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The Magical Geometry of 1D Quantum Liquids

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To my parents, for their unwavering love and support.

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Curriculum Vitæ

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Education

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Publications

1	Eugeniu Plamadeala, T. Pereg-Barnea, and Gil Refael. "*Probing order parameter structure in iron-based superconductors using vortices*". Phys. Rev. B, 81:134513, 2011.
2	Eugeniu Plamadeala, Michael Mulligan, and Chetan Nayak. "*Short-range entangled bosonic states with chiral edge modes and T-duality of heterotic strings*". Phys. Rev. B, 88:045131, 2013.
3	Jennifer Cano, Meng Cheng, Michael Mulligan, Chetan Nayak, Eugeniu Plamadeala, Jon Yard."*Bulk-Edge Correspondence in 2+1-Dimensional Abelian Topological Phases*" . Phys. Rev. B, 89:115116, 2014.
4	Eugeniu Plamadeala, Michael Mulligan, Chetan Nayak. "*Perfect Metal Phases of One-Dimensional and Anisotropic Higher-Dimensional Systems*". Phys. Rev. B, 90, 241101(R), 2014.
5	Eugeniu Plamadeala, Michael Mulligan, Chetan Nayak. "*Transport in a One-Dimensional Hyperconductor*. Phys. Rev. B 93, 125142, 2016.
6	Parsa Bonderson, Meng Cheng, Kaushal Patel, Eugeniu Plamadeala. "*Topological Enrichment of Luttingers Theorem*". arXiv:1601.07902 2016

Abstract

The Magical Geometry of 1D Quantum Liquids

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We investigate the edge properties of Abelian topological phases in two spatial dimensions. We discover that many of them support multiple fully chiral edge phases, with surprising and measurable experimental consequences. Using the machinery of conformal field theory and integral quadratic forms we establish that distinct chiral edge phases correspond to genera of positive-definite integral lattices. This completes the notion of bulk-boundary correspondence for topological phases. We establish that by tuning inter-channel interactions the system can be made to transition between the different edge phases without closing the bulk gap.

Separately we construct a family of one-dimensional models, called Perfect Metals, with no relevant mass-generating operators. These theories describe stable quantum critical phases of interacting fermions, bosons or spins in a quantum nanowire. These models rigorously answer a long-standing question about the existence of stable metallic phases in one and two spatial dimensions in the presence of generic disorder. Separately, they are the first example of a stable phase of an infinite parallel array of coupled Luttinger liquids.

We perform a detailed study of the transport properties of Perfect Metals and show that in addition to violating the Wiedemann-Franz law, they naturally exhibit low power-law dependence of electric and thermal conductivities on temperature ($\sigma \propto 1/T$) all the way to zero temperature. We dub this phenomenological set of properties a hyperconductor because in some sense, hyperconductors are better conductors that superconductors,

which may have thermal conductivities that are exponentially small in temperature.

Contents

Chapter 1

Introduction

1.1 The study of condensed matter physics

The various branches of physics admit a classification in terms of the typical length scale on which the phenomena studied occur. Astrophysics is the study of the very large, with typical length scales on the order of light-years (10^{16} m). Geophysics or planetary physics studies smaller objects, comparable to the size of the Earth (10^{7} m). Particle physics studies the internal structure of atoms (10^{-10} m). Condensed matter physics studies phenomena of intermediate length scales, from the size of nanotubes (10^{-9} m) to that solar cells (0.01 m).

While this is a very wide range of length scales, all the problems we condensed matter physicists concern ourselves with involve many atoms (anywhere from $10,000$ to 10^{21}). This last fact is responsible for the difficulty of the problem - with so many variables around we cannot solve the governing equation of the system - the Schroedinger equation. Put differently, we all know that $\mathbf{F} = m\,\mathbf{a}$, but how do you apply this to a million particles? You cannot write down the problem, let alone solve it. It is for this reason we often call it "Many-Body Physics".

On the other hand, it is precisely the multitude of degrees of freedom of a system that give rise to the richness and beauty of physical phenomena it exhibits. You can know everything there is to know about a single molecule of water, but it is very difficult (and some people think impossible) to predict from that the existence of water waves, or the diversity of snowflakes. The set of characteristics necessary to describe a water wave (its amplitude, wavelength, velocity) are very different from the characteristics of its constituents (the water molecules), and in fact do not even make sense at the molecular scale. We say that water waves are an *emergent property* of large collections of water molecules. That is why, in my opinion, the motto of condensed matter physics is that "more is different" [1].

All the physical systems discussed in this document are of intermediate lengths (10^{-6} m or larger) and have sufficiently many degrees of freedom to qualify for this motto. Indeed, that in the fractional quantum Hall effect (discussed in Chapter 4) a collection of electrons, all of unit charge, can behave as a quantum liquid with waves of fractional charge is an example of emergence at play: many ones sometimes look like fractions.

1.2 Organization

In this section I will outline the organization of this document, as well as the results presented in each chapter.

Lattices In Chapter 2 I give a mathematical introduction to lattices, with an emphasis on integral unimodular lattices in high dimensions. I spell out the connection between integral lattices and K-matrices, and I discuss a series of results from the theory of quadratic forms that are central to the work in the entire rest of this document. In particular, it is established that positive-definite integral unimodular lattices exist with

arbitrarily large minimal norms exist, and that all integral indefinite lattices are related by generalized Lorentz transformations.

Edge phases of bosonic IQH states In Chapter 3 we present a study of the edge phases of 2D Abelian bosonic integer quantum Hall phases. It was known that the bulk topological phases are classified by their chiral central charge, $c = 8k$, which must be a multiple of 8. We show that multiple edge phases can terminate the same bulk, and that the bulk-boundary correspondence is many-to-one. Furthermore, we show that different edge phases are not only allowed but also energetically favorable in certain parameter regimes.

We achieve this by writing down the exact edge theories for the $c = 16$ case in the language of multi-channel Luttinger liquids and studying their stability to channel-mixing and backscattering in RG. The fixed point theories are parametrized by the short-range density-density interactions between the channels. We exhibit a one-parameter family of interactions that can drive the edge of a sample from one phase to the other, while encountering (generically) a KT-transition.

The distinct edge phases are shown to be in one-to-one correspondence with positive-definite even integral unimodular lattices, which also exist only in dimensions that are a multiple of 8.

While any given edge phase is chiral and therefore immune to perturbations (they cannot open a gap), in the presence of counter-propagating edge modes that are gapped out the mixing between the channels becomes important. This is the physical mechanism that makes the transition between the edge phases possible. Mathematically, it is conveniently captured by the notion of stable equivalence of lattices.

To make connection with possible experiments, we exhibit constructions of the two possible $c = 16$ edge theories starting with fermionic degrees of freedom. The difference

between the two theories turns out to be subtle and closely related to whether one can "hear the shape of a drum?"[2]. If the wave equation is a Dirichlet problem for the Laplacian, the question then is whether the spectrum of eigenvalues of the Laplacian (the so-called *fundamental tones*) uniquely identifies the shape of the space which is resonating. In 1964 Milnor constructed two flat 16-manifolds by modding out $\mathbb{R}^{16}$ by the integral unimodular lattices known as Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ (see appendix). He did this with the help of modular forms, which are functions defined on a given lattice and which count the number of vectors of a given length in it. He established that the two lattices have the same modular forms, and therefore that the two 16-manifolds constructed with their help are isospectral. You can't always hear the shape of a drum it turns out (for more pedestrian examples, featuring 2-manifolds see Ref [3]).

The lattices used by Milnor in his isospectrality proof are the very lattices associated with the two distinct edge phases we exhibit for $c = 16$ integer quantum Hall phases of bosons. The addition of vectors in a lattice are the fusion rules of operators in the edge conformal field theory. Furthermore, the lengths of vectors in each corresponding lattice are the scaling dimensions of associated operators. As a result the equality of the two modular forms implies that the edge phases cannot be distinguished through any experiment that is sensitive only to the two-point functions in the theory (e.g. tunneling across a quantum point contact, tunneling to an STM tip). However, as we show in 3.4 the two edge theories differ in their 3- and higher-point functions.

Finally, to show that neither of the two edge phases is a priori preferable to the other we construct phase diagrams of the $c = 16$ edge as a function of various tuning parameters. These suggest that both phases occupy comparable volumes of the parameter space.

Edge phases of 2+1D Abelian topological phases In this chapter we generalize the results of the Chapter 3 to all 2+1-dimensional Abelian topological phases, fermionic and bosonic, long-range entangled and short-range entangled.

Edge phases of bosonic/fermionic IQH states are in one-to-one correspondence with even/odd integral unimodular lattices, and these are all stably equivalent upon enlargement with appropriate (bosonic or fermionic) trivial gapped degrees of freedom.

Here we discover a surprise, which is that integer quantum Hall states of fermions admit edge phases with only bosonic low-lying excitations, and in particular that the $\nu = 8$ state can be terminated by the E_8 conformal field theory. Make makes this finding particularly exciting is that $\nu = 8$ has been seen in experiments, but its edge properties have not yet been carefully investigated.

The case of FQH edge phases is richer. Through the help of some powerful pre-existing theorems we establish that the data that defines a 2+1-dimensional Abelian TQFT (the anyon types, their braiding and exchange statistics, chiral central charge) determine a unique integral lattice. The first implication of this is that all 2+1-dimensional Abelian TQFTs can be written down as Abelian Chern-Simons theories with an appropriate choice of the K-matrix.

A given bulk phase is also shown to generically have multiple physically distinct chiral edge phases. These correspond to distinct positive-definite integral but not unimodular lattices. In contrast to unimodular lattices, the non-unimodular ones of a given dimension are not always stably-equivalent. Those that are said to form a genus. Therefore edge phases of FQH states are in one-to-one correspondence with the notion of genera of integral lattices. We make contact with the theory of quadratic forms in which every lattice in a genus is shown to have the same p-adic symbols. We show an algorithm for computing these symbols and checking whether two lattices are in the same genus and therefore whether they are both admissible edge phases of the same bulk.

Much like in the case of integer quantum Hall states, we establish that in the fractional case certain fermionic bulk topological phases admit edge phases that are either fermionic or bosonic. Moreover, we find that any fermionic bulk phase admits a edge phase with only low-lying bosonic excitations provided we enlarge it with a certain number of Landau levels.

Finally, we construct a set of new examples of edge phases of conventional quantum Hall states, with $\nu = 8, 12, \frac{8}{15}, \frac{16}{5}$.

Perfect Metals and Hyperconductors Using the technology of Abelian Conformal Field Theories associated with integral lattices developed in Chapters 3,4 we construct a family of multi-channel Luttinger liquids with no relevant perturbations. These theories describe stable quantum critical phases of interacting fermions, bosons or spins in one-spatial dimension. Precisely, we show that for certain kinds of frustrated interactions all potentially gap-opening operators flow to weak-coupling in the IR, and the system remains gapless. Quantum critical points that extend into entire phases without protection from any symmetry are exceedingly rare.

In 2D and 3D such an example is the familiar Fermi liquid with a sufficiently distorted Fermi surface to preclude the BCS and Kohn-Luttinger instabilities. However, we were not aware of 1D examples prior to our work. Recently another example has been found in the family of Schulz-Shastry models, but it is unclear whether the interactions in the model are physical[4].

One of the consequences of the Perfect Metal construction is that coupling to all weak disorder or single-site impurities is RG irrelevant. As a result Anderson localization (or its extension to weak interactions) is thwarted. This establishes the Perfect Metals as a first example of one-dimensional metallic states in the presence of generic disorder.

In Chapter 6 we perform a detailed study of the transport properties of a particular

Perfect Metal phase. For the reasons mentioned above all current relaxation mechanisms becomes less efficient at low energies, and conclude that the $T = 0$ state has dissipationless charge and heat current flow. In contrast to s-wave superconductors, where the electronic degrees of freedom are gapped and therefore heat flow is exponentially weak in temperature, hyperconductors become increasingly good heat conductors.

We show that in addition to violating the Wiedemann-Franz law, they naturally exhibit low power-law dependence of electric and thermal conductivities on temperature ($\propto 1/T$). This last feature is notable because it signals a breakdown of the conventional quasiparticle picture of transport and is also present in the poorly understood strange metal phase of the cuprate superconductors. Much effort is currently devoted to going beyond the quasiparticle paradigm. Our model is a concrete and well-controlled example of transport in a non-Fermi liquid and these results may shine light on general principles regarding non-Fermi liquids and transport in strongly-correlated electron systems.

1.3 Permissions and Attributions

1. The content of Chapters 3,5,6 and its associated appendices is the result of collaborations with Michael Mulligan and Chetan Nayak, and has previously appeared in Physical Review B[5, 6, 7]. The content of Chapter 4 and its associated appendices is the result of collaborations with Jennifer Cano, Meng Cheng, Michael Mulligan, Jon Yard and Chetan Nayak. It has previously appeared in Physical Review B[8]. All content is reproduced here with the permission of the American Physical Society: `http://www.aps.org`.

Chapter 2

Lattices

In this chapter I collect a number of facts about lattices and their properties.

Definition 1 (Lattice) *A lattice Λ is a finite-dimensional vector space over the integers.*

Example 2.0.1 (An n-dimensional lattice embedded in $\mathbb{R}^n$)

$$\Lambda = \left\{ \sum_{I=1}^{n} m_I \mathbf{f}^I | m_I \in \mathbb{Z} \right\} = \left\langle \mathbf{f}^1, \mathbf{f}^2, ... \mathbf{f}^n \right\rangle \tag{2.1}$$

$$\mathbf{f}^I \equiv \sum_{a=1}^{n} f_a^I \mathbf{x}_a \in \mathbb{R}^n \tag{2.2}$$

where $\mathbf{x}_a$ is a unit vector in the a-th dimension.

In the usual classification of crystal lattices encountered in the study of solids these are called *Bravais lattices*, or lattices with no basis. In what follows I will use *basis* to mean the generating set of a vector space (the set $\mathbf{f}^1, ...\mathbf{f}^n$).

By its embedding in $\mathbb{R}^n$ a lattice can naturally be endowed with a Euclidean inner product, or any number of choices for a Lorentzian inner product. I will often refer to the signature $(n, m), n, m \in \mathbb{Z}$ of a lattice, or a matrix, or the inner product. This will refer

to the number of space-like and time-like dimensions in the vector space, or the number of positive or negative eigenvalues of a matrix.

The Euclidean inner product is positive-definite, and therefore its signature can be written as $(n, 0)$ is $(0, n)$ depending on the convention. I will use δ^{ab} to refer to the Euclidean inner product and η^{ab} to refer to a general inner product.

For clarity

$$\delta_{ab} = \delta^{ab} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.3}$$

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & 0 & \cdots \\ \vdots & 0 & \pm 1 & 0 \\ 0 & \cdots & 0 & \pm 1 \end{pmatrix} \tag{2.4}$$

N.B. All inner products considered will be assumed to be non-degenerate. This means if $\mathbf{u} \cdot \mathbf{v} = 0, \forall v \in \Lambda$, then $u = 0$.

Definition 2 (Dual Lattice) *Given a lattice Λ embedded in $\mathbb{R}^n$ and an inner product on $\mathbb{R}^n$, we can define the dual lattice Λ^* as the set of all vectors in $\mathbb{R}^n$ that have integer inner products with all elements in Λ.*

$$\Lambda^* = \{\mathbf{v} | \mathbf{v} \in \mathbb{R}^n \text{ and } \mathbf{v} \cdot u \in \mathbb{Z}, \forall u \in \Lambda\} \tag{2.5}$$

Definition 3 (Gram matrix) *The Gram matrix of a lattice Λ in a certain basis $\{\mathbf{e}_I\}$*

is the matrix of inner products of the basis vectors.

Since the inner product as a binary operation is symmetric in the two arguments, the Gram matrix is a symmetric matrix. The determinant of the Gram matrix is the volume of the unit cell of the lattice (the volume form and the determinant both involve the ϵ-tensor).

Example 2.0.2 (Hypercubic lattice) *The n-dimensional hypercubic lattice embedded in $\mathbb{R}^n$, with an inner product of signature $(r, n-r)$, is denoted by $I_{r,n-r}$. A simple basis is given by the columns of the $n \times n$ identity matrix.*

The Gram matrix is

$$G = \begin{pmatrix} 1_r & 0 \\ 0 & -1_{n-r} \end{pmatrix} \tag{2.6}$$

where 1_r is the $r \times r$ identity matrix.

Example 2.0.3 (A_2 root lattice) *A_2 is Lie algebra of $\mathrm{SU}(3)$ and its roots form a two-dimensional lattice with basis vectors*

$$\mathbf{e}_1 = \left(\sqrt{2}, 0\right) \tag{2.7}$$

$$\mathbf{e}_2 = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right) \tag{2.8}$$

The Gram matrix is

$$G = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.9}$$

N.B. *The Cartan matrices of simple Lie algebras are always positive definite, the inner product on the lattice is always Euclidean.*

I will introduce the notation $\mathbf{e}_I \equiv e_I^a \mathbf{x}_a$ for the basis of the dual lattice, such that

$$\mathbf{e}_I \cdot \mathbf{f}^J = \delta_I^J \tag{2.10}$$

Let $(K^{-1})^{IJ}$ be the Gram matrix Λ with basis $\{\mathbf{f}^I\}$. The requirement that our inner product is non-degenerate implies that the Gram matrix is non-singular and therefore invertible.

Then, a concrete construction of a basis for the dual lattice is

$$e_I^a = K_{IJ}\eta^{ab} f_b^J \tag{2.11}$$

The Gram matrix of the dual lattice is

$$G_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J = e_I^a \eta_{ab} e_J^b \tag{2.12}$$

$$= e_I^a K_{IJ} \eta_{ab} \eta^{bc} f_c^J = e_I^a K_{IJ} f_a^J \tag{2.13}$$

$$= K_{IJ} \tag{2.14}$$

N.B. The Gram matrix of a lattice is the inverse of the Gram of its dual. We will typically use K to denote the Gram matrix of a lattice, calling it the K-matrix.

Three equivalent notions The following three notions are equivalent

1. Integral quadratic form
2. Lattice
3. K-matrix

Given a K-matrix of signature $(r, n-r)$ we can always constructed a lattice whose Gram matrix is the given K-matrix. First, let λ_a be the eigenvalues of K_{IJ} with e_I^a

the corresponding eigenvector. We normalize e_I^a such that $K_{IJ} = e_I^a \eta_{ab} e_J^b$ where $\eta_{ab} = \mathrm{sgn}(\lambda_a)\delta_{ab}$. Now regard e_I^a as the components of the vector $\mathbf{e}_I \in \mathbb{R}^n$ with the inner product η_{ab}. The lattice is then the integral span of the eigenvectors of K. If the K-matrix happens to be positive-definite, a basis can be obtained by performing a Cholesky Decomposition. More generally a Singular Value decomposition and a rescaling of the outer matrices by the square roots of the absolute values of the eigenvalues will yield a valid basis.

Vice-versa, given a lattice and an inner product we can simply compute its Gram matrix.

Finally, an integral quadratic form $q : \mathbb{Z} \to \mathbb{R}$ can always be constructed out of a symmetric matrix K: $q(\mathbf{x}) = x_i x_j K_{ij}$

Since all these notions are equivalent, to make things intentionally confusing we will typically operate with K-matrices but refer to them as lattices.

There are two senses in which one might define a change of basis of the lattice. One is rotating the entire set of points in $\mathbb{R}^n$, with no effect on the Gram matrix.

The other is constructing a set of *integer* linear combinations of the old basis vectors.

Example 2.0.4 (Basis change) *A basis change of* Λ *is given by* $\mathrm{SL}(n, \mathbb{Z})$.

$$\mathbf{f}^I = W^I_J \tilde{\mathbf{f}}^J, \text{ where } W \in \mathrm{SL}(n, \mathbb{Z}) \tag{2.15}$$

In the new basis the Gram matrix becomes

$$\tilde{G} = W^T G W \tag{2.16}$$

The reason why $\mathrm{SL}(n, \mathbb{Z})$ is the right group and not $\mathrm{GL}(n, \mathbb{Z})$ is because the inverse of a basis change is also a basis change, but since $\det(W) = 1/\det(W^{-1})$, W and W^{-1}

can both have integral entries only if $\det(W) = \pm 1$.

N.B. Note that this implies we allow $\det(W) = -1$ which is a basis change that also changes the *orientation*.

We will say that two lattices with K-matrices K_1 and K_2 are equivalent iff there is a basis change $W \in \mathrm{SL}(n, \mathbb{Z})$ such that $K_1 = W^T K_2 W$.

Definition 4 (Integral lattice) *A lattice is integral iff every entry in its Gram matrix is integral.*

Note that since basis changes are elements of $\mathrm{SL}(n, \mathbb{Z})$ the Gram matrix is either integral in all bases, or in none.

Definition 5 (Even/Odd lattice) *An integral lattice is even iff every diagonal entry in its Gram matrix is even, otherwise it is odd.*

Definition 6 (Unimodular lattice) *A lattice is unimodular iff its Gram matrix has determinant* ± 1.

An integral lattice is unimodular if and only if its dual lattice is integral. The name "modular" probably comes from the fact that

Earlier we saw the hypercubic lattice $I_{r,n-r}$, which is both integral and unimodular. We saw the root lattice of A_2 which is integral but not unimodular ($\det(K) = 3$).

Note that out of any integral (but not unimodular) lattice one can construct a lattice with unit cell of volume 1 by rescaling all basis vectors.

2.0.1 Unimodular lattices of definite signature

Every unimodular lattice is either odd or even, in the sense of Definition 5. The other properties we can ascribe to one are its dimension/rank and its signature (which is the signature of the inner product on $\mathbb{R}^n$.

The classification of positive-definite unimodular lattices is complete for dimensions $1 - 25$, and many partial results exist for higher dimensions.

Theorem 2.0.1 (Even unimodular lattices) *Even unimodular lattices of positive-definite signature exist only in dimensions divisible by 8 [9].*

A summary of known results about the number of distinct unimodular lattices in different dimension is in Table 2.1. Two lattices are different if they cannot be related by a $\mathrm{SL}(n,\mathbb{Z})$ or, equivalently, $\mathrm{SO}(n,\mathbb{Z})$ transformation.

Classification of even unimodular lattices Given a lattice Λ, one can define the (Jacobi) theta function[11, 9] as follows

$$\theta_\Lambda(z) = \sum_{v\in\Lambda} e^{\pi i z|v|^2} \tag{2.17}$$

where the sum runs over all vectors in the lattice, and $|v|^2 = v \cdot v$ is the norm. (For further confusion we shall sometimes also refer to this as the norm-squared).

Roughly speaking this function counts the number of vectors of each norm in the lattice. It is clear that for integral Λ the theta function is a holomorphic function.

It turns out that the theta function of an even unimodular lattice in n dimensions is a modular form of weight $n/2$[9]. These objects are well understood. This is one of tools used in classifying even unimodular lattices.

Let Γ_8 be the root lattice of the E_8 exceptional algebra. Γ_{16} is the only other even integral unimodular lattice, as is closely related to the root lattice of SO(32) modded by a Z_2 [11].

Remark 2.0.1 (Isospectrality) *Since there is unique modular form of weight 8, the theta functions of the lattices $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} are the same.*

$$\theta_{\Gamma_{16}} = 1 + 480e^{2\pi z} + 129 \times 480e^{4\pi i z} + \dots \tag{2.18}$$

For instance, we can see that there are 480 vectors of norm 2 in each lattice. The theta function equality of the two lattices implies they have exactly the same number of vectors of each norm, a fact that was used by Milnor [12] to prove that $\mathbb{R}^{16}/\Gamma_{16}$ and $\mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8$ are isospectral manifolds.

2.0.2 Unimodular lattices of indefinite signature

Contrary to the positive-definite case, there are even unimodular lattices of signature $(8k + n, n)$. The simplest example occurs in 2 dimensions, is known as $II_{1,1}$ and has K-matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.19}$$

The simplest indefinite odd unimodular lattice is just the $I_{1,1}$ hypercubic, from Example 2.0.2.

The main result about indefinite unimodular lattices is that

Theorem 2.0.2 *There is unique odd unimodular lattice of signature* $(r, n - r)$ *up to* $\mathrm{SO}(r, n - r)$. *It is* $I_{r,n-r}$.
There is a unique even unimodular lattice of signature $(8k + n, n)$ *up to* $\mathrm{SO}(8k + n, n)$. *It is* $II_{8k+n,n}$ *[9].*

This means that all indefinite lattices are classified by: their signature and their parity (odd/even). In the positive-definite case further data was necessary (such as classification of its theta function).

Example 2.0.5 (Equivalence of indefinite lattices) *Let K_1 be (the Gram matrix of) an odd positive-definite unimodular lattice of dimension n and K_2 an even/odd positive-definite unimodular lattice of dimension m. Then the lattice $K_1 \oplus -K_2$ is odd unimodular and indefinite with signature (n, m), aka $I_{n,m}$.*

There is both a basis change $W \in \mathrm{SL}(n+m, \mathbb{Z})$ such that $K_1 \oplus -K_2 = W^T I_{n,m} W$, and a "Lorentz transformation" $O \in \mathrm{SO}(n, m)$ that maps the basis vectors of the $K_1 \oplus -K_2$ into some basis for $I_{n,m}$. This fact becomes important in the construction of Perfect Metals in Chapter 5.

2.0.3 Integral lattices of definite signature

Theorem 2.0.3 (Eisenstein,Hermite [13]) *There exist only finitely many distinct definite integral lattices of dimension n and Gram matrix determinant $d \in \mathbb{N}^+$.*

2-dimensions

Two dimensional lattices are classified by their Gauss reduced form, which can be computed in a finite number of steps outlined here [14].

Higher dimensions

In higher dimensions the classification is done with other methods, and is not complete.

2.0.4 Integral lattices of indefinite signature

One of the questions that comes up in the study of quantum Hall edges is whether a particular K-matrix has null-vectors. Those vectors can subsequently be used to define operators that may lead to a gap opening in the energy spectrum.

We illustrate a few examples of indefinite integral lattices with no null vectors.

N.B. Given an indefinite but invertible K-matrix it is always possible to find *real vectors* whose inner product is null. This follows easily from the fact that the eigenvectors of a real symmetric matrix are orthogonal, so we can take an appropriate linear combination to make the weighted sum of their eigenvalues (of opposite signs) zero.

2-dimensions

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -p \end{pmatrix} \tag{2.20}$$

has no null vectors for any integer p that is not a square.

3-dimensions

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{2.21}$$

has no null vectors for $p \equiv 3 \mod 4$.

4-dimensions

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{2.22}$$

has no null vectors for $p \equiv 7 \mod 8$.

From Lagrange's four-square theorem it follows that the 4-dimensional hypercubic lattice has vectors of whose norm is any positive integer. In particular that means that the lattice $I_{4,1}$ has null-vectors.

A more general result is the following.

Theorem 2.0.4 (5.22 in Ref [15]) *An integral quadratic form represents zero rationally iff it is a p-adic zero form for every prime p.*

If a form represents zero rationally, it represents zero integrally because we can multiply each rational by the least common multiple of all their denominators. In practice one does not need to check that the form represents zero p-adically for all primes, but only those that divide the determinant.

A corollary of this theorem is Meyer's Theorem which states that all indefinite quadratic forms over the rationals in five or more variables represent zero, which implies all indefinite integral lattices in five or more dimensions have null vectors.

2.0.5 Minimal norms

Often one needs to know what is the minimal norm $\mu(\Lambda)$ among all non-zero vectors of a positive-definite lattice Λ.

$$\mu(\Lambda) = \frac{\min_{v\in\Lambda, v\neq 0} \mathbf{v}\cdot\mathbf{v}}{\sqrt[n]{\det(\Lambda)}} \tag{2.23}$$

This is known as the *Short Vector Problem* and is believed to be NP-hard [1].

A simple to prove and very general result is due to Minkowski:

Theorem 2.0.5 (Minkowski Convex Body Theorem [13]) *Any lattice $\Lambda \in \mathbb{R}^n$, regardless of the inner product, contains at least one vector whose Euclidean norm $(v \cdot v)$ is less than or equal to (Hermite's constant) r_m^2, where $r_m^n = \left(\frac{2^n}{\omega_n}\right) \mathrm{vol}(\mathbb{R}^n/\Lambda)$.*

Here ω_n is the volume of the unit radius sphere in n-dimensions. The volume of the fundamental domain of a unimodular lattice is 1 by definition, $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$. In that case the formula simplifies and says that any n-dimensional unimodular lattice has a non-zero vector with norm at most

$$|\mathbf{v}|^2 \leq \frac{4}{\omega_n^{2/n}}$$

We summarize this in Table 2.2.

A lower bound is due to Hlawka, and a much tighter upper bound comes from studies of the sphere packing problem and is due to Rogers.

Theorem 2.0.6 (Hlawka, Rogers, p35 of Ref [13]) *For any n-dimensional lattice*

$$\left(\frac{2\zeta(n)}{\omega_n}\right)^{2/n} \leq \mu(\Lambda) \leq 4\left(\frac{\sigma_n}{\omega_n}\right)^{2/n} \tag{2.24}$$

[1] `https://en.wikipedia.org/wiki/Lattice_problem`

where asymptotically $\sigma_n \sim \frac{n}{e\sqrt{2^n}}$.

Note that there is no guarantee that such a lattice exists among the integral and unimodular ones. In fact, that is not true, as can be seen from Table 2.1. A usefum heuristic to remember is that asymptotically the minimal norm in n-dimensions goes like $\frac{n}{12} + 1$.

Finally, a constructive result:

Theorem 2.0.7 (Lemma 7.1 of Ref [13]) *For each dimension n there exists a lattice Λ_n which maximizes the minimal norm $\mu(\Lambda_n)$ from Eqn 2.23, and furthermore, the Gram matrix of this lattice can be made rational after rescaling by a real number.*

It says that there is indeed a lattice that saturates the maximal possible value of the minimal norm in n-dimensions. The lattice will generically be non-integral. However, computing the maximal value of the minimal norm is a hard problem and values are only known up to dimension 8.

All this is summarized in Figure 2.1

The figure shows that in low dimensions the lattices known are very close to saturating the Rogers bound, and in particular that no lattice with minimal norm larger than 2 exists exists below 10 dimensions. The first such lattice is the laminated Λ_{10}.

A useful result about existence of *integral unimodular* lattices with certain minimal norms is due to Conway and Thompson.

Theorem 2.0.8 (9.5 of Ref [13]) *For any dimension n there exists an odd positive-definite integral unimodular lattice Λ with (minimal norm)*

$$\min_{\mathbf{v}\in\Lambda,\mathbf{v}\neq 0} \mathbf{v}\cdot\mathbf{v} \geq k(n)$$

where $k(n)$ is the closest integer to $\left(\frac{5}{3}\omega_n^{-1}\right)^{2/n}$.

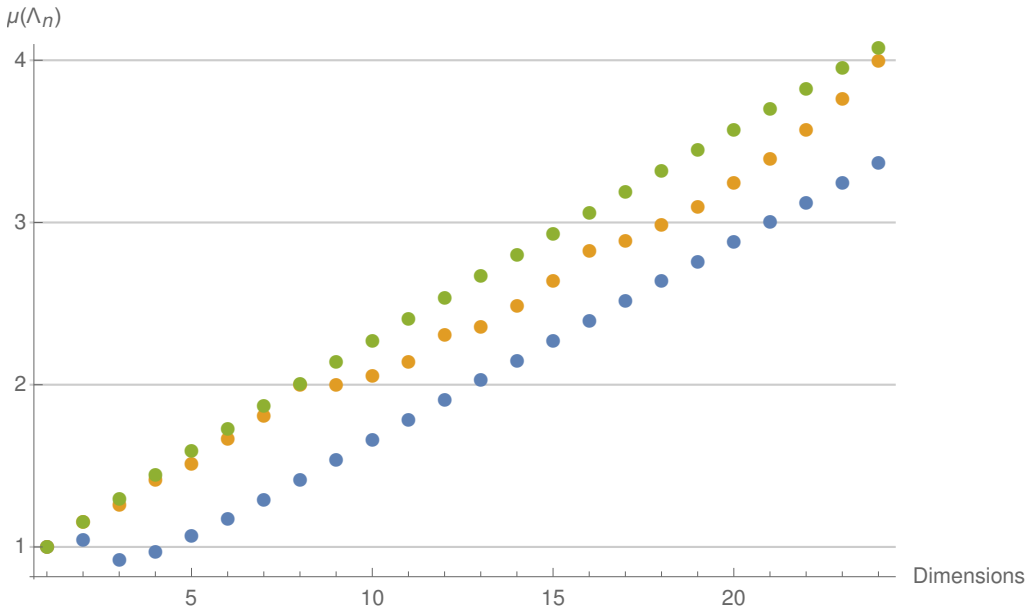

Figure 2.1: Plot of minimal norms by dimension. Lower branch is lower bound due to Minkowski-Hlawka. Upper branch is upper bound by Rogers. The middle branch are the maximal attainable minimal norms up to dimensions 8, and maximal known so far above.

Since $k(n)$ is an increasing function, this implies that given your favorite positive integer k, there is an odd integral unimodular lattice in sufficiently many dimensions with minimal norm k. The statement can be extended to even unimodular lattices as well. For example, the lattice $\Gamma_8 \otimes ... \Gamma_8$, made of the n-fold tensor product of the E_8 root lattice, is even unimodular and has minimal norm 2^n (follows for example from Theorem 9.6 (Steinberg) of Ref [13].

The result above will be important in Chapter 5, where integral lattices with larger norms will be used to build increasingly stable phases.

Dimension	Odd lattices	Odd latt., no roots	Even lattices	Even latt., no roots
1	1	0	0	0
2	1	0	0	0
3	1	0	0	0
4	1	0	0	0
5	1	0	0	0
6	1	0	0	0
7	1	0	0	0
8	1	0	1 (E_8)	0
9	2	0	0	0
10	2	0	0	0
11	2	0	0	0
12	3	0	0	0
13	3	0	0	0
14	4	0	0	0
15	5	0	0	0
16	6	0	2 (E_8^2, D_{16}^+)	0
17	9	0	0	0
18	13	0	0	0
19	16	0	0	0
20	28	0	0	0
21	40	0	0	0
22	68	0	0	0
23	117	1 (shorter Leech)	0	0
24	273	1 (odd Leech)	24 (Niemeier lattices)	1 (Leech lattice)
25	665	0	0	0
26	≥ 2307	1	0	0
27	≥ 14179	3	0	0
28	≥ 327972	38	0	0
29	≥ 37938009	≥ 8900	0	0
30	$\geq 2 \times 10^{10}$	≥ 82000000	0	0
31	$\geq 5 \times 10^{12}$	8×10^{11}	0	0
32	$\geq 8 \times 10^{16}$	$\geq 10^{16}$	$\geq 10^9$	$\geq 10^7$

Table 2.1: Bounds on the number of unimodular lattices up to dimension 32. Data from [10]. Lattices with no roots contain no non-zero vectors of norm 2 or below.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\frac{4}{\omega_n^{2/n}}$	1.28	1.54	1.81	2.06	2.32	2.57	2.82	3.07	3.32	3.57	3.82	4.06	4.31

Table 2.2: Upper bounds on the minimal norms of unimodular lattices up to dimension 14.

Chapter 3

Edge phase transitions in the $E_8 \times E_8$ state

3.1 Introduction

The last decade has seen enormous progress in the understanding of topological phases (see Ref. [16] and references therein) and of symmetry-protected topological (SPT) phases [17, 18, 19, 20]. SPT phases are gapped phases of matter that do not have non-trivial excitations in the bulk; have vanishing topological entanglement entropy [21, 22] or, equivalently, have short-ranged entanglement (SRE); but have gapless excitations at the edge in the presence of a symmetry. In the case of the most famous and best-understood example, 'topological insulators' (see Refs. [23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and references therein), the symmetry is time-reversal. Topological phases (without a modifier) are gapped phases of matter that are stable to arbitrary perturbations; support anyons in the bulk; and have non-zero topological entanglement entropy or, equivalently, have long-ranged entanglement (LRE). They may or may not (depending on the topological

phase) have gapless edge excitations.[1]

However, there is a third possibility: phases of matter that do not support anyons but nevertheless have gapless excitations even in the absence of any symmetry. Thus, they lie somewhere between topological phases and symmetry-protected topological phases, but are neither. Integer quantum Hall states of fermions are a well-known example. Their gapless edge excitations are stable to arbitrary weak perturbations even though they do not support anyons and only have SRE [34, 35]. Although the existence and stability of SRE integer quantum Hall states might seem to be a special feature of fermions, such states also exist in purely bosonic systems, albeit with some peculiar features.

For any integer N, there is an integer quantum Hall state of fermions with SRE, electrical Hall conductance $\sigma_{xy} = N\frac{e^2}{h}$, and thermal Hall conductance $\kappa_{xy} = N\frac{\pi^2 k_B^2 T}{3h}$. [36] In fact, there is only one such state for each N: any two SRE states of fermions at the same filling fraction N can be transformed into each other without encountering a phase transition. [2] (This is true in the bulk; see Section 3.7.2 for the situation at the edge.) Therefore, the state with N filled Landau levels of non-interacting fermions is representative of an entire universality class of SRE states. As a result of its N chiral Dirac fermion edge modes, this is a distinct universality class from ordinary band insulators. These edge modes, which have Virasoro central charge $c = N$ if all of the velocities are equal, are stable to *all* perturbations. If we do not require charge conservation symmetry, then some Hamiltonians in this universality class may not have $\sigma_{xy} = N\frac{e^2}{h}$, but they will

[1] We note that SPT phases can all be adiabatically connected to a trivial ground state if we do not require that the associated symmetry be preserved. Topological phases cannot be. However, if we restrict to Hamiltonians that respect a symmetry then, just as the trivial phase splits into many SPT phases, a non-trivial topological phase could split into multiple phases that could be distinguished, for instance, by their edge excitations. For a discussion of such "symmetry-enhanced topological phases", see Ref [33].

[2] Of course, it may be possible to take a route from one to the other that does cross a phase transition but such a transition can always be avoided. For instance, if we restrict to S_z-conserving Hamiltonians, then a phase transition must be encountered in going from a spin-singlet $N = 2$ state to a spin-polarized one. If we do not make this restriction, however, then this phase transition can be avoided and the two states can be adiabatically-connected.

all have $\kappa_{xy} = c\frac{\pi^2 k_B^2 T}{3h} = N\frac{\pi^2 k_B^2 T}{3h}$.

Turning now to bosons, there are SRE states of bosons with similarly stable chiral edge modes, but only for central charges $c = 8k$. As we discuss, they correspond to even, positive-definite, unimodular lattices. Moreover, while there is a unique such state with $c = 8$, there appear to be two with $c = 16$, twenty-four with $c = 24$, and more than ten million with $c = 32$. [9] Thus, we are faced with the possibility that there are many SRE bosonic states with the same thermal Hall conductance κ_{xy}, presumably distinguished by a more subtle invariant. In this chapter, we show that this is not the case for $c = 16$. The two SRE bosonic states with $c = 16$ edge excitations are equivalent in the bulk: their partition functions on arbitrary closed manifolds are equal. However, there are two distinct chiral edge phases of this unique bulk state. They are connected by an edge reconstruction: a phase transition must be encountered at the edge in going from one state to the other, but this transition can occur solely at the edge and the gap need not close in the bulk. Although we focus on the $c = 16$ case, the logic of our analysis readily generalizes. Therefore, we claim that there is essentially a unique bulk bosonic phase for each $c = 8k$ given by k copies of the so-called E_8-state [19, 20]. However, there are two distinct *fully-chiral* edge phases with $c = 16$, twenty-four with $c = 24$, more than ten million with $c = 32$, and even more for larger c.

One important subtlety arises in our analysis. The two $c = 16$ phases do not, initially, appear to be identical. However, when combined with a trivial insulating phase, the two bulk partition functions can be mapped directly into each other by a change of variables. This is a physical realization of the mathematical notion of *stable equivalence*. In general, an effective description of a phase of matter will neglect many gapped degrees of freedom (e.g., the electrons in inner shells). However, the sequence of gapped Hamiltonians that interpolates between two gapped Hamiltonians may involve mixing with these usually-forgotten gapped degrees of freedom. Therefore, it is natural, in considering a phase

of matter, to allow an arbitrary enlargement of the Hilbert space by trivial gapped degrees of freedom (i.e., by SRE phases without gapless edge excitations). This is useful when, for instance, comparing a trivial insulating phase with p bands with another trivial insulating phase with $q > p$ bands. They can be adiabatically connected if we are allowed to append $q - p$ trivial insulating bands to the latter system. This notion is also natural when connecting different phases of gapless edge excitations. The edge of a gapped bulk state will generically have gapped excitations that we ordinarily ignore. However, they can become gapless – which is a form of edge reconstruction – and interact with the other gapless degrees of freedom, driving the edge into a different phase. However, this does not require any change in the bulk. As we will see, such a purely edge phase transition connects the two seemingly different chiral gapped edges with $c = 16$. By combining a $c = 16$ state with a trivial insulator, we are able to take advantage of the uniqueness of signature $(8k + n, n)$ even unimodular lattices [37], from which it follows that the two phases are the same. This is closely-related to the fact that T-duality exchanges toroidal compactifications of the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ versions of the heterotic string, as explained by Ginsparg [38].

In the remainder of this chapter, we describe the equivalence of the two candidate phases at $k = 2$ from two complementary perspectives. To set the stage, we begin in Section 3.2 with a short introduction to the K-matrix formalism that we use to describe the phases of matter studied in this chapter. In Section 3.3, we provide a bulk description of the equivalence of the two candidate phases at $k = 2$. We then turn to the edge, where we show that there are two distinct chiral phases of the edge. We first discuss the fermionic description of the edge modes in Section 3.4 and then turn to the bosonic description in Section 3.5. There is an (purely) edge transition between these two phases. We discuss the phase diagram of the edge, which is rather intricate, and its relation to the bulk. In Section 3.6, we summarize how the phase diagram can change when some

of the degrees of freedom are electromagnetically charged so that a $U(1)$ symmetry is preserved. We then conclude in Section 3.7 and discuss possible generalizations of this picture.

In Appendix A, we collect basic definitions and explain the notation used throughout the text. In Appendix B, we provide some technical details for an argument used in the main text.

3.2 K-matrix Formalism

3.2.1 Chern-Simons Theory

We will consider $2+1$-dimensional phases of matter governed by bulk effective field theories of the form:

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}K_{IJ}a^I_\mu\partial_\nu a^J_\rho + j^\mu_I a^I_\mu, \tag{3.1}$$

where a^I_μ, for $I = 1, ..., N$ and $\mu = 0, 1, 2$. See Refs. [39] and [40] for a pedagogical introduction to such phases. K_{IJ} is a symmetric, non-degenerate $N \times N$ integer matrix. (Repeated indices should be summed over unless otherwise specified.) We normalize the gauge fields a^I_μ and sources j^μ_I so that fluxes that are multiples of 2π are unobservable by the Aharonov-Bohm effect. Consequently, if we take the sources to be given by prescribed non-dynamical classical trajectories $x^\mu_m(\tau)$ that serve as sources of a^I_μ flux, they must take the form:

$$j^\mu_I = \sum_m n^{(m)}_I \delta(x^\mu - x^\mu_m(\tau))\partial_\tau x^\mu_m, \tag{3.2}$$

for integers $n^{(m)}_I$. The sum over m is a sum over the possible sources x_m.

Therefore, each excitation m of the system is associated with an integer vector $n^{(m)}_I$.

These integer vectors can be associated with the points of a lattice as follows. Let λ_a for $a = 1, \ldots, N$ be the eigenvalues of $(K^{-1})^{IJ}$ with f_a^I the corresponding eigenvectors. We normalize the f_a^I so that $(K^{-1})^{IJ} = \eta^{ab} f_a^I f_b^J$ where $\eta^{ab} = \mathrm{sgn}(\lambda_a)\delta^{ab}$. Now suppose that we view the f_a^I as the components of a vector $\mathbf{f}^I \in \mathbb{R}^{N_+,N_-}$ (i.e., of $\mathbb{R}^N$ with a metric $\eta_{ab} = \mathrm{sgn}(\lambda_a)\,\delta_{ab}$ of signature (N_+, N_-)), where K^{-1} has N_+ positive eigenvalues and N_- negative ones. In other words, the unit vector $\hat{\mathbf{x}}_\mathbf{a} = (0, ..., 0, 1, 0, ..., 0)^{\mathrm{tr}}$ with a 1 in the a-th entry and zeros otherwise is an orthonormal basis of $\mathbb{R}^{N_+,N_-}$ so that $\hat{\mathbf{x}}_\mathbf{a} \cdot \hat{\mathbf{x}}_\mathbf{b} \equiv (\hat{\mathbf{x}}_\mathbf{a})^c \eta_{cd} (\hat{\mathbf{x}}_\mathbf{b})^d = \eta_{ab}$. Then we can define $\mathbf{f}^I \equiv f_a^I \hat{\mathbf{x}}_\mathbf{a}$. Thus, the eigenvectors $\mathbf{f}^I$ define a lattice Γ in $\mathbb{R}^{N_+,N_-}$ according to $\Gamma = \{m_I \mathbf{f}^I | m_I \in \mathbb{Z}\}$; this lattice determines the allowed excitations of the system [41, 42].

The lattice Γ enters directly into the computation of various physical observables. For example, consider two distinct excitations corresponding to the lattice vectors $\mathbf{u} = m_I \mathbf{f}^I$ and $\mathbf{v} = n_J \mathbf{f}^J$ in Γ. If one excitation is taken fully around the other, then the resulting wavefunction differs from its original value by the exponential of the Berry's phase $2\pi (K^{-1})^{IJ} m_I n_J = 2\pi \mathbf{u} \cdot \mathbf{v}$. When the excitations are identical, $\mathbf{u} = \mathbf{v}$, a half-braid is sufficient and a phase equal to $\pi \mathbf{u} \cdot \mathbf{u}$ is obtained.

Of course, any basis of the lattice Γ is equally good; there is nothing special about the basis $\mathbf{f}^I$. We can change to a different basis $\mathbf{f}^I = W^I{}_J \tilde{\mathbf{f}}^J$, where $W \in SL(N, \mathbb{Z})$. (W must have integer entries since it relates one set of lattice vectors to another. Its inverse must also be an integer matrix since either set must be able to serve as a basis. But since $\det(W) = 1/\det(W^{-1})$, W and W^{-1} can both be integer matrices only if $\det(W) = \pm 1$.) This lattice change of basis can be interpreted as the field redefinitions, $\tilde{a}_\mu^I = W^I{}_J a_\mu^J$ and $\tilde{j}_I^\mu W^I{}_J = j_J^\mu$, in terms of which the Lagrangian (3.1) becomes

$$\mathcal{L} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} \tilde{K}_{IJ} \tilde{a}_\mu^I \partial_\nu \tilde{a}_\rho^J + \tilde{j}_I^\mu \tilde{a}_\mu^I, \tag{3.3}$$

where $K = W^T \tilde{K} W$. Therefore, two theories are physically identical if their K-matrices are related by such a similarity transformation.

We note that the low energy phases described here may be further sub-divided according to their coupling to the electromagnetic field, which is determined by the N-component vector t_I:

$$\mathcal{L} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} K_{IJ} a^I_\mu \partial_\nu a^J_\rho + j^\mu_I a^I_\mu - \frac{1}{2\pi} \epsilon^{\mu\nu\rho} t_I A_\mu \partial_\nu a^I_\rho. \tag{3.4}$$

It is possible for two theories with the same K-matrix to correspond to different phases if they have different t_I vectors since they may have different Hall conductances $\sigma_{xy} = \frac{e^2}{h}(K^{-1})^{IJ} t_I t_J$. (It is also possible for discrete global symmetries, such as time-reversal, to act differently on theories with the same K-matrix in which case they can lead to different SPT phases if that symmetry is present.)

In this chapter, we will be interested in states of matter in which all excitations have bosonic braiding properties, i.e., in which any exchange of identical particles or full braid of distinguishable particles leads to a phase that is a multiple of 2π. Hence, we are interested in lattices for which $\mathbf{f}^I \cdot \mathbf{f}^J$ is an integer for all I, J and is an even integer if $I = J$. Hence, K^{-1} is a symmetric integer matrix with even entries on the diagonal. By definition K must also an integer matrix. Since both K and K^{-1} are integer matrices, their determinant must be ± 1. Because $\mathbf{f}^I \cdot \mathbf{f}^I \in 2\mathbb{Z}$ (no summation on I) and $\det(\mathbf{f}^I \cdot \mathbf{f}^J) = \pm 1$, the lattice Γ is said to be an even unimodular lattice.

It is convenient to introduce the (dual) vectors $e^a_I = K_{IJ} \eta^{ab} f^J_b$. If, as above, we view the e^a_I as the components of a vector $\mathbf{e}_I \in \mathbb{R}^{N_+, N_-}$ according to $\mathbf{e}_I \equiv e^a_I \hat{\mathbf{x}}_\mathbf{a}$, then $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. Moreover, $\mathbf{e}_I$ is the basis of the dual lattice Γ^* defined by $\mathbf{f}^I \cdot \mathbf{e}_J = \delta^I{}_J$. Since the lattice Γ is unimodular, it is equal to Γ^*, up to an $SO(N_+, N_-)$ rotation, from which we see that K must be equivalent to K^{-1}, up to an $SL(N, \mathbb{Z})$ change of basis. (In

fact, the required change of basis is provided by the defining relation $e_I^a = K_{IJ}\eta^{ab} f_b^J$.)

Now consider the Lagrangian (3.5) on the spatial torus. For convenience, we assume there are no sources so $\mathbf{j}^\mu = 0$. We can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\mathbf{e}_I \cdot \mathbf{e}_J a_\mu^I \partial_\nu a_\rho^J + j_I^\mu \mathbf{f}^I \cdot \mathbf{e}_J a_\mu^J \qquad (3.5) \\ &= \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\mathbf{a}_\mu \cdot \partial_\nu \mathbf{a}_\rho + \mathbf{j}^\mu \cdot \mathbf{a}_\mu, \qquad (3.6) \end{aligned}$$

where we have defined $\mathbf{a}_\mu \equiv \mathbf{e}_I a_\mu^I$ and $\mathbf{j}^\mu \equiv \mathbf{f}^I j_I^\mu$. Choosing the gauge $\mathbf{a}_0 = 0$, $\partial_i \mathbf{a}_i = 0$, the Lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{2\pi}\mathbf{a}_1 \cdot \partial_t \mathbf{a}_2. \qquad (3.7)$$

Therefore, $\mathbf{a}_1$ and $\mathbf{a}_2$ are canonically conjugate. Although we have gauge-fixed the theory for small gauge transformations, under a large gauge transformation, $a_k^I \to a_k^I + n_{(k)}^I$ where $n_{(k)}^I$ are integers (so that physical observables such as the Wilson loop $e^{i\oint_{C_k} a_k^I}$ about the 1-cycle C_k remains invariant). Therefore, we must identify $\mathbf{a}_j$ and $\mathbf{a}_j + n_{(k)}^I \mathbf{e}_I$ since they are related by a gauge transformation.

Suppose that we write a ground state wavefunction in the form $\Psi[\mathbf{a}_1]$. Then $\mathbf{a}_1$ will act by multiplication and its canonical conjugate $\mathbf{a}_2$ will act by differentiation. To display the full gauge invariance of the wavefunction, $\Psi[\mathbf{a}_1] = \Psi[\mathbf{a}_1 + n^I \mathbf{e}_I]$, it is instructive to expand it in the form:

$$\Psi[\mathbf{a}_1] = \mathcal{N} \sum_{m_I} \Psi_{m_I} e^{2\pi i m_I \mathbf{f}^I \cdot \mathbf{a}_1} \qquad (3.8)$$

where $m_I \in \mathbb{Z}$. This is an expansion in eigenstates of $\mathbf{a}_2$, with the m_I term having the eigenvalue $2\pi i m_I \mathbf{f}^I$. However, by gauge invariance, $\mathbf{a}_1$ takes values in $\mathbb{R}^N/\Gamma^*$. Therefore, we should restrict m_I such that $m_I \mathbf{f}^I$ lies inside the unit cell of Γ^*. In other words, the number of ground states on the torus is equal to the number of sites of Γ that lie

inside the unit cell of Γ^*. This is simply the ratio of the volumes of the unit cells, $|\det(K)|^{1/2}/|\det(K)|^{-1/2} = |\det(K)|$. It may be shown that this result generalizes to a ground state degeneracy $|\det K|^g$ on a genus g surface [43]. Therefore, the theories on which we focus in this chapter have non-degenerate ground states on an arbitrary surface, which is another manifestation of the trivial braiding properties of its excitations.

One further manifestation of the trivial braiding properties of such a phase's excitations is the bipartite entanglement entropy of the ground state [21, 22]. If a system with action (3.1) with $j_I^\mu = 0$ is divided into two subsystems A and B and the reduced density matrix ρ_A for subsystem A is formed by tracing out the degrees of freedom of subsystem B, then the von Neumann entropy $S_A = -\mathrm{tr}\Big(\rho_A \log(\rho_A)\Big)$ takes the form:

$$S_A = \alpha L - \ln \sqrt{|\det(K)|} + \dots \tag{3.9}$$

Here, α is a non-universal constant that vanishes for the action (3.1), but is non-zero if we include irrelevant sub-leading terms in the action (e.g., Maxwell terms for the gauge fields). L is the length of the boundary between regions A and B. The $\dots$ denote terms with sub-leading L dependence. For the theories that we will consider in this chapter, the second term, which is universal, vanishes. For this reason, such phases are called "short-range entangled."

The discussion around Eq. (3.8), though essentially correct as far as the ground state degeneracy is concerned, swept some subtleties under the rug. A more careful treatment [44] uses holomorphic coordinates $\mathbf{a} = \mathbf{a}_1 + iK \cdot \mathbf{a}_2$, in terms of which the wavefunctions are ϑ-functions. Moreover, the normalization $\mathcal{N}$ must account for the fact that the wavefunction Ψ is a function only on the space of $\mathbf{a}_i$ with vanishing field strength (which the $\mathbf{a}_0 = 0$ gauge constraint requires), not on arbitrary $\mathbf{a}_i$. Consequently, it depends on the modular parameter of the torus as $\mathcal{N} = (\eta(\tau))^{-N_+}(\eta(\overline{\tau}))^{-N_-}$ where

$N_\pm$ are the number of positive and negative eigenvalues of K_{IJ}; the torus is defined by the parallelogram in the complex plane with corners at 0, 1, τ, $\tau + 1$ and opposite sides identified; and is $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ is the Dedekind η function, where $q = e^{2\pi i \tau}$. Consequently, the ground state wavefunction transforms non-trivially under the mapping class group of the torus (i.e., under diffeomorphisms of the torus that are disconnected from the identity, modulo those that can be deformed to the identity) which is equal to the modular group $SL(2, \mathbb{Z})$ generated by $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. Under T, which cuts open the torus along its longitude, twists one end of the resulting cylinder by 2π, and then rejoins the two ends of the cylinder to reform the torus, thereby enacting $\tau \to \tau + 1$, the ground state transforms according to $\Psi \to e^{-2\pi i (N_+ - N_-)/24} \Psi$. Therefore, so long as $N_+ - N_- \not\equiv 0 \pmod{24}$, the bulk is not really trivial.

3.2.2 Edge Excitations

The non-trivial nature of these states is reflected in more dramatic fashion on surfaces with a boundary, where there may be gapless edge excitations. For simplicity, consider the disk D with no sources in its interior [45, 46]. The action (3.1) is invariant under gauge transformations $a^I_\mu \to a^I_\mu + -i(g^I)^{-1}\partial_\mu g^I$, where $g^I \in [U(1)]^N$, so long as $g^I = 1$ at the boundary ∂D. In order to fully specify the theory on a disk, we must fix the boundary conditions. Under a variation of the gauge fields δa^J_μ, the variation of the action $S = \int_{\mathbb{R}\times D} L$ (here, $\mathbb{R}$ is the time direction) is

$$\delta S = \frac{1}{2\pi} \int_{\mathbb{R}\times D} \delta a^I_\mu \, K_{IJ} \epsilon^{\mu\nu\rho} \partial_\nu a^J_\rho + \frac{1}{4\pi} \int_{\mathbb{R}\times \partial D} \epsilon^{\mu\nu r} K_{IJ} a^I_\mu \delta a^J_\nu \tag{3.10}$$

Here r is the radial coordinate on the disk. The action will be extremized by $K_{IJ}\epsilon^{\mu\nu\rho}\partial_\nu a^J_\rho = 0$ (i.e. there won't be extra boundary terms in the equations of motion) so long as we take boundary conditions such that $\epsilon^{\mu\nu r} K_{IJ} a^I_\mu \delta a^J_\nu = 0$. We can take boundary condition

$K_{IJ}a_0^I + V_{IJ}a_x^I = 0$, where x is the azimuthal coordinate. Here V_{IJ} is a symmetric matrix that is determined by non-universal properties of the edge such as how sharp it is. The Lagrangian (3.1) is invariant under all transformations $a_\mu^J(x) \to a_\mu^J(x) - i(g^J)^{-1}(x)\partial_\mu g^J(x)$ that are consistent with this boundary condition. Only those with $g^J = 1$ at the boundary are gauge symmetries. The rest are ordinary symmetries of the theory. Therefore, although all bulk degrees of freedom on the disk are fixed by gauge invariance and the Chern-Simons constraint, there are local degrees of freedom at the boundary.

The Chern-Simons constraint $K_{IJ}\epsilon_{ij}\partial_i a_j^J = 0$ can be solved by taking $a_i^I = (U^I)^{-1}\partial_i U^I$ or, writing $U^I = e^{i\phi}$, $a_i^I = \partial_i \phi$, where $\phi \equiv \phi + 2\pi$. This gauge field is pure gauge everywhere in the interior of the disk (i.e. we can locally set it to zero in the interior with a gauge transformation), but it is non-trivial on the boundary because we can only make gauge transformations that are consistent with the boundary condition. Substituting this expression into the action (3.1), we see that the action is a total derivative which can be integrated to give a purely boundary action:

$$S = \frac{1}{4\pi} \int dt\, dx \, \left[K_{IJ}\, \partial_t \phi^I\, \partial_x \phi^J - V_{IJ}\, \partial_x \phi^I\, \partial_x \phi^J \right]. \tag{3.11}$$

The Hamiltonian associated with this action will be positive semi-definite if and only if V_{IJ} has non-negative eigenvalues. If we define $\mathbf{X} \equiv \mathbf{e}_J \phi^J$ or, in components, $X^a \equiv e_J^a \phi^J$, then we can rewrite this in the form

$$S = \frac{1}{4\pi} \int dt\, dx \, \left[\eta_{ab} \partial_t X^a\, \partial_x X^b - v_{ab}\, \partial_x X^a\, \partial_x X^b \right], \tag{3.12}$$

where $v_{ab} \equiv V_{IJ} f_a^I f_b^J$. We see that the velocity matrix v_{ab} parameterizes density-density interactions between the edge modes. Note that the fields X^a satisfy the periodicity conditions $X^a \equiv X^a + 2\pi e_I^a n^I$ for $n^I \in \mathbb{Z}$.

