$$

where $c_i = 1 + \sum_{\underline{x}:\phi_i(\underline{x})=0} (\lambda^*(t))^{(2-\psi(\underline{x}))} \prod_{\substack{\underline{x}'=\{x_k^b\}; \\ b=\phi_k(\underline{x})>0}} \lambda_{\underline{x}'}^*(t) > 0$. Hence for any i and almost all t , the collection $\{\lambda_i^{(j)}(t); j = 1, 2, \dots\}$ uniquely determines $\{\lambda_{\{x_i^k\}}^*(t); k = 1, 2, \dots\}$ (see e.g. ch. 6 of Berman and Fryer, 1972); thus for almost all t , $\{\lambda_i^{(j)}(t); i, j = 1, 2, \dots\}$ uniquely determines $\{\lambda_{\{x_i^k\}}^*(t); k = 1, 2, \dots\}$. Since λ^* is separable, this implies that $\{\lambda_i^j(t)\}$ determines λ^* a.e. and thus the finite-dimensional distributions of N from Theorem 3.1. \square

Under the assumption that the simplified conditional intensity λ^* is completely separable, the ordinary conditional intensity λ can be written directly in terms of the simplified conditional intensity, as in the following result.

Theorem 4.2. Suppose N is a t.m.p.p. with conditional intensity λ and completely separable λ^* . Then for all i and almost all t ,

$$\lambda_i(t) = \frac{2 \lambda_{\{x_i\}}^*(t) \lambda^*(t)}{\lambda^*(t) - \lambda_{\{x_i\}}^*(t)}. \quad (5)$$

Proof. The following proof makes repeated use of the note following Definition 4.2, as well as the idea that for any i , the set \mathcal{X}^* corresponds one-to-one with the collection $\{\underline{x} : \phi_i(\underline{x}) > 0; \phi(\underline{x}) > 1\}$, where the correspondence is simply the augmentation by one of the multiplicity $\phi_i(\underline{x})$.

Fix i . We shall suppress t for brevity, as what follows is true for almost all t .

Consider the collection of all \underline{x} such that the multiplicity $\phi_i(\underline{x})$ of the mark x_i is positive.

Complete separability implies that the sum of $\lambda_{\underline{x}}^*$ over all such \underline{x} is given by

$$\begin{aligned}
\bar{\lambda}_i^* &:= \sum_{\underline{x}: \phi_i(\underline{x}) > 0} \lambda_{\underline{x}}^* \\
&= \lambda_{\{x_i\}}^* + \sum_{\underline{x}: \phi_i(\underline{x}) > 0; \phi(\underline{x}) > 1} \lambda_{\underline{x}}^* \\
&= \lambda_{\{x_i\}}^* + \sum_{\underline{x}: \phi(\underline{x}) > 0} \left(\lambda_{\{x_i\}}^* / \lambda^* \right) \lambda_{\underline{x}}^* \\
&= \lambda_{\{x_i\}}^* + \left(\lambda_{\{x_i\}}^* / \lambda^* \right) \lambda^* \\
&= 2\lambda_{\{x_i\}}^*. \tag{6}
\end{aligned}$$

One may now write λ_i in terms of λ^* , summing over all possible $\underline{x} \in \mathcal{X}^*$, as follows:

$$\begin{aligned}
\lambda_i &= \sum_{j=1}^{\infty} \sum_{\underline{x}: \phi_i(\underline{x})=j} j \lambda_{\underline{x}}^* \\
&= \sum_{j=1}^{\infty} j \lambda_{\{x_i^j\}}^* + \sum_{j=1}^{\infty} \sum_{\underline{x}: j=\phi_i(\underline{x}) < \phi(\underline{x})} j \lambda_{\underline{x}}^* \\
&= \sum_{j=1}^{\infty} j (\lambda^*)^{1-j} (\lambda_{\{x_i\}}^*)^j + \sum_{j=1}^{\infty} \sum_{\underline{x}': \phi_i(\underline{x}')=0} j (\lambda^*)^{-j} (\lambda_{\{x_i\}}^*)^j \lambda_{\underline{x}'}^* \tag{7}
\end{aligned}$$

$$= \left[\sum_{j=1}^{\infty} j \left(\frac{\lambda_{\{x_i\}}^*}{\lambda^*} \right)^j \right] [\lambda^* + (\lambda^* - \bar{\lambda}_i^*)] \tag{8}$$

$$= \frac{(\lambda_{\{x_i\}}^* / \lambda^*)}{[1 - \lambda_{\{x_i\}}^* / \lambda^*]^2} [2\lambda^* - 2\lambda_{\{x_i\}}^*] \tag{9}$$

$$= \frac{2\lambda_{\{x_i\}}^* \lambda^*}{\lambda^* - \lambda_{\{x_i\}}^*},$$

where the observation that $\sum_{\underline{x}': \phi_i(\underline{x}')=0} \lambda_{\underline{x}'}^* = \lambda^* - \bar{\lambda}_i^*$ is used to go from (7) to (8), and the relation $\bar{\lambda}_i^* = 2\lambda_{\{x_i\}}^*$ of (6) is used to go from (8) to (9). \square

We now consider the case where N may have multiple points simultaneously, but where no two such points may occur at the same mark.

