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Investigating number sense development in a mathematics content course for prospective elementary teachers

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Investigating Number Sense Development in a Mathematics Content Course for Prospective Elementary Teachers

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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The Dissertation of Ian Michael Whitacre is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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This dissertation is dedicated to my wife, Karen, who has supported me throughout this whole process.

VITA

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ABSTRACT OF THE DISSERTATION

Investigating Number Sense Development in a Mathematics Content Course for Prospective Elementary Teachers

by

Ian Michael Whitacre

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2012
San Diego State University, 2012

Professor Susan Nickerson, Chair

In order to support children's learning of elementary mathematics meaningfully, elementary teachers need to understand that mathematics deeply and flexibly (Ball, 1990; Ma, 1999). In other words, they need good number sense (Reys & Yang, 1998). However, researchers have found that prospective elementary teachers tend to reason inflexibly, relying heavily on standard algorithms (e.g., Ma, 1999; Newton, 2008; Yang,

2007). Previous research has provided single snapshots or comparisons of pre/post snapshots of number sense. In this study, I analyzed prospective elementary teachers' number sense development.

In earlier work, Nickerson and I created a local instruction theory (Gravemeijer, 1999) for the development of number sense (Nickerson & Whitacre, 2010). In a previous classroom teaching experiment, we found that prospective elementary teachers enrolled in a mathematics content course informed by the local instruction theory developed improved number sense (Whitacre & Nickerson, 2006). They moved from being reliant on the mental analogues of the standard algorithms to reasoning more flexibly in mental computation.

In the present study, I duplicated analyses from the previous study and found similar results. I also moved beyond the previous study by investigating number sense development as a microgenetic, sociogenetic, and ontogenetic process (Saxe & Esmonde, 2005). I asked the following research questions: As prospective elementary teachers participate in a mathematics content course designed to support their development of number sense,

1. How does the number sense of individuals evolve?
2. What ideas come to function as if shared? What classroom mathematical practices emerge and become established?

I approached this study from a situated perspective (Cobb & Bowers, 1999). The emergent perspective informed my approach to the research in terms of taking both social and individual lenses to the analysis of number sense development (Cobb & Yackel, 1996).

I made innovations in the analysis of number sense. I documented collective activity in the class in terms of progressions through classroom mathematical practices. I also analyzed two case studies of individuals' number sense development. These analyses provide insights into the phenomenon of prospective elementary teachers' number sense development, which will inform revisions and elaboration to the local instruction theory.

Chapter 1: Introduction

The focus of this dissertation study was number sense development. The setting was a mathematics content course for prospective elementary teachers, which was designed to support their development of number sense. I approached the investigation of this phenomenon through two lenses, social and psychological. I found that students' number sense improved while they were involved in the content course. Thus, this setting afforded me the opportunity to study processes by which a group of prospective elementary teachers developed number sense. I investigated this phenomenon on the collective level, through an analysis of classroom mathematical practices. I also investigated number sense development on the individual level, primarily through analyses of interviews with a subset of the students. The findings provide new insights into the phenomenon of interest and have implications for both research and practice.

What Is Number Sense and Why Do Teachers Need It?

I use the term *number sense* to describe "an acquired 'conceptual sense-making' of mathematics," which is typical of the use of the term in the mathematics education literature. This is in contrast to the notion of number sense as "a biologically based 'perceptual' sense of quantity" as the term is used in the mathematical cognition literature (Berch, 2005, p. 334). Howden (1989) describes number sense as "good intuition about numbers and their relationships" (p. 11). Reys and Yang (1998) define number sense as "a person's general understanding of number and operations," and they include in their definition "the ability and inclination to use this understanding in *flexible* ways to make mathematical judgments and to develop useful *strategies* for handling numbers and operations" (p. 225-226, emphasis added). Flexibility and the strategies people use are themes in the number sense literature that are especially relevant to this study.

Number sense is recognized as an important goal of mathematics instruction (National Council of Teachers of Mathematics [NCTM], 2000; National Research Council [NRC], 2001). For example, NCTM's *Principles and Standards for School Mathematics* states:

> In grades 3-5, students' development of number sense should continue, with a focus on multiplication and division. Their understanding of the meanings of these operations should grow deeper as they encounter a range of representations and problem situations, learn about the properties of these operations, and develop fluency in whole-number computation. (p. 149)

Despite these recommendations, however, children both within the United States and internationally tend to learn mathematics in a way that emphasizes the use of standard algorithms and does not support their development of number sense (Reys et al., 1999).

A common theme in the literature on number sense is that it cannot be taught directly (e.g., Greeno, 1991; McIntosh, 1998). Rather, it "develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Howden, 1989, p. 11). Teaching in a manner that supports students' number sense development puts added demands on teachers. It requires teachers to be sensitive to their students' mathematical thinking and to make sense of that thinking, and this in turn requires deep understanding of the mathematics itself (Jacobs, Lamb, & Philipp, 2010).

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990). However, prospective and practicing elementary teachers alike often know the procedures of elementary mathematics, but do not understand the material conceptually (Ball, 1990; Ma, 1999; Zazkis & Campbell, 1996). Preservice elementary teachers have been characterized as having poor number sense (Tsao, 2005; Yang, Reys, & Reys, 2009). In particular, they are reliant on precisely those standard algorithms that they do not understand (Ball, 1990; Newton, 2008; Thanheiser, 2010; Yang, 2007).

The terms *prospective teachers* and *preservice teachers* are not used consistently in the mathematics education literature. I use the term *prospective teachers* to refer to undergraduates who have expressed by their choice of major an interest in teaching but still have substantial undergraduate coursework ahead of them. I reserve the term *preservice* teachers for individuals who are currently enrolled in a credential program or have nearly completed their undergraduate coursework and plan to enter a teacher credential program.

If mathematics content courses for prospective elementary teachers typically resulted in substanial improvement in the number sense of those individuals, then prospective and preservice elementary teachers would be clearly distinct student populations, at least with regard to their mathematical reasoning. Unfortunately, however, this is not the case. I believe that the inconsistency in the field in the use of the terms prospective and preservice is symptomatic of a reality in which prospective elementary teachers tend to experience little growth in their mathematical reasoning during their undergraduate education. *Even after* having completed their college mathematics courses, preservice elementary teachers' understanding of mathematics tends to be "rule-bound and compartmentalized" (Ball, 1990, p. 453).

It follows that an important problem in mathematics teacher education is to design content courses for prospective elementary teachers that are effective in promoting number sense development. In previous research, Nickerson and I designed an instructional approach focused on number sense development, and we found that prospective elementary teachers involved in the course experienced substantial growth in their number sense (Nickerson & Whitacre, 2010; Whitacre, 2007; Whitacre & Nickerson, 2006). These were encouraging results. On the other hand, witnessing improvement in the number sense of prospective elementary teachers is a far cry from understanding how that change occurred. In the present study, I set out to investigate number sense development in the same course as both an individual and a collective phenomenon.

Overview

The remainder of this chapter is organized in three major sections. In the first section, I review previous research concerning number sense; this includes pilot research that directly motivated the present study. In the second section, I discuss established research methods and frameworks that are relevant to the present study. This includes methods for analyzing change in individuals' number sense, a methodology for analyzing learning as a collective phenomenon, and a framework for coordinating the individual and collective levels. In the third section, I highlight the particular phenomena that I am interested in studying. Finally, I formally state my research questions, and I discuss their significance.

Previous Research Concerning Number Sense

Previous studies concerning number sense help to lay the foundations for the proposed study. One finds in the literature two established approaches to the analysis of number sense: (1) students' responses to particular sorts of tasks that are intended to require number sense, and (2) characterizations of strategies used by students to solve certain kinds of tasks. In the following paragraphs, I provide examples of each of these approaches. I also review the findings that these methods have produced. I then describe the results of my Master's study, which built on these established methods.

Correctness and Strategies as Indicators of Number Sense

Yang and colleagues developed a multiple-choice survey instrument called the Number Sense Rating Scale (NSRS) as a quantitative measure of students' number sense (Hsu, Yang, & Li, 2001). The NSRS was originally created for use with fifth and sixth grade students. In a series of studies, Yang and colleagues have used the NSRS and similar instruments to assess the number sense of both middle school students and preservice elementary teachers (e.g., Reys & Yang, 1998; Tsao, 2005; Yang, 2003). The tasks on the NSRS and similar instruments are intended to be non-routine. For example, students are asked to estimate a sum or product, rather than to calculate it exactly. These tasks are to be performed mentally. Sample items from the NSRS appear in Figure 1.

24. What is the sum of $\frac{14}{15}$ and $\frac{7}{8}$ *approximately*? a) 1 b) 2 c) 21 d) can't tell without calculating
25. Which answer is the product of 18 and 19 closest to? a) 250 b) 350 c) 450 d) 550

Figure 1. Sample items from the NSRS (Hsu et al., 2001).[1]

[1] The original test instrument, obtained from its authors, was written in Chinese. The NSRS was

Researchers have also analyzed students' number sense on the basis of their strategies for solving interview tasks (Markovits & Sowder, 1994; Yang, 2003, 2005, 2007; Yang et al., 2009). Often, in these reports, the term *strategy* is not explicitly defined. However, the way that the term is used seems to be consistent with Smith's (1995) definition of *strategies* as "patterns of reasoning that provide solutions to classes of items within a particular task context" (p. 12). In practice, patterns of observable behavior, especially computational steps, stand in for patterns of reasoning.

There are various schemes for describing the strategies that individuals use within particular task contexts, such as mental computation (Heirdsfield & Cooper, 2004), computational estimation (Hanson & Hogan, 2000), and fraction comparisons (Smith, 1995). Schemes such as these include standard procedures that are typically taught in schools, as well as nonstandard strategies. From mental computation to fraction comparisons, nonstandard strategies are generally associated with number sense, whereas standard, school-learned procedures are not (e.g., Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Yang, 2007).

Yang and colleagues have assessed students' number sense using tests like the NSRS, as well as related interview tasks. They have also analyzed students' strategies for solving these tasks. Based on these methods, Yang and colleagues have found that Taiwanese middle school students generally exhibit poor number sense (Reys & Yang, 1998; Yang, 2005; Yang & Huang, 2004; Yang, Li, & Lin, 2008). These students are able to perform familiar computations using school-learned procedures, but they perform

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translated to English for the purposes of my Master's study by Yun-Chu "Yvonne" Lai, who at that time was a graduate student at San Diego State University.

poorly on non-routine tasks designed to tap their number sense (Reys & Yang, 1998; Yang, 2005; Yang & Huang, 2004). Similar findings have been reported for elementary and middle school students in various countries, including Australia, Korea, Sweden, and the United States (Reys et al., 1999).

Researchers have also found that preservice teachers exhibit poor number sense and rely heavily on standard written algorithms (Tsao, 2005; Yang, 2007; Yang et al., 2009). Although preservice teachers tend to be more capable of correctly answering test items than their middle-school counterparts, Yang and colleagues found that the majority of their answers were also obtained by written computation using standard algorithms. Both middle school students[2] and preservice teachers have tended to rely on standard written algorithms, despite the fact that the instructions for these assessments explicitly discouraged such approaches (Yang, 2005, 2007; Yang et al., 2009).

The two methods discussed above are prominent in number sense research. Students' abilities to solve non-routine tasks, and the strategies they use for doing so, are established indicators of number sense. Both of these approaches informed the analysis of change in number sense in my Master's study.

A Study of Prospective Elementary Teachers' Improved Number Sense

In Fall Semester 2005, Nickerson and I conducted a classroom teaching experiment (Cobb, 2000) in two sections of a mathematics content course for prospective elementary teachers. I was the instructor. My personal motivation for conducting the study was my previous experience teaching the course and my dissatisfaction with

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[2] The number sense of middle school students is relevant to the proposed study since these students have recently completed elementary school. Thus, their performance indicates what they took away from their elementary mathematics education.

students' learning. In particular, mental computation stood out as an elusive topic. Students were meant to develop number sense, but I had taught them problem-specific strategies. I saw evidence in final exam responses that students' reasoning had not changed outside of their ability to solve problems that were tailor-made for those strategies. I was interested in designing instruction that would actually promote number sense development.

Based on a review of the literature, we developed a conjectured local instruction theory for the development of number sense[3] (Gravemeijer, 1999, 2004). Over the course of the semester, that conjectured local instruction theory was fleshed out as it was enacted in two classrooms. Through subsequent analysis and conversations, as well as additional experience teaching the course, we refined it further. Our local instruction theory for the development of number sense is described in detail in Nickerson and Whitacre (2010).

In my Master's study, I used two established means of analyzing students' number sense: their scores on a multiple-choice instrument, the NSRS, and their mental computation strategies used in interviews. I compared pre/post results to assess change in students' number sense. Both measures showed substantial improvement in the participants' number sense. At the end of the semester, the study participants did not look like typical prospective elementary teachers. They were no longer overly reliant on the standard algorithms; rather, they used various nonstandard mental computation strategies (Whitacre, 2007).

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[3] The construct of local instruction theory, as well our particular local instruction theory, will be discussed in detail in Chapter 2.

In the following paragraphs, I describe the 2005 teaching experiment. I give two examples of classroom activity that point to phenomena that are of particular interest in the proposed study. I also present the findings from analyses of interview data.

Classroom activity. We made the decision to modify the curriculum so that mental computation no longer appeared as an isolated unit to be studied. Rather, *authentic* mental computation *activity*[4] was integrated throughout the course. We identified opportunities in the curriculum for mental computation in service of problem solving. As the instructor, I asked students to perform these mentally, and I led whole-class discussions concerning students' strategies.

Two aspects of the classroom activity concerning students' strategies evolved that seemed crucial to the viability of this instructional approach. A routine developed of naming the strategies that had been discussed. Initially, a strategy was named according to the student who had nominated it, for example, "Karen's strategy." Over time, students gave more meaningful names to these strategies. For example, "Break Up, then Make Up" was the name that one class gave to a strategy that took advantage of the distributive property of multiplication over addition (e.g., 15 x 24 was solved 10 x 24 + 5 x 24). Naming made the strategies objects of discourse and appeared to facilitate *reflective discourse* and *collective reflection* (Cobb, Boufi, McClain, & Whitenack, 1997). It also facilitated comparisons of strategies across time. Instead of discussing how Karen's strategy (presently under discussion) compared to the strategy that So-and-So had shared with us two weeks ago, the class discussed how Karen's strategy compared to Break Up,

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[4] The term *authentic activity* is used in the sense of Brown, Collins, and Duguid's (1989) discussion of situated cognition. The authors define authentic activities as "the ordinary practices of the culture" (p. 34).

then Make Up, which was a familiar strategy, having been named and referred to repeatedly. These comparisons led to negotiation concerning what constituted a different strategy, as well as what about the calculation at hand made one strategy more suitable than another. We kept an official list of the strategies that had been shared in class.

The other key aspect of class discussions of students' strategies seemed to be the way that particular inscriptions were used. When strategies were discussed, the students' mental calculative work was represented on the chalkboard in two ways: in number sentences and using drawings of some sort. The drawings were intended to capture the main idea of a student's strategy and to help others make sense of it. Particular representational tools came to be used repeatedly and in increasingly sophisticated ways. Number line drawings came to be used to represent students' addition and subtraction work, while rectangular area drawings came to be used to represent their multiplication and division work. These representational tools became sources of justification in students' explanations. For example, the empty number line evolved as a tool for recording and making sense of addition and subtraction strategies involving "jumps" from one number to the next. As depicted in Figure 2, one student calculated 1000 - 729 by adding on to 729 in successive jumps, while keeping a running subtotal. Anghileri (2000) discusses such uses of the number line in children's development of number sense. We saw it used similarly by prospective elementary teachers.

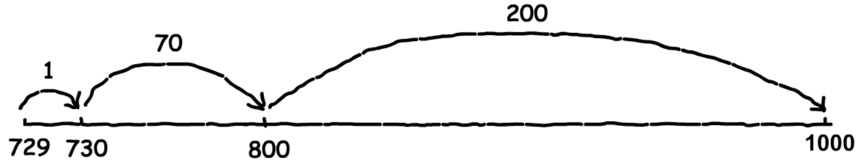

Figure 2. Empty number line drawing for 1000 - 729.

Later in the course, the empty number line was used to justify a new strategy, which the class called "Shifting the Difference." A student used this strategy to calculate 142 - 57 by first subtracting 2 from both the minuend and the subtrahend. This made the calculation 140 - 55 instead, and the student was able to compute this difference readily. Strategies that involve altering the subtrahend tend to be difficult for students because they may be unsure of how to compensate for the adjustment.[5] However, this strategy was justified not in terms of formal properties of subtraction but in terms of properties of the number line. This student interpreted the difference between the two numbers as the distance between their locations on the number line. Under this interpretation, subtracting 2 from each number would not affect the difference because this would correspond to shifting both of the locations two units to the left, which would not affect the distance between them.

Shifting the Difference was a novel strategy for the class. This strategy was more sophisticated than those discussed earlier in the semester in the sense that it involved a reformulation of the given computation. It is an example of a nonstandard strategy that is considered to be indicative of number sense (Markovits & Sowder, 1994). Furthermore, the empty number line seemed to have served a new function. Rather than being used to represent a student's mental calculative work after the fact, it came to be used as an important source of justification in a case in which a new strategy was nontrivial for students and in need of justification. In this way, we saw evidence that the empty number

[5] For example, a preservice teacher in a pilot interview that I conducted thought of rounding 49 to 50 to solve 125 - 49 mentally. She knew that 125 - 50 was 75, but she was not sure how to compensate for her rounding. She decided that 74 should be the answer.

line shifted in function from a *model of* students' informal activity to a *model for* more sophisticated mathematical reasoning (Gravemeijer, 1999).

The classroom activity during discussions of students' strategies was an especially interesting aspect of the teaching experiment. However, the data collected did not afford detailed analyses of this activity. The analysis focused primarily on the mental computation strategies used by 13 of the students in pre/post interviews.

Interview findings. The interviews consisted of whole-number mental computation tasks. Following precedents in the number sense literature, I coded participants' responses according to the strategy that they employed. The codes were developed through constant comparative analysis of all students' responses (Creswell, 1998), as well as being informed by previously established schemes in the literature (e.g., Heirdsfield & Cooper, 2004). A scheme of Markovits and Sowder (1994) was used as an organizing framework to order strategies according to the degree to which they differed from the standard algorithms, an accepted criterion associated with number sense.

Students in the pre-instruction interview were very limited in their mental computation abilities. Numbers were selected deliberately to afford various strategies. For example, 24 and 15 afford various applications of the distributive or associative property to find their product, and numbers such as 99 afford compensation strategies that take advantage of their proximity to friendly, benchmark numbers (e.g., 100). Despite these affordances, most interview participants used only one or two different strategies for performing each of addition, subtraction, and multiplication mentally. When asked if they had other ways of performing the computations mentally, most students responded that they did not. Furthermore, the most common strategies used in the pre-instruction interview were those least indicative of number sense, the mental analogues of the standard paper-and-pencil algorithms. Most of the interview participants came into the class with a go-to way of performing mental computation, which was to picture the work that they would do if using the standard written algorithm. This inflexible approach to mental computation is commonly associated with poor number sense (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Reys & Yang, 1998).

Analysis of the pre-instruction interviews indicated that the participants were generally inflexible mental calculators and that their understandings of the operations appeared to be bound to the standard algorithms. In the post-instruction interviews, by contrast, participants showed improved flexibility. Most used three or four different strategies for each of addition, subtraction, and multiplication. For each operation, participants were less likely to use any particular strategy for the majority of their computations. Furthermore, the strategies used most often were those strategies most indicative of number sense, nonstandard strategies that involved some reformulation of the computation (Markovits & Sowder, 1994). For example, the strategy named Shifting the Difference came to be used by 7 of the 13 interview participants, whereas none had used it in the first interview (Whitacre, 2006).

Data collection in the 2005 study included the use of the NSRS as a quantitative measure of students' number sense. A total of 48 students took the test both pre- and post-instruction, 22 from one class and 26 from the other. In both classes, students' scores increased, and the increases were statistically significant. Cohen's *d* provides a measure of effect size. This is the ratio of the mean difference to the standard deviation of the difference scores. In both classes, the effect size was greater than one standard deviation (Whitacre, 2006).

While these results were encouraging, this research raised new questions. The analyses presented in my Master's thesis were limited in several ways. Strategies were coded in the tradition of research in number sense and mental computation. The grain size for these analyses leaves some relevant aspects of students' reasoning unexamined. Furthermore, having only pre/post data allowed for a contrast between students' responses in the two interviews but did not permit me access to the processes by which students' reasoning had developed. As an instructor of the course, I was privy to various interesting phenomena, both in class and in individual students' work. However, as a researcher, I lacked the data necessary to support more rigorous and in-depth investigations concerning these phenomena.

Analyzing Development

The major phenomenon of interest is number sense development—in particular, the development of number sense for prospective elementary teachers during a first mathematics content course. By *development* here, I mean a gradual process of change that occurs in three strands – microgenesis, sociogenesis, and ontogenesis (Saxe & Esmonde, 2005). Saxe et al. (2009) describe these three strands of genetic analysis as follows:

> *Ontogenesis* focuses on shifts in patterns of thinking over the development of the individual; *microgenesis* involves the construction of meaningful representations in activity; and *sociogenesis* entails the reproduction and alteration of representational forms that enable communication among participants in a community. (p. 208)

These three strands frame the analysis of number sense development in this study. The microgenetic strand concerns instances of mathematical activity, both in class and in interviews. The sociogenetic strand concerns trends over time in the collective activity that occurs in the classroom. The ontogenetic strand concerns development over time in individual students' reasoning.

I take a situated perspective on knowing and learning (Cobb & Bowers, 1999). The emergent perspective informs my approach to the research in terms of taking both social and individual lenses to the analysis of number sense development, as well as taking an interest in the relationship between these (Cobb & Yackel, 1996).

Number Sense Development

The work of Yang and others has contributed to the research literature by producing a body of evidence that suggests that school children and preservice teachers alike tend to have poor number sense. In rare instances, researchers have reported on improvement in the number sense of middle school students involved in instructional interventions (Markovits & Sowder, 1994; Yang, 2003). These articles provide general descriptions of the teaching and then focus on pre/post assessment results. Researchers have also described activity in mathematics courses that purportedly had a positive effect on students' number sense; however, evidence for improvement in students' number sense is lacking (e.g., Yang, 2002, 2006). Kaminski (2002) describes activity in a mathematics content course for preservice elementary teachers in Australia. He offers anecdotal indications of improvement in students' number sense. However, no systematic methods for analyzing collective activity are reported, and neither are measures of individuals' number sense. My extensive review of the literature revealed no reports of investigations of number sense *development*, in the sense of Saxe et al. (2009), for any population.

The analysis of number sense is a challenging endeavor. There are good reasons for associating nonstandard strategies with number sense. On the other hand, if the researcher is true to the meaning of the construct, such methods are insufficient. These strategies are taken as a proxy for the underlying reasoning. A more thorough investigation of number sense must delve into the reasoning behind students' strategies.

The Role of Strategies in Number Sense Development

We can gather from the mathematics education literature a picture of contrast between people with good number sense versus poor number sense in terms of their use of strategies. People with good number sense are flexible, which is to say that they employ a repertoire of different strategies. The choice of strategy depends on the details of the problem at hand, for example, the particular numbers involved in a computation. Individuals with poor number sense, by contrast, tend to exhibit limited flexibility. Furthermore, individuals with good number sense tend to use nonstandard strategies, whereas individuals with poor number sense tend to rely on the standard algorithms that they were taught in school (Greeno, 1991; Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Reys & Yang, 1998).

Mathematics instructors are encouraged to engage students in solving problems and sharing and discussing strategies (NCTM, 2000). Mathematics education researchers value classroom discourse concerning students' strategies (e.g., Schoenfeld, 1987). Furthermore, authors have made particular recommendations concerning the discussion of strategies as a means of supporting number sense development (Greeno, 1991;

Howden, 1989; McIntosh, 1998). However, the literature offers little in the way of answers to the question, "How does number sense develop?" except for the general axioms that it develops "gradually" (Howden, 1989) and that its development is supported by "process-oriented activities" (Yang, 2002). Given the prominence in the literature of strategies as an indicator of number sense, it seemed natural to look to class discussion of students' strategies as a site for number sense development, as well as to study how individuals become more flexible in their reasoning about numbers and operations.

The study reported here extends previous research by looking deeper than the level of strategies to students' justifications for their strategies, by developing analytic tools for analyzing change in individuals' use of strategies, and by investigating how individuals come to use the strategies that they do. In the context of a mathematics class, I investigated how ideas related to mental computation emerged and became established in the classroom community. I did this through the lens of mathematical argumentation by identifying normative ways of reasoning of the classroom community.

On the individual level, I analyzed change in the reasoning of the interview participants. This involved methods similar to those used previously, as well as new analytic tools that I developed for the purpose. In selected case studies, I incoporated additional data into the analysis to analyze individuals' mathematical reasoning in a more nuanced fashion and to shed light on developmental processes. I also examined relationships between individual participants' conceptions and activity and the normative ways of reasoning that developed in the class.

Research Questions

Having described the phenomena of interest and their relations to the literature, I now state my research questions formally:

As prospective elementary teachers participate in a mathematics content course designed to support their development of number sense,

1. How does the number sense of individuals evolve over the course of the semester?
2. What ideas come to function as if shared? What classroom mathematical practices emerge and become established?

Significance

The study reported here represents a contribution to the mathematics education literature in the following ways:

- By investigating the ontogenetic development of number sense, rather than being limited to pre/post snapshots.
- By illuminating the number sense development of prospective elementary teachers, a group whose number sense is known to be poor and who have a particular need for good number sense.
- By looking deeper than the levels of correctness or strategy selection to describe individuals' number sense.
- By investigating the sociogenesis of number sense in a mathematics content course.
- By integrating and extending established frameworks for number sense and mental computation.

The central purpose of this study was to investigate prospective elementary teachers' number sense development by means of genetic analysis (Saxe & Esmonde, 2005). I took a situated perspective to the investigation of number sense. My particular focus was on mental computation as a microcosm of number sense. I took an interest in the normative ways of reasoning related to nonstandard strategies that became established in the classroom community, as well as in how individual students' reasoned in relation to these ideas.

In Chapter 2, I elaborate a theoretical framework, explore the literature related to number sense, and describe the general theory underlying the instructional design and analysis. In Chapter 3, I describe the research design, data corpus, and methods of analysis. Chapters 4, 5, and 6 presents study results. Chapter 4 presents results of analyses similar to those conducted previously. It also introduces some new innovations in number sense research. Chapter 5 presents results of the analysis of collective activity. Chapter 6 shifts back to analysis of individuals' conceptions and activity and moves beyond previously established methods of analyzing number sense. Chapter 7 concludes the dissertation, summarizing and discussing results and implications.

Chapter 2: Literature Review

This literature review is organized in five sections. The first briefly describes the general theoretical perspective that informs the present study. The second section concerns research on number sense in general. This includes characterizations of the number sense construct, as well as past studies concerning number sense. The third section is devoted to specific mathematical content. Number sense is broad. This study focuses on whole-number number sense, particularly whole-number mental computation and understanding of place value. The fourth section concerns instructional design. The fifth section concerns genetic analysis. Special attention is given to the analysis of collective activity.

Theoretical Perspective

The *emergent perspective* (Cobb & Yackel, 1996) influences my approach to research concerning prospective elementary teachers' number sense development, both in terms of this study and the larger research program to which it belongs. From the emergent perspective, learning is a complex process involving a reflexive relationship between individual and collective activity. The emergent perspective represents neither a purely social perspective nor an individual, psychological perspective, but rather an attempt to coordinate the two. The perspective on learning that informs this study can be broadly described as situated (Cobb & Bowers, 1999). Learning occurs in the doing of activities within a culture. The nature of those activities and the culture in which they are situated profoundly shape what is learned. Knowledge becomes meaningful and useful in the practice of *authentic activities*, which "are most simply defined as the ordinary practices of the culture" (p. 34).

This study involves analyses of learning through both the social and psychological lenses. Collective activity in a classroom setting is framed in terms of classroom mathematical practices (Cobb & Yackel, 1996; Rasmussen & Stephan, 2008). These ideas are discussed in some detail in the section titled Analysis of Collective Activity and Individual Learning. My perspective concerning learning on the individual level involves two more specific views: (1) that a learner's prior conceptions serve as resources in the learning process, and (2) that learning involves activity in a zone of proximal development. A knowledge-as-resources view (Hammer, 1996; Smith, diSessa, & Roschelle, 1993) lends itself to an analysis of learning that values students' prior conceptions and uses these to account for how learning proceeds.

Even when focusing on the learning of an individual, it is important to account for the social context in which this learning takes place. In classrooms and other social settings, learners often engage in activity that is supported by aspects of that setting. In particular, the assistance of more knowledgeable others make it possible for the learner to do something that she could not have done alone (Mercer, 1995; Vygotsky, 1978). This activity is possible within the *zone of proximal development* (ZPD), which Vygotsky (1978) defined as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (p. 86). From this perspective, learners first engage in problem solving activity in the ZPD before they become capable of performing that type of activity on their own.

Number Sense

At a conference in 1989 organized to discuss number sense research, the participants considered how to define *number sense* in a way that would distinguish it from related ideas, such as "higher order thinking" (Sowder & Schappelle, 1989). The participants also discussed the inherent difficulties in attempting to assess students' number sense, as well as questions concerning pedagogy to foster students' development of number sense (Sowder & Schappelle, 1989). Nearly a quarter of a century has passed since that number sense conference. In that time, some progress has been made toward answering these central questions (e.g., Greeno, 1991; Markovits & Sowder, 1994). However, there is considerably more work to be done.

This section reviews descriptions of number sense in the mathematics education literature, methods for analyzing number sense, findings concerning the number sense of particular populations, and findings concerning change in number sense for students involved in instructional interventions.

Descriptions of Number Sense

Various definitions and characterizations of *number sense* can be found in the mathematics education literature. Greeno (1991) drew a dichotomy between viewing number sense as a set of skills to be learned, versus taking a "global view of number sense" (p. 173). There are examples of each of these views in the descriptions of number sense in the literature. One influential characterization is the decomposition of number sense into components (Reys et al., 1999). Also common are descriptions of number sense in more general terms, which include aspects of understanding, as well as metacognitive characteristics (e.g., Reys & Yang, 1998; Yang, 2007). Distinct from these is Greeno's (1991) theoretical treatment of number sense as situated knowing in a conceptual domain. Greeno's account seems to be the only truly "global" view of number sense found in the literature. The paragraphs that follow elaborate on the different views of number sense.

The Components View of Number Sense

Based on their review of previous literature, Reys et al. (1999) identified six components of number sense:

1. Understanding of the meaning and size of numbers
2. Understanding and use of equivalent representations of numbers
3. Understanding the meaning and effect of operations
4. Understanding and use of equivalent expressions
5. Flexible computing and counting strategies for mental computation, written computation, and calculator use
6. Measurement benchmarks (p. 62)

It is not entirely clear what is meant by the term *component* here. The authors also use the term "content strand" (p. 62), and relate the above components to "'indicators' of number sense" (p. 61). Based on descriptions of these components, it is safe to say that they can generally be regarded as mathematical understandings. However, note also that the word *use* enters in to certain of these components, so that in that respect there is a sense that number sense is a matter not only of having knowledge but applying it. The notion of components of number sense has influenced empirical research concerning number sense. For example, Hsu et al.'s (2001) Number Sense Rating Scale is organized in five sections, which are meant to correspond to five components of number sense. The decomposition of number sense into a finite set of components seems to be consistent with the kind of skill-set view of number sense that Greeno (1991) described.

In a more general description of number sense, Reys and Yang (1998) define the construct as follows:

> Number sense refers to a person's general understanding of number and operations. It also includes the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity. (p. 225-226)

In subsequent work, Yang and colleagues have given various similar descriptions of number sense (e.g., Yang, 2002, 2007; Yang et al., 2008; Yang et al., 2009). This kind of description seems to capture the gist of the term *number sense*, as it is commonly used in the mathematics education community, which is to say that the description paints a picture of a person who makes sense of mathematics and behaves mathematically in ways that are considered desirable. Note that the above definition contrasts considerably with the Reys et al.'s (1999) list of components in that it does not mention any particular mathematical understandings. Rather, it focuses on habits of mind and ways that a person behaves mathematically.

Given the obvious contrasts between the above definition and the six components of number sense, the former appears consistent with Greeno's (1991) notion of a global view of number sense. However, in practice, this apparently global view has not been applied; when it comes to assessing number sense, the same authors cited above have operationalized number sense as a set of components. For example, particular instances of problem solving have been coded as reflecting number sense or not, based on whether or not one or more components of number sense was evident in the person's solution process (Yang, 2003, 2005, 2007; Yang et al., 2009). Thus, in practice, these researchers have tended to ignore the general, meta-cognitive aspects of their own definition of number sense, focusing primarily on mathematical understandings. In this way, although Yang and colleagues have proposed definitions of the construct of number sense that appear global (e.g., Reys & Yang, 1998), corresponding operational definitions have been based on a components (or skill-set) view.

The Environment Metaphor

In contrast to a components view of number sense, Greeno (1991) characterized number sense as situated knowing in a conceptual domain—the domain of numbers and quantities. From this perspective, a person's knowledge and activities are seen metaphorically as situated within a physical environment. Knowing in an environment consists of knowing how to get around, where to find things, and how to use them. In various conceptual domains, knowing one's way around requires relating concepts and solving problems. Greeno's metaphor relates mathematical properties, such as the distributive property of multiplication over addition, to features of a physical environment. The strategies that an individual uses, then, are ways of making use of those features in order to accomplish one's goals (Greeno, 1991).

To clarify the meanings of the terminology, the *domain* of numbers and quantities is a conceptual domain with known properties that can be found in textbooks. An *environment*, by contrast, is a metaphorical characterization of an individual's perception of that domain. In this dissertation, the term *mathematical environment* is used in the vein of Bowers, Cobb, and McClain (1999), who used the environment metaphor to describe their instructional intent in a teaching experiment. The term *environment* is used in the sense of Greeno's environment metaphor, and the word *mathematical* serves to highlight the fact that the corresponding conceptual domain is a mathematical domain. From the perspective of the environment metaphor, a person's mathematical activity occurs not in the domain of numbers and quantities, but in a unique mathematical environment.

Objects in an environment have certain, inherent constraints and affordances. For example, composite numbers afford applications of the associative property of multiplication, whereas primes do not. According to Greeno, learning in a conceptual domain involves increasing attunement to the constraints and affordances of elements in that domain. The better a person knows the domain, the more attuned she will be to those constraints and affordances. In other words, learning looks like a process whereby the constraints and affordances that one perceives in the environment become increasingly aligned with the actual properties of the corresponding domain (Greeno, 1991).

The environment metaphor represents an attempt to bridge a situated perspective with a cognitive account of number sense:

> The view of number sense presented in this paper is an attempt to characterize conceptual knowledge in the framework of situated cognition. The basic form of situated cognition is an interaction of an agent within a situation, with the agent participating along with objects and other people to co-constitute activity. The agent's connection with the situation includes direct local interaction with objects and other people in the immediate vicinity as well as knowing where he or she is in relation to more remote features of the environment. (Greeno, 1991, p. 200)

The innovative aspect of this perspective was to view the individual's mathematical activity as situated in an environment, as opposed to locating the mathematics in the mind of the individual. In the environment metaphor, the learner acts within a mathematical environment. Teachers and more experienced peers, then, function as old-timers who are familiar with the domain. They can guide the newcomer to become better acquainted with it (Lave & Wenger, 1991). In this way, learning in a conceptual domain relates to cognitive apprenticeship: "knowing in a domain is an activity, and learning in the domain is acquiring the capabilities of understanding and reasoning that the domain affords, a kind of practice" (Greeno, 1991, p. 210).

Greeno's (1991) article has been widely cited. A search on December 7, 2009 showed that it was cited by 63 different articles listed at PsychINFO and by 79 at the Web of Science. However, few articles citing Greeno (1991) are reports of empirical studies concerning number sense. Even amongst those reports of number sense studies that do cite the article, the analytic methods employed do not seem to be informed by the environment metaphor in any substantive way. Evidently, Greeno's theoretical analysis of number sense has had more influence on the thinking of researchers outside of number sense research.

Analyses of Number Sense

Given the ways that number sense has been described in the literature, how has individuals' number sense been analyzed? Researchers have used written test instruments (on paper or computer), as well as interviews to assess individuals' number sense, and these have included both multiple-choice and open-ended items (Markovits & Sowder, 1994; Reys et al., 1999; Reys & Yang, 1998; Yang, 2003, 2005, 2007; Yang & Huang, 2004; Yang et al., 2008; Yang et al., 2009). These have largely been single-snapshot studies of particular populations. Relatively few studies have investigated change in students' number sense, and those that have done so have relied on comparisons of snapshots (Markovits & Sowder, 1994; Whitacre & Nickerson, 2006; Yang, 2002, 2003).

In several studies, Yang and colleagues have used multiple-choice test instruments (without written verbal elaboration of answers) to assess students' number sense (Reys & Yang, 1998; Yang, 2003; Yang & Huang, 2004; Yang et al., 2008). Specifically, these and other authors have used a Number Sense Test (NST) (Reys & Yang, 1998), a Number Sense Rating Scale (NSRS) (Hsu et al., 2001), and a computerized number sense scale (Yang et al., 2008). These measures consist of similar items and have been designed on the basis of a components view. For example, the NSRS consists of five groups of questions, which correspond to the following five components of number sense:

1. Understanding number meanings and relationships
2. Recognizing the magnitude of numbers
3. Understanding the relative effect of operations on numbers
4. Developing computational strategies and being able to judge their reasonableness
5. Ability to represent numbers in multiple ways (Hsu et al., 2001)

(The test instrument can be found in Appendix 1. Questions 1-5 correspond to Component 1, Questions 6-14 Component 2, Questions 15-24 Component 3, Questions 25-31 Component 4, and Questions 32-37 Component 5.)

In terms of assessing number sense development, gain scores on these measures are the natural source of evidence. These may be broken down by number sense component, given the assumption that particular questions are assessing targeted components. This assumption reflects a broader limitation of multiple-choice instruments. If only students' answers are analyzed, researchers must infer that correct answers on test items designed to assess number sense are, in fact, indicative of number sense. Multiple-choice survey instruments have the advantages that they can be administered to large numbers of students and that they afford quantitative analyses. However, analyses based on the results of these tests require a high level of inference in linking subjects' responses to their number sense. These instruments are limited in terms of their ability to capture those characteristics of number sense referred to in authors' descriptions of the construct. Multiple-choice instruments are limited in their ability to assess computational flexibility, and they cannot possibly capture other purported characteristics of number sense, such as "an expectation that numbers are useful and that mathematics has a certain regularity" (Reys & Yang, 1998, p. 226).

In other studies, Yang and colleagues have examined subjects' strategies for solving test and interview items designed to assess their number sense. In addition to coding responses for correctness, these authors have coded subjects' strategies as *Number sense-based* and *Rule-based* (Yang, 2003, 2005, 2007; Yang et al., 2009). Yang's stated criterion for a Number sense-based strategy is whether one or more components of number sense are evident in the person's solution process (Yang, 2003, 2005, 2007). Conspicuously, there is no case in the studies cited above in which Yang coded an incorrect response as Number sense-based, although there is nothing in his stated operational definition that would preclude this possibility (Yang, 2003, 2005, 2007, 2009). Recently, *Partially Number-sense based* has also been included as an intermediate category and has been applied to both correct and incorrect answers (Yang et al., 2009).

Coding subjects' strategies has clear advantages over assessing number sense on the basis of answers alone. The researcher is able to consider how the subject arrived at his or her answer, and these computational strategies come closer to reflecting components of number sense. For example, one component identified by Reys et al. (1999) is "Understanding and use of equivalent representations of numbers" (p. 62). At least the "use" part of this "understanding and use" would seem to be evident in a person's solution strategy.

A potential disadvantage of coding strategies as Number sense-based versus Rule-based is the attribution to the strategy itself of the quality of being reflective of number sense or a lack thereof. Consider, for example, the task of comparing the size of two given fractions. If a student compares two fractions by finding a common denominator, the student's strategy would be coded as Rule-based, as opposed to Number sense-based, since the method of finding a common denominator is commonly taught in schools (Yang, 2003). However, a person with good number sense could certainly employ this strategy at times. As such, it seems misleading to attribute number sense (or the lack thereof) to particular instances of strategy use. The particular mathematical understandings behind the strategy are ignored, as is the broader picture of the person's reasoning about relative size of fractions. This method discredits the value of school-learned procedures, while affording privileged status to all other valid procedures, and does not delve into the understanding behind those procedures, as used by individuals.

The Number sense-based versus Rule-based scheme lacks an illuminative quality with regard to the nature of a person's number sense. Coding strategies as Number sense-based or not results in a picture of individuals as employing number sense or not, or employing number sense with a certain frequency. By contrast, the descriptions of number sense cited earlier portray number sense as richer than an all or nothing matter or than an attribute that can be quantified. Suppose, for example, that a student uses Number sense-based strategies on 9 of 15 interview items, or 60% of the items. On its own, this statistic tells us very little about the person's number sense. Two people who score 60% may have used different strategies. One may have used a wider variety of strategies than the other. One subject may understand her strategies conceptually, while the other only knows them as procedures. Reporting the results in a more detailed fashion, for example, by identifying the particular strategies used or assessing each component of number sense separately, is helpful but still has its limitations. It does not seem to capture "a person's general understanding of number and operations" or the "inclination and an ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information" (Reys & Yang, 1998, p. 225-226).

In a related but less inferential coding scheme, Markovits and Sowder (1994) coded subjects' strategies as *Standard*, *Transition*, *Nonstandard*, and *Nonstandard with Reformulation*. The essential criterion in this scheme is the extent to which the person's procedure is tied to (or departs from) the standard algorithm for the given operation. This scheme is less inferential than that of Yang in the sense that the coding of strategies as Standard, Transition, Nonstandard, or Nonstandard with Reformulation is relatively uncontroversial. Claims regarding subjects' number sense become a separate inferential step, which may be situated in a broader context and take into account additional data. For example, Whitacre and Nickerson (2006) applied Markovits and Sowder's (1994) scheme as an organizing framework for mental computation strategies that they had first coded at a finer grain size. In a pre/post comparison of interview data, the use of this framework revealed a rather dramatic shift in the strategies used by the 13 interview subjects: In the first interview, Standard strategies were most common; in the second interview, the most common category was Nonstandard with Reformulation—the other extreme of the continuum. Along with additional evidence, this qualitative shift supported a claim about improved number sense on the level of overall mental computation performance across a group of 13 interview participants and hundreds of particular computations. Single computations were not coded as number sense-based or not number sense-based.

Beyond associating specific strategies themselves with number sense, coding of strategies had also been used to facilitate analyses of flexibility. Flexibility is considered to be an indicator of number sense (Greeno, 1991; Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Reys & Yang, 1998). An individual's flexibility can be observed in the number of different strategies that she uses for a given operation (Heirdsfield & Cooper, 2004). Change in flexibility may then be assessed on the basis of change in the number of strategies used. Coding of strategies à la Yang (2003) or Markovits and Sowder (1994) is not the ideal grain size for analyses of flexibility. Since there may be several different strategies within a single category, such as Nonstandard, a more fine-grained scheme is better suited to this kind of analysis.

As mentioned previously, there seems to have been no empirical study of number sense or its development that employed methods that were substantively informed by

Greeno's (1991) environment metaphor.[6] This being the case, it is interesting to consider what such a study might look like. In the environment metaphor, the development of number sense takes the form of increasing attunement to constraints and affordances in the domain of numbers and operations. To study an individual's number sense in a manner informed by the environment metaphor would seem to entail a qualitative analysis involving thick description with a particular focus on attunement to constraints and affordances. Such an empirical study could lead to further elaboration of the environment metaphor, and could thereby extend existing theory concerning number sense and its development.

Empirical Findings

Empirical findings concerning number sense may be grouped into two categories: (1) snapshot studies of number sense and (2) studies of change in number sense.

Snapshot studies of number sense. In a series of studies, Yang and colleagues have found that Taiwanese fifth, sixth, and eighth graders generally exhibit poor number sense (Reys & Yang, 1998; Yang, 2005; Yang & Huang, 2004; Yang et al., 2008). These students are able to perform familiar computations using school-learned procedures, but they perform poorly on non-routine tasks designed to tap their number sense, unless they are able to solve the tasks by employing Rule-based methods (Reys & Yang, 1998; Yang & Huang, 2004; Yang, 2005). Yang and colleagues have also found that preservice teachers in Taiwan exhibit poor number sense, relying heavily on standard written algorithms (Yang, 2007; Yang et al., 2009). Although the preservice teachers were more

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[6] Whitacre and Nickerson (2006) interpreted their results in terms of the metaphor, but their methods were similar to those used by researchers in other studies that made no mention of it (e.g., Heirdsfield & Cooper, 2002; 2004; Markovits & Sowder, 1994).

capable of correctly answering number sense test items than their middle-school counterparts, the majority of their answers were also obtained by written computation using standard algorithms. Both the middle school students and preservice teachers have tended to rely on standard written algorithms, despite the fact that the instructions for these assessments explicitly discouraged such approaches (Yang, 2005, 2007; Yang et al., 2009).

Other snapshot studies of "number sense" involve young children or are concerned with the number sense construct that is discussed in the mathematical cognition literature, which is not the number sense of interest in the present study.

Studies of number sense improvement. Against this background picture of the generally poor number sense of students and preservice teachers, there is some literature that suggests that number sense can be improved through instruction. Although there have not been many studies of change in number sense, the results of these are encouraging. Thus, two main points come from this part of the review: (1) number sense can be improved through instruction, and (2) there is more to be learned regarding *how* number sense improves with instruction.

Markovits and Sowder (1994) reported the results of an instructional intervention in a seventh grade mathematics class. Instruction consisted of units on number magnitude, mental computation, and computational estimation. In solving tasks, students were encouraged to invent their own strategies and to try strategies used by other students. In discussion, students were expected to provide informal mathematical justifications of the strategies they used. Students participated in pre, post, and retention written tests and interviews. The written tests emphasized tasks concerning number size, while the interviews focused on mental computation. Students written test scores improved substantially, and these improvements were sustained on the retention test. The researchers coded students' mental computation strategies as Standard, Transition, Nonstandard, and Nonstandard with Reformulation. They found that the frequency of less standard strategies increased from the pre to post interview, and this change was sustained in the retention interview. Markovits and Sowder (1994) argued that these results demonstrated improvement in the participants' number sense.

Yang (2002) describes an activity within one day of class in a sixth-grade classroom in Taiwan. The activity concerned the problem of determining whether 3/8 or 7/13 is closer to 1/2. This task was used by Markovits and Sowder (1994), and it is an item that is typical of number-sense assessment instruments (e.g., Yang, 2007). Students worked in groups on the task, and then presented their solutions to the class. The teacher facilitated a discussion that focused on sense making and mathematical justification. Students' performance on a similar item was evaluated in written pre, post, and retention tests. The percentage of correct responses improved from 35% to 72%, pre to post, and decreased slightly to 66% on the retention test. Yang (2002) claims that these results indicate improvement in the students' number sense, and that the "process-oriented activity" in the lesson was responsible for that improvement.

Yang (2003) reports on a semester-long quasi-experimental study of two fifth grade classes in Taiwan, consisting of 37 and 38 students. The experimental class was taught a special number sense curriculum, whereas the control class was taught the standard curriculum. Measures used were the 37-item NSRS pre/post/retention and a number sense interview with 6 students from each class, also pre/post/retention. Gains on the NSRS were statistically significant for both groups, but gains for the experimental class were greater from pretest to posttest, at 44% versus 10%. Interview results showed students from the experimental class used a higher proportion of Number sense-based strategies in post-instruction and retention interviews.

Kaminski (2002) describes instructional activities that took place in a mathematics content course for preservice elementary teachers in Australia. These activities included performing mental computations and discussing their strategies, as well as reasoning about relative fraction size, as in finding a fraction between two given fractions. The author describes certain in-class phenomena that point to positive changes in students' mathematical understandings or orientations. For example, he reports that students "preferred to gain understanding of mathematical procedures" and that they "frequently attempted to address their weaknesses in areas of mathematical understanding" (Kaminski, 2002, p. 135). It is difficult to distinguish examples of classroom activity from evidence for development (whether ontogenetic or sociogenetic) in Kaminski's account. Furthermore, many of the findings reported appear to be only loosely related to number sense improvement.

Kaminski's (2002) findings that appear most closely tied to number sense are that students "developed and utilized multiple relationships among number" and that students "had a strong preference for written computations but increasing use of mental computation was evidenced" (p. 135). No systematic methods for analyzing classroom activity are reported, and neither are measures of individuals' number sense. The course that Kaminski describes bears considerable similarity to the mathematics content course of interest in the present study. It is evident that the instructor's goals concerned facilitating the development of number sense, and the activities reported appear to be consistent with those goals. However, the analytic methods are unclear and the findings are not closely related to number sense.