This theory has N different dimension-1 fields $\partial_x \phi^I$. The theory also has 'vertex operators', or exponentials of these fields that must be consistent with their periodicity conditions: $e^{im_I \phi^I}$ or, equivalently, $e^{im_I \mathbf{f}^I \cdot \mathbf{X}}$ or, simply, $e^{i\mathbf{u}\cdot\mathbf{X}} = e^{i\eta_{ab}u^a X^b}$ for $\mathbf{u} \in \Gamma$. They have correlation functions:

$$\left\langle e^{i\mathbf{u}\cdot\mathbf{X}} e^{-i\mathbf{u}\cdot\mathbf{X}} \right\rangle = \prod_{b=1}^{N_+} \frac{1}{(x - v_b t)^{y_b}} \prod_{b=N_++1}^{N} \frac{1}{(x + v_b t)^{y_b}} \tag{3.13}$$

In this equation, $y_b \equiv \sum_{a,c,d,e} u_a S_{ab} \eta_{bc} (S^T)_{cd} \eta_{de} u_e$, where S_{ab} is an $SO(N)$ matrix that diagonalizes $\eta_{ab} v_{bc}$. Its first N_+ columns are the normalized eigenvectors corresponding to positive eigenvalues of $\eta_{ab} v_{bc}$ and the next N_- columns are the normalized eigenvectors corresponding to negative eigenvalues of $\eta_{ab} v_{bc}$. The velocities v_b are the absolute values of the eigenvalues of $\eta_{ab} v_{bc}$. Therefore, this operator has scaling dimension

$$\Delta_{\mathbf{u}} = \frac{1}{2} \sum_{b=1}^{N} y_b. \tag{3.14}$$

The scaling dimensions of an operator in a non-chiral theory generally depend upon the velocity matrix v_{ab}. For a fully chiral edge, however, $\eta_{ab} = \delta_{ab}$ and $S_{ab} \in SO(N)$, so $\Delta_{\mathbf{u}} = \frac{1}{2}|\mathbf{u}|^2$.

If the velocities all have the same absolute value, $|v_a| = v$ for all a, then the theory is a conformal field theory with right and left Virasoro central charges $c = N_+$ and $\overline{c} = N_-$. Consequently, we can separately rescale the right- and left-moving coordinates: $(x - vt) \to \lambda(x - vt)$ and $(x + vt) \to \lambda'(x + vt)$. The field $\partial_x X^a$ has right and left scaling dimension $(1, 0)$ for $a = 1, 2, \ldots, N_+$ and dimension $(0, 1)$ for $a = N_+ + 1, \ldots, N$. Meanwhile, $e^{i\mathbf{u}\cdot\mathbf{X}}$ has scaling dimension:

$$(\Delta_{\mathbf{u}}^R, \Delta_{\mathbf{u}}^L) = (\frac{1}{2} \sum_{b=1}^{N_+} y_b, \frac{1}{2} \sum_{b=N_++1}^{N} y_b). \tag{3.15}$$

which simplifies, for the case of a fully chiral edge, to $(\Delta^R_{\mathbf{u}}, \Delta^L_{\mathbf{u}}) = (\frac{1}{2}\mathbf{u}\cdot\mathbf{u}, 0)$.

In a slight abuse of terminology, we will call the state of matter described by Eq. (3.1) in the bulk and Eq. (3.11) on the edge a $c = N_+$, $\overline{c} = N_-$ bosonic SRE phase. In the case of fully chiral theories that have $\overline{c} = 0$, we will sometimes simply call them $c = N$ bosonic SRE phases. Strictly speaking, the gapless edge excitations are only described by a conformal field theory when the velocities are all equal. However, we will continue to use this terminology even when the velocities are not equal, and we will use it to refer to both the bulk and edge theories.

In the case of a $c > 0$, $\overline{c} = 0$ bosonic SRE phase, all possible perturbations of the edge effective field theory Eq. (3.11) – or, equivalently, Eq. (3.12) – are chiral. Since such perturbations cannot open a gap, completely chiral edges are stable. A non-chiral edge may have a vertex operator $e^{i\mathbf{u}\cdot\mathbf{X}}$ with equal right- and left-scaling dimensions. If its total scaling dimension is less than 2, it will be relevant and can open a gap at weak coupling. More generally, we expect that a bosonic SRE will have stable gapless edge excitations if $c - \overline{c} > 0$. Some of the degrees of freedom of the theory (3.11) will be gapped out, but some will remain gapless in the infrared (IR) limit and the remaining degrees of freedom will be fully chiral with $c_{IR} = c - \overline{c}$ and $\overline{c}_{IR} = 0$. Therefore, even if such a phase is not, initially, fully-chiral, the degrees of freedom that remain stable to arbitrary perturbations is fully chiral. Therefore, positive-definite even unimodular lattices correspond to $c > 0$, $\overline{c} = 0$ bosonic SRE phases with stable chiral edge excitations, in spite of the absence of anyons in the bulk.

3.2.3 The Cases $c - \overline{c} = 0, 8, 16$

Positive-definite even unimodular lattices only exist in dimension $8k$ for integer k, [37] so bosonic SRE phases with stable chiral edge excitations must have $c = 8k$. There is a

unique positive-definite even unimodular lattice in dimension 8, up to an overall rotation of the lattice. There are two positive-definite even unimodular lattices in dimension 16; there are 24 in dimension 24; there are more than 10^7 in dimension 32; and even more in higher dimensions. If we relax the condition of positive definiteness, then there are even unimodular lattices in all even dimensions; there is a unique one with signature $(8k+n, n)$ for $n \geq 1$.

In dimension-2, the unique even unimodular lattice in $\mathbb{R}^{1,1}$, which we will call U, has basis vectors $\mathbf{e}_1 = \frac{1}{r}(\hat{\mathbf{x}}_\mathbf{1} + \hat{\mathbf{x}}_\mathbf{2})$, $\mathbf{e}_2 = \frac{r}{2}(\hat{\mathbf{x}}_\mathbf{1} - \hat{\mathbf{x}}_\mathbf{2})$, and the corresponding K-matrix is:

$$K_U = \mathbf{e}_1 \cdot \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.16}$$

This matrix has signature $(1,1)$. (Within this discussion, r is an arbitrary parameter. It will later develop a physical meaning and play an important role in the phase transition we describe.) The even unimodular lattice of signature (n,n) has a block diagonal K-matrix with n copies of K_U along the diagonal:

$$K_{U\oplus U\oplus\ldots\oplus U} = \begin{pmatrix} K_U & 0 & 0 & \ldots \\ 0 & K_U & 0 & \\ 0 & 0 & K_U & \\ \vdots & & & \ddots \end{pmatrix}. \tag{3.17}$$

The unique positive definite even unimodular lattice in dimension-8 is the lattice generated by the roots of the Lie algebra of E_8. We call this lattice Γ_{E_8}. The basis vectors for Γ_{E_8} are given in Appendix A, and the corresponding K-matrix takes the

form:

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.18}$$

The two positive-definite even unimodular lattices in dimension 16 are the lattices generated by the roots of $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$. (The latter means that a basis for the lattice is given by the roots of $SO(32)$, but with the root corresponding to the vector representation replaced by the weight of one of the spinor representations.) We will call these lattices $\Gamma_{E_8} \oplus \Gamma_{E_8}$ and $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. They are discussed further in Appendix A. The corresponding K-matrices take the form:

$$K_{E_8 \times E_8} = \begin{pmatrix} K_{E_8} & 0 \\ 0 & K_{E_8} \end{pmatrix}, \tag{3.19}$$

(for later convenience, we permute the rows and columns of the second copy of E_8 in Eq. (A.5) so that it looks superficially different from the first) and $K_{\mathrm{Spin}(32)/Z_2}$ is

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4
\end{pmatrix}. \tag{3.20}$$

The even unimodular lattice with signature $(8+n, n)$ has K-matrix:

$$K_{E_8 \oplus U \oplus \ldots \oplus U} = \begin{pmatrix}
K_{E_8} & 0 & 0 & \ldots \\
0 & U & 0 & \\
0 & 0 & U & \\
\vdots & & & \ddots
\end{pmatrix}. \tag{3.21}$$

The even unimodular lattice with signature $(16+n, n)$ has K-matrix:

$$K_{E_8\times E_8\oplus U\oplus\ldots\oplus U} = \begin{pmatrix} K_{E_8} & 0 & 0 & \ldots \\ 0 & K_{E_8} & 0 & \\ 0 & 0 & U & \\ \vdots & & & \ddots \end{pmatrix}. \tag{3.22}$$

These lattices are unique, so the matrix,

$$K_{\mathrm{Spin}(32)/\mathbb{Z}_2\oplus U\oplus\ldots\oplus U} = \begin{pmatrix} K_{\mathrm{Spin}(32)/\mathbb{Z}_2} & 0 & \ldots \\ 0 & U & \\ \vdots & & \ddots \end{pmatrix}, \tag{3.23}$$

is equivalent to (3.22) under an $SL(16+2n, \mathbb{Z})$ basis change. This fact will play an important role in the sections that follow.

3.3 Equivalence of the Two $c = 16$ Bosonic SRE Phases

In the previous section, we saw that two theories of the form (3.1) with different $N \times N$ K-matrices are equivalent if the two K-matrices are related by an $SL(N, \mathbb{Z})$ transformation or, equivalently, if they correspond to the same lattice. But if two K-matrices are not related by an $SL(N, \mathbb{Z})$ transformation, is there a more general notion that may relate the theories? A more general notion might be expected if the difference in the number of positive and negative eigenvalues of the two K-matrices coincide. Consider, for instance, the case of an $N_1 \times N_1$ K-matrix and an $N_2 \times N_2$ K-matrix with $N_1 < N_2$. Could there be a relation between them, even though they clearly cannot be related be related by an $SL(N_1, \mathbb{Z})$ or $SL(N_2, \mathbb{Z})$ similarity transformation?

The answer is yes, for the following reason. Consider the theory associated with K_U, defined in Eq. (3.16). Its partition function is equal to 1 on an arbitrary 3-manifold, M_3, as was shown in Ref. [47]:

$$Z(M_3) \equiv \int \mathcal{D}a_I e^{i \int \frac{1}{4\pi} \epsilon^{\mu\nu\rho} (K_U)_{IJ} a^I_\mu \partial_\nu a^J_\rho} = 1. \tag{3.24}$$

One manifestation of the triviality of this theory in the bulk is that it transforms trivially under modular transformations, as we saw earlier. Furthermore, a state with this K-matrix can be smoothly connected to a trivial insulator by local unitary transformations if no symmetries are maintained [17]. We shall not do so here, but it is important to note that, if we impose a symmetry on the theory, then we can guarantee the existence of gapless (non-chiral) excitations that live at the edge of the system [17, 20]. (We emphasize that we focus, in this section, on the bulk and, in this chapter, on properties that do not require symmetry.)

Therefore, we can simply replace it with a theory with no degrees of freedom. We will denote such a theory by $K = \emptyset$ to emphasize that it is a 0×0 K-matrix in a theory with 0 fields and *not* a theory with a 1×1 K-matrix that vanishes. Similarly, the partition function for a theory with arbitrary K-matrix K_A on any 3-manifold M_3 is equal to the partition function of $K_{A \oplus U}$

$$\begin{aligned} \int \mathcal{D}a_I e^{\frac{i}{4\pi} \int \epsilon^{\mu\nu\rho} (K_A)_{IJ} a^I_\mu \partial_\nu a^J_\rho} = & \\ & \int \mathcal{D}a_I \, \mathcal{D}a'_I \Big[e^{\frac{i}{4\pi} \int \epsilon^{\mu\nu\rho} (K_A)_{IJ} a^I_\mu \partial_\nu a^J_\rho} \times \\ & \quad e^{\frac{i}{4\pi} \int \epsilon^{\mu\nu\rho} (K_U)_{IJ} a'^I_\mu \partial_\nu a'^J_\rho} \Big] \\ & = \int \mathcal{D}a_I e^{\frac{i}{4\pi} \int \epsilon^{\mu\nu\rho} (K_{A\oplus U})_{IJ} a^I_\mu \partial_\nu a^J_\rho} \end{aligned} \tag{3.25}$$

Therefore, all of the theories corresponding to even, unimodular lattices of signature (n, n) are, in fact, equivalent when there is no symmetry preserved. There is just a single completely trivial gapped phase. We may choose to describe it by a very large K-matrix (which is seemingly perverse), but it is still the same phase. Moreover, any phase associated with a K-matrix can equally well be described by a larger K-matrix to which we have added copies of K_U along the block diagonal. This is an expression of the physical idea that no phase transition will be encountered in going from a given state to one in which additional trivial, gapped degrees of freedom have been added. Of course, in this particular case, we have added zero local degrees of freedom to the bulk and we have not enlarged the Hilbert space at all. So it is an even more innocuous operation. However, when we turn to the structure of edge excitations, there will be more heft to this idea.

At a more mathematical level, the equivalence of these theories is related to the notion of "stable equivalence", according to which two objects are the same if they become isomorphic after augmentation by a "trivial" object. In physics, stable equivalence has been used in the K-theoretic classification of (non-interacting) topological insulators [48]. In the present context, we will be comparing gapped phases and the trivial object that may be added to either phase is a topologically-trivial band insulator. Heuristically, stable equivalence says that we may add some number of topologically-trivial bands to our system in order to effectively enlarge the parameter space and, thereby, allow a continuous interpolation between two otherwise different states.

We now turn to the two $c = 16$ bosonic SRE phases. Their bulk effective field theories are of the form of Eq. (3.1) with K-matrices given by $K_{E_8 \times E_8}$ and $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. Their bulk properties are seemingly trivial. But not entirely so since, as we noted in Section 3.2, they transform non-trivially under modular transformations.

These two non-trivial theories are, at first glance, distinct. They are associated

with different lattices. For instance, $\Gamma_{E_8} \oplus \Gamma_{E_8}$ is the direct sum of two 8-dimensional lattices while $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ is not. The two K-matrices are not related by an $SL(16,\mathbb{Z})$ transformation.

Suppose, however, that we consider the K-matrices $K_{E_8\times E_8} \oplus U$ and $K_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$ which describe "enlarged" systems. (We use quotation marks because, although we now have theories with 18 rather than 16 gauge fields, the physical Hilbert space has not been enlarged.) These K-matrices are, in fact, related by an $SL(18,\mathbb{Z})$ transformation:

$$W_G^T \, K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U} \, W_G = K_{E_8\times E_8 \oplus U}, \tag{3.26}$$

where W_G is given by:

$$W_G = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-9 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-10 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\
-11 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\
-12 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \\
-13 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 & -5 \\
-14 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & -6 \\
-7 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -3 \\
-8 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & -4 \\
-2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 2
\end{pmatrix}. \quad (3.27)$$

We will explain how W_G is derived in Section 3.5. Here, we focus on its implication: these two theories are equivalent on an arbitrary closed manifold. There is a unique bulk $c = 16$ bosonic SRE phase of matter. However, there appear to be two possible distinct effective field theories for the edge of this unique bulk phase, namely the theories (3.11) with $K_{E_8 \times E_8}$ and $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. In the next section, we explain the relation between these edge theories.

3.4 Fermionic Representations of the Two $c = 16$ SRE Bosonic Phases

In Section 3.3, we saw that there is a unique bulk $c = 16$ bosonic SRE phase of matter. We now turn our attention to the two corresponding edge effective field theories, namely Eq. (3.11) with K_{IJ} given by either $K_{E_8 \times E_8}$ or $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. These two edge theories are distinct, although the difference is subtle. To understand this difference, it is useful to consider fermionic representations [49, 11] of these edge theories.

Consider 32 free chiral Majorana fermions:

$$S = \int dx d\tau \, \psi_j \left(-\partial_\tau + v_a i \partial_x \right) \psi_j, \tag{3.28}$$

where $j = 1, \ldots, 32$. If the velocities v_a are all the same, then this theory naively has $SO(32)$ symmetry, up to a choice of boundary conditions. We could imagine such a $1 + 1$-dimensional theory as the edge of a 32-layer system of electrons, with each layer in a spin-polarized $p + ip$ superconducting state. We will assume that the order parameters in the different layers are coupled by inter-layer Josephson tunneling so that the superconducting order parameters are locked together. Consequently, if a flux $hc/2e$ vortex passes through one of the layers, it must pass through all 32 layers. Then all 32 Majorana fermion edge modes have the same boundary conditions. When two vortices in a single-layer spin-polarized $p + ip$ superconducting state are exchanged, the resulting phase is $e^{-i\pi/8}$ or $e^{3i\pi/8}$, depending on the fusion channel of the vortices (i.e., the fermion parity of the combined state of their zero modes). Therefore, a vortex passing through all 32 layers (which may be viewed as a composite of 32 vortices, one in each layer) is a boson. These bosons carry 32 zero modes, so there are actually 2^{16} states of such vortices – 2^{15} if we require such a vortex to have even fermion parity. (Of course, the above construction

only required 16 layers if our goal was to construct the minimal dimension SRE chiral phase of bosons. [19])

Now suppose that such vortices condense. (Without loss of generality, we suppose that the vortices are in some particular internal state with even fermion parity.) Superconductivity is destroyed and the system enters an insulating phase. Although individual fermions are confined since they acquire a minus sign in going around a vortex, a pair of fermions, one in layer i and one in layer j, is an allowed excitation. The dimension-1 operators in the edge theory are of the form $i\psi_i\psi_j$ where $1 \leq i < j \leq 32$. There are $\frac{1}{2} \cdot 32 \cdot 31 = 496$ such operators. We may choose $i\psi_{2a-1}\psi_{2a}$, with $a = 1, 2, \ldots, 16$ as a maximal commuting subset, i.e. as the Cartan subalgebra of $SO(32)$. The remaining 480 correspond to the vectors of $(\text{length})^2 = 2$ in the lattice Γ_{16}. To see this, it is useful to bosonize the theory (3.28). We define the Dirac fermions $\Psi_I \equiv \psi_{2a-1} + i\psi_{2a}$, with $a = 1, 2, \ldots, 16$ and represent them with bosons: $\Psi_I = e^{iX_a}$. Then the Cartan subalgebra consists of the 16 dimension-1 operators ∂X_a. The operators $e^{i\mathbf{v}\cdot\mathbf{X}}$ with $\mathbf{v} \in \Gamma_{SO(32)} \subset \Gamma_{\text{Spin}(32)/\mathbb{Z}_2}$ and $|\mathbf{v}|^2 = 2$ correspond to the vectors of $(\text{length})^2 = 2$ in the $SO(32)$ root lattice: $\pm\hat{\mathbf{x}}_\mathbf{a} \pm \hat{\mathbf{x}}_\mathbf{b}$ with $1 \leq a < b \leq 16$. In the fermionic language, we see that the relevant perturbations of $i\psi_i\psi_k$ can be gauged away with a spatially-dependent $SO(32)$ rotation and, therefore, do not affect the basic physics of the state.

To complete the description of the $\text{Spin}(32)/\mathbb{Z}_2$ theory, recall that a vortex in a single layer braids non-trivially with the composite vortex that condenses. Such single vortices are confined after condensation of the composite. Therefore, it is impossible to change the boundary conditions of just one of the fermions ψ_i by inserting a single vortex into the bulk; all of the fermions must have the same boundary conditions. The fermion boundary conditions can be changed from anti-periodic to periodic by the operator $e^{i\mu_s\cdot\mathbf{X}} = \exp(i(X_1 + X_2 + \ldots + X_{16})/2)$, where μ_s is the weight of one of the spinor representations of $SO(32)$. This is a dimension-2 operator.

Note that the group Spin(32) is a double-cover of $SO(32)$ that has spinor representations. By disallowing one of the spinor representations and the vector representation (i.e., the odd fermion parity sector), the theory is associated with $\mathrm{Spin}(32)/\mathbb{Z}_2$ but the $\mathbb{Z}_2$ that is modded out is not the the $\mathbb{Z}_2$ that leads back to $SO(32)$. Thus, it is the inclusion of μ_s along with the vectors $\mathbf{v}$ of $SO(32)$ mentioned above that is essential to the description of the fermionic representation of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. If we had chosen not to include μ_s, i.e., if we had not condensed the composite vortex, the resulting theory would have had topological order with a torus ground state degeneracy equal to four. (The $SO(32)$ root lattice has unit cell volume equal to four while the unit cell volume of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ lattice is unity.)

Now suppose that the first 16 layers are coupled by interlayer Josephson tunneling so that their order parameters are locked and the remaining 16 layers are coupled similarly, but the first 16 layers are not coupled to the remaining 16. Then there are independent vortices in the first 16 layers and in the remaining 16 layers. Suppose that both types of vortices condense. Each of these 16-vortex composites is a boson, and superconductivity is again destroyed. Individual fermions are again confined and, moreover, the fermion parity in each half of the system must be even. Therefore, the allowed dimension-1 operators in the theory are $i\psi_i\psi_j$ with $1 \le i < j \le 16$ or $17 \le i < j \le 32$. There are $2 \cdot \frac{1}{2} \cdot 16 \cdot 15 = 240$ such dimension-1 operators. As above, 16 of them correspond to the Cartan subalgebra. The other 224 correspond to lattice vectors $e^{i\mathbf{v}\cdot\mathbf{X}}$ with $\mathbf{v} = \pm\hat{\mathbf{x}}_\mathbf{a} \pm \hat{\mathbf{x}}_\mathbf{b}$ and $1 \le a < b \le 8$ or $9 \le a < b \le 16$. Unlike in the case of $\mathrm{Spin}(32)/\mathbb{Z}_2$, the boundary-condition changing operators $\exp(i(\pm X_1 \pm X_2 \ldots \pm X_8)/2)$ and $\exp(i(\pm X_9 \pm X_{10} \ldots \pm X_{16})/2)$ are dimension-1 operators. There are $2 \cdot 2^7 = 256$ such operators with even fermion parity in each half of the system (i.e., an even number of + signs in the exponential). The corresponding vectors $\mathbf{v} = (\pm\hat{\mathbf{x}}_\mathbf{1} \pm \hat{\mathbf{x}}_\mathbf{2} \ldots \pm \hat{\mathbf{x}}_\mathbf{8})/2$ and $\mathbf{v} = (\pm\hat{\mathbf{x}}_\mathbf{9} \pm \hat{\mathbf{x}}_\mathbf{10} \ldots \pm \hat{\mathbf{x}}_\mathbf{16})/2$ with an even number of + signs together with $\mathbf{v} = \pm\hat{\mathbf{x}}_\mathbf{a} \pm \hat{\mathbf{x}}_\mathbf{b}$ are the 480 different $(\text{length})^2 = 2$

vectors in the $E_8 \times E_8$ root lattice. Consequently, this is the fermionic representation of the $E_8 \times E_8$ theory.

It is unclear, from this fermionic description, how to adiabatically connect the two bulk theories. The most obvious route between them, starting from the $E_8 \times E_8$ theory, is to restore superconductivity, couple the order parameters of the two sets of 16 layers, and then condense 32-layer vortices to destroy superconductivity again. This route takes the system across three phase transitions while the analysis in the previous section showed that they are, in fact, the same phase and, therefore, it should be possible to go from one to the other without crossing any bulk phase boundaries.

As we saw above, there are 480 vectors $\mathbf{u}$ with $|\mathbf{u}|^2 = 2$ in both $\Gamma_{E_8\times E_8}$ and $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. In fact, a result of Milnor[12] (related to hearing the shape of a drum) states that the two lattices have the same number of vectors of *all lengths*: for every $\mathbf{u} \in \Gamma_{E_8\times E_8}$, there is a unique partner $\mathbf{v} \in \Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ such that $|\mathbf{v}|^2 = |\mathbf{u}|^2$. (See Ref. [11] for an elegant presentation of this fact following Ref. [37].) Therefore, the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ edge theories have identical spectra of operator scaling dimensions $\Delta_{\mathbf{u}} = \frac{1}{2}|\mathbf{u}|^2$. Thus, it is impossible to distinguish these two edge theories by measuring the possible exponents associated with two-point functions. However, in the fermionic realization described above, consider one of the 496 dimension-1 operators, which we will call J_i, $i = 1, 2, \ldots, 496$. They are given by ∂X_a and $e^{i\mathbf{u}\cdot\mathbf{X}}$ with $|\mathbf{u}|^2 = 2$ for $\mathbf{u} \in \Gamma_{E_8\times E_8}$ or $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$. In the limit that all of the velocities are equal, these are conserved currents corresponding to the 496 generators of either $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$, but we will use the notation J_i even when the velocities are not equal. It is clear that, in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase, there are J_is that involve both halves of the system, but not in the $E_8 \times E_8$ phase. In other words, in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase, there are two-point functions involving both halves of the system that decay as $\langle J_i(x,0)J_i(0,0)\rangle \propto 1/x^2$. In the $E_8 \times E_8$ phase, such operators J_i only exist acting entirely within the top half or the bottom half of the system.

Moreover, the n-point functions for $n \geq 3$ of the two theories can be different. Consider the following 4-point function in our 32-layer model,

$$\langle J_{i_1}(x_1, t_1) J_{i_2}(x_2, t_2) J_{i_3}(x_3, t_3) J_{i_4}(x_4, t_4) \rangle_c, \tag{3.29}$$

where the subscript c denotes a connected correlation function, and J_{i_1} acts within the first 16 layers and J_{i_2} within the second 16 layers. In the $E_8 \times E_8$ theory, this correlation function vanishes for all choices of i_3, i_4 because there are no dimension-1 operators that act on both halves of the system, i.e. within both the first 16 layers and the second 16 layers. On the other hand, in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory, there will always be choices of i_3, i_4 such that the connected correlation function is non-zero: if $J_{i_1} = i\psi_k\psi_l$ and $J_{i_2} = i\psi_m\psi_n$ with $1 \leq k < l \leq 16$ and $17 \leq m < n \leq 32$ then the connected correlation function is non-zero for $J_{i_3} = i\psi_k\psi_m$ and $J_{i_4} = i\psi_l\psi_n$. Such a correlation function (3.29) corresponds to a measurement of a current J_{i_1} in the top half of the system in response to a probe that couples to J_{i_2} in the bottom half of the system. While such a measurement will give a vanishing result in the absence of other perturbations, it will give a non-vanishing result in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory in the presence of perturbations that couple to J_{i_3} and J_{i_4}. In other words, it is a measurement of J_{i_1} to linear order in external fields that couple to J_{i_2}, J_{i_3}, and J_{i_4}.

Of course, in some other physical realization it may be more difficult to divide these currents into a 'top half' and a 'bottom half', but there will always be correlation functions that distinguish the two edge theories.

3.5 Phase Diagram of the $c - \overline{c} = 16$ Edge.

Since there is a unique bulk $c = 16$ bosonic SRE phase of matter, the two different edge theories corresponding to $K_{E_8 \times E_8}$ or $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ must be different edge phases that can occur at the boundary of the same bulk phase. For this scenario to hold, it must be the case that the transition between these two edge theories is purely an edge transition – or, in other words, an "edge reconstruction" – that can occur without affecting the bulk. Such a transition can occur as follows. The gapless modes in the effective theory (3.11) are the lowest energy excitations in the system. However, there will generically be gapped excitations at the edge of the system that we usually ignore. So long as they remain gapped, this is safe. However, these excitations could move downward in energy and begin to mix with the gapless excitations, eventually driving a phase transition. Such gapped excitations must be non-chiral and can only support bosonic excitations.

A perturbed non-chiral Luttinger liquid is the simplest example of such a gapped mode:

$$S_{\mathrm{LL}} = \frac{1}{4\pi} \int dt\, dx \Big[2\partial_t \varphi\, \partial_x \theta - \frac{v}{g} (\partial_x \theta)^2 - vg (\partial_x \varphi)^2 + u_1^{(m)} \cos(m\theta) + u_2^{(n)} \cos(n\varphi) \Big], \quad (3.30)$$

with Luttinger parameter g and integers m, n. The φ and θ fields have period 2π. The first line is the action for a gapless Luttinger liquid. The second line contains perturbations that can open a gap in the Luttinger liquid spectrum. The couplings $u_1^{(m)}$ and $u_2^{(n)}$ have scaling dimensions $2 - \frac{m^2}{2} g$ and $2 - 2n^2 g^{-1}$, respectively. Let us concentrate on the lowest harmonics which are the most relevant operators with couplings $u_1^{(1)} \equiv u_1$ and $u_2^{(1)} \equiv u_2$. The first operator is relevant if $g < 4$ and the second one is relevant if $g > 1$. At least one of these is always relevant. Given our parameterization of the Luttinger Lagrangian, a

system of hard-core bosons on the lattice with no other interactions or in the continuum with infinite δ-function repulsion has $g = 1$ (see Ref. [50]).

When considering one-dimensional bosonic systems, the above cosine perturbations can be forbidden by, respectively, particle-number conservation and translational invariance. Here, however, we do not assume that there is any symmetry present, so these terms are allowed. The Luttinger action can be rewritten in the same way as the edge theory (3.11):

$$S_{\mathrm{LL}} = \frac{1}{4\pi} \int dt\, dx \Big[(K_U)_{IJ} \partial_t \phi^I\, \partial_x \phi^J - V_{IJ} \partial_t \phi^I\, \partial_x \phi^J + u_1 \cos(\phi_{17}) + u_2 \cos(\phi_{18}) \Big], \quad (3.31)$$

where $I, J = 17, 18$ in this equation and $\phi_{17} = \theta$ and $\phi_{18} = \varphi$. Therefore, we see that the action for a perturbed Luttinger liquid is the edge theory associated with the trivial bulk theory with K-matrix given by K_U that we discussed in Section 3.3. It is gapped unless u_1 and u_2 are fine-tuned to zero or forbidden by a symmetry. However, augmenting our system with this trivial one does increase the number of degrees of freedom at the edge and expands the Hilbert space, unlike in the case of the bulk.

Hence, we consider the edge theory

$$S = \frac{1}{4\pi} \int dt dx \Big[(K_{E_8 \times E_8 \oplus U})_{IJ}\, \partial_t \phi^I\, \partial_x \phi^J - V_{IJ}\, \partial_x \phi^I\, \partial_x \phi^J + u_1 \cos(\phi_{17}) + u_2 \cos(\phi_{18}) + \ldots \Big] \quad (3.32)$$

We can integrate out the trivial gapped degrees of freedom ϕ^{17} or ϕ^{18}, leaving the gapless chiral edge theory associated with $K_{E_8 \times E_8}$. The $\ldots$ represents other non-chiral terms

that could appear in the Lagrangian (i.e., cosines of linear combinations of the fields ϕ^I); they are all irrelevant for $V_{I,17} = V_{I,18} = 0$ for $I = 1, \ldots, 16$; or more accurately, they are less relevant than u_1 or u_2 and so we ignore them to first approximation. However, if we vary the couplings V_{IJ}, then u_1, u_2 could both become irrelevant and some other term could become relevant, driving the edge into another phase.

To further analyze the possible transition, it is useful to rewrite the action in terms of the fields $\mathbf{X} = \mathbf{e}_J \phi^J$:

$$S = \frac{1}{4\pi} \int dt\, dx \left[\eta_{ab} \partial_t X^a\, \partial_x X^b - v_{ab}\, \partial_x X^a\, \partial_x X^b \right. \\ \left. + u_1 \cos(\tfrac{r}{2}(X^{17} + X^{18})) + u_2 \cos(\tfrac{1}{r}(X^{17} - X^{18})) + \ldots \right]. \quad (3.33)$$

where $v_{ab} \equiv V_{IJ} f_a^I f_b^J$, $f_a^I e_J^a = \mathbf{f}^I \cdot \mathbf{e}_J = \delta^I{}_J$, and $\eta_{ab} = (1^{16}, 1, -1)$. Here, $\mathbf{e}_J$ for $J = 1, \ldots, 16$ is a basis of $\Gamma_{E_8} \oplus \Gamma_{E_8}$ given explicitly in Appendix A and c^n refers to the n-component vector where each component equals c. We take $\mathbf{e}_{17} = (0^{16}, \frac{1}{r}, \frac{1}{r})$ and $\mathbf{e}_{18} = (0^{16}, \frac{r}{2}, -\frac{r}{2})$ so that $\mathbf{e}_{17} \cdot \mathbf{e}_{17} = \mathbf{e}_{18} \cdot \mathbf{e}_{18} = 0$ and $\mathbf{e}_{17} \cdot \mathbf{e}_{18} = 1$. When $v_{a,17} = v_{a,18} = 0$ for $a = 1, \ldots, 16$ (or, equivalently, when $V_{I,17} = V_{I,18} = 0$ for $I = 1, \ldots, 16$), the parameter r is related to the Luttinger parameter according to $g = r^2/2$ and u_1, u_2 have renormalization group (RG) equations:

$$\begin{aligned} \frac{du_1}{d\ell} &= \left(2 - \frac{r^2}{4}\right) u_1, \\ \frac{du_2}{d\ell} &= \left(2 - r^{-2}\right) u_2. \end{aligned} \quad (3.34)$$

Hence, one of these two perturbations is always relevant when $v_{a,17} = v_{a,18} = 0$ for $a = 1, \ldots, 16$ and, consequently, $X^{17,18}$ become gapped. The arguments of the cosine follow from the field redefinition $\phi^I = \mathbf{f}^I \cdot \mathbf{X} = (K^{-1})^{IJ} \mathbf{e}_J \cdot \mathbf{X}$. The field $\mathbf{X}$ satisfies the periodicity conditions $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{u}$ for $\mathbf{u} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$. Again, the $\ldots$ refers to other

possible perturbations, i.e., cosines of other linear combinations of the X^as.

In a nearly identical manner, we can construct a theory for $\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U$ in which a non-chiral gapped mode is added to the $\mathrm{Spin}(32)/\mathbb{Z}_2$ edge theory and allowed to interact with it. The only difference is in the parameterization of the U lattice. We choose $\tilde{\mathbf{e}}_{17} = (0^{16}, -r, r)$ and $\tilde{\mathbf{e}}^{18} = (0^{16}, -\frac{1}{2r}, -\frac{1}{2r})$. The action,

$$S = \frac{1}{4\pi} \int dt\, dx \, \big[\eta_{ab} \partial_t \tilde{X}^a \, \partial_x \tilde{X}^b - \tilde{v}_{ab} \, \partial_x \tilde{X}^a \, \partial_x \tilde{X}^b + \tilde{u}_1 \cos(\tfrac{1}{2r}(\tilde{X}^{17} - \tilde{X}^{18})) + \tilde{u}_2 \cos(r(\tilde{X}^{17} + \tilde{X}^{18})) + \ldots \big]. \quad (3.35)$$

Again, the $\ldots$ refers to cosines of other linear combinations of the $\tilde{X}^a$s. When $\tilde{v}_{17,18} = \tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1, \ldots, 16$, the parameter r is related to the Luttinger parameter according to $g = r^{-2}/2$ and $\tilde{u}_1$, $\tilde{u}_2$ have RG equations:

$$\begin{aligned} \frac{d\tilde{u}_1}{d\ell} &= \left(2 - \frac{1}{4r^2}\right)\tilde{u}_1, \\ \frac{d\tilde{u}_2}{d\ell} &= \left(2 - r^2\right)\tilde{u}_2. \end{aligned} \quad (3.36)$$

Hence, one of these two perturbations is always most relevant when $\tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1, \ldots, 16$ and, consequently, $X^{17,18}$ become gapped. The fields $\tilde{\mathbf{X}}$ satisfy the periodicity conditions $\tilde{\mathbf{X}} \equiv \tilde{\mathbf{X}} + 2\pi\mathbf{v}$ for $\mathbf{v} \in \Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$.

We now make use of the fact there is a unique signature $(17, 1)$ even unimodular lattice. It implies that there is an $SO(17, 1)$ rotation O_G that transforms $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ into $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$. Therefore, the fields $O_G\mathbf{X}$ satisfy the periodicity condition $O_G\mathbf{X} \equiv O_G\mathbf{X} + 2\pi\mathbf{v}$ for $\mathbf{v} \in \Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$ or, in components, $(O_G)^a{}_b X^b \equiv (O_G)^a{}_b X^b + 2\pi n^I \tilde{e}^a_I$ for $n^I \in \mathbb{Z}$. Thus, we identify $\tilde{X}^a = (O_G)^a{}_b X^b$. The explicit expression for O_G is provided in Appendix A.

(As an aside, having identified X^a and $\tilde{X}^b$ through the $SO(17, 1)$ transformation O_G,

we can now explain how the $SL(18,\mathbb{Z})$ transformation W_G is obtained. The desired transformation is read off from the relation,

$$\tilde{\phi}^J = \tilde{f}^J_a (O_G)^a{}_b e^b_I \phi^I =: (W_G)_{IJ} \phi^I, \tag{3.37}$$

which follows from equation relating the $\Gamma_{E_8} \oplus \Gamma_{E_8}$ and $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ bases,

$$(O_G)^a{}_b e^b_I = \sum_K m^K_I \tilde{e}^a_K, \tag{3.38}$$

where the m^K_I are a collection of integers. Multiplying both sides of Eq. (3.38) by $\tilde{f}^J_c$ allows us to read off the elements of W_G.)

Therefore, by substituting $\tilde{X}^a = (O_G)^a{}_b X^b$, the action (3.35) could equally well be written in the form:

$$\begin{aligned} S \;\; = \tfrac{1}{4\pi} \int dt\, dx \big[\;\; & \eta_{ab} \partial_t X^a \partial_x X^b \; - \tilde{v}_{ab} (O_G)^a{}_c (O_G)^b{}_d \, \partial_x X^c \, \partial_x X^d \\ & + \tilde{u}_1 \cos(\tfrac{1}{2r}((O_G)^{17}_a X^a - (O_G)^{18}_a X^a)) \\ & + \tilde{u}_2 \cos(r((O_G)^{17}_a X^a + (O_G)^{18}_a X^a)) + \ldots \big], \end{aligned} \tag{3.39}$$

where $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{u}$ for $\mathbf{u} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$. (We have used the defining property, $(O_G)^a{}_b \eta_{ac} (O_G)^c{}_d = \eta_{bd}$, in rewriting the first term in the action (3.35).)

Having rewritten the augmented $\mathrm{Spin}(32)/\mathbb{Z}_2$ action Eq. (3.35) in terms of the $\Gamma_{E_8} \oplus \Gamma_{E_8}$ fields, let us add in two of the available mass perturbations u_1, u_2 written explicitly in Eq. (3.33):

$$S = \frac{1}{4\pi} \int dt\, dx \left[\eta_{ab} \partial_t X^a \, \partial_x X^b - \tilde{v}_{ab} (O_G)^a{}_c (O_G)^b{}_d \, \partial_x X^c \, \partial_x X^d \right. \tag{3.40}$$
$$+ \tilde{u}_1 \cos(\tfrac{1}{2r}((O_G)^{17}_a X^a - (O_G)^{18}_a X^a)) + \tilde{u}_2 \cos(r((O_G)^{17}_a X^a + (O_G)^{18}_a X^a))$$
$$\left. + u_1 \cos(\tfrac{r}{2}(X^{17} + X^{18})) + u_2 \cos(\tfrac{1}{r}(X^{17} - X^{18})) + \ldots \right].$$

So far we have only rewritten Eq. (3.35) and included additional mass perturbations implicitly denoted by "...". If $\tilde{v}_{17,18} = \tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1, \ldots, 16$, then either $\tilde{u}_1$ or $\tilde{u}_2$ is the most relevant operator and the $\tilde{X}^{17}$ and $\tilde{X}^{18}$ fields are gapped out. The remaining gapless degrees of freedom are those of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ edge theory. On the other hand, if $v_{cd} = \tilde{v}_{ab}(O_G)^a{}_c (O_G)^b{}_d$ with $v_{17,18} = v_{a,17} = v_{a,18} = 0$, either u_1 or u_2 is the most relevant operator. At low energies, X^{17} and X^{18} are gapped with the remaining degrees of freedom being those of the $E_8 \times E_8$ theory. We see that the transition between the chiral $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ is mediated by O_G given a starting velocity matrix – this is an interaction driven transition.

Given O_G, we can define a one-parameter family of $SO(17,1)$ transformations as follows. As discussed in Appendix A, O_G can be written in the form $O_G = \eta W(A) \eta W(A')$, where $W(A), W(A')$ are $SO(17,1)$ transformations labelled by the vectors A, A' which are defined in Appendix A as well and η is a reflection. We define $O_G(s) = \eta W(sA) \eta W(sA')$. This family of $SO(17,1)$ transformations, parametrized by $s \in [0,1]$ interpolates between $O_G(0) = I$, the identity, and $O_G(1) = O_G$ or, in components, $(O_G(0))^a{}_b = \delta^a{}_b$, the identity, and $(O_G(1))^a{}_b = (O_G)^a{}_b$. This one-parameter family of transformations defines a one-parameter family of theories:

$$S_4(s) = \frac{1}{4\pi} \int dt\, dx \, \big[\eta_{ab}\partial_t X^a\, \partial_x X^b - v_{ab}(O_G(s))^a{}_c (O_G(s))^b{}_d\, \partial_x X^c\, \partial_x X^d + \tilde{u}_1 \cos(\tfrac{1}{2r}((O_G)^{17}_a X^a - (O_G)^{18}_a X^a)) + \tilde{u}_2 \cos(r((O_G)^{17}_a X^a + (O_G)^{18}_a X^a)) + u_1 \cos(\tfrac{r}{2}(X^{17} + X^{18})) + u_2 \cos(\tfrac{1}{r}(X^{17} - X^{18})) + \ldots\big].$$

These theories are parametrized by s, which determines a one-parameter family of velocity matrices $v_{ab}(O_G(s))^a{}_c(O_G(s))^b{}_d$ (this is the only place where s enters the action). We call this action $S_4(s)$ because there are 4 potentially mass-generating cosine perturbations. Note that the $\tilde{u}_{1,2}$ terms have $O_G = O_G(1)$ in the arguments of the cosines, not $O_G(s)$. As our starting point, we take $v_{17,18} = v_{a,17} = v_{a,18} = 0$ for $a = 1, \ldots, 16$. (For instance, we can take diagonal v_{ab}.) Then, for $s = 0$, this theory is of the form of Eq. (3.33) with two extra mass perturbations parameterized by $\tilde{u}_1$ and $\tilde{u}_2$; however, either u_1 or u_2 is most relevant; and the remaining gapless degrees of freedom are those of the chiral $E_8 \times E_8$ edge theory. For $s = 1$, this theory is of the form of Eq. (3.41) which we know is equivalent to Eq. (3.35) with two extra mass perturbations parameterized by u_1 and u_2; now, either $\tilde{u}_1$, $\tilde{u}_2$ is most relevant; and the remaining gapless degrees of freedom are those of the Spin(32)/$\mathbb{Z}_2$ edge theory. For intermediate values of s, the RG equations for u_1, u_2, $\tilde{u}_1$, $\tilde{u}_2$ are:

$$\begin{aligned}
\frac{du_1}{d\ell} &= \Big[2 - \tfrac{(2s^2+r^2(1-s^2+4s^4))^2}{4r^2}\Big] u_1, \\
\frac{du_2}{d\ell} &= \Big[2 - \tfrac{(1+2r^2s^2)^2}{r^2}\Big] u_2, \\
\frac{d\tilde{u}_1}{d\ell} &= \Big[2 - \tfrac{(4-7s+4s^2+2r^2(s-1)^2(1+s+4s^2))^2}{4r^2}\Big] \tilde{u}_1, \\
\frac{d\tilde{u}_2}{d\ell} &= \Big[2 - \tfrac{(2(s-1)^2+r^2(1+s+3s^2-8s^3+4s^4))^2}{r^2}\Big] \tilde{u}_2.
\end{aligned} \tag{3.41}$$

The expressions in square brackets on the right-hand-sides of these equations, which

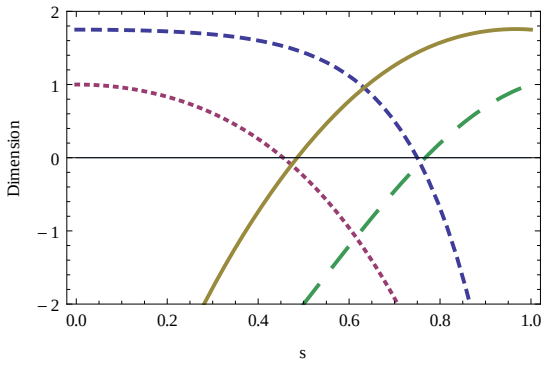

Figure 3.1: The scaling dimensions of $u_{1,2}$ (densely dashed and dotted) and $\tilde{u}_{1,2}$ (thick and dashed), plotted as a function of s at $r = 1$. The $E_8 \times E_8$ phase lives roughly within $0 \leq s < .625$ and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase between $.625 < s \leq 1$.

are equal to $\frac{1}{u_{1,2}} \frac{du_{1,2}}{d\ell}$ and $\frac{1}{\tilde{u}_{1,2}} \frac{d\tilde{u}_{1,2}}{d\ell}$, are the scaling dimensions of $u_{1,2}$ and $\tilde{u}_{1,2}$ near the $u_{1,2} = \tilde{u}_{1,2} = 0$ fixed line.

We plot the weak-coupling RG flows of these operators in Figs. 3.1-3.3 for three different choices of r. First, we notice that, depending upon r, either u_1 or u_2 is most relevant at $s = 0$. At $s = 1$, either $\tilde{u}_1$ or $\tilde{u}_2$ is most relevant. At intermediate values of s, there are several possibilities. Assuming that the most relevant operator determines the flow to low energy (which must have the same value $c - \bar{c} = 16$ as the action (**??**)), we conclude that when either of these two sets of operators is most relevant we expect a mass to be generated for, respectively, the $X^{17,18}$ or $\tilde{X}^{17,18}$ modes, thereby leaving behind either the $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ edge theories at low energies. If there are no relevant operators, then the edge is not fully chiral; it has $c = 17$, $\bar{c} = 1$.

Thus, we see that the two different positive-definite even unimodular lattices in 16 dimensions correspond to two different fully chiral phases at the edge of the same bulk phase. In the model in Eq. (**??**), the transition between them can occur in two possible ways: either a direct transition (naively, first-order, as we argue below) or or via two

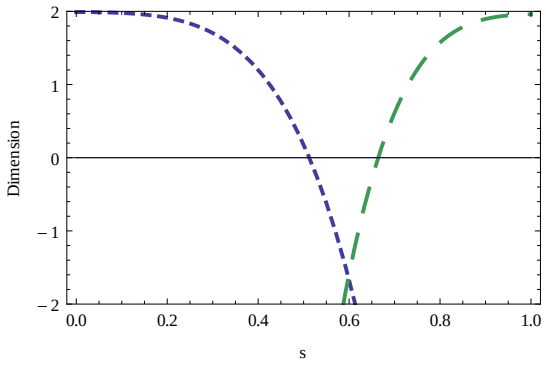

Figure 3.2: The scaling dimensions of u_1 (densely dashed) and $\tilde{u}_2$ (dashed), plotted as a function of s at at $r = .2$. The scaling dimensions of u_2 and $\tilde{u}_1$ lie outside the range of the plot and are not displayed. The system is not fully chiral phase between approximately $s = .5$ and $s = .625$.

Kosterlitz-Thouless-like phase transitions, with an intermediate $c = 17$, $\overline{c} = 1$ phase between the two fully chiral phases. The former possibility occurs (again, assuming that the most relevant operator determines the flow to low energy) when there is always at least one relevant operator. The system is in the minimum of the corresponding cosine, but when another operator becomes more relevant, the system jumps to this minimum as s is tuned through the crossing point. If the most relevant operator is in the set $u_1, u_2, \tilde{u}_1, \tilde{u}_2$, then this means that the crossing point between the larger of $\frac{1}{u_{1,2}} \frac{du_{1,2}}{d\ell}$ and the larger of $\frac{1}{\tilde{u}_{1,2}} \frac{d\tilde{u}_{1,2}}{d\ell}$ occurs when both are positive so that the system goes directly from $E_8 \times E_8$ to $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. However, if there is a regime in which there are no relevant operators, then there will be a stable $c = 17$, $\overline{c} = 1$ phase. (Note that we adhere to a slightly weaker definition of stability than used in the paper [51]; we say that an edge is unstable to gapping out some subset of its modes if a null vector [52] of the K-matrix exists and that the associated operator is relevant in the RG sense. A null vector is simply an integer vector n_I satisfying $n_I (K^{-1})^{IJ} n_J = 0$ or, equivalently,

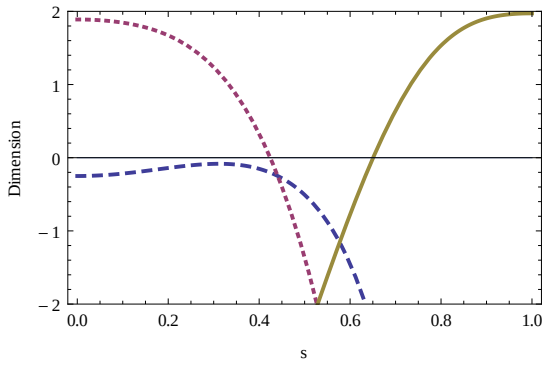

Figure 3.3: The scaling dimensions of $u_{1,2}$ (densely dashed and dotted) and $\tilde{u}_1$ (thick), plotted as a function of s at at $r = 3$. The scaling dimension of $\tilde{u}_2$ lies outside the range of the plot and is not displayed. The system is not fully chiral phase between approximately $s = .425$ and $s = .625$.

a lattice vector k_a satisfying $k_a \eta^{ab} k_b = 0$.) If the crossing point between the larger of $\frac{1}{u_{1,2}} \frac{du_{1,2}}{d\ell}$ and the larger of $\frac{1}{\tilde{u}_{1,2}} \frac{d\tilde{u}_{1,2}}{d\ell}$ occurs when both are negative, then there may be a stable $c = 17$, $\bar{c} = 1$ phase.

However, the model of Eq. (**??**) is not the most general possible model; it is a particular slice of the parameter space in which the only perturbations of the quadratic theory are $u_{1,2}$ and $\tilde{u}_{1,2}$. A more general model will have many potentially mass-generating perturbations:

$$S_{\text{gen}}(s) = S_4(s) + \int dt\, dx \sum_{\mathbf{v} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U} \delta_{|\mathbf{v}|^2,0}\, u_{\mathbf{v},s} \cos(\mathbf{v} \cdot \mathbf{X}) \tag{3.42}$$

where the sum is over vectors $\mathbf{v} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ that have zero norm. This guarantees that these are spin-0 operators that are mass-generating if relevant. In Eq. (**??**), we have chosen 4 particular operators of this form and set the coefficients of the others to zero.

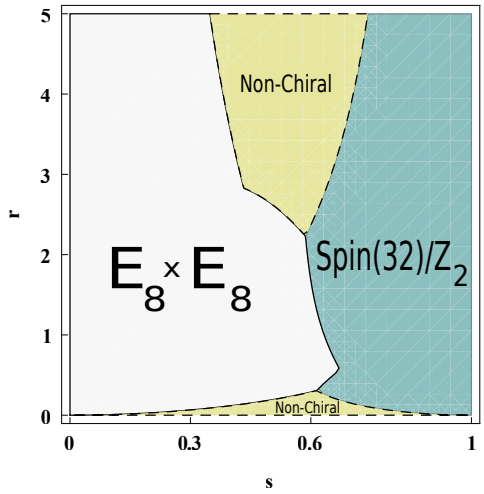

Figure 3.4: Phase diagram of our edge theory as a function of s and r for the theory $S_4(s)$ in which the only non-zero perturbations are $u_{1,2}$ and $\tilde{u}_{1,2}$. The light region is in the $E_8 \times E_8$ phase. The darkest region is in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase. The system is not fully chiral in the intermediately-shaded region. Solid phase boundary lines denote a first-order transition while the dashed phase boundary line indicates a KT transition.

[3] However, to determine if there is a stable non-chiral phase, it behooves us to consider a more general model in order to determine whether the non-chiral phase requires us to set more than one of the potentially mass-generating operators in Eq. (3.42) to zero by hand and so any such critical point is multi-critical.

Of course, there are many possible $\mathbf{v} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ with $|\mathbf{v}|^2 = 0$. But most of them give rise to operators that are highly irrelevant over most of the range of the parameters r and s. However, there are two sets of operators that cannot be ignored. In one set, each operator is highly relevant in the vicinity of a particular value of s (which depends on the operator) in the $r \to 0$ limit and, in the other set, each operator is highly relevant

[3] To lowest-order in $u_{1,2}$ and $\tilde{u}_{1,2}$, this is consistent, but at higher order, these 4 operators will generate some others, and we must consider a more general theory. However, it does not appear that these operators generate any spin-0 operators other than multiples of themselves, which are less relevant than they are.

in the vicinity of a particular value of s in the $r \to \infty$ limit. Consider the operators:

$$\cos(\alpha \tilde{f}^{17}_a R^a{}_b X^b) \ , \ \ \cos(\beta \tilde{f}^{18}_a R^a{}_b X^b) \tag{3.43}$$

where R is an arbitrary $SO(17,1)$ transformation. These operators have spin-0 since $\tilde{\mathbf{f}}^{17,18}$ have vanishing norm, which R preserves. Although they have spin-0 and can, therefore, generate a mass gap, there is no particular reason to think that either one is relevant. Moreover, it is not even likely that either one is an allowed operator. For an arbitrary $SO(17,1)$ transformation, $\tilde{f}^{17}_a R^a{}_b$ will not lie in the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice spanned by the f^Is, so this operator will not be allowed. However, there is a special class of R for which these operators are allowed and are relevant in the vicinity of special points. Let us suppose that $R = O_G(p/q)$ and let us consider $\alpha = q^4$, $\beta = q^2$. [4] Consider the action

$$S_4(s = \tfrac{p}{q}) + u_{18,\frac{p}{q}} \int dt\, dx\, \cos\left[q^2 \tilde{f}^{18}_a \left(O_G(p/q)\right)^a{}_b X^b\right] \tag{3.44}$$

This is a spin-0 perturbation. Moreover, it is an allowed operator for the following reason. We can write

$$q^2 \tilde{f}^{18}_a \left(O_G(p/q)\right)^a{}_b = q^2 (W(p/q))_{18,J} f^J{}_a \tag{3.45}$$

where $(W(s))_{IJ}$ is defined in analogy with W_G: $(W(s))_{IJ} = \tilde{f}^J_a (O_G(s))^a{}_b e^b_I$. The vector $q^2(W(p/q))_{18,J}$ has integer entries, so $q^2 \tilde{f}^{18}_a \left(O_G(p/q)\right)^a{}_b$ is in the lattice $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$. At the point $s = p/q$, its scaling dimension is the same as the scaling dimension of $q^2 \tilde{f}^{18}_a X^a$ at $s = 0$:

$$\frac{d}{d\ell} u_{18,\frac{p}{q}} = \left[2 - q^4 r^2\right] u_{18,\frac{p}{q}} \tag{3.46}$$

[4] This choice of α and β is a sufficient one for generic $s = p/q$; however, certain q accommodate smaller α and β so that the resulting operators are well defined. For example, when q is even, we may take $\alpha = q^2/2$ and $\beta = q^4/4$.

Therefore, for $r < \sqrt{2}/q^2$, the coupling $u_{18,\frac{p}{q}}$ is a relevant mass-generating interaction at $s = p/q$ and, over some range of small r, it is relevant for s sufficiently near p/q. By a similar analysis, $u_{17,\frac{p}{q}}$ is a relevant mass-generating interaction at $s = p/q$ for $r > q^4/(2\sqrt{2})$ and, over some range of large r, it is relevant for s sufficiently near p/q. Therefore, when these couplings are non-zero, the non-chiral phase survives in a much smaller region of the phase diagram. (Making contact with our previous notation, we see that $u_{17,1} = \tilde{u}_1$ and $u_{18,1} = \tilde{u}_2$.)

When one of these interactions gaps out a pair of counter-propagating modes, we are left with a fully chiral $c = 16$ edge theory corresponding to either $E_8 \times E_8$ to $\mathrm{Spin}(32)/\mathbb{Z}_2$. To see which phase we get, consider, for the sake of concreteness, the coupling $u_{18,\frac{p}{q}}$. When it generates a gap, it locks the combination of fields $q^2 \tilde{f}^{18}_{\ a} \left(O_G(p/q)\right)^a_{\ b} X^b = q^2 (W(p/q))_{18,J} f^J_{\ a} X^a$. In the low-energy limit, we may set this combination to zero. Only fields that commute with this combination remain gapless. (Moreover, since we have set this combination to zero, any fields that differ by a multiple of it are equal to each other at low-energy.) Therefore, the vertex operators that remain in the theory are of the form $\exp(n_I f^J_{\ a} X^a)$ where n_I satisfies $n_I (K^{-1})^{IJ} (W(p/q))_{18,J} = 0$. We note that $(W(p/q))_{18,J}$ is non-zero only for $J = 8, 16, 17, 18$. Therefore, $(W(p/q))_{18,J} f^J_{\ a}$ is orthogonal to $\mathbf{e}_1, \ldots, \mathbf{e}_7$ and $\mathbf{e}_9, \ldots, \mathbf{e}_{15}$.