Definition 4.3. λ^* is *singular* if for all \underline{x} such that $\phi_i(\underline{x}) > 1$ for any i , $\lambda_{\underline{x}}^*(t) = 0$ a.e.

Theorem 4.3. Suppose N is a singular t.m.p.p. with conditional intensity λ and separable λ^* . Then for all i , for almost all t ,

$$\lambda_i(t) = \frac{2\lambda_{\{x_i\}}^*(t)\lambda^*(t)}{\lambda^*(t) + \lambda_{\{x_i\}}^*(t)}. \quad (10)$$

Proof. Again, we suppress t ; the following is true for almost all t .

$$\begin{aligned} \lambda_i &= \lambda_{\{x_i\}}^* + \sum_{\underline{x}: \phi_i(\underline{x})=1; \phi(\underline{x})>1} \lambda^*(\underline{x}) \\ &= \lambda_{\{x_i\}}^* + \frac{\lambda_{\{x_i\}}^*}{\lambda^*} \sum_{\underline{x}: \phi_i(\underline{x})=0} \lambda_{\underline{x}}^* \\ &= \lambda_{\{x_i\}}^* + \frac{\lambda_{\{x_i\}}^*}{\lambda^*} (\lambda^* - \lambda_i). \end{aligned}$$

So $\lambda_i \left(1 + \frac{\lambda_{\{x_i\}}^*}{\lambda^*}\right) = 2\lambda_{\{x_i\}}^*$, which establishes (10). \square

We now consider the antithesis of singularity, i.e. the case where N may have multiple points at a given time, but such multiple points must all occur at the same mark.

Theorem 4.4. Suppose N is a t.m.p.p. with conditional intensity λ and that $\lambda_{\underline{x}}^*(t) = 0$ a.e. for all \underline{x} such that $\phi(\underline{x}) > \phi_i(\underline{x})$ for all i . Suppose also that for almost all t , $\lambda_{\{x_i\}}^{*a_i} = (\lambda^*)^{1-a_i} (\lambda_{\{x_i\}}^*)^{a_i}$. Then for each i , for almost all t ,

$$\lambda_i(t) = \frac{\lambda_{\{x_i\}}^*(t) (\lambda^*(t))^2}{(\lambda^*(t) - \lambda_{\{x_i\}}^*(t))^2}.$$

Proof. Again we suppress t , and the following is true a.e. Under the stated conditions, one need only consider elements of \mathcal{X}^* of the form $\underline{x} = \{x_i^k\}$. Hence

$$\begin{aligned} \lambda_i &= \lambda_{\{x_i\}}^* + 2\lambda_{\{x_i^2\}}^* + 3\lambda_{\{x_i^3\}}^* + \dots \\ &= \lambda^* \sum_{k=1}^{\infty} k \left(\frac{\lambda_{\{x_i\}}^*}{\lambda^*}\right)^k \\ &= \frac{\lambda^* (\lambda_{\{x_i\}}^*/\lambda^*)}{(1 - (\lambda_{\{x_i\}}^*/\lambda^*))^2} \end{aligned}$$

$$= \frac{\lambda_{\{x_i\}}^* (\lambda^*)^2}{(\lambda^* - \lambda_{\{x_i\}}^*)^2}. \quad \square$$

5 Transformations.

One way to simplify a point process is via Definition 3.3, i.e. by expanding the mark space. Alternatively, one may randomly transform the process to obtain a Poisson or completely separable process, as in the following three results.

Theorem 5.1. Suppose N is a t.m.p.p. with simplified intensity λ^* , such that for each $\underline{x} \in \mathcal{X}^*$, $\int_0^\infty \lambda_{\underline{x}}^*(u) du = \infty$. Then the time transformation which moves each point of N^* from (t, \underline{x}) to $\left(\int_0^t \lambda_{\underline{x}}^*(s) ds, \underline{x}\right)$ results in a sequence $\{\tilde{N}_{\underline{x}}; \underline{x} \in \mathcal{X}^*\}$ of independent Poisson processes of unit rate.

Proof. Since N^* is completely simple, the result follows from application of the random time change theorem of Meyer (1971). \square

Our final two results involve randomly thinning a point process, where the thinning depends on a uniformly distributed random variable that is independent of the point process. Hence we suppose there exists a white noise process U_t on $(\Omega, \mathcal{F}_t, P)$ with $\{U_t; t \geq 0\}$ independent of N , and where the U_t are i.i.d. uniformly distributed on $(0, 1)$.