Whitacre and Nickerson (2006) described similar instructional activities in a mathematics content course for prospective elementary teachers, designed to support number sense development. These researchers assessed change in students' number sense on the basis of three pre/post measures. They used Hsu et al.'s (2001) NSRS as a quantitative measure of number sense, finding that participants' scores increased to a statistically significant extent (Whitacre, 2006). The researchers also conducted mental computation interviews and coded students' responses based on the strategy used. They then applied Markovits and Sowder's (1994) coding scheme as an organizing framework. They found a shift in the strategies used, from Standard to Nonstandard. They also found that participants' flexibility increased, as seen in differences in the mean numbers of strategies used by each participant for each operation (Whitacre, 2007). Thus, the interview participants tended to use a wider variety of strategies at the end of the semester, and they tended to favor the least standard strategies, which are considered most sophisticated and are associated with number sense (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994).

The above articles are the only reports found in this literature review of studies of number sense improvement.[7] These examples suggest that number sense can be improved through instruction. However, more research is needed that can help us to understand the

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[7] Again, this review excludes studies involving young children and those in the mathematical cognition literature, which concern a different construct.

relationship between mathematics instruction and number sense improvement. In particular, evidence for improvement has been based on pre/post comparisons, rather than investigations of number sense development as a gradual process that occurs in relation to specific instructional activities.

Other findings. Yang et al. (2008) found that number sense test scores were significantly related to mathematics achievement (grades in year 5). In particular, two components—recognizing relative number size and using multiple representations of numbers and operations—were moderately correlated with achievement. Yang and Huang (2004) found that the high written computation scores of Taiwanese sixth-graders differed significantly from their number sense test scores. Skill in written computation did not imply high scores in number sense. In a study of Taiwanese preservice teachers, Tsao (2004) reported that number sense test scores were correlated with both mental computation and written computation test scores.

Summary of the Number Sense Literature

Number sense has been defined in different ways. Greeno's (1991) environment metaphor offers a theoretical treatment of the construct that seems to have promise for empirical research from a situated perspective. The analytic methods that have been used previously have focused on coding students' answers to tasks and on their strategies. These have produced some useful findings, but they have their limitations. Researchers have found that middle school students and preservice teachers generally exhibit poor number sense. However, number sense can be improved with instruction. There is more research to be done concerning number sense development, and the relationship between that development and students' involvement in instructional activities.

Whole-number Sense in Mental Computation and Number Composition

This section presents a review literature at the intersection of whole-number mental computation and number sense. This includes literature that specifically connects mental computation with number sense, as well as literature in which mental computation is treated in a way that is consistent with the number sense literature. The most prominent characteristic that is common to the mental computation and number sense literatures is flexibility. Thus, literature concerning flexible mental computation is discussed. Mental calculative strategies are discussed, together with extant coding schemes for these. The general mathematical understandings that support flexible mental computation are also addressed. There is a brief review of recommendations concerning pedagogy to support number sensible mental computation. Finally, as a summary piece, the major points are framed in terms of the environment metaphor.

The present study focuses on mental computation as a microcosm of whole-number sense. Mental computation relates to number sense in at least two important ways. First, mental computation contributes to number sense development (Anghileri, 2000; Sowder, 1992). Second, mental computation performance reflects number sense (Markovits & Sowder, 1994; McIntosh, Nohda, Reys, & Reys, 1995). Therefore, students' evolving mental computation reasoning can be studied as a window into their number sense development.

Flexible Mental Computation

The most prominent characteristic associated with skilled mental computation is flexibility (Carraher, Carraher, & Schlieman, 1987; Heirdsfield & Cooper, 2004; Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Reys, Nohda, & Emori, 1995; Reys, Rybolt, Bestgen, & Wyatt, 1982; Sowder, 1992). Flexibility is also central to descriptions of number sense (e.g., Greeno, 1991; Reys & Yang, 1998). *Flexibility* in mental computation refers to the ability and inclination to use a variety of mental calculative strategies (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994).

Standard written algorithms exist for addition, subtraction, multiplication, and division. *Inflexible* mental calculators are characterized by a tendency to perform mental computation by visualizing the written work that they would perform using these standard algorithms (Heirdsfield & Cooper, 2004). Such a mental procedure is known in the literature as the *mental analogue of a standard algorithm* (e.g., Markovits & Sowder, 1994), which will be abbreviated *MASA*. When number sense manifests itself in mental computation, MASAs are rarely employed. Instead, flexible mental calculators exhibit facility with a variety of different strategies (Heirdsfield & Cooper, 2004; Hope & Sherrill, 1987; McIntosh, 1998; Reys et al., 1982; Sowder, 1992).

The types of strategies preferred by flexible mental calculators tend to be those that are divorced from the standard written algorithms (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994). For example, the standard addition algorithm involves adding column-wise from right to left and regrouping ("carrying") as necessary. Markovits and Sowder (1994) describe some mental computation strategies as *Transition* strategies, which they define as being "somewhat bound to the standard algorithm" (p. 14). In the case of addition, this would involve adding from right to left but without actually picturing the numbers aligned as in the standard algorithm or mentally performing the written work. A *Nonstandard* strategy, by contrast, is one that is not bound to the standard algorithm. In the case of addition, this could be a strategy that involves adding while keeping a running subtotal, as in computing 124 + 59 by first adding 50 to 124 to obtain 174, and then adding 9 more for a total sum of 183.

In case studies of two Australian third graders, Heirdsfield and Cooper (2002) compared the characteristics of an accurate and flexible mental calculator with those of an accurate but inflexible mental calculator. They found that Clare, the flexible mental calculator, had strong numeration knowledge that supported her accuracy and flexibility. She also made use of knowledge of place value and properties of numbers and operations. She quickly recalled many number facts, and could derive others from recalled facts. Her computational estimation ability helped her to assess the reasonableness of her answers. Clare was also confident in her mental computation ability and in her strategy selection. According to the authors, these characteristics all contributed to her accurate and flexible mental computation performance (Heirdsfield & Cooper, 2002).

Heirdsfield and Cooper (2004) compared a group of accurate and flexible third graders with a group of accurate but inflexible third graders in mental addition and subtraction. The authors used a variety of tasks in multiple interviews in order to compare various aspects of the students' mental computation activity. They identified two distinct general processes used by the two groups.

Given one-step story problems in context, both groups could identify the appropriate operation to perform. For example, students were given the task, "You start with $400 and spend $298 on the CD player. How much is left?" (Heirdsfield & Cooper, 2004, p. 450). Each recognized that this task called for subtraction, or in other words that the solution would be given by the difference between 400 and 298. However, after that, the processes of the two groups differed. For the inflexible calculators, the recognition of the operation to be performed led directly to the use of the MASA for that operation. For the flexible mental calculators, the next step after determining the operation to be performed was to *choose* an efficient strategy for performing it. After the strategy was selected, it was implemented. Finally, the flexible calculators checked their work or assessed the reasonableness of their answers in some way (Heirdsfield & Cooper, 2004).

Flexible mental calculators do not arbitrarily select a strategy, as if at random (Markovits & Sowder, 1994). On the contrary, the choice of strategy is influenced by the particulars of the computation, together with a combination of knowledge of number and operations and beliefs about mental computation strategies (Heirdsfield & Cooper, 2004). The operation and the numbers involved make certain strategies better suited than others. For example, composite numbers afford applications of the associative property of multiplication, as in reformulating 15 x 24 as 30 x 12, whereas primes do not. The MASA for subtraction is better suited to computations that do not require regrouping, as in 36 - 12, as opposed to 36 - 19. Relative ease and efficiency influence the strategies that skilled mental calculators tend to use (Hope & Sherrill, 1987). Furthermore, it is not uncommon that a skilled mental calculator may entertain a strategy that is relatively unfamiliar to her, even inventing an approach to a particular problem that she has never used or seen before (McIntosh, 1998).

The process of flexible mental calculators is characteristic of skilled problem solvers more generally. Schoenfeld (1992) described skilled problem solvers as employing the meta-cognitive habits of planning, monitoring, and control. Planning corresponds to strategy selection. Monitoring refers to keeping track of progress and allowing for the possibility of changing strategies. Control processes involve decisions to change strategies, as well as checking the reasonableness of answers (Schoenfeld, 1992). According to Heirdsfield and Cooper (2004), knowledge of the effect of operation on number, numeration and number facts knowledge, strategies, and beliefs are all involved in the process of choosing a strategy, implementing that strategy, and checking one's mental work. Knowledge of the effects of operations is crucial when rounding and compensation are used. For example, to compute 125 - 49 mentally, it is helpful to recognize that 49 is one less than 50. However, one must then know how to appropriately compensate for rounding 49 to 50. Many prospective and practicing elementary teachers have difficulty determining how to compensate for rounding of the subtrahend in subtraction (Kazemi et al., 2010).

Whitacre (2007) found that prospective elementary teachers beginning a first mathematics content course tended toward inflexibility in mental computation. Most used only one or two strategies for a given operation, with the MASA being most common for each of addition, subtraction, and multiplication. Tsao (2004) assessed preservice elementary teachers' mental computation performance based on their ability to obtain correct answers to test items without the aid of written work. The mean score on this measure was less than 50%. These findings are consistent with reports that preservice elementary teachers generally exhibit poor number sense (Tsao, 2005; Yang, 2007; Yang et al., 2009).

Mental Computation Strategies

Extant coding schemes exist for mental computation strategies. These differ in terms of grain size and are suited to different purposes. Heirdsfield and Cooper's (2004) scheme for mental addition and subtraction strategies is rather fine-grained and comprehensive. It was created on the basis of a literature review. Nunes, Schliemann, and Carraher's (1993) scheme covers each of the basic operations, but the grain size is coarse. This scheme was grounded in a particular data set. Markovits and Sowder's (1994) scheme covers addition, subtraction, and multiplication. Strategies as such are not coded in this scheme. Rather, the criterion of similarity to the MASA is used to categorize strategies. Whitacre's (2007) coding of addition, subtraction, and multiplication strategies was grounded in data from interviews with prospective elementary teachers, while also being informed by extant coding schemes, especially those listed above.

Markovits and Sowder (1994) formulated an overarching framework for mental addition, subtraction, and multiplication strategies. They used the following coding scheme for the strategies used by students in their study:

> *Standard:* The student used a mental analogue of a standard paper-and-pencil algorithm.
>
> *Transition:* The student continued to be somewhat bound to the standard algorithm. However, more attention was given to the numbers being computed and less to algorithmic procedures.
>
> *Nonstandard with no reformulation:* A left-to-right process was used.
>
> *Nonstandard with reformulation:* The numbers were reformulated to make the computation easier. (Markovits & Sowder, 1994, p. 14)

According to Markovits and Sowder (1994), reliance on Standard strategies indicates poor number sense, while the use of Nonstandard strategies indicates good number sense. A student whose go-to strategy is the MASA lacks flexibility. Her understanding of the operation appears tied to the standard algorithm, and she is not making a choice based on the given numbers. By contrast, a student whose understanding of an operation is not dependent upon any particular algorithm has good flexibility. She is free to choose a strategy that takes advantage of the affordances of the given numbers. In terms of a learning trajectory, a student who primarily uses Transition strategies may be regarded as being in transition from algorithm-bound to algorithm independent, thus from inflexible to flexible (Markovits & Sowder, 1994; Whitacre, 2007).

Heirdsfield and Cooper (2002, 2004) used a coding scheme for mental addition and subtraction strategies that they developed based on a review of previous literature. Their scheme consists of five broad categories of strategies: *Counting, Separation, Aggregation, Wholistic,* and *Mental image of pen and paper algorithm*. The categories of Counting and Mental image of pen and paper algorithm are, in fact, codes for particular strategies in this scheme, while the categories of Separation, Aggregation, and Wholistic consist of two to three particular strategy codes. Although this scheme is not explicitly hierarchical like that of Markovits and Sowder (1994), these categories more or less correspond to Transition, Nonstandard, and Nonstandard with Reformulation, respectively. Furthermore, Heirdsfield and Cooper (2004) have argued that Separation, Aggregation, and Wholistic (in that order) represent increasing levels of sophistication, which is in accord with Markovits and Sowder's (1994) hierarchy. Thus, for coding of mental addition and subtraction strategies, these two schemes are quite compatible.

Nunes et al. (1993) distinguish "oral" versus "written" mental computation procedures. This is essentially the Nonstandard versus Standard distinction made by Markovits and Sowder (1994), or the Number sense-based versus Rule-based distinction made by Yang (2007). Nunes et al. (1993) found that oral computation procedures for addition and subtraction tended to involve a *decomposing* strategy, while oral computation procedures for multiplication and division tended to involve *repeated grouping*. *Decomposing* refers to the additive decomposition of numbers, for example treating 35 as 30 and 5. Numbers are decomposed into round numbers that are convenient for the purposes of the computation. Carraher et al. (1987) described *repeated grouping* as "multiplying by means of successive additions or dividing by means of successive subtractions" (p. 93).

The categories of decomposition and repeated grouping are useful in that they emphasize the mathematical properties underlying various strategies. As Nunes et al. (1993) point out, implicit in decomposition is the associative property of addition. For example, it is often useful for the sake of mental addition to treat 137 as 100 + 30 + 7 (where the 7, in turn, may be treated as 2 + 5 or 1 + 6). Similarly, implicit in repeated grouping is the distributive property of multiplication over addition. For example, for ease of mental multiplication, we may treat 13 x 25 as 10 x 25 + 3 x 25.

Note that there is at least one additional distinction that, even from a purely mathematical standpoint, seems necessary. Mental multiplication or division strategies that make use of the associative property of multiplication should be distinguished from those that do not. Nunes et al. (1993) give the example of a child who found 100/4 mentally by two successive divisions by two. Such a strategy is distinct from repeated grouping strategies that consist of "multiplying by means of successive additions or dividing by means of successive subtractions" (p. 43). The authors acknowledged that this example did not fit their definition of repeated grouping, although it was coded as such. The authors suggested that this matter merited further study.

In terms of Markovits and Sowder's scheme, strategies involving the distributive property of multiplication over addition are more similar to the standard multiplication algorithm than are those involving the associative property of multiplication. Whitacre (2007) found that prospective elementary teachers in pre-instruction interviews rarely took advantage of the associative property in mental multiplication, whereas explicit applications of the distributive property were more common.

Whitacre (2007) coded prospective elementary teachers' mental computation strategies using a scheme developed through constant comparative analysis of their particular data set. The coding scheme consists of six valid strategies for addition, eight for subtraction, and seven for multiplication. (There is also an invalid category for each operation.) For each operation, strategies are ordered along the spectrum from Standard to Nonstandard. In this way, Markovits and Sowder's framework is fleshed out to include particular mental computation strategies.

Whitacre (2007) departs somewhat from Markovits and Sowder in defining *Nonstandard with no reformulation*. Whereas Markovits and Sowder defined this category as involving a "left-to-right process," Whitacre found this definition too stringent. A nonstandard strategy without reformulation need not involve a left-to-right process. Right-to-left aggregation is one example (Heirdsfield & Cooper, 2004). On the other hand, the thinking of prospective elementary teachers who used left-to-right processes often seemed to be tied to the standard algorithms. When it came to addition and subtraction, for example, several participants worked from left to right when no regrouping was required, and right to left otherwise. Furthermore, some participants working left-to-right reported visualizing the numbers aligned vertically, as in the standard algorithm. These instances seemed most appropriately categorized as Transition, rather than Nonstandard, strategies.

For reasons grounded in the particular data set, Whitacre (2007) revised the definition of *Nonstandard with no reformulation*, replacing it with the following: "The student was not bound to the standard algorithm, but did work with the given numbers." This revised definition is in keeping with the spirit of Markovits and Sowder's scheme in that the continuum from most standard to least standard is emphasized, as opposed to the particular process employed. Markovits and Sowder's scheme is useful as a general framework that relates mental computation strategies to number sense, whereas a scheme like that of Whitacre (2007) is useful for making fine-grained distinctions between strategies for particular operations.

Understanding Strategies and Algorithms

What sense do students make of the computational methods that they use? The literature provides partial answers to this question. These computational methods can be categorized by operation and/or by type (e.g., standard versus nonstandard). Thanheiser (2009) developed a framework for preservice teachers' conceptions of multidigit whole numbers in the context of the standard addition and subtraction algorithms. The literature does not seem to contain a comparable framework for conceptions related to the multiplication and division algorithms. Children's whole-number computational reasoning has been studied extensively. Carraher and colleagues (1987) emphasized understanding of additive and multiplicative composition of number in their discussion of nonstandard strategies. The literature on Cognitively Guided Instruction (CGI) maps out developmental trajectories of children's computational strategies. These authors also emphasize children's understanding of place value and number composition and relate these to children's strategies. This research helps to answer the question of how students understand their computational strategies. However, the CGI framework concerns less sophisticated strategies than many of those identified by Heirdsfield and Cooper (2004) and used by skilled mental calculators. When it comes to relatively sophisticated nonstandard strategies, it is difficult to identify in the literature direct answers concerning how these are understood.

Addition and subtraction. Thanheiser (2009, 2010) has made an extensive study of the conceptions of multidigit whole numbers that preservice teachers bring to their use of standard written algorithms. Her framework categorizes conceptions of multidigit whole numbers and orders these in terms of levels of sophistication. The defining characteristic of this framework could be described as flexibility in reasoning about the amounts that digits in a number represent. At the lowest level of sophistication, each digit is conceived of as a number of ones, e.g., 529 represents 5 ones, 2 ones, and 9 ones, in their respective columns. The individual may be able to give appropriate names for place values, but she does not see the 2 as representing two tens or the 5 as five hundreds. At the highest level of sophistication, the individual has maximal flexibility in reasoning about reference units, e.g., she can view the 5 as representing 500 ones or 50 tens or 5 hundreds. Thus, this framework describes preservice teachers' understanding of the additive composition of number, especially the canonical composition in terms groups of ones, tens, and hundreds.

Thanheiser (2009) describes preservice teachers' conceptions of multidigit whole numbers in terms of the following four categories:

> *Reference units.* PSTs with this conception reliably conceive of the reference units for each digit in the number and can relate the reference units to one another; in 389, the 3 can be seen as 3 hundreds or 30 tens or 300 ones, and the 8 can be seen as 8 tens or 80 ones.
>
> *Groups of ones*. PSTs with this conception reliably conceive of all digits in terms of groups of ones; 389 would be 300 ones, 80 ones, and 9 ones. PSTs holding this conception do not conceive of the digits in terms of reference units.
>
> *Concatenated digits plus.* PSTs with this conception conceive of at least one of the digits in the number in terms of an incorrect unit type at least some of the time. They therefore struggle when relating the values of the digits in a number to one another. A PST may correctly conceive of groups of 100 ones for a digit in the hundreds place but incorrectly conceive of ones for the tens place (e.g., 389 would be seen as 300 ones, 8 ones, and 9 ones).
>
> *Concatenated digits only*. PSTs holding this conception conceive of all the digits in terms of only ones (e.g., 548 would be 5 ones, 4 ones, and 8 ones). (p. 263)

Thanheiser (2009) found that the less sophisticated conceptions of concatenated digits only and concatenated digits plus were common among preservice teachers, with concatenated digits plus being most common. In further research, Thanheiser (2010) found that preservice teachers often explained the meanings of digits inconsistently across tasks. She further refined the framework to account for consistency or inconsistency in these interpretations. For the present purposes, the original categories of Thanheiser (2009) suffice.

Thanheiser’s framework answers the question of how preservice teachers understand the *standard* addition and subtraction algorithms. It does not speak to understanding of nonstandard strategies. Little is known about prospective elementary teachers’ understanding of nonstandard strategies, perhaps because they rarely use these.

There is, however, research literature concerning children's use of and reasoning about nonstandard strategies.

In their discussion of *written* versus *oral arithmetic*, Carraher et al. (1987) identified *decomposition* as the key heuristic employed in oral (nonstandard) addition and subtraction strategies. For example, a child named Lucia computed 200 – 35, saying, "If it were thirty, then the result would be seventy. But it is thirty-five. So it's sixty-five; one hundred sixty-five" (p. 91). The authors explain that Lucia decomposed 35 into 30 and 5 and decomposed 200 into 100 and 100 in order to accomplish her solution.

The notion of decomposition begins to answer the question of how nonstandard strategies are understood. However, there are a wide variety of strategies that make use of decomposition in some way. Furthermore, justifications for these strategies are generally left implicit, both in the subjects' reported reasoning and in the authors' analyses. We do not know how Lucia would have justified her strategy if asked to do so explicitly.

Suppose a different child reasons that 200 – 30 = 170, and so 200 – 35 = 175. This child likewise decomposed 35 into 30 and 5 and 200 into 100 and 100. However, she reasoned differently about how to compensate for the fact that she had subtracted 30, rather than 35. Perhaps Lucia did not think of her strategy in terms of compensation at all, but in terms of aggregation: I take away 30, and that leaves 170. Then I take away 5 more and end up with 165. Note that this is not clear from her explanation. She might have reasoned in some other way. When it comes to sophisticated strategies like these, the computational steps alone do not allow us sufficient access to the students' reasoning.

The CGI literature also emphasizes understanding of number composition. As children develop an improved understanding of number composition, both canonical and noncanonical, they can leverage this understanding to solve problems using invented strategies. A major contribution of the CGI research is a problem-specific developmental framework for students' strategies. Carpenter, Fennema, Franke, Levi, and Empson (1999) describe story problem types and related trajectories of students' strategies. Broadly, these proceed from Direct Modeling to Counting to Number Facts. Each of these categories includes several specific strategies. Number Facts strategies (which correlate with Transition and Nonstandard strategies) are those of interest to the present study. However, development toward using such strategies with understanding likely occurs differently for prospective elementary teachers than for children, being that the starting points for these two populations are quite different. In CGI, children invent their own strategies prior to learning the standard algorithms. Prospective elementary teachers, by contrast, know and are dependent upon the standard algorithms.

Since far more research has been done concerning the mathematical thinking of children versus prospective/preservice elementary teachers, there is the potential for work with prospective teachers to be informed by the literature on children's thinking. However, particular distinctions of interest to this study are not well addressed in the children's thinking literature. For example, in a synthesis piece, bringing together research from four extensive projects, Fuson et al. (1997) describe a common framework for children's multidigit addition and subtraction strategies and supporting conceptions. In line with other research in this area, number composition is emphasized. In fact, compensation is scarcely mentioned in the report. The authors note a common error in subtrahend compensation, whereby students add when they should subtract, or vice versa. Regarding conceptions related to compensation, the authors state:

> The methods can be easily confused if children do not understand what must remain the same in each method. In addition the total must stay the same; this method can be thought of as just moving some entities from one number to the other to make one number easy to add. In subtraction, the difference must be maintained, so the same number must be added to (or subtracted from) each number. (Fuson et al., 1997, p. 152)

The above statement makes sense from an expert perspective, but its relationship to students' mathematical thinking is unclear. As stated earlier, many prospective elementary are known to have difficulty determining how to compensate appropriately for rounding the subtrahend (Kazemi et al., 2010). Prospective teachers reasoning about a subtraction problem attempt to get the right answer. If they round one or both numbers, they attempt to compensate for doing so. In other words, they seem aware that "the difference must be maintained" and they try to maintain it; however, it is evidently unclear to many prospective teachers *why* maintaining the difference requires that "the same number must be added to (or subtracted from) each number" (Fuson et al., 1997, p. 152).

Multiplication.[8] Multiplication and division have been less extensively researched than have addition and subtraction. Fuson (2003) described the standard multiplication and division algorithms as "complex embedded methods that are not easy to understand or to carry out" (p. 302). As with the standard addition and subtraction algorithms, numbers wise in the multiplication algorithm are aligned place-value wise. This conventional alignment, together with the rule of shifting to the left or placing a zero when moving to the next row in the procedure, enables the algorithm to work "without requiring any understanding of what is actually happening with the ones, tens, and

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[8] I focus on multiplication, rather than division, in this section because it is more relevant to the results presented in Chapters 5 and 6.

hundreds" (Fuson, 2003, p. 302). Such algorithms are efficient and reliable but not very accessible to students.

Authors have identified invented strategies that children use for multiplication when given the opportunity to create their own methods. For example, Baek (1998) describes complete number strategies, partitioning number strategies, and compensating strategies. *Complete number strategies* refer to strategies in which multiplication is interpreted as repeated addition. For example, 6 x 23 represents six 23's, and so we may find the product by adding six 23's together.

Partitioning number strategies refer to strategies in which one or both factors are partitioned before multiplying. Within this category, there are different forms of partitioning. What Beak calls *partitioning a number into nondecade numbers* refers to making use of the associative property of multiplication. For example, 15 x 177 may be solved by computing 3 x 77 first, and then multiplying that result by 5. Formally, the principle at work can be illustrated as follows: 15 x 177 = (5 x 3) x 177 = 5 x (3 x 177). This kind of strategy involves a multiplicative partitioning, which is quite unlike additive partitioning.

Partitioning a number into decade numbers refers to partitioning one factor into tens and ones and computing partial products. For example, 43 x 61 = 40 x 61 + 3 x 61. Such a strategy may involve thinking in terms of repeated addition (i.e., adding up forty-three 61's) but partitioning by tens and ones makes the computation easier. One can compute the two partial products (40 x 61 and 3 x 61) separately and then added these together.

Baek (1998) also identifies the strategy of *partioning both numbers into decade numbers*. He gives the example of a child computing 26 x 39 by partitioning 26 into 20 and 6 and partitioning 39 into 30 and 9. She computed all four partial products explicitly (20 x 30, 20 x 9, 6 x 30, and 6 x 9) and added these together. Baek (1998) also gives the example of a child named James applying this strategy to a two-digit-by-three-digit product to compute 17 x 177. Absent in these descriptions are explanations of just how children are able to appropriately account for partial products. Baek (1998) claims, "James showed his good number sense and flexibility depending on the numbers in the problem as well as his understanding of base ten and multiplication" (p. 156). Certainly, not all invented strategies are valid ones. Vague descriptions of understanding are insufficient to distinguish the reasoning of individuals who correctly account for partial products from the reasoning of those who do not.

Compensating strategies are those in which one or both factors are changed prior to computing. An example of changing both factors is halving and doubling, as in multiplying 5 x 250 by changing the problem to 10 x 125. Like partitioning into nondecade numbers, this example makes use of the associative property. Compensating strategies also applications of the distributive property of multiplication over subtraction. For example, a child computes 17 x 70 by first finding 20 x 70 and then compensating for rounding. To do this, the child must have a way of reasoning that enables her to recognize how much she has to subtract to compensate. In Baek's (1998) example, the child correctly subtracted 3 x 70 from 20 x 70 to get her answer by reasoning about multiplication in terms of repeated addition (i.e., thinking about how many 70's she was supposed to add together). Whitacre (2006) reported instances of prospective elementary teachers using invalid compensation strategies (e.g., subtracting 3, rather than 3 x 70 in the above example). Again, the literature concerning children's invented strategies seems to devote little attention to invalid strategies or to particular ways of reasoning that support children's invention of valid strategies.

There is little literature concerning prospective elementary teachers' understanding of multiplication. Simon (1995) described a teaching experiment related rectangular area in a course for preservice teachers. He documents the teacher's role in responding to students' reasoning over the course of an 8-day instructional sequence in which the class worked to make sense of the relationship between multiplication and area. Zazkis and Campell (1996) investigated preservice teachers' reasoning about multiplication in the context of elementary number theory. They documented preservice teachers' dependence on standard algorithms, as opposed to reasoning about multiplicative composition of number, in tasks such as determining whether a number was divisible by a certain factor. These studies highlight difficulties that preservice teachers have in making sense of multiplication.

Summary of Whole-number Sense in Mental Computation

In summary, good whole-number sense manifests in flexible mental computation. The process of flexible mental calculators is distinct from that of inflexible calculators in that the former make a choice of strategy for a given computation. This choice is guided by perceived constraints and affordances, which depend on the properties of numbers and operations. As a result, flexible mental calculators often employ nonstandard strategies. Inflexible mental calculators, by contrast, tend to rely on the mental analogues of the standard algorithms. Previous research suggests that prospective elementary teachers tend toward inflexibility in mental computation.

Mental computation strategies can be characterized in various ways, which serve different analytic purposes. The schemes reviewed inform the analysis of prospective elementary teachers' mental computation strategies in the present study. Thanheiser (2009) developed a framework to describe preservice teachers' conceptions related to the standard addition and subtraction algorithms. My literature review did not reveal analogous findings concerning multiplication or division. The research literature describes children's reasoning concerning invented strategies for each of the operations. However, less attention has been given to the conceptions supporting students' reasoning when using nonstandard strategies.

Design Research and Instructional Design Theory

This section describes design research and aspects of the instructional design theory of Realistic Mathematics Education. Recommendations for pedagogy to support number sense development are reviewed. The instructional goals for the mathematics content course are described. Then the local instruction theory that guided instruction in the course is presented.

Design Research

Design research is characterized by a reflexive relationship between instructional design and classroom-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gravemeijer, 1994, 2009). A distinguishing characteristic of this paradigm is that it is focused on producing theory, as opposed to curriculum materials. This includes local instruction theories, which will be discussed in detail in the next subsection.

Developmental research occurs in cycles of instructional design, classroom teaching experiments, and analysis of classroom events. These analyses feed back to inform the instructional design process, and so on. In this way, individual studies and analyses are seen as contributing to the broader enterprise of educational development.

Nickerson and Whitacre (2010) developed a local instruction theory, which was revised based on the results one iteration of a classroom teaching experiment. Our previous analysis focused on instructional design and on changes in individual students (shifts in their mental computation behavior, which suggested improved number sense). Analysis of classroom events was constrained due to the limits of the data corpus. The present study constituted a new design research cycle and had a different emphasis, focusing on genetic analysis of number sense development, including the sociogenetic and ontogenetic strands. These analyses contribute to the building of theory concerning prospective elementary teachers' number sense development.

Instructional Design Theory

This subsection focuses on the instructional design aspect of the research. The design of instruction in the course of interest was informed broadly by the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991). Three instructional design heuristics are central tenets of RME: (1) sequences must be experientially real, (2) students should be guided to reinvent significant mathematics for themselves, wherein (3) students and the teacher develop models of informal activity, which become models for mathematical reasoning (Stephan, Bowers, Cobb, & Gravemeijer, 2003).

Emergent models. Gravemeijer, Bowers, and Stephan (2003) explain that the shift from model-of to model-for occurs when "[t]he process of acting with the model changes from one of constructing solutions situated in the context to one of using the model to communicate reasoning strategies" (p. 54). In RME, this process goes hand in hand with the move from informal mathematical activity to more formal mathematical reasoning. Thus, "[t]he shift from *model of* to *model for* can be seen as aligned with shifts in the collective mathematical practices" (p. 54).

In theoretical work building on the tenets of RME, Rasmussen and Marrongelle (2006) introduced the notion of a *transformational record*:

> Transformational records are defined as notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems. (p. 389)

The authors offer transformational records as a *pedagogical content tool* that serves as the instructional counterpart of the emergent models heuristic. That is, on the micro-level of classroom interaction, teachers can record student reasoning in ways that afford the possibility that students will transform these records by later using them to solve new problems. Likewise, teachers may recognize this potential in students' records of their own or each others' thinking and make such records objects of classroom discourse. By facilitating the realization of transformational records, teachers can promote shifts from models-of to models-for (Rasmussen & Marrongelle, 2006).

Hypothetical learning trajectories and local instruction theories. Instructional design involves the development of hypothetical learning trajectories. Simon (1995) introduced the construct of *hypothetical learning trajectory*, which refers to an anticipated learning route. This learning route is hypothetical, whereas *actual learning* *trajectory* refers to the learning route that actually plays in a particular classroom. In a classroom teaching experiment, a hypothetical learning trajectory is developed and enacted in the classroom. The instructional activities and the HLT itself are later revised on the basis of actual learning trajectories.

In some cases, hypothetical learning trajectories are informed by local instruction theories, which guide instruction more broadly. A local instruction theory refers to "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). According to Gravemeijer, local instruction theories are distinguished from hypothetical learning trajectories in two major respects, grain size and situatedness: hypothetical learning trajectories have a finer grain size and are specific to a group of students, while local instruction theories have a coarser grain size and are designed for a particular student population (Gravemeijer, 1999, 2004).

In my ongoing work with Nickerson, we have come to conceptualize the distinctions between LIT and HLTs differently. As far as the LIT for number sense development is concerned, we would say that the LIT spans mathematical topics, whereas HLTs are topic-specific. Our LIT concerns number sense, whereas an HLT may concern a specific mathematical topic like reasoning about fraction magnitude (Whitacre & Nickerson, 2011a). For us, the LIT informs topic-specific HLTs. *As a result* of this more general nature, the duration of the LIT is greater. With regard to situatedness, we would say that HLTs are specific to a student population, while an LIT may be broader. If an HLT applied only to a particular set of students in a classroom, then design research would be quite limited in its application. The findings of a design cycle are valuable because they inform instruction in future classes. Our LIT for number sense development is not specific to prospective elementary teachers. Rather, it applies to student populations that tend to be dependent on standard algorithms and to reason inflexibly about numbers and operations. This could be true of students at various levels.

Pedagogy to Support Number Sense Development

Brown et al. (1989) emphasize the discontinuity between traditional schooling and cognitive apprenticeship. The behavior of just plain folks (JPFs) is not unlike that of practitioners, whereas the behavior of these groups differs markedly from the behavior typical of students in traditional classroom settings. The authors give examples of mathematics teaching that does have the characteristics of cognitive apprenticeship. In Schoenfeld's problem-solving class, the students and the teacher engage with problems that are problematic in ways that traditional school exercises are not. Students' in-class work is more like the work of mathematicians (Schoenfeld, 1987).

The notion of authentic activity aligns with recent recommendations for mathematics pedagogy in general (NRC, 2001) and for promoting number sense in particular (Greeno, 1991; Howden, 1989; Kaminski, 2002; McIntosh, 1998; Sowder & Schappelle, 1994; Yang, 2002). While symptoms of number sense can be identified, researchers generally recommend that instructors not attempt to teach to these directly. There is general theme in the literature that number sense is not amenable to direct instruction and must develop over time through certain kinds of activities (Howden, 1989; Greeno, 1991; McIntosh, 1998; Sowder, 1992).

McIntosh (1998) argues that mental computation should be given greater emphasis in elementary curricula. However, specific strategies should not be taught directly:

> Traditional pedagogical methods would suggest that we should look for the best mental-computation algorithms and teach them. I suggest, however, that doing so may be counterproductive. (p. 44)

The key characteristic identified by Heirdsfield and Cooper (2004) in the process of flexible mental calculators was the habit of making a choice. Individuals who exhibit number sense do not tend use a single, go-to procedure. They decide what approach to take based on the relevant constraints and affordances that they perceive (Greeno, 1991). To be taught a particular algorithm is to be taught not to make such a choice. As Schoenfeld (1992) points out, when strategies are taught directly, “they are no longer heuristics in Pólya’s sense; they are mere algorithms” (p. 354). In their review of literature concerning the learning of mental computation strategies, Varol and Farran (2007) report on a study that found that students who had been taught a specific mental computation algorithm were less likely to invent novel computational strategies.

In terms of Greeno (1991)’s environment metaphor, direct instruction is analogous to giving a newcomer explicit directions for getting from one place to another. If these directions are any good, they should aid the newcomer in reaching the desired destination from a specified starting point. However, memorizing a potentially infinite number of specific routes is impractical and not conducive to coming to know the environment in the way that an experienced resident does. Ultimately, people need to establish their own sense of the lay of the land, to devise their own routes, and to spend time getting lost and finding their way back to their own personal landmarks. The experienced resident may serve as a guide, who can facilitate this process by suggesting activities, devising practical problems, and giving general advice. However, there is no substitute for the person's own goal-oriented exploration of the environment.

To support the gradual development of number sense, Sowder (1992) suggests that "number sense should permeate the curriculum… rather than being relegated to 'special lessons' designed to 'teach number sense'" (p. 386). McIntosh (1998) likewise warns against "restricting our mental-arithmetic sessions to bursts of short, unrelated calculations in which we emphasize accuracy and speed" (p. 47). He suggests instead that, if given time, students will invent novel strategies. After students have been asked to perform a mental calculation, strategies should be shared and discussed. The emphasis of these discussions should not be on whether the answers are correct but on understanding multiple, valid approaches. McIntosh (1998) suggests further than the instructor should take advantage of students' spontaneous interest in each others' strategies:

> Often you will notice a heightening of interest in a method described by a child, which indicates that the algorithm has been understood and appreciated by several children. At such times 'teaching an algorithm' is valid. You might say, 'Let's see if we can all use Amanda's method to do these calculations.' Making such a suggestion is quite different from imposing a single algorithm from outside. The situation is more like what happens when someone shows off a new toy. A child might ask, 'Can I have a go with that?' We all want to try it out a few times to see if we would like it for ourselves. (p. 47)

Thus, there is room for an approach that is authentic and honors student's choice making, while also allowing for opportunities to engage students in thinking about and trying on particular strategies. The recommendations of McIntosh (1998) are echoed repeatedly in the number sense literature (Anghileri, 2000; Greeno, 1991; Howden, 1989; Kaminski, 2002; Sowder, 1992; Varol & Farran; Yang, 2002).

Instructional goals. Beyond general recommendations like those above, instructional design is necessarily informed by goals concerning students' learning. In their developmental research, Cobb and Bowers (1999) state that they have found it useful to frame their instructional goals in terms of Greeno's (1991) environment metaphor. In the case of the study reported by Bowers et al., (1999),

> [the researchers'] instructional intent was that students would eventually come to act in a mathematical environment in which quantities are invariant under certain mathematical transformations (e.g., regrouping 100 as ten 10s). (p. 33)

Such application of the environment metaphor seems to be a useful way of framing one's broad instructional goals, particularly when these concern number sense development. Note the distinction that we can infer from the above quote between the domain of numbers and quantities and a particular mathematical environment: *the* domain of numbers and quantities is *not* the mathematical environment in which students act. Rather, the environment in which students act is influenced by their mathematical experiences in the classroom and their particular interpretations of these.

The broad instructional intent in the mathematics content course is for students to come to act in a mathematical environment in which the properties of numbers and operations afford various calculative strategies, as opposed to one in which mathematical operations map directly to particular algorithms (Nickerson & Whitacre, 2010). The instructional design theory of RME, especially the emergent models heuristic, along with recommendations from the literature, and the environment metaphor, informed the design of instruction in a course intended to promote the development of number sense.

A local instruction theory for the development of number sense. Beyond the broad instructional intent, articulated in terms of the environment metaphor, Nickerson and Whitacre's (2010) local instruction theory is guided more specifically by three major goals. The statement of these goals involves a construct that must first be defined. A *number-sensible strategy* as an approach that an individual chooses from amongst a set of possible approaches, where the choice of approach is based on the constraints and affordances that she perceives. The construct of number-sensible strategy is intended to characterize the manner in which the student approaches the task from the perspective of the student. A number-sensible strategy is in contrast to an approach that is algorithmic or nonstrategic from the student's perspective. As a construct, the notion of number-sensible strategy also contrasts with the way in which the term *strategy* is typically used in the mathematics education literature to describe how students solve problems. The latter is an observer-oriented construct; it characterizes students' activity from the researcher's perspective. In keeping with our actor-oriented perspective (Lobato, 2003), we would not, for example, label valid subtrahend compensation as number-sensible or not number-sensible. To do so would be to disjoin the computational steps from the thinking of the individual who carried out the approach.

Nickerson and Whitacre's (2010) local instruction theory includes three goals concerning student learning: (1) Students will capitalize on opportunities to use number-sensible strategies for problem-solving situations both inside and outside the classroom; (2) Second, students will draw on deep, connected knowledge of number and operations to develop a repertoire of number-sensible strategies; (3) Students will reason with models to build on their understanding and flexibly use number-sensible strategies. Put succinctly, our three major goals are that students come to recognize opportunities to use number-sensible strategies, develop many ways to think about number and operations, and flexibly draw on a repertoire of meaningful strategies (Nickerson & Whitacre, 2010).

With these goals in mind, the local instruction theory for the development of number sense can be briefly described. To reiterate, according to Gravemeijer (2004), a local instruction theory consists of "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (p. 107). In this case, the topic is number sense. The local instruction theory is represented here in the form of three tables, one for each of the learning goals listed above. Each table depicts a progression, which weaves back and forth between the instructional activities and the envisioned learning route. Thus, to read each table, read it row by row, from left to right. Note, however, that this format is not meant to imply discrete chronological stages, wherein one stage must cease before the next begins. Rather, these are ongoing processes, which overlap chronologically. Tables 1, 2, and 3 depict the instructional activities and accompanying envisioned learning route for Goals 1, 2, and 3, respectively.

The rationales for the learning routes are rather involved. See Nickerson and Whitacre (2010) for more details.

This local instruction theory was developed in a previous classroom teaching experiment. The authors developed a conjectured local instruction theory, based on a review of the literature, together with previous experience teaching the mathematics content course. During the semester in which the teaching took place, and then in subsequent analysis, they revised and further elaborated the local instruction theory. Since then, the local instruction theory has been enacted informally in teaching but

Table 1. Route to Goal 1 (Table Adapted from Nickerson & Whitacre, 2010)

Instructional Activities	**Envisioned Learning Route**
Instructor identifies and engineers opportunities for computational reasoning	
Students are invited to use computational reasoning and to reason quantitatively	Many students initially rely on standard algorithms
Students are invited to carry sense making to solutions with nonstandard strategies	Students use their own nonstandard strategies
Students solve problems mentally in a variety of contexts	Students capitalize on opportunities to use number-sensible strategies

Table 2. Route to Goal 2 (Table Adapted from Nickerson & Whitacre, 2010)

Instructional Activities	**Envisioned Learning Route**
Instructor anticipates the nonstandard strategies that students might use	
Class negotiates records of strategies and initiates the practice of naming	Students may initially name strategies in ways tied to specific examples
Instructor and students negotiate differences and relative efficacy of strategies	Students name strategies according to essential characteristics
Instructor and students make strategies objects of discourse	Students develop a repertoire of number sensible-strategies

Table 3. Route to Goal 3 (Table Adapted from Nickerson & Whitacre, 2010)

Instructional Activities	**Envisioned Learning Route**
Instructor anticipates powerful models for reasoning	
Students are asked to explain why the strategies make sense	Students begin to justify strategies meaningfully
Instructor and students negotiate use of models-of shared strategies	Students use models-of strategies to make sense of nonstandard strategies
Students use familiar models-of as models-for reasoning	Students develop the ability to reason with models

formal data collection has not taken place. In this study, the local instruction theory guided instructional planning during the semester. Data were collected to afford detailed analyses of classroom events, as well the learning of particular students. Thus, the present study is not focused on instructional design, as was the previous one. Rather, the focus is on the genetic analysis of number sense development.

Having articulated a perspective regarding what it means to learn, and having designed an instructional approach with specific learning goals in mind, the next problem is that of analyzing the activity that takes place.

Analysis of Collective Activity and Individual Learning

This section presents a review of literature involving the analysis of collective activity. Genetic analysis is described in greater detail, as is the interpretive framework of the emergent perspective. Special attention is given to the notions of emergent models and classroom mathematical practices. Finally, two studies are discussed that attempted to account for the relationship between student learning and collective activity.

Genetic Analysis

Saxe and colleagues have worked on the problem of analyzing the development of over time of meanings in relation to collective practices (Saxe, 1982, 2002; Saxe & Esmonde, 2005). These investigations involve three levels of analysis—ontogenesis, microgenesis, and sociogenesis. In order to elaborate on these three strands and to provide a paradigmatic case of genetic analysis, this subsections describes in some detail a set of studies conducted by Saxe and colleagues (Saxe, 1982; Saxe & Esmonde, 2005) concerning the mathematical practices of an isolated community in Papua New Guinea.

Saxe (1982) reported on the computational strategies of Oksapmin adults from remote villages in Papua New Guinea. The Oksapmin people used a numeration system in which parts of the body (and/or the words for those body parts) represented the numbers 1 to 27. Traditionally, these communities had no system of arithmetic. The numeration system had been used exclusively for counting. However, in recent years, currency had made its way into the community. Saxe examined the computational strategies that Oksapmin adults used for addition and subtraction tasks. He also compared the strategies used by different groups of adults, grouped according to the extent of their experience with exchanges involving currency (Saxe, 1982).

Saxe found that many of the interview subjects developed computational strategies analogous to those used by Western children. He also found that those adults who had more experience dealing with currency, particularly those who had opened trade stores, used more advanced computational strategies than other adults. These strategies, too, were analogous to more advanced mental computation strategies used by (older) Western children and seemed to reflect a profound conceptual advance. For example, adults with less experience with currency performed addition by counting the first set of objects and then continuing the count for the second set. When physical objects were present to be counted, this procedure tended to be effective. However, without physical objects available, the subjects often arrived at incorrect answers due to difficulty in keeping track of when they should stop counting. The more advanced subjects were able to dissociate number words from body parts, as in counting on from one of the two given numbers. This process required referring to one body part by the name of another body part (Saxe, 1982).

In sum, Saxe (1982) found that the Oksapmin people used a numeration system very different from the base-ten system used in much of the world. Their capacity for arithmetic was quite limited. These limitations reflected a culture in which arithmetic had not been a traditional practice and a numeration system that had traditionally been used only for counting. At the same time, there was evidence that the advent of currency had created a need for arithmetic and had led certain members of the community to adapt the numeration system in novel ways in order to serve these purposes.

When Saxe and colleagues returned to Papua New Guinea in 2001, they found that the communities' mathematical practices had changed. This change was signaled by a conspicuous difference in people's use of a particular word: *fu*. As Saxe and Esmonde (2005) explain, two decades before, "*fu* with fists raised meant a completion of all 27 body parts in a count" (p. 172). Now, the meaning of the word had changed so that "when preceded by a body part name, *fu* meant to double the value of the body part" (p. 172).

This development was surprising. Saxe (1982) had reported on a community in which arithmetic had not been a traditional practice and individuals' facility with addition and subtraction tasks had been limited. At that time, arithmetic operations such as doubling were unheard of (Saxe & Esmonde, 2005). The change in the meaning of the word *fu* suggested that the mathematical practices of the Oksapmin community had changed dramatically in the past two decades.

Intrigued by their discovery, Saxe and colleagues sought to investigate the recent history of Oksapmin mathematics and, in particular, the process by which the meaning of the word *fu* had transformed. In examining the recent history of the Oksapmin people, the researchers partitioned the period 1940-2001 into three distinct subperiods of interest. This partitioning emphasized a transition from a subsistence economy to a cash economy. They examined records from each of these periods with a particular focus on the role of currency and other changes in the nature of the economy. Against the background of this general, historical record, the researchers sought to investigate the development of the usage of *fu*.

The investigation of the genetic roots of *fu* involved three major challenges: establishing (1) continuity of form, (2) transitional functions of the form, and (3) the organization of collective practices that support shifting relations between form and function (Saxe & Esmonde, 2005, pp. 188-189). That is, first, the researchers needed to establish that the word *fu* as used in the present day to indicate doubling was, in a genetic sense, the same word *fu* that had once been used to indicate completion of a count, as opposed to a sound-alike word of a different origin. Second, the researchers needed to identify distinct, transitional usages of the word *fu* that might represent landmarks in developmental process. Finally, the researchers needed to identify cultural phenomena that would have supported the move from one of these landmarks to the next – that is, the transition from one meaning or function of *fu* to the next – in order to construct a viable account of the developmental process.

Saxe and Esmonde (2005) were able to address the above challenges and eventually to arrive at an evidence-based account of the development of *fu*. Evidence came from the coordination of the general economic/mathematical history of the Oksapmin with data from several sets of interviews involving stratified samples of the Oksapmin people. Comparisons of samples of individuals from different generations, as well as from more and less Westernized strata, revealed differences in the meaning(s) of *fu* across these groups. For example, for the elders, *fu* tended to denote the number 27 (associated with the left pinky finger) and/or to have the general qualitative meaning of "plenty" or "complete." For younger adults, by contrast, *fu* tended to mean 20, the number associated with the left elbow. How could the same word denote different numbers for these subgroups of the population? The researchers found that the influence of Australian currency in the economic history of the Oksapmin seemed to have introduced aspects of a base-ten numeration system, which became integrated with the native Oksapmin numeration system. In base ten, decade numbers represent complete sets of tens, and this may have led to the use of *fu* to denote the number twenty (Saxe & Esmonde, 2005). From there, *fu* may have taken on a more general meaning of two of something, and this meaning in turn may have led to the present-day doubling meaning.

The above is an extremely abbreviated version of the story of the development of *fu*, as described by Saxe and Esmonde (2005). The authors detail progressive shifts in the usage of the term by painstakingly coordinating interview data and historical records. They create and revise hypotheses, performing additional interview studies in order to test or inform subsequent hypotheses. They arrive ultimately at an account of the development of *fu* that integrates microgenetic sociogenetic, and ontogenetic processes. On the level of microgensis, words such as *fu* are used in various cultural contexts on a day-to-day basis in interactions between Oksapmin people. On the sociogenetic level, patterns of usage of *fu* in particular types of social interactions, such as the exchange of money for goods, become collective practices. In terms of ontogenesis, differences in the meaning of *fu* across generations point to different developmental trajectories, whereby the initial meaning that the word has for a set of individuals inevitably informs their later interpretations or formulations of its use.