Much as in our discussion in Section 3.4 of the difference between the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ edge theories, we again make use of the basic observation that $E_8 \times E_8$ is a product while $\mathrm{Spin}(32)/\mathbb{Z}_2$ has a single component in order to identify the low energy theory. If the vectors $n_I f^I_{\ a}$ with $n_I (K^{-1})^{IJ} (W(p/q))_{18,J} = 0$ (and two vectors differing by a multiple of $q^2 (W(p/q))_{18,J} f^J_{\ a}$ identified) form the $\mathrm{Spin}(32)/\mathbb{Z}_2$ lattice, then there must be a vector $\mathbf{c} = c_I \mathbf{f}^I$ in the lattice with $|\mathbf{c}|^2 = 2$ such that $\mathbf{c} \cdot \mathbf{e}_1 = -\mathbf{c} \cdot \mathbf{e}_7 = \mathbf{c} \cdot \mathbf{e}_9 = 1$ and $\mathbf{c} \cdot \mathbf{e}_2 = \mathbf{c} \cdot \mathbf{e}_3 = \ldots = \mathbf{c} \cdot \mathbf{e}_6 = 0$ and $\mathbf{c} \cdot e_{10} = \mathbf{c} \cdot e_{11} = \ldots = \mathbf{c} \cdot e_{15} = 0$. This is because there exists a set of Cartesian coordinates $\hat{\mathbf{y}}_a$ such that all the vectors in

Spin(32)/$\mathbb{Z}_2$ with (length)$^2 = 2$ are of the form $\pm\hat{\mathbf{y}}_a \pm \hat{\mathbf{y}}_b$ with $a, b = 1, \ldots, 16$, while for $E_8 \times E_8$, vectors of the form $\pm\hat{\mathbf{y}}_a \pm \hat{\mathbf{y}}_b$ must have $a, b = 1, \ldots, 8$ or $a, b = 9, \ldots, 16$. In $E_8 \times E_8$, vectors of (length)$^2 = 2$ cannot "connect" the two halves of the system. If the equations $c_I (K^{-1})^{IJ} (W(p/q))_{18,J} = 0$ and $c_I (K^{-1})^{IJ} c_J = 2$ with $c_1 = -c_7 = c_9 = 1$ and $c_2 = c_3 = \ldots = c_6 = c_{10} = c_{11} = \ldots = c_{15} = 0$ have integer solutions, then the remaining gapless degrees of freedom are in the Spin(32)/$\mathbb{Z}_2$ phase. Otherwise, they are in the $E_8 \times E_8$ phase. We could choose $\mathbf{e}_1$, $-\mathbf{e}_7$, and $\mathbf{e}_9$ as the vectors with unit product with $\mathbf{c}$ because such a $\mathbf{c}$ must exist in Spin(32)/$\mathbb{Z}_2$. (Note, that we could have taken c_7 to be arbitrary, and we would have found that solutions to these equations must necessarily have $c_7 = -1$.) The phase is $E_8 \times E_8$ if and only if such a vector $\mathbf{c}$ is not in the lattice. Of course, it is essential that we can restrict our attention to the two possibilities, $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$, since these are the only two unimodular self-dual lattices in dimension 16.

With the aid of MATHEMATICA, we have found that solutions to the above equations must be of the form $c_I = (1, 0^5, -1, c_8, 1, 0^6, c_8 - 1, q/p(2c_8 - 1), -p/q(2c_8 - 1))$. Since c_I must be an integer vector, both p and q must be odd since $2c_8 - 1$ is odd. Here, as above, we have assumed that p and q are relatively prime. Further, we see that this solution requires $2c_8 = pqm + 1$ for odd m.

This means that the chiral Spin(32)/$\mathbb{Z}_2$ theory is left behind at low energies when both p and q are odd and $u_{18,\frac{p}{q}}$ is the most relevant operator that generates a mass gap for two counter-propagating edge modes. When either p or q is even, the remaining gapless modes of the edge are in the $E_8 \times E_8$ phase. We find the identical behavior for the low energy theory when $u_{17,\frac{p}{q}}$ is the most relevant operator.

When these operators have non-zero coefficients in the Lagrangian, they eliminate a great deal of the non-chiral phase shown in the $u_{1,2}$, $\tilde{u}_{1,2}$-only phase diagram in Fig. 4. The effect is most noticeable as $r \to 0$ and $r \to \infty$ as shown in Fig. 5.

However, there still remain pockets of the non-chiral phase at intermediate values of r and s, where these operators are irrelevant. However, we find that these regions of non-chiral phase are not stable when we include a larger set of operators in the Lagrangian. Consistent with our expectations, it is possible to find a relevant operator in the region around any given point (r, s) in the phase diagram such that the low energy theory remaining after a pair of counter-propagating modes gaps out is $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.

To see how this works, consider, for instance, the point $(r, s) = (3, 3/5)$ that exists in the putative region of non-chiral phase according to Fig. 4. The couplings $u_{17,\frac{p}{q}}$, $u_{18,\frac{p}{q}}$ are all irrelevant there so the system remains non-chiral even when these couplings are turned on. However, we can find a relevant spin-0 operator at this point as follows. It must take the form $\cos(p_a X^a)$, with $p_a \in \Gamma_8 \oplus \Gamma_8 \oplus U$, where $\eta^{ab} p_a p_b = 0$ (this is the spin-0 condition). To compute its scaling dimension, we observe that it can be written in the form $\cos(q_a X^a(s))$, where $X^a(s) \equiv (O_G(s))^a_b X^b$ and $p_b = q_a (O_G(s))^a_b$. In terms of this field, the quadratic part of the action is diagonal in the $X^a(s)$ fields, so their correlation functions (and, therefore, their scaling dimensions can be computed straightforwardly). Since the operator in question has spin-0, its total scaling dimension $\delta^{ab} q_a q_b$ is twice their left-moving dimension or, simply, $|q_{18}|^2$. Therefore, such an operator is relevant if $|q_{18}|^2 < 2$.

$O_G^{-1}(s)$ is simply a boost along some particular direction in the 17-dimensional space combined with a spatial rotation. The eigenvalues of such a transformation are either complex numbers of modulus 1 (rotation) or contraction/dilation by $e^{\pm\alpha}$ (Lorentz boost). Consequently, even if $\delta^{ab} p_a p_b$ is large – which means that $\cos(p_a X^a)$ is highly irrelevant at $s = 0$ – $\delta^{ab} q_a q_b$ can be smaller by as much as $e^{-2\alpha}$, thereby making $\cos(p_a X^a)$ a relevant operator at this value of s (and of r). The maximum possible contraction, $e^{-\alpha}$, occurs when p_a is anti-parallel to the boost. (The maximum dilation, $e^{-\alpha}$, occurs when p_a is parallel to the boost, and there is no change in the scaling dimension when p_a is

perpendicular to the boost.) For a given r, s, we can choose a lattice vector p_a that is arbitrarily close to the direction of the boost, but at the cost of making $\delta^{ab}p_a p_b$ very large. Then $\delta^{ab}q_a q_b \approx e^{-2\alpha}\delta^{ab}p_a p_b$ may not be sufficiently small to be relevant. (The $\approx$ will be an $=$ sign if p_a is precisely parallel to the direction of the boost, however, we are not guaranteed to be able to find an element of the lattice that is precisely parallel.) Alternatively, we can choose a smaller $\delta^{ab}p_a p_b$, but the angle between p_a and the boost may not larger. As explained through an example in Appendix B, we can balance these two competing imperatives and find a p_a so that neither $\delta^{ab}p_a p_b$ nor the angle between p_a and the boost is too large. Then $\frac{1}{2}\delta^{ab}q_a q_b \approx \frac{1}{2}e^{-2\alpha}\delta^{ab}p_a p_b < 2$, so that the corresponding operator is relevant.

The following simple ansatz leads to a relevant operator

$$p_a = nf^7_a + (m-2n)f^8_a + mf^{16}_a + n_{17}f^{17}_a + n_{18}f^{18}_a \tag{3.47}$$

at all candidate non-chiral points in the (r, s) phase diagram that we have checked. We do not have a proof that there is not some region in parameter space where a non-chiral phase is stable, but we have explicitly excluded nearly all of it, as may be seen from the phase diagram in Fig. 3.6 where we have included a selection of the possible operators described here that become relevant at the set of points $(r, s) = (6, p/q)$ for $q = 5$, and we anticipate that this ansatz will enable us to do so for any other point not already excluded. Thus, we expect the non-chiral phase to be entirely removed by this collection of operators combined with those discussed earlier.

Therefore, the phase diagram has a quite rich and intricate structure. From our experience with the above operators, our general expectation is that in the neighborhood of any point $(r, p/q)$, there exists a relevant operator that gaps out a pair of modes leading to the fully chiral $E_8 \times E_8$ theory if p or q is even, while $\mathrm{Spin}(32)/\mathbb{Z}_2$ remains if p and q

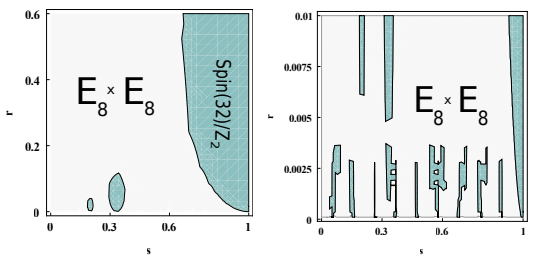

Figure 3.5: The small-r region of the phase diagram of our edge theory as a function of s and r for the theory with non-zero $u_{1,2}$, $\tilde{u}_{1,2}$; $u_{17,\frac{p}{q}}$, $u_{17,\frac{p}{q}}$ for all $p, q \leq 57$. The light region is in the $E_8 \times E_8$ phase. The darker region is in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase. All phase boundary lines denote first-order transitions. The left panel shows the $r < 0.6$ region of the phase diagram, where we see that regions of the two phases are interspersed with each other along the s-axis. In the right panel, we zoom in on the $r < 0.01$ region of the phase diagram and see an even richer intermingling of these two phases as we sweep over s.

are odd.

3.6 Charged Systems

We return to our $c-\overline{c} = 16$ theories and consider the case in which some of the degrees of freedom are charged as a result of coupling to an external electromagnetic field as in Eq. (3.4). Now, there are many phases for a given K, distinguished by different t. They may, as a consequence, have different Hall conductances $\sigma_{xy} = \frac{e^2}{h} t_I (K^{-1})^{IJ} t_J$, which must be even integer multiples of $\frac{e^2}{h}$ since K^{-1} is an integer matrix with even entries on the diagonal.

Let us focus on the minimal possible non-zero Hall conductance, $\sigma_{xy} = 2\frac{e^2}{h}$. We will not attempt to systematically catalog all of these states here, but will examine a few examples with $c = 16$ that are enlightening. By inspection, we see that we have three distinct $\sigma_{xy} = 2\frac{e^2}{h}$ states with K-matrix $K = K_{E_8 \times E_8}$: (1) $t_I = \delta_{I6}$, (2) $t_I = \delta_{I9}$, and (3) $t_I = -2\delta_{I1} + \delta_{I2}$. These states have stable edge modes even if the $U(1)$ symmetry of charge conservation is violated (e.g., by coupling the system to a superconductor), in

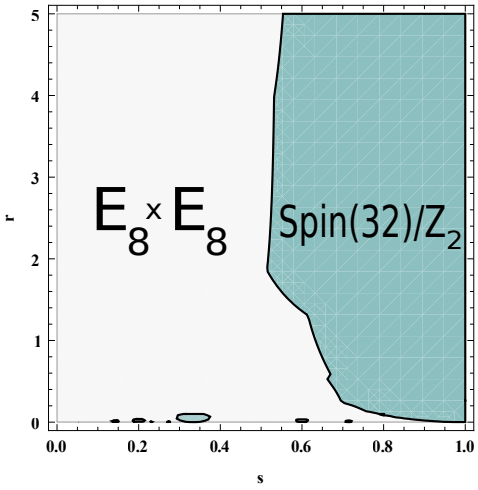

Figure 3.6: Phase diagram of our edge theory as a function of s and r for the theory $S_4(s)$ in which the only non-zero perturbations are $u_{1,2}$ and $\tilde{u}_{1,2}$; $u_{17,\frac{p}{q}}$, $u_{17,\frac{p}{q}}$ for all $p, q \leq 57$; and several $\cos(p_a X^a)$ operators with p_a nearly aligned with the direction of the boost $O_G(s)$, as described in the text. The latter operators were specifically chosen to remove the remaining points of non-chiral phase at $r = 6$, $s = p/q$ for $q = 5$. This set of operators was sufficient to remove all the non-chiral phase displayed previously in Fig. 4. The light region is in the $E_8 \times E_8$ phase. The darker region is in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase. Solid phase boundary lines denote a first-order transition.

contrast to the $\sigma_{xy} = 2\frac{e^2}{h}$ bosonic quantum Hall states discussed in Ref. [53].

As before, we adjoin a trivial system to our system so that the K-matrices are $K = K_{E_8 \times E_8 \oplus U}$. Under the similarity transformation W_G, these states are equivalent to the states with $K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U}$ and, respectively, $t = (0,0,0,0,1,-2,0,0,0,0,0,0,0,0,0,4,-2,2)$, $t = 0,0,0,0,0,0,0,-2,1,0,0,0,0,0,0,4,-2,2$, and $t_I = \delta_{I1}$. Consider the first of these, $K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U}$, $t = (0,0,0,0,1,-2,0,0,0,0,0,0,0,0,0,4,-2,2)$. It is *not* equal to $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ with an additional trivial system adjoined to it because $\tilde{\phi}_{17}$ and $\tilde{\phi}_{18}$ are both charged. In other words, there is a right-moving neutral edge mode $\tilde{\phi}_{17} + \tilde{\phi}_{18}$ and a left-moving charged edge mode $\tilde{\phi}_{17} - \tilde{\phi}_{18}$. This is non-trivial, and there is no charge-conserving perturbation which will give a gap to these modes. The same is true of the second state. In the case of the third state, both ϕ_{17}, ϕ_{18} and $\tilde{\phi}_{17}$, $\tilde{\phi}_{18}$ are neutral. Therefore, there are perturbations that could gap out either of them. Consequently,

we conclude that $K = K_{E_8 \times E_8}$, $t_I = -2\delta_{I1} + \delta_{I2}$ and $K_{\mathrm{Spin}(32)/\mathbb{Z}_2}$, $t_I = \delta_{I1}$ are stably equivalent bulk states with a edge theory phase diagram similar to that in Figure 3.4.

3.7 Discussion

3.7.1 Summary

Bosonic SRE states with chiral edge modes are bosonic analogues of fermionic integer quantum Hall states: they do not support anyons in the bulk, but they have completely stable chiral edge modes. Together, they populate an 'intermediate' class of phases that are completely stable and do not require symmetry-protection, however, they lack non-trivial bulk excitations. Unlike in the fermionic case, such states can only occur when the number of edge modes is a multiple of 8. As we have seen in this chapter, the scary possibility that the number of edge modes does not uniquely determine such a state is not realized, at least for the first case in which it can happen, namely, when there are 16 edge modes. The two phases that are naively different are, in fact, the same phase. This is consistent with the result that all 3-manifold invariants associated with the two phases are the same [44], and we have gone further and shown that it is possible to go directly from one state to the other without crossing a phase boundary in the bulk. However, there are actually two distinct sets of edge excitations corresponding to these adiabatically connected bulk states. We have shown that the phase transition between them can occur purely at the edge, without closing the bulk gap.

Our construction is motivated by the observation that there is a unique even, unimodular lattices with signature $(8k + n, n)$. Consequently, enlarging the Hilbert spaces of seemingly different phases associated with distinct even, unimodular lattices with signature $(8k, 0)$ by adding trivial insulating degrees of freedom associated with even,

unimodular lattices with signature (n, n) leads to the same bulk phase. Since the edge is characterized by additional data, the corresponding edge theories are distinct but are separated by a phase transition that can occur purely on the edge without closing the bulk gap. The details of our construction draw on a similar one by Ginsparg [38] who showed explicitly how to interpolate between toroidal compactifications of $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string theories.

3.7.2 Future Directions

Let us describe a few possible directions for future study.

- We have considered one possible interpolation between the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ theories and, therefore, have only considered a small region of possible parameter space determined by r and s. It would be interesting to carve out in more detail the full 153-dimensional phase space.

- The last phase diagram displayed in Fig. 3.6 includes only a subset of the possible operators that may be added to the edge theory. The operators that have been added are sufficient to lift the non-chiral phase that is naively present and displayed in Fig. 3.4 when only four operators are included. It is possible that consideration of all allowed operators could result in an even more complex phase diagram with a rich topography of interspersed $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ phases.

- The uniqueness of even, unimodular lattices with signature $(8k + n, n)$ implies that a similar route can be taken to adiabatically connect states associated to different positive-definite even unimodular lattices of dimension $8k = 24, 32, \ldots$. However, in these cases, it is possible for states corresponding to different lattices to have different spectra of operator scaling dimensions at the edge, unlike in the

$c = 16$ case, so the situation may be more subtle. The 24-dimensional case may be particularly interesting as the ground state transforms trivially (as reviewed at the end of Section 3.2.1) under modular transformation of the torus.

- It is possible to have an edge in which the interaction varies along the edge so that u_1 is the only relevant operator for $x < 0$ and $\tilde{u}_1$ is the only relevant operator for $x > 0$. The edge will then be in the $E_8 \times E_8$ phase to the left of the origin and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ phase to the right of the origin. It would be interesting to study the defect that will be located at the origin.

- Unimodular lattices occur in the study of four-manifold topology as the intersection form of $H^2(M, \mathbb{Z})$, where M is a four-manifold and $H^2(M, \mathbb{Z})$ is the second cohomology group over the integers. (We assume that M is closed.) In the circumstances when de Rham cohomology can be defined, we can think of the intersection form as follows. Consider all pairs of 2-forms, ω_I, ω_J and construct the matrix, $K_{IJ} = \int_M \omega_I \wedge \omega_J \in \mathbb{Z}$. Even when de Rham cohomology does not make sense, the above matrix can be defined. K_{IJ} is unimodular and symmetric. Interestingly, the cases for which K_{IJ} is even (and, therefore, provide intersection forms of the type studied in this chapter) correspond to non-smooth four-manifolds. The first instance is the so-called E_8 manifold whose intersection form is the E_8 Cartan matrix. Likewise, there exist two distinct four-manifolds, $E_8 \times E_8$ and the Chern manifold, with $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ intersection form, respectively.[54] While these two four-manifolds are not equivalent or homeomorphic, they are cobordic: there exists a five-manifold whose two boundary components correspond to these two four-manifolds. The cobordism can be understood as taking the direct sum of each four-manifold with $S^2 \times S^2$ which has intersection matrix equal to U. A series of surgeries then relates these two connected augmented four-manifolds. In

other words, our work has been a physical implementation of the above cobordism. Is there a deeper connection between four-manifold topology and integer quantum Hall states? We might go further and imagine that any such relation could be generalized to fractional and, possibly, non-abelian states. Further, the introduction of symmetry-protected topological phases in 2+1d could inform the study of four-manifolds, i.e., the stabilizing symmetry of any phase could further refine the possible invariants characterizing any manifold.

- We have concentrated on bosonic systems in this chapter, but very similar considerations apply to fermionic SRE systems with chiral edge modes, which correspond to positive-definite odd unimodular lattices. The conventional integer quantum Hall states correspond to the hypercubic lattices $\mathbb{Z}^N$. However, there is a second positive-definite odd unimodular lattice in dimensions greater than 8 namely $K_{E_8} \oplus I_{N-8}$. In dimensions greater than 11, there is also a third one, and there are still more in higher dimensions. However, there is a unique unimodular lattice with indefinite signature. Therefore, by a very similar construction to the one that we have used here, these different lattices correspond to different edge phases of the $\nu \geq 9$ integer quantum Hall states.

- Finally, stable equivalence is not restricted to topologically ordered states in 2+1d; it would be interesting to see explicitly how it manifests itself in the study of topological phases in other dimensions.

Chapter 4

Bulk-Edge Correspondence in $2+1$-Dimensional Abelian Topological Phases

4.1 Introduction

In the limit of vanishing electron-electron interactions, the edge excitations of an integer quantum Hall state form a multi-channel chiral Fermi liquid. These excitations are stable with respect to weak interactions by their chirality [55]. However, the Coulomb energy in observed integer quantum Hall states is larger than the energy of the lowest gapped edge excitation. Therefore, interactions are not weak in these experiments, and we must consider whether interactions with gapped unprotected non-chiral excitations can alter the nature of the gapless protected chiral edge excitations of an integer quantum Hall state even when the bulk is unaffected.[1]

[1] In fact, the Coulomb energy is often larger than the bulk cyclotron energy, too, so it is not a given that the bulk state is in the same universality class as the non-interacting integer quantum Hall state, but we will assume that this is true in this chapter.

In this chapter, we show that sufficiently strong interactions can drive the edge of an integer quantum Hall state with $\nu \geq 8$ into a different phase in which the edge excitations form a multi-channel chiral *Luttinger liquid* while the bulk remains adiabatically connected to an integer quantum Hall state of non-interacting electrons. This chiral Luttinger liquid is also stable against all weak perturbations, but it is not adiabatically connected to the edge of an integer quantum Hall state of non-interacting electrons even though the bulk of the system is. For $\nu \geq 12$, there are several possible such stable chiral edge phases corresponding to the same bulk phase. The edge excitations of many fractional quantum Hall states, such as the principal Jain series with $\nu = \frac{n}{2pn+1}$ form a multi-channel chiral Luttinger liquid, which is stable against weak perturbations due to its chirality. We show that such edges can also be subject to reconstruction into a different chiral Luttinger liquid as a result of strong interactions with gapped unprotected excitations at the edge. The new chiral Luttinger liquid is also stable against all weak perturbations.

In Chapter 3 we analyzed edge phases of bosonic integer quantum Hall states[5]. We saw that without symmetry, integer quantum Hall states of bosons that only support bosonic excitations in the bulk, not anyons, occur only when the chiral central charge, $c_- = c_R - c_L$, the difference between the number of right- and left-moving edge modes, is a multiple of eight (or, equivalently, when the thermal Hall conductance is $\kappa_{xy} = c_- \frac{\pi^2 k_B^2 T}{3h}$ with $c_- = 8k$ for integers k).[19] There is a unique [37, 13] bulk state for each possible value of $c_- = 8k$, but there are many possible chiral edge phases when the chiral central charge is greater than 8: there are two chiral edge phases for $c_- = 16$, twenty-four chiral edge phases for $c_- = 24$, more than one billion for $c_- = 32$, and larger numbers of such edge phases for $c_- > 32$. The transition between the two possible chiral edge phases was studied in detail in the $c_- = 16$ case. [38, 5]

These fermionic and bosonic quantum Hall states illustrate the fact that the boundary-

bulk correspondence in topological states is not one-to-one. There can be multiple possible edge phases corresponding to the same bulk phase. This can happen in a trivial way: two edge phases may differ by unstable gapless degrees of freedom, so that one of the edge theories is more stable than the other.[56, 57, 51, 33, 58] (One interesting refinement of this scenario is that the additional gapless degrees of freedom can be protected by a symmetry so that, in the presence of this symmetry, both edge phases are stable[20].) However, our focus here is the situation in which there are multiple edge phases, each of which is stable to weak perturbations without any symmetry considerations and none of which is more "minimal" than the others. In other words, in the integer and fractional quantum Hall states that we discuss here – which have the additional property that they are all chiral – all of the edge phases are on the same footing. Although they can bound the same bulk, such edge phases generically have different exponents and scaling functions for transport through point contacts and tunneling in from external leads. In some cases, the differences only show up in three-point and higher edge correlation functions.

In Sections 4.7.1, 4.7.2 of this chapter, we discuss fermionic integer quantum Hall states at $\nu = 8$ and $\nu = 12$, their possible stable chiral edge phases, and the experimental signatures that could distinguish these phases. In Section 4.7.3, we discuss the simplest fractional quantum Hall states with multiple chiral edge phases, which occur at $\nu = 8/7, 8/15, 16/5$ (fermions) and $\nu = 12/23$ (bosons). Some of the edge phases that we construct do not support gapless excitations with the quantum numbers of an electron. When the Hall conductance is non-zero, the edge must have gapless excitations; in a system of electrons, there must be a finite-energy excitation everywhere in the system with an electron's quantum numbers. However, it is not necessary that the electron be among the gapless edge excitations of an electronic quantum Hall state; it may be a gapped excitation at the edge, above the gapless excitations that are responsible for carrying the Hall current.

Given the above statement that the same bulk phase can have multiple distinct chiral edge phases, we should ask what breaks down in the usual relation between bulk topological phases and their associated edge spectra. By the usual relationship, we mean the "integration by parts" of a bulk Abelian Chern-Simons action that gives an edge theory of chiral bosons with the same K-matrix [45, 59]. The answer is simply that the usual relation focuses only upon the lowest energy excitations of a system and ignores higher-energy excitations. These higher-energy excitations are necessarily adiabatically connected to a topologically-trivial band insulator in the bulk and, generically, gapped excitations at the edge. Surprisingly, interactions between these "trivial" modes and the degrees of freedom responsible for the topologically non-trivial state can drive an edge phase transition that leads to a distinct edge phase without closing the bulk gap. We refer to the relationship between these two distinct edge theories associated with the same bulk as *stable equivalence*. At the level of the gapless edge modes, this manifests itself in the form of an edge reconstruction. While the interpolation at the edge necessarily involves strong interactions, these can be understood using standard Luttinger liquid techniques.

The relationship between the edge and the bulk can also be viewed in the following manner. Each quasiparticle in the bulk has a topological twist factor $\theta_a = e^{2\pi i h_a}$, with $0 < h_a < 1$. If the edge is fully chiral, each such quasiparticle corresponds to a tower of excitations. The minimum scaling dimension for creating an excitation in this tower is $\min \Delta_a = h_a + n_a$ for some integer n_a. The other excitations in the tower are obtained by creating additional bosonic excitations on top of this minimal one; their scaling dimensions are larger than the minimal one by integers. But if the edge has a different phase, the minimal scaling dimension operator in this tower may be $\min \Delta_a = h_a + \tilde{n}_a$. Therefore, the spectrum of edge operators can be different, even though the fractional parts of their scaling dimensions must be the same. (In the case of a fermionic topological phase,

we must compare scaling dimensions modulo 1/2, rather than modulo 1. By fermionic topological phase, we mean one which can only occur in a system in which some of the microscopic consitutents are fermions. At a more formal level, this translates into the existence of a fermionic particle which braids trivially with all other particles.)

The purpose of this chapter is to describe the precise conditions under which two different edge phases can terminate the same bulk state, i.e. are stably equivalent. These conditions are intuitive: the braiding statistics of the quasiparticle excitations of the bulk states must be identical and the chiral central charges of the respective states must be equal.

Let us summarize the general relation between bulk Abelian topological states and their associated edge phases in slightly more mathematical terms. Edge phases are described by lattices Λ equipped with an integer-valued bilinear symmetric form B. [41, 60, 39, 61, 62, 63] We collectively write this data as $\mathcal{E} = (\Lambda, B)$. The signature of B is simply the chiral central charge c_- of the edge theory. Given a basis e_I for Λ, the bilinear form determines a K-matrix $K_{IJ} = B(\mathbf{e}_I, \mathbf{e}_J)$. In a bosonic system, the lattice Λ must be even while in a fermionic system, the lattice Λ is odd. (An odd lattice is one in which at least one basis vector has $(\text{length})^2$ equal to an odd integer. The corresponding physical system will have a fermionic particle that braids trivially with all other particles. This particle can be identified with an electron. An even lattice has no such vectors and, therefore, no fermionic particles that braid trivially with all other particles. Hence, it can occur in a system in which none of the microscopic constituents are fermions. Of course, a system, such as the toric code, may have fermionic quasiparticles that braid non-trivially with at least some other particles.) Given the lattice Λ, vertex operators of the edge theory are associated with elements in the dual lattice Λ^*. For integer quantum Hall states, $\Lambda^* = \Lambda$, however, for fractional states $\Lambda \subset \Lambda^*$. The operator product expansion of vertex operators is simply given by addition in Λ^*.

Each bulk phase is characterized by the following data concisely written as $\mathcal{B} = (A, q, c_- \bmod 24)$:[60, 61, 62, 63, 44, 64] a finite Abelian group A encoding the fusion rules for the distinct quasiparticle types, a finite quadratic form q on A that gives the topological spin to each particle type, and the chiral central charge modulo 24. As we will discuss at length, since the map $\mathcal{E} \to \mathcal{B}$ associating edge data $\mathcal{E}$ to a given bulk $\mathcal{B}$ is *not* one-to-one, several different edge phases may correspond to the same bulk phase. We will provide an in-depth mathematical description of the above formalism in order to precisely determine when two distinct edge phases correspond to the same bulk phase. To determine all of the edge phases that can bound the same bulk, one can perform a brute force search through all lattices of a given dimension and determinant. (For low-dimensional cases, the results of such enumeration is in tables in Ref. [65] and in, for instance, G. Nebe's online Catalogue of Lattices.) Moreover, one can use a mass formula described in Section 4.5 to check if a list of edge phases is complete.

We will exemplify the many-to-one nature of the map $\mathcal{E} \to \mathcal{B}$ through various examples. The most primitive example occurs for integer quantum Hall states. For such states, the lattice is self-dual, $\Lambda^* = \Lambda$ so there are no non-trivial quasiparticles. For $c_- < 8$, there is a unique edge theory for the fermionic integer quantum Hall state, however, at $c_- = 8$, there are two distinct lattices: the hypercubic latttice $\mathbb{I}_8$ and the E_8 root lattice. Therefore, the associated gapless edge theories corresponding to each lattice may bound the same bulk state; there exists an edge reconstruction connecting the two edge phases. Fractional states for which A is non-trivial enrich this general structure.

A rather remarkable corollary of our analysis is the following: all rational Abelian topological phases in 2+1 dimensions can be described by Abelian Chern-Simons theory. By rational, we mean that there is a finite number of bulk quasiparticle types, i.e., the group A has finite order. As may be seen by giving a physical interpretion to a theorem of Nikulin [66] the particle types, fusion rules, and topological twist factors determine a

genus of lattices, from which we can define an Abelian Chern-Simons theory. A second result that follows from a theorem of Nikulin [66] is that any fermionic Abelian topological phase can be mapped to a bosonic topological phase, together with some number of filled Landau levels.

The remainder of this chapter is organized as follows. We begin in Section 4.2 by reviewing the formalism used to describe the bulk and boundary excitations of Abelian Hall states. As a means to both motivate the general mathematical structure and because of their intrinsic interest, we provide two examples of stable equivalence in the fractional quantum Hall setting in Section 4.3 and summarize their physically distinct signatures. In Section 4.4, we abstract from these two examples the general method for understanding how distinct edge phases of a single bulk are related via an edge phase transition. In Section 4.5, we explain the bulk-edge correspondence through the concepts of stable equivalence and genera of lattices. In Section 4.6, we explain how fermionic topological phases can be represented by bosonic topological phases together with some number of filled Landau levels. In Section 4.7, we analyze observed integer and fractional quantum Hall states that admit multiple stable, fully chiral edge phases. In Section 4.8, we explain how a number of theorems due to Nikulin, that we use throughout the text, apply to the description of *all* Abelian topological field theories in (2+1)-D. We conclude in Section 4.9. We have three appendices that collect ideas used within the text.

4.2 Preliminaries

4.2.1 Edge Theories

In this section, we review the formalism that describes the edges of conventional integer and Abelian fractional quantum Hall states. We begin with the edges of fermionic in-

teger quantum Hall states. We assume that the bulks of these states are the conventional states that are adiabatically connected to the corresponding states of non-interacting fermions. As we will see in later sections, the edge structure is not uniquely determined, even if we focus solely on chiral edge phases that are stable against all weak perturbations.

All integer quantum Hall states have one edge phase that is adiabatically connected to the edge of the corresponding non-interacting fermionic integer quantum Hall state. This edge phase has effective action $S_0 + S_1$, where

$$S_0 = \int dx dt \, \psi_J^\dagger \left(i\partial_t + A_t + v_J (i\partial_x + A_x) \right) \psi_J \tag{4.1}$$

and $J = 1, 2, \ldots, N$. We shall later study two interesting examples that occur when $N = 8$ or $N = 12$. The operator $\psi_J^\dagger$ creates an electron at the edge in the J^{th} Landau level; v_J is the edge velocity of an electron in the J^{th} Landau level. Inter-edge interactions take the form

$$S_1 = \int dx \, dt \, \big(t_{JK}(x) \, e^{i\left(k_F^J - k_F^K\right)x} \, \psi_J^\dagger \psi_K + \text{h.c.} \\ + v_{JK} \psi_J^\dagger \psi_J \psi_K^\dagger \psi_K + \ldots \big). \tag{4.2}$$

The $\ldots$ in Eq. (4.2) represent higher-order tunneling and interaction terms that are irrelevant by power counting. We neglect these terms and focus on the first two terms. Electrons in different Landau levels will generically have different Fermi momenta. When this is the case, the tunneling term (the first term in Eq. (4.2)) will average to zero in a translationally-invariant system. In the presence of disorder, however, $t_{IJ}(x)$ will be random and relevant (e.g. in a replicated action which is averaged over $t_{IJ}(x)$). Moreover, it is possible for the Fermi momenta to be equal; for instance, in an N-layer system in which each layer has a single filled Landau level, the Fermi momenta will be the same if

the electron density is the same in each layer. Fortunately, we can make the change of variables:

$$\psi_J(x) \to \left(\overline{\mathcal{P}} \exp\left(i \int_{-\infty}^{x} dx' M(x')\right)\right)_{JK} \psi_K(x),$$

where $M(x)$ is the matrix with entries $M_{JK} = t_{JK}(x')\, e^{i\left(k_F^J - k_F^K\right)x'}/\overline{v}$, $\overline{v} = \sum_J v_J/N$, and $\overline{\mathcal{P}}$ denotes anti-path-ordering. When this is substituted into Eq. (4.1), the first term in Eq. (4.2) is eliminated from the action $S_0 + S_1$. This is essentially a U(N) gauge transformation that gauges away inter-mode scattering. An extra random kinetic term proportional to $(v_J - \overline{v})\delta_{IJ}$ is generated, but this is irrelevant in the infrared when disorder-averaged.

The second term in Eq. (4.2) is an inter-edge density-density interaction; v_{JK} is the interaction between edge electrons in the J^{th} and K^{th} Landau levels. This interaction term can be solved by bosonization. The action $S_0 + S_1$ from Eqs. (4.1) and (4.2) can be equivalently represented by the bosonic action

$$S = \int dx\, dt \Bigg(\frac{1}{4\pi} \delta_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} \sum_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \Bigg), \quad (4.3)$$

where $V_{II} \equiv v_I + v_{II}$ (no summation) and $V_{IJ} \equiv v_{IJ}$ for $I \neq J$. The electron annihilation operator is bosonized according to $\psi_J \sim \eta_J e^{i\phi^J}$. Here η_J is a "Klein factor" satisfying $\eta_J \eta_K = -\eta_K \eta_J$ for $J \neq K$, which ensures that $\psi_J \psi_K = -\psi_K \psi_J$. Products of even numbers of Klein factors can be diagonalized and set to one of their eigenvalues, ± 1, if all terms in the Hamiltonian commute with them. They can then be safely ignored. This is the case in all of the models studied in this chapter. This action can be brought into the following diagonal form (setting the external electromagnetic field to zero for

simplicity):

$$S = \int dx\, dt \left(\frac{1}{4\pi} \delta_{IJ} \partial_t \tilde{\phi}^I \partial_x \tilde{\phi}^J - \frac{1}{4\pi} v_I \delta_{IJ} \partial_x \tilde{\phi}^I \partial_x \tilde{\phi}^J \right) \tag{4.4}$$

with an orthogonal transformation $\phi^I = O^I_J \tilde{\phi}^J$ that diagonalizes V_{IJ} according to $O^I_L V_{IJ} O^J_K = \tilde{v}_L \delta_{LK}$. Two-point correlation functions take the form

$$\left\langle e^{i m_I \phi^I} e^{-i m_K \phi^K} \right\rangle = \prod_{J=1}^{N} \frac{1}{(x - \tilde{v}_J t)^{m_I m_K O^I_J O^K_J}}. \tag{4.5}$$

There is no sum over J in the exponent on the right-hand-side of Eq. (4.5). The electron Green function in the I^{th} Landau level is a special case of this with $m_K = \delta_{IK}$.

It is now straightforward to generalize the preceding discussion to the case of an arbitrary Abelian integer or fractional quantum Hall state [59]. For simplicity, we will focus on the case of fully chiral phases in which all edge modes move in the same direction. Such phases do not, in general, have a free fermion representation and can only be described by a chiral Luttinger liquid. They are characterized by equivalence classes of positive-definite symmetric integer *K-matrices* K, and integer *charge vectors* t that enter the chiral Luttinger liquid action according to

$$S_{LL} = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right). \tag{4.6}$$

The fields in this action satisfy the periodicity condition $\phi^I \equiv \phi^I + 2\pi n^I$ for $n^I \in \mathbb{Z}$. Two phases, characterized by the pairs (K_1, t_1) and (K_2, t_2), are equivalent if $K_1 = W^T K_2 W$ and $t_1 = t_2 W$, where $W \in \mathrm{GL}(N, \mathbb{Z})$ since the first and third terms in the two theories can be transformed into each other by the change of variables $\phi^I = W^I{}_J \tilde{\phi}^J$. So long as $W \in \mathrm{GL}(N, \mathbb{Z})$, the periodicity condition satisfied by $\tilde{\phi}^J$ is precisely the same as the periodicity condition satisfied by ϕ^I. The matrix V_{IJ} consists of marginal deformations that do not change the phase of the edge but affect the propagation velocities. (If we

wish, we can think of each phase as a fixed surface under RG flow, and the V_{IJ}s are marginal deformations that parametrize the fixed surface.) All such chiral edge theories are stable to all weak perturbations by the same reasoning by which we analyzed integer quantum Hall edges. The simplest fermionic fractional quantum Hall edge theory is that of the Laughlin $\nu = 1/3$ state, for which $K = (3)$ and $t = (1)$ (a 1×1 matrix and a 1-component vector, respectively). Integer quantum Hall edges are the special case, $K_{IJ} = \delta_{IJ}$ or, allowing for basis changes, $K = W^T W$ with $W \in \mathrm{GL}(N, \mathbb{Z})$.

It is useful to characterize these phases by lattices Λ rather than equivalence classes of K-matrices. Let e_I^a be the eigenvector of K corresponding to eigenvalue λ_a: $K_{IJ} e_J^a = \lambda^a e_I^a$. We normalize e_J^a so that $e_J^a e_J^b = \delta^{ab}$ and define a metric $g_{ab} = \lambda_a \delta_{ab}$. Then, $K_{IJ} = g_{ab} e_I^a e_J^b$ or, using vector notation, $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. We will be focusing mostly on positive-definite lattices, so that g_{ab} has signature $(N, 0)$ but we will occasionally deal with Lorentzian lattices, for which we take g_{ab} has signature $(p, N-p)$. The metric g_{ab} defines a *bilinear form* B on the lattice Λ (and its dual Λ^*) – this just means we can multiply two lattice vectors $\mathbf{e}_I, \mathbf{e}_J$ together using the metric, $\mathbf{e}_I \cdot \mathbf{e}_J = e_I^a g_{ab} e_J^b = B(\mathbf{e}_I, \mathbf{e}_J)$. The N vectors $\mathbf{e}_I$ define a lattice $\Lambda = \{m_I \mathbf{e}_I | m_I \in \mathbb{Z}\}$. The $\mathrm{GL}(N, \mathbb{Z})$ transformations $K \to W^T K W$ are simply basis changes of this lattice, so we can equally well describe edge phases by equivalence classes of K-matrices or by lattices Λ. The conventional edge phases of integer quantum Hall states described above correspond to hypercubic lattices $\mathbb{Z}^N$, which we will often denote by the corresponding K matrix in its canonical basis, $\mathbb{I}_N$. The $\nu = 1/3$ Laughlin state corresponds to the lattice $\Lambda = \mathbb{Z}$ with dual $\Lambda^* = \frac{1}{3}\mathbb{Z}$. [2] The connection of quantum Hall edge phases to lattices can be exploited more easily if we make the following change of variables, $X^a = e_I^a \phi^I$, in terms of which the action takes

[2] This statement assumes the periodicity convention, $\phi \equiv \phi + 2\pi n$, for $n \in \mathbb{Z}$.

the form

$$S = \frac{1}{4\pi}\int dx\, dt\Big(g_{ab}\partial_t X^a \partial_x X^b - v_{ab}\partial_x X^a \partial_x X^b.\Big) \tag{4.7}$$

The variables X^a satisfy the periodicity condition $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{y}$ for $\mathbf{y} \in \Lambda$ and $v_{ab} \equiv V_{IJ} f_a^I f_b^J$, where f_a^I are basis vectors for the dual lattice Λ^*, satisfying $f_a^I e_J^a = e_{La}(K^{-1})^{LI} e_J^a = \delta_J^I$.

Different edge phases (which may correspond to different bulks or the same bulk; the latter is the focus of this chapter) are distinguished by their correlation functions. The periodicity conditions on the fields X^a dictate that the allowed exponential operators are of the form $e^{i\mathbf{v}\cdot\mathbf{X}}$, where $\mathbf{v} \in \Lambda^*$. These operators have scaling dimensions

$$\dim\big[e^{i\mathbf{v}\cdot\mathbf{X}}\big] = \frac{1}{2}|\mathbf{v}|^2. \tag{4.8}$$

They obey the operator algebra

$$: e^{i\mathbf{v_1}\cdot\mathbf{X}} :: e^{i\mathbf{v_2}\cdot\mathbf{X}} :\sim: e^{i(\mathbf{v_1}+\mathbf{v_2})\cdot\mathbf{X}} : \tag{4.9}$$

where $: \cdot :$ denotes normal ordering. Thus, the operator spectrum and algebra is entirely determined by the underlying dual lattice Λ^*.

In a quantum Hall state, there are two complementary ways of measuring some of the scaling exponents. The first is a quantum point contact (QPC) at which two edges of a quantum Hall fluid are brought together at a point so that quasiparticles can tunnel across the bulk from one edge to the other. Even though a single edge is completely stable against all weak perturbations, a pair of oppositely-directed edges will, in general,

be coupled by relevant perturbations

$$S = S_T + S_B + \int dt \sum_{\mathbf{v}\in\Lambda^*} v_{\mathbf{v}}\, e^{i\mathbf{v}\cdot[\mathbf{X}_T - \mathbf{X}_B]}. \tag{4.10}$$

Here, T, B are the two edges, e.g., the top and bottom edges of a Hall bar; we will use this notation throughout whenever it is necessary to distinguish the two edges. The renormalization group (RG) equation for $v_{\mathbf{v}}$ is

$$\frac{dv_{\mathbf{v}}}{d\ell} = \left(1 - |\mathbf{v}|^2\right) v_{\mathbf{v}}. \tag{4.11}$$

If $\mathbf{v}\cdot\mathbf{f}^I t_I \neq 0$, the above coupling transfers $\mathbf{v}\cdot\mathbf{f}^I t_I$ units of charge across the junction and this perturbation will contribute to the backscattered current according to

$$I^b \propto |v_{\mathbf{v}}|^2\ V^{2|\mathbf{v}|^2-1}. \tag{4.12}$$

A second probe is the tunneling current from a metallic lead:

$$\begin{aligned} S &= S_{\text{edge}} + S_{\text{lead}} \\ &\quad + \int dt \sum_{\mathbf{v}\in\Lambda} t_{\mathbf{v}} \Big[\psi^\dagger_{\text{lead}} \partial\psi^\dagger_{\text{lead}} \partial^2\psi^\dagger_{\text{lead}} \ldots\Big] e^{i\mathbf{v}\cdot\mathbf{X}}. \end{aligned}$$

The term in square brackets [...] contains n factors of $\psi^\dagger_{\text{lead}}$ and $n(n-1)/2$ derivatives, where $n = \mathbf{v}\cdot\mathbf{f}^I t_I$ must be an integer. The RG equation for $t_{\mathbf{v}}$

$$\frac{dt_{\mathbf{v}}}{d\ell} = \left(1 - \frac{n^2}{2} - \frac{1}{2}|\mathbf{v}|^2\right) t_{\mathbf{v}}. \tag{4.13}$$

The contribution to the tunneling current from $t_{\mathbf{v}}$ (assuming $n \neq 0$) is

$$I^{\text{tun}} \propto |t_{\mathbf{v}}|^2 \; V^{|\mathbf{v}|^2+n^2-1}. \tag{4.14}$$

Here, we have assumed that the spins at the edge of the quantum Hall state are fully spin-polarized and that tunneling from the lead conserves S_z. If, however, either of these conditions is violated, then other terms are possible in the action. For instance, charge-$2e$ tunneling can take the form

$$t_{\text{pair}} \int dt \, \psi^\dagger_{\text{lead},\uparrow} \psi^\dagger_{\text{lead},\downarrow} \, e^{i\mathbf{v}\cdot\mathbf{X}}, \tag{4.15}$$

where $\mathbf{v} \cdot \mathbf{f}^I t_I = 2$. Then, we have tunneling current

$$I^{\text{tun}} \propto |t_{\mathbf{v}}|^2 \; V^{|\mathbf{v}|^2+1}. \tag{4.16}$$

Generically, two lattices Λ_1 and Λ_2 can be distinguished by the possible squared lengths $|\mathbf{v}|^2$ for $\mathbf{v} \in \Lambda_1^*$. In many cases of interest, the shortest length, which will dominate the backscattered current discussed above, is enough to distinguish two edge phases of the same bulk. However, sometimes, as in the case of the two bosonic integer quantum Hall states with $c = 16$ discussed in Ref. [5] the spectrum of operator scaling dimensions (not just the shortest length, but all lengths along with degeneracies at each length level) is precisely the same in the two theories, so they could only be distinguished by comparing three-point correlation functions. In either case, different edge phases can be distinguished by their correlation functions.

4.2.2 Bulk Theories

In a later section, we will explain how bulk phases correspond to the mathematical notion of a *genus* of lattices, while their associated edge theories are given by lattices within a genus (or in the case of fermionic theories, a pair of genera, one odd and one even). In order to explain the relation between the genus of a lattice and a bulk Abelian phase, we recall some facts about Abelian topological phases.

Suppose that we have a $2+1$d Abelian topological phase associated to a lattice Λ. Choosing a basis $\mathbf{e}_I$ for the lattice Λ, we define $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$ and write a bulk effective action

$$\mathcal{S} = \int d^3x \Big(\frac{1}{4\pi} \epsilon^{\mu\nu\rho} K_{IJ} a^I_\mu \partial_\nu a^J_\rho + \frac{1}{2\pi} j^\mu_I a^I_\mu \Big). \tag{4.17}$$

A particle in this theory carrying charge m_I under the gauge field a_I can be associated with a vector $\mathbf{v} \equiv m_I \mathbf{f}^I$, where $\mathbf{f}_I$ is the basis vector of Λ^* dual to $\mathbf{e}_I$ and satisfying $(K^{-1})^{IJ}\mathbf{e}_J = \mathbf{f}^I$. Recall that because $\Lambda \subset \Lambda^*$, any element in Λ can be expressed in terms of the basis for Λ^*, however, the converse is only true for integer Hall states for which $\Lambda = \Lambda^*$. Particles $\mathbf{v}$, $\mathbf{v}' \in \Lambda^*$ satisfy the fusion rule $\mathbf{v} \times \mathbf{v}' = \mathbf{v} + \mathbf{v}'$ and their braiding results in the multiplication of the wave function describing the state by an overall phase $e^{2\pi i \mathbf{v}\cdot\mathbf{v}'}$. Since this phase is invariant under shifts $\mathbf{v} \to \mathbf{v} + \boldsymbol{\lambda}$ for $\boldsymbol{\lambda} \in \Lambda$, the topologically-distinct particles are associated with elements of the so-called discriminant group $A = \Lambda^*/\Lambda$. The many-to-one nature of the edge-bulk correspondence is a reflection of the many-to-one correspondence between lattices Λ and their discriminant groups A. Equivalent bulk phases necessarily have identical discriminant groups so our initial choice of lattice is merely a representative in an equivalence class of bulk theories.

We now define a few terms. A *bilinear symmetric form* on a finite Abelian group A

is a function $b\colon A \times A \to \mathbb{Q}/\mathbb{Z}$ such that for every $a, a', a'' \in A$,

$$b(a + a', a'') = b(a, a'') + b(a', a'')$$

and $b(a, a') = b(a', a)$. As all bilinear forms considered in this chapter will be symmetric, we will simply call them *bilinear forms* with symmetric being understood. A *quadratic form* q on a finite Abelian group A is a function $q : A \to \mathbb{Q}/\mathbb{Z}$ such that $q(na) = n^2 q(a)$ for every $n \in \mathbb{Z}$, and such that

$$q(a + a') - q(a) - q(a') = b(a, a')$$

for some bilinear form $b\colon A \times A \to \mathbb{Q}/\mathbb{Z}$. In this case, we say that q *refines* b, or is a *quadratic refinement* of b. A bilinear b or quadratic form q is *degenerate* if there exists a non-trivial subgroup $S \subset A$ such that $b(s, s') = 0$ or $q(s) = 0$ for every $s, s' \in S$. Throughout this chapter, all bilinear and quadratic forms will be assumed nondegnerate. Each K-matrix K determines a symmetric bilinear form B on $\mathbb{R}^n$ via $B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T K \mathbf{y}$ that takes integer values on the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Every other lattice $\Lambda \subset \mathbb{R}^n$ on which B is integral can be obtained by acting on $\mathbb{Z}^n$ by the orthogonal group $\{g \in \mathrm{GL}(N, \mathbb{R}) : gKg^T = K\}$ of K. On the other hand, an integral symmetric bilinear form is equivalent to a lattice according to the construction before Eq. (4.7) in Section 4.2.1. We are therefore justified in using the terminology "lattice" and "K-matrix" in place of "integral symmetric bilinear form" throughout this chapter. Every diagonal entry of a K-matrix K is even iff the (length)^2 of every element in the lattice $\mathbb{Z}^N$ is even. We call K *even* if this is the case, and otherwise it is *odd*. Even K-matrices determine integral quadratic forms on $\mathbb{Z}^N$ via $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T K \mathbf{x}$, while for odd K-matrices they are half-integral. When we simply write bilinear or quadratic form or, sometimes, *finite bilinear form* or *finite*

quadratic form, we will mean a nondegenerate symmetric bilinear form, or nondegenerate quadratic form, whose domain is a finite Abelian group. Throughout, we abbreviate the ring $\mathbb{Z}/N\mathbb{Z}$ of integers modulo N as $\mathbb{Z}/N$.

The S-matrix of the theory can be given in terms of the elements of the discriminant group:

$$S_{[\mathbf{v}],[\mathbf{v}']} = \frac{1}{\sqrt{|A|}} e^{-2\pi i \mathbf{v}\cdot\mathbf{v}'} = \frac{1}{\sqrt{|A|}} e^{-2\pi i m_I (K^{-1})^{IJ} m'_J} \tag{4.18}$$

where $\mathbf{v} = m_I \mathbf{f}^I, \mathbf{v}' = m'_J \mathbf{f}^J \in \Lambda^*$ and $|A|$ is the dimension of the discriminant group. The bracketed notation $[\mathbf{v}]$ indicates an equivalence class of elements $[\mathbf{v}] \in \Lambda^*/\Lambda = A$. Our normalization convention is to represent elements in the dual lattice Λ^* with integer vectors m_I. The bilinear form B on Λ^* reduces modulo Λ to define a finite bilinear form on the discriminant group Λ^*/Λ via

$$b([m_I \mathbf{f}^I], [m'_J \mathbf{f}^J]) = B(m_I \mathbf{f}^I, m'_J \mathbf{f}^J) = m_I (K^{-1})^{IJ} m'_J.$$

The topological twists $\theta_{[\mathbf{v}]}$, which are the eigenvalues of the T matrix, are defined by

$$T_{[\mathbf{v}],[\mathbf{v}']} = e^{-\frac{2\pi i}{24} c_-} \, \theta_{[\mathbf{v}]} \, \delta_{[\mathbf{v}],[\mathbf{v}']} \tag{4.19}$$

where

$$\theta_{[\mathbf{v}]} = e^{\pi i \mathbf{v}\cdot\mathbf{v}}. \tag{4.20}$$

Note that Eq. (4.19) implies that the theory is invariant under shifts of c_- by 24 so long as the topological twists $\theta_{[\mathbf{v}]}$ are invariant, but its modular transformation properties, which determine the partition function on 3-manifolds via surgery [67], is sensitive to shifts by $c_- \neq 0 \pmod{24}$.

If the topological twists are well-defined on the set of quasiparticles A, then they must

be invariant under $\mathbf{v} \mapsto \mathbf{v} + \boldsymbol{\lambda}$, where $\boldsymbol{\lambda} \in \Lambda$, under which

$$\theta_{[\mathbf{v}]} \mapsto \theta_{[\mathbf{v}+\lambda]} = \theta_{[\mathbf{v}]}\, e^{\pi i \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}. \tag{4.21}$$

If the K-matrix is even, so that we are dealing with a bosonic theory, $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ is even for all $\boldsymbol{\lambda} \in \Lambda$. If the K-matrix is odd, however – i.e. if the system is fermionic – then there are some $\boldsymbol{\lambda} \in \Lambda$ for which $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ is odd. In this case, the topological twists are not quite well-defined, and more care must be taken, as we describe in Section 4.6. Given the above definition, only T^2 is well-defined.

In a bosonic Abelian topological phase, we can define a finite quadratic form q on the discriminant group, usually called the *discriminant form*, according to

$$q([\mathbf{v}]) = \frac{1}{2}\mathbf{v}^2 = \frac{1}{2} m_I (K^{-1})^{IJ} m_J \bmod \mathbb{Z}, \tag{4.22}$$

where $\mathbf{v} = m_I \mathbf{f}^I$. In a topological phase of fermions, we will have to define q with more care, as we discuss in Section 4.6. Thus, we postpone its definition until then and will only discuss Abelian bosonic topological phases in the remainder of this section. In terms of the discriminant form q, the T-matrix takes the form

$$\theta_a = e^{2\pi i q(a)} \tag{4.23}$$

and the S-matrix takes the form

$$S_{a,a'} = \frac{1}{\sqrt{|A|}} e^{2\pi i (q(a-a') - q(a) - q(-a'))} \tag{4.24}$$

$$= \frac{1}{\sqrt{|A|}} e^{-2\pi i (q(a+a') - q(a) - q(a'))} \tag{4.25}$$

The equation for the S-matrix makes use of the fact that the finite bilinear form b can be

recovered from the finite quadratic form according to $b(a,a') = q(a+a') - q(a) - q(a')$. (It is satisfying to observe that the relation between the bilinear form b and the discriminant form q coincides exactly with the phase obtained by a wave function when two particles are twisted about one another.) While the introduction of the discriminant form may appear perverse in the bosonic context, we will find it to be an essential ingredient when discussing fermionic topological phases.

In any bosonic topological phase, the chiral central charge is related to the bulk topological twists by the following relation [68]:

$$\frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a = e^{2\pi i c_-/8}. \tag{4.26}$$

Here $\mathcal{D} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension, d_a is the quantum dimension of the quasiparticle type a, and θ_a is the corresponding topological twist/spin. $c_- = c - \overline{c}$ is the chiral central charge. In an Abelian bosonic phase described by an even matrix K, the formula simplifies to

$$\frac{1}{\sqrt{|A|}} \sum_{a \in A} e^{2\pi i q(a)} = e^{2\pi i c_-/8} \tag{4.27}$$

since $d_a = 1$ for all quasiparticle types. Here $|A| = \sqrt{|\det K|}$ and $c_- = r_+ - r_-$ is the signature of the matrix, the difference between the number of positive and negative eigenvalues. (We will sometimes, as we have done here, use the term signature to refer to the difference $r_+ - r_-$, rather than the pair (r_+, r_-); the meaning will be clear from context.) Notice that $e^{2\pi i q(a)}$ is just the topological twist of the quasiparticle represented by $a \in \Lambda^*/\Lambda$. This is known as the Gauss-Milgram sum in the theory of integral lattices.

Let us pause momentarily to illustrate these definitions in a simple example: namely, the semion theory described by the K-matrix, $K = (2)$. This theory has discriminant group $A = \mathbb{Z}/2\mathbb{Z} = Z_2$ and, therefore, two particle types, the vacuum denoted by the

lattice vector $[0]$ and the semion $s = [1]$. Recall that our normalization convention is to take the bilinear form on A to be $b([\mathbf{x}], [\mathbf{y}]) = x \cdot \frac{1}{2} \cdot y$; the associated quadratic form is then $q([\mathbf{x}]) = \frac{1}{2} b([\mathbf{x}], [\mathbf{x}])$. The discriminant form, evaluated on the semion particle, is given by $q([1]) = \frac{1}{2} \cdot \frac{1^2}{2}$. The T matrix equals $\exp(-2\pi i/24)\mathrm{diag}(1, i)$, and the S-matrix, $S = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix} \right)$. Evaluating the Gauss-Milgram sum confirms that $c_- = 1$.

In order to determine the discriminant group from a given K-matrix, we can use the following procedure. First, we compute the Gauss-Smith normal form of the K-matrix, which can be found using a standard algorithm[69]. Given K, this algorithm produces integer matrices P, Q, D such that

$$K = PDQ. \tag{4.28}$$

Here both P and Q are unimodular $|\det P| = |\det Q| = 1$, and D is diagonal. The diagonal entries of D give the orders of a minimal cyclic decomposition of the discriminant group

$$A \simeq \prod_J \mathbb{Z}/D_{JJ},$$

with the fewest possible cyclic factors, giving yet another set of generators for the quasiparticles. Although more compact, this form does not directly lend itself towards checking the equivalence of discriminant forms.

Now recall that the bases of Λ and Λ^* are related by K:

$$\mathbf{e}_I = K_{IJ} \mathbf{f}^J \tag{4.29}$$

Substituting the Gauss-Smith normal form, this can be rewritten

$$(P^{-1})^{IL} \mathbf{e}_L = D^{IK} Q_{KJ} \mathbf{f}^J. \tag{4.30}$$

The left-hand side is just a basis change of the original lattice. On the right-hand side, the row vectors of Q that correspond to entries of D greater than 1 give the generators of the cyclic subgroups of the discriminant group. A non-trivial example is given in Appendix C.