Theorem 5.2. Suppose N has simplified intensity λ^* and that for each $i = 1, 2, \dots$, for almost all t , $\bar{\lambda}_i^*(t) > 0$. Let $b(t, x)$ be any strictly positive predictable process on $(0, \infty) \times \mathcal{X}$, independent of $\{U_t\}$, and such that for almost all t , $\sum_{x \in \mathcal{X}} c(t, x) < 1$, where $c(t, x_i) := b(t, x_i) / \bar{\lambda}_i^*(t)$. Let $c(t, x_0) = 0$, and consider the transformation $N \rightarrow \tilde{N}$ where \tilde{N} has a singular point at (t, x_i) provided N^* has a point at (t, \underline{x}) with $\phi_i(\underline{x}) > 0$ and $\sum_{j=0}^{i-1} c(t, x_j) \leq$

$U_t < \sum_{j=0}^i c(t, x_j)$. Then \tilde{N} is a simple \mathcal{F} -adapted marked point process with conditional intensity $\tilde{\lambda}_i(t) = b(t, x_i)$.

Proof. It is clear that \tilde{N} is an \mathcal{F} -adapted point process, and \tilde{N} is simple by construction since for any t , $\sum_{j=0}^{i-1} c(t, x_j) \leq U_t < \sum_{j=0}^i c(t, x_j)$ can be true for at most one i . For any \mathcal{F} -predictable process $Y(t, x)$ on $(0, \infty) \times \mathcal{X}$,

$$\begin{aligned} E \sum_i \int Y(t, x_i) d\tilde{N} &= E \sum_i \sum_{\underline{x}: \phi_i(\underline{x}) > 0} c(t, x_i) \int Y(t, x_i) dN_{\underline{x}}^* \\ &= E \sum_i \sum_{\underline{x}: \phi_i(\underline{x}) > 0} c(t, x_i) \int Y(t, x_i) \lambda_{\underline{x}}^*(t) dt \\ &= E \sum_i \int Y(t, x_i) c(t, x_i) \sum_{\underline{x}: \phi_i(\underline{x}) > 0} \lambda_{\underline{x}}^*(t) dt. \end{aligned}$$

Therefore a version of the conditional intensity of \tilde{N} is given by

$$\begin{aligned} \tilde{\lambda}_i(t) &= c(t, x_i) \sum_{\underline{x}: \phi_i(\underline{x}) > 0} \lambda_{\underline{x}}^*(t) \\ &= \frac{b(t, x_i)}{\bar{\lambda}_i^*(t)} \bar{\lambda}_i^*(t) \\ &= b(t, x_i). \end{aligned} \quad \square$$

The previous results involve rescaling or thinning N in order to form a simple point process.

We now turn our attention to the problem of transforming N instead into a completely separable t.m.p.p.

Theorem 5.3. Suppose a t.m.p.p. N has simplified intensity λ^* and that a strictly positive predictable process $b(t, x)$ can be found that is independent of $\{U_t\}$ and so that for all \underline{x} and almost all t ,

$$(\lambda^*(t))^{1-\phi(\underline{x})} \prod_i b(t, x_i)^{\phi_i(\underline{x})} \leq \lambda_{\underline{x}}^*(t). \quad (11)$$

Let \tilde{N} be a thinned version of N so that \tilde{N}^* has a point at (t, \underline{x}) whenever N^* has a point at (t, \underline{x}) , provided $U_t < \left(\lambda_{\underline{x}}^*(t)\right)^{-1} (\lambda^*(t))^{1-\phi(\underline{x})} \prod_i b(t, x_i)^{\phi_i(\underline{x})}$. Then \tilde{N} is an \mathcal{F} -adapted marked point process with completely separable conditional intensity $\tilde{\lambda}^*$ such that $\tilde{\lambda}_{\{x\}}^*(t) = b(t, x)$.

Proof. That \tilde{N} is an \mathcal{F} -adapted point process is clear as it inherits the necessary properties directly from N , and \tilde{N}^* is simple since the same is true of N^* . Since N^* has conditional intensity λ^* , for any \mathcal{F} -predictable process $Y(t, \underline{x})$ on $(0, \infty) \times \mathcal{X}^*$,

$$\begin{aligned} E \sum_{\underline{x}} \int Y(t, \underline{x}) d\tilde{N}^* &= E \sum_{\underline{x}} \int Y(t, \underline{x}) \left(\lambda_{\underline{x}}^*(t)\right)^{-1} (\lambda^*(t))^{1-\phi(\underline{x})} \prod_i b(t, x_i)^{\phi_i(\underline{x})} dN^* \\ &= E \sum_{\underline{x}} \int Y(t, \underline{x}) \left(\lambda_{\underline{x}}^*(t)\right)^{-1} (\lambda^*(t))^{1-\phi(\underline{x})} \prod_i b(t, x_i)^{\phi_i(\underline{x})} \lambda_{\underline{x}}^*(t) dt. \end{aligned}$$

Hence $(\lambda^*(t))^{1-\phi(\underline{x})} \prod_i b(t, x_i)^{\phi_i(\underline{x})}$ is a conditional intensity for \tilde{N}^* , which in view of Definition 4.2 establishes the desired result. \square

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