Although the story of the shift in the meaning of *fu* from a complete count to doubling can be extracted from the history, that story is not one of community-wide shifts from one homogeneous meaning to another. Rather, the authors found diversity in the meaning of the word throughout its history. In fact, this diversity seems to have been a powerful catalyst for the transitions in its usage, as individuals with different meanings for the word interacted in the context of everyday cultural practices, especially trade. Saxe and Esmonde (2005) note that no one person that they interviewed seemed to have access to the complete range of meanings of *fu*, let alone to the process by which its meanings had transformed over time. Rather, in going about their daily activities with their own meanings and intentions, each had contributed and continued to contribute to the simultaneous reproduction and transformation of cultural practices involving occasions for the use of *fu*, and these had charted the course of the development of its meanings.

The work of Saxe and colleagues provides a paradigmatic example of genetic analysis. Saxe and Esmonde (2005) were able to account for the evolution of particular mathematical practices in the Oksapmin community. This example informs my thinking about the present study. At the same time, as discussed earlier, a classroom community is unique in many respects. As such, its analysis merits somewhat different methods and a particular interpretive framework. We now look to analyses of learning in mathematics classrooms.

Individual and Social Lenses

The present study is informed by the perspective that learning is inherently situated in the social contexts in which it occurs (Cobb & Bowers, 1999). Furthermore, learning takes the form of increasing participation in a community of practice (Lave & Wenger, 1991). This perspective does not prohibit analyses of individual learning. On the contrary, the meanings that individual students develop relate to their ways of participating in the practices of the community (Bowers et al., 1999).

Cobb and Yackel (1996) offer both a perspective on learning and an interpretive framework for analyzing students' learning in classroom settings. From the *emergent perspective*, learning is a complex process involving a reflexive relationship between individual and collective activity. The emergent perspective represents neither a purely social perspective nor an individual, psychological perspective, but rather an attempt to coordinate the two. This perspective is illustrated by the interpretive framework, which is depicted in Table 4.

Table 4. Cobb and Yackel's (1996) interpretive framework (p. 177)

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

From the social perspective, the authors distinguish *classroom social norms, sociomathematical norms,* and *classroom mathematical practices*. Social norms for whole-class discussion include explaining and justifying solutions and indicating agreement or disagreement. Social norms are non-mathematical in the sense that the social norm that solutions should be justified does not entail specific criteria for mathematical justifications. Socio-mathematical norms, by contrast, refer to the normative aspects of whole-class discussions that are specific to students' mathematical activity. Examples include "what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation and justification" (Cobb, 2000, p. 323). Classroom mathematical practices refer to the accepted means of justification and explanation that develop within a classroom community (Cobb & Yackel, 1996).

Cobb and Yackel (1996) hypothesized correspondences between the social constructs in the left column of Table 4 and the psychological constructs in the right column. Each construct in the right column is posited to be the individual, psychological correlate of the construct in the left cell of the same row. Thus, students' beliefs about their role, others' roles, and the general nature of mathematical activity in school are viewed as the psychological correlate of classroom social norms. Beliefs and values that are specifically mathematical are regarded as the psychological correlate of sociomathematical norms. Likewise, individual students' mathematical conceptions and activity are seen as the psychological correlates of classroom mathematical practices.[9]

Analysis of Collective Activity

This subsection concerns analysis of collective activity. This includes the constructs of as-if shared ideas and classroom mathematical practices, as well as two approaches to the analysis of these. Rasmussen and Stephan (2008) provide a useful definition of *collective activity*:

> We define collective activity as the normative ways of reasoning of a classroom community. We stress that collective activity is a social phenomenon in which mathematical or scientific ideas become established in the classroom community through patterns of interaction. (p. 195)

The authors make an analogy to the behavior of a couple. A couple, such as a married couple, can be characterized as a unit. That is, the couple has certain characteristics that have to do with how the two individuals interact. The description of a couple in terms of the way that the pair interacts is distinct from descriptions of the individuals themselves or of their commonalities. It is a characterization of collective activity.

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[9] Rasmussen and Stephan (2008) redefined classroom mathematical practices. Their definition, to which I subscribe, influences my thinking about this hypothesized correspondence. I would say that individual students' conceptions correspond to as-if shared ideas.

Argumentation in analyses of classroom mathematical practices. Cobb and Yackel's (1996) interpretive framework informs the present study. In particular, classroom mathematical practices and their hypothesized psychological correlate, students' conceptions and activity, are of interest. This subsection presents a review of literature concerning classroom mathematical practices. To reiterate, the construct of *classroom mathematical practices* refers to the accepted means of justification and explanation that develop within a classroom community (Cobb & Yackel, 1996).

It is difficult to describe collective activity. In informal settings, people might freely attribute adjectives to couples or classroom communities. However, analyzing collective activity systematically presents challenges. In attributing a normative way of reasoning to a classroom community, a researcher is not making a claim about any one particular individual nor about all individuals that make up that community. Neither is the researcher making a claim about a certain proportion of the individuals that make up the community. Bowers and Nickerson (2001) address this issue in their discussion of a *modal view* of classroom research. A researcher taking a modal view might attribute ways of a reasoning to a community based on a quantitative criterion, say 80% of the class: If at least 80% of students behave in a particular way, then that way of reasoning is attributed to the class as a whole. The authors contrast this modal view with their own:

> we can distinguish between a process that identifies how the mode of the class is thinking or acting at any given time with our approach that involves creating a chronology of the interrelations among students' views as they progress over time. In fact, those students who do not follow the developmental trajectory of the mode often add the spice and initiative needed to propel the evolution of new practices. (Bowers & Nickerson, 2001, p. 3)

Thus, these authors make the point that a chronological characterization of the mode may not enable researchers to account for shifts in classroom mathematical practices. To second this point, Stephan, Cobb, and Gravemeijer (2003) highlighted the distinct ways in which particular students participated in emerging mathematical practices. Ultimately, one student's interpretation, or way of participating in a practice, may become accepted, while another student's interpretation is rejected. Nonetheless, each contributes to the negotiation of the classroom math practice.

Stephan et al. (2003) used Toulmin's (1969) model of argumentation in their analysis of mathematical practices in a first-grade classroom. This model frames the anatomy of an argument as consisting of *claim*, *evidence*, *warrant*, and *backing*[10]. In a given argument, the claim is the assertion being made, or the conclusion that the person is drawing. The person provides evidence, or data, in support of that claim. The warrant of the argument explains how the evidence supports the claim. In the event that the warrant is questioned, backing is provided, which justifies the warrant. Not all of these elements need to be present in an argument. Claim, evidence, and warrant constitute the *core* of an argument, and are the three necessary components. Stephan et al. (2003) used Toulmin's model to analyze students' arguments and to provide operational criteria for the establishment of mathematical practices:

> As we see it, mathematical practices emerge during argumentations in which the participants provide new backings that shift the mathematical interpretations of the community to a new level. When backings for a particular interpretation drop out of the discussions or when alternative

[10] Toulmin (1969) also discussed *rebuttal* and *qualification* as elements of an argument, but these are not relevant to the present discussion since they do not tend to be used in analyses of classroom mathematical practices (e.g., Stephan & Rasmussen, 2002).

> backings are contributed by a student and rejected by the community, we say that a mathematical practice has become established. (p. 73)

By analyzing collective mathematical argumentation, Stephan et al. (2003) describe the process of negotiation by which each of several mathematical practices developed over the course of a teaching experiment.

A key construct in the analysis of mathematical practices is the notion of *taken-as-shared*. According to Bowers et al. (1999), "taken-as-shared understandings refer to the collective knowing of the classroom community" (p. 44). We might think of these as the understandings behind, or connected to, classroom mathematical practices. Stephan and Rasmussen (2002) analyzed classroom mathematical practices in a different equations class. They articulated their operational definition for taken-as-shared as follows:

> We contend that mathematical ideas become taken-as-shared when either (1) the backings and/or warrants for an argumentation no longer appear in students' explanations and therefore the mathematical idea expressed in the core of the argument stands as self-evident, or (2) any of the four parts of an argument (data, warrant, claim, backing) shift position (i.e., function) within subsequent arguments and are unchallenged. For example, when students used a previously justified claim as unchallenged justification (data, warrant or backing) for future arguments, we concluded that the mathematical idea expressed in the claim had become taken-as-shared. When either of these instances occurred, no member of the community rejected the argumentation, and/or if the argumentation was rejected and the student's rejection was rejected, we documented that the mathematical idea had become established. (p. 462)

Stephan and Rasmussen (2002) documented the emergence and establishment of classroom mathematical practices in relation to particular mathematical ideas. They characterized classroom math practices in terms of sets of related ideas expressed and use in the context of certain mathematical activities.

To clarify, for Stephan and Rasmussen (2002), a particular classroom math practice may involve several mathematical ideas. For example, in their analysis of a differential equations class, the authors identified the classroom mathematical practice of *creating and structuring a slope field as it relates to predicting*. This practice involved three distinct mathematical ideas concerning slopes and slope fields. One of these was *invariance of slopes across time*. This idea was introduced at one point during the semester and, as the authors document, later became taken-as-shared. Such ideas emerged as students participated in the activity of creating and structuring slope fields, and the ideas became taken-as-shared as the classroom mathematical practice became established.

In the analysis of classroom math practices described by Stephan et al. (2003), the authors found that "each practice grew out of practices previously established by the classroom community" (p. 100). That is, classroom mathematical practices developed sequentially, each building upon the previous. By contrast, Stephan and Rasmussen (2002) found that "[t]he emergence of classroom mathematical practices can be non-sequential in both time and structure" (p. 486). Non-sequential in time means that classroom mathematical practices can emerge simultaneously. One classroom mathematical practice need not strictly succeed the previous practice. The emergence of classroom mathematical practices being non-sequential in structure means that taken-as-shared mathematical ideas can contribute to more than one classroom mathematical practice. As the authors put it,

> taken-as-shared ideas do not have to always be viewed as an element of only one classroom mathematical practice—they may contribute to the emergence of other practices and form a network of practices instead of a

> sequential chain of practices with distinct taken-as-shared ideas. (Stephan & Rasmussen, 2002, p. 487-488)

A final note concerning argumentation in analyses of classroom math practices: Although the reports cited have focused on the construct of classroom mathematical practices, social norms are nonetheless relevant. The inferences that researchers make in documenting the development of classroom math practices are dependent on characteristics of the classroom culture. In a class in which it is the norm for students to justify their statements mathematically, an inference can be made when a particular statement ceases to require justification; as in the examples cited above, the researchers could claim that a mathematical idea had become taken-as-shared. However, in a class in which mathematical justification occurs rarely or sporadically, such an inference would be unwarranted. Stephan and Rasmussen (2002) note that *in classrooms in which students engage in mathematical argumentation*, "the construct of a classroom mathematical practice is a way to document and characterize the learning of the classroom community" (p. 489). In classrooms with different norms, this may not be the case.[11]

The transition from model-of to model-for. Although originally a design heuristic in RME, the model-of/model-for construct has also been used to describe collective activity after the fact. Researchers have used it describe shifts in activity that occurred as a classroom community transitioned from one mathematical practice to another. This subsection discusses how the model-of/model-for construct has been used

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[11] Izsák et al. (2008) studied students' learning in a rather traditional, teacher-centered classroom. For their purposes, they adapted the definition of classroom mathematical practice to account for the different set of norms that existed in the class.

to analyze collective activity. The particular meaning of the term *model* in these contexts is also addressed.

Bowers et al. (1999) describe the results of a teaching experiment concerning place value in a third-grade classroom. These results are presented in the form of five mathematical practices, which emerged in succession. The development of these practices is framed in terms of shifts in collective activity from model-of to model-for. These authors use particular models as their way of characterizing a mathematical practice. For example, they describe the second mathematical practice as follows:

> As [the students] participated in Mathematical Practice 2, they drew different arrangements that contained the same number of candies. These drawings constituted models *of* their informal activity. Although this modeling activity varied from student to student (i.e., separated arrangements, ordered arrangements, and charts containing numerals), it involved generating arrangements that the students could imagine packing up to the canonical form. (Bowers et al., 1999, p. 55)

Thus, Mathematical Practice 2 is characterized on the basis of a common behavior, whereby drawings served as models-of students' activity. The authors then use the shift from model-of to model-for as the defining characteristic of the transition to a new mathematical practice:

> The transition from Mathematical Practice 2 to Mathematical Practice 3 involved a shift in the students' ways of creating different arrangements from starting from scratch each time to starting by unpacking previous arrangements. In this transition, models *of* packing and unpacking became models *for* quantitative transformations. […] we believe that this indicates a critical reorganization in the classroom community's collective activity. […] although the instructional task of creating different arrangements was the same, the taken-as-shared way of creating and justifying solutions differed. (Bowers et al., 1999, p. 56)

Thus, the shift from model-of to model-for is the defining characteristic of the transition from Mathematical Practice 2 to Mathematical Practice 3. Students' modeling behavior effectively defines these classroom mathematical practices.

Further discussion is called for concerning the meaning of the term *model* in the transition from model-of to model-for. Bowers et al. (1999) articulate an important distinction regarding the models that they discuss:

> it is important to stress that the microworlds themselves did not first serve as models of and then become models for. Instead, in the analysis we have presented, it was the students' actions on boxes, rolls, and pieces as they used the microworlds that functioned as models. This distinction between the microworlds per se and the students' activity with them differentiates the approach we have taken to instructional design from that involving what is sometimes called a *modeling point of view*. In this latter viewpoint, models are viewed as originating in initial, starting-point situations rather than in students' ways of acting and reasoning with them. (p. 61)

Thus, in Bowers et al.'s (1999) terms, tools such as a computer microworld or the empty number line are not themselves models. Rather, models are defined in terms of sets of actions.

In a recent paper, Zandieh and Rasmussen (2010) address the issue of how to define the *model* construct:

> we define models as student-generated ways of organizing their activity with observable and mental tools. By observable tools, we mean things in their environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. By mental tools we mean the ways in which students think and reason as they solve problems – their mental organizing activity. We make no sharp distinction between the diversity of student reasoning and the things in their environment that afford and constrain their reasoning. (p. 58)

The above definition further clarifies the use of the term *model* in the transition from model-of to model-for. Whereas, Bowers et al. (1999) defined models as sets of actions, Zandieh and Rasmussen's (2010) definition highlights the point that modeling is a meaning-making activity. This definition locates models not in the actions themselves but in students' "ways of organizing their activity," which include those actions. Furthermore, these actions involve tools that may be observable and/or mental.

The above definitions of Bowers et al. (1999) and Zandieh and Rasmussen (2010) are not mutually exclusive. On the contrary, the latter adds clarity to the former. The above perspective (which is to say the two definitions taken together) concerning models informs this study. Thus, throughout this document, language such as *an* "empty number line *model*," does not refer to *the* empty number line, which is a familiar (observable) mathematical tool. Rather, it refers to student-generated ways of organizing their activity with that tool. Note that this definition allows for multiple, distinct empty number line models.

To make an analogy, a hammer is a tool. Combing one's hair with a hammer would be an unconventional way of using that tool. There are various more appropriate ways of using hammers, such as for pounding nails into wood and for prying nails out of wood. The set of actions that a person performs with a hammer reflect a certain way of organizing his or her activity with that tool. That is, this set of actions reflects what we might call a *hammer model*. This model would include recognition of the hammer as a tool that has certain purposes, such as pounding nails into wood. It would also include details of how the tool is to be used. For example, when using it for pounding, one should be careful not to pound one's fingers. Appropriate finger positioning and care, then, would be part of a typical hammer model.

The shift from model-of to model-for has been described as issuing in a “new mathematical reality” (Gravemeijer, 1999; Zandieh & Rasmussen, 2010). The new model opens up new possibilities for action. This notion fits well with the environment metaphor. When one perceives new affordances, as in new possibilities for making use of resources in the environment, the environment itself has effectively been transformed.

Accounting for gesturing in analyses of classroom mathematical practices. Beyond analyses using Toulmin’s model and those concerning models of and for, gesturing has also been considered in analyzing the evolution of classroom math practices. Rasmussen, Stephan, and Allen (2004) incorporated the analysis of gesturing into a previous analysis of classroom math practices in a differential equations course (Stephan & Rasmussen, 2002). This approach made a novel contribution to the study of classroom math practices by analyzing gesturing, and it made a contribution to the literature on gesture by examining it as a collective phenomenon. In taking their approach to the analysis of classroom math practices, the authors use the term *gesturing*, as opposed to *gesture*, in order to emphasize the fact that gesturing is a human activity.

Rasmussen et al.’s (2004) analysis led to the identification of gesturing/ argumentation dyads, which are “instance[s] of classroom talk in which a mathematical idea is expressed along with gesturing” (p. 311). The authors found that a gesturing/argumentation dyad can emerge during the establishment of one or more taken-as-shared ideas within a given practice. Furthermore, a gesturing/argumentation dyad could shift function to support the establishment of ideas across other practices. The authors illustrate how the use of a particular way of gesturing, which they called “SLOPE SHIFTING,” evolved. This way of gesturing first appeared in a student group, then was introduced into whole-class discussion and was adopted by the instructor. It was then was used by other students, eventually shifting its argumentative role from claim to data in the same way that mathematical ideas shift argumentative roles (Rasmussen et al., 2004).

Rasmussen et al. (2004) analyzed gesturing retrospectively and then integrated the results of the analysis with the results of a previous analysis of classroom math practices. The authors suggested that attention to gesturing could be incorporated into initial analyses of classroom math practices.

Summary of analyses of classroom mathematical practices. In summary, researchers have used Toulmin's (1969) model of argumentation in analyses of collective activity. Through the use of this model, researchers have operationalized the constructs of taken-as-shared and classroom mathematical practice. This method of analysis takes justification as the key criterion in establishing that a mathematical idea is taken-as-shared status. Multiple taken-as-shared ideas may contribute to a single classroom math practice or to more than one classroom math practice. Classroom math practices may emerge sequentially or non-sequentially in both time and structure. In addition to taken-as-shared mathematical ideas, the notion of model-of and model-for has framed analyses of the emergence of classroom mathematical practices. More recently, gesturing has also been integrated into these analyses.

Coordinating Individual and Social Lenses

Cobb and Yackel (1996) suggested the coordination of the individual and social lenses of the interpretive framework and hypothesized correspondences between constructs. However, they did not articulate a methodology for coordinating analyses of individual students' learning with analyses of collective activity. Exactly how such coordination can be accomplished is an open question in educational research (and not a question that the present study is meant to answer). A few recent reports offer ways of approaching the task (Izsák, Tillema, & Tunç-Pekkan, 2008; Lobato, Ellis, & Muñoz, 2003). The study Izsák et al. (2008), which was briefly described earlier, provides an example.

Recall that Izsák et al. (2008) present an account of a student named Sonya's interpretations of number-line representations of fractions. Sonya's teacher used a routine of making tick marks to draw number lines, consistently proceeding from left to right by eyeballing the size of the intervals. In those instances when her estimates were off, the teacher would erase the 1 and relocate it. The teacher also talked about numbers such as 1 as "benchmarks" and used this language in the context of estimating locations of numbers on number lines.

Sonya interpreted fractions as pairs of whole numbers denoting "n out of m." For example, she used two different-sized fraction strips to represent 1/2 (one out of two pieces) and 1/3 (one out of three pieces). She also came to interpret "benchmarks" such as 1 as "estimates," rather than exact values or locations. This may have been due to an unintended interpretation of the language the teacher had used. In an interview, Sonya labeled 6/6 and 1 as distinct locations on a number-line drawing and did not find this unproblematic. The authors argue that Sonya's n-out-of-m conception and her notion of benchmarks as estimates, together with the teacher's routine of fudging the location of 1, enabled Sonya to treat 1 as though its location was not fixed (Izsák et al., 2008).

Thus, the authors' account coordinates the conceptions that Sonya brought with her to instruction, together with routine and endorsed ways of using tools and language in classroom activity, and Sonya's subsequent interpretations of mathematical ideas and tool-use related to that classroom activity. This coordinated analysis enables the authors to describe how Sonya's conceptions may have interacted with the classroom activities, thus producing a plausible account of the effects of the classroom instruction on Sonya's understanding of fractions and number-line representations. This account amounts to a process explanation (Maxwell, 2005) for how Sonya came to behave mathematically in the way that she did (e.g., locating 6/6 and 1 separately). It bears similarities to Saxe and Esmonde's (2005) account of the evolution of the meaning of *fu* in the Oksapmin community, except that Saxe and Esmonde sought an explanation for a community-level phenomenon, rather than the behavior of a particular individual.

Summary of Analysis of Collective Activity and Individual Learning

Genetic analysis and the interpretive framework of the emergent perspective offer useful and compatible approaches for the study of collective activity and individual learning. Genetic analysis consists of microgenetic, sociogenetic, and ontogenetic strands, which together contribute to an account of development. The interpretive framework highlights classroom mathematical practices and individual students' conceptions and activity and posits a correspondence between these. Classroom math practices describe sets of taken-as-shared ideas related to particular mathematical activities. These have been analyzed in terms of argumentation and models of and for. The work of Izsák et al. (2008) provides an example of the coordination of analyses of classroom activity and individual students' conceptions and activity.

Summary of Literature Review

This study concerns the activity and learning of prospective elementary teachers in a mathematics content course that is designed to support their development of number sense. This phenomenon is viewed from a situated perspective. From this perspective, the nature of the classroom culture and activity profoundly influence students' learning. The normative ways of reasoning that evolve in the classroom community characterize important aspects of the collective activity. These normative ways of reasoning can be described in terms of classroom mathematical practices that emerge and become established. Individual students participate in the classroom math practices in distinct ways, which relate to their mathematical conceptions and to their interpretations of classroom activity.

Prospective elementary teachers come to mathematics content courses with extensive familiarity with the numerical representations and operations involved in elementary mathematics. At the same time, their understanding of the domain tends to be bound to the standard algorithms, so that they behave inflexibly. Rather than perceive a bounty of affordances based on the properties of numbers and operations, these students tend to be constrained by the superficial characteristics of the numerals and algorithms themselves. The construct of number sense grasps at the contrast between procedural competence and flexibility that is grounded in the perception of a mathematical environment with features analogous to the properties of numbers and operations.

Whole-number sense manifests in flexible mental computation, which is distinguished by the tendency to make an informed, purposeful choice of strategy. The strategies used by skilled, flexible mental calculators depend on the particular numbers and operation at hand, so that they tend not to resemble mental analogues of the standard algorithms.

Useful strategies and tools are employed in flexible computational reasoning. However, direct instruction in the use of these cannot equip students to navigate a mathematical environment in the manner of an experienced, knowledgeable resident. One's own road map of the environment develops through exploration and experimentation in the service of authentic, goal-oriented activity. This learning process is facilitated by communication with peers and more knowledgeable others regarding both common and disparate experiences.

Accounting for number sense development involves microgenetic, sociogenetic, and ontogenetic strands of analysis. Microgenesis concerns instances of mathematical activity. Sociogenesis concerns the evolution over time of collective activity. Ontogenesis concerns the evolution over time of individual students' conceptions as students engage in the practices of the classroom community and reflect individually on these activities.

Chapter 3: Methods

In this chapter, I describe the design of the study. I discuss the data collection and analytic methods, as well as the relevance of these to the research questions. I also address issues of validity and reliability.

Introduction

Saxe and colleagues found a striking contrast in the use of the word *fu* by the Oksapmin people from field studies done in 1978 and 1980 versus a return trip in 2001. Like detectives, they then sought to reconstruct the story of the developmental process by which that changed had occurred (Saxe & Esmonde, 2005). This investigation required the coordination of interview data, cultural artifacts, and various historical records. The evidence pointed researchers to the increasingly central role of currency in the community: Social interactions involving monetary exchange had necessitated new arithmetic practices. This fact helped researchers to knit together an account of a gradual and complex developmental process.

Evidence from a previous study with prospective elementary teachers enrolled in a specially designed mathematics content course suggested that their number sense improved significantly in the span of a single semester. There was a statistically significant increase in students' scores on an established quantitative measure of number sense (Whitacre, 2006). Furthermore, interview participants shifted from inflexible to flexible in mental computation, and their preferred strategies shifted from standard to nonstandard (Whitacre, 2007; Whitacre & Nickerson, 2006). Each of these three changes indicated improved number sense (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994; Yang, 2003). Having identified a setting in which prospective elementary teachers' number sense improved, I set out to investigate number sense development.

In contrast to Saxe and Esmonde (2005), my interest did not lie in the particular group of people that was studied previously. On the contrary, I was interested in the phenomenon of prospective teachers' number sense development. As such, I studied number sense development in the mathematics content course, as taught by an experienced instructor in Fall Semester 2010. The instructional approach was guided by the local instruction theory that Nickerson and I developed in our previous work (Nickerson & Whitacre, 2010). As such, I expected similar improvements in students' number sense. The study was designed so that the data collected and the methods of analysis would afford genetic analysis of number sense development.

Overview of Research Design

In this chapter, I describe the overall design of the study. In the sections that follow, I elaborate on the details of the design.

Setting and Participants

The setting for this study was a naturalistic classroom: one section of a first mathematics content course for prospective elementary teachers at a large urban university in the southwestern United States. The instructor for the course was a mathematics educator. She was an experienced instructor of mathematics courses for prospective teachers. All 39 students enrolled in the course were invited to participate in the study. The vast majority of the students were female Liberal Studies majors in their freshman or sophomore year. A total of 34 students completed survey instruments both pre and post. Seven of the students participated in interviews. The interview participants were all female undergraduates.

Research Questions

Before discussing the data corpus and methods, I restate my research questions: As prospective elementary teachers participate in a mathematics content course designed to support their development of number sense,

1. How does the number sense of individuals evolve?
2. What ideas come to function as if shared? What classroom mathematical practices emerge and become established?

Cobb and Yackel (1996) hypothesized correspondences between the social constructs and the psychological constructs of the interpretive framework. Given Rasmussen and Stephan's (2008) redefinition of classroom mathematical practices as consisting of sets of as-if shared ideas, I view individual students' mathematical conceptions as the psychological correlates of as-if shared ideas. The focus of the analysis was number sense development on these two levels, i.e., as both an ontogenetic and a sociogenetic phenomenon.

This study consisted of several phases of analysis: Assessing Change, Microgenetic Analysis, Sociogenetic Analysis, and Ontogenetic Analysis. I assessed change in the prospective teachers' number sense using established pre/post measures, including interviews with seven of them. Pre/post results informed the selection of two case study students from amongst the interview participants. The microgenetic analysis involved classroom data and individual data. The products of this analysis became the data that was analyzed in the sociogenetic and ontogenetic phases. The products of the sociogenetic phase informed the ontogenetic analysis in the sense of pointing to ideas of interest. The assessment of change also informed the ontogenetic analysis in that the pre/post snapshots were conceived as landmark points in developmental trajectories.

This study fit within the tradition of number sense research by employing established quantitative and qualitative methods for the analysis of number sense. It also extended previous research by analyzing the development of number sense through the lens of genetic analysis.

Data Corpus and Data Reduction

The following data were collected for the purposes of the present study:

- Audio/video recordings of whole-class discussions from 14 class sessions
- Still images of selected class work
- Field notes from all class sessions
- Written notes from meetings with the instructor
- Audio/video recordings and field notes from multiple interviews 7 of the students
- Copies of students' homework assignments and exam items related to mental computation
- Students' responses to a survey instrument, the Number Sense Rating Scale (Hsu et al., 2001), which was administered as a pre/post assessment of students' number sense
- Students' responses to the Student Preference Survey (McIntosh, Bana, & Farrell, 1995), which was modified to match the mental computation tasks used in interviews

Below, I describe each data source.

Classroom data. The course can be described as consisting of two major parts. The first part focused on the whole-number domain, and the second on rational numbers. This study analyzed students' number sense development with a whole-number focus. Within the whole-number portion of the course, the following major topics were addressed: quantitative reasoning, place value, meanings for operations, ways of performing operations, and children's mathematical thinking. Video recording began on Day 2. A total of 14 class sessions were videorecorded for the purposes of this study.

Interview data and written work. The Number Sense Rating Scale (NSRS) and Student Preference Survey (SPS) were administered as pre/post assessments of students' number sense. Seven students agreed to participate in whole-number interviews. A total of three whole-number interviews were conducted with each student. The first was a pre-assessment of the participants' number sense, which focused on mental computation. The second was a post-assessment, which used the same tasks as the first. The third interview served to investigate participants' interpretations of particular sets of ideas from class, as well as to invite participants to reflect on their experiences in the class. The data from these interviews was analyzed only for the case study students. The case study students were selected from amongst the interview participants after all data collection and some initial analyses had been completed. Copies of students' written work were collected, but only the case study students' work was analyzed.

Preparation

This study was informed by previous research, notably research in a similar class for my Master's thesis. In the previous study, I analyzed mental computation interviews similar to those used in the present study. The analysis focused on shifts in flexibility and distribution of strategies. I also prepared for the present study by conducting pilot interviews. I interviewed six preservice elementary teachers[12] and made revisions to the mental-computation interview protocol based on the results of the pilot interviews.

Also in preparation for the study, the instructor and I met to discuss instructional planning. The local instruction theory guided the instructional approach broadly. Thoughtful lesson planning in response to students' thinking was, of course, still required. The instructor and I identified opportunities in the curriculum for authentic activity in service of our instructional goals. We engaged in thought experiments, informed by prior teaching, anticipating how the learning might progress, and discussing various contingencies. We met about the planning of each lesson. We also met after class to reflect on lessons and discuss possible next steps. This careful planning and reflection is essential to the design research process (Gravemeijer, 1994).

Data Collection

In this section, I describe the data that was collected and the specific methods of its collection. I also relate each data source to the research question(s) that it addresses. I present these in more or less chronological order of data collection.

On the first day of class, I recruited students to participate in the study. Three levels of participation were offered: (1) consent for surveys and copies of other written work to be analyzed for research purposes, (2) consent to be videotaped while in class, and (3) participation in interviews, with consent that interview data can be analyzed for research purposes.

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[12] For the pilot interviews, I used individuals who had completed their undergraduate mathematics courses and were entering or planning to enter a credential program.

Number Sense Rating Scale

I administered the NSRS to all students in attendance on the second day of class and on the second-to-last day of class. This was used as a pre/post measure of students' number sense (Hsu et al., 2001). The NSRS is a multiple-choice test, which was originally designed to assess the number sense of middle school students. Participants were instructed to answer the test questions mentally. The NSRS is not intended to be a test of procedural knowledge, but of number sense. Disallowing written work is meant to force students to use approaches other than the standard written algorithms, and thus to tap their number sense. One student asked whether using her fingers counted as solving problems mentally. I answered that using fingers was okay, although there probably would not be much opportunity for it. Students were allowed approximately 20 minutes to complete the NSRS.

The results of the posttest, compared to those of the pretest, provided one way of analyzing students' learning over the course of the semester. Gain scores on the NSRS are one way of assessing change in students' number sense (Yang, 2003) and, therefore, contribute to answering Research Question 1.

Student Preference Survey

A second pre/post survey was used as a supplementary data source. The Student Preference Survey (McIntosh et al., 1995) asks students to indicate whether or not they would perform a computation mentally, as opposed to with written work or calculator. I modified the set of computations on the survey to match those that participants were asked to perform in the mental-computation interviews. For purposes that extend beyond the present study, fraction comparison tasks were also added to the survey to assess students' preferences regarding performing those mentally. The modified instrument appears in Appendix 2.

Interviews

I conducted three whole-number interviews with each of the seven interview participants. The first interviews were completed within the first three weeks of the semester. The second interviews took place at mid-semester, after the end of the whole-number portion of the course. The third interviews took place approximately one week after the second interviews.

Two types of semi-structured (Rubin & Rubin, 1995) interviews were conducted. *Mental Computation Interviews* involved mental computation and other tasks designed to investigate the participants' whole-number sense. *Standout Strategies and Tools Interviews* concerned students' interpretations and understandings related to particular strategies and representational tools that were used in class. I describe each particular interview in what follows.

Mental computation interviews. The first and second interviews included sets of whole-number mental computation tasks. Each of these tasks required one of the four basic operations. The tasks used were single-step story problems in a real-world context. These were preferred for the purposes of the study over naked-number arithmetic tasks since story problems are more likely to elicit nonstandard computational strategies (Carraher et al., 1987). For each story problem, participants were asked to solve it mentally and to describe their thinking. The specific set of tasks was a revised version of those that I used in my Master's study (Whitacre, 2006). These appear in Appendix 3.

For each of the four operations, participants were given four specific problems to be solved mentally. The same story was used for each problem, with different number substituted into it. The pairs of numbers were carefully selected for the protocol on the basis of their affordances. For example, numbers for addition were selected on the basis of whether or not regrouping was required and whether or not obvious benchmarks, like 99, or possible benchmarks, like 96, were involved. The problems were ordered such that strategies specific to one problem would be least likely to influence approaches to subsequent problems. So, for example, problems involving obvious benchmarks were given last. The particular numbers used and ordering were piloted in interviews with preservice teachers.

The first and second interviews also included Numeration Tasks and Operations Tasks. The numeration tasks were designed to investigate the participants' conceptions of number composition apart from the context of mental computation. The Operations tasks were designed to investigate how the participants understood the standard written algorithms for addition, subtraction, multiplication, and division.

In the Numeration tasks, participants were asked a set of questions that related to place value and number composition. For example, participants were asked to fill in the blanks in "The number 63 is made up of __ tens and __ ones." After the first response, participants were asked for other possible answers to this question. The intention of the question was to investigate whether participants could see numbers as consisting of tens and ones in different ones. (Some answered 6 tens and 3 ones, then 5 tens and 13 ones, 4 tens and 23 ones, etc.) This understanding was of interest primarily because of its relationship to regrouping in the standard addition and subtraction algorithms. The complete protocol for the Numeration tasks appears in Appendix 4.

In the Operations tasks, participants were asked to enact the standard addition, subtraction, multiplication, and division algorithms and to justify the details of the algorithms. Addition and subtraction were paired, so that questions could be asked that compared the two algorithms, especially the meaning of carried and borrowed 1's. These were adapted from the Ones Task used by Philipp, Schapelle, Siegfried, Jacobs, and Lamb (2008) and by Thanheiser (2009, 2010). The multiplication and division tasks were similar in nature, although they did not include a comparative element. The protocol for the Operations tasks appears in Appendix 5.

Standout strategies and tools interviews. The third interviews provided opportunities to gather additional data concerning the interview participants' reasoning about whole-number operations. By the time of the second interview, several different mental computation strategies had been shared, discussed, and named by the class. I identified strategies that had been justified on the basis of what seemed to be as-if shared ideas for the classroom community. By mid-semester, I had not yet had the opportunity to do the sort of rigorous analysis required to establish whether ideas functioned as-if shared. I identified *standout strategies* as a subset of the nonstandard strategies that had been shared and discussed in class and that were of particular interest in terms of students' learning (especially strategies that prospective elementary teachers do not typically use and that require new understandings).

There were also particular tools that stood out by mid-semester as having been involved in students' sense making relevant to mental computation. Specifically, use of the empty number line seemed to be integral to advancement in students' reasoning about subtraction. Similarly, rectangular area stood out as integral to the advancement of students' reasoning about multiplication. The empty number line and rectangular area were regarded as *standout tools*. In the SST interviews, I investigated how participants used and interpreted these.

Prior to each SST interview, I reviewed the participant's first and second interview responses to identify adopted strategies and other particular topics of interest. Thus, the content of these interviews varied somewhat by participant. They were effectively interviews concerning both strategies and tools that stood out from class (from my perspective) and strategies that seemed to stand out to the individual participant.

In the SST interviews, participants were posed tasks and questions concerning standout strategies and tools related to mental computation. Examples of standout strategies were presented to the interviewee verbally as another student's work (e.g., "Jamie wanted to calculate 142 – 57. He subtracted 2 from both numbers to get 140 – 55. Then he was able to tell that the answer was 85."). The interviewee was asked a set of questions concerning standout strategies.

I asked the following main questions concerning strategies:

- Does this strategy make sense to you? Why or why not?
- Is this a strategy that you use, yourself? If so, please say more. If not, any reason why?

I asked follow-up questions aimed at better understanding the participants' reasoning.

I also asked questions pertaining to the standout tools. These questions were designed to investigate how the interview participants interpreted the standout tools and whether and how they would reason differently about the operations when using the tools. The particular kinds of questions asked were the following:

- How would you record this student's work? (e.g., Jamie's strategy above)
- Explain in detail how your drawing represents this person's work.
- In your opinion, does that drawing capture the idea behind the person's strategy?
- Are there other ways to model the student's work?
- Would it make sense to model her strategy this way instead? (Interviewer presents alternative drawing.) Why or why not?

The intention of these tasks and questions was to investigate individual students' conceptions related to what would likely by as-if shared ideas (Rasmussen & Stephan, 2008). Different students reason with as-if shared ideas in different ways (Bowers et al., 1999). These interviews were designed to provide evidence of individual students' interpretations and conceptions that relate to classroom math practices. I viewed the SST interviews as analogous to the interviews described by Saxe and Esmonde (2005) in which Oksapmin people were asked about the meaning of the word *fu.* People from different cohorts had different meanings for the word, which the researchers could account for based on the people's backgrounds and the ways in which they had encountered its use. At the same time, people from different cohorts could interact and understand one another, despite interpreting the word in distinct ways.

Interview particulars. Interviews took place in private rooms on campus. Some were conducted in a conference room. Others were held in a faculty member's office. All interviews were audio and video recorded. In the mental computation interviews, participants were presented with each task, one by one, verbally. The numbers involved in the story problems were shown to the participant in written form so that the participant would not need to hold them in short-term memory during the computation (Hope & Sherrill, 1987). In the standout strategies and tools interviews, tasks were presented in written and/or verbal form, depending on the particular task. Strategies were always described verbally. If working at the chalkboard or whiteboard, the students would record details of the strategy on the board. If the participant was seated, I would repeat the information as needed.

In the mental-computation interviews, participants were asked questions of three types: main questions, probes, and follow-up questions (Rubin & Rubin, 1995). The main questions were the mental computation tasks described above. Interviewee's responses to these consisted of the solution processes that they described. When it was necessary, I probed for clarification of the details of those processes. For example, "Which did you add first, 20 and 50, or 3 and 4?" Follow-up questions in the computational reasoning interviews took the form of requests for justification. For example, if an interviewee solved 142 - 57 by treating the computation as 140 - 55 instead, I asked why that computation would give the same results as the given one, e.g., "Why can you subtract 2 from each number?" In my note taking, I thought in terms of the interviewee articulating an argument, and I looked for each component of the argument to be there. In these terms, the follow-up questions constituted requests for backing (Toulmin, 1969).

The intent of the probes and follow-up questions was to clarify the nature of the computational steps performed and to investigate whether and how the interviewees could justify their strategies mathematically. I refrained from asking questions concerning mental processes that might not be known to the interviewees. For example, I did not ask questions like, "Why did you decide to do that one differently?" Such questions might have led subjects to invent answers, rather than give accurate reports (Ericsson & Simon, 1993). They might also have influenced interviewees' strategy selection in subsequent tasks.

In the case of the Standout Strategies and Tools Interviews, the main questions were those provided in the bulleted lists above. Probes concerned the particular details of an interviewee's interpretation of a standout strategy or way of using a standout tool. For example, "What does it mean when you draw that arrow from 37 to 100?" Follow-up questions typically took the form of *why* questions, such as "Why do you go to the left when you subtract?"

Relevance of interviews to research questions. The mental computation interviews afforded analyses of change in participants' computational reasoning. As discussed in the literature review, mental computation of whole numbers is seen as a microcosm of whole-number sense. In this way, changes in participants' activity when solving these interview tasks speak to changes in their number sense. As such, these data help me to answer Research Question 1 by assessing change in students' number sense. This data also contributed to the analysis of the ontogenetic strand of number sense development by providing the beginning and ending points of the case study students' developmental trajectories.

The standout strategies and tools interviews provide data concerning students' interpretations and understandings of particular strategies and ways of representing student reasoning that become normative in the classroom community. Based on the hypothesized correspondence between students' conceptions and as-if shared ideas, I expected that this data would be useful in the ontogenetic analyses of case study students' number sense development.

Classroom Data

The class took place in a technologically advanced classroom. The technology served both instruction and data collection purposes. In particular, the classroom was equipped with the following: two walls of whiteboards; a "smart station" including an instructor computer, a connection for a laptop computer, and an external document camera; a smart board; eight small whiteboards; and a collection of student laptops. The instructor's computer and document camera were used to project images onto either of two projector screens. The classroom contained eight student tables, and four or five students were typically seated at each table. This seating arrangement facilitated communication within groups, which was conducive to the kind of classroom culture in which students developed solutions and explanations and often presented these to the class. Each group had a small whiteboard. These facilitated collaborative group work and were often used by students to prepare and present their group's work to the whole class. The whiteboards also made it easy for me to capture still images of group work.

Classroom data included audio/video recordings, still photos taken with a digital camera, and hard copies of selected student work. Three video cameras were used to record classroom activity. There was a built-in video camera situated in a fixed location in the corner of the classroom. A graduate student operated this camera and was responsible for following the instructor throughout the class period. There was also a digital video camera, which was positioned on a tripod within the classroom. Another graduate student operated this camera and was responsible for filming students. During group work, this camera focused on two groups. During whole-class discussion, it focused on the students participating in the discussion. The third camera was a Flip video camera, which was hand-held and portable. I operated the Flip camera. During group work, I used this camera to survey students' work and conversations. During whole-class discussion, I used it to capture interactions from angles that were not available to the other cameras.

As Roschelle (2000) points out, "video is a constructed artifact," as opposed to an objective record (p. 709). The choices that a researcher makes in collecting video have implications for the study. It was a priority in filming to capture a wide-framed *restored* view of classroom activity, as opposed to a narrow-framed *deleted* view (Hall, 2000). A restored view records people's activity in the physical and social circumstances in which it occurs. It enables the research to see, for example, the activity of a person's hands as part of the activity of her whole body, and the activity of her whole body in the context of interaction with materials and communication with other people. A deleted view, by contrast, isolates hands from whole bodies, heads from bodies, speakers from interlocutors, and so on (Hall, 2000). This focus was consistent with the situated perspective that I brought to the analysis of classroom activity (Cobb & Bowers, 1999). At the same time, close-up images of students' written work were collected and proved valuable. These were obtained via digital photos, as well as from the video recordings. All such image files were named according to the day of class, activity, and students.

When I was not recording video with the Flip camera, I took field notes on my laptop computer. Bogdan and Biklen (2007) describe field notes as "the written account of what the researcher hears, sees, experiences, and thinks in the course of collecting and reflecting on the data in a qualitative study" (p. 119). These are distinct from memos, which consists of more in-depth thoughts about an event, usually written after leaving the field (Corbin & Strauss, 2008). I used field notes to record my thoughts about events that stood out from each day of class. These were useful in conversations with the instructor after each class meeting. They also helped me to reorient to the classroom activity and phenomena of interest when it came to analyzing the data.

The classroom data afforded analyses of mathematical argumentation during whole-class discussion. The particular methodology that was used will be discussed in the next section. The classroom data was analyzed for the purposes of addressing Research Question 2. These data enabled me to analyze the microgenetic and sociogenetic strands of number sense development.

Students' Written Work

Students' written work of interest will take a few different forms. I describe each of these in the subsections that follow.

Strategy reflection assignments. The instructor and I collaborated on homework assignments that both served our instructional goals and provided useful data sources. A common type of assignment that was relevant to the analysis was strategy reflections. After a strategy had been discussed in class, students were often given a homework assignment that involved reflecting on the strategy. The details of these assignments varied. Students were sometimes asked to discuss their understanding of the strategy, to apply it to a new problem, to draw a picture to represent the strategy, or to suggest a name for the strategy and to provide a rationale for the suggested name.

These assignments were useful to the instructor because they gave her additional access to students' reasoning concerning strategies discussed in class. I made direct use of this data in the analyses of case studies. These assignments provided crucial information concerning case study students' reasoning about standout strategies at points in between the interviews. As such, this data helped me to study the ontogenetic strand of number sense development in service of addressing Research Question 1.

Computational reasoning journals. Students were periodically assigned to describe instances of their mental computation activity done outside of class. They described the setting in which the activity took place, the particular computation performed, and how it was performed. Students described their reasoning with number sentences, drawings, and written explanations. Students were occasionally asked whether the strategy described was on that they had used before, whether it related to a strategy from class, etc.

These journals provided additional data that was useful for the purposes of the case studies. They provide examples of strategies used by students on tasks that were very open. They chose which instance of mental computation activity to write about, so that these responses varied in terms of the operation, the strategy, the degree of difficulty, and so on. The disadvantage of this data was the openness. For example, my case study of Valerie is focused on multiplication. So, her responses concerning addition strategies are not useful data. On the other hand, the openness of these assignments also had advantages. Students' responses seemed to ring true. That is, they gave the impression of true accounts of events in students' lives in which mental computation strategies were actually used. By contrast, assignments that forced students to engage with a particular strategy provided information about the student's reasoning specific to that strategy, but the strategy may have been one that the student would not use independently. Both types of assignments were useful.

To clarify, the instructor and I both looked at all students' written assignments during the semester. Their responses informed ongoing instructional design. These also helped me to select particular questions to include in the Standout Strategies and Tools Interviews. However, only the case study students' written assignments were analyzed in detail for research purposes. These regular written assignments helped me to analyze ontogenesis, connecting the conceptual dots, so to speak, between the snapshots obtained in pre and post interviews. These data help me to answer Research Question 1.

Exams. Quizzes, midterms, and a final exam were used to assess student learning and to assign grades. These exams also served as data sources for the purposes of the case studies. Like the computational reasoning journals, certain exam items asked targeted questions concerning particular strategies. For example, Item #12 on Midterm 2 asked students to evaluate a child's subtraction strategy as "all right" or not. The item asked students to explain the child's thinking. If they determined that the child's strategy was valid, they were to apply it to another calculation. If they determined that it was not valid, they were asked to explain why. Such questions concerned standout strategies, as well as mathematical ideas of particular relevance to mental computation. As with the other written work described above, these data were used in the analysis of case studies for the purposes of answering Research Question 1.

Data Analysis

Having described the data to be collected and the procedures for its collection, I now turn to a description of how the data were analyzed. I conceived of the analytic process toward answering Research Question 1 as consisting of two pieces: assessing change in number sense by established methods to determine milestones in developmental trajectories, and then analyzing the ontogenesis of number sense in selected case studies to connect the dots between those milestones. The analytic process toward answering Research Question 2 consisted of microgenetic analysis of arguments made in whole-class discussion, followed by sociogenetic analysis to identify as-if shared ideas and describe classroom math practices.

Assessing Change in Number Sense

I describe here the analytic methods involved in assessing change in students' number sense, based on pre/post comparisons. This process involved both quantitative and qualitative methods of analysis for which there were precedents in the literature.

Number Sense Rating Scale. Apart from the use of NSRS pretest scores to compare interview volunteers' scores to those of the class as a whole, analysis of NSRS scores took place after semester's end. I used difference scores (posttest - pretest) as one measure of change in students' number sense. I used a Student's t-test on the mean of the difference scores to test the hypothesis that the mean level of the attribute measured by the test improved (Kutner, Nachtsheim, Neter, & Li, 2005). I verified that the distribution of the data was reasonably normal, so that the test was appropriate.

Student Preference Survey. Descriptions of number sense go beyond understanding of mathematics. For example, Reys and Yang's (1998) description refers to both "ability and inclination" for flexible computational reasoning (p. 225). The SPS addresses the inclination piece. Surveys will be scored 1 point for each Yes response and 0 points for each No response. Class means will be compared pre/post, as described above for the NSRS.

Mental computation interview data. For the purposes of establishing number sense improvement, the mental computation interview data were coded in a manner for which there was a precedent in the literature. Participants' verbal responses to mental computation tasks were coded for strategies based on an extant scheme. I used the same coding scheme that I developed in my Master's study (Whitacre, 2006), since that scheme was grounded in the data from a previous study of prospective elementary teachers. Slight modifications were made with sensitivity to the particular data set, as well as progress in my reasoning about distinctions between strategies. The changes involved collapsing codes, splitting codes, and changing the names of some codes. Three invalid strategies were also named, whereas previously all invalid strategies had simply been categorized as Invalid. The current coding scheme for mental computation strategies appears in the section titled Participants' Mental Computation Strategies.

The extant scheme did not include division strategies, since these were not investigated in the previous study. I extended the scheme to include the strategies that the interview participants use for mental division of whole numbers. Creating strategy codes involved a process of constant comparative analysis (Creswell, 1998). The participants did not use a wide variety of division strategies in either interview. Four valid strategies and one invalid strategy were identified.