4.3 Two Illustrative Examples of Bulk Topological Phases with Two Distinct Edge Phases

The chiral Luttinger liquid action is stable against all small perturbations involving only the gapless fields in the action in Eq. (4.6) (or, equivalently in the integer case, the action in Eq. (4.1)). This essentially follows from the chirality of the theory, but it is instructive to see how this plays out explicitly.[55] However, this does not mean that a given bulk will have only a single edge phase.[70] A quantum Hall system will have additional gapped excitations which we can ignore only if the interactions between them and the gapless excitations in Eq. (4.6) are weak. If they are not weak, however, we cannot ignore them and interactions with these degrees of freedom can lead to an edge phase transition [5].

We will generally describe the gapped excitations with a K-matrix equal to $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We may imagine this K-matrix arising from a thin strip of $\nu = 1$ fluid living around the perimeter of our starting Hall state.[70] For edge phase transitions between bosonic edges theories, we should instead take the gapped modes to be described by a K-matrix equal to $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It is important to realize that the existence of the localized (gapped) edge modes described by either of these K-matrices implies the appropriate modification to the Chern-Simons theory describing the bulk topological order. This addition does not affect the bulk topological order[71]; without symmetry, such a gapped

state is adiabatically connected to a trivial band insulator.

We will illustrate this with two concrete examples. We begin with the general edge action

$$S = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right). \quad (4.31)$$

The first example is described by the K-matrix

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix} \quad (4.32)$$

with $t = (1, -1)^T$. This is not an example that is particularly relevant to quantum Hall states observed in experiments – we will discuss several examples of those in Section 4.7 – but it is simple and serves as a paradigm for the more general structure that we discuss in Sections 4.5 and 4.6.

Let us suppose that we have an additional left-moving and additional right-moving fermion which, together, form a gapped unprotected excitation. The action now takes the form

$$S = \int dx\, dt \left(\frac{1}{4\pi} \left(K_1 \oplus \sigma_z \right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right), \quad (4.33)$$

where we have now extended $t = (1, -1, 1, 1)^T$. The K-matrix for the two additional modes is taken to be σ_z. We will comment on the relation to the σ_x case in Sections 4.4 and 4.5.

If the matrix V_{IJ} is such that the perturbation

$$S' = \int dx\, dt\, u' \cos(\phi_3 + \phi_4) \quad (4.34)$$

is relevant, and if this is the only perturbation added to Eq. (4.33), then the two additional modes become gapped and the system is in the phase (4.32). Suppose, instead, that the only perturbation is

$$S'' = \int dx\, dt\, u'' \cos(\phi_1 - 11\phi_2 + 2\phi_3 + 4\phi_4). \tag{4.35}$$

This perturbation is charge-conserving and spin-zero (i.e., its left and right scaling dimensions are equal). If it is relevant, then the edge is in a different phase. To find this phase, it is helpful to make the basis change:

$$W^T(K_1 \oplus \sigma_z)W = K_2 \oplus \sigma_z, \tag{4.36}$$

where

$$K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \tag{4.37}$$

and

$$W = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ -2 & 3 & 0 & -2 \\ 1 & -7 & 0 & 4 \end{pmatrix}. \tag{4.38}$$

Making the basis change $\phi = W\phi'$, we see that

$$\phi_1 - 11\phi_2 + 2\phi_3 + 4\phi_4 = \phi'_3 + \phi'_4. \tag{4.39}$$

Therefore, the resulting phase is described by (4.37).

To see that these are, indeed, different phases, we can compute basis-independent quantities, such as the lowest scaling dimension of any operator in the two theories.

In the K_1 theory, it is $1/22$ while in the K_2 theory, it is $3/22$. Measurements that probe the edge structure in detail can, thereby, distinguish these two phases of the edge. Consider, first, transport through a QPC that allows tunneling between the two edges of the Hall bar, as described in Sec 4.2.1. In the state governed by K_1, the most relevant backscattering term is $\cos(\phi_2^T - \phi_2^B)$. Applying Eq (4.12), the backscattered current will depend on the voltage according to

$$I_1^b \propto V^{-9/11}. \tag{4.40}$$

An alternative probe is given by tunneling into the edge from a metallic lead. The most relevant term in the K_1 edge phase that tunnels one electron into the lead is $\psi_{\text{lead}}^\dagger e^{i\phi_1^T}$. Applying Eq (4.14) yields the familiar current-voltage relation,

$$I_1^{\text{tun}} \propto V. \tag{4.41}$$

In contrast, in the phase governed by K_2, the most relevant backscattering term across a QPC is given by $\cos(\phi_2'^T - \phi_2'^B)$, which from Eq (4.12) yields the current-voltage relation

$$I_2^b \propto V^{-5/11}, \tag{4.42}$$

while the most relevant single-electron tunneling term is given by $\psi_{\text{lead}}^\dagger e^{-3i\phi_1'^T - i\phi_2'^T}$, which yields the scaling from Eq (4.14)

$$I_2^{\text{tun}} \propto V^3. \tag{4.43}$$

Since the two edge theories given by K_1 and K_2 are connected by a phase transition just on the edge, we may expect they bound the same bulk Chern-Simons theory. Indeed, the bulk quasiparticles can be identified up to ambiguous signs due to their fermionic

nature. First, the discriminant group of the K_1 theory is $\mathbb{Z}/11$. We define a quasiparticle basis for this theory as $\psi_j \equiv (-j, -6j)^T, j = 0, 1, \ldots, 10$. (While the cyclic nature of the group $\mathbb{Z}/11$ implies the identification $(a, b) \equiv (a', b')$ mod $(1, 11)$ for $a, b, a', b' \in \mathbb{Z}$, we choose the above basis in order to ensure charge conservation.) The S matrix is given by $S_{jj'} = \frac{1}{\sqrt{11}} e^{-\frac{72\pi i}{11} jj'}$. For the other theory given by K_2, the discriminant group obviously has the same structure with the generator being $(0, 1)^T$ and the quasiparticles are denoted by ψ'_j. The S matrix is given by $S'_{jj'} = \frac{1}{\sqrt{11}} e^{-\frac{6\pi i}{11} jj'}$. Now we make the following identification:

$$\psi'_j \longleftrightarrow \psi_j. \tag{4.44}$$

This identification preserves the U(1) charge carried by each quasiparticle. The S matrices are also identified:

$$S_{j,j'} = \frac{1}{\sqrt{11}} e^{-\frac{72\pi i}{11} jj'} = \frac{1}{\sqrt{11}} e^{-\frac{6\pi i}{11} jj'} = S'_{jj'}. \tag{4.45}$$

Since the diagonal elements of S are basically T^2, it follows that the topological spins are also identified up to ± 1.

Our second example is

$$K'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \tag{4.46}$$

with $t = (1, 1)^T$. As before, we suppose that a non-chiral pair of modes comes down in energy and interacts strongly with the two right-moving modes described by (4.46). The action now takes the form

$$S = \int dx\, dt \left(\frac{1}{4\pi} \left(K'_1 \oplus \sigma_z \right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right). \tag{4.47}$$

If the matrix V_{IJ} is such that the perturbation

$$S' = \int dx\, dt\, u' \cos(\phi_3 + \phi_4) \tag{4.48}$$

is relevant and this is the only perturbation added to Eq. (4.47), then the two additional modes become gapped and the system is in the phase in Eq. (4.46). Suppose, instead, the only perturbation is the following:

$$S'' = \int dx\, dt\, u'' \cos(\phi_1 + 7\phi_2 + \phi_3 + 3\phi_4). \tag{4.49}$$

This perturbation is charge-conserving and spin-zero. If it is relevant, then the edge is in a different phase. To find this phase, it is helpful to make the basis change

$$W'^T(K_1' \oplus \sigma_z)W' = K_2' \oplus \sigma_z, \tag{4.50}$$

where

$$K_2' = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \tag{4.51}$$

and

$$W' = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -3 & 2 & 0 & 3 \end{pmatrix}. \tag{4.52}$$

Making the basis change $\phi = W'\phi'$, we see that

$$\phi_1 + 7\phi_2 + \phi_3 + 3\phi_4 = \phi_4' - \phi_3'. \tag{4.53}$$

Therefore, the resulting phase is described by (4.51). This is a different phase, as may be seen by noting that the lattice corresponding to Eq. (4.51) is an even lattice while the lattice corresponding to Eq. (4.46) is odd.

The difference between the two edge phases is even more dramatic than in the previous example. One edge phase has gapless fermionic excitations while the other one does not! This example shows that an edge reconstruction can relate a theory with fermionic topological order to one with bosonic topological order. Again, these two edge phases of the $\nu = 8/7$ can be distinguished by the voltage dependence of the current backscattered at a quantum point contact and the tunneling current from a metallic lead. In the K_1' edge phase (4.46), the backscattered current at a QPC is dominated by the tunneling term $\cos(\phi_2^T - \phi_2^B)$; using Eq (4.12) this yields the current-voltage relation

$$I_1^b \propto V^{-5/7}, \tag{4.54}$$

while the single-electron tunneling into a metallic lead is dominated by the tunneling term $\psi_{\text{lead}}^\dagger e^{i\phi_1^T}$, which, using Eq (4.14), yields the familiar linear current-voltage scaling

$$I_1^{\text{tun}} \propto V. \tag{4.55}$$

In the K_2' edge phase (4.51), the backscattered current at a QPC is dominated by the backscattering term $\cos(\phi_2'^T - \phi_2'^B)$, yielding:

$$I_2^b \propto V^{-3/7}. \tag{4.56}$$

The tunneling current from a metallic lead is due to the tunneling of charge-$2e$ objects created by the edge operator $e^{i\phi_1' + 4i\phi_2'}$. If we assume that the electrons are fully spin-polarized and S_z is conserved, then the most relevant term that tunnels $2e$ into the

metallic lead is $\psi^\dagger_{\text{lead}}\partial\psi^\dagger_{\text{lead}}e^{i\phi_1'^T+4i\phi_2'^T}$. Using Eq (4.14) the tunneling current is proportional to a very high power of the voltage:

$$I_2^{\text{tun}} \propto V^7. \tag{4.57}$$

Again, although the theories look drastically different, we can show that the bulk S matrices are isomorphic. First, the discriminant group of the K_1' theory is $\mathbb{Z}/7$ whose generator we can take to be the $(0,4)$ quasiparticle. We label all quasiparticles in this theory as $\psi_j \equiv (0,4j), j = 0,1,\ldots,6$. The S matrix is given by $S_{jj'} = \frac{1}{\sqrt{7}}e^{-\frac{32\pi i}{7}jj'}$. For the other theory given by K_2', the discriminant group is generated by $(0,1)^T$ and we denote the quasiparticles by ψ_j'. The S matrix is given by $S'_{jj'} = \frac{1}{\sqrt{7}}e^{-\frac{4\pi i}{7}jj'}$. Now we make the following identification:

$$\psi_j' \longleftrightarrow \psi_j. \tag{4.58}$$

The S matrices are then seen to be identical:

$$S_{j,j'} = \frac{1}{\sqrt{7}}e^{-\frac{32\pi i}{7}jj'} = \frac{1}{\sqrt{7}}e^{-\frac{4\pi i}{7}jj'} = S'_{jj'}. \tag{4.59}$$

4.4 Edge Phase Transitions

In the previous section, we gave two simple examples of edge phase transitions that can occur between two distinct chiral theories. In this section, we discuss how edge transitions can occur in full generality.

The chiral Luttinger liquid action is stable against all perturbations involving only the gapless fields in the action in Eq. (4.6) (or, equivalently in the integer case, the action in Eq. (4.1)). However, as we have seen in the previous section, strong interactions with

gapped excitations can drive a phase transition that occurs purely at the edge. While the bulk is completely unaffected, the edge undergoes a transition into another phase.

On the way to understanding this in more generality, we first consider an integer quantum Hall state. At the edge of such a state, we expect additional gapped excitations that we ordinarily ignore. However, they can interact with gapless excitations. (Under some circumstances, they can even become gapless.[70]) Let us suppose that we have an additional left-moving and and additional right-moving fermion which, together, form a gapped unprotected excitation. Then additional terms must be considered in the action. Let us first consider the case of an integer quantum Hall edge. The action in Eqs. (4.1) and (4.2) becomes $S_0 + S_1 + S_u$ with

$$S_u = \int dx\, dt \Big(\psi^\dagger_{N+1} \left(i\partial_t + v_{N+1} i\partial_x \right) \psi_{N+1} + \psi^\dagger_{N+2} \left(i\partial_t - v_{N+2} i\partial_x \right) \psi_{N+2} \\ + u \psi^\dagger_{N+1} \psi_{N+2} + \text{h.c.} + v_{I,N+1} \psi^\dagger_I \psi_I \psi^\dagger_{N+1} \psi_{N+1} + v_{I,N+2} \psi^\dagger_I \psi_I \psi^\dagger_{N+2} \psi_{N+2} + \mathcal{L}_{N,L} \Big), \quad (4.60)$$

where ψ_{N+1}, ψ_{N+2} annihilate right- and left-moving excitations which have an energy gap u for $v_{I,N+1} = v_{I,N+2} = 0$. So long as $v_{I,N+1}$ and $v_{I,N+2}$ are small, this energy gap survives, and we can integrate out ψ_{N+1}, ψ_{N+2}, thereby recovering the action $S_0 + S_1$ in Eqs. (4.1) and (4.2), but with the couplings renormalized. However, if $v_{I,N+1}$ and $v_{I,N+2}$ are sufficiently large, then some of the other terms in the action, which we have denoted by $\mathcal{L}_{N,L}$ in Eq. (4.60) may become more relevant than u. These include terms such as

$$\mathcal{L}_{N,L} = u_I \psi^\dagger_I \psi_{N+2} + \text{h.c.} + \ldots . \quad (4.61)$$

In order to understand these terms better, it is helpful to switch to the bosonic representation, where there is no additional overhead involved in considering the general

case of a chiral Abelian state, integer or fractional:

$$S = \int dx\, dt \Bigg(\frac{1}{4\pi} \left(K \oplus \sigma_z\right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \sum_{m_I} u_{m_I} \cos\left(m_I \phi^I\right) + \frac{1}{2\pi} \sum_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \Bigg). \quad (4.62)$$

Here, $I = 1, 2, \ldots, N{+}2$; and $\left(K \oplus \sigma_z\right)_{IJ}$ is the direct sum of K and σ_z: $\left(K \oplus \sigma_z\right)_{IJ} = K_{IJ}$ for $I = J = 1, 2, \ldots, N$, $\left(K \oplus \sigma_z\right)_{IJ} = 1$ for $I = J = N+1$, $\left(K \oplus \sigma_z\right)_{IJ} = -1$ for $I = J = N+2$, and $\left(K \oplus \sigma_z\right)_{IJ} = 0$ if $I \in \{1, 2, \ldots, N\}$, $J \in \{N+1, N+2\}$ or vice-versa. The interaction matrix has $V_{I,N+1} \equiv v_{I,N+1}$, $V_{I,N+2} \equiv v_{I,N+2}$. The m^Is must be integers because the ϕ^Is are periodic. For instance, $m_I = (0, 0, \ldots, 0, 1, -1)$ corresponds to the mass term $u(\psi^\dagger_{N+1}\psi_{N+2} + \text{h.c.})$ in Eq. (4.60), so $u_{m_I} = u$. In the last term, we are coupling all modes equally to the electromagnetic field, i.e. this term can be written in the form $t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu$ with $t_I = 1$ for all I. This is the natural choice, since we expect additional fermionic excitations to carry electrical charge e.

In general, most of the couplings u_{m_I} will be irrelevant at the Gaussian fixed point. An irrelevant coupling cannot open a gap if it is small enough to remain in the basin of attraction of the Gaussian fixed point. However, if we make the coupling large enough, it may be in the basin of attraction of another fixed point and it may open a gap. We will not comment more on this possibility here. However, we can imagine tuning the V_{IJ}s so that any given u_{m_I} is relevant. To analyze this possibility, it is helpful to change to the variables $X^a = e^a_I \phi^I$, in terms of which the action takes the form

$$S = \int dx\, dt \Bigg(\frac{1}{4\pi} \eta_{ab} \partial_t X^a \partial_x X^b - \frac{1}{4\pi} v_{ab} \partial_x X^a \partial_x X^b + \sum_{m_I} u_{m_I} \cos\left(m_I f^I_a X^a\right) + \frac{1}{2\pi} \sum_I f^J_a \epsilon_{\mu\nu} \partial_\mu X^a A_\nu. \Bigg) \quad (4.63)$$

e_I^a and f_a^I are bases for the lattice Λ_{N+2} and its dual Λ_{N+2}^*, where the lattice Λ_{N+2} corresponds to $K \oplus \sigma_z$. The variables X^a satisfy the periodicity condition $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{y}$ for $\mathbf{y} \in \Lambda_{N+2}$. Note that, since one of the modes is left-moving, the Lorentzian metric $\eta_{ab} = \mathrm{diag}(\mathbf{1}_{N-1}, -1)$ appears in Eq. (4.63).

Since f_a^I is a basis of the dual lattice Λ_{N+2}^*, the cosine term can also be written in the form

$$\sum_{\mathbf{v} \in \Lambda_{N+2}^*} u_{\mathbf{v}} \cos\left(\mathbf{v} \cdot \mathbf{X}\right).$$

The velocity/interaction matrix is given by $v_{ab} = V_{IJ} f_a^I f_b^J$. Now suppose that the velocity/interaction matrix takes the form

$$v_{ab} = v\, O^c{}_a \delta_{cd} O^d{}_b, \tag{4.64}$$

where $O \in \mathrm{SO}(N+1,1)$. Then we can make a change of variables to $\tilde{X}^a \equiv O^a{}_b X^b$. We specialize to the case of a single cosine perturbation associated with a particular vector in the dual lattice $\mathbf{v}_0 \equiv p_I \mathbf{f}^I$ which we will make relevant (we have also set $A_\nu = 0$ since it is inessential to the present discussion). Now Eq. (4.63) takes the form

$$S = \frac{1}{4\pi} \int dx\, dt \left(\eta_{ab} \partial_t \tilde{X}^a \partial_x \tilde{X}^b - v \delta_{ab} \partial_x \tilde{X}^a \partial_x \tilde{X}^b + u_{\mathbf{v}_0} \cos\left(p_I f_a^I (O^{-1})^a{}_b \tilde{X}^b \right) \right). \tag{4.65}$$

If this perturbation has equal right and left scaling dimensions (i.e., is spin-zero), then its scaling dimension is simply twice its left scaling dimension with corresponding beta function

$$\frac{du_{\mathbf{v}_0}}{d\ell} = \left(2 - q_{N+2}^2\right) u_{\mathbf{v}_0}, \tag{4.66}$$

where $q_b \equiv p_I f_a^I (O^{-1})^a{}_b$. The transformation O^{-1} can be chosen to be a particular boost in the $(N+2)$-dimensional space $\mathbb{R}^{N+1,1}$. Because q_a is a null vector (i.e., a light-

like vector) in this space, by taking the boost in the opposite direction of the "spatial" components of q_a, we can "Lorentz contract" them, thereby making q_{N+2} as small as desired. Thus, by taking v_{ab} of the form (4.64) and choosing $O \in \mathrm{SO}(N+1,1)$ so that $q_{N+2}^2 < 2$, we can make this coupling relevant.

When this occurs, two modes, one right-moving and one left-moving, will acquire a gap. We will then be left over with a theory with N gapless right-moving modes. The gapless excitations $\exp(i\mathbf{v}\cdot\mathbf{X})$ of the system must commute with $\mathbf{v}_0\cdot\mathbf{X}$ and, since the cosine fixes $\mathbf{v}_0\cdot\mathbf{X}$, any two excitations that differ by $\mathbf{v}_0\cdot\mathbf{X}$ should be identified. Thus, the resulting low-energy theory will be associated with the lattice Γ defined by $\Gamma \equiv \Lambda_\perp/\Lambda_\parallel$, where $\Lambda_\perp, \Lambda_\parallel \subset \Lambda_{N+2}$ are defined by $\Lambda_\perp \equiv \{\mathbf{v}\in\Lambda_{N+2}\,|\,\mathbf{v}\cdot\mathbf{v}_0 = 0\}$ and $\Lambda_\parallel \equiv \{n\mathbf{v}_0\,|\,n\in\mathbb{Z}\}$. If $\mathbf{g}_I$ is a basis for Γ, then we can define a K-matrix in this basis, $\tilde{K}_{IJ} = \mathbf{g}_I\cdot\mathbf{g}_J$. The low-energy effective theory for the gapless modes is

$$S \;=\; \int dx\,dt\left(\frac{1}{4\pi}\tilde{K}_{IJ}\partial_t\phi^I\partial_x\phi^J \;-\; \frac{1}{4\pi}\tilde{V}_{IJ}\partial_x\phi^I\partial_x\phi^J \;+\; \frac{1}{2\pi}\tilde{t}_I\epsilon_{\mu\nu}\partial_\mu\phi^I A_\nu\right). \quad (4.67)$$

When $\mathbf{v}_0 = (0,0,\ldots,0,1,-1)$ is the only relevant operator, ϕ^{N+1} and ϕ^{N+2} are gapped out. Therefore, $\Gamma = \Lambda$ and $\tilde{K}_{IJ} = K_{IJ}$. However, when other operators are present, Γ could be a different lattice $\Gamma \ncong \Lambda$, from which it follows that $\tilde{K}_{IJ} \neq K_{IJ}$ (and, $\tilde{K} \neq W^T K W$ for any W).

We motivated the enlargement of the theory from K to $K\oplus\sigma_z$ by assuming that an additional pair of gapped counter-propagating fermionic modes comes down in energy and interacts strongly with the gapless edge excitations. This counter-propagating pair of modes can be viewed as a thin strip of $\nu = 1$ integer quantum Hall fluid or, simply, as a fermionic Luttinger liquid. Of course, more than one such pair of modes may interact strongly with the gapless edge excitations, so we should also consider enlarging the K-matrix to $K\oplus\sigma_z\oplus\sigma_z\ldots\oplus\sigma_z$. We can generalize this by imagining that we can

add any one-dimensional system to the edge of a quantum Hall state. (This may not be experimentally-relevant to presently observed quantum Hall states, but as a matter of principle, this is something that could be done without affecting the bulk, so we should allow ourselves this freedom.) Any clean, gapless 1D system of fermions is in a Luttinger liquid phase (possibly with some degrees of freedom gapped). Therefore, $K \oplus \sigma_z \oplus \sigma_z \ldots \oplus \sigma_z$ is actually the most general possible form for the edge theory.

One might wonder about the possibility of attaching a thin strip of a fractional quantum Hall state to the edge of the system. Naively, this would seem to be a generalization of our putative most general form $K \oplus \sigma_z \oplus \sigma_z \ldots \oplus \sigma_z$. To illustrate the issue, let us consider a bulk $\nu = 1$ IQH state and place a thin strip of $\nu = 1/9$ FQH state at its edge. The two edges that are in close proximity can be described by the following K-matrix:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -9 \end{pmatrix}. \tag{4.68}$$

As discussed in Ref. [51], this edge theory can become fully gapped with charge-non-conserving backscattering. Then we are left with the outer chiral edge of the thin strip, which is described by $K = (9)$, which can only bound a topologically ordered $\nu = 1/9$ Laughlin state. The subtlety here is that a thin strip of the fractional quantum Hall state has no two-dimensional bulk and should be considered as a purely one-dimensional system. Fractionalized excitations, characterized by fractional conformal spins only make sense when a true 2D bulk exists. If the width of the strip is small, so that there is no well-defined bulk between them, then we can only allow operators that add an integer number of electrons to the two edges. We cannot add fractional charge since there is no bulk which can absorb compensating charge. Thus the minimal conformal spin of any operator is 1/2. In other words, starting from an one-dimensional interacting electronic

system, one cannot change the conformal spin of the electron operators. So attaching a thin strip of FQH state is no different from attaching a trivial pair of modes.

In a bosonic system, we cannot even enlarge our theory by a pair of counter-propagating fermionic modes. We can only enlarge our theory by a Luttinger liquid of bosons or, equivalently, a thin strip of $\sigma_{xy} = \frac{2e^2}{h}$ bosonic integer quantum hall fluid [20, 51, 53]. Such a system has K-matrix equal to σ_x, which only has bosonic excitations. Equivalently, bosonic systems must have even K-matrices – matrices with only even numbers along the diagonal – because all particles that braid trivially with every other particle must be a boson. Since the enlarged matrix must have the same determinant as the original one because the determinant is the ground state degeneracy of the bulk phase on the torus [39], we can only enlarge the theory by σ_x, the minimal even unimodular matrix. Therefore, in the bosonic case, we must enlarge our theory by $K \to K \oplus \sigma_x$.

In the fermionic case, we must allow such an enlargement by σ_x as well. We can imagine the fermions forming pairs and these pairs forming a bosonic Luttinger liquid which enlarges K by σ_x. In fact, it is redundant to consider both σ_z and σ_x: for an odd matrix K, $W(K \oplus \sigma_z)W^T = K \oplus \sigma_x$, where

$$W = \begin{pmatrix} 1 & 0 & \cdots & 0 & y_1 & -y_1 \\ 0 & 1 & \cdots & 0 & y_2 & -y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_N & -y_N \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ x_1 & x_2 & \cdots & x_N & s & 1-s \end{pmatrix} \tag{4.69}$$

Here the vector $\mathbf{x}$ has an odd length squared, i.e. $\mathbf{x}^T K \mathbf{x}$ is odd; by definition of K odd, such an $\mathbf{x}$ must exist. The vector $\mathbf{y}$ is defined as $\mathbf{y} = -K\mathbf{x}$ and the integer s by

$s = \frac{1}{2}(1 - \mathbf{x}^T K \mathbf{x})$. Thus $K \oplus \sigma_x$ is $\mathrm{GL}(N+2, \mathbb{Z})$-equivalent to $K \oplus \sigma_z$ and our previous discussion for fermionic systems could be redone entirely with extra modes described by σ_x. However, if K is even, then $K \oplus \sigma_x$ is not $\mathrm{GL}(N+2, \mathbb{Z})$-equivalent to $K \oplus \sigma_z$.

We remark that although σ_z enlargement and σ_x enlargement are equivalent for fermionic states when topological properties are concerned, they do make a difference in charge vectors: the appropriate charge vector for the σ_z block should be odd and typically taken to be $(1,1)^T$. However the charge vector for the σ_x block must be even and needs to be determined from the similarity transformation.

To summarize, a quantum Hall edge phase described by matrix K_1 can undergo a purely edge phase transition to another edge phase with $\mathrm{GL}(N, \mathbb{Z})$-inequivalent K_2 (with identical bulk) if there exists $\tilde{W} \in \mathrm{GL}(N+2k, \mathbb{Z})$ such that

$$K_2 \oplus \sigma_x \oplus \ldots \oplus \sigma_x = \tilde{W}^T \left(K_1 \oplus \sigma_x \oplus \ldots \oplus \sigma_x\right) \tilde{W}. \tag{4.70}$$

for some number k of σ_xs on each side of the equation. In a fermionic system with K_1 odd, an edge phase transition can also occur to an even matrix K_2 if

$$K_2^{\text{even}} \oplus \sigma_z \oplus \ldots \oplus \sigma_x = \tilde{W}^T \left(K_1^{\text{odd}} \oplus \sigma_x \oplus \ldots \oplus \sigma_x\right) \tilde{W}. \tag{4.71}$$

4.5 Stable Equivalence, Genera of Lattices, and the Bulk-Edge Correspondence for Abelian Topological Phases

4.5.1 Stable Equivalence and Genera of Lattices

In the previous section, we saw that a bulk Abelian quantum Hall state associated with K_1 has more than one different stable chiral edge phase if there exists $\mathrm{GL}(N,\mathbb{Z})$-inequivalent K_2 and $\tilde{W} \in \mathrm{GL}(N+2k,\mathbb{Z})$ such that

$$K_2 \oplus \sigma_x \oplus \ldots \oplus \sigma_x = \tilde{W}^T \left(K_1 \oplus \sigma_x \oplus \ldots \oplus \sigma_x\right) \tilde{W}. \tag{4.72}$$

This is an example of a stable equivalence; we say that K_1 and K_2 are *stably equivalent* if, for some n, there exist signature (n,n) unimodular matrices L_i such that $K_1 \oplus L_1$ and $K_2 \oplus L_2$ are *integrally equivalent*, i.e. are $\mathrm{GL}(N+2n,\mathbb{Z})$-equivalent. If there is a choice of L_is such that both are even, we will say that K_1 and K_2 are "σ_x-stably equivalent" since the L_is can be written as direct sums of σ_xs. We also saw in Eq. 4.71 that when K_1 is odd and K_2 is even, we will need L_2 to be an odd matrix. We will call this "σ_z-stable equivalence" since L_2 must contain a σ_z block. We will use U to denote the signature $(1,1)$ even Lorentzian lattice associated with σ_x. Then σ_x-stable equivalence can be restated in the language of lattices as follows. Two lattices Λ_1, Λ_2 are σ_x-stably equivalent if $\Lambda_1 \oplus U \cdots \oplus U$, and $\Lambda_2 \oplus U \cdots \oplus U$ are isomorphic lattices. Similarly, U_z will denote the Lorentzian lattice associated with σ_z. Occasionally, we will abuse notation and use σ_x and σ_z to refer to the corresponding lattices U, U_z.

Stable equivalence means that the two K-matrices are equivalent after adding "trivial" degrees of freedom – i.e. purely 1D degrees of freedom that do not require any

change to the bulk. This is analogous to the notion of stable equivalence of vector bundles, according to which two vector bundles are stably equivalent if and only if isomorphic bundles are obtained upon joining them with trivial bundles.

We now introduce the concept of the *genus* of a lattice or integral quadratic form. Two integral quadratic forms are in the same *genus*[72, 65] when they have the same signature and are equivalent over the *p-adic integers* $\mathbb{Z}_p$ for every prime p. Loosely speaking, equivalence over $\mathbb{Z}_p$ can be thought of as equivalence modulo arbitrarily high powers of p, i.e. in $\mathbb{Z}/p^n$ for every n. The importance of genus in the present context stems from the following statement of Conway and Sloane [65]:

Two integral quadratic forms K_1 and K_2 are in the same genus if and only if $K_1 \oplus \sigma_x$ and $K_2 \oplus \sigma_x$ are integrally equivalent.

Proofs of this statement are, however, difficult to pin down in the literature. It follows, for instance, from results in Ref. [72] about a refinement of the genus called the spinor genus. Below, we show how it follows in the even case from results stated by Nikulin[66]. This characterization of the genus is nearly the same as the definition of σ_x-stable equivalence given in (4.72), except that Eq. (4.72) allows multiple copies which is natural since a physical system may have access to multiple copies of trivial degrees of freedom. Its relevance to our situation follows from the following theorem that we demonstrate below:

Two K-matrices K_1 and K_2 of the same dimension, signature and type are stably equivalent if and only if $K_1 \oplus \sigma_x$ and $K_2 \oplus \sigma_x$ are integrally equivalent, i.e. only a single copy of σ_x is needed in Eq. (4.72).

Thus any edge phase that can be reached via a phase transition involving multiple sets of trivial 1D bosonic degrees of freedom (described by K-matrix σ_x) can also be reached through a phase transition involving only a single such set. We demonstrate

this by appealing to the following result stated by Nikulin[66] (which we paraphrase but identify by his numbering):

Corollary 1.16.3: *The genus of a lattice is determined by its discriminant group A, parity, signature (r_+, r_-), and bilinear form b on the discriminant group.*

Since taking the direct sum with multiple copies of σ_x does not change the parity, or bilinear form on the discriminant group, any K_1 and K_2 that are σ_x-stably equivalent are in the same genus. The theorem then follows from the statement[65] above that only a single copy of σ_x is needed.

In the even case, the theorem follows directly from two other results found in Nikulin[66]:

Corollary 1.13.4: *For any even lattice Λ with signature (r_+, r_-) and discriminant quadratic form q, the lattice $\Lambda \oplus U$ is the only lattice with signature $(r_+ + 1, r_- + 1)$ and quadratic form q.*

Theorem 1.11.3: *Two quadratic forms on the discriminant group are isomorphic if and only if their bilinear forms are isomorphic and they have the same signature* (mod 8).

If lattices Λ_1 and Λ_2 are in the same genus, they must have the same (r_+, r_-) and bilinear form b. According to Theorem 1.11.3, they must have the same quadratic form, namely $q([\mathbf{x}]) = \frac{1}{2}b([\mathbf{x}], [\mathbf{x}])$, which is well-defined in the case of an even lattice. Then, Corollary 1.13.4 tells us that $\Lambda_1 \oplus U$ is the unique lattice with signature $(r_+ + 1, r_- + 1)$ and quadratic form q. Since $\Lambda_2 \oplus U$ has the same signature $(r_+ + 1, r_- + 1)$ and quadratic form q, $\Lambda_1 \oplus U \cong \Lambda_2 \oplus U$. Thus, we see that any two even K-matrices in the same genus are integrally-equivalent after taking the direct sum with a single copy of σ_x. Of course, our previous arguments that used Nikulin's Corollary 1.16.3 and the characterization of genus from Conway and Sloane[65] are stronger since they apply to odd matrices.

4.5.2 Bulk-Edge Correspondence

Since the quadratic form $q([\mathbf{u}])$ gives the T and S matrices according to Eqs. (4.23) and (4.25), we can equally-well say that the genus of a lattice is completely determined by the particle types, T-matrix, S-matrix, and right- and left-central charges. For a bosonic system, the genus completely determines a bulk phase. Conversely, a bulk topological phase almost completely determines a genus: the bulk phase determines $(c_+ - c_-)$ mod 24 while a genus is specified by (c_+, c_-). However, if the topological phase is fully chiral, so that it can have $c_- = 0$, then it fully specifies a family of genera that differ only by adding central charges that are a multiple of 24, i.e. $3k$ copies of the E_8 state for some integer k (see Section 4.7.1 for a discussion of this state). Thus, up to innocuous shifts of the central charge by 24, we can say that

A bulk bosonic topological phase corresponds to a genus of even lattices while its edge phases correspond to the different lattices in this genus.

The problem of detemining the different stable edge phases that can occur for the same bosonic bulk is then the problem of determining how many distinct lattices there are in a genus.

In the fermionic case, the situation is more complicated. A fermionic topological phase is determined by its particle types, its S-matrix, and its central charge (mod 24). It does not have a well-defined T-matrix because we can always change the topological twist factor of a particle by -1 simply by adding an electron to it. According to the following result of Nikulin, these quantities determine an odd lattice:

Corollary 1.16.6: *Given a finite Abelian group A, a bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$, and two positive numbers (r_+, r_-), then, for sufficiently large r_+, r_-, there exists an odd lattice for which A is its discriminant group. b is the bilinear form on the discriminant group, and (r_+, r_-) is its signature.*

Since the S-matrix defines a bilinear form on the Abelian group of particle types, this theorem means that the quantities that specify a fermionic Abelian topological phase are compatible with an odd lattice. Clearly, they are also compatible with an entire genus of odd lattices since σ_x stable equivalence preserves these quantities. Moreover, by Corollary 1.16.3, there is only a single genus of odd lattices that are compatible with this bulk fermionic Abelian topological phase. However, Corollary 1.16.3 leaves open the possibility that there is also a genus of even lattices that is compatible with this fermionic bulk phase, a possibility that was realized in one of the examples in Section 4.3. This possibility is discussed in detail in Section 4.6. However, the general result that we can already state, up to shifts of the central charge by 24 is

A bulk fermionic topological phase corresponds to a genus of odd lattices while its edge phases correspond to the different lattices in this genus and, in some cases (specificed in Section 4.6), to the different lattices in an associated genus of even lattices.

In principle, one can determine how many lattices there are in a given genus by using the Smith-Siegel-Minkowski mass formula [65] to evaluate the weighted sum

$$\sum_{\Lambda \in g} \frac{1}{|\mathrm{Aut}(\Lambda)|} = m(K) \tag{4.73}$$

over the equivalence classes of lattices in a given genus g. Each equivalence class of forms corresponds to a lattice Λ. The denominator is the order of the automorphism group $\mathrm{Aut}(\Lambda)$ of the lattice Λ. The right-hand-side is the *mass* of the genus of K, which is given by a complicated but explicit formula (see Ref. [65]).

Given a K-matrix for a bosonic state, one can compute the size of its automorphism group[3], which gives one term in the sum in (4.73). If this equals the mass formula on

[3]For generic K-matrices without any symmetries, the automorphism group often only consists of two elements: $W = \pm I_{N\times N}$.

the right-hand-side of Eq. (4.73), then it means the genus has only one equivalence class. If not, we know there is more than one equivalence class in the genus. Such a program shows [15] that, in fact, all genera contain more than one equivalence class for $N > 10$, i.e. all chiral Abelian quantum Hall states with central charge $c > 10$ have multiple distinct stable chiral edge phases. For $3 \leq N \leq 10$, there is a finite set of genera with only a single equivalence class [73]; all others have multiple equivalence classes. The examples of $\nu = 16$ analyzed in Ref. [5] and $\nu = 12/23$ that we gave in Section 4.7 are, in fact, the rule. Bosonic chiral Abelian quantum Hall states with a single stable chiral edge phase are the exception, they can only exist for $c \leq 10$ and they have been completely enumerated[73].

This does not tell us how, given one equivalence class, to find other equivalence classes of K-matrices in the same genus. However, one can use the Gauss reduced form [65] to find all quadratic forms of given rank and determinant by brute force. Then we can use the results at the end of previous Section to determine if the resulting forms are in the same genus.

4.5.3 Primary Decomposition of Abelian Topological Phases

According to the preceding discussion, two distinct edge phases can terminate the same bulk phase if they are both in the same genus (but not necessarily only if they are in the same genus in the fermionic case). It may be intuitively clear what this means, but it is useful to be more precise about what we mean by "the same bulk phase". In more physical terms, we would like to be more precise about what it means for two theories to have the same particle types and S- and T-matrices. In more formal terms, we would like to be more precise about what is meant in Nikulin's Theorem 1.11.3 by isomorphic quadratic forms and bilinear forms. In order to do this, it helps to view an

Abelian topological phase in a somewhat more abstract light. When viewed from the perspective of an edge phase or, equivalently, a K-matrix, the bulk phase is determined by the signature (r_+, r_-), together with the bilinear form on the discriminant group Λ^*/Λ induced by the bilinear form on the dual lattice Λ^* determined by K. As we have seen, this data uniquely specifies a nondegenerate quadratic form $q\colon \Lambda^*/\Lambda \to \mathbb{Q}/\mathbb{Z}$ on the discriminant group. Therefore, we may view the genus more abstractly in terms of an arbitrary finite Abelian group A and a quadratic form $q\colon A \to \mathbb{Q}/\mathbb{Z}$, making no direct reference to an underlying lattice. We will sometimes call such a quadratic form a *finite* quadratic form to emphasize that its domain is a finite Abelian group. The elements of the group A are the particle types in the bulk Abelian topological phase.

Now suppose we have two bulk theories associated with Abelian groups A, A', quadratic forms $q\colon A \to \mathbb{Q}/\mathbb{Z}$, $q'\colon A' \to \mathbb{Q}/\mathbb{Z}$ and chiral central charges c_-, c'_-. These theories are the same precisely when the chiral central charges satisfy $c_- \equiv c'_-$ mod 24, and when the associated quadratic forms are *isomorphic*. This latter condition means that there exists a group isomorphism $f\colon A' \to A$ such that $q' = q \circ f$. Note that if the quadratic forms are isomorphic then the chiral central charges must be equal (mod 8) according to the Gauss-Milgram sum. However, the bulk theories are the same only if they satisfy the stricter condition that their central charges are equal modulo 24.

The implications of this become more apparent after observing that any Abelian group factors as a direct sum $A \simeq \oplus_p A_p$ over primes dividing $|A|$, where $A_p \subset A$ is the *p-primary* subgroup of elements with order a power of p. Any isomorphism $f\colon A' \to A$ must respect this factorization by decomposing as $f = \oplus_p f_p$, with each $f_p\colon A'_p \to A_p$. Furthermore, every finite quadratic form decomposes into a direct sum $q = \oplus_p q_p$ of *p-primary* forms; we call q_p the *p-part* of q. This ultimately leads to a physical interpretation for p-adic integral equivalence: if p is odd, two K-matrices are p-adically integrally equivalent precisely when the p-parts of their associated quadratic forms are isomorphic. Additional subtleties arise

when $p = 2$ but, as we will see, these are the reason for the distinction between σ_x- and σ_z-equivalence.

The image of a given finite quadratic form q is a finite cyclic subgroup $N_q^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ isomorphic to $\mathbb{Z}/N_q$, where N_q is the *level* of the finite quadratic form q. The level is the smallest integer N such that q factors through $\mathbb{Z}/N$, implying that the topological spins of particles in A_q are N_qth roots of unity. Because the level of the direct sum of finite quadratic forms is the least common multiple of the levels of the summands, the level of $q = \oplus_p q_p$ is equal to the product $N_q = \prod_p N_{q_p}$ of the levels of the q_p. If p is odd, the level of q_p is the order of the largest cyclic subgroup of A_p, while it is typically twice as big for q_2. Physically, this means that the entire theory uniquely factors into a tensor product of anyon theories such that the topological spins of the anyons in the pth theory are pth-power roots of unity. This decomposition lets us express a *local-to-global principle* for finite quadratic forms: q and q' are isomorphic iff q_p and q'_p are for every p. Indeed, if one views prime numbers as "points" in an abstract topological space[4], this principle says that q and q' are *globally equivalent* (at all primes) iff they are locally equivalent at each prime dividing $|A|$.

Further information about the prime theories is obtained by decomposing each A_p into a product

$$A_p \simeq \prod_{m=0}^{m_p} (\mathbb{Z}/p^m)^{d_{p^m}} \tag{4.74}$$

of cyclic groups, where $d_{p^0}, \ldots, d_{p^{m_p-1}} \geq 0$ and $d_{p^{m_p}} > 0$. When p is odd, there is a 1-1 correspondence between bilinear and quadratic forms on A_p because multiplication by 2 is invertible in every $\mathbb{Z}/p^m$. Furthermore, given a quadratic form q_p on A_p for odd p, we claim there always exists an automorphism $g \in \mathrm{Aut}(A_p)$ that fully diagonalizes q_p

[4]This space is known as $\mathrm{Spec}(\mathbb{Z})$. Rational numbers are identified with functions on this space according to their prime factorizations.

relative to a fixed decomposition (4.74) such that

$$q_p \circ g = \bigoplus_m \big(\underbrace{q^+_{p^m} \oplus \ldots \oplus q^+_{p^m} \oplus q^{\pm}_{p^m}}_{d_{p^m} \text{ terms}} \big), \tag{4.75}$$

where

$$q^+_{p^m}(x) = \frac{1}{p^m} 2^{-1} x^2 \bmod \mathbb{Z},$$

$$q^-_{p^m}(x) = \frac{1}{p^m} u_p 2^{-1} x^2 \bmod \mathbb{Z}$$

and u_p is some fixed non-square modulo p^n. A dual perspective is that, given q_p, it is always possible to choose a decomposition (4.74) of A_p relative to which q_p has the form of the right-hand-side of (4.75). However, not every decomposition will work for a given q_p because $\mathrm{Aut}(A_p)$ can mix the different cyclic factors. For example, $\mathrm{Aut}((\mathbb{Z}/p)^d) \simeq \mathrm{GL}(d, \mathbb{Z}/p)$ mixes the cyclic factors of order p. There will also be automorphisms mixing lower-order generators with ones of higher order, such as the automorphism of $\mathbb{Z}/3 \oplus \mathbb{Z}/9 = \langle \alpha_3, \alpha_9 \rangle$ defined on generators by $\alpha_3 \mapsto \alpha_3$ and $\alpha_9 \mapsto \alpha_3 + \alpha_9$. Physically, this means that the anyon theory associated to A_p further decomposes into a tensor product of "cyclic" theories, although now such decompositions are not unique because one can always redefine the particle types via automorphisms of A_p.

4.5.4 p-adic Symbols

Two K-matrices are p-adically integrally equivalent iff the diagonalizations of the p-parts of their associated finite quadratic forms coincide. The numbers d_{p^m} and the sign of the last form in the mth block thus form a complete set of invariants for p-adic integral equivalence of K-matrices. This data is encoded into the *p-adic symbol*, which is written as $1^{\pm d_{p^0}} p^{\pm d_{p^1}} (p^2)^{\pm d_{p^2}} \cdots$ (terms with $d_{p^m} = 0$ are omitted) and can be computed using

Sage[74]. Two K-matrices are p-adically integrally equivalent iff their p-adic symbols coincide.

The p-adic symbol can be computed more directly by noting that K-matrices are equivalent over the p-adic integers when they are equivalent by a *rational* transformation whose determinant and matrix entries do not involve dividing by p. Such transformations can be reduced modulo arbitrary powers of p and give rise to automorphisms of the p-part A_p of the discriminant group. Given a K-matrix K, there always exists a p-adically integral transformation g putting K into *p-adically block diagonalized*[65] form

$$gKg^T = K_{p^0} \oplus pK_{p^1} \oplus p^2K_{p^2} \oplus \cdots , \tag{4.76}$$

where $\det(K_{p^m})$ is prime to p for every m.

A more direct characterization of the genus can now be given: Two K-matrices are in the same genus iff they are related by a *rational* transformation whose determinant and matrix entries are relatively prime to twice the determinant, or rather, to the level N of the associated discriminant forms. Such a transformation suffices to simultaneously p-adically block-diagonalize K over the p-adic integers for every p dividing twice the determinant, and a similar reduction yields the entire quadratic form on the discriminant group, with some extra complications when $p = 2$. Such a non-integral transformation mapping two edge theories as $g(\Lambda_1) = \Lambda_2$ does not, however induce fractionalization in the bulk since it reduces to an isomorphism between the discriminant groups $\Lambda_1^*/\Lambda_1 \to \Lambda_2^*/\Lambda_2$. For example, the $\nu = 12/11$ K-matrices (4.32) and (4.37) are related by the following rational transformation that divides by 3:

$$\begin{pmatrix} 1 & 0 \\ -1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}.$$

One might be tempted to look at this transformation and conclude that one of the particle types on the left-hand-side has undergone fractionalization and divided into 3 partons (due to the $-1/3$ entries in the matrix), thereby leading to the phase on the right-hand-side. But in mod 11 arithmetic, the number 3 is invertible, so no fractionalization has actually occurred.

When $p \neq 2$, the p-adic symbol can be directly computed from any such p-adic block diagonalization, as the term $(p^m)^{\pm d_{p^m}}$ records the *dimension* $d_{p^m} = \dim(K_{p^m})$ and *sign* $\pm$ of $\det(K_{p^m})$, the latter being given by the *Legendre symbol*

$$\left(\frac{\det(K_{p^m})}{p}\right) = \begin{cases} +1 & \text{if } p \text{ is a square mod } p \\ -1 & \text{if } p \text{ is not a square mod } p. \end{cases}$$

In this case, it is further possible to p-adically diagonalize all of the blocks K_{p^m}, in which case there exists a p-adically integral transformation g that diagonalizes the form $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T K^{-1}\mathbf{x}$ on the dual lattice Λ^* such that its reduction modulo Λ takes the form (4.75).

When $p = 2$, it is possible that only some of the blocks K_{2^m} in the decomposition (4.76) can be 2-adically diagonalized[65] (we call these blocks *odd*). The remaining *even* blocks can only be block diagonalized into 2×2 blocks of the form $\left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$ with b odd, or rather, some number of copies of σ_x and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. As with odd p, the 2-adic symbol associated to such a block diagonalization records the dimensions d_{2^m} of the blocks, together with the *signs* of the determinants $\det(K_{2^m})$, which are given by the *Jacobi symbols*

$$\left(\frac{2}{\det(K_{2^m})}\right) = \begin{cases} +1 & \text{if } \det(K_{2^m}) \equiv \pm 1 \bmod 8 \\ -1 & \text{if } \det(K_{2^m}) \equiv \pm 3 \bmod 8 \end{cases}$$

and record whether or not $\det(K_{p^m})$ is a square mod 8. In addition to this data, the 2-

K-matrix	p-adic symbols			quadratic form
$\left(\begin{smallmatrix}1&0\\0&7\end{smallmatrix}\right)$ $\left(\begin{smallmatrix}2&1\\1&4\end{smallmatrix}\right)$	1_0^{+2} 1_{even}^{+2}	$1^{+1}7^{+1}$ $1^{+1}7^{+1}$		q_7^+
$\left(\begin{smallmatrix}1&0\\0&11\end{smallmatrix}\right)$ $\left(\begin{smallmatrix}3&1\\1&4\end{smallmatrix}\right)$	1_4^{-2}	$1^{+1}11^{+1}$		q_{11}^+
$\left(\begin{smallmatrix}3&0\\0&5\end{smallmatrix}\right)$ $\left(\begin{smallmatrix}2&1\\1&8\end{smallmatrix}\right)$	1_0^{+2} 1_{even}^{+2}	$1^{-1}3^{+1}$ $1^{-1}3^{+1}$	$1^{-1}5^{+1}$ $1^{-1}5^{+1}$	$q_3^+ \oplus q_5^+$
$\left(\begin{smallmatrix}2&3\\3&16\end{smallmatrix}\right)$ $\left(\begin{smallmatrix}4&1\\1&6\end{smallmatrix}\right)$	1_{even}^{+2}	$1^{+1}23^{+1}$		q_{23}^+
K_{A_4} $5 \oplus \mathbb{I}_3$	1_{even}^{-4} 1_0^{-4}	$1^{+3}5^{+1}$ $1^{+3}5^{+1}$		q_5^+
K_{E_8} $\mathbb{I}_8$	1_{even}^{+8} 1_0^{+8}			0
$K_{E_8} \oplus \mathbb{I}_4$ $\mathbb{I}_{12}$ $K_{D_{12}^+}$	1_4^{+12}			0
$\left(\begin{smallmatrix}2&\\&2\end{smallmatrix}\right)$	2_{even}^{+2}			$q_{2,2}^+$
K_{D_4}	$1_{\text{even}}^{-2}2_{\text{even}}^{-2}$			$q_{2,2}^-$
$\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$	2_{even}^{-2}	$1^{+1}3^{+1}$		$q_{2,2}^- \oplus q_3^+$

Table 4.1: Here we list the p-adic symbols and discriminant quadratic forms for various K-matrices appearing in this chapter, beginning with the canonical 2-adic symbol in every case, followed by the symbols for each prime dividing the determinant. Each block contains inequivalent-but-stably-equivalent matrices. The last few rows contain K-matrices giving rise to some of the exceptional 2-adic quadratic forms mentioned in the text.

adic symbol also records the parities as well as the traces $\mathrm{Tr}K_{2^m} \bmod 8$ of the odd blocks. An additional complication is that a given K-matrix can be 2-adically diagonalized in more than one way, and while the dimensions and parities of the blocks will be the same, the signs and traces of the odd blocks – and thus the 2-adic symbols – can be different. While this makes checking 2-adic equivalence more difficult, it is nonetheless possible to define a *canonical* 2-adic symbol[65] that *is* a complete invariant for 2-adic equivalence. We record these canonical 2-adic symbols for many of the K-matrices considered in this chapter in Table 4.1.

The reason for the additional complexity when $p = 2$ is because multiplication by 2 is not invertible on the 2-primary part $(\mathbb{Q}/\mathbb{Z})_2$ of $\mathbb{Q}/\mathbb{Z}$. This implies that if q refines a bilinear form on a 2-group then so does $q + \frac{1}{2} \bmod \mathbb{Z}$, and sometimes these refinements are not isomorphic. For example, there is only one nondegenerate bilinear form $b_2(x, y) = \frac{xy}{2} \bmod \mathbb{Z}$ on $\mathbb{Z}/2$, with two non-isomorphic quadratic refinements $q_2^{\pm}(x) = \pm\frac{x}{4} \bmod \mathbb{Z}$. Each of these refinements has level 4 and corresponds respectively to the semion $K = (2)$ and its conjugate $K = (-2)$. These give the S and T matrices

$$S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T_2^{\pm} = e^{\mp\frac{2\pi i}{24}}\begin{pmatrix} 1 & \\ & \pm i \end{pmatrix}.$$

On $\mathbb{Z}/2 \times \mathbb{Z}/2$, there are two isomorphism classes of nondegenerate bilinear forms. The first class is represented by

$$(b_2 \oplus b_2)(x, y) = \tfrac{1}{2}(x_1y_1 + x_2y_2) \bmod \mathbb{Z}$$

and has the S-matrix

$$S_2 \otimes S_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

All the refinements in this case have level 4 and are given by tensor products of semions. Up to isomorphism, this gives three refinements $q_2^+ \oplus q_2^+$, $q_2^+ \oplus q_2^-$ and $q_2^- \oplus q_2^-$, determined by the K-matrices $\left(\begin{smallmatrix} 2 & \\ & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & \\ & -2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -2 & \\ & -2 \end{smallmatrix}\right)$ with $c_- = 2, 0, -2$ respectively.

The second class of bilinear forms on $\mathbb{Z}/2 \times \mathbb{Z}/2$ contains the single form

$$b_{2,2}(x, y) = \tfrac{1}{2}(x_1 y_2 + x_2 y_1) \bmod \mathbb{Z}$$

and gives the S-matrix

$$S_{2,2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It is refined by two isomorphism classes $q_{2,2}^{\pm}$ of quadratic forms with T-matrices $T_{2,2}^{\pm} = \mathrm{diag}(1, \pm 1, \pm 1, -1)$ (these have level 2, the exception to the rule), up to the usual phase of $-2\pi i c_-/24$. The form $q_{2,2}^+$ is given by the K-matrix $\left(\begin{smallmatrix} & 2 \\ 2 & \end{smallmatrix}\right)$ and corresponds to the toric code. The form $q_{2,2}^-$ is given by the K-matrix

$$K_{D_4} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

of $\mathrm{SO}(8)_1$, or equivalently, by the restriction of the quadratic form associated to the K-matrix $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$ to the 2-part of its discriminant group $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3$. Again, these are distinguished by their signatures, which are 0 and 4 mod 8. The 2-adic diagonalizations of these K-matrices contain examples of even blocks, as illustrated in to even blocks in Table 4.1.

Further complexity arises for higher powers of 2: There are two bilinear forms $b_4^{\pm}$ on $\mathbb{Z}/4$, and four $b_{2^m}^{1,3,5,7}$ on each $\mathbb{Z}/2^m$ when $m \geq 3$. There are also four quadratic forms $q_{2^m}^{1,3,5,7}$ on $\mathbb{Z}/2^m$ for every $m \geq 2$, all with level 2^{m+1}. Therefore, the bilinear forms $b_4^{\pm}$ have two refinements each, while the rest have unique refinements. On top of all this, even more complexity arises from the fact that factorizations of such forms is not typically unique. It is therefore less straightforward to check equivalence of 2-adic forms. It is nonetheless still possible to define a canonical 2-adic symbol[65] that is a complete invariant for 2-adic equivalence of K-matrices. However, this symbol carries strictly more information than the isomorphism class of the 2-part of the discriminant form because it knows the parity of K. To characterize the even-odd equivalences that we investigate in the next section, the usual 2-adic equivalence is replaced with equivalence of the 2-parts of discriminant forms as in the odd p case above.

The 2-adic symbol contains slightly more information than just the equivalence class of a quadratic form on the discriminant group. This is evident in our even-odd examples, for which all p-adic symbols for odd p coincide, with the only difference occurring in the 2-adic symbol. It is however clear that two K-matrices K_{even} and K_{odd} of different parities are stably equivalent precisely when either $K_{\mathrm{even}} \oplus 1$ and $K_{\mathrm{odd}} \oplus 1$ are in the same genus, or otherwise, when $K_{\mathrm{even}} \oplus \sigma_z$ and $K_{\mathrm{odd}} \oplus \sigma_z$ are in the same genus. A detailed study of the 2-adic symbols in this context will appear elsewhere.

4.6 Stable Equivalence between Odd and Even Matrices: Fermionic Bulk States with Bosonic Edges Phases

We now focus on the case of fermionic systems, which are described by odd K-matrices (i.e., matrices that have at least one odd number on the diagonal). We ask: Under what circumstances is such a K-matrix equivalent, upon enlargement by σ_z (or σ_x, since it makes no difference for an odd matrix), to an even K-matrix enlarged by σ_z:

$$K_{\text{odd}} \oplus \sigma_z = W^T (K_{\text{even}} \oplus \sigma_z) W ? \tag{4.77}$$

This question can be answered using the theory of quadratic refinements.[44, 64]

As we have alluded to earlier, the naive definition of a quadratic form on the discriminant group breaks down for odd matrices. To be more concrete, $\frac{1}{2}\mathbf{u}^2 \pmod 1$ is no longer well-defined on the discriminant group. In order to be well-defined on the discriminant group, shifting $\mathbf{u}$ by a lattice vector $\boldsymbol{\lambda} \in \Lambda$ must leave $q(\mathbf{u})$ invariant modulo integers, so that $e^{2\pi i q(\mathbf{u})}$ in Eq. (4.23) is independent of which representative in Λ^* we take for an equivalence class in $A = \Lambda^*/\Lambda$. When K is odd, there are some vectors $\boldsymbol{\lambda}$ in the original lattice Λ such that

$$q(\mathbf{u} + \boldsymbol{\lambda}) \equiv q(\mathbf{u}) + \frac{1}{2} \bmod 1. \tag{4.78}$$

Physically, such a vector is just an electron ($\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$ is an odd integer). One can attach an odd number of electrons to any quasiparticle and change the exchange statistics by -1. In a sense, the discriminant group should be enlarged to $A \oplus (A + \boldsymbol{\lambda}_{\text{odd}})$: quasiparticles come in doublets composed of particles with opposite fermion parity, and therefore opposite topological twists. The Gauss-Milgram sum over this enlarged set of quasiparticles is

identically zero, which is a clear signature that the Abelian topological phase defined by an odd K-matrix is not a TQFT in the usual sense.

While the T matrix is not well-defined for a fermionic theory, the S matrix, which is determined by the discriminant bilinear form $b([\mathbf{v}],[\mathbf{v}'])$, makes perfect sense. This is because a full braid of one electron around any other particle does not generate a non-trivial phase.