Once participants' responses were coded for the strategy used, I tabulated the coded data. I recorded the number of distinct strategies used by each participant for each arithmetic operation in each interview. I noted the particular strategies used and the number of times each was used. Particular statistics of interest were the mean difference in the numbers of different strategies used for each operation and overall for addition, subtraction, multiplication, and division. This is a measure of change in flexibility. Conceptually, what I by *flexibility* is a participant's tendency to select a particular strategy that is suited to the given numbers and operation, rather than automatically performing the operation with a go-to procedure (Heirdsfield & Cooper, 2004). Operationally, what I mean by *flexibility* in terms of the analysis of interview data is simply the number of different strategies that a participant uses for a given operation. The interview tasks were designed to elicit these strategic choices by varying the affordances of the numbers involved. While it remains possible that a participant could arbitrarily choose to use one strategy or another for a particular computation (and, hence, potentially to use one strategy or many strategies across a set of computations), the literature suggests that this is unlikely. Flexible mental calculators do make choices on the basis of the given numbers, with ease of computation generally being a high priority (e.g., Heirdsfield & Cooper, 2004). Therefore, if an interview participant *is* flexible, it is unlikely that she would choose to use the same strategy repeatedly across varied pairs of numbers, some of which make the use of that strategy cumbersome.

Flexibility was categorized using a scheme that developed in the course of the research. Heirdsfield and Cooper (2004) contrasted the processes of flexible and inflexible mental calculators: flexible mental calculators make a choice of strategy, whereas inflexible mental calculators do not. While this distinction is useful, I found that many participants were not strictly inflexible or fully flexible. They were *semiflexible*. They made choices based on the given numbers, but they did this in limited ways. Participants who chose between only two possible alternatives (typically the MASA and one nonstandard strategy) were considered semiflexible.

For each operation, each participant was classified as Inflexible, Semiflexible, or Flexible. Change from the first to second interview was assessed on the basis of whether a participant moved from one category to another. This scheme was preferred over using the number of strategies directly as a measure of flexibility due to an emphasis on the participants' process. There is a noteworthy shift in process that occurs when someone moves from not making a choice of strategy to making one. It is also a noteworthy shift when someone moves from making a dichotomous choice to choosing from a repertoire of at least three strategies. However, whether someone uses three or four strategies seems far less significant. In both cases, I consider the person's process to be essentially the same, and I categorize it as Flexible.

The strategies used by the interview participants were also categorized according to Markovits and Sowder's (1994) framework. All strategies for a given operation were ordered based on the extent to which they departed from the standard algorithm. I pooled the data and counted all instances of Standard, Transition, Nonstandard, and Nonstandard with reformulation that occurred in the pre and post in response to the basic Bobo tasks. I then compared the distribution of strategies across categories from pre to post.

Summary of analytic methods for assessing change in number sense. In summary, change in students' number sense was assessed by methods for which there were precedents in the literature. These consisted of gain scores on an established multiple-choice instrument, change in flexibility in mental computation, and shift in strategy preferences. By using these established methods, I situated the study within the tradition of research concerning number sense. In the remainder of the study, I extended previous research concerning number sense by investigating its development in terms of genetic analysis.

Analyzing Development: Microgenesis

I describe here the methods involved in the microgenetic analysis of number sense development. This involved the analysis of classroom data, interview data, and students' written work. However, a common analytic method was used across each of the types of data. This involved the construction of argumentation diagrams.

Argumentation diagrams. In approaching this analysis, I took argumentation to be a useful lens through which to analyze mathematical activity. I begin with an example from whole-class discussion, for reasons that will become apparent. However, the arguments made by case study students in interviews were analyzed similarly.

On Day 12, students considered children's subtraction strategies. Valerie made an argument for the validity of one of these. The child had computed 364 – 79 by adding 1 to both numbers to get 365 – 80, and then adding 20 to both number to get 385 – 100. The child could then tell that the difference was 285. (This is an example of Shifting the

Difference.) Valerie's argument related the subtraction strategy to distance and movement along a number line:

> *Valerie*: Okay, so we thought about it in terms of, when you're subtracting, you're trying to find the distance between two numbers. So, we thought of it kind of in terms of a number line. So, if you—uh, for the first one—when he, or whoever the student was, made it 365 and 80. So, you started off with 79 and 364. So, 364 moved up one to 365 and also, likewise the 79 moved up to 80. So, the distance didn't change between the numbers. So, originally it was right here, and they both moved up one on a number line. So, the distance between them is the same. So, similarly when you have 385 and 100, you just added 21. So, if you took the numbers from their original position and moved them each up 21 spaces, the shift would be the same and the distance between both numbers is the same.

Figure 3 represents Valerie's argument. Valerie claimed that the child's strategy was valid, or legitimate. She did not make this statement explicitly, but it was the gist of her argument. As data, Valerie pointed out that the same amount had been added to both numbers. This was a factual statement concerning the steps that the child had performed, based on the information that students had been given in a handout. The warrant in Valerie's argument was that, in general, adding the same amounts to both numbers would not affect the difference. With respect to the core of Valerie's argument, it had a basic modus ponens structure. That is, given A, she claimed B, and her warrant was that A implied B. However, Valerie's warrant had not been used previously in the class. It was a new and previously unjustified idea. The backing in Valerie's argument was particularly of interest because it explained why she thought her warrant was true. This is where reasoning in terms of the number line came in. Valerie reasoned about the difference in subtraction as a distance between number-locations. In line with that interpretation, she

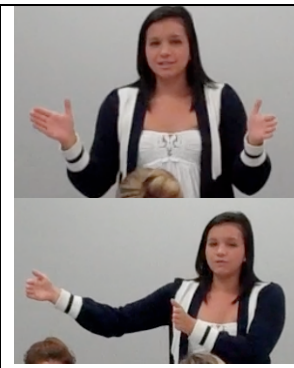
<table>
<tr><td>Claim: The child’s strategy is legitimate
[Valerie’s argument supports the validity of the child’s strategy]</td><td rowspan="4">
Valerie holds her hands approximately equi-distant. She moves both hands to her right as she talks about the numbers “moving up.”</td></tr>
<tr><td>Data: The child added the same amounts to the minuend and subtrahend
(“you took the numbers from their original position and moved them each up 21 spaces”)</td></tr>
<tr><td>Warrant: Adding the same amounts to the minuend and subtrahend does not change the difference
(“So, if you took the numbers from their original position and moved them each up 21 spaces, the shift would be the same and the distance between both numbers is the same.”)</td></tr>
<tr><td>Backing: Reasoning about difference as distance between number-locations; Reasoning about adding in terms of movement along a number line
(“So, the distance didn’t change between the numbers. So, originally it was right here, and they both moved up one on a number line. So, the distance between them is the same.”)</td></tr>
</table>

Figure 3. Argument A12.20: Valerie argues for the validity of Shifting the Difference.

reasoned about adding as shifting those number-locations to the right. It was on this basis that she justified her warrant.

It was often the case in the class that students displayed their whiteboards when they presented arguments to the class. In this way, they shared inscriptions that supported their arguments. These typically included both records of computations and some manner of drawing. In the case of the example above, Valerie’s gesturing effectively took the place of such inscriptions. She explicitly stated that she was reasoning in terms of a number line, and she used her hands to designate number-locations and to illustrate the shifting actions that she described.

I include the still images in the right column of the argumentation diagram to identify gesturing that appears relevant to the ideas expressed in an argument. In the above example, Valerie’s verbal explanation was quite explicit, so that we need not refer to her gesturing to understand how she was reasoning. I see the evidence in Valerie's gesturing as corroborating my interpretation of her argument. In some cases, students were less clear with their words, and so gesturing provided important clues to how they were reasoning. In other cases, gesturing did not seem to convey relevant mathematical ideas and was not included in the argumentation diagrams. In general, my approach was not to separate out speech, gesture, and inscriptions. Rather, I took all of these as evidence of how the person was reasoning. The argumentation diagram is my best attempt to convey that reasoning by making the content and structure of the argument explicit. Toulmin's model of argumentation has been used in previous analyses of classroom math practices (Stephan & Rasmussen, 2002), and gesturing has also been incorporated into these analyses (Rasmussen et al., 2004).

Argumentation diagrams of interview responses. The interview participants' responses to the mental computation tasks were coded according to the strategy used. The case study students' responses were analyzed in a more fine-grained fashion, which included regarding their responses as instances of argumentation activity. I constructed argumentation diagrams to describe the case study students' reasoning in response to the mental computation and Operations tasks, as well as to relevant tasks from the SST interview.

Indentifying models involved in computational reasoning. Originally, I had planned to code classroom data in terms of both argumentation and models. In the tradition of earlier analyses of classroom math practices, I considered documenting these in terms of successive shifts from model-of to model-for. In the process of the sociogenetic analysis, I decided that this was impractical. I made this decision for a few reasons: (1) The methodology of Rasmussen and Stephan (2008) proved to be robust. It incorporated gesturing and inscriptions, so that ideas that I might otherwise have characterized in terms of models were captured; (2) analysis in terms of models did not seem well suited to account for the complexity of students' activity. I conceptualize classroom math practices as sets of as-if shared ideas, as opposed to single ideas. It did not seem plausible for a single model to characterize an entire classroom math practice.

Summary of microgenetic analysis of number sense development. Data were analyzed by means of constructing argumentation diagrams. These products of the microgenetic analyses then became data that was used in the sociogenetic and ontogenetic analyses of number sense development.

Analyzing Development: Sociogenesis

I describe here the sociogenetic analysis of number sense development, which addressed Research Question 2. I looked at trends in the collective argumentation activity in order to analyze sociogenesis. In particular, I applied the criteria for identifying as-if shared ideas. These normative ways of reasoning then informed descriptions of general mathematical activity in terms of classroom math practices.

Classroom mathematical practices. The methodology of Rasmussen and Stephan (2008) was used to analyze collective activity. This methodology involves coding arguments using Toulmin's (1969) model, which describes the anatomy of an argument in terms of claim, data, warrant, and backing. The claim is the assertion being made. The data is evidence offered in support of the claim. The warrant explains how the data supports the claim. Backing serves to justify the validity of the warrant.

The methodology is a three-phase process.[13] The phases are as follows: (1) Whole-class discussions are transcribed. Researchers watch video of each discussion and identify any claims that are made. For each claim, an argumentation scheme is constructed, which explicitly identifies each of the components of the argument. This analysis yields a chronological argumentation log. (2) Researchers look across the argumentation log to identify ideas that *functioned as if shared* in whole-class discussion. Criteria for ideas functioning as if shared are (i) warrants or backings dropping off, (ii) an element of an argument shifting roles (e.g., from claim to warrant), and (iii) repeated use of data or warrants in support of different claims (Cole et al., 2011). (3) The as-if shared ideas are then organized according to related mathematical activities to describe classroom mathematical practices.

I followed the three-phase approach described above. I identified in the reduced data set a total of 208 mathematical claims for which some justification was given. For each of these claims, I constructed an argumentation diagram. More specifically, my analysis was initially done separately for three content-specific strands of activity: place value, addition and subtraction, and multiplication and division. I identified whole-class discussions as primarily related to activity in one of these three strands. I analyzed the addition and subtraction data first, coding all 48 arguments belonging to that set. Then I analyzed the place value data, which consisted of 70 arguments. I made multiple passes through the coding of each discussion and each particular argument, refining my coding

[13] I classify the first phase as microgenetic analysis, but for clarity of presentation, I include it in the description of the methodology here.

iteratively. Once I was satisfied that the arguments had been coded consistently, I proceeded to Phase 2.

In Phase 2, I applied the criteria for identifying ideas that functioned as if shared. I did this by creating tables of occurrences of ideas that had been used repeatedly in arguments. I tabulated occurrences of each idea as claim, data, warrant, or backing. I did this conscientiously, not blindly. In other words, I did not apply the criteria automatically or without question. Rather, if one or more of the criteria appeared to be satisfied, I examined the details of how that idea was used in each argument and took care to consider whether I believed that the idea in question did, in fact, function as if shared. (I discuss this process in more detail in the section titled Reliability and Validity.)

Together, the place value and addition/subtraction strands formed a coherent, larger strand. When it came to categorization of sets of as-if shared ideas into classroom mathematical practices, I integrated these two smaller strands of activity. In the strand of activity focused on place value, addition, and subtraction, I identified five classroom math practices. This phase of categorization is naturally theory-laded. It requires researchers to make distinctions, and researchers value certain kinds of distinctions over others. In my case, my interest in the sociogenesis of number sense informed the classification of ideas and activity. I was interested in the kinds of strategies that were being used by the class and on what basis these were justified. The classroom math practices that I identified and the names that I gave to these reflect that focus.

After the analysis of the addition and subtraction and place value data was complete, I proceeded to analyze the multiplication and division data. I followed the same three-phase process. I identified 90 claims in the multiplication and division strand.

I coded these arguments, again making multiple passes through the data until I was satisfied with all coding decisions. This data set consisted largely of arguments related to multiplication, as opposed to division. Of the five days in which multiplication and division ideas were the focus of classroom activity, only one was devoted to division. Furthermore, there was little coherence between the multiplication activity and the division activity. I did not see a unified story of development emerging. As a result, I made the decision to focus the analysis on multiplication activity. I identified as-if shared multiplication ideas and categorized these into classroom mathematical practices, as described above. One classroom math practice was conceived more generally because it did not seem to be operation-specific. Thus, I regard the first classroom math practice (which had been identified in the analysis of the addition and subtraction strand) as belonging to both strands of activity. Within the strand exclusively related to multiplication, I identified three additional classroom math practices.

Note that I had originally envisioned identifying as-if shared strategies and models. As discussed earlier, the models piece proved unnecessary and somewhat incongruent with the analysis. In lieu of as-if shared strategies per se, I identified as-if shared mathematical ideas. In many cases, these ideas were used to describe and to justify strategies. However, in my current conceptualization, strategies themselves do not function as-if shared. Students may refer to strategies by name, and these names may function as warrants in students' arguments. This is essentially what I previously had in mind when I talked about "taken-as-shared strategies."

Summary of methods for analyzing the sociogenesis of number sense. The analysis of the sociogenesis of number sense involved the products of the microgenetic analysis of classroom data. These were analyzed to identify trends that occurred over periods of time. I applied the criteria for to identify as-if shared ideas. I then grouped these ideas according to commonalities among them and to the nature of the general mathematical activity in which they were involved in order to describe classroom mathematical practices. The results of this analysis directly answer Research Question 2.

Analyzing Development: Ontogenesis

I describe here the methods for analyzing the ontogenesis of number sense, in service of answering Research Question 1. I describe these methods after those for the analysis of microgenesis and sociogenesis since the former analyses support this analysis. The microgenetic analysis of participants' mental computation strategies afforded basic descriptions of their reasoning during the interviews. When it came to the case studies, these responses and others were analyzed in greater detail in order to understand as best I could the case study students' reasoning at these milestone points (essentially before and after instruction).

The products of the sociogenetic analysis pointed me to normative ways of reasoning from the classroom activity. Particular ideas from class and certain classroom events stood out as relevant to each case study. Detailed ontogenetic analysis was restricted to the two interview participants who are selected for case studies. I selected the case of Brandy's developing reasoning about place value, addition, and subtraction and the case of Valerie's developing reasoning about multiplication. These cases were selected for distinct reasons, which I elaborate on in Chapter 6. Brandy's case was one of exceptional improvement, as she changed from Inflexible to Flexible in both addition and subtraction. Valerie's case afforded an illuminating, fine-grained analysis of her reasoning about multiplication with a particular focus on partial products.

For this analysis, I drew on the following data sources: case students' pre/post respones to the basic Bobo mental computation tasks, case students' pre/post responses to other pre/post interview tasks, case students' responses to tasks and questions from the SST interview, case students' homework and exam responses, and case students' relevant contributions to whole-class discussion. The results of the sociogenetic analysis also helped me to situate case students' reasoning in the chronology of the development of collective activity and to relate their reasoning to ideas expressed in class.

The ontogenetic analyses took the form of case study research (Yin, 1994). The analytic process consisted of the following phases:

- Identifying and comparing the case students' pre/post strategy ranges
- Identifying the case students' pre/post scaffolded strategy ranges
- Identifying ways of reasoning evident in case students' responses to other interview tasks, especially the Operations tasks
- Documenting the case students' reasoning about particular ideas via written responses and occasional contributions to whole-class discussion
- Identifying themes in the case students' reasoning over periods of time and identifying points at which the case students' reasoning appeared to change
- Relating changes in the case students' reasoning to both classroom events and the students' prior knowledge

- Developing plausible explanations of the case students' trajectories of number sense development (in the vein of Izsák et al., 2008)

In the following paragraphs, I describe the specifics of each of these phases of the ontogenetic analysis.

Identifying and comparing the case students' pre/post strategy ranges. This phase was accomplished in the analysis of microgenesis described earlier. I identified, for each of the interview participants, which particular strategies were used in the pre/post mental computation interviews. These were used to describe the participants' strategy ranges pre/post. Comparing these ranges indicate which strategies the case student had adopted. This basic comparison also gave initial indications of ways in which the case students' reasoning appeared to have changed.

Identifying the case students' pre/post scaffolded strategy ranges. Using participants' responses to the Scaffolded Alternatives Tasks (for the operation(s) relevant to the case study), I identified the strategies that the case student used when asked for alternative ways of performing the same computation. These were used to describe the case students' scaffolded strategy ranges pre/post. Those strategies belonging to the case students' scaffolded range and not their proper range provided indications of additional knowledge that may have served as resources in the learning process.

Identifying ways of reasoning evident in case students' responses to other interview tasks, especially the Operations tasks. Using case students' responses to the other interview tasks, I gathered further evidence of their ways of reasoning by analyzing these responses as arguments. For example, in the Operations tasks, students were asked to perform the standard algorithms and explain and justify the details of these. Brandy made arguments concerning carrying in addition, borrowing in subtraction, and so on. Valerie made arguments concerning the details of the standard multiplication algorithm. The justifications that the case students offered for these details indicate ways of reasoning about the algorithms, and these ways of reasoning changed from pre to post.

Documenting the case students' reasoning about particular ideas via written responses and occasional contributions to whole-class discussion. I examined the case students' homework responses concerning strategies from class to identify how they reasoned about these. I also analyzed journal responses in which students reported using mental computation outside of school. In addition, I identified instances from whole-class discussion in which case study students shared their reasoning. These data were used to describe the case students' reasoning in each instance. In the next phase, I looked chronologically across these instances.

Identifying themes in the case students' reasoning over periods of time and identifying points at which the case students' reasoning appeared to change. Taking students' reasoning identified in the previous analyses as data, I identified common themes across those instances. I also identified contrasts in students' reasoning, such as a change in interpretation of a strategy or a first instance of using an idea or strategy. I focused especially on adopted strategies because I would need to explain in the final phase how the student came to adopt the particular strategy.

Relating changes in the case students' reasoning to both classroom events and the students' prior knowledge. Using all relevant data, I attempted to account for changes in students' reasoning based on the interplay between students' prior knowledge and classroom events. Essentially, for each instance in which the student expressed a new or different idea, I asked the question, "Where did this idea come from?" and attempted to identify antecedents in the data. Particularly, in Valerie's case, additional evidence of interest came from the SST interview since there was an aspect of her reasoning at the endpoint that was still somewhat puzzling. So, I drew comparisons and contrasts between her relevant responses from the second and third interviews.

Developing plausible explanations of the case students' trajectories of number sense development. In this phase, I took a step back from the previous to construct a chronological account that integrated the results of the previous phases. These accounts were forward-looking, by which I mean that themes in students' early reasoning were highlighted with an eye toward contrasting them with later themes, as well as explaining how the change came about. I view these accounts as similar to the analsysis presented by Izsák et al. (2008) of Sonya's interpretations of her teacher's solutions to fraction addition problems in terms of Sonya's n-out-of-m conception of fractions.

Summary of ontogenetic analysis of number sense development. The ontogenetic analysis of the case students' number sense development will involve a variety of data sources, most of which derive from the previous microgenetic and sociogenetic analyses. This analysis will also incorporate data from the Standout Strategies and Tools Interviews, which are designed to directly investigate participants' interpretations and understandings of the standout strategies and tools from class. This analysis will proceed from identifying the strategies and models that a case student used in pre/post computational reasoning interviews, to tracking trends in these over time, to determining whether or not established strategies and models from class were adopted by the student, to attempting to account for this based on students' interpretations and understandings, and finally to developing plausible explanations for the case students' trajectories of number sense development.

Reliability and Validity

In this section, I describe measures that I took in the interest of the validity and reliability of the results of the study. This study involved mixed methods. However, it is fundamentally a study that asks process questions, as opposed to variance questions (Maxwell, 2005). Therefore, I conceive of the study as a whole as qualitative, and I frame the discussion of reliability and validity in the tradition of qualitative research.

Reliability

In qualitative analysis, internal reliability in the form of inter-rater reliability is a concern. Internal reliability refers to "the question of whether, within a single study, multiple observers would agree about what happened" (LeCompte & Preissle, 1993, p. 337). According to LeCompte and Preissle (1993), the best way to guard against threats to internal reliability is to actually involve multiple researchers in the analysis process. In this study, I analyzed all of the data. During the analysis process, I met with several other researchers to discuss both general issues that arose in coding and particular interpretations of data. I met regularly with Dr. Nickerson about all aspects of the research. I had meetings with Dr. Chris Rasmussen to discuss issues that arose in the analysis of collective activity. I also had conversations with Dr. Megan Wawro and researcher George Sweeney about general issues involved in that analysis. I met with Dr. Randy Philipp and researchers Bonnie Schappelle and John Siegfried to discuss aspects of Brandy's reasoning for the purposes of the first case study.

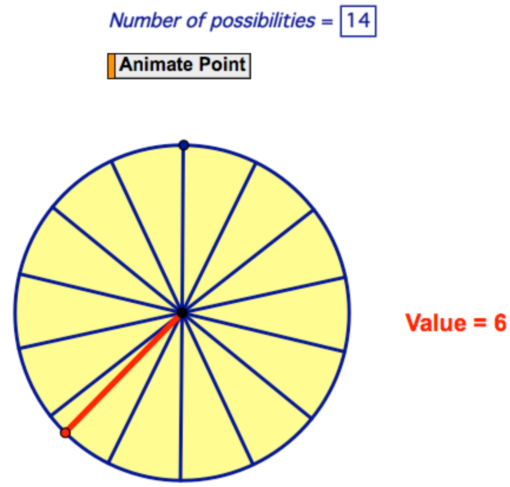
After my analyses were completed, I enlisted two researchers to code subsets of the data in order to assess the reliability of the coding. George Sweeney, currently a doctoral candidate, assisted with the analysis of collective activity. He had extensive previous experience with the methodology of Rasmussen and Stephan (2008). Dr. Susan Nickerson assisted in the analysis of participants' mental computation strategies. Dr. Nickerson and I had done previous research concerning prospective elementary teachers' whole-number mental computation strategies. In the next few pages, I discuss the details and results of the reliability checks.

Reliability of mental computation strategy coding. Interview participants' mental computation strategies for the basic Bobo tasks were coding according to the scheme that I developed for the purpose. There were seven interview participants, and each was interviewed pre and post. Thus, there were 14 mental computation interviews in the data set. These were numbered from 1 to 14 based on alphabetical order of pre and then post interview participants (see Figure 4). Four of these 14 interviews were randomly selected to be double-coded as a reliability check.

I designed a simple Geometer's Sketchpad (GSP) sketch for the purpose of random selection of data. The sketch (Figure 4) represents a spinner. The user determines the number of possible outcomes of a spin. In this case, that number was 14. The red line segment represents the spinner needle. The segment is determined by the location of a point on the circle, which is connected to the point at the center of the circle. An animation button sends the point on the circle to a random location on the circle, using GSP's built-in pseudo-random number generator.

1. Angela pre
2. Brandy pre
3. Maricela pre
4. Natalie pre
5. Trina pre
6. Valerie pre
7. Zelda pre
8. Angela post
9. Brandy post
10. Maricela post
11. Natalie post
12. Trina post
13. Valerie post
14. Zelda post

List of mental computation interview data



Spinner.gsp

Figure 4. Random selection of interview data.

Dr. Nickerson and I discussed the question of sampling and agreed that the random sample of four interviews should include two pre interviews and two post interviews. So, the sampling was performed with this constraint. It was done with replacement, so that renumbering of interviews was not required. As a result, several additional spins had to be made since they resulted in either a repeated outcome or an outcome belonging to the wrong interview (post, when pre was needed).

The outcomes of each spin were recorded and are shown in Table 5. The following four interviews were selected: Valerie pre, Trina post, Zelda post, and Brandy pre. Videos of each interview were given to Dr. Nickerson for coding, using the scheme that I had provided. There four interviews, four operations per interview, and four computations per operation. Thus, there were 4 x 4 x 4 = 64 responses to be coded for mental computation strategy. Dr. Nickerson coded these and then compared her coding to mine. We agreed on all 64 coding decisions. So, on a random sample of 4/14 (≈ 28.6%) of the data set, we agreed on 100% of coding decisions.

Table 5. Outcomes of Spins to Select a Random Subset of the Interview Data

Trial	Outcome	Result
1st	6: Valerie pre	First pre seletion.
2nd	12: Trina post	First post selection.
3rd	14: Zelda post	Second post selection.
4th	6: Valerie pre	Duplicate. Spin again.
5th	11: Natalie post	Need pre. Spin again.
6th	14: Zelda post	Duplicate. Spin again.
7th	9: Brandy post	Need pre. Spin again.
8th	2: Brandy pre	Second pre selection.

I attribute this extraordinarily high degree of reliabilty to the appropriateness of the coding scheme to this particular data set, the relative ease of this type of coding, and the fact that Dr. Nickerson was close to the research and familiar with the strategy codes. Interview participants' verbal responses typically ranged from 1 to 2 minutes, including the interviewer's probes and follow-ups and the participant's responses to these. (Some responses took several minutes, but the mean was between 1 and 2 minutes.)

The interviewer was specifically concerned with being able to accurately identify the participant's strategy, and his probes were primarily concerned with clarifying details when necessary. Thus, participants' descriptions of their strategies tended to be clear and detailed. Furthermore, the scheme was such that difficult coding decisions were rare because strategies were adequately distinguished operationally. Hence, the coder's simple task was to take 1-2 minutes worth of good data and apply to it a single code.

Reliability of the analysis of collective activity. The reliability of the analysis of collective activity was assessed similarly. The methodology of Rasmussen and Stephan (2008) consists of three phases: (1) coding of arguments, (2) applying the criteria to identify as-if shared ideas, and (3) categorizing as-if shared ideas as belonging to classroom mathematical practices. The results of Phase 1 become the data that is analyzed in Phase 2, and the results of Phase 2 become the data that is analyzed in Phase 3. For this reason, coding of arguments in Phase 1 was the focus of the reliability check.

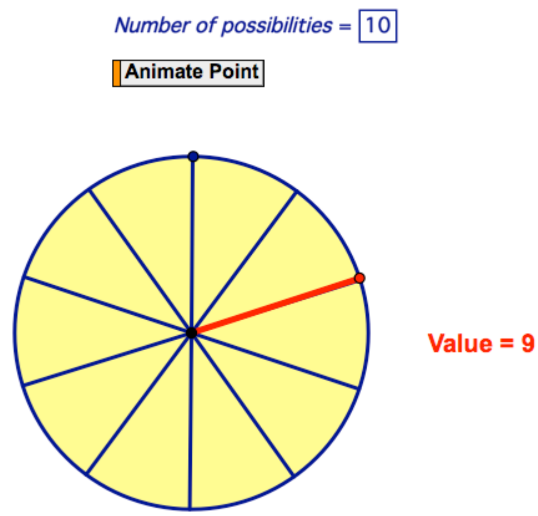
George Sweeney was a qualified coder of arguments, having had extensive experience with the methodology. At the same time, he was an outside researcher whose research had been conducted in inquiry-oriented linear algebra. He was not intimately familiar with the mathematics or with students' mathematical thinking in the mathematics content course, although he had taught the course once in Fall Semester 2007. (The reliability check was done in Spring 2012.) Thus, George received training to prepare him code data from a different sort of mathematics class.

I sent George a sample video clip and transcript excerpt, involving only a single argument, and asked him to code it. We then discussed the argument and compared our thinking about the coding. He also asked more general questions regarding how I thought about aspects of the coding process. A few days later, I sent George another sample of data, this one consisting of three arguments. We compared coding and had a similar conversation. Although not assessed formally, we agreed on most coding decisions pertaining to these arguments. At that point, we agreed that we were ready to do a reliability check.

To select a suitable subset of the data, I first identified whole-class discussions consisting of substantial numbers of arguments within a particular content strand. By contrast, there were days in which content was touched on but only a small number of arguments relevant to that strand were made. (For example, there was one argument belonging to the multiplication/ division strand made on Day 4. By contrast, there were 17 arguments belonging to that strand of activity made on Day 14.) This process resulted in a list of 10 days worth of content strand-specific discussions (see Figure 5). Five of these belonged to the place value, addition, and subtraction strand, and five belonged to the multiplication and division strand. Although not the entire data set, these days accounted for 191 of the 208 arguments that I had coded, or 92% of the data set.

1. PV Day 6
2. PV Day 7
3. PV Day 8
4. AS Day 11
5. AS Day 12
6. MD Day 13
7. MD Day 14
8. MD Day 15
9. MD Day 16
10. MD Day 17

Days of focal class activity



Spinner.gsp

Figure 5. Random selection of classroom data.

Using the spinner, I randomly selected one of the ten days of class. The result was Day 16, which was focused on multiplication ideas. I sent George the video and transcript of Day 16. The video was edited to include only whole-class discussions. George coded the data, and then I sent him my coding of the same data. He compared how each argument had been coded and assigned scores. He then tabulated these to get a total reliability score. Scoring was based on the number of components of an argument. For example, an argument consisting of claim, data, and warrant was worth 3 points, whereas an argument that also included backing was worth 4 points due to the additional component.

According to my coding, there were 18 arguments made on Day 16. Initially, George and I agreed on 39 of 52 components of these arguments, or 75% of coding decisions. We met via Skype to discuss the coding of each argument. In most cases in which our coding did not match exactly, the differences were matters of (a) coder discretion, i.e., there is not always one right way to code an argument, or (b) my coding being more fine-grained due to attunement to particular student ideas. As we discussed arguments, we either came to agreement or could not reach agreement. After discussion, we agreed on how to code 48 of 52 components (92%).

The frequency of initial inter-rater agreement achieved on Day 16 was not unreasonable. This type of coding is difficult. In contrast with the coding of mental computation interview data for strategies, coding of arguments was far more fine-grained and was not as simple as applying an established scheme. It required greater sensitivity to nuanced of students' mathematical thinking. However, 75% agreement was less than the 80% that was desired, and it seemed unsatisfactory. So, I asked George to code an additional day of class. I purposively selected Day 17 for the second reliability check because it involved similar content to Day 16, and I attributed many of our cases of initial disagreement to George's lack of familiarity with the nuances of prospective elementary teachers' mathematical thinking in the context of an elementary mathematics content course. Having coded Day 16 and discussed these arguments, I expected that George would be well prepared to code another set of arguments involving multiplication ideas.

According to my coding, there were 14 arguments made on Day 16. George and I agreed on 32 of the 37 components (86.5%) of this set of arguments. We again met via Skype to discuss the coding of each argument. This time around, the cases in which our coding did not match exactly were mostly due to coder discretion. Of the 5 differences in coding of components, I believe that 2 could have been alleviated with further training. I attribute one instance to George inferring a warrant without sufficient evidence. Upon discussion, he agreed with me that this evidence was lacking. The other instance was one in which George's coding emphasized a rational-number idea that was not of interest to the present analysis. The other 3 discrepancies were instances of coder discretion. For example, in one instances, George had coded as data in an argument the idea that I saw as the warrant, and vice versa. We both had sensible reasons for our coding. Upon discussion, he agreed with my version of the coding.

In the end, 2 of 10 days worth of content-specific discussion were subjected to a reliability check. In particular, a total of 32 arguments, or 15.4% of the total data set were double-coded. The level of initial agreement on the first round reliability check (Day 16) was 75%. In the second round (Day 17), 86.5% initial agreement was achieved. George and discussed why we thought the frequency of agreement had improved on Day 17. We agreed that it was due to two factors: (1) his improved understanding of my approach to the coding of arguments from this class generally, as a result of our discussion of coding decisions regarding the Day 16 data; and (2) his increased familiarity with content-specific ideas and subtleties of students' reasoning, which also resulted from his coding of Day 16, together with our subsequent discussion.

If George were to code discussions of addition and subtraction ideas or of place value ideas, I would expect on the basis of the above results that our initial agreement would be somewhere in the neighborhood of 75–80%. As with the coding of multiplication discussions, if he was given a second day of discussion belonging to the same content strand, I would expect a similar jump in reliability to somewhere in the 80–90% range.

Validity

The validity of an account concerns its accuracy, or as Maxwell (2005) puts it, "How might you be wrong?" (p. 105). Thus, validity is tied to the kinds of claims that result from the study. In order to guard against validity threats in qualitative research, one first must identify potential threats. Below, I discuss issues of validity and the measures that I took generally. I then discuss validity threats with respect to the different analyses that I conducted and their respective findings.

General discussion of validity. The three principal ways that my design guarded against threats to validity were by collecting rich data, by searching for discrepant evidence and negative cases, and via triangulation (Maxwell, 2005). The qualitative analogues to external validity are *comparability* and *translatability*. Other researchers should be able to compare the results of a study with the results of other studies, and other researchers should be able to make sense of the theoretical constructs, definitions, and methods used (LeCompte & Preissle, 1993). As discussed below, comparability and translatability are strengths of this study.

The most inferential aspects of my methodological approach are also those for which I will have the richest data from the most sources. I searched for discrepant evidence to guard against making inferences that would have been contradicted by that evidence. In the analysis of collective activity, a large number of arguments were analyzed, and the coding in terms of argumentation diagrams included multiple passes in which coding decisions were refined and related arguments were compared for consistency of coding. In the ontogenetical analyses, a variety of data sources were used to piece together accounts of the case students' developing reasoning. Much of this data is presented in these accounts, both in raw form (transcript or scan of student work) and in coded form (argumentation diagrams).

In the following subsections, I discuss validity issues specific to the various analyses.

Analysis of interview data. It was clear from the reliability check that the analysis of mental computation interview data was reliable. Thus, I would say with confidence that the participants did, in fact, use the strategies that I claim they used in the interviews (where *strategies* here refers to codes belonging to the scheme that I devised). So, how might I be wrong? A potential threat arises in the broader conclusions that might be drawn from these results. I do not claim that the seven interview participants constituted a representative sample of the students in the class, let alone of the population of prospective elementary teachers. In fact, I would not apply a statistical sampling framing even to this part of the study.

We know that the interview volunteers represented a range of ability levels (as results presented in Chapter 4 indicate). At the same time, as a group, they tended to rely on MASAs. In other words, in terms of the measures used, the interview participants looked the way the literature suggests a group of prospective elementary teachers would look. The participants also performed similarly to those who participated in the 2005 study. This is all to say that, while I do not regard them as representative of the population, the data suggest that the interview participants were not misrepresentative of the population. That is, the set of volunteers offered me the potential to study prospective elementary teachers' number sense development on the individual level. (By contrast, if the interview volunteers had turned out to be a group of skilled, flexible mental calculators at the beginning of the course, then there would have been serious concerns regarding that potential.)

The analysis of the mental computation interview data laid the foundations for the further analyses, which were of primary interest. This analysis was used to ask whether the number sense of at least some of the prospective elementary teachers improved substantially over the course of the semester. If so, then the class and the interview participants provided an appropriate setting for the further analyses. In addition, the analysis of the mental computation interview data aided in the selection of case studies.

What about specific claims that I make concerning the flexibility of the participants when performing each operation mentally? How could these claims be wrong? Since a small number of computations were used to investigate the range of strategies available to the participants, there is the potential that categorizing of participants as Inflexible or Semiflexible, as opposed to Flexible, could underrepresent their flexibility. In particular, it could be that a participant uses the MASA for all four addition computations in the interview, but she would have used some other strategy if given just the right pair of numbers. Thus, if more computations had been used, her ability to use another strategy might have been uncovered.

The above threat was well addressed by the study design. The pairs of numbers given were chosen to represent a range of cases, which would lend themselves to different strategies. Benchmark numbers, like 99 and 25, were used in the later computations for each operation since these are most likely to elicit specialized strategies. (The data bore this out. In the first interview, it was often he case that participants used the MASAs, except for computations involving benchmark numbers.) Given the range of affordances of the given numbers, it becomes highly unlikely that a person making a choice of strategy based on the given numbers would end up using the same strategy for each computation.

If eight or ten computations had been given for each operation, it is possible that participants who I categorized as Inflexible would at some point have used a second strategy. It is also possible, and in fact more likely, that individuals who I categorized as Flexible would have used wider ranges of strategies if given the opportunity. (Clearly, if a person has five or more strategies that they use for a given operation, then observing four computations would not give the person the opportunity to display their full range of strategies.) A design involving more computations, then, would have lent itself to a more fine-grained scheme. Instead of categorizing participants' operation-specific reasoning into one of three categories (Inflexible, Semiflexible, or Flexible) I might have used five or six categories instead. Given that prospective elementary teachers tend to reason inflexibly in mental computation, the scheme that was used seems appropriate. I was interested in distinguishing between people who would use one, two, or three different strategies for the range of problems given for each operation. For the purposes of the study, there seemed to be little risk in grouping together students who were highly unlikely to make a choice of strategy with those who would never make a choice of strategy. Similarly, there was little risk in grouping together students who used three or four strategies with those who might have used five or more if given the opportunity.

Comparability and translatability are strengths of the assessment of change in number sense because the analysis involved established methods from the number sense literature. In particular, I coded participants' strategies based on a scheme that is similar to other coding schemes for mental computation strategies. These results were then used to analyze change in the participants' flexibility and in the group's overall tendency to use nonstandard strategies. These measures are similar to those that have been used by other researchers. As more research is done concerning the number sense of prospective elementary teachers, as well as other populations, researchers will be able to use these or similar measures and to compare the results.

Analysis of collective activity. In terms of the analysis of collective activity, the relevant threat would be the possibility of drawing erroneous conclusions in Phase 2 (identifying as-if shared ideas) as a result of inconsistencies in Phase 1 coding (of arguments). Specifically, coding mistakes could result in false positives (Type 1 errors), i.e., mistakenly identifying ideas as functioning as if shared when they did not actually function as if shared. Coding mistakes could also result in false negatives, i.e., mistakenly concluding that an idea did not function as if shared when, in fact, it did function as if shared. These seem to be the greatest threats.

In applying the methodology, I was acutely aware that I was using it in a different sort of mathematics class, and I was concerned with these threats. So, I made a concered effort to guard against them. I also made the choice to err on the side of false negatives, rather than false positives. I decided that mistakenly identifying an idea as functioning as if shared was worse than failing to identify an as-if shared idea as functioning as if shared. I attempted to guard against making either type of error by not trusting the criteria for as-if shared ideas. I was concerned that, given differences between the math content course and an inquiry-oriented differential equations or linear algebra course, false positives could result from superficial differences in arguments. Maybe an idea would occur as data in one argument and warrant in another due to some characteristic of the nature of the task or the structure of the argument that would not actually constitute evidence for an idea functioning as-if shared. I guarded against this threat in Phase 1 by making consistency of coding a priority. In refiningy my coding of arguments, I looked across arguments that were similar in content and/or structure and asked whether my coding was consistent with respect to those similarities. I also traced particular ideas that occurred repeatedly in arguments and asked whether the same idea was truly at work in each of them. In some cases, these tests of consistency led to new distinctions between ideas and revisions to the coding of sets of arguments.

In Phase 2, I guarded against making mistakes by applying a commonsense notion of functioning as-if shared and testing candidate ideas against it. After I had compiled the argumentation log for a given content strand, I created a table of occurrences of ideas. For each idea that occurred more than once in the argumentation log, I tabulated its occurrences as claim, data, warrant, or backing by listing the argument numbers under the appropriate column. Once this table was filled in, I proceeded to consider whether each idea functioned as if shared. If the table showed that an idea had occurred as more than one type of component (e.g., as both data and warrant), then Criterion 2 (the shift criterion) was apparently satisfied. However, I did not conclude that the idea functioned as if shared. Rather, I marked it as a *candidate* for satisfying Criterion 2. I then went back to the details of each instance of that idea to ask myself whether a legitimate shift in the function of that idea had taken place. If I could describe in layman's terms how the idea was used differently in a pair of arguments, and in the later instance the idea was used in service of more advanced mathematical activity, *then* I considered the idea as legitimately satisfying the criterion. Thus, by not applying the as-if shared criteria blindly, but instead being careful and conscientious in my approach, I strived to avoid errors.

Case studies. Validity threats arising from the case studies take two forms: (1) the validity of the accounts of the cases themselves, and (2) the validity of generalizations that might be made from those cases. In other words, these concern the qualitative analogues of internal validity and external validity (Merriam, 1998). The accounts of the two cases presented in Chapter 6 are sufficiently detailed and transparent, in terms of the evidence presented and the inferences made, that the reader should be able to judge their validity. My purpose in these analyses was to understand the processes by which the case students' reasoning developed. The themes that I identified emerged from the analyses as I attempted to make sense of patterns and changes in Brandy and Valerie's reasoning. These accounts are characterized by thick description, seeking out discrepant evidence, and triangulation (Maxwell, 2005).

Yin (1994) distinguishes *analytic generalization* from *statistical generalization*. The former refers to generalization to theory, which is appropriate to case study research. The latter refers to generalization to a larger population, which is not appropriate to case study research. Qualitative researchers think differently about the notion of generalizability: “A single case or small, nonrandom purposeful sample is selected precisely because the researcher wishes to understand the particular in depth, not to find out what is generally true of the many” (Yin, 1994, p. 224). It is in the extent to which the details of the case overlap with others that understanding these becomes useful. Consumers of the research can judge for themselves the applicability of particular findings from a case study (Merriam, 1998; Yin, 1994).

From among the interview participants, and informed by the previous analyses, I selected two cases. These were not selected as typical cases of prospective elementary teachers developing number sense. Rather, they were selected as interesting cases with the potential to contribute to what is known about the phenomenon. In other words, they were selected for their potential to inform theory. The relevant external validity threat, then, lies in the possibility of making inappropriate generalizations to the local instruction theory. For the purposes of the dissertation study itself, I was careful not to make broad generalizations on the basis of characteristics of the case studies. For the most part, I limited my claims to the cases themselves. In terms of the larger research program, generalizing from cases to theory is certainly of interest, but this will be done cautiously.

Summary of Reliability and Validity

In the interest of reliability and validity, I followed recommendations concerning the design and conduct of qualitative research. Other researchers were involved in various stages of the process, including conversations concerning interpretations of data, discussions of methodological issues, and reliability checks. I collected rich data, searching for discrepant evidence and negative cases, and used triangulation when possible. I identified potential validity threats and took measures to address them. The kinds of conclusions that were drawn from the analyses were appropriate to the design of the study.

Conclusion

I conceptualized the research design in terms of four broad analytic tasks: Assessing Change, Microgenetic Analysis, Sociogenetic Analysis, and Ontogenetic Analysis. Change in the prospective teachers' number sense was assessed using established pre/post measures. These results informed the selection of case study students. The microgenetic analysis involved classroom data and individual data. The products of this analysis became the data to be analyzed in the sociogenetic and ontogenetic phases. The products of the sociogenetic phase informed the ontogenetic analysis in the sense of pointing to ideas of interest. The assessment of change also informed the ontogenetic analysis in that the pre/post snapshots were conceived as milestones in developmental trajectories.

Chapter 4 presents the results of pre/post analyses of survey data and mental computation interview data. The first section essentially duplicates previous study results, showing that the number sense of the participants improved. The second section introduces new innovations in the analysis of number sense development. Chapter 5 presents the results of the sociogenetic analysis. I describe the progression through classroom mathematical practices in the two major content strands. Chapter 6 presents the results of the two case study analyses. These ontogenetic analyses parallel the sociogenetic in terms of being organized around the same content strands. In Chapter 7, I summarize and discuss the study results. I also address implications for teaching and directions for future research.

Chapter 4: Results Part 1: Old and New Analyses

Research Question 1 asks: As prospective elementary teachers participate in a mathematics content course designed to support their development of number sense, how does the number sense of individuals evolve? As a first step toward answering this question, I describe ways in which the number sense of the study participants changed, both qualitatively and quantitatively, based on pre/post comparisons. I also present a new analytic tool and the results of a microgenetic analysis that afford progress in the investigation of number sense development.

The section titled Participants' Improved Number Sense presents results of the two survey instruments, the NSRS and SPS. These were both administered to 34 students pre- and post-instruction. I present results of quantitative analyses and interpret these with respect to the first research question. I also present results of the analysis of data from mental computation interviews with the seven interview participants as: (1) analysis of change in the participants' flexibility and (2) analysis of the types of strategies (Standard, Transition, Nonstandard, or Nonstandard with Reformulation) used by the participants for each operation.

The section titled Mental Computation Interview Results presents a new analytic tool in the analysis of number sense development: *strategy range profiles*. I characterize various types of profiles that occurred and then report on the specific strategy ranges of each participant. This section also includes the results of the analysis of strategy-arguments, including several examples of these. The strategy-arguments highlight mathematical ideas that are important for understanding how the participants thought about their strategies. These ideas are of particular relevance to the sociogenetic and ontogenetic analyses to be presented in Chapters 5 and 6, respectively.

Participants' Improved Number Sense

This section presents results of pre/post analyses of the participants' number sense. This includes results of surveys administered to the class. It also includes results of analyses of mental computation interviews conducted with seven participants. Most of these results constitute a duplication of results of the previous study. They provide evidence of the study participants' improved number sense.

Survey Results: Number Sense Rating Scale

A total of 34 students completed the NSRS both pre- and post-instruction. Of the 37 points possible on the test, the mean score on the pretest was 24 pts (65%). The mean score on the posttest was 29.4 pts (79%). The difference between the mean scores was statistically significant ($p < 0.0001$), and the effect size was greater than one standard deviation (Cohen's $d = 1.27$). The distribution of students' gain scores appear in Figure 6, as do the details of the hypothesis test.

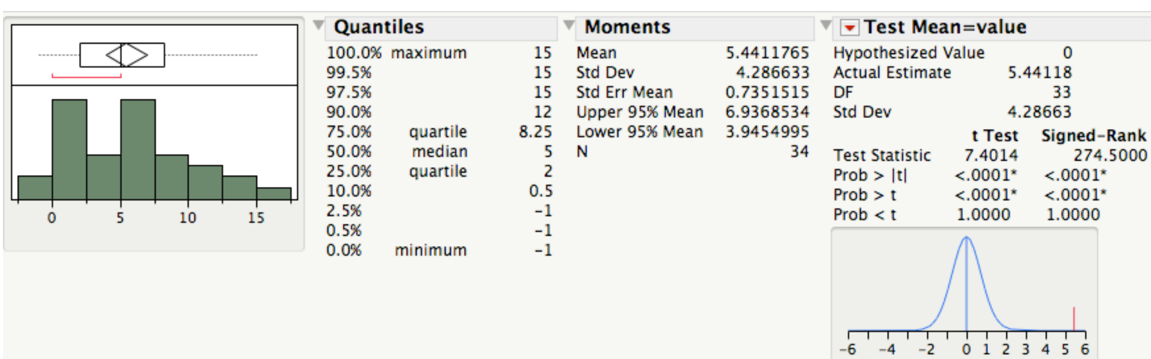


Figure 6. Distribution of difference scores in Fall 2010.

For the purposes of further analyses, it is useful to group the study participants based on their pretest scores. Pretest scores ranged from 11 to 35 points (30%–95%), with the majority of participants (25 of 34) scoring in the range from 20 to 29 points (54%–78%). We partition these scores into three categories: below 22, from 22 to 27, and above 27. Partitioning in this way yields approximately equal-sized groups, which we will refer to as *low*, *middle*, and *high*. The distribution of scores into the low, middle, and high groups is given in Table 6.

Table 6. Low, Middle, and High NSRS Pretest Scores

Group	Range	Count
Low	< 22	11
Middle	22-27	11
High	> 27	12

We will revisit the low, middle, and high groupings in Chapter 6 in the discussion of case studies. For the purposes of the present chapter, these groupings help in comparing the learning gains of students in the class involved in the study to those of students in previous classes.

As part of pilot data collection for the dissertation study, the NSRS was administered to students in all three sections of the same math content course that were offered during the semester preceding the study. In that semester, a total of 56 students took the NSRS both pre and post. For that group of students, the mean NSRS score also increased from pre to post to a statistically significant extent. However, the amount of improvement was not as substantial, at less than one standard deviation (Cohen's d = 0.95). (See Figure 7).