Given a bilinear form b, a systematic approach for defining a quadratic form that is well-defined on the discriminant group comes from the theory of quadratic refinements. The crucial result is that a given bilinear form can *always* be lifted to a quadratic form q on the discriminant group. The precise meaning of "lifting" is that there exists a well-defined discriminant quadratic form such that $b([\mathbf{v}],[\mathbf{v}']) = q([\mathbf{v}+\mathbf{v}']) - q([\mathbf{v}]) - q([\mathbf{v}'])$.[44, 64] With q, the topological twists are well-defined: $e^{2\pi i q(\mathbf{u})} = e^{2\pi i q(\mathbf{u}+\lambda)}$ for all $\mathbf{u} \in \Lambda^*$ and $\lambda \in \Lambda$. We will give a constructive proof for the existence of such a q, given any odd K-matrix.

Once the existence of such a quadratic form $q([\mathbf{v}])$ is established, we can evaluate the Gauss-Milgram sum (4.27) and determine c_- mod 8. We then appeal to the following result of Nikulin [66]:

Corollary 1.10.2: *Given an Abelian group A, a quadratic form q on A, and positive integers (r_+, r_-) that satisfy the Gauss-Milgram sum for q, there exists an even lattice with discriminant group A, quadratic form q on the discriminant group, and signature (r_+, r_-), provided $r_+ + r_-$ is sufficiently-large.*

Using Corollary 1.10.2, we immediately see that an even lattice characterized by $(A, q, c_- \text{ mod } 8)$ exists, whose Gram matrix is denoted by K_{even}. Recall that the chiral central charge c_- is equal to the signature $\sigma = r_+ - r_-$ of the lattice. Next we show that K_{even} is σ_z-stably equivalent to the odd matrix we started with: namely, (4.77) holds for

this K_{even}. Since K_{even} and K share the same discriminant group and S matrix, they are stably equivalent upon adding unimodular lattices, according to Theorem 1. 1. 9. In other words, there exist unimodular matrices U and U' such that

$$K \oplus U \simeq K_{\text{even}} \oplus U'. \tag{4.79}$$

Apparently U' must be odd. We now add to both sides of the equation the conjugate of U' denoted by $\overline{U'}$:

$$K \oplus (U \oplus \overline{U'}) \simeq K_{\text{even}} \oplus (U' \oplus \overline{U'}). \tag{4.80}$$

On the right-hand side, $U' \otimes \overline{U'}$ is equivalent to $\sigma_z \oplus \sigma_z \oplus \cdots \sigma_z$. On the left-hand side, $U \oplus \overline{U'}$ can be transformed to the direct sum of $\mathbb{I}_n$ where $n = \sigma(U) - \sigma(U') = \sigma(K_{\text{even}}) - \sigma(K)$ and several $\sigma_{z/x}$'s. Here $\mathbb{I}_n$ is the $|n| \times |n|$ identity matrix and when n is negative we take it to be $-\mathbb{I}_{|n|}$. If $n \neq 0 \bmod 8$, then K_{even} has a different chiral central charge as K. Therefore we have arrived at the following theorem:

For any odd K matrix, $K \oplus \mathbb{I}_n$ is σ_z-stably equivalent to an even K-matrix for an appropriate n.

The physical implication is that by adding a certain number of Landau levels the edge phase of a fermionic Abelian topological phase is always stably equivalent to a purely bosonic edge phase which has no electron excitations in its low-energy spectrum.

The possible central charges of the bosonic edge theory are $c_{\text{ferm}} + n + 8m$ for $m \in \mathbb{Z}$. We can consider a fermionic system with an additional $8m + n$ Landau levels, where m is the smallest positive integer such that $8m + n > 0$. Such a fermionic theory has precisely the same discriminant group as the original fermionic theory and, consequently,

is associated with precisely the same bosonic system defined by the refinement $q([\mathbf{u}])$. So even if the original fermionic theory does not have a stable chiral edge phase with only bosonic excitations, there is a closely-related fermionic theory with some extra filled Landau levels which does have a chiral edge phase whose gapless excitations are all bosonic. A simple example of this is given by the $\nu = 1/5$ Laughlin state, which has $K = 5$. The corresponding bosonic state has $c = 4$, so the $\nu = 1/5$ Laughlin state does not have a chiral edge phase whose gapless excitations are all bosonic. However, the central charges do match if, instead, we consider the $\nu = 3 + \frac{1}{5} = 16/5$ state. This state *does* have a bosonic edge phase, with K-matrix

$$K_{A_4} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \tag{4.81}$$

corresponding to $\mathrm{SU}(5)_1$. (Ordinarily, the Cartan matrix for SU(5) is written with -1s off-diagonal, but by a change of basis we can make them equal to $+1$.)

In the following we demonstrate concretely how to obtain a particular discriminant quadratic form q, starting from the odd lattice given by K. We already know that the naive definition $\frac{1}{2}\mathbf{u}^2 (\mathrm{mod}\ 1)$ does not qualify as a discriminant quadratic form. In order to define a quadratic form on the discriminant group, we first define a quadratic function $Q_{\mathbf{w}}(\mathbf{u})$ according to:

$$Q_{\mathbf{w}}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^2 - \frac{1}{2}\mathbf{u}\cdot\mathbf{w}, \tag{4.82}$$

for $\mathbf{w} \in \Lambda^*$. Such a linear shift preserves the relation between the quadratic function (T

matrix) and the bilinear form (S matrix):

$$Q_{\mathbf{w}}(\mathbf{u}+\mathbf{v}) - Q_{\mathbf{w}}(\mathbf{u}) - Q_{\mathbf{w}}(\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}. \tag{4.83}$$

(Notice that $\mathbf{u}\cdot\mathbf{v}$ is the symmetric bilinear form $b(\mathbf{u},\mathbf{v})$ in Stirling's thesis [64]). Notice that at this stage $Q_{\mathbf{w}}$ is not yet a quadratic form on A, being just a quadratic function.

If, for any $\boldsymbol{\lambda}\in\Lambda$, $Q_{\mathbf{w}}$ satisfies $Q_{\mathbf{w}}(\mathbf{u}+\boldsymbol{\lambda}) \equiv Q_{\mathbf{w}}(\mathbf{u}) \bmod 1$ or, in other words,

$$\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}\cdot\mathbf{w} \bmod 2. \tag{4.84}$$

then we can define the following quadratic form on the discriminant group:

$$q([\mathbf{u}]) = Q_{\mathbf{w}}(\mathbf{u}).$$

Expanding $\mathbf{w}$ in the basis of the dual lattice $\mathbf{w} = w_I\mathbf{f}^I$ and expanding $\boldsymbol{\lambda}^I\mathbf{e}_I$, we find that this condition is satisfied if we take $w_I \equiv K_{II} \bmod 2$. Thus, for a Hall state expressed in the symmetric basis, we may identify $\mathbf{w}$ with twice the spin vector $s_I = K_{II}/2$.[75, 76]

A central result of Ref. [44] is that such a $\mathbf{w}$ leads to a generalized Gauss-Milgram sum:

$$\frac{1}{\sqrt{|A|}} e^{\frac{2\pi i}{8}\mathbf{w}^2} \sum_{\mathbf{u}} e^{2\pi i Q_{\mathbf{w}}(\mathbf{u})} = e^{2\pi i\sigma/8}, \tag{4.85}$$

where, in order for the notation to coincide, we have replaced the chiral central charge with the signature σ on the right-hand-side of the above equation. Note that the choice of $\mathbf{w}$ here is not unique. We can check that the modified Gauss-Milgram sum holds for $\mathbf{w}+2\boldsymbol{\lambda}^*$ where $\boldsymbol{\lambda}^*\in\Lambda^*$. First note that

$$Q_{\mathbf{w}+2\boldsymbol{\lambda}^*}(\mathbf{u}) \;=\; \frac{1}{2}\mathbf{u}^2 - \frac{1}{2}\mathbf{u}\cdot\mathbf{w} - \mathbf{u}\cdot\boldsymbol{\lambda}^* \;=\; Q_{\mathbf{w}}(\mathbf{u}-\boldsymbol{\lambda}^*) - \frac{1}{2}\boldsymbol{\lambda}^{*2} - \frac{1}{2}\boldsymbol{\lambda}^*\cdot\mathbf{w}, \tag{4.86}$$

while at the same time

$$(\mathbf{w}+2\boldsymbol{\lambda}^*)^2 = \mathbf{w}^2 + 4\boldsymbol{\lambda}^*\cdot\mathbf{w} + 4\boldsymbol{\lambda}^{*2}. \tag{4.87}$$

Therefore,

$$e^{\frac{2\pi i}{8}(\mathbf{w}+2\boldsymbol{\lambda}^*)^2}\sum_{\mathbf{u}} e^{2\pi i Q_{\mathbf{w}+2\boldsymbol{\lambda}^*}(\mathbf{u})} = e^{\frac{2\pi i}{8}\mathbf{w}^2}\sum_{\mathbf{u}} e^{2\pi i Q_{\mathbf{w}}(\mathbf{u}-\boldsymbol{\lambda}^*)} = e^{2\pi i\sigma/8}. \tag{4.88}$$

One can freely shift $\mathbf{w}$ by $2\lambda^*$. Consequently, $\mathbf{w}$ is really an equivalence class in $\Lambda^*/2\Lambda^*$.

In Appendix D, we further prove that such a representative $\mathbf{w}$ can always be chosen to lie in the original lattice Λ. We denote such a $\mathbf{w}$ by $\mathbf{w}_0$. The advantage of such a choice can be seen from the expression

$$e^{2\pi i Q_{\mathbf{w}_0}(\mathbf{u})} = e^{\pi i\mathbf{u}^2}e^{\pi i\mathbf{u}\cdot\mathbf{w}_0}$$

the topological twists. Since $\mathbf{w}_0$ now lives in Λ, we have $\mathbf{u}\cdot\mathbf{w}_0 \in \mathbb{Z}$ and $e^{\pi i\mathbf{u}\cdot\mathbf{w}_0} = \pm 1$. This corroborates our intuition that one can salvage the Gauss-Milgram sum in the case of odd matrices by inserting appropriate signs in the sum.

In addition, we can prove that our quadratic function now defines a finite quadratic form because $Q_{\mathbf{w}_0}(n\mathbf{u}) \equiv n^2 Q_{\mathbf{w}_0}(\mathbf{u}) \bmod \mathbb{Z}$. To see why this is true, we use the definition of q:

$$\begin{aligned} Q_{\mathbf{w}_0}(n\mathbf{u}) &= \frac{n^2}{2}\mathbf{u}^2 - \frac{n}{2}\mathbf{u}\cdot\mathbf{w}_0 \\ &\equiv \left(\frac{n^2}{2}\mathbf{u}^2 - \frac{n^2}{2}\mathbf{u}\cdot\mathbf{w}_0\right) \bmod \mathbb{Z}. \end{aligned} \tag{4.89}$$

The second equality follows from the elementary fact that $n^2 \equiv n \,(\mathrm{mod}\ 2)$ together with $\mathbf{u}\cdot\mathbf{w}_0 \in \mathbb{Z}$. Therefore the definition $q([\mathbf{u}]) = Q_{\mathbf{w}_0}(\mathbf{u}) \bmod \mathbb{Z}$ is well-defined.

Having found the discriminant quadratic form $q(\mathbf{u})$, the generalized Gauss-Milgram sum now can be re-interpreted as the ordinary Gauss-Milgram sum of a bosonic Abelian topological phase. As aforementioned, there exists a lifting to an even lattice with the signature $\sigma' \equiv (\sigma - \mathbf{w}_0^2) \bmod 8$ where σ is the signature of the odd matrix K and thus the number of Landau levels we need to add is $n = -\mathbf{w}_0^2 \mod 8$.

Hence, we have the sufficient condition for the existence of an even lattice that is stably equivalent to a given odd lattice: $\sigma' = \sigma$, or $\mathbf{w}_0^2 \equiv 0 \bmod 8$.

An obvious drawback of this discussion is that it is not constructive (which stems from the non-constructive nature of the proof of Nikulin's theorem [66]): we do not know how to construct uniquely the even matrix corresponding to a given discriminant group, quadratic form q, and central charge c. The distinct ways of lifting usually result in lattices with different signatures.

4.7 Novel Chiral Edge Phases of the Conventional Bulk Fermionic $\nu = 8, 12, \frac{8}{15}, \frac{16}{5}$ states

Now that the general framework has been established, in this section we consider a few experimentally relevant examples and their tunneling signatures.

4.7.1 $\nu = 8$

The integer quantum Hall states are the easiest to produce in experiment and are considered to be well understood theoretically. But surprisingly, integer fillings, too, can exhibit edge phase transitions. The smallest integer filling for which this can occur is at $\nu = 8$, because eight is the smallest dimension for which there exist two equivalence classes of unimodular matrices. One class contains the identity matrix, $\mathbb{I}_8$, and the other

contains K_{E_8}, defined in Appendix A.7, which is generated by the roots of the Lie algebra of E_8. K_{E_8} is an even matrix and hence describes a system whose gapless excitations are all bosonic[20, 5] (although if we consider the bosons to be paired fermions, it must contain gapped fermionic excitations.) Yet, counterintuitively, it is stably equivalent to the fermionic $\mathbb{I}_8$; for W_8 defined in Appendix A.7,

$$W_8^T(K_{E_8} \oplus \sigma_z)W_8 = \mathbb{I}_8 \oplus \sigma_z, \tag{4.90}$$

This is an example of the general theory explained in Section 4.6, but it is an extreme case in which both phases have only a single particle type – the trivial particle. The chiral central charges of both phases are equal and so Nikulin's theorem guarantees that the two bulk phases are equivalent (when the bosonic E_8 state is understood to be ultimately built out of electrons) and that there is a corresponding edge phase transition between the two chiral theories.

The action describing the $\mathbb{I}_8$ state with an additional left- and right-moving mode is

$$S = \int dx\, dt \left(\frac{1}{4\pi} \left(\mathbb{I}_8 \oplus \sigma_z\right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} \sum_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right). \tag{4.91}$$

The charge vector is implicitly $t_I = 1$ for all I. As we have shown in previous sections, the basis change $\phi' = W_8\phi$ makes it straightforward to see that if the perturbation

$$S' = \int dx dt u' \cos\left(\phi'_9 \pm \phi'_{10}\right) \tag{4.92}$$

is the only relevant term, then the two modes ϕ'_9 and ϕ'_{10} would be gapped and the system would effectively be described by K_{E_8}.

As in the previous examples, measurements that probe the edge structure can distinguish the two phases of the edge. Consider, first, transport through a QPC that allows

tunneling between the two edges of the Hall bar. In the $\nu = 8$ state with $K = \mathbb{I}_8$, the backscattered current will be proportional to the voltage

$$I^b_{\mathbb{I}_8} \propto V \tag{4.93}$$

because the most relevant backscattering operators, $\cos(\phi^T_I - \phi^B_I)$, correspond to the tunneling of electrons. In contrast, when $K = K_{E_8}$, there is no single-electron backscattering term. Instead, the most relevant operator is the backscattering of charge-$2e$ bosons – i.e. of pairs of electrons – from terms like $\cos(\phi'^T_1 - \phi'^T_4 - \phi'^B_1 + \phi'^B_4)$, which yields different current-voltage relation

$$I^b_{E_8} \propto V^3. \tag{4.94}$$

An alternative probe is given by tunneling into the edge from a metallic lead. In the $K = \mathbb{I}_8$ case, the leading contribution is due to electrons tunneling between the lead and the Hall bar from the terms $\psi^\dagger_{\text{lead}} e^{i\phi^T_I}$, yielding

$$I^{\text{tun}}_{\mathbb{I}_8} \propto V. \tag{4.95}$$

However, in the K_{E_8} case there are no fermionic charge-e operators to couple to the electrons tunneling from the lead. Instead, the leading term must involve two electrons from the lead tunneling together into the Hall bar. The amplitude for this event may be so small that there is no detectable current. If the amplitude is detectable, then we consider two cases: if the quantum Hall state is not spin-polarized or if spin is not conserved (e.g. due to spin-orbit interaction), then the leading contribution to the tunneling current is from terms like $\psi^\dagger_{\text{lead},\downarrow}\psi^\dagger_{\text{lead},\uparrow} e^{i\phi'^T_1 - i\phi'^T_4}$, which represents two electrons of opposite spin

tunneling together into the Hall bar, yielding

$$I_{E_8}^{\text{tun}} \propto V^3. \tag{4.96}$$

If the quantum Hall state is spin-polarized, and tunneling from the lead is spin-conserving, then the pair of electrons that tunnels from the lead must be a spin-polarized p-wave pair, corresponding to a tunneling term like $\psi^\dagger_{\text{lead},\downarrow}\partial\psi^\dagger_{\text{lead},\downarrow}e^{i\phi_1'^T - i\phi_4'^T}$ in the Lagrangian, and we instead expect

$$I_{E_8}^{\text{tun}} \propto V^5. \tag{4.97}$$

Another important distinction between the two edge phases is the minimal value of electric charge in the low-energy sector, which can be probed by a shot-noice measurement [77, 78], as was done in the $\nu = 1/3$ fractional quantum Hall state [79, 80]. The $\mathbb{I}_8$ phase has gapless electrons, so the minimal charge is just the unit charge e. However, the E_8 edge phase is bosonic and consequently the minimal charge is at least $2e$ (i.e. a pair of electrons). (Electrons are gapped and, therefore, do not contribute to transport at low temperatures and voltages.) Quantum shot noise, generated by weak-backscattering at the QPC is proportional to the minimal current-carrying charge and the average current. So we expect a shot-noise measurement can also distinguish the two edge phases unambiguously.

4.7.2 $\nu = 12$

In dimensions-9, -10, and -11, there exist two unique positive definite unimodular lattices, whose K-matrices are (in the usual canonical bases) $\mathbb{I}_{9,10,11}$ or $K_{E_8} \oplus \mathbb{I}_{1,2,3}$. In each dimension, the two lattices, when enlarged by direct sum with σ_z, are related by the similarity transformation of the previous section. However in dimension-12, a new

lattice appears, D_{12}^{+}, defined in Appendix A.7. One salient feature of this matrix is that it has an odd element along the diagonal, but it is not equal to 1, which is a symptom of the fact that there are vectors in this lattice that have odd (length)^2 but none of them have (length)^2=1. The minimum (length)^2 is 2. Upon taking the direct sum with σ_z, the resulting matrix is equivalent to $\mathbb{I}_{12} \oplus \sigma_z$ – and hence to $K_{E_8} \oplus \mathbb{I}_4 \oplus \sigma_z$ using the transformation of the previous section – by the relation $W_{12}^T (K_{D_{12}^+} \oplus \sigma_z) W_{12} = \mathbb{I}_{12} \oplus \sigma_z$, where W_{12} is defined in Appendix A.7.

Consider the action of the $\nu = 12$ state with two additional counter propagating gapless modes and with the implicit charge vector $t_I = 1$:

$$S = \int dx\, dt \left(\frac{1}{4\pi} \left(\mathbb{I}_{12} \oplus \sigma_z \right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} \sum_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right). \quad (4.98)$$

The matrix W_{12} suggests a natural basis change $\phi' = W_{12}\phi$ in which the perturbation

$$S' = \int dx dt u' \cos\left(\phi'_9 \pm \phi'_{10} \right) \quad (4.99)$$

can open a gap, leaving behind an effective theory described by $K_{D_{12}^+}$.

It is difficult to distinguish the $\mathbb{I}_{12}$ edge phase from the $E_8 \oplus \mathbb{I}_4$ phase because both phases have charge-e fermions with scaling dimension-1/2. However, both of these edge phases can be distinguished from the D_{12}^+ phase in the manner described for the $\nu = 8$ phases in the previous subsection. At a QPC, the most relevant backscattering terms will have scaling dimension 1; one example is the term $\cos(\phi_{11}'^T - \phi_{11}'^B)$, which yields the current-voltage relation

$$I^b_{D_{12}^+} \propto V^3. \quad (4.100)$$

This is the same as in the E_8 edge phase at $\nu = 8$ because the most-relevant backscattering operator is a charge-$2e$ bosonic operator with scaling dimension 2. There is a charge-e

fermionic operator $\exp(i(\phi_2'^T + 2\phi_{12}'^T))$, but it has scaling dimension $3/2$. Its contribution to the backscattered current is $\propto V^5$, which is sub-leading compared to the contribution above, although its bare coefficient may be larger. However, if we couple the edge to a metallic lead via $\psi^\dagger_{\text{lead}} \exp(i(\phi_2'^T + 2\phi_{12}'^T))$, single-electron tunneling is the dominant contribution for a spin-polarized edge, yielding

$$I^{\text{tun}}_{D^+_{12}} \propto V^3, \tag{4.101}$$

while pair tunneling via the coupling $\psi^\dagger_{\text{lead}} \partial \psi^\dagger_{\text{lead}} e^{i\phi_{11}'^T}$ gives a sub-leading contribution $\propto V^5$. If the edge is spin-unpolarized, pair tunneling via the coupling $\psi^\dagger_{\text{lead},\uparrow} \psi^\dagger_{\text{lead},\downarrow} e^{i\phi_{11}'^T}$ gives a contribution with the same V dependence as single-electron tunneling.

4.7.3 Fractional Quantum Hall States with Multiple Edge Phases

In Section 4.3, we discussed the $\nu = 8/7$ state, which has two possible edge phases. Our second fermionic fractional quantum Hall example is

$$K_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \tag{4.102}$$

with $t = (1,1)^T$. We again assume that a pair of gapped modes interacts with these two modes, and we assume that they are modes of oppositely-charged particles (e.g. holes), so that $t = (1,1,-1,-1)^T$. Upon enlarging by σ_z, we find that $K_1 \oplus \sigma_z = W^T (K_2 \oplus \sigma_z) W$, where

$$K_2 = \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix} \tag{4.103}$$

and

$$W = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 8 & 0 & 3 \end{pmatrix}. \tag{4.104}$$

If the following perturbation is relevant, it gaps out a pair of modes:

$$S' = \int dx\, dt\, u' \cos(-3\phi_1 - 5\phi_2 + \phi_3 + 3\phi_4). \tag{4.105}$$

Under the basis change (4.104), $-3\phi_1 - 5\phi_2 + \phi_3 + 3\phi_4 = \phi'_3 + \phi'_4$, so the remaining theory has K-matrix (4.103).

In the K_1 edge phase (4.102), the backscattered current at a QPC is dominated by the tunneling term $\cos(\phi_2^T - \phi_2^B)$, which yields

$$I_1^b \propto V^{-3/5}, \tag{4.106}$$

while the tunneling current from a metallic lead is dominated by the single-electron tunneling term $\psi_{\text{lead}}^\dagger e^{3i\phi_1^T}$, which yields

$$I_1^{\text{tun}} \propto V^3. \tag{4.107}$$

In the K_2 edge phase (4.103), the backscattered current at a QPC is dominated by the tunneling term $\cos(\phi_2'^T - \phi_2'^B)$, yielding

$$I_1^b \propto V^{-11/15}, \tag{4.108}$$

while the tunneling current from a metallic lead is dominated by the pair-tunneling term

$\psi^\dagger_{\text{lead}}\partial\psi^\dagger_{\text{lead}}e^{i\phi_1'^T-7i\phi_2'^T}$, which assumes a spin-polarized edge, and yields

$$I_2^{\text{tun}} \propto V^{11}. \tag{4.109}$$

As we discussed in Section 4.6, the $\nu = 16/5$ state can have two possible edge phases, one with

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \tag{4.110}$$

which is essentially the edge of the $\nu = 1/5$ state, together with 3 integer quantum Hall edges. The other possible phase has

$$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \tag{4.111}$$

Upon enlarging by a pair of gapped modes, the two matrices are related by $K_1 \oplus \sigma_z = W^T(K_2 \oplus \sigma_z)W$, where

$$W = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ -1 & 1 & 0 & -4 & 0 & 2 \\ 1 & -1 & 1 & 6 & 0 & -3 \\ -1 & 1 & -1 & -8 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 & -2 \\ -1 & 1 & -1 & -10 & 1 & 5 \end{pmatrix} \tag{4.112}$$

If the gapped modes are oppositely charged holes, then the following perturbation carries no charge:

$$S' = \int dxdtu' \cos(-\phi_1 + \phi_2 - \phi_3 - 5\phi_4 + \phi_5 + 3\phi_6) \tag{4.113}$$

If this perturbation is relevant, it will gap out a pair of modes and leave behind an effective theory describe by the K-matrix (4.111),

The two edge phases of the $\nu = 16/5$ state can be distinguished by the voltage dependence of the current backscattered at a quantum point contact and the tunneling current from a metallic lead. In the K_1 edge phase, the backscattered current at a QPC is dominated by the quasiparticle backscattering term $\cos(\phi_4^T - \phi_4^B)$, yielding the current-voltage relation

$$I_1^b \propto V^{-3/5}. \tag{4.114}$$

In the K_2 edge phase, there are several terms that are equally most-relevant, including, for example $\cos(\phi_1'^T - \phi_1'^B)$, which yield the current-voltage relation

$$I_2^b \propto V^{3/5}. \tag{4.115}$$

Meanwhile, in the K_1 edge phase, single-electron tunneling from a metallic lead given by, for example, $\psi_{\text{lead}}^\dagger e^{i\phi_1^T}$, yields the dependence

$$I_1^{\text{tun}} \propto V, \tag{4.116}$$

while in the K_2 edge phase there are only pair-tunneling terms; one such term for a spin-polarized edge is $\psi_{\text{lead}}^\dagger \partial \psi_{\text{lead}}^\dagger e^{i\phi_1'^T + i\phi_4'^T}$, which yields

$$I_2^{\text{tun}} \propto V^5. \tag{4.117}$$

We now consider an example of a bosonic fractional quantum Hall state with $\nu = 12/23$,

$$K_1^b = \begin{pmatrix} 2 & 3 \\ 3 & 16 \end{pmatrix} \tag{4.118}$$

and $t = (1,1)^T$. (This is a natural choice of charge vector for bosonic atoms in a rotating trap. For paired electrons in a magnetic field, it would be more natural to have $t = (2,2)^T$) By a construction similar to the one discussed in the fermionic cases of $\nu = 8, 12, 8/7, 8/15$ and the bosonic integer quantum Hall cases of $\nu = 8, 16$, this state has another edge phase described by

$$K_2^b = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix} \tag{4.119}$$

and $t = (1,-1)^T$. As in the previous cases, the two edge phases can be distinguished by transport through a QPC or tunneling from a metallic lead.

4.8 Some Remarks on Genera of Lattices and Bulk Topological Phases

The focus in this chapter is on the multiple possible gapless edge phases associated with a given bulk topological phase. However, having established that the former correspond to lattices while the latter correspond to genera of lattices (or, possibly, pairs of genera of lattices), we note here that some results on genera of lattices published by Nikulin in Ref. [66] have direct implications for bulk topological phases. We hope to explore these relations more thoroughly in the future.

We begin by noting that the data that determine a genus of lattices is precisely the data that determine a $2+1$-D Abelian topological phase. Recall that the elements of

the discriminant group A of a lattice form the particle content of an Abelian topological phase. We can turn this around by noting that the particle content and fusion rules of any Abelian topological phase can be summarized by an Abelian group A whose elements are the particle types in the theory and whose multiplication rules give the fusion rules of the theory. The fusion rules take the form of the multiplication rules of an Abelian group because only one term can appear on the right-hand-side of the fusion rules in an Abelian topological phase. Meanwhile, specifying the S-matrix for the topological phase is equivalent to giving a bilinear form on the Abelian group A according to $S_{[\mathbf{v}],[\mathbf{v}']} = \frac{1}{\sqrt{|A|}} e^{-2\pi i b([\mathbf{v}],[\mathbf{v}'])}$. A quadratic form q on the Abelian group A determines the topological twist factors or, equivalently, the T-matrix of an Abelian topological phase according to $\theta_{[\mathbf{v}]} = e^{2\pi i q([\mathbf{v}])}$. Finally, the signature of the form, the number of positive and negative eigenvalues r_+ and r_- of the quadratic form q, determines the right and left central charges, according to $c_R = r_+$ and $c_L = r_-$. The chiral central charge $c_- = c_R - c_L$ is given by $c_- = r_+ - r_-$ which, in turn, determines the modular transformation properties of states and, consequently, the partition functions of the bulk theory on closed 3-manifolds (e.g. obtained by cutting a torus out of S^3, performing a Dehn twist, and gluing it back in). The signature is determined (mod 8) by the quadratic form q, according to the Gauss-Milgram sum:

$$\frac{1}{\sqrt{|A|}} \sum_{a \in A} e^{2\pi i q(a)} = e^{2\pi i c_-/8}$$

We now consider Nikulin's Theorem 1.11.3, given in Section 4.5 and also his result

Proposition 1.11.4: *There are at most* 4 *possible values for the signature* (mod 8) *for the quadratic forms associated with a given bilinear form on the discriminant group.*

Theorem 1.11.3 (given in Section 4.5) states that the S-matrix and $r_+ - r_-$ (mod 8)

completely and uniquely determine the T-matrix, up to relabellings of the particles that leave the theory invariant. In Section 4.6 we show constructively that such a T-matrix exists in the fermionic case. Proposition 1.11.4 tells us that, for a given S-matrix, there are at most 4 possible values for the signature $r_+ - r_-$ (mod 8) and, therefore, at most 4 possible T-matrices. One way to interpret this is that the elements of the T-matrix are the square roots of the diagonal elements of the S-matrix; therefore, they can be determined, up to signs from the S-matrix. There are, at most, four consistent ways of doing this, corresponding to, at most, four possible values of the Gauss-Milgram sum.

Then, Theorem 1.10.2, stated in Section 4.6, tells us that the quadratic form defines an even lattice. Thus, to any fermionic Abelian topological phase, we can associate a bosonic Abelian topological phase with the same particle types, fusion rules, and S-matrix. The bosonic phase has a well-defined T-matrix, unlike the fermionic phase. In addition, we have:

Theorem 1.3.1: *Two lattices S_1 and S_2 have isomorphic bilinear forms on their discriminant groups if and only if there exist unimodular lattices L_1, L_2 such that $S_1 \oplus L_1 \cong S_2 \oplus L_2$.*

In other words, two lattices have isomorphic bilinear forms if they are stably equivalent under direct sum with arbitrary unimodular lattices, i.e. if we are allowed to take direct sums with arbitrary direct sums of σ_x, σ_z, 1, and K_{E_8}. One example of this is two lattices in the same genus. They have the same parity, signature, and bilinear form and are stably equivalent under direct sum with σ_x, as required by the theorem. However, we can also consider lattices that are not in the same genus. The example that is relevant to the present discussion is a pair of theories, one of which is fermionic and the other bosonic. They have the same S-matrix but may not have the same chiral central charges.

The theorem tells us that the difference can be made up with unimodular theories. But since σ_x and σ_z do not change the chiral central charge, the unimodular lattices given by the theorem must be hypercubic lattices. (In the fermionic context, the E_8 lattice is σ_z-stably equivalent to the 8-dimensional hypercubic lattice.) In other words, *every fermionic Abelian topological phase is equivalent to a bosonic Abelian topological phase, together with some number of filled Landau levels.*

Finally, we consider Nikulin's Corollary 1.16.3, given in Section 4.5, which states that the genus of a lattice is determined by its parity, signature, and bilinear form on the discriminant group. Recall that the parity of a lattice is even or odd according whether its K-matrix is even or odd. The even case can occur in a purely bosonic system while the odd case necessarily requires "fundamental" fermions, i.e. fermions that braid trivially with respect to all other particles. Therefore, specifying the parity, signature, and bilinear form on an Abelian group A is equivalent to specifying (1) whether or not the phase can occur in a system in which the microscopic constituents are all bosons, (2) the S-matrix, and (3) the chiral central charge. (According to the previous theorem, the T-matrix is determined by the latter two.) This is sufficient to specify any Abelian topological phase. According to Corollary 1.16.3, these quantities specify a genus of lattices. Thus, given any Abelian topological phase, there is an associated genus of lattices. We can take any lattice in this genus, compute the associated K-matrix (in some basis) and define a $\mathrm{U}(1)^{r_+ + r_-}$ Chern-Simons theory. A change of basis of the lattice corresponds to a change of variables in the Chern-Simons theory. Different lattices in the same genus correspond to different equivalent $\mathrm{U}(1)^{r_+ + r_-}$ Chern-Simons theories for the same topological phase. Therefore, it follows from Corollary 1.16.3 that *every Abelian topological phase can be represented as a* $\mathrm{U}(1)^N$ *Chern-Simons theory.*

4.9 Discussion

A theoretical construction of a bulk quantum Hall state typically suggests a particular edge phase, which we will call K_1. The simplest example of this is given by integer quantum Hall states, as we discussed in Sections 4.2 and 4.7. However, there is no reason to believe that the state observed in experiments is in this particular edge phase K_1. This is particularly important because the exponents associated with gapless edge excitations, as measured through quantum point contacts, for instance, are among the few ways to identify the topological order of the state [43, 16]. In fact, such experiments are virtually the only way to probe the state in the absence of interferometry experiments [81, 82, 83, 84, 85, 86, 87] that could measure quasiparticle braiding properties. Thus, given an edge theory K_2 that is deduced from experiments, we need to know if a purely edge phase transition can take the system from K_1 to K_2 – in other words, whether the edge theory K_2 is consistent with the proposed theoretical construction of the bulk state. We would also like to predict, given an edge theory K_2 deduced from experiments, what other edge phases $K_3, K_4, \ldots$ might be reached by tuning parameters at the edge, such as the steepness of the confining potential. In this chapter, we have given answers to these two questions.

The exotic edge phases at $\nu = 8, 12$ discussed in this chapter may be realized in experiments in a number of materials which display the integer quantum Hall effect. These include Si-MOSFETs [88], GaAs heterojunctions and quantum wells (see, e.g. Refs. [89], [90] and references therein), InAs quanutm wells [91], graphene [92], polar $\mathrm{ZnO/Mg_xZn_{1-x}O}$ interfaces [93]. In all of these systems, edge excitations can interact strongly and could be in an E_8 phase at $\nu = 8$ or the D_{12}^+ phase or the $E_8 \oplus \mathbb{I}_4$ phase at $\nu = 12$. To the best of our knowledge, there are no published studies of the detailed properties of edge excitations at these integer quantum Hall states.

The novel edge phase that we have predicted at $\nu = 16/5$ could occur at the $\nu = 3 + 1/5$ state that has been observed[94] in a 31 million cm^2/Vs mobility GaAs quantum well. This edge phase is dramatically different than the edge of the $\nu = 1/5$ Laughlin state weakly-coupled to 3 filled Landau levels. Meanwhile, a $\nu = 8/15$ state could occur in an unbalanced double-layer system (or, possibly, in a single wide quantum well) with $\nu = 1/3$ and $1/5$ fractional quantum Hall states in the two layers. Even if the bulks of the two layers are very weakly-correlated, the edges may interact strongly, thereby leading to the alternative edge phase that we predict. Finally, if an $\nu = 8/7$ state is observed, then, as in the two cases mentioned above, it could have an edge phase without gapless fermionic excitations.

We have focussed on the relationship between the K-matrices of different edge phases of the same bulk. However, in a quantum Hall state, there is also a t-vector, which specifies how the topological phase is coupled to the electromagnetic field. An Abelian topological phase specified by a K-matrix splits into several phases with inequivalent t-vectors. Therefore, two different K-matrices that are stably equivalent may still belong to different phases if the corresponding t-vectors are are not related by the appropriate similarity transformation. However, in all of the examples that we have studied, given a (K, t) pair, and a K' stably equivalent to K, we were always able to find a t' related to t by the appropriate similarity transformation. Said differently, we were always able to find an edge phase transition driven by a charge-conserving perturbation. It would be interesting to see if there are cases in which there is no charge-conserving phase transition between stably-equivalent K, K' so that charge-conservation symmetry presents an obstruction to an edge phase transition between K, K'.

When a bulk topological phase has two different edge phases, one that supports gapless fermionic excitations and one that doesn't, as is the case in the $\nu = 8$ integer quantum Hall state and the fractional states mentioned in the previous paragraph, then a

domain wall at the edge must support a fermionic zero mode. For the sake of concreteness, let us consider the $\nu = 8$ IQH edge. Suppose that the edge of the system lies along the x-axis and the edge is in the conventional phase with $K = \mathbb{I}_8$ for $x < 0$ and the K_{E_8} phase for $x > 0$. The gapless excitations of the edge are fully chiral; let us take their chirality to be such that they are all right-moving. A low-energy fermionic excitation propagating along the edge cannot pass the origin since there are no gapless fermionic excitations in the E_8 phase. But since the edge is chiral, it cannot be reflected either. Therefore, there must be a fermionic zero mode at the origin that absorbs it.

We discussed how the quadratic refinement allows us to relate a given fermionic theory to a bosonic one. One example that we considered in detail related $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$ to $K_2 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$. Both of these states are purely chiral. However, we noted that we are not restricted to relating purely chiral theories; we could have instead considered a transition between the $\nu = 1/7$ Laughlin edge and the non-chiral theory described by $K = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. This transition does not preserve chirality, but the chiral central charges of the two edge theories are the same. It can be shown that there exist regions in parameter space where the non-chiral theory is stable – for example, if the interaction matrix, that we often write as V, is diagonal, then the lowest dimension backscattering operator has dimension equal to 4. Even more tantalizingly, it is also possible to consider the $\nu = 1/3$ Laughlin edge which admits an edge transition to the theory described by $K' = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus \mathbb{I}_{3\times 3}$. The upper left block is simply the conjugate or (-1) times the Cartan matrix for $\mathrm{SU}(3)_1$. About the diagonal V matrix point, the lowest dimension backscattering term is marginal; it would be interesting to know if stable regions exist.

The theory of quadratic refinements implies that any fermionic TQFT can be realized as a bosonic one, together with some filled Landau levels, as we discussed as the end of Sec. 4.8. In particular, it suggests the following picture: a system of fermions forms a weakly-paired state in which the phase of the complex pairing function winds $2N$ times around the Fermi surface. The pairs then condense in a bosonic topological phase. The winding of the pairing function gives the additional central charge (and, if the fermions are charged, the same Hall conductance) as N filled Landau levels. The remarkable result that follows from the theory of quadratic refinements is that all Abelian fermionic topological phases can be realized in this way.

In this chapter, we have focused exclusively on fully chiral states. However, there are many quantum Hall states that are not fully chiral, such as the $\nu = 2/3$ states. The stable edge phases of such states correspond to lattices of indefinite signature. Once again, bulk phases of bosonic systems correspond to genera of lattices while bulk phases of fermionic systems correspond either to genera of lattices or to pairs of genera – one even and one odd. Single-lattice genera are much more common in the indefinite case than in the definite case [65]. If an n-dimensional genus has more than one lattice in it then $4^{[\frac{n}{2}]}d$ is divisible by $k^{\binom{n}{2}}$ for some non-square natural number k satisfying $k \equiv 0$ or $1 \pmod 4$, where d is the determinant of the associated Gram matrix (i.e. the K-matrix). In particular, genera containing multiple equivalence classes of K-matrices must have determinant greater than or equal to 17 if their rank is 2; greater than or equal to 128 if their rank is 3; and $5^{\binom{n}{2}}$ or $2 \cdot 5^{\binom{n}{2}}$ for, respectively, even or odd rank $n \geq 4$.

Quantum Hall states are just one realization of topological phases. Our results apply to other realizations of Abelian topological states as well. In those physical realizations which do not have a conserved U(1) charge (which is electric charge in the quantum Hall case), there will be additional U(1)-violating operators which could tune the edge of a system between different phases.

Although we have, in this chapter, focussed on Abelian quantum Hall states, we believe that non-Abelian states can also have multiple chiral edge phases. This will occur when two different edge conformal field theories with the same chiral central charge are associated with the same modular tensor category of the bulk. The physical mechanism underlying the transitions between different edge phases associated with the same bulk is likely to be the same as the one discussed here. In this general case, we will not be able to use results on lattices and quadratic forms to find such one-to-many bulk-edge correspondances. Finding analogous criteria would be useful for interpreting experiments on the $\nu = 5/2$ fractional quantum Hall state.

Chapter 5

Perfect Metal Phases of One-Dimensional and Anisotropic Higher-Dimensional Systems

5.1 Introduction

Do stable zero-temperature metallic phases exist in one or two dimensions? A system of non-interacting fermions will always be localized at $T = 0$ in dimensions $\mathrm{D} = 1, 2$ in the presence of generic types of impurities [95] [1]. Localization can be avoided if the fermions have sufficiently strong attractive interactions, but then they form a superconductor (SC) rather than a metal [97, 98]. A system of charged bosons is similarly known to have insulating and superconducting phases [99]. Although the critical point between insulating and superconducting phases is metallic in both cases, it is not known in either case whether a stable metallic phase exists. Such a metallic phase of fermions would

[1] With spin-orbit interactions, there can be a metallic phase in 2D, but any time-reversal symmetry-breaking perturbation will lead to localization [96].

necessarily be a non-Fermi liquid since a Fermi liquid becomes localized [100, 101].

In addition, we consider a second, related question: if an infinite array of 1D Luttinger liquids is coupled, is there a completely stable, albeit anisotropic, non-Fermi liquid phase? At the turn of the millennium, it was shown that inter-chain interactions could stabilize "sliding Luttinger liquid phases" against many types of interactions [102, 103, 104, 105, 106, 107, 108, 109]. On physical grounds, one could argue that any other perturbation would be negligibly small and, therefore, would not have any effect until extremely low temperatures were reached. But, as a matter of principle, it is not known whether "sliding Luttinger liquid phases" are actually stable against *all* perturbations. Therefore, as a question of principle, it is not known whether there is a completely stable zero-temperature 1D multi-channel Luttinger liquid phase or an anisotropic 2D phase of coupled Luttinger liquids.

In this chapter, we answer both questions in the affirmative. We show that there are 1D phases of interacting fermions and bosons that are stable against all weak perturbations. These phases do not depend upon a symmetry for their stability, unlike the edges of symmetry-protected topological phases [31, 110, 17, 18]. They are stable not only against all types of disorder, but also against coupling to an external 3D superconductor. Since long-ranged order is impossible in 1D [111, 112, 113], the absence of proximity-induced superconductivity is a reasonable definition of 'non-superconducting'. Due to its extreme stability, we call such a phase a *perfect metal*. If we form an array of perfect metal wires, such an array is a highly-anisotropic 2D non-Fermi liquid metal or Bose metal [114].

These results are based on a relation that we demonstrate between special values of the interaction parameters of a 1D system with N channels of fermions (bosons) and N-dimensional odd (even) unimodular lattices. Vectors in such a lattice correspond to the different possible chiral excitations of the system, and the square of the length of a vector

is twice the scaling dimension of the operator that creates the corresponding excitation. A non-chiral excitation is made of excitations of both chiralities; at special values of the interaction parameters, its scaling dimension is the sum of the scaling dimensions of the two chiral operators. Small changes in the interactions away from these special values mix the two chiralities, thereby causing small changes in the scaling dimensions. Systems that correspond to so-called non-root unimodular lattices with no short vectors are stable to all weak perturbations because all such interactions are irrelevant in the renormalization group sense. The lowest dimension in which such an odd lattice exists is $N = 23$ (the shorter Leech lattice); for even lattices it is $N = 24$ (the Leech lattice) [65].

5.2 Setup

The stable metallic phases that we describe in this chapter are constructed from 1D electronic systems in which the current-current and density-density interactions have been chosen in a particularly novel way. Such phases can be accessed by perturbing the conventional action describing N channels of free fermions in 1D:

$$S_0 = \int dtdx \Big[\psi^\dagger_{R,I} i(\partial_t - v_I \partial_x) \psi_{R,I} + \psi^\dagger_{L,I} i(\partial_t + v_I \partial_x) \psi_{L,I} \Big],$$

where the operator $\psi^\dagger_{R,I}$ ($\psi^\dagger_{L,I}$) creates a right-moving (left-moving) fermion excitation about the Fermi point $k_{F,I}$ ($-k_{F,I}$) in channel $I = 1, \ldots, N$. The velocity of the I^{th} channel of fermions is v_I. The leading quadratic perturbations couple $\Psi^{SC}_{IJ} = \psi_{R,I}\psi_{L,J}$ to an external 3D charge-$2e$ SC at wavevector $k_{F,I} - k_{F,J}$ or the charge-density-wave (CDW) order parameter $\rho^{2k_F}_{IJ} = \psi^\dagger_{R,I}\psi_{L,J}$ to a periodic electric potential at wavevector $k_{F,I} + k_{F,J}$. Both perturbations are relevant at the free fermion fixed point and generically lead to a gapped ground state that explicitly breaks translation invariance and/or charge

conservation.

The leading fermion-fermion interactions are density-density and current-current interactions, parametrized by the symmetric matrix $U_{I,J}$, with $I, J = 1, \ldots, 2N$:

$$S_{\text{int}} = \int dtdx \Big[U_{I,J} \psi^\dagger_{R,I} \psi_{R,I} \psi^\dagger_{R,J} \psi_{R,J} + U_{I+N,J+N} \psi^\dagger_{L,I} \psi_{L,I} \psi^\dagger_{L,J} \psi_{L,J} + 2U_{I,J+N} \psi^\dagger_{R,I} \psi_{R,I} \psi^\dagger_{L,J} \psi_{L,J} \Big], \quad (5.1)$$

where we assume throughout that the interaction is short-ranged. These quartic interactions are marginal at tree level (in Feynman diagrammatic language). This corresponds to a calculation that is zeroth order in the couplings, and is equivalent to power-counting. If the interactions are added to the free fermion action, the scaling dimensions of the quadratic SC and CDW perturbations, and also all higher-body fermion interaction terms will generally change. Generally, attractive density-density interactions drive SC perturbations more relevant, while repulsive interactions favor the CDW instability. Forward-scattering interactions that couple densities of the same chirality mix the collective modes and renormalize their velocities.

5.3 Luttinger Liquids and Non-Root Unimodular Lattices

To study the perfect metal, it is convenient to use the Luttinger liquid formalism. Thus, we introduce a single, chiral boson ϕ_I (ϕ_{I+N}) for each chiral fermion $\psi_{R,I}$ ($\psi_{L,I}$). Our N-channel fermion system can be described by the following bosonic effective action:

$$S = \frac{1}{4\pi} \int dt dx \Big[K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J \Big]. \tag{5.2}$$

K_{IJ} is a $2N \times 2N$ symmetric integer matrix. Density-density and current-current interactions are parameterized by the symmetric, positive semi-definite $2N \times 2N$ matrix $V_{IJ} = v_I \delta_{IJ} + U_{IJ}$ (with $v_I \equiv v_{I-N}$ for $I > N$). In addition, we must supplement the action with a periodicity condition $\phi_I \sim \phi_I + 2\pi m_I$, for $m_I \in \mathbb{Z}$.

The free fermion fixed point is described within this formalism by choosing $K = K_{\text{ferm}} = \mathbb{I}_N \oplus -\mathbb{I}_N$ and $V_{IJ} = v_I \delta_{IJ}$, where $\mathbb{I}_N$ is the $N \times N$ identity matrix. The operators $\psi^\dagger_{I,R} = \frac{1}{\sqrt{2\pi a}} e^{-i\phi_I} \eta_I$ and $\psi^\dagger_{I,L} = \frac{1}{\sqrt{2\pi a}} e^{i\phi_{I+N}} \eta_{I+N}$ create, respectively, right- and left-moving fermions in the I^{th} channel; a is a short-distance cutoff, and the Klein factors η_I satisfy $\eta_J \eta_K = -\eta_K \eta_J$ for $J \neq K$. The density j^0_I and current j^1_I in the I^{th} channel are given by $j^\mu_I = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi_I$ with $\epsilon^{01} = -\epsilon^{10} = 1$.

A system of hard-core bosons can be re-expressed in terms of fermions by a Jordan-Wigner transformation and then be bosonized as above, but with $K_{\text{boson}} = \sigma_x \oplus \sigma_x \oplus \ldots \oplus \sigma_x$, where $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

It is important to observe that there is still some redundancy in the expression for the Luttinger liquid action. The field redefinition $\phi_I = W_{IJ} \tilde{\phi}_J$ preserves the periodicity conditions of the fields so long as $W \in \text{GL}(2N, \mathbb{Z})$. However, this redefinition transforms the action in Eqn. (5.2) into an action of the same form, but with $\tilde{K} = W^T K W$ and $\tilde{V} = W^T V W$.

This seemingly innocuous observation has a surprising consequence. Consider the operator $\cos(m_I \phi_I)$. It is a local operator that can be added to the Hamiltonian if it is bosonic, which is the case when $\frac{1}{2} m_I (K^{-1})_{IJ} m_J$ is an integer. It could, potentially, open a gap if its right and left scaling dimensions are equal, i.e., if $m_I (K^{-1})_{IJ} m_J = 0$. This operator is an irrelevant perturbation if its scaling dimension is greater than two,

in which case, it will not open a gap in the infrared at weak coupling. The operator has scaling dimension $\frac{1}{2}\sum_{I=1}^{2N}(m_I)^2$ when $V = v_I\delta_{IJ}$. Suppose, instead, that $\tilde{K} = W^TKW$ and $\tilde{V} = W^TVW$ are block-diagonal (the former can always be accomplished with a change of basis and the latter will be relaxed later):

$$\tilde{K} = \begin{pmatrix} \tilde{K}_R & 0 \\ 0 & -\tilde{K}_L \end{pmatrix}, \; \tilde{V} = \begin{pmatrix} \tilde{V}_R & 0 \\ 0 & \tilde{V}_L \end{pmatrix}, \tag{5.3}$$

with positive-definite $\tilde{K}_{R,L}$ and $\tilde{V}_{R,L}$. Then, the field redefinition $\phi_I = W_{IJ}\tilde{\phi}_J$ allows us to compute the scaling dimension of $\cos(m_I\phi_I) = \cos(m_IW_{IJ}\tilde{\phi}_J)$: $(\Delta_m^R, \Delta_m^L) = (\frac{1}{2}\tilde{m}^R\tilde{K}_R^{-1}\tilde{m}^R, \frac{1}{2}\tilde{m}^L\tilde{K}_L^{-1}\tilde{m}^L)$, where $\tilde{m}_J = \tilde{m}_J^{R(L)} = m_IW_{IJ}$ for $J = 1, \ldots, N$ ($J = N+1, \ldots, 2N$). If the off-diagonal blocks in V are non-zero, then the total scaling dimension $\Delta_m^R + \Delta_m^L$ will generally change, but the spin $\Delta_m^R - \Delta_m^L$ will remain the same.

To understand how a perfect metal phase could exist, in which all such operators are irrelevant, it is useful to express the above ideas more geometrically. As described in the Supplementary Online Material, we can associate the N-dimensional integral lattices $\tilde{\Gamma}_{R,L}$, with positive-definite inner products, to the matrices $\tilde{K}_{R,L}$. The K-matrices are the Gram matrices of the lattices and basis changes in the lattice transform the K-matrices according to $K \to \tilde{K} = W^TKW$. Since $1 = \det(K_{\text{ferm/boson}}) = \det(\tilde{K}) = \det(\tilde{K}_R)\det(\tilde{K}_L)$, we conclude that $|\det(\tilde{K}_{R,L})| = 1$. Therefore, $\tilde{\Gamma}_{R,L}$ are unimodular lattices. Consequently, the full matrix $\tilde{K} = \tilde{K}_R \oplus -\tilde{K}_L$ is associated with the unimodular lattice $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L$ of signature (N, N). An operator $\cos(\tilde{m}_J\tilde{\phi}_J)$ can be associated with a vector $(\tilde{\mathbf{v}}_R, \tilde{\mathbf{v}}_L) \in \tilde{\Gamma}_R \oplus \tilde{\Gamma}_L$, where $\tilde{\mathbf{v}}_{R,L} = \tilde{m}_I^{R,L}\tilde{\mathbf{f}}_{R,L}^I$ and $\tilde{\mathbf{f}}_{R,L}^I$ are bases for $\tilde{\Gamma}_{R,L}$ satisfying $\tilde{\mathbf{f}}_{R,L}^I \cdot \tilde{\mathbf{f}}_{R,L}^J = (\tilde{f}_{R,L}^I)_a(\tilde{f}_{R,L}^J)_a = (\tilde{K}_{R,L}^{-1})^{IJ}$ with $a = 1, \ldots, N$. The scaling dimension of the operator is $(\Delta_m^R, \Delta_m^L) = \left(\frac{1}{2}|\tilde{\mathbf{v}}_R|^2, \frac{1}{2}|\tilde{\mathbf{v}}_L|^2\right)$ for block-diagonal $\tilde{V}$, as in Eqn. (5.3).

Therefore, if there are no $\tilde{\mathbf{v}}_{R,L} \in \tilde{\Gamma}_{R,L}$ such that $|\tilde{\mathbf{v}}_R|^2 = |\tilde{\mathbf{v}}_L|^2$ and $\frac{1}{2}|\tilde{\mathbf{v}}_R|^2 + \frac{1}{2}|\tilde{\mathbf{v}}_L|^2 \leq 2$,

or, simply, $|\tilde{\mathbf{v}}_R|^2 = |\tilde{\mathbf{v}}_L|^2 \leq 2$, then there are no relevant or marginal spin-0 perturbations of the Luttinger liquid action Eqn. (5.2) with the choice of couplings in Eqn. (5.3). If, moreover, there are no such $\mathbf{v}_{R,L}$, even if $|\tilde{\mathbf{v}}_R|^2 \neq |\tilde{\mathbf{v}}_L|^2$, then there are no marginal or relevant perturbations of any kind. A lattice Γ is called a non-root lattice if all $\mathbf{v} \in \Gamma$ satisfy $|\mathbf{v}|^2 > 2$ (a vector with $|\mathbf{v}|^2 = 2$ is called a root vector). Therefore, we have reduced the problem of finding a metallic state that is stable against all spin-0 perturbations to the problem of finding a non-root unimodular lattice $\tilde{\Gamma}_R$ whose Gram matrix $\tilde{K}_R$ is related to $K = K_{\text{ferm}}$ (for a system composed out of fermions) or $K = K_{\text{boson}}$ (for a system composed out of bosons) according to $\tilde{K}_R \oplus -\tilde{K}_L = W^T K W$ for some $W \in \text{GL}(2N, \mathbb{Z})$ and unimodular $\tilde{K}_L$. This also guarantees the irrelevance of almost all local chiral perturbations, with some exceptions that we discuss further below (even though such perturbations cannot open a gap).

At this point, we make use of two fortuitous mathematical facts. The first is that there is a unique signature (N, N) unimodular lattice of each parity, up to $\text{SO}(N, N)$ rotations acting on the basis vectors, where a lattice is said to have even parity if the norm-squared of all vectors is even and said to have odd parity otherwise [65]. Therefore, any difference between the Gram matrices of two such lattices can only be due to a difference in choice of basis. Consequently, all signature (N, N) unimodular K-matrices of the same parity are $\text{GL}(2N, \mathbb{Z})$-equivalent [115, 116]. In particular, there exists a $W \in \text{GL}(2N, \mathbb{Z})$ such that $W^T K_{\text{fermion/boson}} W = \tilde{K}_R \oplus -\tilde{K}_L$ for any positive-definite odd/even unimodular lattice $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L$ with Gram matrix $\tilde{K}_R \oplus -\tilde{K}_L$. The second fact is that there exist positive-definite unimodular lattices that contain no roots. In fact, for any integer n, there exists an N-dimensional positive-definite unimodular lattice whose shortest vector $|\mathbf{v}|^2 = n$ [13]. The minimal possible dimension N increases with n. For $n = 3$, the minimal $N = 23$ (the shorter Leech lattice), while for $n = 4$, the minimal $N = 24$ (the even Leech lattice). The Gram matrices K_{sL} and K_{L} of these two lattices

are given in the Supplementary Material.

To summarize, there is a unique signature (N, N) unimodular lattice, up to SO(N,N) transformations. All associated signature (N, N) unimodular K-matrices give the same operator spectrum of conformal spins since these are SO(N,N) invariants. However, each unimodular K-matrix gives a different spectrum of scaling dimensions because these are not SO(N,N) invariants. Non-root unimodular lattices are associated with theories with no relevant cosine operators.

5.4 Shorter Leech Liquid

We first consider the case in which $\tilde{K}_R = \tilde{K}_L = K_{sL}$, which we call the *symmetric shorter Leech liquid*. We will call block diagonal $\tilde{V}$, shown in Eqn. (5.3), the *decoupled surface*. On the decoupled surface, the minimum scaling dimension of an operator is 3/2 if it is completely chiral and 3 if it is spin-0. Small changes in $\tilde{V}$ can only change these scaling dimensions slightly, so there is a finite region of parameter space in which all potential gap-generating perturbations are irrelevant. For block diagonal $\tilde{V}$, we can compute the scaling dimensions of various perturbations using the GL$(46, \mathbb{Z})$ transformation W_s, given explicitly in the Supplementary Online Material, that satisfies $W_s^T K_{\text{ferm}} W_s = K_{sL} \oplus -K_{sL}$. Note that there are many possible GL$(46, \mathbb{Z})$ transformations satisfying $W_s^T K_{\text{ferm}} W_s = K_{sL} \oplus -K_{sL}$ and, therefore, many different possible matrices V that lead to the same block diagonal $\tilde{V}$. The W_s that we construct in the Supplementary Online Material is not symmetrical between right- and left-movers, which means that our choice of velocities and interactions is not parity-invariant. Although this facilitated our calculations, it is not essential for any of our conclusions.

Table 5.1 lists the scaling dimensions of the electron creation operators $\psi^\dagger_{R/L,I}$; inter-channel exchange operators $J^\perp_{R/L;I,J} = \psi^\dagger_{R/L,I}\psi_{R/L,J}$; SC and CDW order parameters $\rho^{2k_F}_{IJ}$

and Ψ_{IJ}^{SC}; and quartic inter-channel interactions in the particle-hole channel, $\mathcal{O}_{IJ}^{\text{p.-h.}} \equiv \psi_{L,I}^\dagger \psi_{R,I} \psi_{R,J}^\dagger \psi_{L,J}$, and particle-particle channel, $\mathcal{O}_{IJ}^{\text{p.-p.}} \equiv \psi_{R,I} \psi_{L,I} \psi_{L,J}^\dagger \psi_{R,J}^\dagger$. We have indicated the channel indices at which the minimal scaling dimension is obtained for each operator. Note that the operator $\rho_{IJ}^{2k_F}$ scatters a left-moving fermion in channel J to a right-moving fermion in channel I. As noted in the Table 1 caption, the inter-channel $I = 2, J = 4$ CDW order parameter has lower scaling dimension than in any single other channel. We also see that the most relevant operator is the $2k_F$ charge-density-wave order parameter in channel 5. All of these operators have very high scaling dimensions. The most relevant operator with 4 fermion fields is $\psi_{R,2} \psi_{L,2}^\dagger \psi_{R,4}^\dagger \psi_{L,3}$, with scaling dimension 10. Note that operators of this form destabilize the sliding Luttinger liquid phase in large parts of the phase diagram [108].

The lowest dimension operators are very complicated combinations of the original electrons. From the θ-function for the shorter Leech lattice [65], we can see that there are 4600 fermionic dimension-3/2 operators of each chirality. One simple (in the tilded basis) dimension-3/2 chiral operator is $e^{i\tilde{\phi}_1}$, but this has a very complicated form in terms of fermion operators (given in the Supplementary Online Material) and has total electric charge -201. There are $(4600)^2$ dimension-3 operators. A relatively simple dimension-3 operator (given in the Supplementary Online Material) is a combination of 10 fermion creation and 12 fermion annihilation operators.

There are also dimension-$(1,0)$ and $(0,1)$ fields $\partial\tilde{\phi}_I$. These shift the Fermi momenta. By coupling such operators together, we can change the matrix $\tilde{V}_{IJ}$, which is a marginal deformation of the phase. If such a deformation moves the system off the decoupled surface, it will change the scaling dimensions of cosine operators, but will leave their conformal spins unchanged. On the decoupled surface, there are dimension-$(2,0)$ and $(0,2)$ chiral operators – in fact, 93150 of each [65]. An example is given in the Supplementary Online Material. They are strictly marginal, due to their chirality, and, so long as they

	$\Delta^{\psi}_{R,I}$	$\Delta^{\psi}_{L,I}$	$\Delta^{J_\perp}_{R,IJ}$	$\Delta^{J_\perp}_{L,IJ}$	$\Delta^{2k_F}_{IJ}$	Δ^{SC}_{IJ}	$\Delta^{\text{p.-h.}}_{IJ}$	$\Delta^{\text{p.-p.}}_{IJ}$
s	11/2	17/2	23	13	5	28	17	21
a	9/2	1/2	20	1	5	5	113	5

Table 5.1: The scaling dimensions of various physical operators in the symmetric (s) and asymmetric (a) shorter Leech liquids. The scaling dimensions depend on the channel indices I, J. We have listed the minimal possible scaling dimensions, which are attained by $\psi_{R,4}$, $\psi_{L,3}$; $J^{\perp}_{R;2,4}$, $J^{\perp}_{L;3,4}$; $\rho^{2k_F}_{5,5}$; $\Psi^{\text{SC}}_{2,4}$; $\mathcal{O}^{\text{p.-h.}}_{2,5}$; $\mathcal{O}^{\text{p.-p.}}_{3,4}$ in the symmetric case and $\psi_{R;2}$ $\psi_{L;21}$; $J^{\perp}_{R;2,5}$, $J^{\perp}_{L;21,22}$; $\rho^{2k_F}_{2,21}$; $\Psi^{\text{SC}}_{2,21}$; $\mathcal{O}^{\text{p.-h.}}_{2,5}$; $\mathcal{O}^{\text{p.-p.}}_{2,5}$ in the asymmetric case.

are sufficiently small, they will not make any of the irrelevant operators relevant. Hence, they do not destabilize the shorter Leech liquid, but their coefficients can be non-zero and they can play a role in determining physical properties on the decoupled surface. Off the decoupled surface, such an operator will have scaling dimension $(2+\alpha, \alpha)$ or $(\alpha, 2+\alpha)$ and will, therefore, be irrelevant. These observations also apply to the other perfect metals described in this chapter.

5.5 Asymmetric Shorter Leech Liquid

We now consider the case in which $\tilde{K}_R = K_{sL}$ but $\tilde{K}_L = \mathbb{I}_{23}$, which we call the *asymmetric shorter Leech liquid*. On the decoupled surface, the minimum scaling dimension of a right-moving chiral operator is 3/2, but a left-moving chiral operator can have dimension-1/2. While the minimal dimension of a spin-0 operator is 3, as in the case of the symmetric shorter Leech liquid, there are again harmless strictly marginal operators of dimension-$(3/2, 1/2)$ on the decoupled surface.

On the decoupled surface, we can compute the scaling dimensions of various perturbations using the $\text{GL}(46, \mathbb{Z})$ transformation W_a that satisfies $W_a^T K_{\text{ferm}} W_a = K_{sL} \oplus -\mathbb{I}_{23}$ and is given explicitly in the Supplementary Online Material. They are given in Table 5.1. It is unclear whether the asymmetric shorter Leech liquid can be adiabatically

connected to the symmetric one through a sequence of perfect metal Hamiltonians.

5.6 Region of Stability of Perfect Metals

As we tune the interactions away from the decoupled surface of any perfect metal phase associated with a non-root unimodular lattice, some of the irrelevant perturbations will decrease in scaling dimension and will, eventually, become relevant. The parameter space is too large for us to fully map out the region of stability of either the symmetric or asymmetric shorter Leech liquids. However, as a representative example, consider the one-parameter family of symmetric Leech theories with $\tilde{V}(\lambda) = vM_s^T O_s^T(\lambda) O_s(\lambda) M_s$. Here, v is a velocity scale, $O_s(\lambda)$ is an $\mathrm{SO}(23,23)$ rotation, and M_s is a matrix satisfying $\tilde{V}(0) = M_s^T M_s$; both are given in the Supplementary Online Materials. $\tilde{V}(0)$ is of the form given in Eqn. (5.3) with $\tilde{V}_R = \tilde{V}_L = K_{sL}$. The minimal scaling dimension of a spin-0 operator is $3e^{-2\lambda}$, which becomes relevant at $\lambda \approx 0.203$, where the largest change in an element of $\tilde{V}$ is $1.25v$.

This illustrates that while the perfect metal phase exists only in special regions of the parameter space of $\tilde{V}$, none of the couplings require fine-tuning.

5.7 Discussion

Thus far, we have focused on fermionic systems. However, the same basic strategy applies to bosonic ones as well. The bosonic system associated with the Leech lattice, the lowest dimension non-root even unimodular lattice, is stable against all weak spin-0 perturbations, since their minimal scaling dimension is 4. We will call this phase the Leech liquid. If we consider systems with more channels, then even the minimal dimension chiral perturbations are irrelevant. In 48 dimensions, there are 4 lattices with minimal

norm 6. Moreover, in the $n = 8k$ channel asymmetric fermionic case, it is possible for the right-moving sector to be associated with an even lattice so that all right-moving excitations are bosonic.

While the channel numbers in our construction might seem large, recall that they correspond to the experimentally controllable number of filled spin-polarized sub-bands of a narrow wire.

Perfect metals are described by conformal field theories (CFTs) with no primary operators of low scaling dimension. CFTs with a large gap in the spectrum of operator scaling dimensions must have large central charge, according to Hellerman's inequality $0 < \Delta_1 < (c+\bar{c})/12+3/2\pi$ [117]. This may explain why our phases have a large number of channels. According to the AdS_3/CFT_2 correspondence [118], such CFTs correspond to weakly-curved gravity duals without light BTZ black holes [119].

If we couple a Fermi liquid lead to a point in the middle of a symmetric shorter Leech wire then, on the decoupled surface, the tunneling conductance will be $G_{\text{tun}} \sim T^{10}$ due to the high scaling dimension of electron operators; in an asymmetric shorter Leech wire, it will be Ohmic, $G_{\text{tun}} \sim T^0$, as in a Fermi liquid, due to the left-moving sector. These exponents vary continuously as we move away from the decoupled surface. Other properties are proportional to high powers of T due to the high scaling dimensions of the operators in Table 5.1.

An array of 1D symmetric shorter Leech or Leech liquids forms an anisotropic 2D perfect metal. Since the minimal scaling dimension of any quasiparticle creation operator in each 1D wire is 3/2 (fermions, shorter Leech) or 2 (bosons, Leech), all couplings between wires are irrelevant except for the marginal couplings between densities and currents on the different perfect metal wires. The irrelevance of tunneling operators precludes the possibility of charge transport between wires, but density-density and current-current interactions will enable inter-wire energy transport. Although inter-wire density-density

and current-current interactions can change the dimensions of cosine operators, the latter are highly irrelevant in the limit of decoupled wires, so there is a non-zero range of parameter space within which couplings between cosine operators remain irrelevant.

An array of asymmetric shorter Leech liquids presents an even more interesting possibility. The left-moving channels are chiral Fermi liquids at the decoupled point, and interwire couplings will drive a crossover to a 2D chiral Fermi surface. On the other hand, the right-moving channels are chiral shorter Leech liquids, and inter-wire tunneling operators are irrelevant. Such a system could combine 2D Fermi liquid properties with 1D shorter Leech liquid properties and exhibit interesting non-Fermi liquid behavior.

Chapter 6

Transport in a One-Dimensional Hyperconductor

6.1 Introduction

6.1.1 Goal of this chapter

In this chapter, we study transport in the one-dimensional non-Fermi liquid introduced in Ref. [[120]]. This metallic phase is very different from a Fermi liquid: in addition to anomalous single-electron properties, it is a perfect metal at zero-temperature, with infinite DC conductivity even in the presence of impurities, unlike a Fermi liquid. We call such a material a "hyperconductor," to distinguish it from a superconductor, since a hyperconductor does not have a Meissner effect at zero temperature; its electrical conductivity is finite at any non-zero temperature; and its thermal conductivity diverges as the temperature approaches zero. The goal of this chapter is to compute the temperature and frequency dependence of the electrical and thermal conductivity of a hyperconductor at low temperature. The temperature dependence of the conductivities depends on whether

the Fermi surface is commensurate with the lattice. In the commensurate case, *both* the electrical σ and thermal κ conductivities behave as a power law: $\sigma, \kappa \propto 1/T^{1-2(2-\Delta_X)}$ with the special case $\Delta_X = 2$ occurring along a surface in parameter space. This constitutes a violation of the Wiedemann-Franz "law," which states that the ratio $\kappa/\sigma T$ is constant, and is due to differing relaxation mechanisms of the electrical and thermal currents. In the incommensurate case, there is a range of temperatures over which both σ and κ diverge exponentially, although with differing algebraic prefactors, as $T \to 0$; at the lowest temperatures, $\sigma \propto \kappa/T \propto 1/T^{2-2(2-\Delta_X)}$. The above temperature dependences reflect the non-Fermi liquid physics of this hyperconductor. As a concrete and well-controlled example of transport in a non-Fermi liquid, these results may shine light on general principles regarding non-Fermi liquids and transport in strongly-correlated electron systems.

6.1.2 General remarks about metallic transport

Transport provides one of the most important characterizations of a physical system. It is often said that the DC electrical conductivity is the first property to be measured when a new material is investigated. However, this is usually followed by noting that it is often the last property to be understood, highlighting the subtle nature of transport properties, when compared with thermodynamic ones.[121] This is one of the difficulties involved in understanding metallic states whose low-temperature behavior is not controlled by the Fermi liquid fixed point but by some other fixed point – generally called a 'non-Fermi liquid'. Experimental systems that are candidate non-Fermi liquid metals have primarily been identified by the occurrence of DC conductivity exhibiting unusual temperature dependence. Perhaps the most famous example is the normal state of the cuprate high-temperature superconductors[122, 123] around optimal doping, where the

DC electrical conductivity $\sigma \sim 1/T$ over a large range of temperatures T. It is difficult to construct models that show such behavior; non-Fermi liquids[124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145] (e.g., fermion-gauge field systems) often have more pronounced anomalies in single-particle properties, but more conventional behavior in transport.[146] See Refs. [[147]] and [[148]] for two counterexamples.

The rate at which the conductivity of a metal approaches its zero-temperature value is determined by the available relaxation mechanisms, which are, in turn, reflective of the nature of the zero-temperature metallic state. In a clean Fermi liquid, umklapp scattering provides the leading low-temperature momentum-relaxation mechanism and results in the familiar contribution, $\delta\rho_{xx}(T) \propto T^2$, in spatial dimensions $D > 1$, [149, 150] to the DC electrical resistivity. [1] In 3D, an electron-phonon interaction contributes $\delta\rho_{xx} \propto T^5$ below the Debye temperature, while $\rho_{xx}(T) \propto T$ is found above the Debye temperature.[150] Similar behavior is found for the scattering of electrons by other collective bosonic modes. However, at the lowest temperatures, which is inevitably below the Debye temperature or its analogues for other collective bosonic modes, the resistivity vanishes faster than linearly in almost all theoretical models.

One way to understand this is as follows. In a metal, the resistivity generally vanishes at low temperatures as $\rho \sim 1/\tau_{\rm tr}$, where $\tau_{\rm tr}$ is the decay rate for the current, usually called the transport lifetime. On dimensional grounds, $1/\tau_{\rm tr} \propto (gT^{-\Delta_g})^2 \cdot T$ where g is the coupling constant that dominates the relaxation of the current and Δ_g is its scaling dimension. (For umklapp-dominated relaxation, g is the strength of umklapp scattering process and Δ_g is its scaling dimension, with $\Delta_g = 2 - \Delta_X$ if X is the umklapp scattering operator specified in Eq. 6.24. For disorder-dominated relaxation, g^2 is the variance of the disorder and $2\Delta_g$ is its scaling dimension, with $2\Delta_g = 3 - 2\Delta_X$ if X is the

[1] In $D = 1$, the umklapp scattering may result in a linear dependence upon temperature [[151]].

operator that is coupled to disorder in Eq. 6.27.) If the coupling g is an irrelevant perturbation, $\Delta_g < 0$, (including the case of a marginally irrelevant perturbation) at the zero-temperature metallic fixed point, then the resistivity vanishes faster than linearly with T, which is the usual case. If, on the other hand, g is a relevant or marginally relevant perturbation, $\Delta_g > 0$, then the fixed point is not stable, and the ultimate low-temperature behavior is determined by some other fixed point. Hence, $\rho \propto T$ can only occur in a model that contains a strictly marginal operator, $\Delta_g = 0$, that relaxes the current. This, in turn, implies that an observed $\rho \propto T$ is either an intermediate temperature behavior that does not survive to the lowest of temperatures, as in the case of electron-phonon scattering above the Debye temperature, or it is a consequence of physical processes encapsulated by a strictly marginal operator. See Refs. [[152, 153, 154]] for related scaling arguments.

The 23-channel Luttinger liquid parameter regime that was called the 'asymmetric shorter Leech liquid' in Ref. [[120]] has many such marginal operators. This model is a 1D hyperconductor, in the sense defined above: its electrical and thermal conductivities diverge at zero temperature in the presence of arbitrary (perturbative) electron-electron and disorder-mediated interactions. However, the temperature and frequency dependence of these transport coefficients is interesting because of the presence of these marginal operators. The purpose of this chapter is to explore this dependence.

In the presence of conservation laws, there is an important caveat to the scaling considerations given above.[155, 151, 156, 157, 158, 159, 160] Some theoretical models may have conservation laws that prevent the electrical and/or thermal currents from fully relaxing, thereby leading to infinite conductivities. Some care is required in these cases, since approximate calculations of transport relaxation times $\tau_{\rm tr}$ may give finite answers due to the failure of these approximations to properly account for these conservation laws. An additional complication is that the Fermi momentum k_F and the reciprocal

lattice vectors **G** enter into (pseudo)-momentum conservation for low-energy excitations. As a result, these momentum scales, which are nominally short-distance or ultraviolet scales, may enter into the low-temperature, low-frequency response.[161] Conservation laws, together with these momentum scales, may conspire to modify the simple scaling form $1/\tau_{\rm tr} \propto (gT^{-\Delta_g})^2 \cdot T$ to $1/\tau_{\rm tr} \propto (gT^{-\Delta_g})^2 \cdot T \cdot f(p/T)$, where $f(x)$ is a scaling function that could have, for instance, the asymptotic form $f(x) \sim e^{-x}$ for large x and p is some characteristic momentum (e.g. a combination of the Fermi momentum and reciprocal lattice vectors) that is relevant to the relaxation of the current. One possible consequence is that the Wiedemann-Franz law may be implied by scaling, but need not be realized because of symmetry considerations.

6.1.3 Organization of this chapter

The remainder of this chapter is organized as follows. In Sec. 6.2, we review the construction of the hyperconductor of Ref. [[120]]. In Sec. 6.3, we discuss the relation between conservation laws and dissipative transport with an eye towards the application to the hyperconductor phases. In Sec. 6.4, we calculate the electrical and thermal conductivities of the hyperconductor at both commensurate and incommensurate filling for a pure system with umklapp scattering and a weakly disordered system. The memory matrix formalism provides the calculational tool of this section. We conclude and outline future plans in Sec. 6.5. We include three appendices that provide details for the calculations underlying the results presented in Sec. 6.4.

6.2 Review of the 1D Hyperconductor

In this section, we give a highly condensed review of the derivation of the hyperconductor of Ref. [[120]] in order to establish notation that is used in the remainder of this

chapter. For the most part in this chapter, when we use the term, hyperconductor, we specifically have in mind the example previously called the 1D 'asymmetric shorter Leech liquid,' however, we emphasize that the notion is more general and we are merely studying one particular realization. The reader interested in the details of this construction is directed to Ref. [[120]].

The 1D hyperconductor that is the subject of this chapter obtains from the low-energy effective theory of a particular interacting model of electrons in a 1D quantum wire. We can regard the bands with different values of the transverse momentum, as well as the two spin states of the electron, as separate channels. The simplest example then, and the one we will study in this chapter has $N = 23$ channels of spinless fermions Ψ_I.

At low energies, the non-relativistic fermions can be linearized into a theory of $N = 23$ channels of chiral linearly-dispersing spinless (Dirac) fermions, with a left and a right mover in each channel. Their complete action is given by:

$$S_{\text{lin}} = S_0 + S_{\text{int}} \tag{6.1}$$

$$S_0 = \int_{t,x} \left[\psi^\dagger_{R,I} i(\partial_t + v_I \partial_x) \psi_{R,I} + \psi^\dagger_{L,I} i(\partial_t - v_I \partial_x) \psi_{L,I} \right] \tag{6.2}$$

$$\begin{aligned} S_{\text{int}} = \int_{t,x} \Big[& U_{I,J} \psi^\dagger_{R,I} \psi_{R,I} \psi^\dagger_{R,J} \psi_{R,J} \\ & + U_{I+N,J+N} \psi^\dagger_{L,I} \psi_{L,I} \psi^\dagger_{L,J} \psi_{L,J} \\ & + 2U_{I,J+N} \psi^\dagger_{R,I} \psi_{R,I} \psi^\dagger_{L,J} \psi_{L,J} \Big] \end{aligned} \tag{6.3}$$

where the operator $\psi^\dagger_{R,I}$ ($\psi^\dagger_{L,I}$) creates a right-moving (left-moving) fermion excitation about the Fermi point $k_{F,I}$ ($-k_{F,I}$) in channel $I = 1, \ldots, N$ and we have used the shorthand $\int_{t,x} \equiv \int dt dx$. The velocity of the I^{th} channel of fermions is v_I. It is important to

keep in mind that the linear regime only includes momenta smaller than some cutoff Λ, where $\Lambda \ll k_F$

As the real symmetric matrix $U_{I,J}$ for $I, J = 1, \ldots, 2N$ specifying the density-density interaction is varied, the system explores the parameter space of a 23-channel Luttinger liquid. As discussed in Ref. [[120]], there is an open set of $U_{I,J}$ for which all potentially-gap-opening or potentially-localizing perturbations to Eq. 6.1 are irrelevant; this entire parameter regime is the hyperconductor phase. The calculations of Ref. [[120]] that establish the existence of this phase as well as the following transport calculations rely on the bosonic representation of Eq. (6.1):

$$S_{\rm b} = \frac{1}{4\pi} \int_{t,x} \Big[K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J \Big]. \tag{6.4}$$

with $K = K_{\rm ferm} = -\mathbb{I}_N \oplus \mathbb{I}_N$, $V_{IJ} = v_I \delta_{IJ} + U_{IJ}$, $\mathbb{I}_N$ the $N \times N$ identity matrix, and $I, J = 1, \ldots, 2N$ in Eq. (6.4). The operators $\psi^\dagger_{I,R} = \frac{1}{\sqrt{2\pi a}} e^{i\phi_I} \gamma_I$ and $\psi^\dagger_{I,L} = \frac{1}{\sqrt{2\pi a}} e^{-i\phi_{I+N}} \gamma_{I+N}$ create, respectively, right- and left-moving fermions in the $I^{\rm th}$ channel; a is a short-distance cutoff, and the Klein factors γ_I satisfy $\gamma_J \gamma_K = -\gamma_K \gamma_J$ for $J \neq K$. The bosonic fields satisfy the equal-time commutation relations $[\phi_I(x), \Pi_J(y)] = i\delta_{I,J}\delta(x-y)$, where the canonical momenta $\Pi_I = \frac{1}{2\pi} K_{IJ} \partial_x \phi_J$. (The index on the fields $\Psi_{I,R/L}$ runs from $1, \ldots, N$, while the index on the bosonic fields ϕ_I runs from $1, \ldots 2N$.)

The hyperconductor construction is based on the observation that under an $SL(2N, \mathbb{Z})$ basis change, $\phi_I \equiv W_{IJ} \tilde{\phi}_J$, it is possible to transform K to the Gram matrix $\tilde{K} = W^T K W = -\tilde{K}_R \oplus \tilde{K}_L$ of a signature (N, N) lattice of the form $-\tilde{\Lambda}_R \oplus \tilde{\Lambda}_L$ where $\tilde{\Lambda}_R$, $\tilde{\Lambda}_L$ are positive-definite unimodular[2] N-dimensional lattices. For $N \geq 23$, there exist *non-root* positive-definite unimodular lattices – i.e., lattices such that all vectors $\mathbf{v}$ in the lattice satisfy $|\mathbf{v}|^2 > 2$ – and there exist matrices W that transform $K_{\rm ferm}$ to the corre-

[2]The unimodularity follows from the unimodularity of $K_{\rm ferm} = -\mathbb{I}_N \oplus \mathbb{I}_N$ and the determinant-preserving property of W.

sponding Gram matrices. If, in this basis, $\tilde{V} = W^T V W$ is block diagonal (i.e., does not mix right-movers and left-movers), then all potentially gap-opening or localizing operators $\cos(\tilde{m}_I \tilde{\phi}_I)$ are irrelevant when $\tilde{\Lambda}_R$ *or* $\tilde{\Lambda}_L$ is non-root, where $\tilde{m}_J = m_I W_{IJ}$. Stability persists for a small but finite range of values of any parameters in the model (i.e., away from block diagonal $\tilde{V}$), including the chemical potentials in each channel, the velocities, and all the inter-channel and inter-spin interactions. In the hyperconductor phase considered in this chapter, $\tilde{\Lambda}_R$ is the so-called shorter Leech lattice, the unique non-root unimodular integral lattice in 23 dimensions, while $\tilde{\Lambda}_L$ is $\mathbb{Z}^{23}$, the ordinary hypercubic lattice, which is *not* a non-root lattice. This phase was called the *asymmetric shorter Leech liquid*. (See Refs. [162, 8] for a fuller discussion of the mathematical technology underlying the hyperconductor construction.)

For simplicity, we perform the calculations in this chapter using an interaction matrix $\tilde{V}_{IJ}$ in the transformed basis that is simply proportional to the positive-defined matrix $\tilde{K}_R \oplus \tilde{K}_L$, so that all of the eigenmodes have equal velocities v. We similarly assume, for simplicity, that $k_{F,I} = k_F$ for all I.

The salient feature of the asymmetric shorter Leech hyperconductor that is relevant to this chapter is the existence of a large number of marginal backscattering operators of the form $\cos\left(\tilde{m}_I \tilde{\phi}_I\right)$ when $\tilde{V} = W^T V W$ is block diagonal and $\tilde{\Lambda}_R$ and $\tilde{\Lambda}_L$ are, respectively, the shorter Leech lattice and $\mathbb{Z}^{23}$. In conformal field theory[163] (CFT) terminology, these operators have different right and left scaling dimensions $(\Delta_R, \Delta_L) = \left(\frac{3}{2}, \frac{1}{2}\right)$. If $\tilde{V}$ is moved slightly away from block diagonal, then the scaling dimensions of any such operator will be shifted to $(\Delta_R, \Delta_L) = \left(\frac{3}{2} + y, \frac{1}{2} + y\right)$, where y will depend on the particular operator in question. For block diagonal $\tilde{V}$, these scaling dimensions are protected by their chirality: their RG equations do not contain higher-order terms.[164] (See Appendix E.4 for a review of this argument.) As a result, transport coefficients exhibit anomalous power-law dependence all the way to zero temperature. For block diagonal $\tilde{V}$, this is

manifested as DC electrical resistivity $\rho_{\rm DC} \propto T$ all the way to zero temperature.

6.3 Symmetry and Transport

In this section, we describe some of the complications associated with computing the transport properties of a 23-channel Luttinger liquid. Most of the material in this section has been described elsewhere (see below for references) but, for the sake of completeness, we give a review of transport that is tailored to the application of the formalism described in the next section. The reader that is interested primarily in our results may wish to skip this rather technical section on a first reading of this chapter.

6.3.1 Conservation Laws

The conservation of total electrical charge and total energy,

$$\frac{d}{dt}Q = \frac{d}{dt}H = 0, \tag{6.5}$$

(where Q and H are the total electrical charge and energy operators) make it possible for those quantities to diffuse, thereby leading to finite electrical and thermal conductivities. If, however, the charge or energy *currents*, respectively $\mathbf{J}^e$ or $\mathbf{J}^T$, were conserved,

$$\frac{d}{dt}\mathbf{J}^e = 0 \quad \text{or} \quad \frac{d}{dt}\mathbf{J}^T = 0, \tag{6.6}$$

then the electrical or thermal conductivity would be infinite. Even if the charge and energy currents were not themselves conserved, the electrical or thermal conductivity would still be infinite, if there were some other conserved quantities with non-zero 'overlap' (in a sense to be made precise in Eq. (6.29)) with the charge or energy current. Hence,

finite conductivities only occur when the corresponding currents have no overlap with any conserved quantities.[165, 166, 156]

In addition to total charge and energy there are other globally conserved quantities (we will interchangeable call them charges) for the fixed point action of a hyperconductor in Eq. (6.4). There are 47 conservation laws at the asymmetric shorter Leech fixed point that are important for transport: the charges of the right- and left-movers in each channel as well as the total energy.[3] We now discuss these conservation laws, as well as the relaxation mechanisms due to irrelevant perturbations of the fixed point that are required to make these conductivities finite.

Continuous translation symmetry of the parent non-relativistic theory, whose low-energy effects are captured by S_{lin}, gives a globally conserved charge (total momentum), here written in fermionic language:

$$P = P_0 + P_D, \tag{6.7}$$

$$P_0 = k_F \sum_I \left(N_I^R - N_I^L \right), \tag{6.8}$$

$$P_D = \int_x \left[\psi_{R,I}^\dagger (i\partial_x \psi_{R,I}) + \psi_{L,I}^\dagger (i\partial_x \psi_{L,I}) \right], \tag{6.9}$$

where N_I^R, N_I^L are, respectively, the number operators of the right-moving and left-moving Dirac fermions in channel I:

$$N_I^{R,L} = \int_x \psi_{R/L,I}^\dagger \psi_{R/L,I}. \tag{6.10}$$

P_D, as suggestively named, is the momentum of a Dirac fermion theory also described

[3] In fact, there are an infinite number of conserved charges of the Luttinger liquid action describing the hyperconductor fixed point which take the form of products of the chiral current operators defined Eq. (6.15). These additional charges have vanishing overlap with the chiral currents and momentum operator to lowest order in the scattering interaction, and so make subleading contributions to the conductivity and will be ignored.

by S_{lin}, but where $\psi^\dagger_{R,I}$ $\left(\psi^\dagger_{L,I}\right)$ creates a right-moving (left-moving) fermion about zero momentum instead of the Fermi point $k_{F,I}\,(-k_{F,I})$. From the perspective of the low-energy theory, the total momentum operator P arises from two separately conserved emergent symmetries of S_{lin}: the first is generated by a chiral rotation of the right- and left-moving fermions by the "angle" k_F while the second is generated by continuous translations in the linearized Dirac theory. P_0 accounts for the large momenta $\sim k_F$, while P_D accounts for deviations from the Fermi surface.

These expressions can be rewritten in bosonic form:

$$N^R_I = \frac{1}{2\pi}\int_x \partial_x \phi_I, \tag{6.11}$$

$$N^L_I = \frac{1}{2\pi}\int_x \partial_x \phi_{N+I}, \tag{6.12}$$

and

$$P_D = \frac{1}{4\pi}\int_x K_{IJ}\partial_x\phi_I\partial_x\phi_J. \tag{6.13}$$

The fermionic and bosonic expressions for $P = P_0 + P_D$ are the integrals over all space of the component T^{tx} of the energy-momentum tensor derived via Noether's theorem from, respectively, the fermionic Eq. (6.1) and bosonic Eq. (6.4) forms of the effective action.

The fixed point action S_{b} has emergent $U(1)^N_L \times U(1)^N_R$ chiral symmetries ($\phi_I \to \phi_I + c_I$) generated by the charges $Q^{R/L}_I$:

$$Q^{R,L}_I = eN^{R/L}_I. \tag{6.14}$$

The continuity equation for each chiral charge and the equations of motion for the bosonic

fields allow us to obtain the corresponding currents:

$$J^e_{R,I} = \frac{e}{2\pi} V_{IJ} \int_x \partial_x \phi_J, \tag{6.15}$$

$$J^e_{L,I} = -\frac{e}{2\pi} V_{N+I,J} \int_x \partial_x \phi_J. \tag{6.16}$$

The total electrical and thermal currents are then given by:

$$J^e = \sum_{I=1}^{N} \left(J^e_{R,I} + J^e_{L,I} \right), \tag{6.17}$$

$$J^T = -\frac{1}{4\pi} \sum_{I,J,L=1}^{2N} V_{IJ} K_{II} V_{IL} \int_x \partial_x \phi_J \partial_x \phi_L, \tag{6.18}$$

where the Hamiltonian,

$$H = \frac{1}{4\pi} \int_x V_{IJ} \partial_x \phi_I \partial_x \phi_J, \tag{6.19}$$

and corresponding thermal continuity equation gives J^T. We study the case when all of the eigenvalues of V_{IJ} are the same, so that the Dirac momentum P_D is equal to the thermal current J^T.

Particle-hole symmetry breaking band-curvature effects couple the electrical and thermal currents to one another. For completeness, we give, in fermionic form, the corresponding corrections to the expressions for the currents:

$$\delta J^e = g \frac{e}{m} P_D, \tag{6.20}$$

$$\begin{aligned} \delta J^T = \frac{g}{m} \sum_I \int_x \Big[& \left(\partial_t \psi^\dagger_{R,I} \right) \partial_x \psi_{R,I} + \left(\partial_x \psi^\dagger_{R,I} \right) \partial_t \psi_{R,I} \\ & + \left(\partial_t \psi^\dagger_{L,I} \right) \partial_x \psi_{L,I} + \left(\partial_x \psi^\dagger_{L,I} \right) \partial_t \psi_{L,I} \Big]. \end{aligned} \tag{6.21}$$

In an operator formalism, the time derivative of the fermion operator above is computed by taking the commutator of the fermion operator with the Hamiltonian H. If the fermions have quadratic dispersion, so that there are no higher-order corrections to these expressions for the currents, the action is Galilean-invariant. The band curvature corrected electrical current then gives the expected relation between the total electrical current and total momentum, $J^e + \delta J^e = \frac{e}{m} P$. Band curvature effects that do not break particle-hole symmetry introduce corrections to J^e that are odd in the ϕ_I and corrections to J^T that are even in the ϕ_I. These and other corrections due to band curvature are interesting and deserve further study (see Ref. [167] for a review), however, we focus upon the linearly dispersing regime in this chapter.

To summarize, the fixed point action S_{b} has 47 individually conserved quantities, $Q_I^{R,L}$ and P_D, that generally have non-zero overlap with the electrical and thermal currents. One linear combination of these conserved quantities, the total electrical charge $Q = \sum(Q_I^{R,} + Q_I^L)$, will always[4] remain conserved, but it has no overlap with either the electrical or thermal currents and so it does not prevent their decay. The other 46 conservation laws must be broken in order for the system to have finite electrical and thermal conductivities.

6.3.2 Relaxation Mechanisms

To see the relation between the conductivity and conservation laws, it is helpful to consider the most general expression for the real part of the optical conductivity:[159]

$$\sigma'(\omega, T) = 2\pi D(T)\delta(\omega) + \sigma_{\text{reg}}(\omega, T), \tag{6.22}$$

[4]Assuming that the system is not coupled to an external superconductor to violate charge conservation or driven to violate energy conservation.

where $D(T)$ is the so-called Drude weight. If $D(T)$ is finite, it signals that the DC conductivity is infinite. Using Mazur's inequality,[165, 166] Zotos, Naef, and Prelovsek pointed out in Ref. [156] the following implication of conserved charges for electrical charge transport:

$$D(T) \geq \frac{1}{2LT} \frac{\sum_k \langle J^e Q_k \rangle^2}{\langle Q_k^2 \rangle}, \tag{6.23}$$

where L is the length of the system. The angled brackets denote the thermodynamic average and the right-hand side of Eq. (6.23) is independent of time because the Q_k are conserved quantities. This inequality says that in the presence of conserved charges Q_k which have non-zero overlap with J^e, the electrical current does not completely relax, and the system has dissipationless charge flow even at finite temperature T. (See Eq. (6.29) for an equivalent notion of an 'overlap' which is the one that we adopt in this chapter.) A similar inequality and conclusion applies for the thermal current J^T.

It follows that to fully relax the electrical and thermal currents a system must break all conservation laws, apart from the conservation of total charge and total energy, which have vanishing overlap with the electrical and thermal currents. At zero-temperature and zero frequency, the fixed point theory S_{b} determines the response of the system. Since this theory has the 47 conservation laws described above, it has infinite conductivity. Note that, in a time-reversal invariant 23-channel Luttinger liquid, we would only need to break 24 conservation laws since the time-reversal symmetric conserved quantities would ordinarily have vanishing overlap with the electrical current; but the asymmetric Leech liquid hyperconductor is not time-reversal invariant.

At finite temperature and frequency, irrelevant perturbations can have an effect on the response functions of the system. The bulk of this chapter is a discussion of the effects of such perturbations. In particular, we answer two questions: Which operators

can relax the currents? Which are the most important ones?

In order to break the conservation of the Dirac momentum P_D and the chiral electrical currents $\{J^e_{R/L,I}\}$, we need to include physical processes that (1) break continuous translation symmetry with respect to the low-energy effective theory $S_{\rm b}$ and (2) break particle number conservation within each channel, but (3) conserve total charge and energy. Umklapp scattering at incommensurate fillings and disorder break continuous momentum conservation and generally break the conservation of the chiral currents in individual channels, and so we focus on them here.

Umklapp processes scatter some number of right-movers into left-movers so that the total momentum change is a reciprocal lattice vector. The most general umklapp term is specified by a vector of integers $m^{(\alpha)}_I, I = 1, \ldots, 2N$:

$$\begin{aligned} H_{\rm u} &= \sum_\alpha H^{\rm u}_\alpha \\ &= \sum_\alpha \left[h^{\rm u}_\alpha + \text{h.c.} \right] \\ &= -\sum_\alpha \lambda_\alpha \int_x \left[\frac{1}{a^2} e^{i m^{(\alpha)}_I k_{F,I} x - i p^{(\alpha)} G x} e^{i m^{(\alpha)}_J \phi_J} + \text{h.c.} \right], \end{aligned} \tag{6.24}$$

where λ_α is the coupling constant, G is a basis vector of the reciprocal lattice, a is a short-distance cutoff,[5] and the Einstein summation convention is employed. Here, the operator X to which we referred in our general remarks in Sec. 6.1.2 is $X = e^{i m^{(\alpha)}_J \phi_J}$. The most important umklapp processes at low energies are those for which the corresponding operators $X = e^{i m^{(\alpha)}_J \phi_J}$ have the lowest scaling dimension. In the asymmetric shorter

[5]Typically, the presence of a multiplicative prefactor proportional to a power of the short-distance cutoff a is understood when writing vertex operators of the form, $\exp\left(i m^{(\alpha)}_J\right)$. We retain it here when writing vertex operators of scaling dimension equal to 2 to avoid confusion. See Ref. [[163]] for further details.

Leech hyperconductor studied in this chapter, such operators have scaling dimension $(\Delta_R, \Delta_L) = (3/2, 1/2)$, so they are marginal. The integer $p^{(\alpha)}$ is the "order" of the umklapp process, or the number of Brillouin zone foldings after which the momentum $m_I^{(\alpha)} k_{F,I}$ is again in the first Brillouin zone. Thus, $p^{(\alpha)}$ is actually fixed by $m_I^{(\alpha)} k_{F,I}$, but we will retain it as a formally free parameter. At commensurate filling, there is always a $p^{(\alpha)}$ such that $m_I^{(\alpha)} k_{F,I} = p^{(\alpha)} G$, but we work more generally. Without loss of generality, we may take the difference $m_I^{(\alpha)} k_{F,I} - p^{(\alpha)} G \in [0, 2\pi)$ where the lattice constant has been set to unity. Charge conservation is maintained by requiring equal numbers of creation and annihilation operators: $\sum_{I=1}^{N} m_I^{(\alpha)} = \sum_{I=1}^{N} m_{N+I}^{(\alpha)}$.

While any single umklapp process H_α^{u} might break the conservation of individual currents (e.g., $[H_\alpha^{\mathrm{u}}, J_{R/L,I}^e] \neq 0$), a linear combination of currents might still be conserved.[158] (The linear combination corresponding to total charge is always conserved, however, it has no overlap with the total electrical current.). That is why our model generally requires at least 46 carefully chosen umklapp processes, i.e., $m_I^{(\alpha)}$ vectors to break all conservation laws. Such a requirement is not unreasonable. In the spirit of effective field theory, we expect all operators consistent with symmetry to be present in the low-energy effective action. We simply focus on the minimal set of scattering processes that dominate the low-energy physics. See the accompanying Mathematica file for explicit expressions of the $m_I^{(\alpha)}$ that we choose to study.

To study whether some linear combination (other than the total charge) $a_I J_I$ with $J_I = J_{R,I}^e$ for $I = 1, \ldots N$ and $J_I^e = J_{L,I-N}^e$ for $I = N+1, \ldots, 2N$ is also conserved, we compute the equal-time commutators:

$$[H_\alpha^{\mathrm{u}}, a_I J_I^e] = i a_I b_I^\alpha h_\alpha^{\mathrm{u}} + \mathrm{h.c.}, \tag{6.25}$$

where the vectors b_I^α are defined by,

$$b_I^\alpha = \Big(e\lambda_\alpha \mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IJ}\Big)m_J^{(\alpha)}, \tag{6.26}$$

and we define $\mathrm{sgn}(X) = +1$ for $X \geq 0$ and $\mathrm{sgn}(X) = -1$ for $X < 0$. We ask whether there exist solutions $a_I = \mathbf{a} \in \mathbb{R}^{2N} - \{\mathbf{0}\}$, such that $\forall \alpha, a_I b_I^\alpha = 0$. All umklapp operators preserve total $U(1)$ electrical charge, therefore the vectors $m_I^{(\alpha)}$ specifying them can span at most a $2N-1$ dimensional space. The linear equations, $a_I b_I^\alpha = 0$, say that $\mathbf{a}$ is orthogonal to this space. It follows that when the number of linearly independent umklapp terms N_U ($\alpha = 1, \ldots, N_U$) equals $2N-1$, $\mathbf{a}$ lies in the 1-dimensional space corresponding to total charge, and so no non-trivial conserved linear combination of the currents exists.

Disorder can also relax the electrical and thermal currents by violating conservation laws. A generic disorder-mediated backscattering term takes the form:

$$\begin{aligned} H_{\mathrm{dis}} &= \sum_\alpha \lambda_\alpha^{\mathrm{dis}} H_\alpha^{\mathrm{dis}} \\ &= \sum_\alpha \lambda_\alpha^{\mathrm{dis}} \int_x \Big[\xi_\alpha(x)\frac{1}{a^2}e^{im_I^{(\alpha)}\phi_I} + \mathrm{h.c.}\Big], \end{aligned} \tag{6.27}$$

where α indexes the various backscattering terms specified by $m_I^{(\alpha)} \in \mathbb{Z}$. At low temperatures, the most important backscattering processes are again due to the dimension $(\frac{3}{2}, \frac{1}{2})$ operators $e^{im_I^{(\alpha)}\phi_I}$ introduced in Eq. (6.24). However, due to randomness in $\xi_\alpha(x)$, their effect is weaker than that of uniform umklapp terms. (In the general remarks in Sec. 6.1.2, the operator $X = e^{im_I^{(\alpha)}\phi_I}$ in Eq. (6.27).)

For simplicity, we will take all the couplings $\lambda_\alpha^{\mathrm{dis}} = \lambda^{\mathrm{dis}}$ equal and $\overline{\xi_\alpha(x)\xi_\beta^*(x')} = \delta_{\alpha\beta}D\delta(x-x')$ with $\overline{\xi_\alpha}(x) = 0$, where the overline denotes disorder averaging. Then, we use the replica trick to integrate out the disorder, thereby obtaining the following term

in the replicated action:

$$S_{\text{dis}-\text{avg}} = (\lambda^{\text{dis}})^2 D \sum_{A,B} \sum_{\alpha} \int_{t,t'} \int_{x} \frac{1}{a^4} e^{i m_I^{(\alpha)} (\phi_I^A(t) - \phi_I^B(t'))}. \tag{6.28}$$

For a dimension $(\frac{3}{2}, \frac{1}{2})$ operator $e^{i m_I^{(\alpha)} \phi_I}$, the coupling $(\lambda^{\text{dis}})^2 D$ of the interaction in the replicated theory has scaling dimension equal to -1. Hence, the interaction is irrelevant and its effects are formally subleading compared to the uniform umklapp terms considered above. However, in the commensurate case, umklapp terms commute with P_D; disorder is the leading effect that violates conservation of P_D, thereby leading to finite thermal conductivity. Meanwhile, in the incommensurate case, the effects of uniform umklapp terms are exponentially-suppressed at low temperatures, and disorder becomes the leading effect that relaxes both electrical and thermal currents at low temperatures.

In summary: for a pure system at commensurate filling, the Dirac momentum P_D is not relaxed, however, there is no overlap between the chiral electrical currents J_I^e and P_D when particle-hole symmetry is preserved. Thus, we need 45 umklapp operators to relax the electrical current. When particle-hole symmetry is broken by band-curvature corrections at commensurate filling, $\langle J^e P_D \rangle \neq 0$, so both the electrical and thermal conductivities diverge. When the filling is incommensurate or disorder is present, particle-hole symmetry is broken, so there is generally an overlap between the electrical currents and the Dirac momentum. However, P_D does not generally commute with an umklapp process at incommensurate filling or a disorder-mediated scattering interaction, thereby allowing momentum relaxation. In this case, both the electrical and thermal transport coefficients can be finite in the presence of 46 scattering interactions. The additional interaction arises from the additional conserved charge P_D. To see this one must generalize the previous argument by writing the commutator in Eq. (6.25) as a total derivative.

6.3.3 Memory Matrix

The details of the memory matrix formalism can be found in Refs. [[168, 169, 158, 170, 171]]; we merely observe that it is well-suited for computing transport coefficients in the hydrodynamic regime: when there are globally conserved quantities (energy, electrical charge) that propagate diffusively. Unlike a direct application of the Kubo formulae it makes the role of these conservation laws transparent. In essence, it is a reorganization of the perturbative expansion of the current-current correlation functions of interest.[159]

We choose as a complete basis of conserved quantities the set

$$\{\mathcal{Q}_p\} = \{J^e_{R,1}, ...J^e_{R,N}, J^e_{L,1}, ...J^e_{L,N-1}, P_D\}$$

. $J^e_{L,N}$ can be excluded because total charge is always conserved, so a correlation function involving J^L_N can be obtained from an expression involving the other currents. There is a notion of a symmetric inner product on the vector space of conserved quantities provided by the static susceptibility matrix:

$$\begin{aligned}\hat{\chi}_{pq} =& (\mathcal{Q}_p|\mathcal{Q}_q) \\ \equiv& \frac{1}{L} G^R_{\mathcal{Q}_p\mathcal{Q}_q}(\omega = 0).\end{aligned} \tag{6.29}$$

The retarded Green's functions $G^R_{\mathcal{Q}_p\mathcal{Q}_q}(\omega)$ are calculated at temperature T (left implicit in the definitions below) and evaluated at real frequency ω. (Recall that there is no momentum dependence in the static susceptibility matrix $\hat{\chi}_{pq}$ because the conserved charges are obtained by integrating densities over all space.) Thus, the static susceptibility may be used to define the notion of an 'overlap' between two conserved quantities. Note that the real-time thermodynamic correlation functions involved in Mazur's inequality Eq. (6.23) are non-zero if and only if the corresponding static susceptibilities are non-zero.

The memory matrix $\hat{M}(\omega)$ has contributions from each separate umklapp and disorder-mediated scattering process, both labeled by α. We schematically write this as:

$$\hat{M}(\omega) = \sum_{\alpha} \Big(\lambda_\alpha^2 \hat{\mathcal{M}}_\alpha^{\mathrm{u}}(\omega) + (\lambda_\alpha^{\mathrm{dis}})^2 D \hat{\mathcal{M}}_\alpha^{\mathrm{dis}}(\omega) \Big), \tag{6.30}$$

$$(\hat{\mathcal{M}}^{\mathrm{u}})_\alpha^{pq} = \frac{1}{L} \frac{\langle F_{p,\alpha}^{\mathrm{u}}; F_{q,\alpha}^{\mathrm{u}} \rangle_\omega - \langle F_{p,\alpha}^{\mathrm{u}}; F_{q,\alpha}^{\mathrm{u}} \rangle_{\omega=0}}{i\omega}, \tag{6.31}$$

$$(\hat{\mathcal{M}}^{\mathrm{dis}})_\alpha^{pq} = \frac{1}{L} \frac{\langle F_{p,\alpha}^{\mathrm{dis}}; F_{q,\alpha}^{\mathrm{dis}} \rangle_\omega - \langle F_{p,\alpha}^{\mathrm{dis}}; F_{q,\alpha}^{\mathrm{dis}} \rangle_{\omega=0}}{i\omega}. \tag{6.32}$$

Here, $F_{q,\alpha}^{\mathrm{u}} = \frac{i}{\lambda_\alpha} [H_\alpha^{\mathrm{u}}, \mathcal{Q}_q]$, $F_{q,\alpha}^{\mathrm{dis}} = \frac{i}{\lambda_\alpha^{\mathrm{dis}} \sqrt{D}} \left[H_\alpha^{\mathrm{dis}}, \mathcal{Q}_q \right]$, and $\mathcal{Q}_q$ is a conserved charge (either $J_{R/L,I}^{\mathrm{e}}$ or P_D). $\langle F_{p,\alpha}^{\mathrm{u}}; F_{q,\alpha}^{\mathrm{u}} \rangle_\omega$ and $\langle F_{p,\alpha}^{\mathrm{dis}}; F_{q,\alpha}^{\mathrm{dis}} \rangle_\omega$ are retarded finite-temperature Green's functions evaluated to leading order using S_{b} in Eq. (6.4). λ_α and $\lambda_\alpha^{\mathrm{dis}}$ parameterize the umklapp scattering and coupling to disorder, respectively, and D is the disorder variance of Gaussian-correlated disorder. As mentioned above, we take $\lambda_\alpha = \lambda$ and $\lambda_\alpha^{\mathrm{dis}} = \lambda^{\mathrm{dis}}$ for all α for simplicity. $\hat{\mathcal{M}}^{\mathrm{u}}$ contains the contributions to the memory matrix from umklapp scattering, while $\hat{\mathcal{M}}^{\mathrm{dis}}$ contains the contributions from the disorder-mediated interaction. We stress that the form of the memory matrix given above is correct to leading order in the scattering interaction. See Refs. [[168, 169, 158, 170, 171]] for further discussion.

The label α also specifies the momentum mismatch of an incommensurate scattering process,

$$\Delta k_\alpha \equiv m_I^{(\alpha)} k_{F,I} - p^{(\alpha)} G \in [0, 2\pi), \tag{6.33}$$

for unit lattice constant, and the vector of integers $m_I^{(\alpha)}$ that defines the umklapp process. The vectors $m_I^{(\alpha)}$, in turn, help determine, along with the matrix V_{IJ}, the right and left scaling dimensions (Δ_R, Δ_L) of the operators entering scattering interactions in Eqs. (6.24) and (6.27). Recall that we choose to take the Fermi vectors in all channels to be

equal, $k_{F,I} = k_F$.

The conductivities associated to the various charges $\mathcal{Q}_p$ are encoded in the matrix,

$$\hat{\sigma}(\omega) = \hat{\chi}\left(\hat{N} + \hat{M}(\omega) - i\omega\hat{\chi}\right)^{-1}\hat{\chi}, \tag{6.34}$$

where

$$(\hat{N})_{pq} \equiv \hat{\chi}_{p\dot{q}} = \Big(\mathcal{Q}_p, i[\sum_{\alpha}(H^{\mathrm{u}}_{\alpha} + H^{\mathrm{dis}}_{\alpha}), \mathcal{Q}_q]\Big). \tag{6.35}$$

We show in Appendix E.3 that, at least to quadratic order in the umklapp λ and disorder λ^{dis} couplings, $\hat{N} = 0$.

The electrical conductivity σ is determined by the $(2N-1)\times(2N-1)$ submatrix $\hat{\sigma}_{J^e_I, J^e_J}$. The thermoelectric conductivity $\tilde{\alpha}$ is determined by the $(2N-1)$-dimensional vector $\hat{\sigma}_{J^e_I, P_D}/T$. The thermal conductivity $\kappa = \frac{\hat{\sigma}_{P_D,P_D}}{T} - \frac{\tilde{\alpha}^2 T}{\sigma}$. For commensurate fillings and in the disorder-dominated regime, the thermoelectric conductivity can be ignored to leading order so that the thermal conductivity is equal to the $P_D - P_D$ component of $\hat{\sigma}$.

6.4 Hyperconductor Transport

We now assemble the conductivity matrix $\hat{\sigma}$. The first ingredient is the static susceptibility matrix, which takes the following form:

$$\hat{\chi}_{J^e_I J^e_J} = \frac{e^2}{4\pi}\mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IJ}, \tag{6.36}$$

$$\hat{\chi}_{J^e_I P_D} = 0, \tag{6.37}$$

$$\hat{\chi}_{P_D P_D} = \frac{N\pi^2 T^2}{6}, \tag{6.38}$$

where there is no sum over I and J and we have computed to zeroth order in any perturbation to $S_{\rm b}$. See Appendix E.1 for details on the calculation of the static susceptibilty matrix and the auxiliary Mathematica file for the explicit expression for V_{IJ}. See Appendix E.2 for details on the evaluation of the memory matrix elements.

In the following two sections, we study the contributions to the conductivity in systems at commensurate and incommensurate fillings in the presence of both umklapp scattering and disorder. For the most part, we focus upon the decoupled surface subspace within the hyperconductor phase, however, we provide the more general expressions for the DC conductivities where appropriate.

6.4.1 Commensurate Fillings

If the electron filling is commensurate with the lattice, k_F divided by the reciprocal lattice basis vector is a rational fraction, and so the momentum mismatch Δk_α in any umklapp scattering process may vanish. Umklapp scattering interactions with $\Delta k_\alpha = 0$ provide the dominant contribution to the electrical conductivity matrix. Thus, we consider $S_{\rm b}$ together with 45 umklapp terms, all with $\Delta k_p^{(\alpha)} = 0$. As argued earlier, the most important umklapps are those with total scaling dimension $(\Delta_R, \Delta_L) = (3/2, 1/2)$.

DC Conductivity

We first note that $F^{\rm u}_{P_D,\alpha}$ vanishes when $\Delta k^{(\alpha)} = 0$, along with all the memory matrix elements involving it. This tells us that the dynamics of the electrical current-carrying excitations decouple from the thermal carriers (with P_D remaining conserved) at commensurate fillings without disorder. In computing the electrical conductivity, it is sufficient to choose $\{J^e_I\}$ as the complete basis of hydrodynamic modes. The conservation of P_D in the linearly-dispersing regime also implies that the thermal conductivity κ is

infinite in a pure system since $\left(P_D|J^T\right) \neq 0$. At commensurate fillings, disorder is the leading effect that causes finite thermal conductivity, as we discuss.

To obtain the DC conductivity at commensurate fillings, we need the memory matrix elements obtained in Appendix E.2.2:

$$(\hat{\mathcal{M}}^{\mathrm{u}})_{\alpha}^{J_I^e J_J^e}(T) = \frac{\pi^4}{32} U_{J_I^e,\alpha} U_{J_J^e,\alpha} T, \tag{6.39}$$

where the finite, non-zero coefficients, $U_{J_I^e,\alpha}U_{J_J^e,\alpha} \propto e^2$ are defined in Eq. (E.20). This immediately gives the DC electrical conductivity,

$$\sigma(T) \propto \frac{e^2}{\lambda^2}\frac{1}{T}. \tag{6.40}$$

As promised, the electrical resistivity vanishes linearly in temperature. Note that the dimensionless proportionality constants in Eq. (6.40) and in subsequent conductivity formulas are finite and non-zero.[6]

We have neglected band curvature terms in the preceding and subsequent calculations by working with the linearized action in Eq. (6.4). Their inclusion does not lead to finite thermal conductivity since any non-oscillatory term will commute with P_D. However, particle-hole symmetry-breaking band curvature terms will mix P_D and J_I^e, thereby leading to infinite electrical conductivity so long as P_D is conserved.

Disorder, on the other hand, does cause P_D to decay. While it gives a subleading contribution to the electrical conductivity in the commensurate case – disorder contributes the $\mathcal{O}(T^2)$ correction in Eq. (6.50) to the DC electrical memory matrix elements – it is

[6]In general, inversion of the 46×46 memory matrix is computational difficult and so a precise determination of the overall numerical constant prefactors is currently out of reach. Nevertheless, we have checked that the memory matrix is generically non-singular and so we may safely understand the contributions to the relaxation of the various currents by scaling out any dimensionful quantities from the memory matrix. The remaining numerical matrix then merely contributes a finite constant whose overall value we do not determine.

the leading contribution to the relaxation rate of the thermal conductivity:

$$\kappa(T) \propto \left(\frac{1}{D(\lambda^{\rm dis})^2}\right)\frac{1}{T}, \tag{6.41}$$

where we have used the static susceptibility matrix in Eq. (6.38), the disorder memory matrix elements in Eq. (6.52), and the fact that κT is equal to the $P_D - P_D$ component of the conductivity tensor $\hat{\sigma}$ when the thermoelectric coefficient vanishes (to leading order).

Eqs. (6.40) and (6.41) constitute a violation of the Wiedemann-Franz "law." Marginal umklapp scattering is the leading low-temperature relaxation mechanism for the electrical current, while $\mathcal{O}(1)$ irrelevant disorder is the leading relaxation mechanism for the thermal current at commensurate fillings. In this case, the Lorentz ratio,

$$\mathcal{L} = \frac{\kappa}{\sigma T} \propto \frac{\lambda^2}{e^2 D(\lambda^{\rm dis})^2}\frac{1}{T} \tag{6.42}$$

diverges as $T \to 0$.

Remaining within the hyperconductor phase, but departing from the decoupled surface, the exponents for the electrical and thermal conductivities will generally be modified to the form: $\sigma \propto 1/T^{1-2(2-\Delta_X)}$ and $\kappa \propto 1/T^{1-2(2-\Delta_X)}$, where deviations of Δ_X from 2 encode the shift of the scaling dimensions of the scattering processes away from marginality.

AC Conductivity

The AC conductivities at commensurate fillings are found similarly. From Appendix E.2.2,

$$(\hat{\mathcal{M}}^{\rm u})_{\alpha}^{J_I^e J_J^e}(\omega) = U_{J_I^e,\alpha} U_{J_J^e,\alpha}\Big[\frac{\pi^2}{32}\omega + i\frac{\pi}{16}\omega\log(a_2\omega)\Big], \tag{6.43}$$

where a_2 is proportional to the short-distance cutoff a. Therefore, the AC electrical conductivity at $T \ll \omega$ takes the form:

$$\sigma(\omega) \propto \frac{e^2}{i\omega\Big(c_1 + c_2 \log(a_2\omega)\Big) + c_3\omega}, \tag{6.44}$$

for constants c_1, c_2 and c_3. The finite contribution to the real part of the electrical AC resistivity has given the Drude peak finite width.

Disorder is required for finite AC thermal conductivity. Using the memory matrix element in Eq. (6.52), we find:

$$\kappa(T/\omega \ll 1) \propto \frac{T^3}{ic_4\omega T^2 + c_5 D\omega^4}, \tag{6.45}$$

for constants c_4 and c_5.

6.4.2 Incommensurate Fillings

When the filling is incommensurate, there is no scattering process for which $\Delta k_\alpha = 0$. In this case both the electrical and thermal conductivities are generally finite and so we use the charge basis $\{\mathcal{Q}_p\} = \{J^e_{R,1}, ...J^e_{R,N}, J^e_{L,1}, ...J^e_{L,N-1}, P_D\}$. Band-curvature corrections contribute subleading terms to the temperature dependence and will not be considered.

The Δk_α associated to the 46 umklapp scattering processes defined by the $m_I^{(\alpha)}$ vectors are all generally different from one another. Nevertheless, we set $\Delta k_\alpha = \Delta k$ for all α in the presentation of the results below.

DC Conductivity

The memory matrix elements for umklapp scattering at incommensurate filling is provided in Appendix E.2.2 whose results we quote below.

Infinitesimally close to commensurate filling, $\omega \leq \Delta k \ll T$, we may borrow our previous results computed precisely at commensurate filling with the understanding that $\Delta k \neq 0$ in the expression for $F^{\mathrm{u}}_{P_D,\alpha}$ in Eq. (E.16). The leading contribution to the electrical conductivity is unchanged from Eq. (6.40). However, the thermal conductivity is now finite even in the absence of disorder,

$$\kappa(T) \propto \frac{T^2}{\lambda^2 \Delta k^2}. \tag{6.46}$$

As expected, the thermal conductivity is divergent as commensurability is restored, $\Delta k \to 0$. The Lorentz ratio is a decreasing function of T^2 in the regime $\Delta k \ll T$ as the temperature is decreased.