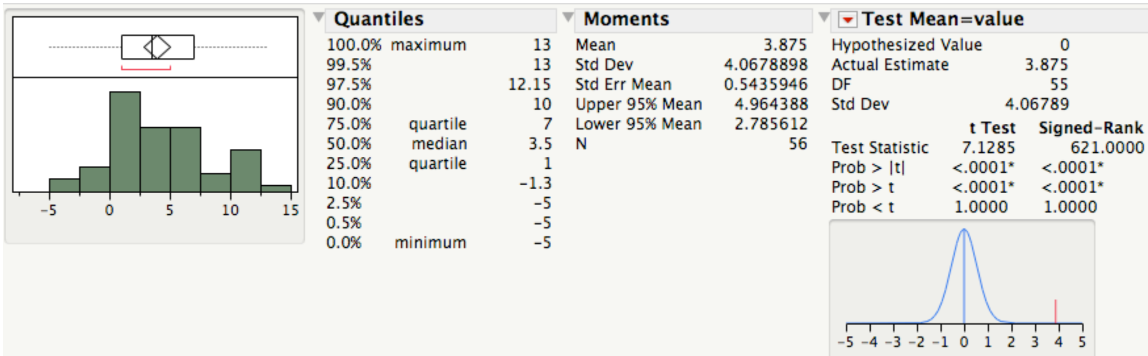


Figure 7. Distribution of difference scores in spring 2010.

In both groups of participants, gains were much greater for students with lower pretest scores. Thus, low-scoring students who improve substantially carry a lot of weight in the mean gain score. As it turns out, using the same cutoff scores for low, middle, and high as described above, there were approximately equal numbers of students in each group in spring. There were actually more students in the low group (21) than in the middle (17) or high groups (18). Nonetheless, as Table 7 shows, the mean gain score was greater for the fall study participants than for the spring participants in each group—low, middle, and high.

Table 7. Comparison of Fall 2010 and Spring 2010 Gain Scores by Group

Group	Spring 2010			Fall 2010		
	Mean	Std Dev	Count	Mean	Std Dev	Count
Low	6.67	3.75	21	9.36	4.57	11
Middle	3.24	3.33	17	4.27	2.87	11
High	1.22	3.02	18	2.92	2.23	12

The data in Table 7 show that students in all three groups improved more during the fall teaching experiment than did their counterparts in the spring classes. These results help to establish (a) that study participants' number sense improved and (b) that their number sense improved to an extent that merits attention.

We now take a closer look at the NSRS results for the class that participated in the teaching experiment.

NSRS Results by Number Domain. The NSRS consists of approximately equal numbers of questions focused on whole numbers, fractions, and decimals. For our purposes, we will simply distinguish whole-number items from rational-number items. Of the 37 test items, 12 were whole-number items and 25 were rational-number items. On the whole-number items, the mean gain was 4.33 correct responses. That is, on average, 4.33 more students answered a given whole-number item correctly on the posttest. The mean gain on rational-number items was 5.32 correct responses. Thus, the class improved substantially on both sets of items. Improvement on rational-number items was greater.

I offer two examples of whole-number items. Item 9 asked students to reason about the difference between two products. (See Figure 8.) This was a relatively difficult item, which only 15 of 34 students (44%) answered correctly on the pretest. On the posttest, 26 of 34 students (76%) answered it correctly. This result suggests improvement in students' reasoning about products of whole numbers, particularly their ability to account for the effect of changing one of the products.

93 x 134 = 12462. How much greater than 12462 is the product of 93 and 135? a) 93 b) 134 c) 135 d) I can't tell without calculating

Figure 8. NSRS Item #9.

Item 25 asked students to choose the closest estimate of the product of 18 and 19 (see Figure 9). On the pretest, 16 of 34 students (47%) answered this item correctly. On the posttest, 23 of 34 students (68%) answered correctly. Given that this test was multiple-choice and that written work was not allowed, we can only conjecture as to how students' reasoning changed. For example, students might have reasoned that this product was close to but less than 400 by comparing it to the product of 20 and 20. Alternatively, students may have developed the ability to perform mental multiplication easily enough that they computed the exact product of 18 and 19 and then answered on that basis. How students' reasoning specifically changed will be addressed in detail later.

25. Which answer is the product of 18 and 19 closest to?
a) 250 b) 350 c) 450 d) 550

Figure 9. NSRS Item #25.

Participants' scores on the NSRS pretest results were on par with those previously seen. Posttest results revealed significant improvement in students' scores. This test was not used to substantively investigate students' reasoning. It was used as an independent, baseline measure that afforded comparisons to this class and previous ones, as well as comparison of the interview participants to the class at large.

Results: Student Preference Survey

The same 34 students who took the NSRS pre and post also took the SPS (See Appendix 2) on both occasions. The SPS gave students a list of computations, four for each operation, (e.g., "78 – 34") and asked whether the student would perform the given computation mentally. Students responded by circling either "Yes" or "No." The "Yes" responses on the pre-survey varied according to the operation. Prior to instruction, students circled "Yes" for 93% of the addition computations and 90% of the subtraction computations. By contrast, they circled “Yes” for only 15% of multiplication computations and 40% of division computations. These data are summarized in Table 8. The disparity between responses by operation suggests that students entering the course saw the mental division tasks as more difficult than the addition or subtraction tasks and that they saw the multiplication tasks as most difficult.

Table 8. SPS Pre Survey “Yes” Counts by Operation

Operation	Yes mean pre	Yes as %
Addition	31.5	92.7%
Subtraction	30.5	89.7%
Multiplication	5	14.7%
Division	13.5	39.7%

On the post survey, students circled “Yes” for more than 90% of the addition and subtraction computations. They circled “Yes” for slightly more than 50% of the multiplication and division computations. These results are summarized in Table 9. Students’ post responses indicate that they saw the addition and subtraction computations as being of comparable difficulty to one another, and the multiplication and division computations as also being of comparable difficulty to one another. They viewed the multiplication and division computations as much more difficult than the addition and subtraction computations but not as difficult as they had regarded them in the pre survey.

There was no change in the frequency of “Yes” responses for addition, and only a slight increase in the case of subtraction. The frequency of “Yes” responses to the division computations increased by 12.5% (of the 34 students). The frequency of “Yes”

Table 9. SPS Post Survey "Yes" Counts by Operation

Operation	Yes count post mean	Yes count as %
Addition	31.5	92.7%
Subtraction	31.75	93.4%
Multiplication	18	52.9%
Division	17.75	52.2%

responses to the multiplication computations increased by 38.2% (of the 34 students). These increases were statistically significant. The results of the pre/post comparison are summarized in Table 10.

Table 10. Mean Change in "Yes" Responses by Operation

Operation	Yes Pre %	Yes Post %	Change (Post – Pre)	Prob < |t|
Addition	92.7%	92.7%	0%	1
Subtraction	89.7%	93.4%	3.7%	0.49
Multiplication	14.7%	52.9%	38.2%	0.0053*
Division	39.7%	52.2%	12.5%	0.0313*

According to the SPS results, most students in the class reported that they were willing to perform addition and subtraction of two- and three-digit numbers mentally at the beginning of the course. Though students saw the multiplication and division computations as more difficult than addition and subtraction, both pre and post, they more willing to mentally perform multiplication and division by the end of the course. The largest change came in students willing to perform multiplication. For example, only 4 of the 34 students circled “Yes” for 15 x 24 on the pre survey. More than 4 times as many students (17) circled “Yes” on the post survey.

It would not be fair to assume that students would, in fact, perform about 90% of the addition and subtraction computations mentally if these arose in everyday life. However, the differences in responses by operation suggest that they came to view the multiplication and division computations as less difficult by the end of the course than they had at the beginning.

We saw statistically significant increases in scores on both the NSRS and SPS. Increases in NSRS scores point to improved number sense. In particular, the increase in scores on a subset of the problems provides evidence of improved whole-number sense. Furthermore, the NSRS gains were greater than in Spring 2010, which suggests that something special happened in the fall class in terms of students’ number sense improvement. The SPS results suggest that students came to see mental multiplication and division as more accessible than they did initially.

The advantage to the SPS and NSRS is that they easily generated pre/post data for 34 students. The disadvantage to these instruments is that they provide limited insights into students’ reasoning. In what follows, we look at *how* a subset of the students performed mental computation and reasoned about those computations.

Mental Computation Interview Results

Seven students participated in whole-number mental computation interviews. I list the participants’ NSRS scores in order to provide a rough picture of how these seven students compared to the entire class, based on a common measure of their number sense. What follows are the results of analyses of the mental computation interview data, with the coding scheme strategies for the participants' strategies. The section ends with results concerning change in interview participant flexibility and change in the distributions of utilized strategies along the spectrum from Standard to Nonstandard.

Interview Participants' NSRS Scores

Seven students participated in pre/post mental computation interviews: Angela, Brandy, Zelda, Natalie, Trina, Maricela, and Valerie.[14] Table 11 presents the interview participants' NSRS pretest scores, identified as belonging to the Low, Middle, or High group. The table also shows the participants' gain score (post – pre), the mean gain score for the group, and the predicted gain score based on the participants' pretest score. (The predicted gain score is based on a linear regression model.)

On the basis of their pretest scores on the NSRS, the interview participants appear to reasonably represent the larger population of students in the class. There were two from the Low group, three from the Middle group, and two from the High group. The interview participants tended to improve more on the NSRS than did the larger population of students in the class. One possible explanation for this would be that their participation in the interviews positively influenced students' learning.

Participants' Mental Computation Strategies

For basic coding of addition, subtraction, and multiplication strategies, the previously developed scheme (Whitacre, 2006) was used and required only slight modifications. Codes for division strategies were developed through constant

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[14] These names are pseudonyms.

Table 11. Interview Participants' NSRS Scores and Gains

Student	Pre Score	Group	Actual Gain (Post – Pre)	Mean Gain for Group	Predicted Gain
Angela	18	Low	13	9.36	9.00
Brandy	21	Low	8	9.36	7.21
Zelda	23	Middle	9	4.27	6.02
Natalie	24	Middle	8	4.27	5.42
Trina	26	Middle	4	4.27	4.23
Maricela	28	High	5	2.92	3.04
Valerie	29	High	3	2.92	2.45

comparative analysis, and these were added to the scheme. Tables 12, 13, 14, and 15 present names and descriptions of the mental addition, subtraction, multiplication, and division strategies (respectively) that were used by the interview participants during the main mental computation tasks.

With this scheme in place, I report on the strategies that the interview participants used for the particular mental computation tasks.

Strategies Used Pre and Post

The mental addition strategies that participants used for the main Bobo tasks are presented in Tables 16 and 17. As Table 16 shows, the participants used the MASA for almost all mental addition computations in the first interview. Only three of the seven participants used any non-MASA strategy. By contrast, Table 17 shows that the participants used a wide variety of addition strategies for these same computations in the

Table 12. Participants' Mental Addition Strategies

Strategy	Description
MASA	The student used the mental analogue of the standard (US) addition algorithm. Language such as "carry the one" often accompanied the use of this strategy. Students generally used non-place-value language.
Right to Left (RtoL)	The student added place-value wise from right to left but did not necessarily picture the digits aligned, as in the standard algorithm. In contrast to the MASA, the student used place-value language.
Left to Right (LtoR)	The student added place-value wise from left to right. Typically, she used place-value language.
Aggregation (Agg)	The student began with one of the two addends and added the other one on in convenient chunks, generally working from big to small and keeping a running subtotal.
Giving	The student altered the problem such that part of one addend (usually a small number of ones) was added ("given") to the other prior to finding their sum.
Single Compensation (SC)	The student altered one of the two addends (usually rounding up or down to the nearest multiple of ten) prior to performing the addition. The student added the rounded

(table continues)

Table 12. continued

Strategy	Description
	numbers and then compensated for rounding.
Double Compensation (DC)	The student altered both addends (usually rounding them up or down to the nearest multiple of ten) prior to performing the addition. The student added the rounded numbers and then compensated for rounding.

Table 13. Participants' Mental Subtraction Strategies

Strategy	Description
MASA	The student used the mental analogue of the standard (US) subtraction algorithm. Language such as "borrowing" often accompanied the use of this strategy.
Right to Left	The student subtracted place-value-wise from right to left but without visualizing the numbers aligned as in the standard algorithm.
Left to Right	The student subtracted place-value-wise from left to right.
Aggregation	The student either (a) began with the subtrahend and added onto it in convenient chunks until the minuend was reached or (b) began with the minuend and subtracted off the subtrahend in convenient chunks. The student kept a cumulative mental record of the amount added or

(table continues)

Table 13. continued

Strategy	Description
	subtracted. In the case of adding on to the subtrahend, this amount gave the difference. In the case of subtracting from the minuend, the result gave the difference.
Minuend Compensation (MC)	The student altered the minuend (often rounding up or down to a multiple of ten) prior to performing the subtraction. The student found the difference between the subtrahend and rounded minuend and then compensated appropriately for rounding.
Invalid Subtrahend Compensation (Invalid SC)	The student altered the subtrahend (often rounding up or down to the nearest multiple of ten) prior to performing the subtraction. The student found the difference between the minuend and rounded subtrahend and then compensated for rounding. However, she compensated incorrectly: she subtracted from the difference to compensate for having added to the subtrahend, or she added to the difference to compensate for having subtracted from the subtrahend.
Valid Subtrahend Compensation	The student altered the subtrahend (often rounding up or down to the nearest multiple of ten) prior to performing

(table continues)

Table 13. continued

Strategy	Description
(Valid SC)	the subtraction. The student found the difference between the minuend and rounded subtrahend and then compensated for rounding. Specifically, she compensated correctly: she added to the difference to compensate for having added to the subtrahend, or she subtracted from the difference to compensate for having subtracted from the subtrahend.
Shifting the Difference	The student added the same amount to, or subtracted the same amount from, both the minuend and subtrahend. She then found the difference between the rounded numbers.

Table 14. Participants' Mental Multiplication Strategies

Strategy	Description
Invalid Partial Products (Invalid PP)	The student computed partial products and then added these together, but the correct set of partial products was not used. Typically, the student computed only the ones x ones and tens x tens.
MASA	The student used the mental analogue of the standard (US) multiplication algorithm.

(table continues)

Table 14. continued

Strategy	Description
	Students spoke in terms of digits, rather than using place-value language, and often made reference to the details of the written algorithm.
Partial Products (Valid PP)	The student decomposed one or both factors place-value-wise and then applied the distributive property of multiplication over addition. Most students who employed Partial Products used place-value language.
Nonstandard Additive Distribution (NAD)	The student decomposed one of the factors non-place-value-wise, and then applied the distributive property of multiplication over addition. Thus, the partial products added were not those that one would add using the Partial Products strategy or the standard algorithm.
Subtractive Distribution (SD)	The student applied the distributive property of multiplication over subtraction.

(table continues)

Table 14. continued

Strategy	Description
	This strategy was often used when there was a benchmark number slightly greater than either of the factors.
Quarters	For multiplication of 25, the student grouped 25's into 100's. For example, she computed 25 x 16 by reasoning that four 25's make 100, and 16 is 4 x 4, so that makes 4 hundreds.
Double Compensation (DC)	The student rounded both factors and then multiplied. She then performed two distinct compensation steps to account for the effects of rounding.

Table 15. Participants' Mental Division Strategies

Strategy	Description
Invalid Partial Quotients (Inv PQ)	The student partitioned the divisor, computed partial quotients (dividend divided by part of the divisor) and then added these together.

(table continues)

Table 15. continued

Strategy	Description
MASA	The student used the mental analogue of the standard (US) division algorithm. Students spoke in terms of digits, rather than using place-value language, and often made reference to the details of the written algorithm, e.g., "bringing down a zero."
Count by	The student counted by multiples of the divisor, either down from the dividend to zero or up from zero to the dividend.
Quarters	The student used a special case of a factorization strategy involving 25's. For example, to divide 600 by 25, she reasoned that there are four 25's in 100, and six 100's in 600. Therefore, the number of 25's is 4 x 6 = 24.
Subtractive Distribution (SD)	The student applied the multiplication strategy Subtractive Distribution, but worked backwards to determine the missing factor (the quotient).

Table 16. Participants' Mental Addition Strategies by Item, First Interview

Participant	**$37 & $52**	**$64 & $87**	**$96 & $157**	**$38 & $99**
Angela	MASA	MASA	MASA	DC
Brandy	MASA	MASA	MASA	MASA
Maricela	MASA	MASA	MASA	MASA
Natalie	MASA	MASA	MASA	MASA
Trina	MASA	MASA	LtoR	Giving
Valerie	MASA	MASA	MASA, (SC)	MASA, (SC)
Zelda	MASA	MASA	MASA	MASA

Table 17. Participants' Mental Addition Strategies by Item, Second Interview

Participant	**$37 & $52**	**$64 & $87**	**$96 & $157**	**$38 & $99**
Angela	MASA	MASA	Giving	Giving
Brandy	RtoL	LtoR	LtoR	SC
Maricela	MASA	MASA	MASA	MASA
Natalie	MASA (Agg)	MASA	SC	SC
Trina	RtoL	LtoR	LtoR	Giving
Valerie	MASA	MASA	MASA	SC
Zelda	DC	DC	SC	SC

second interview. Recall that participants were not asked for alternative strategies during this part of the interview. Alternative strategies are regarded as *Scaffolded Alternatives*, considered differently, since having the answer likely supported other ways of reasoning aobut the problem. In the event of a spontaneous alternative, it appears in parentheses after the code for the first strategy used.

Similar results occurred in subtraction. As a group, the 7 interview participants used the MASA almost exclusively for their mental subtraction computations in the first interview. Only 3 of the 7 participants used a non-MASA subtraction strategy, and only 2 of the 3 used a *valid* non-MASA subtraction strategy. In the second interview, by contrast, a wide variety of subtraction strategies were used. In fact, 6 of the 7 participants used at least one valid non-MASA subtraction strategy. These results are presented in Tables 18 and 19.

Table 18. Participants' Mental Subtraction Strategies by Item, First Interview

Participant	**$34 & $78**	**$52 & $178**	**$45 & $82**	**$49 & $125**
Angela	MASA	MASA	MASA	Valid SC
Brandy	MASA	MASA	MASA	MASA
Maricela	MASA	MASA	MASA	MASA
Natalie	MASA	MASA	MASA	Invalid SC
Trina	MASA	MASA	LtoR	LtoR
Valerie	MASA	MASA	MASA	MASA
Zelda	MASA	MASA	MASA	MASA

Table 19. Participants' Mental Subtraction Strategies by Item, Second Interview

Participant	$34 & $78	$52 & $178	$45 & $82	$49 & $125
Angela	MASA	MASA	Agg	Valid SC
Brandy	MASA	Agg (MASA)	MC	Valid SC
Maricela	MASA	MASA	Agg	Agg
Natalie	MASA	MASA	MC	Valid SC
Trina	LtoR	LtoR	MASA	Valid SC
Valerie	MASA	MASA	Agg	Agg
Zelda	MASA	MASA	MASA	Invalid SC

In multiplication, unlike addition and subtraction, participants used a variety of strategies in the first interview. The change seen from pre to post has less to do with flexibility and more to do with *which* strategies participants used. The group of participants used a total of five distinct, valid strategies in the first interview. However, MASA and Invalid Partial Products together were used for 15 of the 28 computations performed. In the second interview, by contrast, these were used by only 1 of the 7 participants. Instead, the valid Partial Products strategy became much more common (up from 2 to 12 instances), and Subtractive Distribution was used more often (up from 7 to 10 instances). These results appear in Tables 20 and 21.

In mental division, participants were reliant on the MASA in the first interview, using it for 21 of the 28 quotients computed. Other valid strategies were rare. In the second interview, the MASA was used less frequently (down from 21 to 16 instances),

Table 20. Participants' Mental Multiplication Strategies by Item, First Interview

Participant	**15 x 24**	**19 x 21**	**25 x 16**	**99 x 15**
Angela	MASA	SD	MASA	SD
Brandy	Invalid PP	n/a	Quarters	SD
Maricela	MASA	MASA	MASA	MASA
Natalie	MASA	Valid PP	NAD	SD
Trina	Valid PP	SD	Quarters	SD
Valerie	Invalid PP	Invalid PP	Invalid PP	Invalid PP
Zelda	MASA	MASA	MASA	SD

Table 21. Participants' Mental Multiplication Strategies by Item, Second Interview

Participant	**15 x 24**	**19 x 21**	**25 x 16**	**99 x 15**
Angela	Valid PP	PP	Valid PP	SD
Brandy	Valid PP	SD	Quarters	SD
Maricela	Valid PP	PP	SD	SD
Natalie	Valid PP	Estimate	Valid PP	SD
Trina	DC	SD	Quarters	SD
Valerie	Invalid PP	MASA	Valid PP (Quarters)	SD
Zelda	Valid PP	Valid PP	Valid PP	SD

and all 7 participants used the Quarters strategy to compute 275 ÷ 25 (up from 3 to 7). Overall, however, there was less change change seen in participants' mental division than in the other operations. These results appear in Tables 22 and 23.

Table 22. Participants' Mental Division Strategies by Item, First Interview

Participant	**420 ÷ 14**	**570 ÷ 30**	**512 ÷ 16**	**275 ÷ 25**
Angela	MASA	MASA	MASA	MASA
Brandy	MASA	Count by	n/a	Quarters
Maricela	MASA	MASA	MASA	MASA
Natalie	Invalid	MASA	MASA	Quarters
Trina	MASA	MASA	MASA	Quarters
Valerie	MASA	MASA	MASA	MASA
Zelda	MASA	MASA	MASA	Invalid

Table 23. Participants' Mental Division Strategies by Item, Second Interview

Participant	**420 ÷ 14**	**570 ÷ 30**	**512 ÷ 16**	**275 ÷ 25**
Angela	MASA	MASA	MASA	Quarters
Brandy	Invalid PQ	SD	Estimate	Quarters
Maricela	MASA	MASA	MASA	Quarters
Natalie	MASA	MASA	MASA	Quarters
Trina	MASA	MASA	MASA	Quarters
Valerie	MASA	MASA	MASA	Quarters
Zelda	MASA	n/a	Invalid PQ	Quarters

The results presented so far amount to a straightforward report of the strategies used by the interview participants, and these results begin to paint a picture of positive change. The upcoming sections present more detailed analyses focusing on two particular dimensions of the participants' mental computation, flexibility and strategy ranges. *Flexibility* refers to the variety of distinct, valid strategies used by the participant for a given operation. This is a categorical measure based on quantitative data: participants are categorized as Inflexible, Semiflexible, or Flexible depending on the number of distinct, valid strategies that they used for a given operation. *Strategy ranges* are qualitative. These describe the set of strategies the participant used by illustrating the distribution of these stategies along the Standard-to-Nonstandard spectrum.

Participant's Improved Flexibility

Overall, the participants became more flexible from the first to the second interview. However, the amount of change varied both by individual and by operation. For the operation addition, 6 of the 7 participants were Inflexible or Semiflexible in the first interview. By the second interview, 6 of the 7 participants were Semiflexible or Flexible. For the subtraction operation, the change was similar. All participants were Inflexible or Semiflexible in the first interview. By the second interview, 6 of 7 were Semiflexible or Flexible. In multiplication, 3 of the 7 participants were Inflexible in the first interview. By the second interview, all 7 were either Semiflexible or Flexible. In division, 4 participants were Inflexible in the first interview, and 3 were Semiflexible. By the second interview, all were Semiflexible. Thus, there was improvement in flexibility for each of the operations. These results are summarized in Tables 24 and 25.

Table 24. Participants' Flexibility by Operation in the First Interview

Participant	Addition	Subtraction	Multiplication	Division
Angela	Semiflexible	Semiflexible	Semiflexible	Inflexible
Brandy	Inflexible	Inflexible	Inflexible	Semiflexible
Maricela	Inflexible	Inflexible	Inflexible	Inflexible
Natalie	Inflexible	Inflexible	Flexible	Semiflexible
Trina	Flexible	Semiflexible	Flexible	Semiflexible
Valerie	Inflexible	Inflexible	Inflexible	Inflexible
Zelda	Inflexible	Inflexible	Semiflexible	Inflexible

Table 25. Participants' Flexibility by Operation in the Second Interview

Participant	Addition	Subtraction	Multiplication	Division
Angela	Semiflexible	Flexible	Semiflexible	Semiflexible
Brandy	Flexible	Flexible	Flexible	Semiflexible
Maricela	Inflexible	Semiflexible	Semiflexible	Semiflexible
Natalie	Semiflexible	Semiflexible	Flexible	Semiflexible
Trina	Flexible	Flexible	Flexible	Semiflexible
Valerie	Semiflexible	Semiflexible	Flexible	Semiflexible
Zelda	Flexible	Inflexible	Semiflexible	Semiflexible

For the sake of succinctly representing changes in flexibility, the three levels—Inflexible, Semiflexible, and Flexible—are assigned the values 0, 1, and 2, respectively. Shifts in flexibility from the first to the second interview are then described by the difference between the two scores (second – first). Thus, for each operation, each participant has a flexibility change score. These scores appear in Table 26.

Table 26. Participants' Flexibility Change Scores by Operation

Student	Addition	Subtraction	Multiplication	Division
Angela	0	+1	0	+1
Brandy	+2	+2	+2	0
Maricela	0	+1	+1	+1
Natalie	+1	+1	0	0
Trina	0	+1	0	0
Valerie	+1	+1	+2	+1
Zelda	+2	0	0	+1

These results demonstrate that the interview participants became more flexible in mental computation. Each participant became more flexible in at least one operation, and six of the seven participants became more flexible in two or three operations.

Shift from Standard toward Nonstandard

Interview participants' strategies were also coded as belonging to one of the four categories, Standard, Transition, Nonstandard, or Nonstandard with Reformulation. For each operation, the Standard strategy (the MASA) was the most common strategy used in the first interview. Participants were posed a total of 112 mental computations in the first interview (7 participants, 4 operations, 4 problems per operation). In total, there were 102 instances of valid strategy use. Participants used MASAs for 80 of these. Only in the case of multiplication were other strategies used with substantial frequency. Overall, the participants' were reliant on the standard algorithms. These data appear in Table 27.

Table 27. Strategy Use by Category in the First Interview

Operation	Standard	Transition	Nonstandard	Nonstd w/Ref.
Add. Pre	25	1	0	2
Sub. Pre	24	2	0	1
Mult. Pre	10	0	4	8
Div. Pre	21	1	3	0
Total Pre	80	4	7	11

In the second interview, there were 104 instances of valid strategy use. Participants used MASAs for 39 of these. Instances of Nonstandard strategy use (with and without reformulation) totaled 56. The MASA for multiplication was used only once. These data appear in Table 28. Figure 10 represents the total instances of strategies used in each category, pre and post. The light gray graph shows how common the MASAs were in the first interview. The blue (or darker gray) graph shows that the participants used a more balanced set of strategies in the second interview.

This shift from Standard toward Nonstandard is similar to that seen in the 2005 study. As in the previous study, the picture that emerges from the second interview is a desirable one. In some cases, the mental analogues allow for quick and easy computation. However, the first interview results reflect over-reliance on these strategies. The

Table 28. Strategy Use by Category in the Second Interview

Operation	Standard	Transition	Nonstandard	Nonstd w/Ref.
Add. Post	11	6	0	11
Sub. Post	12	2	6	6
Mult. Post	1	0	14	11
Div. Post	15	1	7	1
Total Post	39	9	27	29

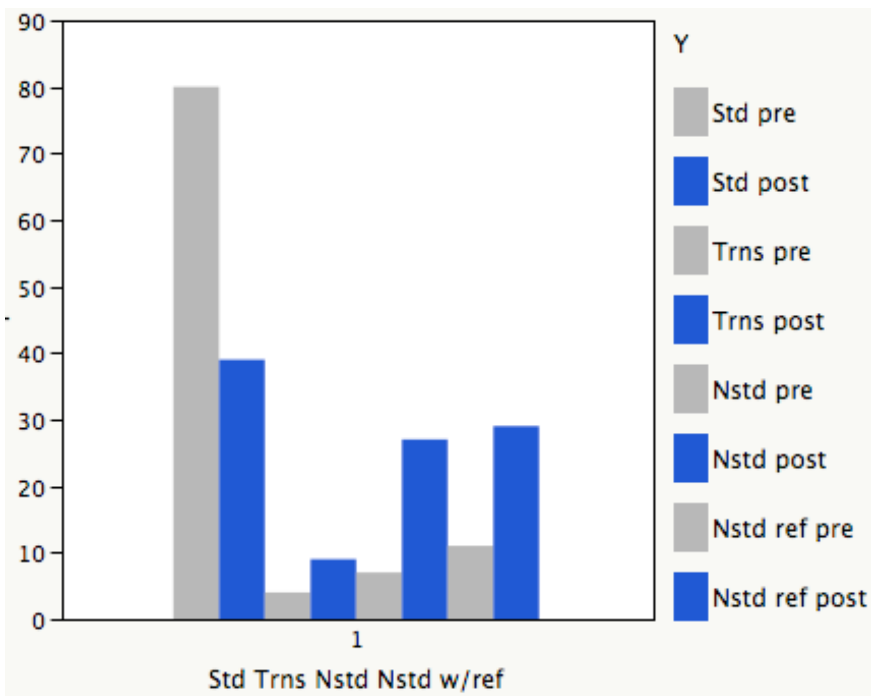


Figure 10. Shift from standard toward nonstandard.

participants were posed problems with various affordances. In the first interview, they rarely capitalized on these affordances. In the second interview, they often did.

Summary of Participants' Improved Number Sense

The participant's number sense was examined in several ways. All 34 participants took the Number Sense Rating Scale (NSRS) and Student Preference Survey (SPS) pre and post-instruction. Students' mean score increased by more than one standard deviation and the mean gain score was greater than in nonintervention classes of a previous semester. Students in all the groups—Low, Middle, and High–showed improvement. The Student Preference Survey results indicate that students came to see multiplication and division as less difficult than they had at the beginning of the semester. Students reported increased willingness to perform computations mentally, especially in the case of multiplication. The inclination to actually apply one's understanding of numbers and operations is an aspect of number sense. Thus, these results provide additional evidence of improved number sense.

Interview participants demonstrated increased flexibility in whole-number mental computation. (i.e., the interview participants came to use a wider variety of strategies for a given operation). In Chapter 6, we revisit the issue of flexibility and shift our attention to flexibility within problems, addressing participants' abilities to use a variety of strategies for the same computation.

In addition to becoming more flexible, the interview participants came to use nonstandard strategies far more often. That is, the distribution of strategies used shifted decidedly towards the nonstandard end of the spectrum. Nonstandard strategies are associated with number sense because they require students to recruit their understanding of numbers and operations in reasoning about how to perform the computation. In the next section, I present participants' justifications for their nonstandard strategies. In this way, the study does not take the use of nonstandard strategies as a proxy for understanding. On the contrary, participants' relevant mathematical understandings were directly investigated.

The results presented thus far provide evidence *that* the participants' number sense improved in a second instantiation of a classroom teaching experiment. Five years after the previous study, with a new group of students and a different instructor teaching the course, prospective elementary teachers are developing improved number sense in a course informed by the local instruction theory for number sense development.

The results presented thus far raise the question. To begin to address the question of *how* the study participants' number sense improved, the Chapter 5 presents an analysis of collective activity during the whole-number portion of the course. In order to integrate an analysis of collective activity and individual development, we revisit the pre/post mental computation activity of the seven interview participants through the lens of new analytic tools.

New Analytic Tools

Analytic tools are introduced, which will be used extensively for an understanding of the *development* of number sense. I begin by characterizing the range of strategies that interview participants used for mental computation. Then, the construct of strategy-arguments, a product of the microgenetic phase of analysis, is introduced. We identify the strategy-arguments that were articulated in interviews, focusing on the mathematical ideas that were used to justify participants' nonstandard strategies. These new tools and constructs afford more illuminating characterizations of change in the participants' conceptions and activity.

The previous analysis of interview participant's number sense was made in terms of two measures: (1) flexibility and (2) the distribution of strategies across the categories of Standard, Transition, Nonstandard, and Nonstandard with Reformulation. Flexibility was assessed at the individual level, and it was based on a count of distinct strategies available to the participant for a given operation. In the analysis of distribution of strategies, participants' responses were pooled, so that the shift that we observed from Standard toward Nonstandard was a characterization of change in the group of seven interview participants, rather than at the individual level.

For the purposes of ontogenetic analysis, we require a new analytic tool that affords a finer-grained examination of individuals' conceptions and activity.

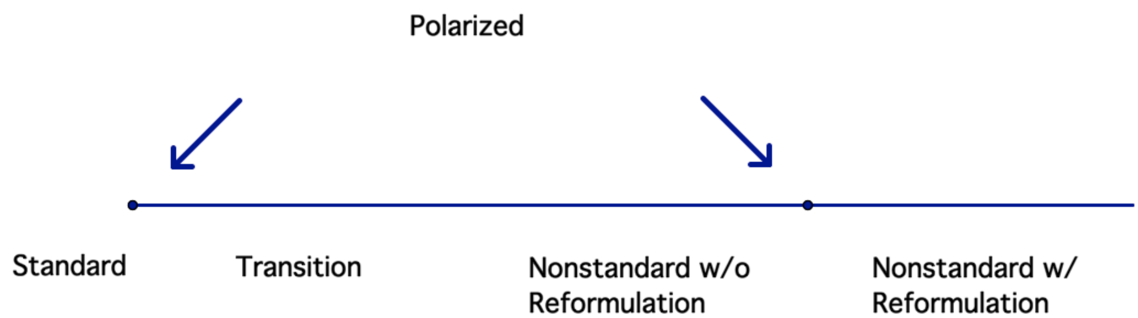
Participants' Strategy Ranges

We describe the ranges of strategies that participants used for mental computation in terms of the spectrum from Standard to Nonstandard. For each characterization of a strategy distribution along that spectrum, we define a *Strategy Range Profile.* The strategy range profiles that arose in the data are described in Table 29. The figures accompanying each profile depict the participants' strategy range pictorially along the Standard-to-Nonstandard spectrum.

All strategy range profiles are profiles of participants' ranges of *valid* strategies. Invalid strategies are not profiled. For participants who used invalid strategies, I indicate parenthetically that their range is limited (i.e., the participant's valid strategy range is a subset of her total strategy range).

Table 29. Strategy Range Profiles

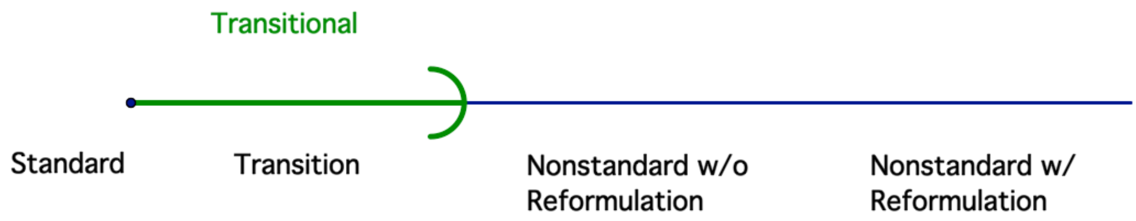
MASA-bound: The student uses the MASA for all computations for a given operation.



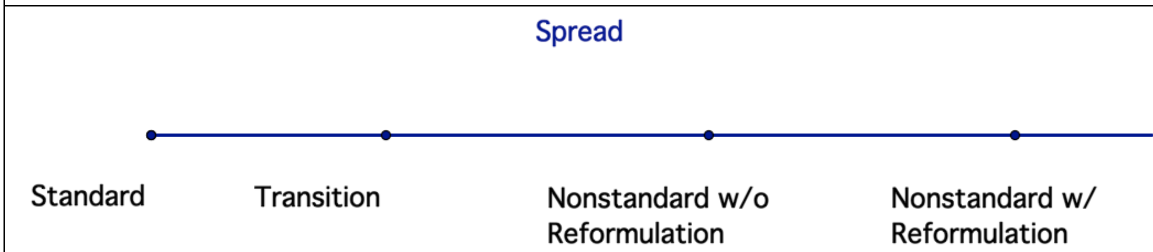
Polarized (X/Y): The student uses two distinct, valid strategies, and these are at opposite ends of the spectrum. Specifically, the student uses the MASA for some computations and a Nonstandard stategy for others. The computations for which a Nonstandard strategy is used tend to involve an obvious benchmark number, like 99. (X and Y refer to the counts of instances of the MASA versus the Nonstandard strategy.) If the nonstandard strategy is used in the special case in which a benchmark number is involved in the computation, this is coded by the notation *(benchmark)* following *Polarized (X/Y)*.

(table continues)

Table 29. continued



Transitional (X/Y): The student uses transition strategies. She may also use the MASA, but she does not use any nonstandard strategy. (X and Y refer to the counts of instances of the MASA versus Transition strategies.)

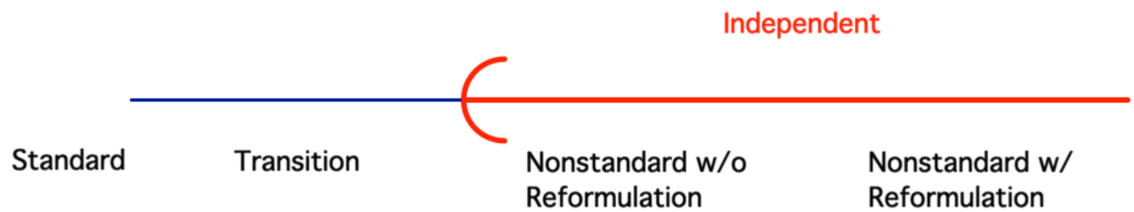


Spread: Student student uses the MASA, along with at least two distinct non-MASA strategies. These may or may not include Transition strategies.

Transition+ (X/Y): Student does not use the MASA. She uses all Transition and Nonstandard strategies. (X and Y refer to the counts of instances of Transition versus

(table continues)

Table 29. continued

Nonstandard strategies. Note: the student may use 2, 3, or 4 distinct strategies.)
 **Independent (Z):** Student uses Nonstandard strategies exclusively. (Z denotes the number of distinct nonstandard strategies that the student uses.)

Shifts in participants' strategy ranges. In this section, participants' strategy ranges are contrasted pre/post in order to characterize how these changed. To illimunate these contrasts, we also describe the nature of the change. The codes below are used to explicitly identify new strategies that a participant used or conditions under which nonstandard strategies were used:

> *Adopted* [Strategy]. The student adopted [Strategy]. That is, she had not used [Strategy] in the first interview, but did use [Strategy] at least once in the second interview. Adopted strategies included Aggregation (Agg.), Compensation (Comp.), Partial Products (PP), and Subtractive Distribution (SD).
>
> *Dropped Invalid*. The student has used an invalid strategy for the given operation in her first interview, and she did not use this strategy in her second interview. A valid strategy was used in its place.
>
> *Dropped MASA*. The student has used the MASA for the given operation in her first interview, and she did not use the MASA in her second interview. A Transition or Nonstandard strategy was used in its place.
>
> *Stable.* The student used the same range of strategies for the given operation in both interviews.

Benchmark. The student applied a new strategy to a special case of a computation, one involving a benchmark.

Benchmark sensitivity. The student expanded her benchmark tolerance. That is, she took a strategy that was previously only used in the case of an obvious benchmark (e.g., 99 in addition) and applied it to a less specialized case (e.g., 96 in addition).

Place value (PV). The student shows greater awareness of place value in reasoning about the operation. Typically, this shift coincides with students moving from reliance on the MASA to wider use of Transition strategies.

With the above scheme in place, changes in participants' strategy ranges can be described. Table 30 shows the specific strategy ranges used by each participant for addition computations in the first and second interview. In the first interview, 5 of the 7 participants were MASA-bound. Only one of these participants remained MASA-bound in the second interview.

Table 30. Shifts in Participants' Addition Strategy Ranges

Participant	Range Pre	Range Post	Shift Description
Angela	Polarized (3/1)	Polarized (2/2)	Benchmark sensitivity
Brandy	MASA-bound	Transition+(3/1)	PV, Benchmark
Maricela	MASA-bound	MASA-bound	Stable
Natalie	MASA-bound	Polarized (2/2)	Adopted Comp., Benchmark
Trina	Spread	Transition+(3/1)	PV
Valerie	MASA-bound	Polarized (3/1)	Benchmark
Zelda	MASA-bound	Independent (2)	Adopted Comp.

The shift descriptions shed some light on how the participants' reasoning changed. The most common change involved participants' recognition of benchmark numbers. They became more likely to recognize and take advantage of the affordances of numbers such as 99 that are close to benchmarks. There were also two participants whose improved strategy ranges reflect an improved understanding of place value, as well as two who adopted Compensation strategies. Brandy and Zelda stand out as compelling cases. Brandy moved from MASA-bound to Transition+. In the second interview, she used Transition strategies for 3 of the 4 addition computations, and she used Single Compensation to find the sum of 38 and 99. Brandy exhibited an improved understanding of place value, as well as the ability to recognize and take advantage of the affordances of benchmark numbers. Zelda moved from MASA-bound to Independent. She adopted Compensation as a way of reasoning in computing sums, and her approach to the mental addition tasks was transformed. In the second interview, she used Single and Double Compensation to solve each addition problem. She rounded, added, and then compensated appropriately for her rounding moves.

Table 31 describes the strategy ranges used by each participant for subtraction computations in the first and second interview. As with addition, 5 of the 7 participants were MASA-bound in the first interview, and all but one of these shifted to using a wider range of strategies. The improvement in participants' subtraction strategy ranges seems to be accounted for mostly by adoption of new strategies. Four adopted Aggregation, and two adopted (valid) Compensation strategies.

Table 32 shows the specific strategy ranges used by each participant for multiplication computations in the first and second interview. In contrast to addition and

Table 31. Shifts in Participants' Subtraction Strategy Ranges

Participant	Range Pre	Range Post	Shift Description
Angela	Polarized (3/1)	Spread	Adopted Agg.
Brandy	MASA-bound	Transition+(1/3)	Adopted Agg. And Comp., PV
Maricela	MASA-bound	Polarized (2/2)	Adopted Agg.
Natalie	MASA-bound	Polarized (2/2)	Adopted Comp.
Trina	Transitional	Spread	Adopted Comp.
Valerie	MASA-bound	Polarized (2/2)	Adopted Agg.
Zelda	MASA-bound	MASA-bound	Stable

subtraction, students' strategy ranges for multiplication in the first interview were more varied and less limited overall. At the same time, however, five of the seven participants were MASA-bound, Polarized, or Limited. By the second interview, five of seven participants were Independent and not Limited. Table 32 also provides descriptions of the shifts in participants' multiplication strategy ranges. Five of the seven participants adopted the valid Partial Products strategy. Two of these five participants also adopted Subtractive Distribution.

Table 33 describes the strategy ranges used by each participant for division computations in the first and second interview. The table also characterizes the pre/post shifts in division strategy ranges. When it came to division, participants' strategy ranges were especially weak in the first interview. Six of the seven participants were MASA-bound and/or Limited. The strongest strategy range was Trina's, and even she used the

Table 32. Shifts in Participants' Multiplication Strategy Ranges

Participant	Range Pre	Range Post	Shift Description
Angela	Polarized (2/2) (benchmark)	Independent (2)	Adopted PP
Brandy	Independent (2) – Limited	Independent (3)	Dropped Invalid, Adopted PP
Maricela	MASA-bound	Independent (2)	Adopted PP and SD
Natalie	Spread	Independent (2) – Limited	Adopted Estimation
Trina	Independent (3)	Independent (3)	Adopted Comp.
Valerie	Limited (no valid strategy)	Spread – Limited	Adopted PP, Quarters, SD
Zelda	Polarized (3/1) (benchmark)	Independent (2)	Adopted PP

MASA for three of the four division computations. By the second interview, none of the participants were MASA-bound, only two were Limited. Five were Polarized and not Limited. This is modest improvement but improvement nonetheless. Moving from MASA-bound or Limited to Polarized is a step in the direction of reasoning flexibly about computing quotients.

Overall, these qualitative pre/post comparisons indicate that the participants adopted additional strategies, which facilitated their improved flexibility in mental

Table 33. Shifts in Participants' Division Strategy Ranges

Participant	Range Pre	Range Post	Shift Description
Angela	MASA-bound	Polarized (3/1) (benchmark)	Adopted Quarters
Brandy	Spread – Limited	Independent (2) – Limited	Adopted SD, Estimation
Maricela	MASA-bound	Polarized (3/1)	Adopted Quarters
Natalie	Polarized (2/1) (benchmark) – Limited	Polarized (3/1)	Dropped Invalid
Trina	Polarized (3/1)	Polarized (3/1)	Stable
Valerie	MASA-bound	Polarized (3/1)	Adopted Quarters
Zelda	MASA-bound – Limited	Polarized (1/1) (benchmark) – Limited	Adopted Quarters

computation. The comparison of strategy ranges helps to illuminate finer-grained distinctions in changes in flexibility. To illustrate this point, Table 34 categorizes the 14 participant-operation pairs for addition and subtraction in terms of flexibility pre and post. This provides a quick and somewhat telling description of change but a blunt one.

Strategy range profiles provide a useful grain size at which to view these changes because these make qualitative distinctions concerning the nature of participants' strategy ranges (Table 35). Number sense development, in the area of mental computation and

Table 34. Flexibility Change Counts for Mental Addition and Subtraction

Flexibility Pre	**Flexibility Post**		
	Inflexible	Semiflexible	Flexible
Inflexible	2	5	3
Semiflexible		1	2
Flexible			1

Table 35. Strategy Range Shift Counts for Addition and Subtraction

Pre Range	**Post Range**					
	MASA-bound	Polarized	Transitional	Spread	Transition+	Independent
MASA-bound	2	5			2	1
Polarized		1		1		
Transitional				1		
Spread					1	
Transition+						
Independent						

otherwise, is a complex phenomenon that proceeds differently for different people. Although most participants' strategy ranges were MASA-bound at the beginning of the course, participants' strategy ranges changed in different ways.

It was most common for students to move from MASA-bound to Polarized. This was the case for 5 of 10 instances of strategy ranges that were initially inflexible (in addition and subtraction). Participants changed from being dependent on the MASA—not making any choice based on the given numbers—to making a dichotomous choice between the MASA and a single alternative strategy. It may be that prospective elementary teachers' strategy ranges tend to move from MASA-bound to Polarized on the way to becoming Flexible (especially Transition+ or Independent).

We know how participants' strategy ranges changed, but how were their new strategies understood? What mathematical ideas provided the foundations for the participants' enhanced flexibility in reasoning about numbers and operations? We now consider the justifications that the interview participants provided for their mental computation strategies in order to better understand participant's understanding of the strategies they used.

Strategies as Arguments

Participants' responses to the mental computation interview tasks are now regarded as acts of argumentation. Through this lens, participants' *strategies as arguments* are identified for each of addition, subtraction, multiplication, and division. The warrants and backings of these arguments involve mathematical ideas of particular interest to this study since these help to account for how the participants understood their strategies, especially their new and nonstandard strategies.

Interview participants' strategy-arguments were produced in both cases in which the interviewer asked the participant to justify her mental computation strategy and where justification was offered spontaneously. Justification for the standard algorithms was investigated separately, in tasks involving writing. Some Transition strategies (e.g., Right to Left addition and subtraction) are not included in the set of strategy-arguments since these are so similar to the MASA.

Across the four operations, a total of 17 strategy-arguments were identified. For reasons of length, these will not all be presented in detailed Toulmin-scheme form. I summarize the justifications that students made for their strategies and provide in a table one or two examples of strategy-arguments for each operation. The left column of each table describes the argument in general terms, while the right column offers a specific example from an interview.

Participants' addition strategy-arguments. The descriptions of participants' addition strategy-arguments involve the following mathematical ideas:

> *Reasoning in terms of canonical number composition.* The student reasoned about the addends and sum as consisting of ones and tens (and hundreds, if applicable), which could be dealt with separately.
>
> *Borrowing to Build.* The student reasoned that she could obtain a sum by subtracting an amount from one added and adding that amount to the other addend.
>
> *Reasoning about addition as an associative operation.* The student expressed the idea (either formally or informally) that addition behaves associatively.
>
> *Reasoning about compensation in terms of inverse operations.* The student reasoned about compensation in terms of adding to undo subtraction or subtracting to undo addition.

> *Reasoning in terms of a balance of rounded amounts.* The student expressed the idea that adding some amount and subtracting some amount from the given addends results in a net change to the sum.

Four addition strategy-arguments were identified. The justifications that occurred in participants' addition strategy-arguments are summarized in Table 36. If a single entry appears in the left column, the idea occurred as a warrant in students' arguments. If two entries appear (separated by an ampersand), the first idea was used as warrant and the second as backing.

Table 36. Participants' Justifications for Mental Addition Strategies

Strategy	Ideas used in Justification
Left to Right	Reasoning in terms of canonical number composition
Levelling ("Borrow to Build")	Borrowing to Build & Reasoning about addition as an associative operation
Single Compensation	Reasoning about compensation in terms of inverse operations
Double Compensation	Reasoning about compensation in terms of inverse operations & Reasoning in terms of a balance of rounded amounts

As an example of an addition strategy-argument, Figure 11 presents the Double Compensation argument from Zelda's second interview. The transcript of her response follows:

Claim: Student asserts a sum Interviewer: So, what if Bobo sells two oboes, and he makes $37 on one sale and $52 on the other sale. How much would he make on those two sales put together? Zelda: … it's $89.
Data: Student describes computational steps consistent with Double Compensation Zelda: Okay. I would just add them up. Um, I would just do like forty plus fifty, is ninety. And then I had to, um, add three to get the forty and then subtract two. So, then I subtracted one from the final answer, so it's $89.
Warrant: Reasoning about compensation in terms of inverse operations Zelda: … that still leaves like one extra that I just like borrowed, or like added. So, then if I subtract at the end, then I get the right answer.
Backing: Reasoning in terms of a balance of rounded amounts Zelda: Because since, because I added three and subtracted two, that still leaves like one extra that I just like borrowed, or like added.

Figure 11. Example of the double compensation addition argument.

> Interviewer: So, what if Bobo sells two oboes, and he makes $37 on one sale and $52 on the other sale. How much would he make on those two sales put together?
>
> Zelda: Okay. I would just add them up. Um, I would just do like forty plus fifty, is ninety. And then I had to, um, add three to get the forty and then subtract two. So, then I subtracted one from the final answer, so it's eighty-nine dollars.
>
> Interviewer: Okay. So, you said you added three and then subtracted two. So, why do you have to subtract one at the end?
>
> Zelda: Because since, because I added three and subtracted two, that still leaves like one extra that I just like borrowed, or like added. So, then if I subtract at the end, then I get the right answer. (Zelda, personal communication, November 3, 2010)

Participants' subtraction strategy-arguments. The descriptions of participants' subtraction strategy-arguments involve the following additional mathematical ideas:

> *Reasoning about subtraction as a cumulative process.* The student reasons about the difference between numbers as an amount that can be partitioned and accounted for cumulatively.
>
> *Reasoning about the difference as a distance between.* The student treats the difference between the minuend and subtrahend as a distance between number-locations.

Reasoning about minuend compensation straightforwardly. The student reasons about minuend compensation as straightforward.

Compensating for effect. The student reasons about compensation on the basis of the effect of a rounding step on the difference, rather than on the basis of the action itself.

Reasoning about compensation straightforwardly. The student reasons as though compensation is straightforward, not distinguishing the effects of rounding the subtrahend versus minuend.

Reasoning about subtrahend compensation as doing the opposite. The student reasons that increasing the subtrahend decreases the difference and/or that decreasing the subtrahend increases the difference.[15]

There were four distinct subtraction strategy-arguments used by the participants. Of these, three were valid, and one was invalid. The justifications that occurred in participants' subtraction strategy-arguments are summarized in Table 37.

I offer two detailed examples of subtraction-strategy arguments. Figure 12 presents the Invalid Subtrahend Compensation Argument. Figure 13 presents its valid counterpart, the Valid Subtrahend Compensation Argument.

Natalie reasoned that she had added a 1 "to the problem" and so, to compensate, she subtracted 1 from the difference of the rounded amounts (Natalie, personal communication, September 14, 2010). She did not distinguish the role of minuend and subtrahend. This is contrast to Trina's argument in Figure 13.

Whereas Natalie reasoned that she had added a 1 "to the problem," Trina took into account that she had added 1 to the amount of money that Bobo spent on the oboe. She knew that this decreased the difference, or the amount of money that Bobo made by

[15] The phrasing "doing the opposite" is my attempt as a researcher to contrast this reasoning with that of students who treated subtrahend compensation straightforwardly. Students did not necessarily draw this contrast themselves. They reasoned about subtrahend compensation appropriately, and my shorthand description of that reasoning is "doing the opposite."

Table 37. Participants' Justifications for Mental Subtraction Strategies

Strategy	Ideas Used in Justification
Missing Addend	Reasoning about subtraction as a cumulative process & Reasoning about the difference as a distance between
Minuend Compensation	Reasoning about compensation in terms of inverse operations & Reasoning about minuend compensation straightforwardly
Invalid Subtrahend Compensation	Reasoning about compensation in terms of inverse operations & Reasoning about compensation straightforwardly
Valid Subtrahend Compensation	Compensating for effect & Reasoning about subtrahend compensation as doing the opposite

Claim: Student asserts a difference Natalie asserted that \$125 – \$49 = \$74
Data: Student describes rounding subtrahend up and compensating by subtracting from difference Natalie reported the following computations: 49 + 1 = 50, 125 – 50 = 75, and 75 – 1 = 74
Warrant: Reasoning about compensation in terms of inverse operations Natalie: I think I have to subtract 1. So, \$74, actually.
Backing: Reasoning about compensation straightforwardly Natalie: I'm technically adding a 1 *to the problem*. But then I have to subtract the 1 again in order to correct what I did originally.

Figure 12. The invalid subtrahend compensation argument.

Claim: Student asserts a difference Trina asserted that \$125 – \$49 = \$76
Data: Student describes adding to the subtrahend, identify the difference, and then adding to the difference Trina reported the following computations: 49 + 1 = 50, 125 – 50 = 75, and 75 + 1 = 76
Warrant: Reasoning about subtrahend compensation as doing the opposite Trina: He actually bought the oboe for \$49 and not \$50… So, I added 1 to the final answer of 75, which gave me 76.
Backing: Compensating for effect Trina: By me changing it to 50, I like pretended he used more money than he did.

Figure 13. Example of the valid subtrahend compensation argument.

selling the oboe. So, she had to compensate for having added 1 to the subtrahend by adding 1 to the difference of the rounded amounts.

Participants' multiplication strategy-arguments. The following codes for mathematical ideas will be used to describe components of the participants' multiplication strategy-arguments:

> *Pairing tens and ones.* The student reasons about partial products in terms of pairing up corresponding place values, i.e., multiplying tens by tens and ones by ones.
>
> *Reasoning about multiplication in terms of repeated addition.* The student reasons about the product of *m* and *n* in terms of counting *m* copies of *n*. This involves distinguishing the roles of multiplier and multiplicand (implicitly or explicitly).
>
> *Reasoning about products in terms of partial products*. The student reasons about products as consisting of partial products. This involves pairing parts of one factor (or rounded amounts) with the other factor (or parts of thereof).
>
> *Grouping 25's.* The student uses the associative property of multiplication implicitly in the special case of multiplying by 25. (To compute 24 x 25, many students reasoned that 4 x 25 = 100, and there were 6 groups of 4 in 24, so the product was 600.)

There were five distinct multiplication strategy-arguments identified in the participants' responses. Of these, four were valid, and one was invalid. The justifications in participants' multiplication strategy-arguments are summarized in Table 38.

I offer two detailed examples of multiplication-strategy arguments. Figure 14 presents the Valid Partial Products Argument. Figure 15 presents the Subtractive Distribution Argument.

Table 38. Participants' Justifications for Mental Multiplication Strategies

Strategy	Ideas Used in Justification
Invalid Partial Products	Pairing tens and ones
Valid Partial Products	Reasoning about multiplication in terms of repeated addition
Subtractive Distribution	Reasoning about products in terms of partial products & Reasoning about multiplication in terms of repeated addition
Double Compensation	Reasoning about products in terms of partial products & Reasoning about multiplication in terms of repeated addition
Quarters	Grouping 25's

Both of the above strategy-arguments were based on reasoning about multiplication in terms of repeated addition. Consistent with the story context, students treated the multiplier and multiplicand as having distinct roles. Angela thought of 21 x 19 in terms of "adding 21 nineteen times" (Angela, personal communication, November 2,

Claim: Student asserts a product Angela asserted that 21 x 19 = 399
Data: Student describes computational steps consistent with Valid Partial Products Angela reported the following computations: 21 x 10 = 210, 21 x 9 = 189, and 210 + 189 = 399
Warrant: Reasoning about multiplication in terms of repeated addition Angela: I like to think of it like you're adding 21 nineteen times. So, like, for the first one, you're doing 21 ten times… and then… for the second part of the problem… you're doing 21 nine times… and then you're adding them together. So, in total it's the same. You're still… adding 21 nineteen times.

Figure 14. Example of the valid partial products argument.

Claim: Student asserts a product Brandy asserted that 99 x 15 = 1485
Data: Student describes computational steps consistent with Subtractive Distribution Brandy reported rounding 99 to 100. She knew that 100 x 15 = 1500. She said that the answer was "around 1500." Then she subtracted 15 from 1500 to get an exact answer of 1485.
Warrant: Reasoning about products in terms of partial products Brandy: Oh, I would take 15 away from 1500 'cause… when you round 99 to 100, you're multiplying 15 an additional time. So, 1500 minus 15 would equal 1485… since, when I'm rounding 99 to 100 to make it easier, I'm like adding an extra 15 that I didn't have before. So, I'd have to subtract the 15 from the cost.
Backing: Reasoning about multiplication in terms of repeated addition Brandy: When you do 99 times 15, you're multiplying 15 ninety-nine times.

Figure 15. Example of the subtractive distribution argument.

2010). Similarly, Brandy thought of 99 x 15 in terms of "multiplying 15 ninety-nine times" (Brandy, personal communication, November 4, 2010). In this way, she realized that rounding 99 to 100 increased the product by 15 because the 1 that was added to 99 was being multiplied by 15 and, hence, represented one more 15.

Participants' division strategy-arguments. The following additional codes for mathematical ideas will be used to describe components of the participants' division strategy-arguments:

> *Separating the divisor into tens and ones.* The student reasons that the divisor can be partitioned into tens and ones, and partial quotients can be computed by dividing the dividend by each part of the divisor.

> *Reasoning about division as a repeated-grouping process.* The student reasons that the dividend consists of a number of copies of the divisor, and these can be counted in convenient groupings.

There were four distinct division strategy-arguments identified in the participants' responses. Of these, three were valid, and one was invalid. The justifications that occurred in participants' division strategy-arguments are summarized in Table 39.

The only prevalent division strategies were MASA and Quarters. Figure 16 presents the Quarters Argument, which became prevalent in the second interview. Angela reasoned about 275 ÷ 25 as asking the question, "How many 25's are in 225?" She knew that there were four 25's in 100 and three 25's in 75, and she used these facts to count the number of 25's in 275.

Table 39. Participants' Justifications for Mental Division Strategies

Strategy	**Ideas Used in Justification**
Invalid Partial Quotients	Separating divisor into tens and ones
Count by	Reasoning about division as a repeated-grouping process
Quarters	Reasoning about division as a repeated-grouping process
Subtractive Distribution	Reasoning about products in terms of partial products & Reasoning about division as a repeated-grouping process

Claim: Student asserts a quotient Angela asserted that 275 ÷ 25 = 11
Data: Student describes computational steps consistent with the Quarters division strategy Angela: I know that there are four 25's in 100. Then how many are in 200? There's eight. And in 75 there's three.
Warrant: Reasoning about division as a repeated-grouping process Angela treats 275 ÷ 25 as asking the question, "How many 25's are in 225?"

Figure 16. Example of the quarters argument.

Summary of participants' strategy-arguments. For each operation, there were a small number of strategy-arguments identified. The mathematical ideas that occurred as warrants and backings in these arguments stand out as keys to understanding the participants' number sense development. In the next chapter, we will examine the collective activity that occurred around mental computation and place value in the whole-number portion of the course. There are connections between the ideas that came to function as if shared for the class and the ideas that the interview participants used in their new and nonstandard strategy-arguments.

Conclusion

In this chapter, I presented results of "old and new" analyses related to prospective elementary teachers' number sense. I presented results of two survey instruments, the NSRS and the SPS, which were administered pre/post to 34 of the 39 students in the class. These results indicate improvement in the number sense of the study participants. Students were more able to correctly answer mathematical questions designed to tap their number sense. They also came to view mental multiplication and division as less difficult than they had initially.

I also presented results of analyses of mental computation interview data. As expected, the seven study participants tended to be reliant on the MASAs in the first interview. Some made choices of strategy depending on the particular numbers given, but participants tended to be limited in their flexibility. Across all four operations, most participants were inflexible, some were semiflexible, and few were flexible. From the first to the second interview, participants became more flexible. Furthermore, the distribution of strategies used by participants shifted toward the Nonstandard end of the spectrum, so that those strategies that are associated with good number sense were used more often.

In the second section, I presented new tools and results, concerning strategy ranges and strategy-arguments. Both of these analyses were grounded in data from the particular interview participants but also represent contributions to the field more broadly.

I introduced the construct of a *strategy range*, as well as six particular *strategy range profiles* that I identified. I also described change in the participants' strategy ranges from pre to post. This analysis represents progress over previous analyses of change in flexibility by making qualitative distinctions between the ranges of strategies used and coordinating these with the categories of Inflexible, Semiflexible, and Flexible.

Finally, I reported on participants' strategy arguments. These resulted from an analysis in which I viewed interview participants' descriptions of their strategies as instances mathematical argumentation. In the interview setting, participants performed mental computations and described their computational steps. They were also asked to justify the nonstandard strategies that they used. The warrants and backings that students offered provide insight into how they understood their valid and invalid nonstandard strategies. Many of these ideas figure prominently in the sociogenetic and ontogenetic analyses to be presented in Chapters 5 and 6.

Chapter 5: Results Part 2: Sociogenesis of Number Sense

The results presented in Chapter 4 provide evidence that the study participants changed in an interesting way: their number sense improved. On the basis of data collection early in the semester, the students in the course looked like prospective elementary teachers, as described in the literature. In particular, they were reliant on the standard algorithms. On the basis of data collection after the whole-number portion of the course, the students looked different. They no longer resembled typical prospective elementary teachers. Instead, the participants exhibited flexibility in reasoning about numbers and operations.

In the vein of Saxe and colleagues' investigation of the evolution of the meaning of *fu* for the Oksapmin people, I sought to make sense of the pre/post contrast in students' number sense by means of genetic analysis. This chapter focuses on collective activity related to place value, number composition, and whole-number operations. The analysis presented is sociogenetic, as it concerns a chronological progression through normative ways of reasoning in the classroom community. This sociogenetic analysis is made possible by the microgenetic analyses of individual instances of argumentation. The story presented carries the broader sociogenetic framing. Within that story, instances of argumentation are presented in order to provide examples of meaning making at the micro level and to demonstrate the normative nature of particular ways of reasoning. Because of the large number of as-if shared ideas, not all of these can be traced. I attempt to tell a coherent story by focusing on a subset of the ideas.

The social lens of the emergent perspective consists of three layers: social norms, sociomathemetical norms, and classroom mathematical practices. Although the focus of this chapter is on classroom mathematical practices, I first briefly address social norms.

Social Norms

Before presenting the analysis of normative ways of reasoning, some attention to social norms is required. As discussed in Chapter 3, the methodology of Rasmussen and Stephan (2008) assumes a classroom environment in which students engage in mathematical argumentation. In such an environment, mathematical arguments are evaluated on the basis of their merit, according to a set of sociomathematical norms that are negotiated by members of the classroom community. The criteria for evaluating whether ideas function as if shared assume such an environment.

The design of the study did not include a rigorous investigation of the social norms that developed in the class. So, this chapter does include comprehensive reporting on those norms. However, the data do provide indications of social norms that are in keeping with the assumptions of the methodology. A large number of student arguments were made during whole-class discussion, which suggests a classroom environment characterized by mathematical argumentation. Furthermore, these arguments tended to include warrants or backings, so that the classroom culture was such that students provided justifications for their mathematical ideas. In addition, some of these arguments were sufficiently complex that I had to use expanded argumentation schemes to describe them adequately. This fact shows that students engaged with the mathematics in sophisticated ways that sometimes involved different layers of justification or different types of justifications.

Indications of the Social Norms in the Class

Table 40 presents counts of argument characteristics. The argument count is the total number of arguments (claims for which some manner of justification was provided) belonging to each strand of activity. The column labeled “Class member(s)” refers to who made the arguments: Instructor alone (I), one or more students (S), or co-constructed by Instructor and one or more students (C). “Justification” refers to which components of arguments were present. In order to be counted, arguments had to include both claim and data. The table shows the numbers of arguments that also included a warrant (Warr.) and those that included warrant and backing (Back.). “Structure” refers to whether or not arguments had Expanded (Exp.) structures, i.e., they included subarguments. Arguments that did not include subarguments are called Basic.

Table 40. Counts of Argument Characteristics

Content	Arg.	Class member(s)			Justification		Structure	
Strand	Count	I	S	Co	Warr.	Back.	Basic	Exp.
PV	70	10	19	41	48	8	60	10
AS	48	16	29	3	27	21	36	12
MD	90	6	73	11	65	14	79	11
Totals	208	32 (15%)	121 (58%)	55 (26%)	140 (67%)	43 (21%)	175 (84%)	33 (16%)

As Table 40 shows, students made the majority of the arguments (58%) in the class. Also, many arguments (26%) were co-constructed. Only 15% of the arguments made in whole-class discussion were made by the Instructor alone. To be clear, my definition of whole-class discussion includes lecture. Thus, the data indicate that very little lecturing occurred in the class. Far more often, students presented arguments.

The majority of arguments (67%) included warrants, and many (21%) also included backing. The large number of warranted and backed arguments indicates discussions in which class members accompanied their claims with reasons why. By contrast, in a traditional mathematics class, students might be called upon to give answers and possibly to explain the work that they had performed to obtain answers but would not often be asked to justify their solutions. This would correspond to students often providing claims or data but not warrants or, especially, backings.

Of the 208 total arguments, 33 of them (16%) had expanded structures. This indicates class members making complex arguments, consisting of one or more subarguments. In particular, students were the ones making these complex arguments. Of the 33 expanded arguments, 22 of them (2/3) were made by students, and 10 were co-constructed (i.e., the instructor made a substantive contribution to the argument). Only one expanded argument was made by the Instructor alone. Slicing this data differently, of the total 121 *student* arguments, 22 of these had expanded structures. Wawro (2012) reports that she needed to use expanded argumentation schemes to document collective activity in inquiry-oriented linear algebra. In her data set, 22 of 118 arguments had expanded structures. Wawro attributes the need for these expanded structures to the complexity of students' arguments in a class in which students are transitioning to formal proof. I saw a similar frequency of these more complex arguments in an elementary mathematics content course. Even though the mathematics was elementary, the level of students' engagement with that mathematics was rather advanced.

Summary of Social Norms

Without formally analyzing social norms, which would have been an ambitious undertaking and beyond the scope of this dissertation, I do see compelling evidence for social norms consistent with the assumptions of the methodology of Rasmussen and Stephan (2008). The mathematics content course studied was a class in which students engaged in mathematical argumentation, the vast majority of their arguments included warrants and/or backings, and the percentage of student arguments involving expanded structures was on par with an inquiry-oriented linear algebra course.

Classroom Mathematical Practices: Place Value, Addition, and Subtraction

I identified the following five classroom mathematical practices in the strand of activity around place value, addition, and subtraction:

CMP1. Assuming the authority of the standard algorithms

CMP2. Making Sense of Place Value

CMP3. Making Sense of Standard Algorithms and Transition Strategies

CMP4. Reasoning Flexibly about Addition

CMP5. Reasoning Flexibly about Subtraction

This section presents the progression through these five CMPs. For each CMP, I describe the mathematical activities in which students engaged. I describe the as-if shared ideas belonging to the CMP and explain the FAIS criteria satisfied by each. I also present selected classroom vignettes, which serve as examples of collective activity and also illustrate shifts over time in the roles of particular ideas in argumentation.

CMP1: Assuming the Authority of the Standard Algorithms

The first CMP that emerged and became established was unexpected. It involved assuming the authority of the standard algorithms. Early instances of students' mental computations that were shared with the class included usage of the mental analogues of the standard algorithms. When nonstandard strategies were discussed, these were often represented using notation that followed the conventions of the standard algorithms. Furthermore, when students shared nonstandard strategies early in the semester, these required justification, whereas usage of the standard algorithms was accepted without justification. During place value activity involving adding and subtracting in base three using multilink cubes, the class followed the standard convention of proceeding place-value wise from right to left. Overall, in CMP1, the standard algorithms and conventions associated with these functioned authoritatively.

CMP1 was unique among the group of CMPs identified. It included only one as-if shared idea: Reasoning about the operations in terms of the standard algorithms. Early on in the course, the legitimacy of these algorithms went unquestioned. The nature of this as-if shared idea is also rather unique. Typically, in an inquiry-oriented course, ideas come to function as if shared by first being justified and later not requiring justification. However, in previous studies of classroom mathematical practices, the mathematical ideas of interest to researchers have been new ideas for students. In a course for prospective elementary teachers, the students come in with a lot of familiarity with the mathematics. In this case, they already knew the standard algorithms. Every student likely assumed legitimacy of these algorithms before the class began. Still, it does not necessarily follow that these would function as-if shared. Functioning as-if shared is a characterization of the role that the algorithms played in mathematical discussions early in the course.

Assuming the authority of the standard algorithms: FAIS Criteria. The authoritative function of the standard algorithms spanned content strands.[16] It was not limited to addition and subtraction. Within the place value strand, multiplication and division computations also arose. For example, in discussions of problems in the Andrew's Apple Farm context, division was repeatedly used. In one instance, a student described dividing 1705 apples by 8 to find the number of baskets that 1705 apples would fill (given that 8 apples filled on basket). For this and other division computations in service of solving these problems, the standard division algorithm was used, and its validity went unquestioned. The focus of the discussion was on interpreting quotients and remainders in terms of the context.

Reasoning about the operations in terms of the standard algorithms satisfied Criterion 3. That is, it was used repeatedly across multiple days and in service of different claims. On Days 2, 3, and 7, a total of 8 arguments were made that assumed the authority of a standard arithmetic algorithm.[17] Criterion 3 is especially appropriate to the nature of this idea. Criteria 1 and 2 concern changes in the function of an idea. When an idea shifts from requiring justification to not requiring justification, or to being used to justify a new claim, its function has changed. Criterion 3, by contrast, concerns the status quo. An idea functions in a certain way, and continues to function in that same way over a period of

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[16] I include CMP1 as part of the place value, addition, and subtraction strand. However, in contrast to the other CMPs in this strand, I am including in it activity related to multiplication and division.

[17] Day 5 was not included in the analysis because the activity that day focused on decimal, rather than whole-number, ideas. There were additional instances on Day 5 of procedures functioning authoritatively.

time. It was not the case that the standard algorithms shifted from requiring justification to not requiring it. Rather, they were used without justification from the start. In fact, later in the semester, the class discussed why the algorithms worked, and students arrived at justifications for these.

Assuming the authority of the standard algorithms: Vignettes. I present two consecutive vignettes from early in the course. Because the relevant FAIS criterion was Criterion 3, it would require many arguments to demonstrate that relying on the standard algorithms functioned as if shared. I include the following vignettes not to provide that evidence but to illustrate how the authoritative nature of the standard algorithms was leveraged productively to motivate the need for justification of nonstandard strategies. This involves a move that the instructor made early in the course, and it is noteworthy in terms of the instructor's role in influencing the direction of the collective activity related to mental computation.

Vignette 1.1. On Day 3, students were given a story problem that involved comparing the heights of two pairs of siblings. In the course of solving the problem, computations arose, and the instructor had asked students to attempt these mentally. Students recorded mental computations on the whiteboards at their tables. In the whole-class discussion, the quantitative relationships involved in the height problem and the need to perform certain computations were discussed first. Then the details of those computations were discussed.

Aaron shared his group's methods for performing two subtraction computations, 64 – 43 and 70 – 19. For the first computation, he used the standard subtraction algorithm. This computation was discussed briefly, and the validity of Aaron's approach went unquestioned. Aaron and the Instructor co-constructed Argument A3.1 since she verbalized part of the data in the argument (albeit simply by reading off what he had written). Figure 17 describes the argument. The following is the transcript of the exchange.

Claim: 64 – 43 = 21 [Answer written on screen]	 *Aaron and Instructor both gesturing down the tens column*
Data: 4 – 3 = 1; 6 – 4 = 2 [Evidence from written record, gesturing, and utterances]	
Warrant: (Implicit) Using the standard subtraction algorithm [Aaron's speech, writing, and gesturing reflected the conventions of the standard subtraction algorithm. Instructor participated in the reenactment of the algorithm by gesturing down the tens column and reciting the second step ("six minus four is two").]	

Figure 17. Argument that 64 – 32 = 21 using the standard algorithm.

[Aaron writes at the board: 64 – 43 = 21 (aligned, standard notation)]

Aaron: Uhhh, I don't really get how you, like, want me to explain it.

Instructor [laughs]: Because I did it in my head, he says.

Aaron: Yeah.

Instructor: So, how did you think about this? You have recorded on your board,

Aaron: Just basically subtracted from four, or the three from the four [gesturing down ones column]

Instructor: And then there's six minus four is two? [Both gesturing down tens column] Okay.

Vignette 1.2. Aaron then presented his group's method for computing 70 – 19. He reported that they subtracted 10 from 70, obtaining 60. Then they subtracted 9 from 60, for an answer of 51. Aaron's written work is depicted in Figure 18. In the exchange, the Instructor challenged the legitimacy of Aaron's approach, and he offered backing for it.

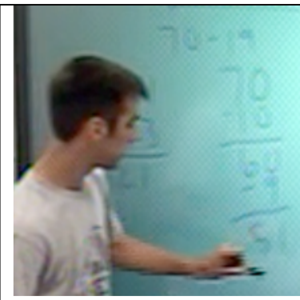
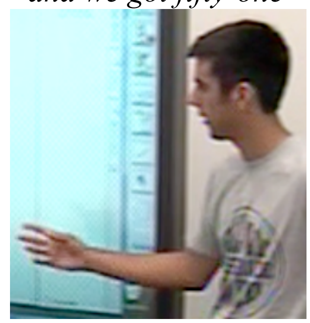
<table>
<tr><td>Claim: 70 – 19 = 51
[Answer on screen and utterances: “and we got fifty-one”]</td><td rowspan="4">
“and we got fifty-one”

“you’re taking the ten out first”</td></tr>
<tr><td>Data: 70 – 10 = 60; 60 – 9 = 51
[Evidence from written work and utterances]</td></tr>
<tr><td>Warrant: Reasoning about subtraction as a cumulative process (“subtracted ten first, and got sixty… And then we subtracted nine from that”)</td></tr>
<tr><td>Backing: Reasoning about subtraction as a take-away process (“Just because you’re still subtracting the whole nineteen. It’s just you’re taking the ten out first.”)
[Phrasing equates subtraction and “taking out”]</td></tr>
</table>

Figure 18. Arg. 3.2a: Aaron describes and justifies his group’s subtraction aggregation strategy.

Aaron: Well, we, um, trying to find the difference between Olivia and Oscar, we took seventy

Instructor: Alright, let’s move on to that one

Aaron: and subtracted ten first, and got sixty *[writing on screen]*

Instructor: Could you write the problem up here and remind us what that

computation was? 70 minus 19

[Aaron writes 70 – 19 on screen]

Aaron: And then we subtracted nine from that, and we got fifty-one

[Instructor made comments about computational approaches that other groups had used. Then she asked Aaron to justify his group’s method.]

Instructor: Now, this second computation, some of you did it as you would do the long-division [sic] algorithm, and Aaron is talking about doing it this way. So, one question is, how come—how do you know it works like this? You took seventy minus ten. That’s not the same thing as you would do if you were, you know [Instructor enacts standard subtraction algorithm on screen] taking this, and borrowing like this, and—this isn't exactly the same algorithm. How come this one gives the same answer, and why are you confident that that works? Can you

Aaron: Just because you're still subtracting the whole nineteen. It's just you're taking the ten out first.

Instructor: Taking the ten out first, and then the nine? Does anybody have any questions for Aaron about that? No? Okay.

The above exchange was coded as a two-part argument. In A3.2a, Aaron describes and justifies his group's method. In A3.2b, Instructor argues that Aaron's strategy requires justification.[18] With respect to A3.2a, the Instructor made a request for backing, which Aaron provided. However, in A3.2b, the instructor's request for backing is framed as the claim, and it was accompanied with an explicit justification.

Aaron's argument made use of a take-away meaning for subtraction as backing for subtraction aggregation. He argued that subtracting ten and then subtracting nine was equivalent to subtracting nineteen because a total of nineteen was subtracted. He took subtraction to mean "taking out," and this idea justified the cumulative nature of his subtraction strategy: By taking out ten and then taking out nine, he had taken out nineteen. In this way, the take-away meaning was essential to justifying his group's method. Interestingly, in the problem context, subtraction did not appear as a take-away process. It was being used to compare heights. In fact, students made arguments that reflected this comparison meaning. (Those arguments do not belong to the reduced data set because they concern the quantitative reasoning involved in solving the problem, rather than reasoning specifically about the computations.)

---

[18] This numbering convention serves two purposes: (1) it signifies that the arguments were related, and (2) in terms of the argumentation log, it reminds me that these two arguments should be regarded as having occurred at the same time. Thus, no criterion for an idea functioning as if shared could be satisfied by looking chronologically at part a and then part b. In terms of the development of collective activity over days of class, these arguments are considered to have occurred at the same moment.

Figure 19 represents the Instructor's argument. In this exchange, the Instructor explicitly justified her request for backing by contrasting Aaron's strategy with an accepted way of performing subtraction. There was a clear contrast in status between his two approaches, as far as she was concerned. The first, using the standard subtraction algorithm, was readily accepted. The second, a nonstandard strategy, required justification. This precedent was established early, and led to discourse concerning nonstandard mental computation strategies that consistently included mathematical justification.

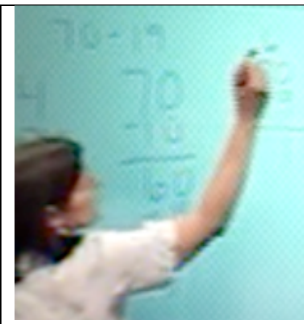
<table>
<tr><td>Claim: Aaron's strategy requires justification</td><td rowspan="3">
"and borrowing like this"</td></tr>
<tr><td>Data: Aaron's strategy differs from the standard algorithm ("You took seventy minus ten. That's not the same thing as you would do if you were, you know [Instructor enacts standard subtraction algorithm on screen] taking this, and borrowing like this, and—this isn't exactly the same algorithm.")</td></tr>
<tr><td>Warrant: Nonstandard strategies require justification ("this isn't exactly the same algorithm. How come this one gives the same answer, and why are you confident that that works?")</td></tr>
</table>

Figure 19. Arg. 3.2b: Instructor argues that Aaron's strategy requires justification.

Assuming the authority of the standard algorithms: Conclusion. In CMP1, the standard arithmetic algorithms functioned authoritatively. Their use went unquestioned, whereas nonstandard strategies required justification. Also, the class tended to follow the notational conventions of the standard algorithms, even when recording nonstandard computational work. The authoritative nature of the standard algorithms was unexpected. However, it was leveraged productively as students worked to make sense of and justify nonstandard strategies.

CMP2: Making Sense of Place Value

CMP2 consisted of the following set of as-if shared ideas, all of which fit into the general theme of making sense of place value:

PV1. Packing baskets, bushels, and trucks Andrew's way:

a. 8 apples fill a basket

b. 64 apples fill a bushel

c. 512 apples fill a truck

PV2. Reasoning in terms of canonical number composition

PV3. Forming groups (and groups of groups) of eight items

PV4. Using and interpreting Andrew's bookkeeping notation

PV5. Using and interpreting place-value notation in base ten

PV6. Forming groups (and groups of groups) of three items

PV7. Using and interpreting place-value notation in base three

These ideas emerged and became established between Days 6 and 8. They were focused around activity in the Andrew's Apple Farm context and activity in base three using multilink cubes. Below, I describe the activities in which students engaged. I present evidence for the as-if shared nature of a subset of these ideas. I also present two illustrative vignettes.

Making sense of place value: Activity. On Days 6 and 7, students solved a variety of tasks in the Andrew's Apple Farm context. They were told that on Andrew's farm, he had a specific way of packing apples: eight apples filled one basket, eight baskets filled one bushel, and eight bushels filled one truck. Students solved problems of two main forms: (1) Given some number of trucks, bushels, baskets, and loose apples, find the total number of apples that Andrew picked, and (2) Given that Andrew picked a certain number of apples, find the number of trucks, bushels, and baskets that would be filled, and the number of loose apples that would remain. Students figured out the numbers of apples per bushel and per truck, based on what they knew about Andrew's way of grouping items. They used this information to solve tasks of both types. Many groups drew pictures of apples, baskets, bushels, and trucks, which served to support their arguments (e.g., Figure 20).

The activity related to the Andrew's Apple Farm tasks involved working informally in base eight and relating bases eight and ten. All of this was grounded in the apple farm context, and it was not apparent in the discussion of these tasks that students were aware that they were dealing with base eight. This activity spanned Day 6 and part of Day 7. Students were also introduced to Andrew's special way of recording the

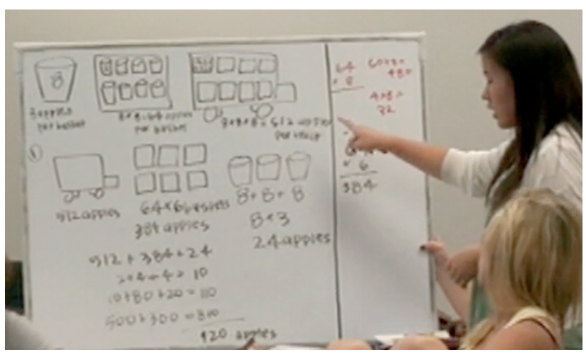


Figure 20. Trina presents her group's solution to an Andrew's Apple Farm problem.

numbers of apples that he had picked, and they made conversions between Andrew's notation (which was in base eight) and base ten.

Later on Day 7, the class began to talk explicitly about different bases and to count and represent numbers in base three. Activity in base three was not related to a story-problem context but instead involved multilink cubes. The Instructor and three students stood in front of the board and physically grouped cubes by threes to count and represent numbers. Students then performed similar activity within their groups, with each student playing the role of one of the four relevant place values (ones, threes, nines, or twenty-sevens). Students converted from base-ten to base-three numerals, using grouping of cubes to help them. Students also counted from zero to twenty-seven with the cubes, grouping them appropriately in base three and recording each number with a base-three numeral.

Making sense of place value: FAIS Criteria. In this section, I describe the FAIS criteria satisfied by each the ideas belonging to CMP2. Packing baskets, bushels, and trucks Andrew's way came to function as-if shared on Day 7. Specifically, PV1a, PV1b, and PV1c each satisfied Criterion 2 by shifting the roles that they played in argumentation. The number of apples per basket was given in the description of the Andrew's Apple Farm context. Andrew's convention of packing eight apples per basket was used as data in arguments on Days 6 and 7 but also occurred as backing in an argument on Day 7. The numbers of apples per bushel and per truck were not given in the description of the context. Students solved for these based on what they knew about Andrew's method of packing apples. Thus, initially, the ideas that Andrew packed 64 apples per bushel and 512 apples per truck occurred as claims. Subsequently, these were used as data in arguments concerning the total numbers of apples that Andrew had picked on a particular day.[19]

[19] The numbers of apples that Andrew packed in each basket, bushel, and truck may seem like somewhat trivial ideas. I include them in CMP2 because they satisfied FAIS criteria and because these ideas were important to students' sense making activity as they worked informally in base eight. In base

Reasoning in terms of canonical number composition came to function as-if shared on Day 7. PV2 satisfied Criterion 2. It occurred as a warrant in five arguments and as data in one. By *reasoning in terms of canonical number composition*, I mean reasoning about a number as a linear combination of powers of the base. For example, in base ten, the number 235 can be decomposed as 200 + 30 + 5, and more explicitly as 2(100) + 3(10) + 5(1). Students reasoned about numbers in this way in their arguments on Days 6 and 7 concerning both the Andrew's Apple Farm tasks and their work in base three.

The as-if shared ideas PV3 and PV6 were related. The class reasoned about forming groups (and groups of groups) of eight items in activity related to Andrew's Apple Farm tasks. The class reasoned about forming groups (and groups of groups) of three items in activity in base three. Both of these ideas occurred many times and could be regarded as satisfying Criterion 2 or Criterion 3. PV3 occurred on Days 6 and 7 as data in five arguments, as warrant in one argument, and as backing in one argument. PV6 occurred a total 19 times between Days 7 and 8. It was used both as data and as warrant. These ideas were integral to students' activity in bases eight and three as they grouped and regrouped place-value-wise.

The as-if shared ideas PV4, PV5, and PV7 were also related. These involved using and interpreting place-value notation. The class did this informally in base eight with Andrew's bookkeeping notation, as well as formally in bases ten and three. PV4 and PV5 satisfied Criterion 2, having been used as both data and warrant in argumentats. PV7 satisfied Criterion 3, occuring 20 times in arguments on Days 7 and 8.

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ten, it is fundamental to understanding place value to know that the places to the left of the ones are the tens, hundreds, thousands, and so on. Analogously, in base eight, students needed to know that the places to the left of the ones were the eights, sixty-fours, and five hundred-twelves.

Making sense of place value: Vignettes.I present three vignettes that illustrate activity that involved students making sense of place value. The first involves base-ten activity from Day 6. The second involves activity from the latter portion of Day 6 involving the Andrew's Apple Farm context. The third involves base-three activity on Day 7.

Vignette 2.1. Day 6 began with a continued discussion of base ten and a set of tasks related to base-ten place value. In one of these, students were asked how many hundreds were in 53,908. Two different arguments were made for the same claim, that there were 539 hundreds. Muriel's argument was the following:

> *Muriel*: I thought about how many hundreds go into nine hundred, and I got nine. And I thought about how many hundreds go into three thousand, and I got three hundred. Or what—got thirty. I got thirty. And then how many hundreds went into fifty thousand, and I got five hundred.
>
> *Instructor*: And then added them together?
>
> *[Muriel nods]*

Muriel's argument (Figure 21) involved reasoning about numbers as being composed of powers of ten. In particular, her approach to the task was to decompose 53,908 into 50,000 and 3,000 and 900. She considered the numbers of hundreds in each of these, and then added all the hundreds together to get her solution.

Vignette 2.2. In initial activity in the Andrew's Apple Farm context, students worked in their groups to find the total number of apples that Andrew picked, given that those apples filled one truck, six bushels, and three baskets. Trina presented her group's solution. Figure 22 represents Trina's argument.

Claim: There are 539 hundreds in 53,908
Data: Reasoning with hundreds as the reference unit (*Muriel:* I thought about how many hundreds go into nine hundred, and I got nine. And I thought about how many hundreds go into three thousand, and… I got thirty. And then how many hundreds went into fifty thousand, and I got five hundred.)
Warrant: Reasoning in terms of canonical number composition 53,900 = 50,000 + 3,000 + 90 500 + 30 + 9 = 539 (*Instructor*: And then added them together?)

Figure 21. Co-constructed argument that there are 539 hundreds in 53,908.

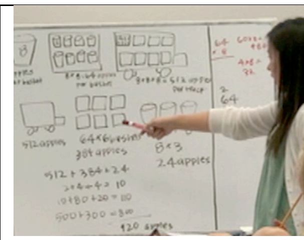
Claim: 1 truck, 6 bushels, and 3 baskets adds up to 912 apples [Written work on whiteboard]	
Data: 512 apples fill a truck; 64 apples fill a bushel; 8 apples fill a basket; 6 bushels = 6 x 64 apples = 384 apples; 3 baskets = 3 x 8 apples = 24 apples [Written work and utterances]	
Warrant: Reasoning about the total number of apples as a linear combination of trucks, bushels, and baskets [Written work shows 512 + 384 + 24 = 920] (*Trina*: And then we added up those three numbers)	*"enough apples to load one truck…"*

Figure 22. Trina's argument that Andrew picked 912 apples

> *Trina*: Okay. So, in number one, they said that there's—Andrew picked enough apples to load one truck, six bushels, and three baskets. And so we drew out what he had. And then we looked up here. And this says each truck had five hundred twelve apples. We said he has five hundred twelve apples in the truck. And since we know that, in each bushel, there's sixty-four apples, and Andrew picked six bushels, we just did sixty-four times six and got three eighty-four. And then he picked three baskets, and we know that there are eight apples in each basket. So, we did eight times three and got twenty-four. And we added up those three numbers.

Trina solved for the total number of apples as a linear combination of trucks, bushels, and baskets that Andrew had filled. Informally, her solution was a conversion from base eight to base ten. Andrew had grouped the apples by eights, and Trina and her group found out what that same number of apples would look like if grouped by tens instead. However, at this point in the activity, students were not aware that they were converting from one base to another. They were simply trying to count the total number of apples. Thus, the linear combination idea that Trina used was distinct from the idea that Muriel used in her argument, the difference being that, at this point, students were not yet using base-eight notation explicitly.

In later activity in the Andrew's Apple Farm context, the class did use Andrew's bookkeeping notation, which was essentially a base eight notation. For example, if Andrew picked enough apples to fill 1 truck, 6 bushels, and 3 baskets, with 0 apples remaining, he would record the apples he had picked that day as $1_{truck}6_{bushels}3_{baskets}0_{apples}$ or just 1630, for short. Students solved problems by converting numbers of apples from Andrew's notation to base ten, and vice versa.

Vignette 2.3. On Day 7, students counted and represented numbers in base three. After students had worked on counting up to 27 in their small groups, they counted together as a class and discussed some of the reasoning involved. The following exchange occurred at the end of the counting discussion and regarded the transition from 26 (which is written as 222_three) to 27 (which is written as 1000_three):

> *Instructor*: First, two two two base three. What number does that represent?
>
> *Molly*: Two ones, two groups of three, and two groups of nine.
>
> *Instructor*: Two ones, two groups of three, two groups of nine. Okay, so that's twenty-six in base ten. And what happened here? [Instructor points to "1000" written on board.]
>
> *Molly*: You add a one to the two ones, and that makes a group of three. And then you have three groups of three.
>
> *Instructor*: Then you have three groups of three.

> *Molly*: Yeah, and that's nine. So, then you take that and the other two groups of nine, and that makes three groups of nine. So, that equals one twenty-seven.

In the above exchange, Molly justified the transition from 222_three to 1000_three in terms of a succession of regrouping actions that occurred as a result of adding one to 222_three: Adding one created a group of three ones, which was passed on the threes place. Then there were three threes, which formed a group of nine, and it was passed on to the nines place. Then there were three nines, which formed a group of twenty-seven, and it was passed on to the twenty-sevens place. Molly effectively made an argument that the numeral after 222 in base three was 1000. (See Figure 23.)

Claim: Counting in base three, the numeral that comes after 222 is 1000 [Claim was made by the class and recorded by the Instructor. Then Molly made her argument.]
Data: Using and interpreting place-value notation in base-three (222 represents "Two ones, two groups of three, and two groups of nine," and 1000 represents "one twenty-seven," zero nines, zero threes, and zero ones.)
Warrant: Forming groups (and groups of groups) of three items (*"*You add a one to the two ones, and that makes a group of three. And then you have three groups of three… and that's nine. So, then you take that and the other two groups of nine, and that makes three groups of nine. So, that equals one twenty-seven.")

Figure 23. Arg. P7.38: Molly argues that 1000 comes after 222 in base three.

Making sense of place value: Conclusion. In CMP2, students engaged in sense making activity related to place-value ideas. This activity involved grouping by eights and by threes, reasoning in terms of canonical number composition, and using and interpreting place-value notation, informally in base eight and formally in bases ten and three.

CMP3: Making Sense of Standard Algorithms and Transition Strategies

CMP3 involved making sense of standard algorithms and Transition strategies for addition and subtraction. It consisted of the following six as-if shared ideas:

PV5. Using and interpreting place-value notation in base ten

PV7. Using and interpreting place-value notation in base three

PV8. Regrouping from right to left

PV9. Regrouping in order to subtract

AS1. Separating numbers canonically in order to add

AS6. Reasoning about subtraction as a take-away process

Two of these as-if shared ideas, PV5 and PV7, overlapped with CMP2. The set of ideas belonging to CMP3 spanned addition and subtraction activity in bases three and ten. This CMP groups together those aspects of students' activity that involved performing and recording addition and subtraction computations when standard algorithms and Transition strategies were used.

Making sense of standard algorithms and transition strategies: Activity. On Day 8, students built on their counting and grouping activities in base three and advanced to performing addition and subtraction and producing written records of these operations. They used base-three notation to record problems and solutions, but answers were obtained by working with the cubes. As the activity progressed, students eventually solved problems without using the cubes. By the end of class, they solved addition and subtraction problems in bases four and five, as well. Written records of students worked involved the standard notation for addition and subtraction. However, discussions of solutions referred to numbers of cubes, the geometry of groups of cubes, and regrouping actions, rather than to procedural moves, like carrying or borrowing. The class discussed how regrouping moves should be recorded. Regrouping in subtraction was recording in two different ways.

On Day 9, various mental computation strategies were shared and discussed. Muriel used a Separation (Left to Right) strategy. On Day 11, students engaged in activities related to children's thinking about subtraction computations.

Making sense of standard algorithms and transition strategies: FAIS Criteria. The evidence that PV5 and PV7 functioned as-if shared was presented in the description of CMP2 and will not be repeated here.

PV8 and PV9 were related. Both involved regrouping. Regrouping from right to left occurred in packing apples, counting and representing numbers in base three, and in performing and reasoning about addition. Regrouping from right to left satisfied Criteria 2 and 3. It occurred 3 times as data and 12 times as warrant. Regrouping in order to subtract satisfied Criterion 2. It was used as both data and warrants.

AS1, separating numbers in order to add, occurred in arguments related to mental addition. This idea satisfied Criterion 1. It occurred on Day 8 with backing. When it was used again on Day 9, it went without backing and this usage was unquestioned. In fact, the Instructor explicitly acknowledged that the strategy had been used before.

AS6, reasoning about subtraction as a take-away process, occurred in mental subtraction arguments on Days 3, 11, and 12. This idea satisfied Criterion 2 since it was used as both data and backing in students' arguments. It was also repeatedly used to justify Subtraction Aggregation strategies (as in Aaron's argument from Day 3).

Making sense of standard algorithms and transition strategies: Vignettes. I present three vignettes related to students making sense of the standard algorithms and Transition strategies for addition and subtraction. The first vignette involves adding in base three. The second involves applying grouping ideas to solve an addition problem in base five. The third vignette is an example of a student using a Transition strategy for addition in base ten.

Vignette 3.1. On Day 8, the class added and subtracted in base three, using the multilink cubes. In the vignette below, they solve the problem 101_three + 122_three. As the transcript begins, the class members standing in front of the board are already holding 101_three. They talk through and act out the addition and regrouping process:

> *Instructor*: Let's say we're going to add 112 base three. Help me. How do I proceed? What would you do?
>
> *Students*: Take two.
>
> *Instructor*: I'm going to take two more? [Instructor takes two cubes from the table.]
>
> *Student*: You've gotta pass them.
>
> *Instructor*: Oh, but somebody says I have to pass them on? Ok, there you go. [Instructor passes a group of three cubes to the threes place.]
>
> *Instructor*: So, what would you do next?
>
> *Aaron*: She gets the other one.
>
> *Instructor*: She gets one more? We're adding to that. [The person in the threes place receives on more cube.]
>
> *Instructor*: Anything for her to do more?
>
> *Students*: No.
>
> *Instructor*: Alright. How about you, Aaron?
>
> *Aaron*: I get another one.

Instructor: You get another one.

Instructor: So, anything, we have a way to represent this sum?

Students: Two two zero.

Instructor: Two two zero base three? Can I record that like this?
[Instructor writes "220_{three}" on board]

In the above vignette, the as-if shared nature of certain ideas is apparent. The attention of the whole class is on the class members acting out the addition up at the board. Students are calling out instructions regarding next steps, and there seems to be agreement on each of these. Furthermore, justification for these instructions is generally left implicit. For example, the Instructor starts out holding one little cube, and she receives two more. A student calls out, "You've gotta pass them," and the Instructor groups and passes her three cubes. No justification was provided for the claim that she has to pass them.[20] Regrouping from right to left, according to a particular group size, functioned as-if shared at this point.

As the above vignette illustrates, on Day 8, in the context of physically counting, adding together, and grouping multilink cubes, certain justifications dropped out of students' arguments. However, when it came to no longer using the cubes to solve problems and instead working problems out in writing, including in different bases, these justifications resurfaced.

Vignette 3.2. In the following vignette, the class discussed the solution to an addition problem in base five. Students had solved a base-five addition problem in their groups. The class discussed the problem:

[20] In a sense, the justification is evident. The Instructor is holding three cubes, and three cubes form a group of three. However, in earlier activity, class members made explicit the ideas that three ones form one three and that threes belong in the threes place.

Instructor: Three, four, one, base five [writes on screen]. Now we're in a new special group, right? Five. Okay. You ready? So, here's the issue that I heard as I came around. Everybody said, great, one plus zero: one. [Instructor writes 1 in the ones place.] Four plus two: six. How am I going to record that?

Students: Write a 1 and carry a 1.

Instructor [writes 1 in the fives place and carries 1 to the twenty-fives place]: Why would you write a 1 and record a 1 in the column to the left?

Students: You have five.

Students: That's a group of five.

Instructor: I hear lots of great voices. Only one person, that's good though. It sounded like you're all saying the same thing, but. Yeah? Go ahead, I'm sorry. Now nobody's talking. Go ahead.

Student: Because you have to move a group of five because you can't have five groups. So you have to move it over.

Instructor: So, you imagine that if you had six, you said you can't—once you have a group of five, it moves over here. Right?

Student: And then you have one left over.

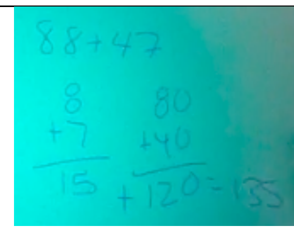
Instructor: So, these are our ones, our strips, it's a square now and it moves over here now, right? And then you had five here.

Student: So, then you have to move that one over.