As the temperature is lowered, we enter the regime $T \ll \Delta k$ in which the DC electrical and thermal memory matrix elements take the asymptotic low-temperature form:

$$(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_{\alpha}(T) = \frac{\pi^2}{32} U_{p,\alpha} U_{q,\alpha} \frac{\Delta k^2}{T} e^{-\frac{\Delta k}{2T}}. \tag{6.47}$$

The resulting DC electrical and thermal conductivities for $T \ll \Delta k$:

$$\begin{aligned} \sigma(T) &\propto \frac{e^2}{\lambda^2} \frac{T}{\Delta k^2} e^{\frac{\Delta k}{2T}}, \\ \kappa(T) &\propto \frac{1}{\lambda^2} \frac{T^4}{\Delta k^4} e^{\frac{\Delta k}{2T}}. \end{aligned} \tag{6.48}$$

In this case, the Lorentz ratio,

$$\mathcal{L} \propto \frac{T^2}{e^2 \Delta k^2}, \tag{6.49}$$

vanishes as $T \to 0$ in the absence of disorder. If we had considered instead a more generic model in which the Fermi momenta were not identical, the Δk would then no longer be same. This would imply that the leading contribution to the memory matrix in Eq. (6.47) would be dominated by the contribution with minimal Δk.

Disorder, if present, eventually dominates the low-temperature transport. The disorder DC electrical and thermal memory matrix elements derived in Appendix E.2.3:

$$(\hat{\mathcal{M}}^{\text{dis}})_{\alpha}^{J_I^e J_I^e} = \frac{2\pi^3}{3} \tilde{U}_{J_I^e,\alpha} \tilde{U}_{J_J^e,\alpha} T^2, \tag{6.50}$$

$$(\hat{\mathcal{M}}^{\text{dis}})_{\alpha}^{J_I^e P_D} = 0, \tag{6.51}$$

$$(\hat{\mathcal{M}}^{\text{dis}})_{\alpha}^{P_D P_D} = \frac{8\pi^5}{5} \tilde{U}_{P_D,\alpha} \tilde{U}_{P_D,\alpha} T^4, \tag{6.52}$$

where the coefficients $\tilde{U}_{p,\alpha}\tilde{U}_{q,\alpha}$ are defined in Eqs. (E.28). For generic, perturbative values of the couplings, the disorder-dominated regime occurs when the exponentially-vanishing contribution to the memory matrix in Eq. (6.47) is overcome by the disorder-dominated contribution above. The resulting electrical and thermal conductivities in the presence of disorder for temperatures $T \ll \Delta k$:

$$\begin{aligned} \sigma(T) &\propto \frac{e^2}{D(\lambda^{\text{dis}})^2} \frac{1}{T^2}, \\ \kappa(T) &\propto \frac{1}{D(\lambda^{\text{dis}})^2} \frac{1}{T}. \end{aligned} \tag{6.53}$$

Away from the decoupled surface, the low-temperature results will be modified as follows:

$\sigma = \kappa/T \propto 1/T^{2-2(2-\Delta_X)}$. In this regime, the Lorentz ratio,

$$\mathcal{L} \propto \frac{1}{e^2}, \tag{6.54}$$

is constant, although the gapless metallic phase is certainly *not* a Fermi liquid. The Wiedemann-Franz law is satisfied at the lowest of temperatures for incommensurate fillings because disorder is the dominant relaxation mechanism at incommensurate fillings for *both* the electrical and thermal currents.

AC Conductivity

The AC conductivity at incommensurate filling follows straightforwardly from the previous analysis. For $T \leq \Delta k \ll \omega$, the AC electrical conductivity is unchanged from the previous result in Eq. (6.44). In fact, the real part of the AC electrical resistivities can be found from inversion of the DC electrical conductivities in Sec. 6.4.2 by the replacement $T \to \omega$ in all algebraic prefactors and so we shall not write them out explicitly.

Let us now concentrate on the real part of the AC thermal conductivities. For $T \ll \Delta k \ll \omega$,

$$\kappa(\omega) \propto \frac{1}{\lambda^2}\frac{T^3}{\Delta k^2 \omega}. \tag{6.55}$$

For $T < \omega \ll \Delta k$ with $T \ll (\Delta k^2/\omega)\exp(\frac{\omega-\Delta k}{2T})$ and in the absence of disorder the thermal conductivity is dominated by incommensurate umklapp scattering,

$$\kappa(\omega) \propto \frac{1}{\lambda^2}\frac{T^3\omega}{\Delta k^4} e^{\frac{\Delta k-\omega}{2T}}, \tag{6.56}$$

where we used Eq. (E.27). Notice the divergent thermal conductivity as $T \to 0$. Finally,

in the disorder-dominated regime with $T^2 \ll D\omega^3$,

$$\kappa(\omega) \propto \frac{1}{D}\frac{T^3}{\omega^4}. \tag{6.57}$$

6.5 Conclusions

In this chapter, we have determined the DC and AC electrical and thermal conductivity of the one-dimensional hyperconductor phase introduced in Ref. [[120]] in the presence of umklapp and disorder-mediated scattering. For instance, we have shown that this metallic phase exhibits a DC conductivity $\sigma \sim 1/T^{1-2(2-\Delta_X))}$ down to $T = 0$ without fine-tuning at commensurate fillings, thereby manifesting the non-Fermi liquid nature of the phase. In addition, we have discussed the relation between conservation laws and transport which has allowed us to provide examples of violations of the Wiedemann-Franz law. As a simple example, the thermal conductivity is only finite in the presence of disorder, while the electrical conductivity can be finite in a pure system at commensurate filling with only umklapp scattering. More generally, we have seen how differing relaxation mechanisms for the electrical and thermal currents can result in violations of the Wiedemann-Franz law.

The power-law $\sigma \sim 1/T$ obtains along the 'decoupled surface' of the hyperconductor when the interactions determined by $\tilde{V}_{IJ}$ – see Sec. 6.2 – are block diagonal at commensurate fillings. On this surface, $\Delta_X = 2$. The hyperconductor phase survives within a finite window off the decoupled surface by the addition of off-diagonal terms to $\tilde{V}_{IJ}$ mixing right-moving and left-moving hyperconductor excitations. Departing from the decoupled surface, but remaining within the hyperconductor phase, the relaxation of the current is controlled by 46 umklapp scattering operators with conformal dimensions $\left(\frac{3}{2}+\delta, \frac{1}{2}+\delta\right)$ so that $\Delta_X = 2 + 2\delta$, with δ determined by the distance from the decou-

pled surface. The conductivity will generally behave $\sigma \sim 1/T^{1-2(2-\Delta_X)}$ with $\Delta_X > 2$ down to $T = 0$. For $\Delta_X < 2$, the zero-temperature perfect metal fixed point is unstable. However, the relevant perturbations are chiral and, therefore, cannot open a gap. At low temperatures, they may strongly renormalize the velocities, shift the Fermi momenta, or otherwise modify the ground state (without opening a gap) in such a manner that the dangerous processes can no longer occur. In the marginal case, $\Delta_X = 2$, such an instability presumably occurs at sufficiently large marginal coupling.

The large marginal coupling limit of this hyperconductor regime is an interesting testing ground for Hartnoll's recently conjectured[172] lower bound on the diffusion constant, $D \geq \hbar v_F^2/(k_B T)$. This bound applies to systems in the "incoherent" metallic regime where there is no overlap between the electrical current and momentum operator. If satisfied, this lower bound implies an upper bound on the coefficient of the linear in temperature DC electrical resistivity that we found at commensurate fillings.

The distinction between a hyperconductor and a superconductor is that a hyperconductor does not have long-ranged order.[7] This distinction is not apparent in zero-temperature electrical transport, which is infinite in both cases. (It does manifest itself in the differential tunneling conductance, which vanishes algebraically with voltage in the hyperconductor but is strongly suppressed at voltages below the energy gap in a superconductor – it would be zero but for Andreev reflection.) However, the difference between a hyperconductor and a superconductor is clearer in low-temperature transport. In a superconductor, the electrical resistivity vanishes for all temperatures below the critical temperature, but in a hyperconductor, the resistivity increases smoothly, with the temperature dependence described above. In the incommensurate case, the resitivity is exponentially-small in temperature over a wide range of temperatures, has the feature

[7] In the case of a 1D system, long-ranged order is impossible. However, a 1D superconductor develops long-ranged order when in contact with a 3D superconductor, while a 1D hyperconductor does not. It resists the development of a proximity effect due to weak coupling to a 3D superconductor.

of very small (albeit not vanishing) resistivity without the threat of a sudden large jump at a critical temperature. While a superconductor conducts electrical current without dissipation even in the presence of impurities for $T < T_c$, a hyperconductor has non-zero resistivity for $T > 0$, but strongly suppressed – in the hyperconductor studied here, the impurity contribution is suppressed by a factor $(T/T_F)^{2\Delta_X - 2}$ with $\Delta_X \geq 2$. Meanwhile, a hyperconductor has radically different thermal transport than a superconductor. In a superconductor, thermal currents are only carried by excited quasiparticles and phonons. Therefore, the thermal conductivity divided by the temperature vanishes with decreasing temperature. In particular, the electronic contribution to the thermal conductivity of an s-wave superconductor has activated form. In a hyperconductor, on the other hand, the thermal conductivity diverges as a power-law at the lowest temperatures and diverges exponentially with inverse temperature over a wide range of temperatures. Thus, the hyperconductor phase, though neither a superconductor nor a superfluid, has an electrical conductivity that approaches that of the former and a thermal conductivity that approaches that of the latter.

In the future, we plan to understand the 2D metallic phase that emerges from an array of hyperconductor wires. This wire array should exhibit diffusive finite-temperature transport both along and transverse to the wires and be stable to weak disorder. this chapter makes clear the reason why finite conductivities obtain along the wires. To understand the latter two statements, we need only observe that such an array forms a sort of 'chiral transverse Fermi liquid' in the sense that only half of the Fermi surface excitations can hop between wires at the lowest of energies, reminiscent of the chiral metals studied in Refs. [[173, 174, 175]] (see Ref. [[176]] for related work). In these works [173, 174, 175], it was found that a collection of wires, each hosting a chiral Fermi liquid (obtained from the edge excitations of a collection of integer quantum Hall systems layered in a transverse direction), exhibits diffusive transport transverse to the wires

and does not localize. One important difference between these constructions and the 2D hyperconductor is the diffusive, as opposed to ballistic, finite-temperature transport exhibited by the hyperconductor along the wires.

Chapter 7

Future Directions

The most interesting basic question that is important to answer is whether Perfect Metals can exist in wires with fewer channels. Here we have some bounds already. For example, from Theorem 2.0.8 no unimodular lattice can have minimal norm larger than 2 in less than 10 dimensions, and larger than 4 in less than 24 dimensions. That suggests that a 12 channel wire ($2 \times 12 = 24$) may support a Perfect Metal phase in which absolutely all interactions are irrelevant at weak coupling.

We can weaken the condition a bit and require than only mass-generating perturbations are irrelevant. This translates to the condition that an operator have equal left and right scaling dimensions (conformal spin zero). In that case 10 channels may be enough to force one of the two sectors to have norms bigger than 2. However, the effect of relevant spin non-zero operators in the IR is unclear and concerning. Even if they perturbatively their couplings do not enter into the β functions of mass generating couplings, we can nevertheless never be sure because they will grow strong and we won't be able to compute scaling dimensions in a strongly interacting theory.

A related direction of investigation is whether non-chiral edge phases can be stabilized at boundaries of 2+1 topological phases. Chapter 4 is an investigation into the

possible chiral phases. The work of Haldane[52] discusses asymmetric non-chiral phases which cannot be simplified to a chiral theory by locking counter-propagating modes. It effectively reduces to the inexistence of null-vectors in the indefinite integral lattice associated with the edge theory. Levin[51] considers non-chiral edges with equal numbers of right and left-movers and only asks whether it can be gapped out in principle, with arbitrary interactions. Both these considerations are of a topological flavor. Instead what is unclear is whether one can construct an non-chiral edge theory (with signature $(n, 1)$, for example) which is dynamically stable in the sense that the mass-generating operators (or all) are irrelevant.

We know that no integral unimodular lattices with no roots exist in less than 23-dimensions, so we must expand the search space beyond these. The Perfect Metal construction rotates the $I_{n,n}$ hypercubic into an indefinite lattice that decomposes into two definite integral unimodular ones. It is interesting to know whether an interesting indecomposable lattices can be reached from $I_{n,n}$.

Alternatively it is important to understand whether an indefinite hypercubic lattice of signature (n, m) can ever be transformed, through $\mathrm{SO}(n, m)$, into a decomposable lattice that is made of unimodular but not integral lattices, or even non-unimodular individually but only together.

One can observe that the rotating to $\Gamma_8 \oplus -\Gamma_8$ produces a theory in which all cosine operators are at most marginal. That however requires fine-tuning. We investigated whether there are infinitesimal $\mathrm{SO}(8, 8)$ rotations that deform the lattice so as to increase all the scaling dimensions, but the answer is no.

While the Perfect Metals require no fine-tuning, we do not know what fraction of the parameter space is in a PM phase. What is clear is that in the shorter Leech case there are infinitely many choices for the $W \in \mathrm{SL}(46, \mathbb{Z})$ that give a PM phase, and they generically correspond to different values of the final charge vector. As a result these

phases are truly distinct. Since each occupies a finite volume in parameter space, it is plausible that a significant fraction of it is in some PM phase. Note that the parameter space is actually not compact because $\mathrm{SO}(23, 23)$ is not.

In the context of quantum Hall edges, the story of the $\nu = 2/3$ edge (Ref [177]) raises another set of interesting questions: can disorder, which will generically be there, play a stabilizing role and drive the system into each of the allowed chiral phases.

A natural follow-up to the work of Chapter 4 is to ask whether non-Abelian 2+1 topological phases admit multiple chiral edge phases, and how to classify them. This line of reasoning is particularly exciting because it may provide much more accessible experimental signatures for identifying the correct bulk phase description from the many existing.

Appendix A

Lattices and Matrices

In this appendix, we collect formulas for the various lattice vectors and matrices we use throughout the main text.

To fix some notation, consider the standard basis for $\mathbf{R}^N$,

$$\hat{x}_I = \begin{pmatrix} 0 \cdots 0 & 1 & 0 \cdots 0 \end{pmatrix}^t, \tag{A.1}$$

where the 1 appears in the I-th row for $I = 1, ..., N$. The root lattice Γ_G of any rank N Lie group G is defined in terms of linear combinations of the $\hat{x}_I$. Given a basis $\mathbf{e}_I$ for the lattice, we may construct the Cartan matrix or K-matrix, $(K_G)_{IJ} = e^a_I \eta_{ab} e^b_J$ where η is the diagonal matrix $\mathrm{diag}(\mathbf{1}^M, -\mathbf{1}^{N-M})$ and $\mathbf{1}^P$ is the P-component vector with every entry equal to unity. The Cartan matrix summarizes the minimal data needed to specify a Lie group. Geometrically, a diagonal entry $(K_G)_{II}$ is equal to the length-squared of the root I and an off-diagonal entry $(K_G)_{IJ}$ gives the dot product between roots I and J and so can be interpreted as being proportional to the cosine of the angle (in $\mathbf{R}^N$) between the two roots. Given the inverse $(K_G^{-1})^{IJ}$, we may define dual lattice vectors $f^I_a = (K_G^{-1})^{IJ} \eta_{ab} e^b_J$ that satisfy $f^I_a e^a_J = \delta^I_J$.

A.1 Γ_{E_8}

A basis for the root lattice Γ_{E_8} of the rank 8 group E_8 is given by

$$\begin{aligned} \mathbf{e}_I &= \hat{\mathbf{x}}_\mathbf{I} - \hat{\mathbf{x}}_\mathbf{I+1}, \text{ for } I = 1, ...6, \\ \mathbf{e}_7 &= -\hat{\mathbf{x}}_\mathbf{1} - \hat{\mathbf{x}}_\mathbf{2}, \\ \mathbf{e}_8 &= \frac{1}{2}(\hat{\mathbf{x}}_\mathbf{1} + ... + \hat{\mathbf{x}}_\mathbf{8}). \end{aligned} \tag{A.2}$$

The associated K-matrix takes the form,

$$K_{E8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{A.3}$$

The inner product is Euclidean so $\eta_{ab} = \delta_{ab}$.

A.2 $\Gamma_{E_8} \oplus \Gamma_{E_8}$

The rank 16 Lie group $E_8 \times E_8$ is equal to two copies of E_8. We take as our lattice basis for $\Gamma_{E_8} \oplus \Gamma_{E_8}$,

$$\begin{aligned} \mathbf{e}_I &= \hat{\mathbf{x}}_\mathbf{I} - \hat{\mathbf{x}}_\mathbf{I+1}, \text{ for } I = 1, ...6, \\ \mathbf{e}_7 &= -\hat{\mathbf{x}}_\mathbf{1} - \hat{\mathbf{x}}_\mathbf{2}, \end{aligned}$$

$$\begin{aligned}
\mathbf{e}_8 &= \frac{1}{2}(\hat{\mathbf{x}}_\mathbf{1} + ... + \hat{\mathbf{x}}_\mathbf{8}), \\
\mathbf{e}_{8+I} &= \hat{\mathbf{x}}_{\mathbf{9+I}} - \hat{\mathbf{x}}_{\mathbf{10+I}}, \text{ for } I = 1, ..., 6, \\
\mathbf{e}_{15} &= \hat{\mathbf{x}}_\mathbf{15} + \hat{\mathbf{x}}_\mathbf{16}, \\
\mathbf{e}_{16} &= -\frac{1}{2}(\hat{\mathbf{x}}_\mathbf{9} + ... + \hat{\mathbf{x}}_\mathbf{16}).
\end{aligned} \tag{A.4}$$

The associated K-matrix, $K_{E_8 \oplus E_8}$, takes the form,

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.$$

The inner product is again taken to be $\eta_{ab} = \delta_{ab}$.

A.3 $\Gamma_{\mathrm{Spin}(32)/Z_2}$

A basis for the root lattice $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ of the rank 16 Lie group $\mathrm{Spin}(32)/\mathbb{Z}_2$ is given by,

$$\begin{aligned} \tilde{e}_I &= \hat{\mathbf{x}}_{\mathbf{I+1}} - \hat{\mathbf{x}}_{\mathbf{I+2}}, \text{ for } I = 1, ..., 14, \\ \tilde{e}_{15} &= \hat{\mathbf{x}}_{\mathbf{15}} + \hat{\mathbf{x}}_{\mathbf{16}}, \\ \tilde{e}_{16} &= -\frac{1}{2}(\hat{\mathbf{x}}_{\mathbf{1}} + ... + \hat{\mathbf{x}}_{\mathbf{16}}). \end{aligned} \tag{A.5}$$

The associated K-matrix, $K_{\mathrm{Spin}(32)/Z_2}$

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4
\end{pmatrix}.$$

The inner product is given by $\eta_{ab} = \delta_{ab}$.

A.4 $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$

To write a basis for the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice, we must enlarge the dimension of our previous $\Gamma_{E_8} \oplus \Gamma_{E_8}$ lattice by two. Thus, we take as our lattice basis,

$$\mathbf{e}_I \;=\; \hat{\mathbf{x}}_{\mathbf{I}} - \hat{\mathbf{x}}_{\mathbf{I+1}}, \text{ for } I = 1, ...6,$$

$$
\begin{aligned}
\mathbf{e}_7 &= -\hat{\mathbf{x}}_{\mathbf{1}} - \hat{\mathbf{x}}_{\mathbf{2}}, \\
\mathbf{e}_8 &= \frac{1}{2}(\hat{\mathbf{x}}_{\mathbf{1}} + ... + \hat{\mathbf{x}}_{\mathbf{8}}), \\
\mathbf{e}_{8+I} &= \hat{\mathbf{x}}_{\mathbf{9+I}} - \hat{\mathbf{x}}_{\mathbf{10+I}}, \text{ for } I = 1, ..., 6, \\
\mathbf{e}_{15} &= \hat{\mathbf{x}}_{\mathbf{15}} + \hat{\mathbf{x}}_{\mathbf{16}}, \\
\mathbf{e}_{16} &= -\frac{1}{2}(\hat{\mathbf{x}}_{\mathbf{9}} + ... + \hat{\mathbf{x}}_{\mathbf{16}}), \\
\mathbf{e}_{17} &= \tfrac{1}{r}\hat{\mathbf{x}}_{\mathbf{17}} + \tfrac{1}{r}\hat{\mathbf{x}}_{\mathbf{18}}, \\
\mathbf{e}_{18} &= \tfrac{r}{2}\hat{\mathbf{x}}_{\mathbf{17}} - \tfrac{r}{2}\hat{\mathbf{x}}_{\mathbf{18}}.
\end{aligned} \tag{A.6}
$$

The associated K-matrix,$K_{E_8 \oplus E_8 \oplus U}$, takes the form,

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

The inner product is taken with respect to $\eta_{ab} = (\mathbf{1}^{17}, -1)$.

A.5 $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$

We must again enlarge the dimension of $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ by two in order to write a basis for $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2} \oplus U$,

$$\begin{aligned}
\tilde{e}_I &= \hat{\mathbf{x}}_{\mathbf{I+1}} - \hat{\mathbf{x}}_{\mathbf{I+2}}, \text{ for } I = 1, ..., 14, \\
\tilde{e}_{15} &= \hat{\mathbf{x}}_{\mathbf{15}} + \hat{\mathbf{x}}_{\mathbf{16}}, \\
\tilde{e}_{16} &= -\frac{1}{2}(\hat{\mathbf{x}}_{\mathbf{1}} + ... + \hat{\mathbf{x}}_{\mathbf{16}}), \\
\tilde{e}_{17} &= -r\hat{\mathbf{x}}_{\mathbf{17}} + r\hat{\mathbf{x}}_{\mathbf{18}}, \\
\tilde{e}_{18} &= -\tfrac{1}{2r}\hat{\mathbf{x}}_{\mathbf{17}} - \tfrac{1}{2r}\hat{\mathbf{x}}_{\mathbf{18}}.
\end{aligned} \tag{A.7}$$

The associated K-matrix,

$$K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

The inner product is taken with respect to $\eta_{ab} = (\mathbf{1}^{17}, -1)$.

A.6 $SO(17,1)$ and $SL(18,\mathbb{Z})$ Transformations

There exist two distinct even, self-dual 16-dimensional lattices, $\Gamma_{E_8}\oplus\Gamma_{E_8}$ and $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}$, that cannot be rotated into each other via an $SO(16)$ transformation [37]. However, if we augment each lattice by U, we obtain a Lorentzian lattice of signature $(17,1)$, i.e., the augmented lattice has the inner product $\eta_{ab} = \mathrm{diag}(\mathbf{1}^{17}, -1)$. Such lattices are unique up to an $SO(17,1)$ rotation. Following [38], the $SO(17,1)$ transformation, O_G, relating the $\Gamma_{E_8}\oplus\Gamma_{E_8}\oplus U$ and $\Gamma_{\mathrm{Spin}(32)/\mathbb{Z}_2}\oplus U$ lattices is given by,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\,0\,0\,0\,0\,0\,0 & -\frac{1}{2r} - \frac{-1+r^2}{2r} & \frac{1}{2r} - \frac{1+r^2}{2r} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{r} & -\frac{1}{r} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\,0\,0\,0\,0\,0\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,1\,0\,0\,0\,0\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\,1\,0\,0\,0\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\,0\,1\,0\,0\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\,0\,0\,1\,0\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0\,1\,0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0\,0\,1 & 0 & 0 \\
\frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & -\frac{r}{2} + \frac{1-r^2}{r} & r - \frac{1-r^2}{r} & 0\,0\,0\,0\,0\,0\,0 & \frac{1}{2} + \frac{\left(1-r^2\right)\left(-1+r^2\right)}{r^2} & -\frac{1}{2} - r^2 + \frac{1-r^2}{r^2} \\
\frac{-r}{2} & \frac{-r}{2} & \frac{-r}{2} & \frac{-r}{2} & \frac{-r}{2} & \frac{-r}{2} & \frac{-r}{2} & \frac{r}{2} + \frac{1+r^2}{r} & -r - \frac{1+r^2}{r} & 0\,0\,0\,0\,0\,0\,0 & -\frac{1}{2} + r^2 - \frac{1+r^2}{r^2} & \frac{1}{2} + \frac{\left(1+r^2\right)^2}{r^2}
\end{pmatrix}.
$$

O_G acts on basis vectors as

$$O_{G\ b}^{a} e_I^b = \sum_J m_I^J \tilde{e}_J^a, \tag{A.8}$$

where m_I^J are a collection of integers.

Because O_G lies in the component of $SO(17,1)$ connected to the identity transformation, we may build O_G from a series of infinitesimal transformations beginning at $\mathbf{1}$. First, we rewrite,

$$O_G = \eta W(A) \eta W(A'), \tag{A.9}$$

where

$$W(A) = \exp\left[\frac{1}{2}\begin{pmatrix} 0 & A & -A \\ -A^t & 0 & 0 \\ -A^t & 0 & 0 \end{pmatrix}\right], \quad \text{with} \tag{A.10}$$

$$A = \frac{2}{r}\left(0^7, -1, 1, 0^7\right), \tag{A.11}$$

$$A' = -2r\left(\left(\frac{1}{2}\right)^8, 0^8\right). \tag{A.12}$$

We then introduce the (infinitesimal) parameter s by rescaling $A, A' \to sA, sA'$ and defining,

$$O_G(s) = \eta W(sA) \eta W(sA'). \tag{A.13}$$

(While the resulting matrix does not fit between the margins of this page, the expression is not beautiful.)

Substituting the transformation Eq. (A.8) into the periodicity condition, $X^a \equiv X^a +$

$2\pi n^I e^a_I$, for the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice, we find:

$$(O_G)^a{}_b X^b \equiv (O_G)^a{}_b X^b + 2\pi \tilde{n}^J \tilde{e}^a_J, \tag{A.14}$$

where we have defined the integer vector $\tilde{n}^J = \sum_I n^I m^J_I$. However, Eq. (A.14) is simply the periodicity obeyed by $\tilde{X}^a$. Therefore, we identify $\tilde{X}^a = (O_G)^a{}_b X^b$. Having identified X^a and $\tilde{X}^b$ through the $SO(17,1)$ transformation O_G, we can obtain the $SL(18,\mathbb{Z})$ transformation W_G that relates $K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U}$ and $K_{E_8 \oplus E_8 \oplus U}$ by conjugation. The desired transformation is read off from the relation,

$$\tilde{\phi}^J = \tilde{f}^J_a (O_G)^a{}_b e^b_I \phi^I =: (W_G)_{IJ} \phi^I, \tag{A.15}$$

which follows immediately from Eq. (A.8). We find:

$$W_G = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-9 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-10 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\
-11 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\
-12 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \\
-13 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & -5 \\
-14 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & -6 \\
-7 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -3 \\
-8 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & -4 \\
-2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 2
\end{pmatrix}.$$

This matrix satisfies $W_G^T K_{\mathrm{Spin}(32)/\mathbb{Z}_2 \oplus U} W = K_{E_8 \oplus E_8 \oplus U}$.

A.7 Relevant large matrices

Here we define matrices referred to in 4.7:

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{A.16}$$

$$W_8 = \begin{pmatrix} -5 & -5 & -5 & 5 & 5 & 5 & 5 & 5 & 8 & 16 \\ -10 & -10 & -10 & 9 & 9 & 9 & 9 & 9 & 15 & 30 \\ -8 & -8 & -8 & 8 & 7 & 7 & 7 & 7 & 12 & 24 \\ -6 & -6 & -6 & 6 & 6 & 5 & 5 & 5 & 9 & 18 \\ -4 & -4 & -4 & 4 & 4 & 4 & 3 & 3 & 6 & 12 \\ -2 & -2 & -2 & 2 & 2 & 2 & 2 & 1 & 3 & 6 \\ -7 & -7 & -6 & 6 & 6 & 6 & 6 & 6 & 10 & 20 \\ -4 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & 5 & 10 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -3 & -4 \\ -2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 4 & 7 \end{pmatrix} \tag{A.17}$$

$$K_{D_{12}^+} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \tag{A.18}$$

$$W_{12} = \begin{pmatrix} 11 & 6 & 6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & 0 & 22 \\ -9 & -4 & -5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 18 \\ -18 & -9 & -9 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 0 & 36 \\ -16 & -8 & -8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 0 & 32 \\ -14 & -7 & -7 & 7 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 & 28 \\ -12 & -6 & -6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 0 & 24 \\ -10 & -5 & -5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 0 & 20 \\ -8 & -4 & -4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 0 & 16 \\ -6 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 0 & 12 \\ -4 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 0 & 8 \\ -2 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 4 \\ 3 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -4 \end{pmatrix} \tag{A.19}$$

Appendix B

Mass-Generating Operators for bosonic IQH edge transitions

We consider spin-0 operators that take the form $\cos(p_a X^a)$, with $p_a \in \Gamma_8 \oplus \Gamma_8 \oplus U$ and $\eta^{ab} p_a p_b = 0$. Even if $\frac{1}{2}\delta^{ab} p_a p_b > 2$, which means that $\cos(p_a X^a)$ is irrelevant at $s = 0$, this operator may become relevant at an intermediate value of s. At general s, the scaling dimension of the operator is $\frac{1}{2}\delta^{ab} q_a q_b = |q_{18}|^2$, where $q_b = p_a (O_G^{-1}(s))^a{}_b$. In writing the scaling dimension in terms of q_{18} only, we have used the fact that q_b is a null vector in $\mathbb{R}^{17,1}$ ($\eta^{ab} q_a q_b = q_1^2 + ... + q_{17}^2 - q_{18}^2 = 0$). Thus, $\cos(p_a X^a)$ will become relevant at s if $p_a (O_G^{-1}(s))^a_{18}$ is sufficiently Lorentz contracted so that $q_{18}^2 < 2$.

If the direction of the boost $O_G^{-1}(s)$ happened to be along the 1-direction, then we know that the only components of p_a affected by the boost are the 1st and 18th component; they are contracted/dilated according to:

$$\begin{pmatrix} p_1 \\ p_{18} \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} p_1 \\ p_{18} \end{pmatrix}. \tag{B.1}$$

Therefore, multiples of the eigenvectors $(1, \pm 1)^T$ with eigenvalues $\exp(\mp\alpha)$ have components that are maximally contracted/dilated. If the boost took the above simple form, it would be simple to choose a vector p_a whose 18th component after the boost was maximally contracted. This vector would determine the most relevant operator at a given point in the (r, s) phase diagram.

Unfortunately, $O_G^{-1}(s)$ is defined in terms of a rather complicated combination of rotations and boosts, and so it is not a priori obvious which spatial direction to choose in order to maximize the possible contraction, i.e., it is difficult to know the direction $\mathbf{v}$ of the boost. However, we know that we can view the $O_G^{-1}(s)$ transformation as: $O_G^{-1}(s) = M^T \Lambda M$, where M is a rotation that aligns $\mathbf{v}$ along the 1-direction and Λ is a boost along the 1-direction. (Both of these transformations, of course, depend upon the initially chosen r and s.) To find null vectors whose components maximally contract, we need only consider the eigenvector of $O_G^{-1}(s)$ given by $M^{\mathrm{tr}}(1, 0^{16}, 1)^{\mathrm{tr}}$ with eigenvalue $\exp(-\alpha)$, for some constant α depending upon r and s. For $(r, s) = (3, 3/5)$ we find that this maximally contracting eigenvector takes the simple (approximate) form:

$$p_a = .3 f^7{}_a + (.1 - .6) f^8{}_a + .1 f^{16}_a + f^{17}_a - .9 f^{18}_a. \tag{B.2}$$

While the components of this vector are maximally contracted under $O_G^{-1}(s)$ in the sense discussed above, it is certainly not an element of $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ since the coefficients are not integral. We can find a vector with very large components that is nearly parallel to this vector, but it will be irrelevant because $O_G^{-1}(s)$ cannot contract it by enough at $(r, s) = (3, 3/5)$.

However, we can find a shorter lattice vector that is sufficiently aligned with the maximally contracting vector, but of lower starting dimension so that we obtain a relevant

operator at the point of interest. Indeed, if we take the ansatz:

$$p_a = n f^7_{\ a} + (m - 2n) f^8_{\ a} + m f^{16}_{\ a} + n_{17} f^{17}_{\ a} + n_{18} f^{18}_{\ a}, \tag{B.3}$$

it is straightforward to find n, m, n_{17} and n_{18} determining a relevant spin-0 operator at (r, s). At $(r, s) = (3, 3/5)$, we may take $n = 1, m = 2, n_{17} = 2$ and $n_{18} = -3$. We lack a proof that this ansatz is sufficient to exclude all possible non-chiral points in the (r, s) phase diagram. However, we have yet to find a point (r, s) for which this ansatz is unsuccessful. Thus, we expect the non-chiral phase to be entirely removed by this collection of operators combined with those discussed earlier. (Note, we expect the resulting chiral phase for this operator to be $\mathrm{Spin}(32)/\mathbb{Z}_2$.)

Appendix C

Using the Gauss-Smith Normal Form to find the Discriminant Group

We now apply the method described in Section 4.5 to the $\mathrm{SO}(8)_1$ theory,which is given by the following K matrix:

$$K = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{C.1}$$

It is not clear, simply by inspection, what vectors correspond to generators of the fusion group.

The Gauss-Smith normal form is

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{C.2}$$

Hence, the fusion group of the theory is $\mathbb{Z}/2 \times \mathbb{Z}/2$.

and the Q matrix

$$Q = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{C.3}$$

So the fusion group is generated by the two quasiparticles corresponding to $(2, 0, 0, 1)$ and $(1, 0, 0, 0)$. We can then compute the S, T matrices and the result agrees with what is known (all nontrivial quasiparticles are fermions and they have semionic mutual braiding statistics with each other).

Another useful piece of information from the Smith normal form is that the discriminant group for a 2×2 K-matrix

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{C.4}$$

with $\gcd(a, b, c) = 1$ and $d = |ac - b^2|$ is $\mathbb{Z}/d$. More generally, it is $\mathbb{Z}/f \times \mathbb{Z}/(d/f)$ when $\gcd(a, b, c) = f$.

Appendix D

Construction of special $\mathbf{w} \in \Lambda$

In this appendix we proove that a special $\mathbf{w} \in \Lambda$ exists such that $\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} \equiv \boldsymbol{\lambda} \cdot \mathbf{w} \bmod 2$ for all $\boldsymbol{\lambda} \in \Lambda$.

We begin by showing that for any K-matrix, there exists a set of integers w_J such that

$$K_{II} \equiv \sum_{J=1}^{N} K_{IJ} w_J \bmod 2, \text{ for all } I \tag{D.1}$$

where N is the dimension of the K-matrix.

Assume the K-matrix has $M \leq N$ rows that are linearly independent mod 2; denote these rows $R_1, ...R_M$ and define the set $R = \{R_i\}$. The linear independence of the R_i implies that Eq (D.1) is satisfied for these rows, i.e., there exists a set of integers $(w_0)_J$ satisfying

$$K_{II} \equiv \sum_{J=1}^{N} K_{IJ} (w_0)_J \bmod 2, \text{ for all } I \in R \tag{D.2}$$

For a row $I \notin R$, the elements of the I^{th} row in K can be written as a linear combination of the rows in R:

$$K_{IJ} \equiv \sum_{R_i \in R} c_{IR_i} K_{R_i J} \bmod 2, \text{ for } I \notin B \tag{D.3}$$

where the $c_{IR_i} \in \{0, 1\}$ are coefficients. It follows that for $I \notin R$:

$$\begin{aligned} K_{II} &\equiv \sum_{R_i \in R} c_{IR_i} K_{R_i I} \equiv \sum_{R_i \in R} c_{IR_i} K_{IR_i} \\ &\equiv \sum_{R_i, R_j \in R} c_{IR_i} c_{IR_j} K_{R_i R_j} \equiv \sum_{R_i \in R} c_{IR_i}^2 K_{R_i R_i} \\ &\equiv \sum_{R_i \in R} c_{IR_i} K_{R_i R_i} \bmod 2 \end{aligned} \tag{D.4}$$

Furthermore, for $I \notin R$

$$\begin{aligned} \sum_{J=1}^{N} K_{IJ}(w_0)_J &\equiv \sum_{J=1}^{N} \sum_{R_i \in R} c_{IR_i} K_{R_i J}(w_0)_J \\ &\equiv \sum_{R_i \in R} c_{IR_i} K_{R_i R_i} \bmod 2 \end{aligned} \tag{D.5}$$

Hence, for $I \notin R$, $K_{II} \equiv \sum_{J=1}^{N} K_{IJ}(w_0)_J \bmod 2$. Since this equation already holds for $I \in R$, we have shown that w_0 is a solution to Eq (D.1).

It follows that for any choice of $\boldsymbol{\lambda} = \lambda_J \mathbf{e}_J \in \Lambda$,

$$\begin{aligned} \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} &= \sum_{I,J=1}^{N} \lambda_I \lambda_J K_{IJ} \equiv \sum_{I=1}^{N} \lambda_I K_{II} \\ &\equiv \sum_{I=1}^{N} \lambda_I \sum_{J=1}^{N} K_{IJ}(w_0)_J \equiv \boldsymbol{\lambda} \cdot \boldsymbol{w}_0 \bmod 2 \end{aligned} \tag{D.6}$$

where $\mathbf{w}_0 = (w_0)_J \mathbf{e}_J$ is a vector in Λ.

Appendix E

Transport in hyperconductors

E.1 Static Susceptibility Matrix

The static susceptibility matrix $\hat{\chi}_{pq} = \frac{1}{L} G^R_{\mathcal{Q}_p \mathcal{Q}_q}(\omega = 0)$ where the conserved charges $\mathcal{Q}_p$ and $\mathcal{Q}_p$ of the action S_b involved in the retarded Green's function $G^R_{\mathcal{Q}_p \mathcal{Q}_q}$ are either one of the chiral electrical currents,

$$\begin{aligned} J^e_I =& \frac{e}{2\pi} \mathrm{sgn}(N-I) \int_x V_{IJ} \partial_x \phi_J \\ =& \frac{e}{2\pi} \mathrm{sgn}(N-I) \int_x V_{IJ} O_{Ja} \partial_x X_a, \end{aligned} \tag{E.1}$$

or the Dirac momentum,

$$\begin{aligned} P_D =& -\frac{1}{4\pi} \int_x \mathrm{sgn}(N-I) \partial_x \phi_I \partial_x \phi_I \\ =& -\frac{1}{4\pi} \mathrm{sgn}(N-a) \int_x \partial_x X_a \partial_x X_a. \end{aligned} \tag{E.2}$$

In the above equations, $x \in (-L, L)$ with the length of the system $L \to \infty$, $\mathrm{sgn}(Z) = +1$ for $Z \geq 0$ and $\mathrm{sgn}(Z) = -1$ for $Z < 0$, and $J^e_I = J^e_{R,I}$ for $I = 1, \ldots, N$ and $J^e_I = J^e_{L,N-I}$

for $I = N+1, \ldots, 2N$ with $N = 23$. Note that I is not summed over on the right-hand side of Eq. (E.1). We have introduced the fields $\phi_I = O_{Ia} X_a$ with $O_{Ia} \in SO(23,23)$ that diagonalize the action $S_{\rm b}$, tuned via the interaction matrix V_{IJ} to the asymmetric Leech liquid point,

$$\begin{aligned} S_{\rm b} =& \frac{1}{4\pi} \int_{t,x} \Big[{\rm sgn}(N-I) \partial_t \phi_I \partial_x \phi_I - V_{IJ} \partial_x \phi_I \partial_x \phi_J \Big] \\ =& \frac{1}{4\pi} \int_{t,x} \Big[{\rm sgn}(N-a) \partial_t X_a \partial_x X_a - v \partial_x X_a \partial_x X_a \Big]. \end{aligned} \tag{E.3}$$

Henceforth, we set the velocity $v = 1$. To isolate the leading temperature and frequency dependence of the conductivity, we need only compute the static susceptibility with respect to $S_{\rm b}$.

The bosonic action $S_{\rm b}$ enjoys the particle-hole symmetry $\phi_I \to -\phi_I$, $X_a \to -X_a$. Thus, the retarded Green's functions $G^R_{J^e_I P_D} = 0$ when computed with respect to $S_{\rm b}$ and so we focus upon the $J^e_I - J^e_J$ or $P_D - P_D$ static susceptibilities. Scattering interactions at incommensurate fillings, interactions mediated by disorder, and higher-derivative band structure corrections to $S_{\rm b}$ generally break particle-hole symmetry and, thus, induce a non-zero overlap between the electrical currents and the momentum. We ignore such overlaps as they contribute higher-order corrections to the conductivity than that to which we choose to work. At commensurate fillings and in the absence of higher-derivative corrections, particle-hole symmetry is preserved.

To compute the retarded correlator, we exploit the relation $G^R_{\mathcal{Q}_p \mathcal{Q}_q}(\omega) = G^E_{\mathcal{Q}_p \mathcal{Q}_p}(i\omega_E \to \omega + i\delta)$ with the infinitesimal $\delta > 0$ between the retarded Green's function and the Euclidean Green's function at Euclidean frequency ω_E. The frequency ω of the retarded correlator has been analytically continued to the upper-half plane. We shall often simply set $\delta = 0$ without mention. Thus, the static susceptibility $\hat{\chi}_{pq} = \frac{1}{L} G^E_{\mathcal{Q}_p \mathcal{Q}_p}(\omega_E = 0)$.

We begin with the $J_I^e - J_J^e$ components of the static susceptibility,

$$\begin{aligned}\hat{\chi}_{J_I^e J_J^e} \equiv & \frac{1}{L}\lim_{\omega_E\to 0}\int_\tau e^{i\omega_E\tau}\Big\langle J_I^e(\tau)J_J^e(0)\Big\rangle \\ = & \frac{e^2 M_{IJ}^{ab}}{4\pi^2 L}\lim_{\omega_E\to 0}\int_{\tau,x,y} e^{i\omega_E\tau}\Big\langle X_a'(\tau,x)X_b'(0,y)\Big\rangle,\end{aligned} \tag{E.4}$$

where $X'(\tau,x) \equiv \partial_x X(\tau,x)$,

$$\begin{aligned}M_{IJ}^{ab} = & \mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IK}V_{JL}O_{Ka}O_{Lb} \\ = & \mathrm{sgn}(N-I)\mathrm{sgn}(N-J)(O^{-1})_{aI}(O^{-1})_{bJ},\end{aligned} \tag{E.5}$$

the Euclidean time $\tau \in [0, 1/T]$ and the brackets denote the thermal average at temperature T. In simplifying Eq. (E.5), we have made use of the identity $O_{Ia}V_{IJ}O_{Jb} = \delta_{ab}$. Because S_b is diagonal when expressed in terms of the X_a fields, the only non-zero correlators in Eq. (E.4) occur when $a = b$ and we obtain the well-known result,[163]

$$\langle X_a'(\tau,x)X_b'(0,0)\rangle = -\,\delta_{ab}\Big(\frac{\pi T}{\sinh\Big(\pi T(x - \mathrm{sgn}_a i\tau)\Big)}\Big)^2, \tag{E.6}$$

where we have used the short-hand, $\mathrm{sgn}_a = \mathrm{sgn}(N-a)$. It will be convenient to calculate a slightly more general Fourier transform than Eq. (E.4) by replacing the exponent in Eq. (E.6), $2 \to 2h$ with h assumed to be half-integral. Thus, we consider

$$
\begin{aligned}
&\frac{1}{L}\int_{\tau,x,y} e^{i\omega_E\tau}\Big(\frac{\pi T}{\sinh\Big(\pi T(x-y-\mathrm{sgn}_a i\tau)\Big)}\Big)^{2h} \qquad \text{(E.7)}\\
=&-\frac{(\pi T)^{2h}}{2L}\int_{x_+,x_-,\tau} e^{i\omega_E\tau}\frac{1}{\Big(\sinh\Big(\pi T(x_- -\mathrm{sgn}_a i\tau)\Big)^{2h}}\\
=&-\pi^{2h}(2T)^{2h-1}\int_{x_-} e^{\mathrm{sgn}_a 2\pi T x_-}\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{\zeta^{\frac{\omega_E\tau}{2\pi T}+h-1}}{(\zeta-e^{\mathrm{sgn}_a 2\pi T x_-})^{2h}}\\
=&-\frac{T^{2h-1}}{2\omega_E}\frac{(2\pi)^{2h}}{(2h-1)!}\prod_{i=1}^{2h-1}\Big(\frac{\omega_E}{2\pi T}+h-i\Big).
\end{aligned}
$$

In the first line, we made the change of variables, $x_\pm = x \pm y$ and then integrated over x_+; in the second line, we made the change of variable $\zeta = \exp(2\pi T i\tau)$, performed the contour integration about the circle $|\zeta| = 1$, and then integrated over x_-. Thus, we find for the current-current static susceptibility:

$$
\begin{aligned}
\hat{\chi}_{J^e_I J^e_J} =&\frac{e^2}{4\pi}\sum_{a=1}^{2N} M^{aa}_{IJ}\\
=&\frac{e^2}{4\pi}\mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IJ}, \qquad \text{(E.8)}
\end{aligned}
$$

where I, J are not summed over and we used the relation $(O^{-1})^T.(O^{-1}) = V$.

Following an analogous procedure, we now calculate the $P_D - P_D$ static susceptibility,

$$\begin{aligned}\hat{\chi}_{P_D P_D} \equiv & \frac{1}{L}\lim_{\omega_E \to 0}\int_\tau e^{i\omega_E \tau}\Big\langle P_D(\tau)P_D(0)\Big\rangle \\ = & \frac{2}{16\pi^2 L}\mathrm{sgn}(N-a)\mathrm{sgn}(N-b) \\ & \times \int_{\tau,x,y} e^{i\omega_E\tau}\Big\langle X'_a(\tau,x)X'_b(0,y)\Big\rangle^2 \\ = & \frac{1}{8\pi^2 L}\int_{\tau,x,y} e^{i\omega_E\tau}\Big\langle X'_a(\tau,x)X'_a(0,y)\Big\rangle^2,\end{aligned} \tag{E.9}$$

where we used Wick's theorem in going from the first to the second line and the fact that the only non-zero correlators occur when $a = b$ in going from the second to the third line. We may now borrow the general result in Eq. (E.7) by setting $h = 2$ to conclude:

$$\hat{\chi}_{P_D P_D} = \frac{N\pi^2 T^2}{6}. \tag{E.10}$$

E.2 Memory Matrix Elements

Recall the definition of the memory matrix reviewed Sec. 6.3.3 which we repeat here for convenience. The memory matrix $\hat{M}(\omega)$ (the temperature dependence is left implicit) is defined as follows:

$$\hat{M}(\omega) = \sum_\alpha \Big(\lambda_\alpha^2 \hat{\mathcal{M}}^u_\alpha(\omega) + (\lambda^{\mathrm{dis}}_\alpha)^2 D\hat{\mathcal{M}}^{\mathrm{dis}}_\alpha(\omega)\Big), \tag{E.11}$$

$$(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_\alpha = \frac{1}{L}\frac{\langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{u}}_{q,\alpha}\rangle_\omega - \langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{u}}_{q,\alpha}\rangle_{\omega=0}}{i\omega}, \tag{E.12}$$

$$(\hat{\mathcal{M}}^{\mathrm{dis}})^{pq}_\alpha = \frac{1}{L}\frac{\langle F^{\mathrm{dis}}_{p,\alpha}; F^{\mathrm{dis}}_{q,\alpha}\rangle_\omega - \langle F^{\mathrm{dis}}_{p,\alpha}; F^{\mathrm{dis}}_{q,\alpha}\rangle_{\omega=0}}{i\omega}. \tag{E.13}$$

Here, $F^{\rm u}_{q,\alpha} = \frac{i}{\lambda_\alpha}[H^{\rm u}_\alpha, \mathcal{Q}_q]$, $F^{\rm dis}_{q,\alpha} = \frac{i}{\lambda^{\rm dis}_\alpha\sqrt{D}}\left[H^{\rm dis}_\alpha, \mathcal{Q}_q\right]$, and $\mathcal{Q}_q$ is a conserved charge (either $J^e_{R/L,I}$ or P_D). $\langle F^{\rm u}_{p,\alpha}; F^{\rm u}_{q,\alpha}\rangle_\omega$ and $\langle F^{\rm dis}_{p,\alpha}; F^{\rm dis}_{q,\alpha}\rangle_\omega$ are retarded finite-temperature Green's functions evaluated using $S_{\rm b}$. λ_α and $\lambda^{\rm dis}_\alpha$ parameterize the umklapp scattering and coupling to disorder, respectively, and D is the disorder variance of the Gaussian-correlated disorder, $\overline{\xi_\alpha(x)} = 0$, $\overline{\xi_\alpha(x)\xi^*_\beta(y)} = D\delta_{\alpha\beta}\delta(x-y)$. For simplicity, we take $\lambda_\alpha = \lambda$ and $\lambda^{\rm dis}_\alpha = \lambda^{\rm dis}$ for all α. $\hat{\mathcal{M}}^{\rm u}$ contains the contributions to the memory matrix from umklapp scattering, while $\hat{\mathcal{M}}^{\rm dis}$ contains the contributions from the disorder-mediated interaction. We stress that the form of the memory matrix given above is correct to leading order in the scattering interaction. See Refs. [[168, 169, 158, 170, 171]] for further discussion.

E.2.1 Evaluation of the $F^{\rm u,dis}_{p,\alpha}$

To compute the $F^{\rm u,dis}_{p,\alpha}$ commutators, we make use of the equal-time commutators:

$$\left[e^{im^{(\alpha)}_J\phi_J(x)}, \frac{\phi'_I(y)}{2\pi}\right] = m^{(\alpha)}_I \mathrm{sgn}_I \delta(x-y) e^{im^{(\alpha)}_J\phi_J(x)}. \tag{E.14}$$

We find for the commutators $F^{\rm u}_{p,\alpha}$ of the $\mathcal{Q}_p$ with the umklapp scattering operators:

$$\begin{aligned} F^{\rm u}_{J^e_I,\alpha} &= -2e\ \mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IJ}m^{(\alpha)}_J \\ &\quad\times \int_x \frac{1}{a^2}\sin\Big(\Delta k_\alpha x + m^{(\alpha)}_K\phi_K\Big), \end{aligned} \tag{E.15}$$

$$F^{\rm u}_{P_D,\alpha} = 2\Delta k_\alpha \int_x \frac{1}{a^2}\sin\Big(\Delta k_\alpha x + m^{(\alpha)}_K\phi_K\Big), \tag{E.16}$$

where the momentum mismatch $\Delta k_\alpha \equiv \sum_I m^{(\alpha)}_I k_F - p^{(\alpha)}G$, G is a basis vector for the reciprocal lattice, and we have taken the Fermi momenta in all channels to be equal. Recall that a is a short-distance cutoff. We see that the Dirac momentum P_D commutes

with the umklapp operators when $\Delta k_\alpha = 0$, i.e., when the translation symmetry of the low-energy effective theory is preserved. The result for $[H^{\rm u}_\alpha, P_D]$ is found using the integration by parts,

$$\begin{aligned}\int_x e^{i\Delta k_\alpha x}\frac{m_K^{(\alpha)}}{2}\{\phi'_K, e^{im_L^{(\alpha)}\phi_L}\} &\equiv -i\int_x e^{i\Delta k_\alpha x}\partial_x e^{im_L^{(\alpha)}\phi_L} \\ &= -\Delta k_\alpha \int_x e^{i\Delta k_\alpha x + im_L^{(\alpha)}\phi_L},\end{aligned} \tag{E.17}$$

where we have dropped the boundary term and have defined the derivative operator on the right-hand side of the first line via a symmetric ordering prescription: $\partial_x \exp(im_I^{(\alpha)}\phi_I) \equiv \frac{i}{2}m_J^{(\alpha)}\Big(\partial_x\phi_J \exp(im_I^{(\alpha)}\phi_I) + \exp(im_I^{(\alpha)}\phi_I)\partial_x\phi_J\Big)$.

The commutators $F^{\rm dis}_{p,\alpha}$ of the $\mathcal{Q}_p$ with the disorder-mediated interactions are computed in a similar fashion:

$$\begin{aligned}F^{\rm dis}_{J^e_I,\alpha} &= \frac{ie}{\sqrt{D}}{\rm sgn}(N-I){\rm sgn}(N-J)V_{IJ}m_J^{(\alpha)} \\ &\quad\times\int_x\Big[\xi_\alpha(x)\frac{1}{a^2}e^{im_K^{(\alpha)}\phi_K} - {\rm h.c.}\Big],\end{aligned} \tag{E.18}$$

$$F^{\rm dis}_{P_D,\alpha} = -\frac{1}{\sqrt{D}}v^2\int_x\Big[\Big(\partial_x\xi_\alpha(x)\Big)\frac{1}{a^2}e^{im_K^{(\alpha)}\phi_K} + {\rm h.c.}\Big]. \tag{E.19}$$

We see that the umklapp commutators in Eqs. (E.15, E.16) may be obtained from the disorder commutators in Eqs. (E.18, E.19) by substituting $\xi_\alpha(x) = \exp(i\Delta k_\alpha x)$.

E.2.2 Evaluation of the $(\hat{\mathcal{M}}^{\rm u})^{pq}_\alpha$

We begin with the evaluation of the retarded two-point correlation functions $\langle F^{\rm u}_{p,\alpha}; F^{\rm u}_{q,\beta}\rangle_\omega$. To leading order in the umklapp (and disorder) perturbations, these correlators are only non-zero when $\alpha = \beta$ because of the linear independence of the $m_I^{(\alpha)}$ so we set $\alpha = \beta$ in

the remainder. Also, notice that $\langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{dis}}_{q,\beta}\rangle_\omega = 0$ because the disorder we study has zero mean, $\overline{\xi_\alpha(x)} = 0$. We simplify the following expressions by introducing the coefficients:

$$\begin{aligned} U_{J^e_I,\alpha} =& -2e\mathrm{sgn}(N-I)\mathrm{sgn}(N-J)V_{IJ}m^{(\alpha)}_J, \\ U_{P_D,\alpha} =& 2v^2\Delta k_\alpha. \end{aligned} \tag{E.20}$$

We see that $U_{P_D,\alpha} = 0$ for commensurate fillings when $\Delta k_\alpha = 0$ because translation invariance in the low-energy effective theory S_{lin} (interpreted as Dirac fermions created about zero-momentum) is preserved, resulting in divergent thermal conductivity.

Just as in Appendix E.1, we compute the retarded correlators by Fourier transforming the Euclidean real-space correlation functions and then analytically continuing the Matsubara frequencies ω_E to real frequencies ω by way of the formula, $G^R_{F^{\mathrm{u}}_{p,\alpha}F^{\mathrm{u}}_{q,\alpha}}(\omega) = G^E_{F^{\mathrm{u}}_{p,\alpha}F^{\mathrm{u}}_{q,\alpha}}(i\omega_E \to \omega + i\delta) \equiv \langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{u}}_{q,\alpha}\rangle_{\omega_E\to -i\omega+\delta}$.

Thus, the Fourier transformed Euclidean correlation functions take the form:

$$\begin{aligned} &\frac{1}{L}\langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{u}}_{q,\alpha}\rangle_{\omega_E} \\ =&\frac{U_{p,\alpha}U_{q,\alpha}}{L}\frac{1}{a^4}\int_{x,y,\tau} e^{i\omega_E\tau}\Big\langle \sin\Big(\Delta k_\alpha x + m^{(\alpha)}_K\phi_K(\tau,x)\Big)\sin\Big(\Delta k_\alpha y + m^{(\alpha)}_L\phi_L(0,y)\Big)\Big\rangle \\ =&\frac{U_{p,\alpha}U_{q,\alpha}}{4L}\int_{x,y,\tau} e^{i\omega_E\tau}\Big[e^{i\Delta k_\alpha(x-y)}\Big\langle \frac{e^{im^{(\alpha)}_K\phi_K(\tau,x)}}{a^2}\frac{e^{-im^{(\alpha)}_L\phi_L(0,y)}}{a^2}\Big\rangle \\ &+e^{-i\Delta k_\alpha(x-y)}\Big\langle \frac{e^{-im^{(\alpha)}_K\phi_K(\tau,x)}}{a^2}\frac{e^{im^{(\alpha)}_L\phi_L(0,y)}}{a^2}\Big\rangle\Big] \\ =&\frac{U_{p,\alpha}U_{q,\alpha}}{2L}\int_{x,y,\tau} e^{i\omega_E\tau}\cos\Big(\Delta k_\alpha(x-y)\Big)\frac{(\pi T)^4}{\sinh^3\Big(\pi T((x-y)-i\tau)\Big)\sinh\Big(\pi T((x-y)+i\tau)\Big)}, \end{aligned} \tag{E.21}$$

where $x, y \in (-L, L)$ with $L \to \infty$ and $\tau \in [0, 1/T]$. The first equality follows from

direct substitution of Eqs. (E.15, E.16); for the second equality, we have only retained the non-zero terms in the product; for the third equality, we have used the standard thermal real-space Euclidean two-point function of a dimension $(\Delta_R, \Delta_L) = (3/2, 1/2)$ vertex operator $\frac{1}{\alpha^2}\exp(im_J^{(\alpha)}\phi_J)$.[163] It is a great simplification of the calculation that all vertex operators considered have the same scaling dimension. If only a fraction of the operators necessary to relax the currents had dimension $(3/2, 1/2)$ and the remaining operators were of higher dimension, it would be straightforward to calculate their effects by methods similar to those presented here. These operators would give subleading contributions to the memory matrix leading to slower relaxation of some conserved currents. As a result these operators would give the dominant contributions to the matrix of conductivities.

Similar to Appendix E.1, we evaluate Eq. (E.21) by first making the change of variables $x_\pm = x \pm y$ and $\xi = e^{2\pi i T\tau}$. We assume a short-distance cutoff $0 < a < |x - y|$. The integral over x_+ factors out, canceling the $1/L$ prefactor, and we are left with the following integral to evaluate:

$$\begin{aligned}
&\frac{1}{L}\langle F^{\mathrm{u}}_{p,\alpha}; F^{\mathrm{u}}_{q,\alpha}\rangle_{\omega_E} \\
=&-4\pi^4 T^3 U_{p,\alpha}U_{q,\alpha}\int_{x_-} e^{-2\pi T x_-}\cos(\Delta k_\alpha x_-)\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{\zeta^{\frac{\omega_E}{2\pi T}+1}}{(\zeta - e^{-2\pi T x_-})^3(\zeta - e^{2\pi T x_-})} \\
=&\frac{\pi^2 T U_{p,\alpha}U_{q,\alpha}}{4} \\
\times&\int_a^\infty dx_- \frac{e^{-\omega_E x_-}\cos(\Delta k_\alpha x_-)}{\sinh^3(2\pi T x_-)}\Big[4\pi^2T^2 + \omega_E^2\sinh^2(2\pi T x_-) + \pi T\omega_E\sinh(4\pi T x_-)\Big].
\end{aligned} \tag{E.22}$$

Next, we Wick rotate, $\omega_E \to -i\omega + \delta$, Eq. (E.22) to obtain the retarded Green's function,

$$\begin{aligned}
&\frac{1}{L}\langle F^{\rm u}_{p,\alpha}; F^{\rm u}_{q,\alpha}\rangle_\omega \\
=&\frac{\pi^2 T U_{p,\alpha}U_{q,\alpha}}{4}\int_a^\infty dx_- \frac{e^{-\delta x_- + i\omega x_-}\cos(\Delta k_\alpha x_-)}{\sinh^3(2\pi T x_-)} \\
\times&\Big[4\pi^2T^2 + (-i\omega+\delta)^2\sinh^2(2\pi T x_-) + \pi T(-i\omega+\delta)\sinh(4\pi T x_-)\Big]. \qquad \text{(E.23)}
\end{aligned}$$

The remaining integral in Eq. (E.22) can be evaluated exactly to obtain the memory matrix elements $(\hat{\mathcal{M}}^{\rm u})^{pq}_\alpha$ defined in Eq. (E.12). The exact expression is rather complicated and so we shall examine it in various low-frequency and low-temperature limits for both commensurate and incommensurate fillings. To study the low-frequency and low-temperature behavior of $(\hat{\mathcal{M}}^{\rm u})^{pq}_\alpha$, we first perform two expansions. First, we expand the result as the short-distance cutoff $a \to 0$, keeping only the singular and finite non-zero terms. Any $a \to 0$ singularities are a reflection of the short-distance divergences of the correlation function. Second, we expand to linear order in δ, however, we find it sufficient to study the resulting expression at $\delta = 0$ as the real part of the memory matrix is generally non-zero at finite ω and finite T.

Commensurate Fillings

For commensurate fillings we set $\Delta k_\alpha = 0$. For $\omega/T \ll 1$, the expression for the memory matrix element at commensurate fillings has the following behavior,

$$(\hat{\mathcal{M}}^{\rm u})^{pq}_\alpha\Big(\frac{\omega}{T}\ll 1\Big) = U_{p,\alpha}U_{q,\alpha}\Big[\frac{\pi^4}{32}T + i\frac{\pi\omega}{16}\log(a_1 T)\Big], \qquad \text{(E.24)}$$

where we have dropped all $\mathcal{O}(\delta)$ terms and absorbed all constants via a redefinition of the cutoff $a \to a_1$. We shall make these multiplicative redefinitions of the short-distance cutoff $a \to a_i$ in each of the following expressions. In the opposite regime when $T/\omega \ll 1$,

we find the following expression for the memory matrix elements at commensurate filling,

$$(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_{\alpha}\Big(\frac{T}{\omega}\ll 1\Big) = U_{p,\alpha}U_{q,\alpha}\Big[\frac{\pi^2}{32}\omega + i\frac{\pi}{16}\omega\log(a_2\omega)\Big], \tag{E.25}$$

where $a_1 \neq a_2$.

Incommensurate Fillings

When the filling is incommensurate, $\Delta k_\alpha \neq 0$. We shall study the memory matrix for frequencies and temperatures $\omega, T \ll \Delta k_\alpha$.

For $\omega/T \ll 1$, the expression for the memory matrix elements at incommensurate fillings have the following behavior,

$$\begin{aligned}(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_{\alpha}\Big(\frac{\omega}{T}\ll 1\Big) =& U_{p,\alpha}U_{q,\alpha}\Big[\frac{\pi^2}{32}\Big(\frac{(\Delta k_\alpha)^2}{T}+4\pi^2 T\Big)e^{-\frac{\Delta k_\alpha}{2T}}\\ &+i\frac{\pi\omega}{16}\log(a_3\Delta k_\alpha)\Big],\end{aligned} \tag{E.26}$$

where we have only retained the leading term present for $T \to 0$. Precisely at $T = 0$ (but first $\omega \to 0$), the real part of the $(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_{\alpha}\Big(\frac{\omega}{T}\ll 1\Big)$ vanishes when $\Delta k_\alpha \neq 0$ and we obtain a purely imaginary memory matrix which implies a finite Drude weight. When $T/\omega \ll 1$, the incommensurate memory matrix takes the form,

$$\begin{aligned}(\hat{\mathcal{M}}^{\mathrm{u}})^{pq}_{\alpha}\Big(\frac{T}{\omega}\ll 1\Big) =& U_{p,\alpha}U_{q,\alpha}\Big[\frac{\pi^2}{16}\Big(\frac{(\Delta k_\alpha)^2}{\omega}+\omega\Big)e^{\frac{\omega-\Delta k_\alpha}{2T}}\\ &+\frac{i\pi}{32}\Big(\omega\log\Big(a_4^2((\Delta k_\alpha)^2-\omega^2)\Big)\\ &+\frac{(\Delta k_\alpha)^2}{\omega}\log\Big(1-\frac{\omega^2}{(\Delta k_\alpha)^2}\Big)\Big)\Big].\end{aligned} \tag{E.27}$$

While we have studied the memory matrix for incommensurate fillings in the limit $\omega, T \ll \Delta k_\alpha$, we have checked that the initial expression obtained before taking the

low-frequency or low-temperature limits reverts to the commensurate values by taking $\Delta k_\alpha = 0$.

E.2.3 Evaluation of the $(\hat{\mathcal{M}}^{\rm dis})^{pq}_\alpha$

Because the same vertex operators are used in both the umklapp and disorder-mediated interactions, the calculation of the disorder memory matrix elements $(\hat{\mathcal{M}}^{\rm dis})^{pq}_\alpha$ will be very similar to that of the previous section. We begin with the evaluation of the retarded two-point correlation functions $\langle F^{\rm dis}_{p,\alpha}; F^{\rm dis}_{q,\alpha}\rangle_\omega$ which we determine by analytically continuing the Euclidean correlator $\langle F^{\rm dis}_{p,\alpha}; F^{\rm dis}_{q,\alpha}\rangle_{\omega_E}$. We again simplify the ensuing expressions by introducing the coefficients,

$$\begin{aligned}\tilde{U}_{J^e_I,\alpha} =& i\, e\, {\rm sgn}(N-I){\rm sgn}(N-J)V_{IJ}m^{(\alpha)}_J,\\ \tilde{U}_{P_D,\alpha} =& -v^2,\end{aligned} \tag{E.28}$$

that occur in the disorder commutators in Eqs. (E.18,E.19).

Unlike the correlators of the commutators involved in the umklapp calculation, we need to examine each of $\langle F^{\rm dis}_{J^e_I,\alpha}; F^{\rm dis}_{J^e_J,\alpha}\rangle_{\omega_E}$, $\langle F^{\rm dis}_{J^e_I,\alpha}; F^{\rm dis}_{P_D,\alpha}\rangle_{\omega_E}$, and $\langle F^{\rm dis}_{P_D,\alpha}; F^{\rm dis}_{P_D,\alpha}\rangle_{\omega_E}$ in turn. First consider:

$$\begin{aligned}&\frac{1}{L}\langle F^{\rm dis}_{J^e_I,\alpha}; F^{\rm dis}_{J^e_J,\alpha}\rangle_{\omega_E=i\omega+\delta}\\ &= -\frac{(\pi T)^4\tilde{U}_{J^e_I,\alpha}\tilde{U}_{J^e_J,\alpha}}{LD}\int_{x,y,\tau} e^{i\omega_E\tau}\frac{\xi_\alpha(x)\xi^*_\alpha(y)+\xi^*_\alpha(x)\xi_\alpha(y)}{\sinh^3\Big(\pi T((x-y)-i\tau)\Big)\sinh\Big(\pi T((x-y)+i\tau)\Big)}\\ &= \frac{\pi^2 T\tilde{U}_{J^e_I,\alpha}\tilde{U}_{J^e_J,\alpha}}{4LD}\int_{x_+}\int_a^\infty dx_-\frac{e^{(-\delta+i\omega)x_-}}{\sinh^3(2\pi T x_-)}\Big[\xi_\alpha(x)\xi^*_\alpha(y)+\xi^*_\alpha(x)\xi_\alpha(y)\Big]\\ &\times\Big[4\pi^2T^2+(-i\omega+\delta)^2\sinh^2(2\pi T x_-)+\pi T(-i\omega+\delta)\sinh(4\pi T x_-)\Big],\end{aligned} \tag{E.29}$$

where $x_\pm = x \pm y$ and we have performed identical manipulations to those explained in the previous section to evaluate Eqs. (E.21), (E.22), and (E.23).

To explicitly evaluate the integrals over x_+ and x_- in Eq. (E.29), we must choose a form for the functions $\xi_\alpha(x)$ parameterizing the disorder. As we have discussed, we have chosen to consider zero-mean Gaussian-correlated disorder, $\overline{\xi(x)} = 0, \overline{\xi_\alpha(x)\xi^*_\alpha(y)} = D\delta(x-y)$. To make contact with the pure system calculation of umklapp scattering at incommensurate fillings, we comment that this form of the disorder may be obtained by choosing a disorder potential, $\xi_\alpha(x) = \int_{\Delta p_\alpha} \tilde{\xi}(\Delta p_\alpha) e^{i\Delta p_\alpha x}$ with $\tilde{\xi}(\Delta p_\alpha) = 1$. We see that incommensurate fillings can be understood as a particular disorder realization with $\tilde{\xi}(\Delta p_\alpha) = \delta(\Delta p_\alpha - \Delta k_\alpha)$.

Before integrating over x_+ and x_- in Eq. (E.29), we first disorder average. This allows us to again factor out the x_+ integral to cancel the $1/L$ prefactor and also to replace the product of disorder potentials $\xi_\alpha(x)$ inside the first brackets by $2D\delta(x-y)$, where the $\delta(x-y)$ is understood to evaluate all terms containing $x_- = a$, the short-distance cutoff. We find:

$$\begin{aligned}\frac{1}{L}\langle F^{\rm dis}_{J^e_I,\alpha}; F^{\rm dis}_{J^e_J,\alpha}\rangle_\omega \\ &= \frac{\pi^2 T \tilde{U}_{J^e_I,\alpha}\tilde{U}_{J^e_J,\alpha}}{2}\frac{e^{(-\delta+i\omega)a}}{\sinh^3(2\pi T a)} \\ &\times\Big[4\pi^2T^2 + (-i\omega+\delta)^2\sinh^2(2\pi T a) + \pi T(-i\omega+\delta)\sinh(4\pi T a)\Big] \quad \text{(E.30)}\end{aligned}$$

Next, consider $\frac{1}{L}\langle F^{\rm dis}_{J^e_I,\alpha}; F^{\rm dis}_{P_D,\alpha}\rangle_\omega$. The calculation of this correlator is identical to the previous one except that the overall coefficient now involves the $\tilde{U}_{J^e_I,\alpha}\tilde{U}_{P_D,\alpha}$ and the

product of disorder potentials in the first line of Eq. (E.29) is replaced:

$$\begin{aligned}\xi_\alpha(x)\xi^*_\alpha(y)+\xi^*_\alpha(x)\xi_\alpha(y)\to&\xi_\alpha(x)\partial_y\xi^*_\alpha(y)-\xi^*_\alpha(x)\partial_y\xi_\alpha(y)\\=&\partial_y\Big(\xi_\alpha(x)\xi^*_\alpha(y)-\xi^*_\alpha(x)\xi_\alpha(y)\Big).\end{aligned}\tag{E.31}$$

Upon disorder averaging, the term in the parentheses in Eq. (E.31) vanishes. Thus, we find:

$$\frac{1}{L}\langle F^{\rm dis}_{J^e_I,\alpha};F^{\rm dis}_{P_D,\alpha}\rangle_\omega=0.\tag{E.32}$$

There is no overlap to leading order in the disorder-variance D between the electrical and thermal currents.

Finally, we evaluate $\frac{1}{L}\langle F^{\rm dis}_{P_D,\alpha};F^{\rm dis}_{P_D,\alpha}\rangle_\omega$ by replacing in Eq. (E.29):

$$\begin{aligned}\tilde U_{J^e_I,\alpha}\tilde U_{J^e_J,\alpha}\to&\tilde U_{P_D,\alpha}\tilde U_{P_D,\alpha},\\ \xi_\alpha(x)\xi^*_\alpha(y)+\xi^*_\alpha(x)\xi_\alpha(y)\to&\partial_x\xi_\alpha(x)\partial_y\xi^*_\alpha(y)+\text{h.c.}\end{aligned}\tag{E.33}$$

Disorder averaging, performing the integration by parts with respect to $\partial_{x/y}=\partial_{x_+}\pm\partial_{x_-}$, discarding all boundary terms, and evaluating $x_-=a$, we find:

$$\begin{aligned}\frac{1}{L}\langle F^{\rm dis}_{P_D,\alpha};F^{\rm dis}_{P_D,\alpha}\rangle_\omega=&\frac{\pi^2T\tilde U_{P_D}\tilde U_{P_D}}{2}\partial_{x_-}\partial_{x_-}\Big[\frac{e^{(-\delta+i\omega)x_-}}{\sinh^3(2\pi Tx_-)}\Big[4\pi^2T^2+(-i\omega+\delta)^2\sinh^2(2\pi Tx_-)\\&+\pi T(-i\omega+\delta)\sinh(4\pi Tx_-)\Big]\Big]\Big|_{x_-=a}\end{aligned}\tag{E.34}$$

Equipped with the above correlation functions, we may now evaluate the memory matrix elements $(\hat{\mathcal{M}}^{\rm dis})_\alpha^{J_I^e J_I^e}$ and $(\hat{\mathcal{M}}^{\rm dis})_\alpha^{P_D P_D}$. As before, we determine these memory matrix elements by expanding about the limit $a \to 0$ and subsequently expanding about $\delta = 0$. It is sufficient to set $\delta = 0$. In summary, we find:

$$(\hat{\mathcal{M}}^{\rm dis})_\alpha^{J_I^e J_I^e} = \tilde{U}_{J_I^e,\alpha}\tilde{U}_{J_J^e,\alpha}\Big[\frac{2\pi^3}{3}T^2 + \frac{\pi}{6}\omega^2 - i\frac{3\pi}{24}\frac{\omega}{a}\Big], \tag{E.35}$$

$$(\hat{\mathcal{M}}^{\rm dis})_\alpha^{J_I^e P_D} = 0, \tag{E.36}$$

$$(\hat{\mathcal{M}}^{\rm dis})_\alpha^{P_D P_D} = \tilde{U}_{P_D,\alpha}\tilde{U}_{P_D,\alpha}\Big[\frac{8\pi^5}{5}T^4 + \frac{2\pi^3}{3}T^2\omega^2 + \frac{\pi}{15}\omega^4 + i\frac{\pi}{15}\frac{\omega}{a^3}\Big]. \tag{E.37}$$

We notice that the logarithmic singularities that occurred in the umklapp memory matrix elements for $a = 0$ are replaced by power-law singularities. Such singularities reflect the short-distance divergences as correlation function insertion points become coincident. They only occur in the imaginary part of the memory matrix elements at finite frequencies. Our prescription is to remove such power-law divergences by hand and concentrate on the real parts of the memory matrix elements that determine the long-wavelength response of the system. This prescription leads to agreement with related computations[178, 179] studying the tunneling conductance between quantum wires at a single point contact.

E.3 $\hat{N}$ Matrix

In this appendix, we show that $\hat{N} = 0$ to quadratic order in the umklapp λ and disorder $\lambda^{\rm dis}$ couplings using rather general considerations. Recall the definition:

$$(\hat{N})_{pq} \equiv \hat{\chi}_{p\dot{q}} = \Big(\mathcal{Q}_p, i[\sum_\alpha (H^{\rm u}_\alpha + H^{\rm dis}_\alpha), \mathcal{Q}_q]\Big). \tag{E.38}$$

E.3.1 Umklapp Contributions

First, consider the contribution to $\hat{N}$ from umklapp processes $H^{\rm u}_\alpha$. Observe that $i[H^{\rm u}_\alpha, \mathcal{Q}_q] = \lambda F^{\rm u}_{q,\alpha}$ and $i[H^{\rm dis}_\alpha, \mathcal{Q}_q] = \sqrt{D}\lambda^{\rm dis}$, where $\mathcal{Q}_q \in \{J^e_I, P_D\}$, so that by using the definition of the static susceptibility and conventions in Appendix E.1:

$$(\hat{N})_{pq} = \frac{\lambda}{L} \lim_{\omega_E \to 0} \int_\tau e^{i\omega_E \tau} \langle \mathcal{Q}_p(\tau) F^{\rm u}_{q,\alpha}(0)\rangle, \tag{E.39}$$

and likewise for the disorder contribution studied momentarily where the bracket denotes the Euclidean correlation function at temperature T. At leading order in λ, the above two-point function $\langle \mathcal{Q}_p(\tau) F^{\rm u}_{q,\alpha}(0)\rangle$ vanishes when computed with respect to $S_{\rm b}$; more specifically, $\langle \partial_x \phi_I(\tau, x) e^{i m^{(\alpha)}_J \phi_J(0,y)}\rangle = 0$ and $\langle \partial_x \phi_I(\tau, x) \partial_x \phi_I(\tau, x) e^{i m^{(\alpha)}_J \phi_J(0,y)}\rangle = 0$ when computed with respect to $S_{\rm b}$. At quadratic order, λ^2, there is the correction,

$$\delta(\hat{N})_{pq} = \frac{\lambda^2}{L} \lim_{\omega_E \to 0} \int_{\tau,\tau',z} e^{i\omega_E \tau} \langle \mathcal{Q}_p(\tau) F^{\rm u}_{q,\alpha}(0) H^{\rm u}_\alpha(\tau', z)\rangle. \tag{E.40}$$

The above correlation function, computed with respect to $S_{\rm b}$ factorizes, into the sum of two three-point functions:

$$\begin{aligned}
&\frac{\lambda^2}{L}\int_{\tau,\tau',z} e^{i\omega_E\tau}\langle \mathcal{Q}_p(\tau)F^{\mathrm{u}}_{q,\alpha}(0)H^{\mathrm{u}}_{\alpha}(\tau',z)\rangle \\
\propto&\frac{i\lambda^2(\pi T)^5}{L}\int_{\tau,\tau',x,y,z} e^{i\omega_E\tau}\Big[\frac{C_1 e^{-i\Delta k_\alpha X_{zy}}-C_2 e^{i\Delta k_\alpha X_{zy}}}{\sinh\Big(\pi T(X_{zy}+i\tau')\Big)}\Big] \\
\times&\frac{1}{\sinh^h\Big(\pi T(X_{xy}-i\tau)\Big)\sinh^h\Big(\pi T(X_{xz}-i\tau+i\tau')\Big)\sinh^{3-h}\Big(\pi T(X_{zy}-i\tau')\Big)},
\end{aligned} \tag{E.41}$$

for constants $C_1 = (-1)^h C_2$ (whose precise magnitude will not be required) equal to the operator product coefficients for the fusion, $\mathcal{Q}_p \exp(im_I^{(\alpha)}\phi_I) \sim \exp(im_I^{(\alpha)}\phi_I)$, and $h = 1$ when $\mathcal{Q}_p = J_I^e$ and $h = 2$ when $\mathcal{Q}_p = P_D$. Above, we have introduced the "difference coordinates" $X_{xy} = x - y, X_{xz} = x - z, X_{zy} = z - y$. At $\omega_E = 0$, we notice that the integrand is odd under the reflection of all spatial and temporal coordinates followed by the shifts, $\tau, \tau' \to \tau - 1/T, \tau' - 1/T$. Therefore, the integral is zero at $\omega_E = 0$ and the quadratic contribution to $\hat{N}$ from umklapp processes vanishes.

E.3.2 Disorder Contributions

Next, consider the contributions to $\hat{N}$ from disorder-mediated processes $H^{\mathrm{dis}}_{\alpha}$. The term linear in λ^{dis} again vanishes for the same reason as before. At quadratic order, we consider Eq. (E.40) with the superscript u replaced by dis. The form of the resulting three-point function is very similar to that which appears in Eq. (E.41). The difference is due to the disorder ξ_α appearing in the disorder commutators Eqs. (E.18, E.19) and $H^{\mathrm{dis}}_{\alpha}$. For $F^{\mathrm{dis}}_{q,\alpha} = F^{\mathrm{dis}}_{J^e_I,\alpha}$, we disorder average and insert $\delta(y - z)$ into integrand in Eq. (E.41) at $\omega_E = 0$: when $\mathcal{Q}_p = J_I^e$, the three-point function vanishes using the above reflection and translation argument; when $\mathcal{Q}_p = P_D$, the three-point function vanishes identically

after setting $y = z$ and using $C_1 = C_2$ for $h = 2$. For $F^{\rm dis}_{q,\alpha} = F^{\rm dis}_{P_D,I}$, we disorder average, replace the relative minus sign between C_1 and C_2 by $(+1)$, and insert $\partial_y \delta(y-z)$ into the integrand in Eq. (E.41): when $\mathcal{Q}_p = J^e_I$, the integrand vanishes identically similar to P_D before; when $\mathcal{Q}_p = P_D$, we may again apply the reflection and translation argument to conclude that the integral vanishes at $\omega_E = 0$. Thus, we may safely ignore the $\hat{N}$ matrix in our transport calculations.

E.4 Exact Marginality Along the 'Decoupled Surface'

In this Appendix, we argue perturbatively for the exact marginality, along the decoupled surface, of the dimension $(\Delta_R, \Delta_L) = (3/2, 1/2)$ operators used to relax the electrical and thermal currents. Our argument strictly applies in the scaling limit in which only classically marginal and relevant interactions are retained in the low-energy effective theory with irrelevant interactions being set to zero.

Recall from Sec. 6.2 that the decoupled surface is a subspace within the hyperconductor phase in which the interaction matrix $\tilde{V}_{IJ}$ is block diagonal. The scaling dimensions of operators are independent of $\tilde{V}_{IJ}$ when the theory lies on the decoupled surface; however, scaling dimensions vary continuously with $\tilde{V}_{IJ}$ upon departing from the decoupled surface.

We consider the collection of operators $\mathcal{O}_\alpha = \cos\left(m^{(\alpha)}_I \phi_I\right)$ with scaling dimension equal to $(3/2, 1/2)$ along the decoupled surface whose coupling constants we denote by g_α . These operators are exactly marginal if their beta function β_{g_α} vanishes to all orders

in the couplings of the theory,

$$\dot{g}_\alpha = \beta_{g_\alpha}, \tag{E.42}$$

where the dot denotes a variation of the coupling with respect to the renormalization group scale. We will understand the contributions to β_{g_α} as arising from corrections to scaling (i.e., conformal perturbation theory) of the zero-temperature two-point function,

$$\langle \mathcal{O}_\alpha(z,\bar{z})\mathcal{O}_\alpha(0)\rangle \sim z^{-1}\bar{z}^{-3}, \tag{E.43}$$

for $z = x + i\tau, \bar{z} = x - i\tau$ computed with respect to the fixed point action S_b in Eq. (6.4).[163]

First, observe that $\mathcal{O}_\alpha$ has unit spin, $\Delta_R - \Delta_L$, under the $SO(2) = U(1)$ rotation symmetry of the Euclidean theory. When the action is perturbed, $S_\mathrm{b} \to S_\mathrm{b} + g_\alpha \int \mathcal{O}_\alpha$, the $SO(2)$ symmetry is broken. We may view g_α as a spurion – a "field" that transforms oppositely to the operator it multiplies so that the product is an $SO(2)$ singlet – of this broken rotational symmetry. This means that g_α may be understood to have spin-(-1). With this understanding, we may constrain the form of β_{g_α}.

The left-hand side of Eq. (E.42) is linear in g_α and so the equality implies that β_{g_α} also carries spin-(-1). Thus, we must determine what spin-1 combination of operators could possibly contribute to β_{g_α}.[164] Working in the scaling limit where all irrelevant operators are ignored allows us to disregard any contribution from high-dimension operators with negative spin. There are no marginal spin-(-1) operators because the lowest scaling dimension of a right-moving vertex operator is equal to 3/2. There do exist spin-(-1) relevant and spin-(-2) marginal operators which are quadratic and quartic in the fermions of the left-moving sector along with marginal spin-0, i.e., dimension $(1,1)$ operators, and

spin-2 operators in addition to the marginal $\mathcal{O}_\alpha$ operators. Perturbations by spin-(-1) operators can be absorbed by a field redefinition of the left-moving fermion sector and so we ignore such deformations.

A general contribution to the $\mathcal{O}_\alpha$ two-point function contains N_{-2} spin-(-2) insertions, N_0 spin-0 insertions, N_2 spin-2 insertions, and $N_{\mathcal{O}_\beta}$ $\mathcal{O}_\beta$ insertions. Note that we are collectively referring to all additional insertions of the $\mathcal{O}_\beta$ operators as $N_{\mathcal{O}_\beta}$. In order for $\beta_{\mathcal{O}_\alpha}$ to carry spin equal to -1, we require the number of insertions of various operators to satisfy:

$$2N_{-2} - N_{\mathcal{O}_\beta} - 2N_2 = -1. \tag{E.44}$$

Thus, $N_{\mathcal{O}_\beta}$ should be odd.

All operators in the left-moving sector can be built from products of the fermion operators and their spatial derivatives. Since the left-moving sector is describable in terms of interacting chiral fermions, fermion parity constrains any non-zero contribution to the $\mathcal{O}_\alpha$ two-point function to contain an even number of left-moving fermion operators:

$$4N_{-2} + N_{\mathcal{O}_\beta} + 2N_2 + 2N_0 \in 2\mathbb{Z}. \tag{E.45}$$

The first contribution to the left-hand side of Eq. (E.45) assumes an operator quartic in the fermion operators. An operator that is only quadratic with a single spatial derivative acting on one of the fermions might also contribute. However, this has no effect on the conclusion that the parity of the left-hand side must be even.

Eq. (E.45) is not consistent with Eq. (E.44) as the former requires $N_{\mathcal{O}_\beta}$ to even. The only resolution is that the $\mathcal{O}_\alpha$ operators are exactly marginal in the scaling limit and so $\beta_{g_\alpha} = 0$. There is likewise no renormalization of the Luttinger liquid parameters of $S_{\rm b}$

due to the spin-1 $\mathcal{O}_\alpha$ operators.

Exact marginality of the dimension $(3/2, 1/2)$ operators and the Luttinger parameters along the decoupled surface is a consequence of the chirality or spin-1 nature of the $\mathcal{O}_\alpha$ operators which is ultimately due to the asymmetric nature of the left-moving and right-moving excitations in the asymmetric shorter Leech liquid underlying the hyperconductor studied in this chapter. The de-coupled renormalization group equations described above should be contrasted with those of the Kosterlitz-Thouless transition that involve a dimension $(1, 1)$ vertex operator and the Luttinger parameter.[169] It is this difference that results in the logarithmic corrections to scaling in the expressions for the conductivities in the work of Giamarchi on transport in a 1D Luttinger liquid.[155]

Bibliography

[1] P. W. Anderson *et al.*, "More is different," *Science*, vol. 177, no. 4047, pp. 393–396, 1972.

[2] M. Kac, "Can one hear the shape of a drum?," *The american mathematical monthly*, vol. 73, no. 4, pp. 1–23, 1966.

[3] C. Gordon, D. L. Webb, and S. Wolpert, "One cannot hear the shape of a drum," *Bulletin of the American Mathematical Society*, vol. 27, no. 1, pp. 134–138, 1992.

[4] B. Dóra and R. Moessner, "Luttinger liquid with complex forward scattering: Robustness and berry phase," *Phys. Rev. B*, vol. 93, p. 075127, Feb 2016.

[5] E. Plamadeala, M. Mulligan, and C. Nayak, "Short-range entangled bosonic states with chiral edge modes and t duality of heterotic strings," *Phys. Rev. B*, vol. 88, p. 045131, Jul 2013.

[6] E. Plamadeala, M. Mulligan, and C. Nayak, "Perfect metal phases of one-dimensional and anisotropic higher-dimensional systems," *Phys. Rev. B*, vol. 90, p. 241101, Dec 2014.

[7] E. Plamadeala, M. Mulligan, and C. Nayak, "Transport in a one-dimensional hyperconductor," *Phys. Rev. B*, vol. 93, p. 125142, Mar 2016.

[8] J. Cano, M. Cheng, M. Mulligan, C. Nayak, E. Plamadeala, and J. Yard, "Bulk-edge correspondence in (2+1)-dimensional abelian topological phases," *Phys. Rev. B*, vol. 89, p. 115116, 2014.

[9] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices, and Groups*, (Springer-Verlag, New York, 1999).

[10] O. D. King, "A mass formula for unimodular lattices with no roots," *Mathematics of Computation*, vol. 72, pp. 839–863, 2003.

[11] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory. Vol. 1: Introduction*, (Cambridge University Press, 1987).

[12] J. Milnor, "Eigenvalues of the laplace operator on certain manifolds," *Proc Natl Acad Sci USA*, vol. 51, no. 4, p. 542, 1964.

[13] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*. Springer-Verlag, 1973.

[14] D. A. Buell, *Binary Quadratic Forms*. Springer, New York, 1989.

[15] G. Watson, "Transformations of a quadratic form which do not increase the class-number," *Proc. London Math. Soc.*, vol. 12, p. 577587, 1962.

[16] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, "Non-abelian anyons and topological quantum computation," *Rev. Mod. Phys.*, vol. 80, p. 1083, 2008.

[17] X. Chen, Z.-X. Liu, and X.-G. Wen, "Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations," Phys. Rev. **B**, vol. 84, p. 235141, 2011.

[18] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, "Symmetry protected topological orders and the group cohomology of their symmetry group," Phys. Rev. **B**, 2013.

[19] A. Kitaev, http://online.kitp.ucsb.edu/online/topomat11/kitaev.

[20] Y.-M. Lu and A. Vishwanath, "Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach," Phys. Rev. **B**, vol. 86, p. 125119, 2012.

[21] A. Kitaev and J. Preskill, "Topological entanglement entropy," *Phys. Rev. Lett.*, vol. 96, p. 110404, Mar 2006.

[22] M. Levin and X.-G. Wen, "Detecting topological order in a ground state wave function," *Phys. Rev. Lett.*, vol. 96, p. 110405, Mar 2006.

[23] C. L. Kane and E. J. Mele, "Z_2 topological order and the quantum spin hall effect," *Phys. Rev. Lett.*, vol. 95, p. 146802, Sep 2005.

[24] C. L. Kane and E. J. Mele, "Quantum spin hall effect in graphene," *Phys. Rev. Lett.*, vol. 95, p. 226801, Nov 2005.

[25] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).

[26] L. Fu, C. L. Kane, and E. J. Mele, "Topological insulators in three dimensions," *Phys. Rev. Lett.*, vol. 98, p. 106803, Mar 2007.

[27] J. E. Moore and L. Balents, "Topological invariants of time-reversal-invariant band structures," *Phys. Rev. B*, vol. 75, p. 121306, Mar 2007.

[28] R. Roy, "Topological phases and the quantum spin hall effect in three dimensions," *Phys. Rev. B*, vol. 79, p. 195322, May 2009.

[29] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, "Topological field theory of time-reversal invariant insulators," *Phys. Rev. B*, vol. 78, p. 195424, Nov 2008.

[30] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, "Classification of topological insulators and superconductors in three spatial dimensions," *Phys. Rev. B*, vol. 78, p. 195125, Nov 2008.

[31] M. Z. Hasan and C. L. Kane, "*Colloquium*: Topological insulators," *Rev. Mod. Phys.*, vol. 82, pp. 3045–3067, Nov 2010.

[32] X.-L. Qi and S.-C. Zhang, "Topological insulators and superconductors," *Rev. Mod. Phys.*, vol. 83, pp. 1057–1110, Oct 2011.

[33] Y.-M. Lu and A. Vishwanath, arXiv:1302.2634.

[34] R. B. Laughlin, "Quantized hall conductivity in two dimensions," *Phys. Rev. B*, vol. 23, pp. 5632–5633, May 1981.

[35] B. I. Halperin, "Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential," *Phys. Rev. B*, vol. 25, pp. 2185–2190, Feb 1982.

[36] C. L. Kane and M. P. A. Fisher, "Quantized thermal transport in the fractional quantum hall effect," *Phys. Rev. B*, vol. 55, pp. 15832–15837, Jun 1997.

[37] J. P. Serre, *A Course in Arithmetic*, (Springer, Berlin, 1973).

[38] P. H. Ginsparg, "Comment on Toroidal Compactification of Heterotic Superstrings," *Phys. Rev. D*, vol. 35, p. 648, 1987.

[39] X. G. Wen and A. Zee, "Classification of Abelian quantum Hall states and matrix formulation of topological fluids," *Phys. Rev. B*, vol. 46, no. 4, pp. 2290–301, 1992.

[40] X. G. Wen *Adv. Phys.*, vol. 44, p. 405, 1995.

[41] N. Read, "Excitation structure of the hierarchy scheme in the fractional quantum Hall effect," *Phys. Rev. Lett.*, vol. 65, pp. 1502–1505, Sep 1990.

[42] B. Blok and X. G. Wen, "Effective theories of the fractional quantum hall effect: Hierarchy construction," *Phys. Rev. B*, vol. 42, pp. 8145–8156, Nov 1990.

[43] X. G. Wen, "Topological orders in rigid states," *Int. J. Mod. Phys. B*, vol. 4, no. 2, pp. 239–71, 1990.

[44] D. Belov and G. W. Moore, May 2005. arXiv:hep-th/0505235.

[45] S. Elitzur, G. W. Moore, A. Schwimmer, and N. Seiberg, "Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory," *Nucl. Phys. B*, vol. 326, p. 108, 1989.

[46] X.-G. Wen, "Theory of the edge states in fractional quantum Hall effects," *Int.J.Mod.Phys. B*, vol. 6, pp. 1711–1762, 1992.

[47] E. Witten, July 2003. arXiv:hep-th/0307041.

[48] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).

[49] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl.Phys. B*, vol. 256, p. 253, 1985.

[50] Thierry Giamarchi, *Quantum Physics in One Dimension*, (Oxford University Press, 2004).

[51] Michael Levin, arXiv:1301.7355.

[52] F. D. M. Haldane, "Stability of chiral luttinger liquids and abelian quantum hall states," *Phys. Rev. Lett.*, vol. 74, pp. 2090–2093, Mar 1995.

[53] T. Senthil and M. Levin, "Integer quantum hall effect for bosons," *Phys. Rev. Lett.*, vol. 110, p. 046801, Jan 2013.

[54] Michael Hartley Freedman, J. Differential Geometry **17**, 357 (1982).

[55] X. G. Wen, "Gapless boundary excitations in the quantum hall states and in the chiral spin states," *Phys. Rev. B*, vol. 43, pp. 11025–11036, May 1991.

[56] F. D. M. Haldane, "Stability of chiral luttinger liquids and abelian quantum hall states," *Phys. Rev. Lett.*, vol. 74, pp. 2090–2093, Mar 1995.

[57] H.-C. Kao, C.-H. Chang, and X.-G. Wen, "Binding transition in quantum hall edge states," *Phys. Rev. Lett.*, vol. 83, pp. 5563–5566, 1999.

[58] C. L. Kane, M. P. A. Fisher, and J. Polchinski, "Randomness at the edge: Theory of quantum hall transport at filling ν=2/3," *Phys. Rev. Lett.*, vol. 72, pp. 4129–4132, Jun 1994.

[59] X.-G. Wen, "Theory of the edge states in fractional quantum Hall effects," *Int. J. Mod. Phys. B*, vol. 6, pp. 1711–1762, 1992.

[60] J. Frohlich and A. Zee, "Large scale physics of the quantum hall fluid," *Nucl. Phys. B*, vol. 364, 1991.

[61] J. Frohlich, U. M. Studer, and E. Thiran, *On Three Levels*, Springer US, 225-232, 1994.

[62] J. Frohlich and E. Thiran, "Integral quadratic forms, kac-moody algebras, and fractional quantum hall effect. an ade-o classification," *Journal of Statistical Physics*, vol. 76, p. 209, 1994.

[63] J. Frohlich, C. A.H., F. Gabbiani, T. Kerler, C. King, P. Marchetti, U. Studer, and E. Thiran, "The fractional quantum hall effect, chern-simons theory, and integral lattices," *Proc. of ICM.*, vol. 94, 1995.

[64] S. D. Stirling, *Abelian Chern-Simons theory with toral gauge group, modular tensor categories, and group categories*. PhD thesis, The University of Texas at Austin, 2008. arXiv:0807.2857.

[65] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices, and Groups*, (Springer-Verlag, New York, 1999).

[66] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications," *Math. USSR Izv.*, vol. 14, p. 103, 1980.

[67] E. Witten, "Quantum field theory and the Jones polynomial," *Comm. Math. Phys.*, vol. 121, pp. 351–399, 1989.

[68] A. Kitaev, "Anyons in an exactly solved model and beyond," *Ann. Phys.*, vol. 321, pp. 2 – 111, 2006.

[69] H. Cohen, *A Course in Computational Algebraic Number Theory*. New York: Springer, 1996.

[70] C. d. C. Chamon and X. G. Wen, "Sharp and smooth boundaries of quantum hall liquids," *Phys. Rev. B*, vol. 49, pp. 8227–8241, Mar 1994.

[71] E. Witten, "Quantum field theory and the jones polynomial," *Comm. Math. Phys.*, vol. 121, p. 351, 1989.

[72] J. W. S. Cassels, *Rational Quadratic Forms*. London: Academic Press, 1978.

[73] D. Lorch and M. Kirschmer, "Single-Class Genera of Positive Integral Lattices," *LMS Journal of Computation and Mathematics*, vol. 16, pp. 172–186, 2013.

[74] W. Stein *et al.*, *Sage Mathematics Software (Version 5.12)*. The Sage Development Team, 2013. `http://www.sagemath.org`.

[75] X. G. Wen and A. Zee, "Shift and spin vector: New topological quantum numbers for the hall fluids," *Phys. Rev. Lett.*, vol. 69, pp. 953–956, Aug 1992.

[76] X. G. Wen, cond-mat/1212.5121.

[77] C. L. Kane and M. P. A. Fisher, "Nonequilibrium noise and fractional charge in the quantum hall effect," *Phys. Rev. Lett.*, vol. 72, p. 724, Jan 1994.

[78] C. d. C. Chamon, D. E. Freed, and X. G. Wen, "Tunneling and quantum noise in one-dimensional luttinger liquids," *Phys. Rev. B*, vol. 51, p. 2363, Jan 1995.

[79] R. De Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, and D. Mahalu, "Direct observation of a fractional charge," *Nature*, vol. 389, p. 162, 1997.

[80] L. Saminadayar, D. C. Glattli, Y. Jin, and B. Etienne, "Observation of the e/3 fractionally charged laughlin quasiparticle," *Phys. Rev. Lett.*, vol. 79, p. 2526, 1997.

[81] C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen, "Two point-contact interferometer for quantum Hall systems," *Phys. Rev. B*, vol. 55, no. 4, pp. 2331–43, 1997.

[82] E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, "A Chern-Simons effective field theory for the Pfaffian quantum Hall state," *Nucl. Phys. B*, vol. 516, no. 3, pp. 704–18, 1998. cond-mat/9711087.

[83] P. Bonderson, A. Kitaev, and K. Shtengel, "Detecting non-Abelian statistics in the $\nu = 5/2$ fractional quantum Hall state," *Phys. Rev. Lett.*, vol. 96, no. 1, p. 016803, 2006.

[84] A. Stern and B. I. Halperin, "Proposed experiments to probe the non-Abelian $\nu = 5/2$ quantum Hall state," *Phys. Rev. Lett.*, vol. 96, no. 1, p. 016802, 2006.

[85] R. L. Willett, L. N. Pfeiffer, and K. W. West, "Measurement of filling factor 5/2 quasiparticle interference with observation of charge e/4 and e/2 period oscillations," *PNAS*, vol. 106, pp. 8853–8858, 2009.

[86] R. L. Willett, L. N. Pfeiffer, and K. W. West, "Alternation and interchange of e/4 and e/2 period interference oscillations consistent with filling factor 5/2 non-Abelian quasiparticles," *Phys. Rev. B*, vol. 82, p. 205301, 2010.

[87] R. L. Willett, L. N. Pfeiffer, and K. W. West 2013. unpublished.

[88] K. v. Klitzing, G. Dorda, and M. Pepper, "New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance," *Phys. Rev. Lett.*, vol. 45, pp. 494–497, Aug 1980.

[89] R. Prange and S. M. Girvin, eds., *The Quantum Hall effect*. New York: Springer-Verlag, 1990.

[90] S. Das Sarma and A. Pinczuk, eds., *Perspectives in quantum Hall effects : novel quantum liquids in low-dimensional semiconductor structures*. New York: Wiley, 1997.

[91] S. Brosiga, K. Ensslin, B. Brar, M. Thomas, and H. Kroemer, "Landau and spin levels in inas quantum wells resolved with in-plane and parallel magnetic fields," *Physica B*, vol. 256-258, pp. 239–242, 1998.

[92] A. F. Young, C. R. Dean, L. Wang, H. Ren, P. Cadden-Zimansky, K. Watanabe, T. Taniguchi, J. Hone, K. L. Shepard, and P. Kim, "Spin and valley quantum Hall ferromagnetism in graphene," *Nature Physics*, vol. 8, pp. 550–556, July 2012.

[93] A. Tsukazaki, A. Ohtomo, T. Kita, Y. Ohno, H. Ohno, and M. Kawasaki, "Quantum Hall Effect in Polar Oxide Heterostructures," *Science*, vol. 315, pp. 1388–, Mar. 2007.

[94] J. P. Eisenstein, K. B. Cooper, L. N. Pfeiffer, and K. W. West, "Insulating and fractional quantum Hall states in the first excited Landau level," *Phys. Rev. Lett.*, vol. 88, no. 7, p. 076801, 2002.

[95] P. A. Lee and T. V. Ramakrishnan, "Disordered electronic systems," *Rev. Mod. Phys.*, vol. 57, pp. 287–337, Apr 1985.

[96] Hikami, S., Larkin, A.I., and Nagaoka, Y., "Spin-orbit interaction and magnetoresistance in the two dimensional random system," *Prog. Theor. Phys.*, vol. 63, pp. 707–710, 1980.

[97] T. Giamarchi and H. J. Schulz, "Anderson localization and interactions in one-dimensional metals," *Phys. Rev. B*, vol. 37, pp. 325–340, Jan 1988.

[98] D. Belitz and T. R. Kirkpatrick, "The anderson-mott transition," *Rev. Mod. Phys.*, vol. 66, pp. 261–380, Apr 1994.

[99] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, "Boson localization and the superfluid-insulator transition," *Phys. Rev. B*, vol. 40, pp. 546–570, Jul 1989.

[100] V. Dobrosavljević, E. Abrahams, E. Miranda, and S. Chakravarty, "Scaling theory of two-dimensional metal-insulator transitions," *Phys. Rev. Lett.*, vol. 79, pp. 455–458, Jul 1997.

[101] S. Chakravarty, L. Yin, and E. Abrahams, "Interactions and scaling in a disordered two-dimensional metal," *Phys. Rev. B*, vol. 58, pp. R559–R562, Jul 1998.

[102] Kivelson, S. A. and Fradkin, E., and Emery, V. J., "Electronic liquid-crystal phases of a doped mott insulator," *Nature*, vol. 393, pp. 550–553, 1998.

[103] L. Golubović and M. Golubović, "Fluctuations of quasi-two-dimensional smectics intercalated between membranes in multilamellar phases of dna-cationic lipid complexes," *Phys. Rev. Lett.*, vol. 80, pp. 4341–4344, May 1998.

[104] L. Golubović and M. Golubović, "Erratum: Fluctuations of quasi-two-dimensional smectics intercalated between membranes in multilamellar phases of dna-cationic lipid complexes [phys. rev. lett. 80, 4341 (1998)]," *Phys. Rev. Lett.*, vol. 81, pp. 5704–5704, Dec 1998.

[105] C. S. O'Hern and T. C. Lubensky, "Sliding columnar phase of dna-lipid complexes," *Phys. Rev. Lett.*, vol. 80, pp. 4345–4348, May 1998.

[106] C. S. O'Hern, T. C. Lubensky, and J. Toner, "Sliding phases in models, crystals, and cationic lipid-dna complexes," *Phys. Rev. Lett.*, vol. 83, pp. 2745–2748, Oct 1999.

[107] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, "Quantum theory of the smectic metal state in stripe phases," *Phys. Rev. Lett.*, vol. 85, pp. 2160–2163, Sep 2000.

[108] A. Vishwanath and D. Carpentier, "Two-dimensional anisotropic non-fermi-liquid phase of coupled luttinger liquids," *Phys. Rev. Lett.*, vol. 86, pp. 676–679, Jan 2001.

[109] R. Mukhopadhyay, C. L. Kane, and T. C. Lubensky, "Sliding luttinger liquid phases," *Phys. Rev. B*, vol. 64, p. 045120, Jul 2001.

[110] X.-L. Qi and S.-C. Zhang, "Topological insulators and superconductors," *Rev. Mod. Phys.*, vol. 83, pp. 1057–1110, Oct 2011.

[111] P. C. Hohenberg, "Existence of long-range order in one and two dimensions," *Phys. Rev.*, vol. 158, pp. 383–386, Jun 1967.

[112] N. D. Mermin and H. Wagner, "Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic heisenberg models," *Phys. Rev. Lett.*, vol. 17, pp. 1133–1136, Nov 1966.

[113] S. R. Coleman, "There are no Goldstone bosons in two-dimensions," *Commun.Math.Phys.*, vol. 31, pp. 259–264, 1973.

[114] O. I. Motrunich and M. P. A. Fisher, "d-wave correlated critical bose liquids in two dimensions," *Phys. Rev. B*, vol. 75, p. 235116, Jun 2007.

[115] E. Plamadeala, M. Mulligan, and C. Nayak, "Short-range entangled bosonic states with chiral edge modes and duality of heterotic strings," *Phys. Rev. B*, vol. 88, p. 045131, Jul 2013.

[116] J. Cano, M. Cheng, M. Mulligan, C. Nayak, E. Plamadeala, and J. Yard, "Bulk-edge correspondence in (2+1)-dimensional abelian topological phases," *Phys. Rev. B*, vol. 89, p. 115116, Mar 2014.

[117] S. Hellerman, "A Universal Inequality for CFT and Quantum Gravity," *JHEP*, vol. 1108, p. 130, 2011.

[118] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.*, vol. 2, pp. 231–252, 1998.

[119] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys.Rev.Lett.*, vol. 69, pp. 1849–1851, 1992.

[120] E. Plamadeala, M. Mulligan, and C. Nayak, "Perfect metal phases of one-dimensional and anisotropic higher-dimensional systems," *Phys. Rev. B*, vol. 90, p. 241101, 2014.

[121] S. A. Kivelson, "Achieving confusion - the puzzle of bad metals." Lecture presented at the KITP Program "Holographic Duality and Condensed Matter Physics", October 2011.

[122] E. W. Carlson, V. J. Emery, S. A. Kivelson, and D. Orgad, *Concepts in High Temperature Superconductivity*. Springer-Verlag, 2002.

[123] P. A. Lee, N. Nagaosa, and X.-G. Wen, "Doping a mott insulator: Physics of high-temperature superconductivity," *Rev. Mod. Phys.*, vol. 78, p. 17, 2006.

[124] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, "Phenomenology of the normal state of cu-o high-temperature superconductors," *Phys. Rev. Lett.*, vol. 63, p. 1996, 1989.

[125] P. W. Anderson, ""luttinger-liquid" behavior of the normal metallic state of the 2d hubbard model," *Phys. Rev. Lett.*, vol. 64, p. 1839, 1990.

[126] B. I. Halperin, P. A. Lee, and N. Read, "Theory of the half-filled landau level," *Phys. Rev. B*, vol. 47, p. 7312, 1993.

[127] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, "Low-energy properties of fermions with singular interactions," *Phys. Rev. B*, vol. 50, p. 14048, 1994.

[128] J. Polchinski, "Low-energy dynamics of the spinon-gauge system," *Nucl. Phys. B*, vol. 422, p. 617, 1994.

[129] C. Nayak and F. Wilczek, "Non-fermi liquid fixed point in 2 + 1 dimensions," *Nucl. Phys. B*, vol. 417, p. 359, 1994.

[130] C. Nayak and F. Wilczek, "Renormalization group approach to low temperature properties of a non-fermi liquid metal," *Nucl. Phys. B*, vol. 430, p. 534, 1994.

[131] S. Chakravarty, R. E. Norton, and O. F. SyljuAAAAsen, "Transverse gauge interactions and the vanquished fermi liquid," *Phys. Rev. Lett.*, vol. 74, p. 1423, 1995.

[132] Dobrosavljević, V. and Abrahams, Elihu and Miranda, E. and Chakravarty, Sudip, "Scaling theory of two-dimensional metal-insulator transitions," *Phys. Rev. Lett.*, vol. 79, p. 455, 1997.

[133] S. Chakravarty, L. Yin, and E. Abrahams, "Interactions and scaling in a disordered two-dimensional metal," *Phys. Rev. B*, vol. 58, p. R559, 1998.

[134] V. Oganesyan, S. A. Kivelson, and E. Fradkin, "Quantum theory of a nematic fermi fluid," *Phys. Rev. B*, vol. 64, p. 195109, 2001.

[135] W. Metzner, D. Rohe, and S. Andergassen, "Soft fermi surfaces and breakdown of fermi-liquid behavior," *Phys. Rev. Lett.*, vol. 91, p. 066402, 2003.

[136] A. Abanov and A. Chubukov, "Anomalous scaling at the quantum critical point in itinerant antiferromagnets," *Phys. Rev. Lett.*, vol. 93, p. 255702, 2004.

[137] M. J. Lawler, D. G. Barci, V. Fernández, E. Fradkin, and L. Oxman, "Nonperturbative behavior of the quantum phase transition to a nematic fermi fluid," *Phys. Rev. B*, vol. 73, p. 085101, 2006.

[138] S.-S. Lee, "Low-energy effective theory of fermi surface coupled with u(1) gauge field in $2+1$ dimensions," *Phys. Rev. B*, vol. 80, p. 165102, 2009.

[139] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, "Controlled expansion for certain non-Fermi-liquid metals," *Phys. Rev. B*, vol. 82, p. 045121, 2010.

[140] M. A. Metlitski and S. Sachdev, "Quantum phase transitions of metals in two spatial dimensions. i. ising-nematic order," *Phys. Rev. B*, vol. 82, p. 075127, 2010.

[141] M. A. Metlitski and S. Sachdev, "Quantum phase transitions of metals in two spatial dimensions. II. Spin density wave order," *Phys. Rev. B*, vol. 82, p. 075128, 2010.

[142] S. A. Hartnoll, D. M. Hofman, M. A. Metlitski, and S. Sachdev, "Quantum critical response at the onset of spin-density-wave order in two-dimensional metals," *Phys. Rev. B*, vol. 84, p. 125115, 2011.

[143] T. Faulkner, N. Iqbal, J. McGreevy, and D. Vegh, "Holographic non-fermi-liquid fixed points," *Phil. Trans. R. Soc. A*, vol. 369, p. 1640, 2011.

[144] A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, "Non-fermi-liquid fixed point in a wilsonian theory of quantum critical metals," *Phys. Rev. B*, vol. 88, p. 125116, 2013.

[145] A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, "Non-Fermi-liquid behavior of large-N_B quantum critical metals," *Phys. Rev. B*, vol. 89, p. 165114, 2014.

[146] Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, "Gauge-invariant response functions of fermions coupled to a gauge field," *Phys. Rev. B*, vol. 50, p. 17917, 1994.

[147] T. Faulkner, N. Iqbal, H. Liu, J. McGreevy, and D. Vegh, "Strange metal transport realized by gauge/gravity duality," *Science*, vol. 329, p. 1043, 2010.

[148] S. A. Hartnoll, R. Mahajan, M. Punk, and S. Sachdev, "Transport near the ising-nematic quantum critical point of metals in two dimensions," *Phys. Rev. B*, vol. 89, p. 155130, 2014.

[149] J. M. Ziman, *Electrons and Phonons: The Theory of Transport Phenomena in Solids*. Oxford Univ. Press, 2001.

[150] N. W. Ashcroft and N. Mermin, *Solid State Physics*. Brooks/Cole, 1976.

[151] T. Giamarchi and A. J. Millis, "Conductivity of a luttinger liquid," *Phys. Rev. B*, vol. 46, p. 9325, 1992.

[152] P. Phillips and C. Chamon, "Breakdown of one-parameter scaling in quantum critical scenarios for high-temperature copper-oxide superconductors," *Phys. Rev. Lett.*, vol. 95, p. 107002, 2005.

[153] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, "Towards strange metallic holography," *JHEP*, vol. 04, p. 120, 2010.

[154] S. A. Hartnoll and A. Karch, "Scaling theory of the cuprate strange metals," *Phys. Rev. B*, vol. 91, p. 155126, 2015.

[155] T. Giamarchi, "Umklapp process and resistivity in one-dimensional fermion systems," *Phys. Rev. B*, vol. 44, p. 2905, 1991.

[156] X. Zotos, F. Naef, and P. Prelovsek, "Transport and conservation laws," *Phys. Rev. B*, vol. 55, p. 11029, 1997.

[157] A. Rosch and N. Andrei, "Conductivity of a clean one-dimensional wire," *Phys. Rev. Lett.*, vol. 85, pp. 1092–1095, Jul 2000.

[158] E. Shimshoni, N. Andrei, and A. Rosch, "Thermal conductivity of spin-1/2 chains," *Phys. Rev. B*, vol. 68, p. 104401, 2003.

[159] J. Sirker, R. G. Pereira, and I. Affleck, "Conservation laws, integrability, and transport in one-dimensional quantum systems," *Phys. Rev. B*, vol. 83, p. 035115, 2011.

[160] C. Karrasch, J. H. Bardarson, and J. E. Moore, "Finite-temperature dynamical density matrix renormalization group and the drude weight of spin-1/2 chains," *Phys. Rev. Lett.*, vol. 108, p. 227206, 2012.

[161] C. Nayak and F. Wilczek, "Physical properties of metals from a renormalization group standpoint," *Int. J. Mod. Phys. B*, vol. 10, p. 847, 1996.

[162] E. Plamadeala, M. Mulligan, and C. Nayak, "Short-range entangled bosonic states with chiral edge modes and t duality of heterotic strings," *Phys. Rev. B*, vol. 88, p. 045131, 2013.

[163] P. Ginsparg, "Applied Conformal Field Theory," *ArXiv: hep-th/9108028*, 1991.

[164] J. L. Cardy, "Critical exponents of the chiral potts model from conformal field theory," *Nucl. Phys. B*, vol. 389, p. 577, 1993.

[165] P. Mazur, "Non-ergodicity of phase functions in certain systems," *Physic (Amsterdam)*, vol. 43, p. 533, 1969.

[166] M. Suzuki, "Ergodicity, constants of motion, and bounds for susceptibilities," *Physica (Amsterdam)*, vol. 51, p. 277, 1971.

[167] A. Imambekov, T. L. Schmidt, and L. I. Glazman, "One-dimensional quantum liquids: Beyond the luttinger liquid paradigm," *Rev. Mod. Phys.*, vol. 84, p. 1253, 2012.

[168] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions*. Frontiers in Physics 47, W.A. Bemjamin, 1975.

[169] T. Giamarchi, *Quantum Physics in One Dimension*. International Series of Monographs on Physics (Book 121), Oxford University Press, 2004.

[170] S. A. Hartnoll, "Lectures at the arnold sommerfeld center for theoretical physics," 2013.

[171] A. Lucas and S. Sachdev, "Memory matrix theory of magnetotransport in strange metals," *Phys. Rev. B*, vol. 91, p. 195122, 2015.

[172] S. A. Hartnoll, "Theory of universal incoherent metallic transport," *Nature Phys.*, vol. 11, p. 54, 2015.

[173] J. T. Chalker and A. Dohmen, "Three-dimensional disordered conductors in a strong magnetic field: Surface states and quantum hall plateaus," *Phys. Rev. Lett.*, vol. 75, p. 4496, 1995.

[174] L. Balents and M. P. A. Fisher, "Chiral surface states in the bulk quantum hall effect," *Phys. Rev. Lett.*, vol. 76, p. 2782, 1996.

[175] L. Balents, M. P. A. Fisher, and M. R. Zirnbauer, "Chiral metal as a ferromagnetic super spin chain," *Nucl. Phys. B*, vol. 483, p. 601, 1997.

[176] S. Sur and S.-S. Lee, "Chiral non-fermi liquids," *Phys. Rev. B*, vol. 90, p. 045121, 2014.

[177] C. L. Kane, M. P. A. Fisher, and J. Polchinski, "Randomness at the edge: Theory of quantum hall transport at filling ν=2/3," *Phys. Rev. Lett.*, vol. 72, pp. 4129–4132, Jun 1994.

[178] C. L. Kane and M. P. A. Fisher, "Transmission through barriers and resonant tunneling in an interacting one-dimensional electron gas," *Phys. Rev. B*, vol. 46, p. 15233, 1992.

[179] P. Fendley, A. W. W. Ludwig, and H. Saleur, "Exact nonequilibrium transport through point contacts in quantum wires and fractional quantum hall devices," *Phys. Rev. B*, vol. 52, p. 8934, 1995.