Instructor: Now you don't get to hold onto it, you hand it to the person who holds the cubes, right?

In Vignette 3.2, class members co-constructed an expanded argument concerning the solution to an addition problem in base five. Base five had not been discussed previously, but students generalized from their work in other bases. Significantly, justifications that had previously dropped off when working in base three and using the cubes resurfaced in the context of doing written addition work and dealing with an unfamiliar base. Every step that the class described in the solution to the base-five addition problem made reference to regrouping actions involving people as place values and physical objects. In this way, place-value notation and the standard algorithms took on these meanings for the class.

Vignette 3.3. In the following vignette from Day 9, Muriel used a Transition strategy for addition. Specifically, she used Left to Right Separation. Muriel wrote her work on the screen, showing her data (the steps she performed) and her claim that 88 + 47 = 135. (See Figure 24). Nancy and another student contributed the warrant of the argument.

<table>
<tr><td>Claim: 88 + 47 = 135
[Written work on screen]</td><td rowspan="3">

Written record of Muriel's addition work</td></tr>
<tr><td>Data: 8 + 7 = 15; 80 + 40 = 120; 15 + 120 = 135
[Written work on screen]</td></tr>
<tr><td>Warrant: Separating numbers into tens and ones in order to add (“She broke it into ones and tens… and then combined it”)</td></tr>
</table>

Figure 24. Arg. 9.6: Muriel's Separation strategy is accepted without backing.

Muriel: [Writes on board: 8 + 7 = 15 / 80 + 40 = 120 / 15 + 120 = 135]

Instructor: [pointing to board] Okay, and so, what did she do here? What do you see her doing?

Student: She broke it into ones and tens.

Nancy: and then combined it.

Student: added the tens, added the ones, and then combined it.

Instructor: She broke it into ones and tens? And added the tens up, added the ones up and combined. We've done that before in here, yeah? We've had people share this.

In Vignette 3.3, the Instructor asserted that Muriel's strategy had been previously established (“We've done that before in here, yeah? We've had people share this.”). I interpret her assertion as making explicit the acceptability of the warrant that the students had provided. The idea of separating numbers into tens and ones and then combining them did not require backing at this point. It was true that Muriel's strategy had been previously established. Amelia had used Left to Right Separation on Day 8 to add 95 and 27. However, I would conjecture that the as-if shared quality of Muriel's strategy was less specific. The class had engaged in a great deal of activity during Days 6–8 that involved grouping numbers by ones and tens (or ones and threes, or ones and eights) and collecting up and counting items. At this broader grain size, Muriel's strategy fit right in with the activity in which students had engaged over the past three days of class. I would expect that any Standard or Transition strategy for addition or subtraction would have been accepted without backing on Day 9.

Making Sense of Standard Algorithms and Transition Strategies: Conclusion. CMP3 involved making sense of standard algorithms and Transition strategies for addition and subtraction. In the interest of brevity and coherence, the examples presented involved addition, but the relevant ideas were similar for subtraction. In both addition and subtraction, the class used and interpreted place-value notation and performed regrouping operations. Students separated numbers canonically in order to add. In subtraction, reasoning in terms of take-away was an important idea. (See CMP5 more about this.)

CMP4: Reasoning Flexibly about Addition

CMP4 involved the following three as-if shared ideas that were used in argumentation concerning mental addition:

AS3. "Borrowing to Build"

AS4. Reasoning about numbers in terms of noncanonical composition

AS5. Reasoning about addition as a cumulative process of increase

These ideas occurred in arguments that related to using, justifying, representing, and comparing nonstandard addition strategies. Previously established ideas, such as canonical number composition, did not tend to be explicit in these arguments. The mathematical ideas that were foregrounded in students' arguments specifically concerned the nonstandard aspects of the mental addition strategies that students discussed.

Reasoning Flexibly about Addition: Activity. Addition computations arose in the course of place value activity, as well as in activities concerning children's mathematical thinking. On Day 8, Trina introduced and justified a Levelling strategy (Heirdsfield & Cooper, 2004) to solve the computation 95 + 27 in the context of place-value activity. On Day 9, Trina's strategy was discussed further, and students suggested names for it. The name that became official for the class was *Borrow to Build*. (See Vignette 4.1.) On Day 11, students watched a video of a child named Connor solving a subtraction problem. As part of the discussion, students discussed how Connor might solve a related addition problem, based on what they knew about his mathematical thinking.

Activity relevant to CMP4 included discussions focused on introducing, justifying, naming, comparing, and representing mental addition strategies. Aggregation, Levelling, and Compensation strategies were used and discussed. Number composition was often a central aspect of these discussions. However, the strategies discussed did not involve decomposing numbers in the conventional fashion into tens and ones. Rather, addends were decomposed conveniently in order to form friendly numbers.

Reasoning Flexibly about Addition: FAIS Criteria. AS3, *Borrowing to Build*, satisfied Criterion 2. This idea shifted from being used as a warrant in Trina's argument on Day 8 to being used as data in later arguments after Borrow to Build had become an established addition strategy. AS4, *reasoning about numbers in terms of noncanonical composition*, satisfied Criterion 2. Its role shifted from warrant to data on Day 11. AS5*, reasoning about addition as a cumulative process of increase*, satisfied Criterion 2. Its role shifted from claim on Day 9 to warrant on Day 11.

Reasoning flexibly about addition: Vignettes. I present four vignettes related to students' reasoning flexibly about addition. In Vignette 4.1, Trina introduced a nonstandard addition strategy, which class members may have interpreted in terms of aggregation or compensation. In Vignette 4.2, Valerie introduced a compensation strategy. In Vignette 4.3, students compared these two addition strategies. In Vignette 4.4, students reasoned about a child's mathematical thinking and constructed a mathematically valid justification for an aggregation strategy.

Vignette 4.1. On Day 8, students solved addition and subtraction problems in a variety of bases, including base ten. Before discussing the algorithmic work for computing 95 + 27, the Instructor asked the students to perform this computation mentally. A few strategies were discussed. In Vignette 4.1, Trina shared her Levelling strategy. Aaron and another student contributed to the justification of Trina's strategy:

> *Trina*: You take—you want to make 95 into 100. So, you just take away 5 from 27, making the 95 a hundred. And then you'll have 22 left from the 27. And then you'll add 100 and 22.
>
> *Instructor*: What do you think? Is this strategy something that's going to work for whatever addition problem she picks? Or, is it just peculiar to this problem? Go ahead.

> *Trina*: Because 95 is really close to 100, and 100's an easy number to work with.
>
> *Instructor*: So, it occurred to you because 95 is very close to 100, and 100 is a really easy number to add onto. Yeah? So there might be other kinds, other numbers like this that you'd use this with, right? So it sounds like Trina can see usefulness there.
>
> *Instructor*: Um, why does this work? Somebody besides Trina, anybody else use this strategy? Somebody else here? Eric used this strategy? Eric, why does this work?
>
> *Aaron*: You're just taking 5 from 27 and giving it to 95. So, you still—you're just moving the 5.
>
> *Instructor*: Okay, so you're saying you moved the 5.
>
> *Instructor*: You were raising your hand. Did you want to add something?
>
> *Student*: It's kind of like taking a part and putting it in a different place.
>
> *Instructor*: What are you taking apart here?
>
> *Student*: From 27, you're taking a part and giving it to the 95. You're still adding all the same numbers, just in different places.

From my perspective, the above discussion of Trina's strategy involved two aspects: Justification and strategy selection. Class members discussed why the strategy worked, in terms of its mathematical validity. They also discussed characteristics of the problem that made the strategy a nice choice—in particular, the proximity of 95 to 100. I coded these as two related arguments, A8.1a and A8.1b. Part a, which involves the description and justification of Trina's strategy, appears in Figure 25.

Based on observations of the class, as well as interviews with students, it seems to me that students may interpret Levelling strategies in terms of either Aggregation or Compensation. Under the Aggregation interpretation, students reason that a total of 27 needs to be added to 95. Trina added 5 from the 27 first, which resulted in a subtotal of

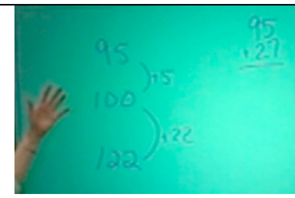
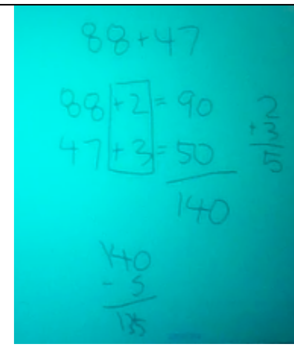
<table>
<tr><td>Claim: 95 + 27 = 122
[Trina’s description and Instructor’s written record on screen]</td><td rowspan="4">
Instructor’s written record of Trina’s strategy</td></tr>
<tr><td>Data: 27 – 5 = 22; 95 + 5 = 100;
100 + 22 = 122
(“So, you just take away 5 from 27, making the 95 a hundred. And then you’ll have 22 left from the 27. And then you’ll add 100 and 22.”)</td></tr>
<tr><td>Warrant: Moving/giving part of one addend to the other addend (“You’re just moving the 5”)</td></tr>
<tr><td>Backing: Reasoning about addition as an associative operation (“You’re still adding all the same numbers, just in different places.”)</td></tr>
</table>

Figure 25. Arg. A8.1a: Describing and justifying Trina’s strategy.

100, and then she added the remaining 22, so her final answer was 122. Under the Compensation interpretation, Trina changed the numbers prior to performing addition. She changed the problem from 95 + 27 to 100 + 22, and then she added. Since the addends were changed in a way that did not affect the sum, no end compensation was necessary.

The inscription that the Instructor produced to record Trina’s reasoning (right column of Figure 26) suggested an Aggregation interpretation. Using a pre-ENL drawing, she recorded Trina’s work as beginning with 95 and adding on 27 by first adding 5 and then adding 22 more. It is not entirely clear how Trina or the other students thought about this strategy on Day 8. The justifications offered appeared to be more along the lines of the Compensation interpretation.

Between Days 8 and 9, students were given a follow-up homework assignment that involved trying to make sense of Trina’s strategy, applying it to a different problem, and suggesting a name for the strategy. On Day 9, these ideas were discussed. Trina’s strategy was named Borrow to Build.

<table>
<tr><td>Claim: 88 + 47 = 135
(“the answer is 135”)</td><td rowspan="4">

Valerie’s record of her strategy, on screen</td></tr>
<tr><td>Data: 88 + 2 = 90; 47 + 3 = 50; 90 + 50 = 140; 2 + 3 = 5;
140 – 5 = 135
[Written work and utterances matching each sub-computation]</td></tr>
<tr><td>Warrant: Rounding and then compensating
(“I added 2 and 3, so I made both of them whole numbers… And then… I need to take 5 away”)</td></tr>
<tr><td>Backing: Reasoning about the balance of two rounding moves; reasoning about compensation as requiring the inverse operation (“Since I added a 3 and a 2, 2 plus 3 is 5, so I need to take 5 away from 140.”)</td></tr>
</table>

Figure 26. Arg. 9.2a: Valerie describes and justifies her double compensation strategy.

Vignette 4.2. In the moments before Vignette 4.2, the class was discussing how Trina’s strategy would apply to another computation. Valerie volunteered a contribution of a different sort. She said that she had thought about the task differently, and she offered to share her own strategy for computing 88 + 47:

> *Valerie*: When I read the strategy, I took it a different way.
>
> *Instructor*: Oh, good. Tell us about that.
>
> *Valerie*: I solved the problem a different way than what you just described.
>
> … *[Valerie explains that she did not understand the homework instructions. She did something different than what was asked]* …
>
> *Instructor*: Can you put it on the board up here?
>
> *Valerie*: Sure. So, for example, I used [writes “88 + 47” on screen]. So, that was my example.
>
> *Instructor*: Could you hold on a second? Hey you guys, this is a new problem. Can you do this mentally? And just make a note about how you thought about it, before you listen to her. Okay? 88 plus 47 is what she has written up here. Do it mentally yourselves before you hear about Valerie’s explanation. Think about how you’d do that problem.
>
> *[Pause while students think]*

> *Instructor*: Okay, alright. Go ahead, Valerie. Did you have time? Everybody had a chance? Great.
>
> *Valerie*: [standing at screen] So, I know in the first problem, Trina, she made the numbers whole, I mean, one of the numbers whole, so it would be easier to deal with [writes "88 + 2 = 90"], but I made both of them that way [writes "47 + 3 = 50"]. So, I added 2 and 3 [draws a rectangle around the "+2" and "+3"], so I made both of them whole numbers, so I didn't have to go through addition. And so I added 90 and 50 and got 140. And then, because I added a 3 and a 2, 2 plus 3 is 5, so I needed to take 5 away from 140, and the answer is 135.

In Arg. 9.2a, Valerie introduced and justified a Double Compensation strategy for addition. It was evident that Valerie saw her strategy as distinct from Borrow to Build ("I solved the problem a different way"). In particular, she made a distinction regarding forming friendly numbers: "Trina, she made the numbers whole, I mean, one of the numbers whole, so it would be easier to deal with, but I made both of them that way." Valerie's use of the word *whole* seemed to indicate round numbers, particularly decades. In Valerie's view, Trina (solving 95 + 27) had made one of the numbers whole by making 95 into 100. Valerie, by contrast, made both of the given numbers whole. She made 88 into 90, and she made 47 into 50. So, Valerie saw her strategy as distinct from Trina's, but it was not clear whether the rest of the class saw it that way. The focus of the initial discussion of Valerie's strategy concerned its validity.

Vignette 4.3. The Instructor then directed the class to consider the issue of whether Valerie's strategy was the same as Borrow to Build or different. Zelda made an argument that Valerie's strategy was different. However, she saw the distinction differently than Valerie had:

> *Instructor*: Is it the same as Trina's method or is it a different method?
>
> *Students*: Different method.

> *Instructor*: Different method. What's different about this? Can somebody say how they see this as different from Trina's method? Aaron and Trina have their hands up, both of them. Oh, okay. Sorry. Thanks.
>
> *Zelda*: You're not borrowing from one of the numbers. You're kind of adding two different numbers to both of them. So, it's not really like the Borrow and the Build method.
>
> *Instructor*: Maybe this isn't Borrow to Build, because what would you be borrowing from? Is that what you're saying? You're not borrowing from one of the numbers.
>
> *Zelda*: Right.
>
> *Instructor*: Yeah? Anybody have a different way they thought about that?
>
> *[No other arguments were voiced. There was apparent agreement that Valerie's strategy was different from Borrow to Build. The discussion moved forward.]*

Zelda argued (Figure 27) that Valerie's strategy was different from Borrow to Build. Like Valerie, she mentioned that both numbers were rounded, rather than just one. However, for Zelda, the key distinction between the two strategies seemed to be the origins of the amounts added. In Borrow to Build, part of one of the addends was taken and given to the other addend. So, the amount addend came from one of the given numbers. In Valerie's strategy, by contrast, amounts were added to the given numbers that came from elsewhere. (In other words, by adding 2 and 3 to the given addends, Valerie temporarily changed the sum. As a result, she had to compensate in the end. In Trina's strategy, end compensation was unnecessary because the sum was unchanged.)

Vignette 4.4. On Day 11, the class watched a video of a child named Connor solving the subtraction problem 25 – 8. Connor used an Aggregation strategy. He said, "Well, I knew that 5 plus 3 was 8, and it was 25. So, minus 5 is 20, and then minus 3 more is 17." After extensive discussion of Connor's thinking (which will be described

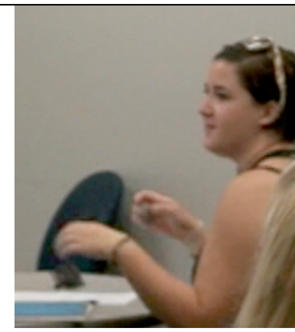
Claim: Valerie's strategy is different from Borrow to Build ("Different method.")	
Data: Valerie added to both addends without borrowing from either of them ("You're not borrowing from one of the numbers. You're kind of adding two different numbers to both of them.")	
Warrant: Borrow to Build involves borrowing from one of the addends ("You're not borrowing from one of the numbers… *So*, it's not really like the Borrow and the Build method.")	
Backing: (Implicit criterion) It matters where the amounts added originate. This is a relevant characteristic in distinguishing strategies.	*Zelda brings up both hands and makes a two-fisted tossing gesture as she says, "adding two different numbers…"*

Figure 27. Arg. 9.4: Zelda argues that Valerie's strategy is different from Borrow to Build

further in CMP5), the Instructor asked the class how they thought Connor would solve the addition problem 25 + 8.

> *Instructor*: How do you think Connor would solve, um, a problem liiiike—let's take a minute and think about this. How would Connor solve a problem like 25 plus 8 mentally? Can I write this down here? [writes on screen "25 + 8"] 25 plus 8. Just take a minute and think about that. How might Connor solve that? 25 plus 8 mentally.
>
> [Ss work quietly]
>
> *Instructor*: What do you think about how Connor might solve that mentally? Yes?
>
> *Natalie*: I think it's very similar to the subtraction problem [Multiple Ss talking]
>
> *Instructor*: Wait, um. Sorry, I want to make sure everybody's listening to you, okay? Everybody ready?
>
> *Natalie*: I think it's similar to the subtraction problem. I think he would write down the 8 and the 5 plus 3. And then he would put 25 plus 5 is 30, and then 30 plus 3 is 33.
>
> *Instructor*: Okay. Is that similar to—can somebody say what she said? Very quickly. Is that similar to how you thought about it?

Students: Yeah.

Instructor: Somebody else say what she said. Well, what—go ahead.

Mandy: Well, you add 5 to 25, and you get 30. And then you add an additional 3, for, and you get 33. And that's calculating a total 8 to 25.

Instructor: So, both of those strategies relied on him being able to see that 8 was—that you could decompose 8 into 5 and 3. Right?

Like many arguments that occurred in these discussions, this one had two aspects. In my view, Argument 11.7a concerned how Connor would solve the addition problem. The ideas relevant to that argument concerned what was known about Connor's thinking and concerned similarities between the given addition problem and the subtraction problem that Connor had solved in the video. Argument 11.7b concerned the details of and justification for Connor's hypothetical addition strategy itself. In other words, viewing that strategy in a vacuum, how does it work and why is it legitimate? I focus here on part b of the argument, in which the mathematical ideas are foregrounded (Figure 28).

Claim: 25 + 8 = 33
Data: 25 + 5 = 30; 30 + 3 = 33 ("I think he would write down the 8 and the 5 plus 3. And then he would put 25 plus 5 is 30, and then 30 plus 3 is 33.")
Warrant: Reasoning about addition as a cumulative process of increase ("Well, you add 5 to 25, and you get 30. And then you add an additional 3")
Backing: Reasoning about 8 as being composed of 5 and 3 ("Well, you add 5 to 25, and you get 30. And then you add an additional 3, for, and you get 33. And that's calculating a total 8 to 25." "So, both of those strategies relied on him being able to see that 8 was—that you could decompose 8 into 5 and 3.")

Figure 28. Arg. 11.7b: Argument that 25 + 8 = 33 by aggregation.

Class members co-constructed an argument to the effect that 25 + 8 = 33 on the basis of Addition Aggregation. They argued that it was legitimate to start with 25 and add on to it in convenient chunks. Specifically, they decomposed 8 into 5 and 3, and added 5 to 25 first. This gave a subtotal of 30. Then adding the additional 3 gave a total of 33. In many arguments related to both Aggregation and Compensation, students took advantage of number composition in convenient ways like this, rather than decomposing numbers canonically, into tens and ones. Whereas Borrow to Build may have been interpreted by students in terms of Aggregation or Compensation, the strategy discussed in Arg. 11.7 was clearly an Aggregation strategy. Class constructed a mathematically valid Addition Aggregation argument, which involved as-if shared ideas belonging to CMP4.

Reasoning flexibly about addition: Conclusion. In CMP4, students reasoned flexibly about computing sums. Activity involved a popular nonstandard strategy that the class named “Borrow to Build.” Students reasoning about numbers in terms of noncanonical composition characterized use of this strategy. Aggregation strategies were also used and discussed, and arguments related to these were grounded in reasoning about addition as a cumulative process of increase. Students engaged in using, justifying, representing, and comparing nonstandard addition strategies. Previously established ideas, such as canonical number composition, did not tend to be explicit in these arguments. Rather, students’ arguments featured mathematical ideas that were specific to the nonstandard aspects of aggregation and compensation strategies.

CMP5: Reasoning Flexibly about Subtraction

CMP5 involved activity around nonstandard subtraction strategies and a set of normative ways of reasoning related to these. The following ideas made up that set:

AS2. Reasoning about subtraction as a cumulative process of decrease

AS6. Reasoning about subtraction as a take-away process

AS7. Reasoning about adding and subtracting in terms of movement along a number line

AS8. Reasoning about difference as distance between

Reasoning about subtraction as a take-away process had come to function as-if shared earlier and was part of CMP3. Subtraction Aggregation strategies had been used and discussed beginning on Day 3. But the idea of subtraction as a cumulative process of decrease did not come to function as-if shared until Day 9. AS7 and AS8 were related to activity with the empty number line. These came to function as if shared on Day 12. All of these ideas were integral to the development of the more general activity of reasoning flexibily about subtraction.

Reasoning Flexibly about Subtraction: Activity. Activity related to CMP5 involved using, justifying, naming, comparing, and representing nonstandard subtraction strategies. Students performed subtraction mentally, reasoned about each other's strategies, interpreted children's thinking, and created and discussed drawings that conveyed the mathematics behind particular strategies. The empty number line was used extensively on Day 12, and reasoning in terms of number-line motion and differences as distances between figured heavily in students' subtraction arguments.

Reasoning Flexibly about Subtraction: FAIS Criteria. AS2, reasoning about subtraction as a cumulative process of decrease, satisfied Criterion 1. In fact, it satisfied Criterion 2 as well, but Criterion 1 seems most appropriate to the way that the idea functioned. Aaron had introduced this idea in his Subtraction Aggregation argument on Day 3 (Vignette 1.1). On Day 9, another student presented a more sophisticated subtraction strategy, which used Subtraction Aggregation as ancillary and without justification. This idea was also used in arguments on Day 11 concerning Connor's subtraction strategy.

When students provided explicit backing for Subtraction Aggregation, it was justified on the basis of AS6, reasoning about subtraction as a take-away process. This idea could be regarded as satisfying Criterion 2 or Criterion 3. There seemed to be a legitimate shift in its function from backing to data on Day 11, and so the shift criterion seems appropriate. On the other hand, reasoning about subtraction as a take-away process was used as backing in a total of five arguments on three different days of class, and so it could qualify for the repeated-use criterion. There is an argument for this since it is typical for students to think about subtraction in terms of taking away, and this is likely an idea that every student brought to the class. In any case, AS6 functioned as-if shared.

AS7 and AS8 evolved in function similarly. Both came to function as-if shared on Day 12, satisying Criterion 2. Both shifted from being used as rather commonplace data (not requiring justification) to being used as warrant or backing in arguments for which these ideas were fundamental to the justification of a novel strategy.

Reasoning Flexibly about Subtraction: Vignettes.I focus the examples related to CMP5 on activity from Day 12 because it was rich with discussion of subtraction ideas. I present three vignettes. These involve students making sense of and representing children's nonstandard subtraction strategies.

Vignette 5.1. The Operation Meanings, Strategies, and Algorithms unit included several tasks related to children's mathematical thinking. On Day 12, students were given three examples of children's reasoning about the computation 364 – 79. The first child's strategy was described as follows:

364 – 100 = 264

100 – 79 = 21

264 + 21 = 285

In Arg A12.12, two students justified this strategy. In Arg A12.13, Maybee present her illustration of the strategy, using the ENL. In Arg A12.14 and A12.15, the class discussed ways of modifying the illustration and offered justifications for their suggestions. In Arguments A12.16 and A12.17, the child's strategy was compared to another strategy, the resolution being that it was different. In Arg A12.18, the name "Back and Forth Subtraction" was suggested and accepted as a name for the strategy.

> *Instructor*: Okay, you had a chance to look at these and I heard some excellent discussion as I went around. The first one, people made some sense of. Why doesn't somebody take an opportunity to see if they can talk about what the child did, and let's see if we can make sense of the mathematics underlying that approach. And then make an argument if the method always works.
>
> *Instructor*: So, anybody? Thoughts on the first one? First of all, somebody tell me the steps the child did. In number one, the steps in number one? What did the child do in number one?
>
> *Betty*: From what it looks like, they subtracted 100, just thinking, you know, that's really easy to do, to get 264. But then because they had subtracted 100, they found the difference between 100 and 79, and that's 21. So, then they added that in at the end.
>
> *Instructor*: So then they added that back in? What do you think? Is that mathematically valid? Why did she add instead of subtracting that? Why add 21 at the end? She's subtracting, then you're adding.
>
> *Torrin*: She subtracted more than she needed.
>
> *Instructor*: She subtracted more than she needed.
>
> *Torrin*: To make it like easier to do mentally, to like visually see. And then she had to add back the 21 because she took away an extra 21 that wasn't necessary.

Instructor: So, she took away an extra 21 that wasn't necessary and she had to add it back? Anybody have, does that sound like the way to think about it? Did anybody have anything to add to that?

Figure 29 represents Betty's argument.

<table>
<tr><td>Claim: The first child's strategy is legitimate
[Torrin's justification supported the conclusion that the strategy was "mathematically valid"]</td><td rowspan="3"></td></tr>
<tr><td>Data: 364 – 100 = 264; 100 – 79 = 21; 264 + 21 = 285
[Computational steps given and Betty's verbal description]</td></tr>
<tr><td>Warrant: Compensating for rounding
("they subtracted 100, just thinking, you know, that's really easy to do, to get 264. But then because they had subtracted 100, they found the difference between 100 and 79, and that's 21. So, then they added that in at the end.")</td></tr>
<tr><td>Backing: Reasoning about subtraction as a take-away process; Reasoning about compensation based on the effects of rounding ("She subtracted more than she needed… And then she had to add back the 21 because she took away an extra 21 that wasn't necessary.")</td><td>[Betty gestures like she's putting something into a container as she says, "then they added that in at the end."]</td></tr>
</table>

Figure 29. Argument A12.12.

Vignette 5.2. Following Argument A12.12, Maybee presented her ENL illustration for the child's strategy (Figure 30). The class discussed Maybee's illustration and suggested some modifications to it. In Arg A12.14a, Melinda argued that Maybee's illustration would be improved if it showed the difference explicitly as the distance between 285 and 364.

Figure 30. Maybee's initial illustration.

> *Melinda:* Um, I think it would be—I really like that one, but I think that it would be even better if, like, somewhere the 79 was shown, or like, the difference between 285 and 364. Because otherwise it kinda looks a little bit more random. But, so maybe somewhere show the distance to show that it's 79.

In her argument (Figure 31), Melinda equated difference and distance between in her argument. She did not justify this meaning for the difference but rather took it as a given. Melinda's argument was accepted. In subsequent discussion of Maybee's illustration, a line segment was added that spanned the distance from 285 to 364. In response to another student's argument, an arrow was added to this segment to show that the net change depicted in the diagram was decrease, or movement to the left. Figure 32 depicts the final version of Maybee's illustration. The class then discussed names for this strategy, and "Back and Forth Subtraction" became the official name. Instructor made an argument for the appropriateness of this name, using gesturing to highlight the back-and-forth nature of the strategy as depicted on the ENL.

> *Instructor*: Can I make a suggestion? How about something like Back and Forth Subtraction? Does that kind of convey, you know, the fact that you add on and then take away. Something like that? … Because there's something about that illustration, especially when I see that empty number line. Back and Forth Subtraction… I think this number line illustrates this pretty nicely, the back and forth part.

Vignette 5.3. The second child's strategy for computing 364 – 79 was presented as depicted in Figure 33. In Arg A12.19, Trina offered a justification for this strategy. In Arg A12.20, Valerie presented her group's justification for the strategy. In Arg A12.21, Amelia presented her drawing, which was related to Valerie's justification.

> *Instructor*: So, I heard some discussion that this method might always work or it might not always work. That was one key discussion as I went around. And some people thought that maybe this method didn't always

<table>
<tr><td>Claim: Maybee’s diagram could be improved by showing the difference of 79.
(“I really like that one, but I think that it would be even better if, like, somewhere the 79 was shown”)</td><td rowspan="3">
M’s gesturing highlights the difference of 79 as the distance between the number-locations 364 and 285 in the diagram</td></tr>
<tr><td>Data: The diagram does not make the difference obvious; the difference between 285 and 364 is the distance between them, which is 79.
(“…if, like, somewhere the 79 was shown, or like, the difference between 285 and 364… so maybe somewhere show the distance to show that it’s 79.”)</td></tr>
<tr><td>Warrant: (Implicit criterion) Diagrams should make details like the difference explicit
(“I think that it would be even better if, like, somewhere the 79 was shown, or like, the difference between 285 and 364. Because otherwise it kinda looks a little bit more random.”)</td></tr>
</table>

Figure 31. Argument A12.14a: Melinda argues that Maybee’s diagram would be improved by making the difference of 79 explicit.

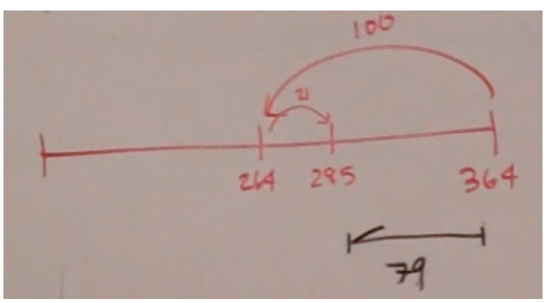


Figure 32. Back and Forth Subtraction: (left) The final version of Maybee’s illustration; (right) Instructor gesturing movement on the number line reminiscent of the illustration.

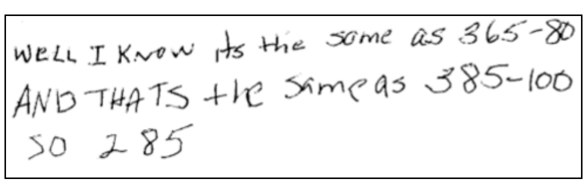

WELL I KNOW its the same as 365-80
AND THATS the same as 385-100
SO 285

Figure 33. Second child’s strategy for 364 – 79.

> work or that it worked better with some number than others. Those are two different questions aren't they? Does this method always work?
>
> *Trina*: I was going to say, when I first thought about it, I didn't understand how, by making both the numbers like one bigger would make it equal out in the end. But as I drew out my diagram, I noticed that by making the number you're subtracting from one bigger and making the number you are subtracting one bigger, you're really like adding one and then subtracting one more, so it like evens out.

Figure 34 represents Trina's argument. Valerie also argued for the validity of the second child's strategy. However, her argument was different than Trina's. Trina reasoned in terms of Compensation and argued that the two rounding moves, adding to the minuend and adding to the subtrahend, evened out. Valerie's group thought in terms of a number line instead. They interpreted adding to the minuend and subtrahend as shifting number-locations to the right, while mainting the distance between them.

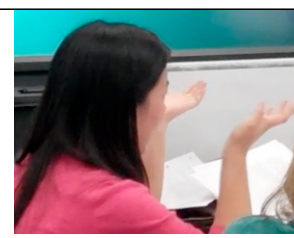
<table>
<tr><td>Claim: The second child's strategy is legitimate
[Trina's argument supports the conclusion that the child's strategy "always works"]</td><td rowspan="4">
Trina turns her hands palm-up and performs a balancing action, raising one hand while lower the other as she says, "evens out"</td></tr>
<tr><td>Data: Making equal additions to the minuend and subtrahend
("making both the numbers like one bigger")</td></tr>
<tr><td>Warrant: Reasoning about the net effect of two rounding moves
("so it like evens out.")</td></tr>
<tr><td>Backing: Adding one to the minuend increases the difference; Adding one to the subtrahend decreases the difference.
("by making the number you're subtracting from one bigger and making the number you are subtracting one bigger, you're really like adding one and then subtracting one more")</td></tr>
</table>

Figure 34. Argument A12.19.

> *Valerie*: Okay, so we thought about it in terms of, when you're subtracting, you're trying to find the distance between two numbers. So, we thought of it kind of in terms of a number line. So, if you—uh, for the first one—when he, or whoever the student was, made it 365 and 80. So, you started off with 79 and 364. So, 364 moved up one to 365 and also, likewise the 79 moved up to 80. So, the distance didn't change between the numbers. So, originally it was right here, and they both moved up one

> on a number line. So, the distance between them is the same. So, similarly when you have 385 and 100, you just added 21. So, if you took the numbers from their original position and moved them each up 21 spaces, the shift would be the same and the distance between both numbers is the same.
>
> *Nancy*: And the numbers would be easier.
>
> *Valerie*: Yeah.

The backing in Valerie's argument (Figure 35) brought together two ideas. Reasoning about adding and subtracting in terms of movement along a number line had been used previously in arguments concerning Back and Forth Subtraction. Reasoning about the difference in subtraction in terms of a distance between number-locations had also been used previously, as in Melinda's argument. Although both of these ideas occurred in arguments concerning Back and Forth Subtraction, they were used independently. Valerie used both ideas in conjunction to support her warrant that adding the same amounts to the minuend and subtrahend did not change the difference. She reasoned that adding to the minuend and to the subtrahend shifted both of them to the right. Since the amounts added were the same, the shift was the same. Therefore, the distance between the number-locations did not change.

Amelia's argument (Figure 36) was similar. In fact, she cited Valerie's argument and provided much less explicit detail herself. Her contribution was a drawing that illustrated the fact that the difference remained constant when the same amount was added to the minuend and subtrahend:

> *Amelia*: It's basically what she said, just illustrated. Like, they're the same distance apart. Like you just add twenty to each. So, it just shows that they're exactly the same.

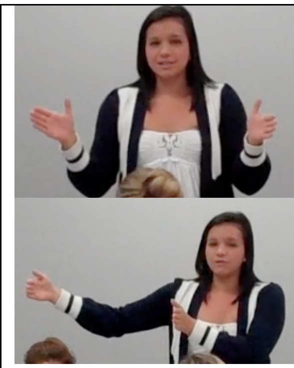
<table>
<tr><td>Claim: The second child’s strategy is legitimate
[Valerie’s argument supports the validity of the child’s strategy]</td><td rowspan="4">
Valerie holds her hands approximately equi-distant. She moves both hands to her right as she talks about the numbers “moving up.”</td></tr>
<tr><td>Data: The child added the same amounts to the minuend and subtrahend
(“you took the numbers from their original position and moved them each up 21 spaces”)</td></tr>
<tr><td>Warrant: Adding the same amounts to the minuend and subtrahend does not change the difference
(“So, if you took the numbers from their original position and moved them each up 21 spaces, the shift would be the same and the distance between both numbers is the same.”)</td></tr>
<tr><td>Backing: Reasoning about difference as distance between number-locations; Reasoning about adding in terms of movement along a number line
(“So, the distance didn’t change between the numbers. So, originally it was right here, and they both moved up one on a number line. So, the distance between them is the same.”)</td></tr>
</table>

Figure 35. Argument A12.20.

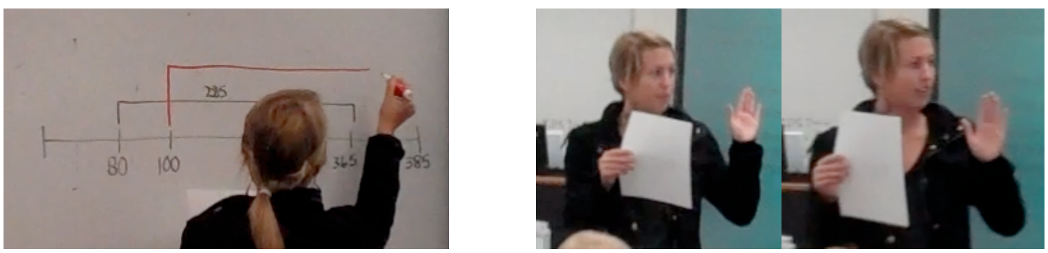


Figure 36. (left) Amelia’s drawing showing that 365 – 80 is equal to 385 – 100; (right) Amelia gestures a shifting action as she says, “they’re the same distance apart.”

As Vignette 5.4 illustrates, on Day 12, the idea of Difference as Distance Between came to function as if shared. This idea had been used as data in Melinda’s argument (A12.14a). In Valerie’s argument, this idea functioned in a new way as it was used to justify a novel subtraction strategy. This shift in the way that the idea functioned is reflected in the contrast between Maybee’s illustration and Amelia’s (see Figure 37). In Maybee’s illustration, the difference of 79 had the meaning of a net action of moving to the left 79 units. It was included to show the distance of 79 between the number-locations

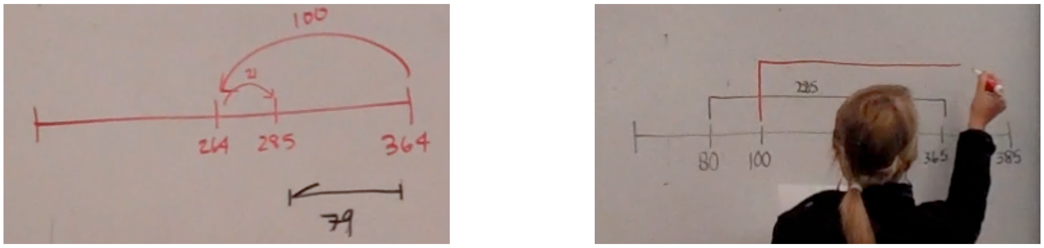


Figure 37. (left) Maybee's final illustration: 364 and 285 are locations, and 79 is represented as the directed distance between them; (right) Amelia's illustration: 80 and 365 are locations, as are the pair 100 and 385. The distance between each pair is 285.

285 and 364. In Amelia's illustration, by contrast, 285 had the meaning of the distance between number-locations. The action involved was not an action of subtracting by moving left. Rather, it was a shifting action, whereby both number-locations were moved until they reached a place that made the distance between them was readily apparent. (Amelia did not explicitly illustrate the original locations of 79 and 364. Her illustration focused on the bigger shift from 80 and 365 to 100 and 385.)

Reasoning flexibly about subtraction: Conclusion. In CMP5, students used and reasoned about nonstandard subtraction strategies. Reasoning about subtraction as a take-away process had come to function as-if shared earlier. In CMP5, it was integral to students' making sense of subtraction aggregation and compensation strategies. Reasoning about subtraction as a cumulative process of decrease also played a key role in students' sense making. Also, the empty number line was used productively as students justified subtraction compensation strategies based on reasoning about adding and subtracting in terms of movement along a number line and reasoning about differences as distances between.

Summary of Development from CMP1 to CMP5

The progression from CMP1 to CMP5 represents a case of number sense development, in the sense of sociogenesis. In terms of collective activity, the class progressed from (CMP1) relying on the standard algorithms to (CMP2) making sense of place value to (CMP3) using an understanding of place value to make sense of the standard addition and subtraction algorithms, as well as Transition strategies, to (CMP4) reasoning flexibly about addition and (CMP5) reasoning flexibly about subtraction. As Figure 38 illustrates, this progression through classroom mathematical practices constitutes an *actual learning route*, which parallels the envisioned learning route described by Nickerson and Whitacre (2010).

Actual learning route	CMP1	CMP2 & CMP3	CMP4 & CMP5	
Envisioned learning route	Standard	Transition	Nonstandard	Nonstandard w/Reformulation

Figure 38. Correspondence between CMPs and envisioned learning route.

Viewing the progression through CMPs in terms of the Standard-to-Nonstandard framework, CMP1 corresponded to the Standard category. In this initial stage of collective activity, the class relied on standard algorithms. CMP2 and CMP3 correspond to the Transition category. By making sense of place value and relating that understanding to the standard algorithms, the class transitioned from using the MASAs for addition and subtraction to using Right to Left and Left to Right strategies. CMP4 and CMP5 correspond to the Nonstandard categories (with and without reformulation). The class reasoned flexibly about addition and subtraction by making sense of and using various nonstandard strategies.

The details that the analysis revealed serve to flesh out number sense development as a sociogenetic process. These details include the particular ideas that functioned as-if shared for the classroom community, as well as how those ideas were used in arguments that were integral to advancing the mathematical activity (Rasmussen, Zandieh, King, & Teppo, 2005) of the class. In further research that is beyond the scope of this dissertation, the results of this analysis will feed back to inform revisions and elaboration to Nickerson and Whitacre's (2010) local instruction theory for the development of number sense.

Classroom Mathematical Practices: Multiplication

In this section, I describe activity related to multiplication that took place during the multiplication and division strand of the course. Students performed multiplication and division computations in the service of problem solving during Days 2–8 prior to the lessons focused on multiplication and division ideas, which occurred during Days 13–17. The totality of activity related to multiplication and division constitutes the multiplication and division strand.

In the section titled Classroom Mathematical Practices: Place Value, Addition, & Subtraction, CMP1, CMP2, and CMP3 were related to both addition and subtraction. CMPs 4 and 5 involved students reasoning flexibly about addition and subtraction, respectively. The analysis revealed a progression in students' activity related to both operations. This section, by contrast, will focus exclusively on multiplication. This is because the vast majority of students' activity during Days 13–17 concerned multiplication, rather than division. There were as-if shared ideas related to division, and students justified the standard division algorithm and a Transition strategy for division. However, there was insufficient division-focused activity and insufficient advancement in reasoning about division in order to tell a story of sociogenesis involving division ideas. Therefore, in this section, I present a story of the sociogenesis of number sense that is specific to multiplication.

I identified the following four classroom mathematical practices in the strand of activity related to multiplication:

CMP1. Assuming the authority of the standard algorithms

CMP6. Separating, multiplying, and adding with single-digit multipliers

CMP7. Making sense of separating, multiplying, and adding with double-digit multipliers

CMP8. Reasoning flexibly in computing and estimating products

This section presents the progression through these four CMPs. For each CMP, I describe the mathematical activities in which students engaged. I describe the as-if shared ideas belonging to the CMP and explain the FAIS criteria satisfied by each. I also present selected classroom vignettes, which serve as examples of collective activity and illustrate shifts over time in the roles of particular ideas in argumentation.

CMP1: Assuming the Authority of the Standard Algorithms

In the activity involving addition and subtraction, the first classroom mathematical practice involved computing with the standard algorithms, assuming the authority of these algorithms, and following the conventions of the algorithms. Since this CMP has been described in some detail, I include a single vignette related to multiplication. The multiplication strand spanned a longer time frame than addition and subtraction. Multiplication was used during Days 6–8, but multiplication ideas did not become the focus until Day 13. The standard multiplication algorithm functioned with authority, despite not having been justified mathematically, through the early part of Day

14. Vignette 1.3 represents the last case of the standard multiplication algorithm functioning in this way. Later, students justified the algorithm. (See Vignette 7.4.)

Vignette 1.3. On Day 14, students worked on a problem concerning rectangular area in the context of carpeting a room:

> A sorority needs to buy new carpet for the floor of their meeting room. The room measures 23 feet by 23 feet. They need to know how much carpet to buy. What is the area of the room?

The Instructor first asked student to mentally compute the area of the room. Initially, the class agreed on an incorrect answer of 409 square feet. After a few minutes discussing the problem in their groups, many students had changed their answers. In the subsequent whole-class discussion, a student reported a revised answer of 529 square feet. Kim made an argument for this answer simply by stating that she had used the standard algorithm. (Many other students had done the same.) There was no disagreement, and 529 became the new consensus answer. Figure 39 represents Kim's argument.

Claim: A room measuring 23ft by 23ft has an area of 529 square feet
Data: 23 x 23 = 529 (*Instructor*: …when you multiplied it out, you got 529)
Warrant: Reasoning about operations in terms of the standard algorithms (*Kim*: Standard algorithm.)

Figure 39. Arg M14.2: Kim's argument concerning the product of 23 and 23.

> *Instructor*: Now, all I want right now is an argument for the answer being 529. How many people think it's 409? How many people think the answer is 409? We changed our minds, oh. And what changed your mind? Does everybody else say 529? How many people think 529? Now the class says 529. What changed your minds? Why isn't it 409?
>
> *Kim*: Standard algorithm.
>
> *Instructor*: Because you did the standard algorithm and when you multiplied it out, you got 529. Alright we've got something important to talk about here. How come the answer isn't 409?

Subsequently, a great deal of productive activity occurred around the Carpet Problem. (That activity will be described in some detail in the description of CMP7.) In the prelude to Vignette 1.3, students attempted to solve the problem by mental multiplication. However, when the answer obtained by a nonstandard strategy did not agree with the answer obtained by the standard algorithm, the alternative was immediately dismissed. The activity shifted to making sense of why the nonstandard strategy that many students had used did not work. At this point, the standard multiplication algorithm had not been justified mathematically, yet it functioned authoritatively. As in the addition and subtraction strand, the authority of this algorithm would be leveraged productively to motivate the need to make sense of the reasoning behind nonstandard strategies. (See Vignette 7.2.)

CMP6: Separating, Multiplying, and Adding with Single-Digit Multipliers

CMP6 involved a relatively simple case of mental multiplication that seemed readily accessible to students. This classroom math practice consisted of the following set of as-if shared ideas:

M1. Separating, multiplying, and adding

M2. Reasoning about products in terms of partial products

M3. Reasoning about multiplication in terms of repeated addition

M4. Recognizing a groups-of structure

M1–M4 represent a core set of as-if shared ideas that supported students' initial mental multiplication activity. These ideas were also used in more sophisticated ways as students' activity advanced from CMP6 to CMPs 7 and 8.

Separating, multiplying, and adding with single-digit multipliers: Activity. Mental multiplication activity first occurred during the place value unit. Students multiplied mentally in the Andrew's Apple Farm context on Day 6. They also used mental multiplication in arithmetic involving different bases on Days 7 and 8.

The central mathematical activity belonging to CMP6 was mentally multiplying using a strategy that the class called "Separate, Multiply, Add" (SMA). Students did this readily and correctly in cases of a one-digit number multiplied by a two-digit number. They would *separate* the two-digit factor into tens and ones, *multiply* the one-digit factor by those numbers of tens and ones to compute two partial products, and then *add* the partial products (M1). *Reasoning about products in terms of partial products* (M2) supported students' sense making in this initial mental multiplication activity. Students used SMA mentally in many cases. It was also used in written form, and computations performed mentally were often recorded in writing. Numerical records involved notation similar to the standard algorithm, except that the partial products were called out explicitly.

Multiplication became a focus in the course on Day 13. Students discussed ideas related to multiplication in the context of classifying and solving story problems. Students argued for why multiplication was the appropriate operation to use to solve various story problems. They did this on the basis of recognizing a groups-of structure (M4) in those problems and applying a meaning for multiplication as repeated addition (M3). Students also continued to multiply mentally using SMA.

Separating, multiplying, and adding with single-digit multipliers: FAIS Criteria. M1, separating, multiplying, and adding, came to function as-if shared on Day 8. It satisfied Criterion 2 by shifting from warrant to data. It was then used extensively as data in students' arguments. The class named this strategy "Separate, Multiply, Add" on Day 8. Students used the abbreviation SMA for this strategy, and this was an acronym for the class (i.e., they often pronounced SMA as a word: *smô*).

M2, reasoning about products in terms of partial products, was integral to students' advancing mathematical activity during CMPs 6–8. M2 actually had several different applications. The specific as-if shared idea belonging to CMP6 was M2a: reasoning about products in terms of partial products *in the case of a single-digit multiplier*. In later activity, this idea was extended to other cases, which will be addressed in the discussion of CMPs 7 and 8. M2a came to function as-if shared on Day 13. It satisfied Criterion 2.

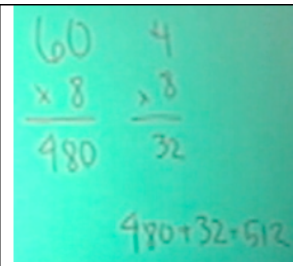
M3, reasoning about multiplication in terms of repeated addition, is a fundamental idea that students likely brought with them to the course. This idea came to function as-if shared on Day 13. It also satisfied Criterion 2. M3 occurred as backing on Day 4. It shifted to being used as a warrant in two arguments that occurred early on Day 13. Its function then shifted again, and it was used as data in later arguments on Days 13, 16, and 17.

M4, recognizing a groups-of structure, refers specifically to students recognizing a quantitative structure involving a number of groups and a number of items per group—*in connection to multiplication*. There were many instances of references to groups of items during the place value unit. However, in arguments involving M4, students explicitly used the presence of this structure as justification for the idea that multiplying the number of groups by the number of items per group would give the total number of items. M4 came to function as-if shared on Day 13, satisfying Criterion 1. The warrant for it dropped off.

Separating, multiplying, and adding with single-digit multipliers: Vignettes. I present three vignettes related to CMP6. The first vignette illustrates students' using separate, multiply, and add with single-digit multipliers and reasoning in terms of partial products during the place-value unit. The second vignette illustrates students' conceptions of the separating, multiplying, and adding strategy as they participate in a strategy-naming discussion. The third vignette shows how students argued for the appropriateness of multiplication on the basis of recognizing groups-of structure and reasoning in terms of repeated addition.

Vignette 6.1. On Day 6, students computed 64 x 8 (i.e., computing 8^3) to find the number of apples per truck on Andrew's Apple Farm. Jenny presented on behalf of her group (Figures 40 and 41), explaining the strategy that they had used:

Figure 40. Jenny makes separating and combining gestures as she says, "we just broke down sixty-four and then put it back together again,"

<table>
<tr><td>Claim: 64 x 8 = 512</td><td rowspan="3">

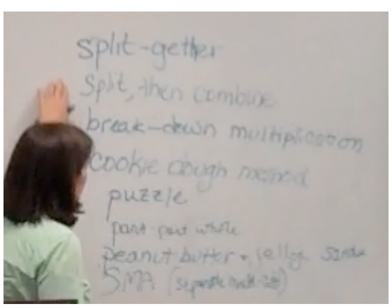
Jenny's recreated record of her strategy on the screen</td></tr>
<tr><td>Data: 60 x 8 = 480; 4 x 8 = 32; 480 + 32 = 512
Reasoning about digits in multiplication in terms of their place values: 6 in 64 represents 60
[Utterances and written work on screen]</td></tr>
<tr><td>Warrant: Reasoning about products in terms of partial products—single digit multiplier
(Jenny: So, we just broke down sixty-four and then put it back together again to get five-hundred-and-twelve.)</td></tr>
</table>

Figure 41. Arg M6.1: Jenny describes her group's strategy for computing the product of 64 and 8.

> *Jenny*: So, first we kinda separated sixty-four into two parts, the sixty and the four, kinda like we were talking about earlier, like sixty [sic] tens and four ones. And then we timesed it by eight and then added those two numbers together. So, we just broke down sixty-four and then put it back together again to get five-hundred-and-twelve. [Jenny repeatedly makes gestures that suggest separating and combining.]

On Day 7, students computed 49 x 7 to determine the size of the cubes place in base seven. As part of a homework assignment prior to class on Day 8, students were asked to reflect on the strategy that Zelda had used to compute 49 x 7 on Day 7 (and that other students had used previously).

Vignette 6.2. On Day 8, the class discussed possible names for the strategy. Eight possible names came out in whole-class discussion: (See Figure 42.)



1. Split-gether
2. Split, then combine
3. Break-down multiplication
4. Cookie dough method
5. Puzzle
6. Part-part-whole
7. Peanut butter and jelly sandwich
8. SMA (separate, mult, add)

Figure 42. Suggested names for a popular multiplication strategy: (left) Instructor listed students' suggestions on the whiteboard; (right) A typed list of the suggested names.

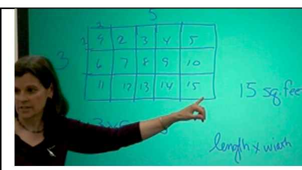
The first suggestion was *split-gether*, which the student explained as a combination of the words *splitting* and *together*. Students suggested *Split, then combine* and *Break-down multiplication*, but their arguments were not made explicit. Amelia suggested the named *Cookie dough method*. She explained:

> Amelia: I have one that is like, totally different from these. I didn't have like, split or anything. I called it the cookie dough method. Because, when you make cookie dough, you split the flour and salt and the baking soda into one, like, bowl and then you put the sugar and everything else in a different bowl, and then you combine them.

Maricela suggested *Peanut butter and jelly sandwich,* "Because on the bread, on one side you put peanut butter and on the other side you put jelly and then you put them together." Another student suggested *Puzzle*, "Because you're taking it apart like a puzzle and you have to figure out how it goes together." Torrin suggested *Separate, Multiply, Add.*

Instructor pointed out that all the names suggested conveyed what she called the "split-then-combine idea." The suggested names seemed to fall into two categories: (1) those that rather straightforwardly described splitting, then combining, and (2) those that metaphorically described splitting, then combining. The class eventually settled on a literal name and one that made the process between the splitting and combining explicit: Separate, Multiply, Add. A student argued that this name was fitting because it clearly described the process involved in the strategy.

Vignette 6.3. On Day 13, students discussed different types of multiplication story problems. Groups wrote and presented story problems, and the class discussed how these were related to multiplication. One story problem discussed was the following: *My backyard garden is 3 feet by 5 feet. How big is my garden?* Natalie solved this problem by drawing a 3-by-5 grid. She said that she had gotten her answer of 15 square feet by counting the squares. The Instructor asked the class how multiplication was related to this problem. Three arguments were offered. Libby said that you could solve by multiplying because length times width would give area, referencing a formula. The Instructor pressed further. Melinda argued that you multiply to find area because adding the side lengths would give perimeter instead. The Instructor pressed further. Finally, Trina argued that multiplication was the appropriate operation by viewing the rows and columns in terms of groups and reasoning about multiplication in terms of repeated addition. Trina's argument appears in Figure 43.

<table>
<tr><td>Claim: You multiply to find area</td><td rowspan="3">

"It's multiplication because you're adding how many in each row?"</td></tr>
<tr><td>Data: Recognizing a groups-of structure
(Instructor: There's three rows with five in each row.)</td></tr>
<tr><td>Warrant: Reasoning about multiplication in terms of repeated addition
(Trina: Doesn't 3 times 5 really mean like you're adding 3 five times?)</td></tr>
</table>

Figure 43. Arg M13.10: Trina argues as to why you multiply to compute area.

> *Trina*: Doesn't 3 times 5 really mean like you're adding 3 five times?
>
> *Instructor*: Okay, so some of you are thinking about this as not that different in some ways from a repeated addition problem. It's multiplication because you're adding how many in each row? There's three rows with five in each row.

Subsequently, students made similar arguments for multiplication concerning a variety of story problems. Two pairs of ideas characterized CMP6: (1) Students reasoned about multiplication in terms of partial products and computed products by separating, multiplying, and adding; (2) Students reasoned about multiplication in terms of repeated addition, and students viewed multiplication as the appropriate operation to use to find the total number of items in quantitative structures that consisted of a number of groups and a number of items per group.

Separating, multiplying, and adding with single-digit multipliers: Conclusion. In CMP6, students separated, multiplied, and added with single-digit multipliers. This activity involved reasoning about products in terms of partial products in a relatively simple case. Students also made arguments concerning situations in which it made sense to multiply. They did this by recognizing in those situations a groups-of structure and reasoning about multiplication in terms of repeated addition.

CMP7: Separating, Multiplying, and Adding with Double-Digit Multipliers

In moving from CMP6 to CMP7, the class transitioned from the case of a single-digit multiplier to a double-digit multiplier. CMP7 involved the following set of as-if shared ideas related to separating, multiplying, and adding with double-digit multipliers:

M1. Separating, multiplying, and adding

M2. Reasoning about products in terms of partial products

M3. Reasoning about multiplication in terms of repeated addition

M5. Reasoning about products in terms of rectangular area

M6. Reasoning about digits in multiplication in terms of their place values

M7. Removing and replacing zeroes

As-if shared ideas M1–M3 overlap with CMP6. However, as will be illustrated, these ideas were applied in more advanced ways in the activity belonging to CMP7. As-if shared ideas M5–M7 were new. M5 was particularly important to the advancement of students' mathematical activity.

Separating, multiplying, and adding with double-digit multipliers: Activity. Moving from mentally computing with single-digit multipliers to double-digit multipliers constituted a very significant transition in activity. Initially, students attempted to compute two-digit-by-two-digit products in an invalid way. Making sense of the appropriate partial products in the two-by-two case involved the application of previously established ideas in new ways, as well as the incorporation of a new way of thinking about products: in terms of rectangular area. On Day 13, multiplying to find area was justified on the basis of groups-of structure and repeated addition. On Day 14, students reasoned in depth about a rectangular area context, carpeting a room. They made drawings relating partial rectangles to partial products, and these connections supported their ability to evaluate the validity of multiplication strategies.

Separating, multiplying, and adding with double-digit multipliers: FAIS Criteria. M1–M3 FAIS criteria were discussed in the description of CMP6.

M5, reasoning about products in terms of rectangular area, satisfied Criterion 2. Rectangular area shifted from being a context in which multiplication was applied to being used as a tool for making sense of multiplication strategies. In this shift, the role of M5 moved from warrant to backing. In further activity, it was integral to the development of more sophisticated ways of reasoning about products. (See CMP8.)

M6 refers to reasoning about digits *in multiplication* in terms of their place values. As described in the first part, students often reasoned about digits in addition and subtraction in terms of place values. Doing so was part of the story of the class making sense of standard algorithms and Transition strategies. I regard M6 as a distinct idea because it was not trivial for students to bring their place value knowledge to bear on reasoning about multiplication *in the 2-digit-by-2-digit case.* In fact, it was in a student's argument on Day 14, which related the four partial products in the standard multiplication algorithm to partial rectangles, that idea M6 came to function as-if shared. It had occurred as data when students used SMA with single-digit multipliers, but its function shifted to warrant when it was used to make sense of the standard algorithm.

Separating, multiplying, and adding with double-digit multipliers: Vignettes. All of the following vignettes relate to separating, multiplying, and adding with double-digit multipliers and are from Day 14. On Day 14, the Instructor posed the Carpet Problem, which asked students to find the area of a room measuring 23 feet by 23 feet. She directed students to solve the problem mentally. One student offered an (incorrect) answer of 409 square feet, and initially there was no disagreement amongst students.

In the first vignette, Valerie argues for the validity of her (invalid) multiplication strategy. In the process, she makes a valid subargument, using the as-if shared idea of removing and replacing zeros (M7). The second vignette illustrates how students began to reason about multiplication in terms of rectangular area as a way of making sense of partial products in the case of a double-digit multiplier. The third vignette represents a transitional point at which students were able to construct a valid strategy in the case of a double-digit multiplier by decomposing only factor. They treated the product as consisting of two partial products in a valid way, but they incorrectly argued that it was not possible to decompose both factors. Finally, in the fourth and fifth vignettes, students made sense of all four partial products reason in the case of a double-digit multiplier by reasoning about products in terms of partial products (M2), reasoning about products in terms of rectangular area (M5), *and* reasoning about digits in multiplication in terms of their place values (M6).

Vignette 7.1. At the beginning of Day 14, the Instructor posed the following problem:

> A sorority had to buy carpet for the floor of their meeting room. The room measures 23 feet by 23 feet. They need to know how much carpet to buy. What's the area of the room? Actually your computation in terms of how you thought about it and then I'll take some answers and see what we get. Yeah. You should be doing it mentally, guys. Think about how we could capture it, individually and mentally. Then you must be finished. Ready? What answer?

One or more students called out 409 square feet as the answer. No other answers were offered. By a show of hands, it appeared that every student agreed with this answer. Rather than raise any issue concerning the consensus answer of 409, the Instructor simply asked students to come up with a justification for it. In the course of their group activity, many students changed their minds about the area of the room. In the next whole-class discussion, students explained how they had obtained their initial answer of 409 and also agreed on the correct answer of 529:

> *Instructor*: Are you ready to talk about it? Ready to talk about it. Okay when we started, several of you suggested 409 square feet as the answer and there wasn't any disagreement, but as I walk around, I've heard other suggestions. So now I'm going to add to this. What do you think?
>
> *Student*: 529.
>
> *Instructor*: 529. Anything else come up as you started to think about your justification for this? All right, I want to hear an argument. Tell me how you came up with 409, those of you who did and many of you did. Yes, ma'am.
>
> *Valerie*: So you want me to talk about how it got it.
>
> *Instructor*: I do.

Valerie: Okay, so I made both 23’s into 20 and I took off the zeros and I made it 2 times 2, which is 4, and then I added back all the zeros so it’s 400. And then I did 3 times 3 is 9 and so I added 400 and 9 and got 409.

Instructor: Okay, can somebody summarize their argument for how they came up with 409 or say that it’s the same or phrased a little bit differently? An argument for 409? So it sounds like you had, I’m going to go to the smart board here. Let’s see if we can capture this, okay? 23 times 23, I need some help. Okay, she said—Valerie, I’m just trying to recreate what you said okay? So you talked about taking 2 times 2, but then recognizing that they were 10’s and so it was actually 20 times 20 and 400?

Valerie: Yeah.

Instructor: And then you did what?

Valerie: And then I did 3 times 3.

Instructor: 3 times 3, 9. Those of you who got 409, is that similar to how you thought about it? Yeah.

Valerie’s argument involved reasoning about partial products in terms of pairing up tens and ones. Figure 44 describes her argument. Evidently, the same strategy was used by most (and possibly even all) of the students initially since no other answers were suggested.

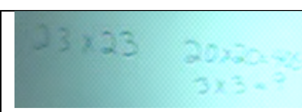
<table>
<tr><td>Claim: A room measuring 23ft by 23ft has an area of 409 square feet
(Valerie: [I] got 409.)</td><td rowspan="3"></td></tr>
<tr><td>Data: 20 x 20 = 400*; 3 x 3 = 9; 400 + 9 = 409
(Valerie: Okay, so I made both 23’s into 20… so it’s 400. And then I did 3 times 3 is 9 and so I added 400 and 9 and got 409.)</td></tr>
<tr><td>Warrant: (Implicit) Reasoning about partial products in terms of pairing up tens and ones</td></tr>
</table>

Figure 44. Arg M14.1a: Valerie’s primary argument concerning the product of 23 and 23.

Initially, the class agreed with Valerie’s solution. However, Kim’s argument (Figure 45), based on the FAIS idea assuming the authority of the standard algorithm

Claim: 20 x 20 = 400
Data: 2 x 2 = 4; (Implicit: there were 2 zeroes), so 400 (4, followed by 2 zeroes) (I made it 2 times 2, which is 4… so it's 400.)
Warrant: Removing and replacing zeroes (*Valerie*: I took off the zeroes… and then I added back all the zeroes)
Backing: Reasoning about removing and replacing zeroes in terms of multiplying by 10 and dividing by 10 (*Instructor:* So you talked about taking 2 times 2, but then recognizing that they were 10's and so it was actually 20 times 20 and 400?)

Figure 45. Arg M14.1b: Co-constructed ancillary argument concerning annexing zeroes.

(M1), trumped it (Vignette 6.1). The initial consensus answer of 409 square feet was rejected, and the correct answer of 529 square feet was accepted.

Vignette 7.2. Instructor asked students to think about why Valerie's strategy had not worked: "Alright, we've got something important to talk about here. How come the answer isn't 409? This sounds like she had a pretty good strategy here." After a few minutes of group work, whole-class discussion reconvened. Aaron's group presented a drawing and made an argument concerning missing area. Libby and the Instructor contributed to the argument:

> *Aaron*: Okay. So, we drew the box,
>
> *Instructor*: Can everybody see that, Aaron, or do you need to step back a little bit?
>
> *Aaron*: which is 23—so the whole box is 23 by 23. And she only did 20 by 20, so we drew 20 by 20 and we colored it in because that's what she did. Then she did 3 by 3 so that would be that down there, but she's missing 20 feet of the 3 on both sides.
>
> *Instructor*: So, I see some heads nodding. Can somebody else say in their own words, maybe more than one of you, but somebody say in your own words, again, an answer to that question why 409 isn't going to be enough carpeting; 409 square feet isn't enough carpeting here. Why is this not enough carpeting to cover the sorority floor? Somebody else try. This is a really good opportunity. Try saying the explanation in your words. You can do it. Go ahead.

Libby: Okay. So, it was the 20 by 20 that's that whole area and then the 3 by 3, but then you're missing a whole chunk on both sides, which is the area of 20 by 3 so you have to multiply those and then add those to it as well because you have to cover those spaces.

Instructor: So these spaces aren't covered.

Libby: Yeah, and those are 20 by 3.

Instructor: Let's hold this up a little bit more. And this is 3 feet along here because remember this entire thing was 23. This is 3 feet, and if this entire space is 23, think of this chunk as 20 and 3. So, what they're saying is there's an area of carpet here that's 20 feet long and 3 feet wide that isn't accounted for in that, right? And, likewise, up here only it's 3 feet wide and 20 feet long there. Does that make sense?

Students: Yes.

Figure 46 represents Aaron and Libby's argument. Aaron's group's drawing represented the 23-by-23 foot floor partitioned into 4 smaller rectangular areas. They related the dimensions and areas of the two shaded rectangles to the two partial products that Valerie had computed. In this way, class members argued that Valerie's strategy did not account for the whole area of the room. Two partial products had been ignored.

Vignette 7.3. In Vignette 7.2, students justified why Valerie's strategy did not give the whole area of the floor. The Instructor then asked students to figure out how to modify her strategy so that it would account for the whole area. Samantha's group conjectured that separating both factors was invalid. Instead, only one 23 should be broken apart.

Samantha: Like, we know you have to figure it out. You can't, you have to times it by 23 still, instead of splitting up both numbers. So, we just split up one 23. So, 23 times 20 is 460, and then 23 by 3 is 69, and we added those two together to get 529.

Instructor: So, can I just write this down over here? You thought about 23 times 23 as 23 times 20 and 23 times 3?

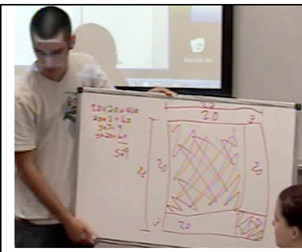
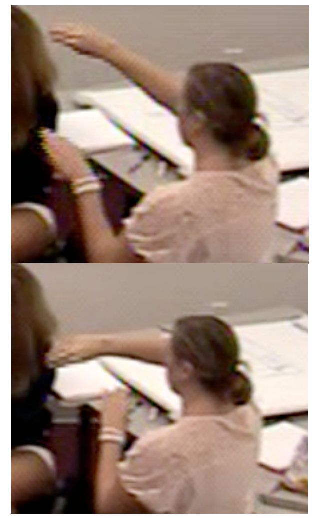
Claim: 409 square feet is not enough carpet to cover the floor (*Instructor*: 409 square feet isn't enough carpeting here.)	 *"whole box is 23 by 23."*
Data: Valerie counted the 20 x 20 square and the 3 x 3 square. But there's also a 20 x 3 rectangle and a 3 x 20 rectangle. (*Libby*: Okay. So, it was the 20 by 20 that's that whole area and then the 3 by 3, but then you're missing a whole chunk on both sides, which is the area of 20 by 3 so you have to multiply those and then add those to it as well because you have to cover those spaces.) [Libby spans two approximately equal distances with her hands, the first vertical and the second horizontal, as she says, "missing a whole chunk on both sides."]	
Warrant: Reasoning about rectangular area as consisting of partial rectangles (*Aaron*: Okay. So, we drew the box, which is 23—so the whole box is 23 by 23. And she only did 20 by 20, so we drew 20 by 20 and we colored it in because that's what she did. Then she did 3 by 3 so that would be that down there, but she's missing 20 feet of the 3 on both sides.)	*"missing a whole chunk on both sides"*

Figure 46. Arg M14.4: Co-constructed argument that 409 square feet is not enough carpet to cover the floor.

Samantha: Yes.

Instructor asked the class to think about this new strategy. She suggested that they sketch out the carpet that Samantha's strategy would account for. Many groups made similar drawings. Michaela and Torrin presented their group's solution and explain why this works.

> *Torrin*: So, we broke it up. We broke up one side to have 20 feet and 3 feet so you can see the two different areas, and we multiplied 23 and 20 and got 460 for the first big area, and we multiplied 23 and 3 and got 69, and we added them and got 529.
>
> *Michaela*: So, you can see the 23 times 20 and the 23 times 3.

Instructor: You can see each of those.

Michaela: And it makes a whole, so you needed to add.

Instructor: Okay, so these two pieces of carpet we agree would cover a square that’s 23 by 23, yeah? Everybody agree with that? Terrific. Thank you.

Figure 47 represents Michaela and Torrin’s argument.

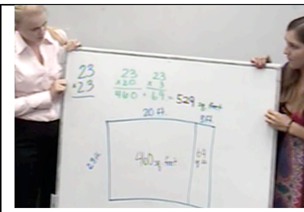
Claim: 23 by 20 and 23 by 3 areas of carpet are the right amount to cover the whole floor (*Instructor*: Okay, so these two pieces of carpet we agree would cover a square that’s 23 by 23)	
Data: A whole 23 by 23 foot square can be split into two smaller rectangles that measure 23 by 20 and 23 by 3 (*Torrin*: So, we broke it up. We broke up one side to have 20 feet and 3 feet so you can see the two different areas, and we multiplied 23 and 20 and got 460 for the first big area, and we multiplied 23 and 3 and got 69, and we added them and got 529.)	
Warrant: Reasoning about rectangular area as consisting of partial areas (*Michaela*: So, you can see the 23 times 20 and the 23 times 3… And it makes a whole)	

Figure 47. Arg M14.7: Michaela and Torrin present a justification for Samantha’s strategy.

Samantha’s strategy was identified as an example of SMA.

Instructor: So, why is this SMA? What makes this this strategy? What makes it this strategy?

Student: By breaking 23 into 20 and 3.

Instructor: You separate 23 into 20 and 3 and then.

Student: You multiply them both by 23.

Instructor: You multiply them both by 23.

Student: And you add them.

Instructor: And you add the products together.

Thus, Samantha’s strategy was related to previous instances of SMA. In the past, SMA had only been used in cases of a single-digit multiplier. Now, it was being used with a double-digit multiplier. In each case, only one factor was separated, so that there were two partial products to compute and add together.

Vignette 7.4. Meanwhile, another group had made a discovery as they were thinking about 23 times 23. They presented a way of thinking about this product as involving four partial products (Figure 48), by relating it to the standard algorithm.

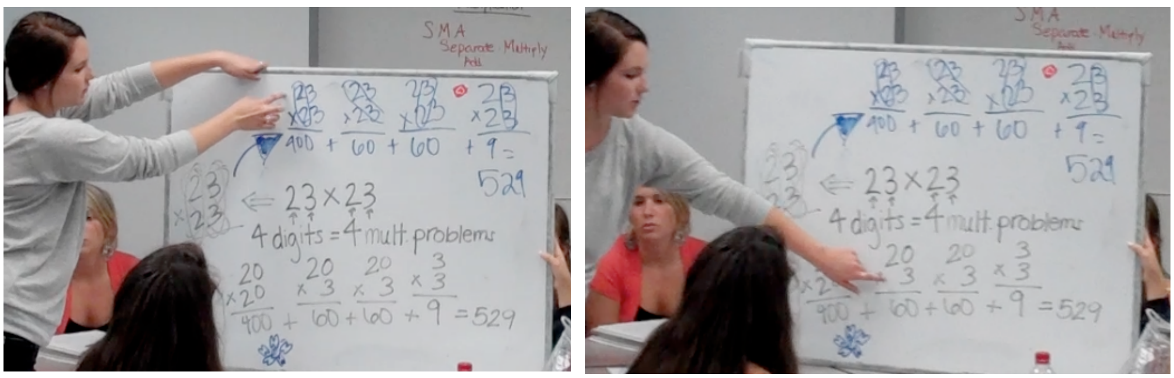


Figure 48. Gwen describes the standard multiplication algorithm in terms of partial products.

> *Instructor*: Now, are you guys ready back here? Alright, now they made an interesting connection, didn’t they? Can somebody from your group walk us through this?
>
> *Student*: So we thought about it, 20 times 23 we thought that we would have to do four different multiplications in total because there’s four digits. And we thought about like the standard algorithm of it. We split it up into what you’re actually doing, and you’re multiplying like the 3 times the 3 and then the 3 times the 2 and the 20—the 2—times the 3 and then the 20 times the 20. So, we wrote it down here: 20 times 20 is 400, 20 times 3 is 60, 20 times 3 is 60, and 3 times 3 is 9. You add them all together, and it’s 529.

The Instructor then added to Gwen’s argument (Figure 49) by pointing out a connection to the diagram that Samantha’s group had presented earlier.

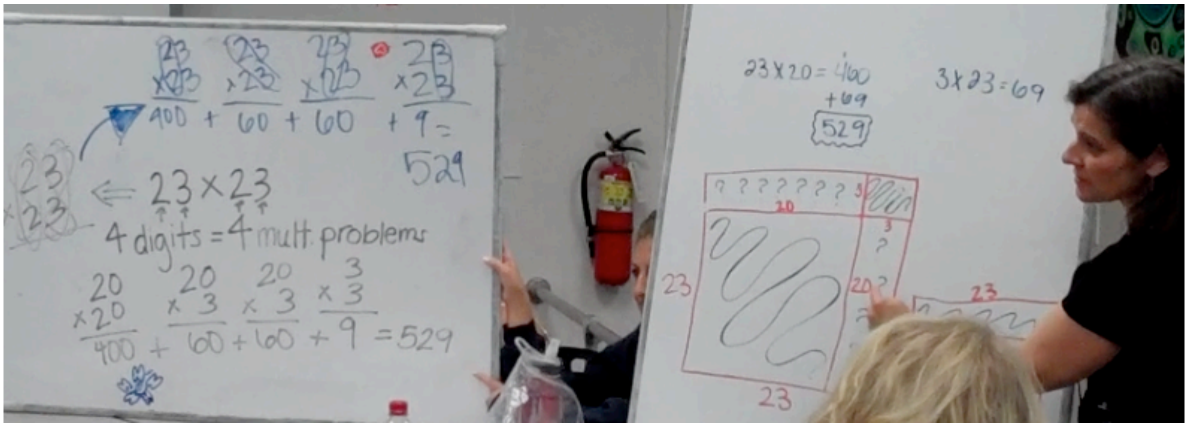


Figure 49. Instructor compares Gwen's group's numerical representation with Samantha's group's rectangular area drawing.

> *Instructor*: So, you remember our picture. And if we look at these two things side by side, where do you see there 3 times 3 in the standard algorithm? We said in the area of this corner here, right? How about, she has two different multiplications here, the 20 times 3. Sorry, I guess the next thing, yeah, you have 20 times 3 next, and we have a piece of carpet that corresponds to that, right, the same area. And then she has 3 times 20 and then this group has 20 times 20.

Samantha's group had represented the partial products pictorially, in terms of partial rectangles. Gwen's group had represented them numerically. There was a correspondence between the two representations based on reasoning about products in terms of rectangular area (Figure 50).

Gwen's group's spontaneous discovery related the standard double-digit multiplication algorithm to partial products for the first time in the class. In this way, they were able to sensibly account for all four partial products. The class first considered these ideas in the context of area. In the following vignette, later on Day 14, as-if shared ideas M2, M5, and M6 were used as students considered children's multiplication work.

Vignette 7.5. One child solved 62 x 54 by explicitly computing all four partial

Claim: In the standard algorithm, you're actually computing four partial products and adding them together (*Student:* And we thought about like the standard algorithm of it. We split it up into what you're actually doing…)
Data: Using the standard algorithm, you do bottom 3 times top 3, bottom 3 times top 2, bottom 2 times top 3, and bottom 2 times top 2 [Drawing highlights the four multiplication steps in the standard algorithm] Reasoning about products in terms of partial products: The steps in the standard algorithm correspond to computing the products 20 x 20, 20 x 3, 3 x 20, and 3 x 3 (*Student:* you're multiplying like the 3 times the 3 and then the 3 times the 2 and the 20—the 2—times the 3 and then the 20 times the 20.)
Warrant: Reasoning about digits in multiplication terms of their place values: The 2 in each 23 is interpreted as representing 20
Backing: Reasoning about products in terms of rectangular area—specifically, reasoning about partial products in terms of partial rectangles (*Instructor*: So, you remember our picture. And if we look at these two things side by side, where do you see there 3 times 3 in the standard algorithm? We said in the area of this corner here, right? How about, she has two different multiplications here, the 20 times 3. Sorry, I guess the next thing, yeah, you have 20 times 3 next, and we have a piece of carpet that corresponds to that, right, the same area. And then she has 3 times 20 and then this group has 20 times 20.)

Figure 50. Arg M14.9: Co-constructed argument concerning partial products in the standard multiplication algorithm.

products and adding them together. Students justified this strategy by relating it to partial rectangles.

> *Instructor*: So, there's some very similar things here. Several of you drew a nice diagram and you have some numerical sentences to accompany that. And I'm looking at this table maybe. Would you be willing to give an explanation? Because an explanation accompanies these. Remember diagrams aren't self-evident, so we need a justification. Is this child's strategy correct?
>
> *Students*: Yeah.
>
> *Instructor*: Yes? Does everybody think so? Anybody think this child's strategy is not correct? I seem to see all around the room that we believe this. Alright, would you guys share this?
>
> *[Students get situated to present.]*
>
> *Torrin*: Okay, so, the problem was 62 times 54. So, we drew a box and broke it up into 60 and 2 like he did and 50 and 4, and then we multiplied

each piece. So, we started with 60 and 50 and got 3,000. And then we multiplied 50 and 2 and got 100 and then 60 times 4 and that's 240 and 2 times 4 is 8. So those are all the different areas and we add them to get the total, which is 3,348, which is what he got.

Instructor: So I see a couple things here. I see that you bring down numerically basically the steps that he did. He said 50 times 60 or he or she did, we'll say he. 50 times 60 is 3,000. 60 times 4, and you see all those pieces add up. Tell me how the diagram helped you think about this.

Torrin: Because you can see it as a whole. You can see all the different parts and how they add together, like the portions.

Michaela: It's all one. It's all one big box. It's kind of like the area problem even though there's no specific area mentioned in that problem.

In their argument (Figure 51), Michaela and Torrin made sense of the child's strategy by reasoning about the total product in terms of rectangular area. They drew a "box" and related each of the partial products in the child's work to partial rectangles in their picture.

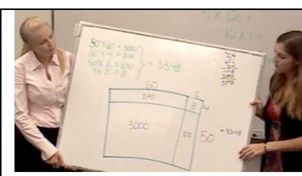
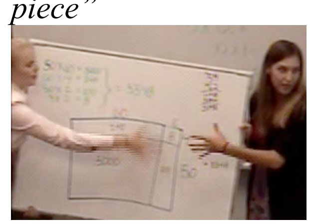
<table>
<tr><td>Claim: The child's strategy is legitimate</td><td rowspan="2">
"we multiplied each piece"</td></tr>
<tr><td>Data: Computing all four partial products/areas explicitly
60 x 50 = 3000, 50 x 2 = 100, 60 x 4 = 240, 2 x 4 = 8, and then added them up to get 3348.
[Child's work, Torrin and Instructor's utterances, diagram]</td></tr>
<tr><td>Warrant: Reasoning about partial products in terms of partial rectangles
The partial products that the child found correspond to partial areas of a 62 by 54 rectangle
(Torrin: Because you can see it as a whole. You can see all the different parts and how they add together, like the portions.)</td><td rowspan="2">
"It's all one bix box."</td></tr>
<tr><td>Backing: Reasoning about products in terms of rectangular area
(Michaela: It's kind of like the area problem even though there's no specific area mentioned in that problem.)</td></tr>
</table>

Figure 51. Arg M14.14: Michaela and Torrin relate rectangular area to partial products.

Separating, multiplying, and adding with double-digit multipliers: Conclusion. In CMP7, students made sense of partial products in the case of a double-digit multiplier. Activity involved bringing together the previously established as-if shared ideas of reasoning about multiplication in terms of repeated addition, separating, multiplying, and adding, and reasoning about products in terms of partial products. These were coordinated with new as-if shared ideas of reasoning about products in terms of rectangular area and reasoning about digits in multiplication in terms of their place values. These ideas were essential to students' ability to solve the puzzle of separating, multiplying, and adding with double-digit multipliers.

CMP8: Reasoning Flexibly in Computing and Estimating Products

CMP8, reasoning flexibly in computing and estimating products, involved the following set of as-if shared ideas:

M2. Reasoning about products in terms of partial products

M5. Reasoning about products in terms of rectangular area

M8. Comparing weights

M9. Halving and doubling

M2 and M5 overlapped with previous CMPs. They belong to CMP8 because they were integral to the development of the mathematical activity that characterized CMP8. As-if shared ideas M8 and M9 were unique to CMP8. Therefore, they will be the focus of discussion here.

Reasoning flexibly in computing and estimating products: Activity. On Days 16 and 17, students reasoned about estimation of products. They evaluated children's estimation strategies. They also compared possible estimates of products and argued over which would give the closer estimate. CMP8 involved the most sophisticated mathematical activity in the multiplication strand. It was focused on the idea of *weight.* This is not a commonly known mathematical topic, so I include a brief explanation below.

The mathematics. The term *weight* was used in the class to describe the amount that part of a factor contributes to a product. For example, in the product 32 x 83, the 2 in 32 "weighs more" than the 3 in 83, meaning that it contributes more to the product. If we compare two possible estimates of this product (as students did in a task on Day 16), 32 x 80 versus 30 x 83, we find that 32 x 80 is the closer estimate. We can see this by thinking in terms of partial products. If we take 32 x 80 as the estimate, we have lost the partial product 32 x 3. If we take the 30 x 83 as our estimate, we have lost 2 x 83. Clearly, then, 32 x 80 is the closer estimate because 3 x 32 is less than 2 x 83.

Reasoning about products in terms of partial products (M2) was integral to activity concerning weight. Reasoning about products in terms of rectangular area (M5) often supported students' partial-products arguments. These ideas also contributed to students' making a new observation about relationships between products, which led to a specialized strategy for computing products (M9).

Reasoning flexibly in computing and estimating products: FAIS Criteria. M2 and M5 were discussed earlier.

M8, *comparing weights*, is shorthand for coordinating the comparison of parts of a factor (including amounts added in rounding) with the comparison of corresponding multiplicands. In other words, M8 is specific to comparing weights *in terms of partial products*. This idea became as-if shared on Day 16. As discussed in Chapter 3 in the section concerning reliability, comparing weights can be seen as satisfying Criterion 1, 2, or 3. Comparing weights was used as data in students' arguments 12 times, spanning Days 16 and 17. In one instance on Day 16, the warrant for this idea dropped off. Also, in the last argument made on Day 16, M8 shifted from data to warrant. Thus, comparing weights can be seen as satisfying Criterion 1, 2, and 3.

Reasoning flexibly in computing and estimating products: Vignettes. The three vignettes that follow are focused around the idea of comparing weights. In Vignette 8.1, students consider an estimation task and begin to reason in terms of weight. In Vignette 8.2, students apply this reasoning to other tasks and the class explicitly discusses weight in reference to several examples. In Vignette 8.3, an idea that arose in the context of discussing weight becomes the focus and evolves into a new, nonstandard multiplication strategy.

Vignette 8.1. Day 16 began with students evaluating the estimation strategies of several children for the product 36 x 55. The class discussed Maria's strategy in detail. Students were told that Maria reasoned about the estimation as follows: "Rounding both up would make it a little too big, so I'll round 36 to 40 and 55 to 50. 40 x 50 = 2000." Katelyn argued that Maria's estimate was a good one because her rounding moves more or less cancelled each other out. Students were asked which of Maria's rounding moves had more effect. Several students made arguments.

Melinda argued that rounding 55 down to 50 had more effect than rounding 36 up to 40 because the difference of 5 was greater than a difference of 4, "so you're changing the number by more." Amelia argued for the opposite claim, that rounding 36 to 40 had more effect:

Amelia: This might be like a far stretch, but when you do 55 times 36, that's fifty-five 36's. So when you take away 5, you're taking away five 36's, whereas when you're adding 4 to 36, you're adding four 55's. And four 55's is larger than five 36's.

Instructor: What do you think about what Amelia says?

Students: Yayyy!

Instructor: Heads nodding. Yay? You like that logic?

Amelia's argument (Figures 52 and 53) was accepted over Melinda's argument. Amelia's argument was supported by two as-if shared ideas: Reasoning about products in terms of partial products (M2) and reasoning about multiplication in terms of repeated addition (M3). I would conjecture that the immediate and enthusiastic acceptance of her argument was owing to the fact that it was supported by normative ways of reasoning. Figure 52 represents Amelia's argument.

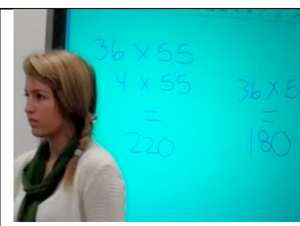
Claim: Rounding 36 to 40 has more effect than rounding 55 to 50	
Data: Comparing weights 4 x 55 > 5 x 36*	
Warrant: Reasoning about products in terms of partial products (applied to rounded amounts) [Amelia views the rounded amounts, 4 and 5, as representing partial products.]	

Figure 52. Arg M16.4a: Amelia's argument about the differential effects of rounding moves.

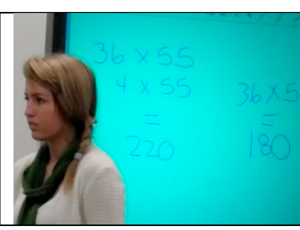
Claim: 4 x 55 > 5 x 36	
Data: Reasoning about multiplication in terms of repeated addition (*Amelia:* …when you do 55 times 36, that's fifty-five 36's…)	

Figure 53. Arg M16.4b: Amelia's subargument that 4 x 55 was greater than 5 x 36.

Students spent time in groups reflecting on Amelia's argument and coming up with their own justifications for the agreed-upon conclusion that rounding 36 to 40 had more effect. Two groups presented drawings and made arguments. The first, like Amelia's argument, involved reasoning about multiplication in terms of repeated addition. The second involved reasoning about products in terms of rectangular area.

Vignette 8.2. The Instructor asked the class to mentally determine which estimate of 32 x 83 would be closer to the exact answer, 32 x 80 or 30 x 83. Initially, many students raised their hands in support of each option. Initial arguments were made in support of each conclusion. Gwen had originally said that 30 x 83 would be the better estimate. Her reasoning was, "Because you're taking away less… you're taking, for the 32 times 80, you're taking away three extra numbers whatever and then for 30 times 83, you're only taking away two." Amelia made an argument that involved reasoning in terms of partial products. Students took a few minutes to discuss this question in their groups. When whole-class discussion reconvened, Gwen had changed her mind, and she made a new argument:

> *Instructor*: Yeah, would you mind? You don't need the whiteboard. Why don't you just kind of verbally share out what your thinking was.
>
> *Gwen*: Okay. Well for the 32 times 80, there are three more 80's that you're not accounting for in the problem so you have to multiply 3 times the 32 and then for the 30 times 83, there's two 30's that you're not accounting for so you have to multiply those 2 times the 83 and since you would think that 2 is less than 3, so 30 times 83 might work, but 83 is a lot greater than 32, so two, 83's is still a lot larger than three, 32's.

Although they concerned somewhat different questions, Gwen's argument (Figures 54 and 55) about estimates of the product of 32 and 83 was very similar to Amelia's

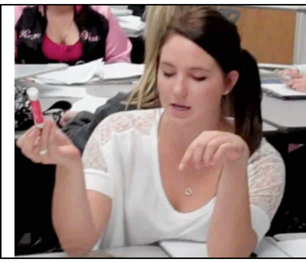
Claim: Reasoning about rounded amounts additively is invalid	
Data: Comparing weights It's true that 3 > 2, but we're comparing 3 x 32 and 2 x 83. 3 x 32 < 2 x 83*	
Warrant: Reasoning about products in terms of partial products	

Figure 54. Arg M16.10a: Gwen's argument about the closer estimate of 32 x 83.

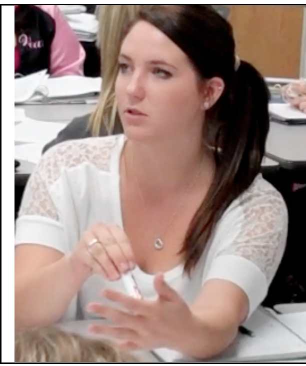
Claim: 3 x 32 < 2 x 83	
Data: Reasoning about multiplication in terms of repeated addition (*Gwen*: Okay. Well for the 32 times 80, there are three more 80's that you're not accounting… and then for the 30 times 83, there's two 30's that you're not accounting for… 83 is a lot greater than 32, so two 83's is still a lot larger than three, 32's.)	

Figure 55. Arg M16.10b: Gwen's subargument that 3 x 32 was less than 2 x 83.

argument about the effects of Maria's rounding moves. The class had yet to give a name to this idea, but both Amelia and Gwen's arguments involved comparing weights.

The Instructor introduced the term *weight*, making reference to Gwen's argument as an example. She also introduced a Geometer's Sketchpad sketch, which was used by the class in a discussion of the weights of numbers of units in various products. Students argued that in 13 x 12, the 3 weighed more; in 44 x 66, the 4 and the 6 weighed the same; in 25 x 35, the 5 in 25 weighed more; and in 13 x 26, the 3 and the 6 weighed the same. For the most part, these arguments were similar in form to Gwen's argument. Students compared weights by comparing the appropriate partial products. Many arguments made reference to the sketch and described partial products in terms of areas. Toward the end of class, students made more general claims about cases of products (e.g., when the numbers of units are the same) and how the weights of the numbers of units compared in each case.

Vignette 8.3. Halving and doubling. Although mental multiplication activity in the class mostly involved applications of the distributive property of multiplication over addition (as in SMA), the associative property of multiplication was also arose in strategies and discussions. To trace the idea of halving and doubling, we briefly step back to Day 14. This story then intertwines with weight beginning on Day 16.

On Day 14, students considered children's multiplication work. Activity included watching a video clip from an interview with a child named Javier and discussing his thinking about multiplication computations. In the video, Javier computes 6 x 12 to find the number of eggs in 6 cartons (6 dozen). His primary strategy involves the distributive property. He says that 5 x 12 is 60, and then he adds 12 more to get an answer of 72. The class discussed this strategy, identifying it as an example of SMA. In the next segment of the video, the interviewer asks Javier how he knew that 5 x 12 was 60. Javier explains that he knows 12 x 10 is 120, and half of 120 is 60. The class discussed this piece of Javier's reasoning:

> *Instructor*: Wow, what'd he do, you guys?
>
> *Trina*: He doubled the 5 to 10, and after he finished
>
> *Instructor*: He doubled the 5 to 10, and after he got the—after he finished, Tricia says
>
> *Trina*: He split it in half.
>
> *Student*: Divided by 2
>
> *Instructor*: He split it in half or divided by 2, yeah? So he doubles the 5 and when he's finished he divides by 2.

Javier used a strategy that involved doubling one factor while then halving the second factor. The discussion of this idea on Day 14 was brief. However, related ideas arose on Days 16 and 17.

In the discussion of weight on Day 16, students tended to compute partial products in order to compare them (e.g., 4 x 55 = 220, 36 x 5 = 180, and 180 < 220). However, Muriel suggested a different way of comparing partial products to assess weight. The class considered the product 13 x 26 and discussed how the weights of the 3 in 13 and the 6 in 26 compared. Muriel noticed that in 13 x 26, "13 is half of 26, and 3 is half of 6." She asserted that this meant the 3 and 6 were of equal weight. However, Muriel was not able to articulate a clear mathematical justification. She offered as a warrant, "Knowing what's half of what." Her observation was confirmed with the rectangular-area sketch.

On Day 17, the class revisited weight and particularly Muriel's observation. The Instructor asked students how they could tell without calculating that 3 x 26 was equal to 6 x 13. Melinda made an argument (Figure 56) that leveraged reasoning about products in terms of rectangular area. The Instructor contributed by revoicing Melinda's argument and illustrating it concretely (Figure 57). The transcript follows:

> *Melinda:* So I started by drawing like an area box thing for 3 by 26. So that's the 3; this is the 26. Can I make it bigger?
>
> *Instructor*: Yeah, if you use the eraser, it will. Go ahead. I don't know.
>
> *Melinda*: Yeah, that's a really bad rectangle. And then to like kind of compare it to this box, I kind of drew them together. I don't know, I'll show you right now. So to make 6, I added 3 with 3 and then for 13, I did half of 26. So this altogether would be 6, and this part is 13 because it's half, obviously. And so to show which areas were which, I shaded this one this way and then I shaded this 3 by 26 this way and then you see that

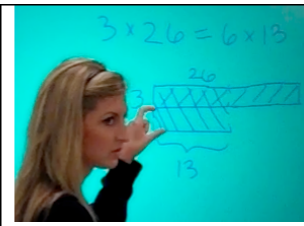
Claim: 3 x 26 = 6 x 13	
Data: Both products consist of two 3-by-13 rectangles (*Melinda*: So to make 6, I added 3 with 3 and then for 13, I did half of 26. So this altogether would be 6, and this part is 13 because it's half, obviously. And so to show which areas were which, I shaded this one this way and then I shaded this 3 by 26 this way and then you see that they're all like equal.)	
Warrant: Rearranging partial rectangles (*Melinda*: So then since they're equal, no matter which way you like kind of arrange them, they'll be in the same area like groups of two.)	

Figure 56. Arg M17.3: Melinda argues that 3 x 26 is equal to 6 x 13.

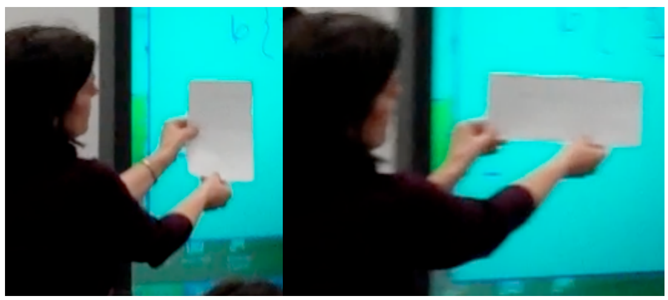


Figure 57. Instructor revoiced Melinda's argument, using two equal-sized pieces of paper to illustrate how they could be rearranged, as in Melinda's drawing.

they're all like equal. So then since they're equal, no matter which way you like kind of arrange them, they'll be in the same area like groups of two.

Instructor: Do you have any questions for her? Does everybody understand that idea?

Students: Yes.

Instructor: I heard her say something like this. She's got something; you can pretend that's like 3 by 26, and then she's looking at essentially these 3 by 13 inch rectangles and saying I can do 6 by 13 or I can do, right? You're kind of talking about how that area is conserved even though it's rearranged like this. It's a nice way to look at it, huh? Okay. Great. Thank you.

Subsequently, halving one factor and doubling the other was discussed as a strategy for mental multiplication. A student suggested that it would be useful to apply it to 4 x 16. One could transform 4 x 16 into 8 x 8, which is a known product. She explained, “Well, 4 times 16 isn’t a very easy thing to do, but if you times 4 by 2. If you times 4 by 2, then you get 8 and if you divide 16 by 2, you get 8.” The class went on to name this strategy. *Double, then Split* was suggested as a name, but it was rejected when Jenny argued that the strategy did not have to involve doubling: “The only thing about doubled then split is that sometimes it won’t always be two, right? Like you could split up, like say you have 15, you could do 5 and 3; that’s technically not doubled.” *Equal Area Shifting* was adopted as the official name based on arguments that it did not have the limitation of being too specific and it made reference to the area picture.

Reasoning flexibly in computing and estimating products: Conclusion. In CMP8, students reasoned flexibly about computing and estimating products. They continued to reason about products in terms of partial products and in terms of rectangular area, and these ideas supported a new way of reasoning, comparing weights. Discussions of weight, in turn, led to the articulation and justification of a new as-if shared multiplication strategy that involved the associative property.

Summary of Development of Multiplication Activity

The progression through multiplication CMPs represents a second case of sociogenesis of number sense. In terms of collective activity around multiplication, the class progressed from (CMP1) assuming the authority of the standard algorithm to (CMP6) separating, multiplying, and adding with single-digit multipliers to (CMP7) separating, multiplying, and adding with double-digit multipliers to (CMP8) reasoning flexibly in computing and estimating products.

Figure 58 describes the actual learning route traversed by the class with regard to reasoning about computing and estimating products. Viewing the progression through CMPs in terms of the Standard-to-Nonstandard framework, CMP1 corresponds to the Standard category. In this initial stage of collective activity, the class assumed the authority of the standard multiplication algorithm.

Actual learning route	CMP1	CMP6	CMP7	CMP8
Envisioned learning route	Standard	Transition	Nonstandard	Nonstandard w/Reformulation

Figure 58. Correspondence between CMPs and envisioned learning route.

In CMP6, students used SMA in the case of single-digit multipliers and reasoned about multiplication in terms of repeated addition. Having reflected on the actual learning route, I view CMP6 as corresponding to the Transition category. I had not previously thought about single-digit SMA as a Transition strategy for multiplication. The interview tasks in this study and the previous study all involved 2-by-2-digit products. However, it was evident from the class activity that single-digit SMA was readily accessible to students. It followed easily from reasoning about multiplication in terms of repeated addition, which is a commonplace mathematical idea that students bring with them to the course. Single-digit SMA served as a Transition strategy in students' progression from dependence on the standard algorithm to reasoning flexibly about multiplication. Through separating, multiplying, and adding, students came to reason about multiplication in terms of partial products, which helped to lay the groundwork for making sense of multiplication in the 2-by-2 case.

I view CMP7 as corresponding to the category Nonstandard without Reformulation. Double-digit SMA is a Nonstandard multiplication strategy. In CMP7, students used previously established ideas concerning partial products, together with reasoning about products in terms of rectangular area, to make sense of 2-by-2-digit multiplication as involving four partial products. This enabled them to both justify the standard algorithm and move beyond it by correctly applying SMA to the case of a double-digit multiplier.

CMP8 corresponds to the category Nonstandard with Reformulation. Students carried their reasoning in terms of partial products and rectangular area further in reasoning about estimates of products by comparing weights. Students also used and justified halving and doubling, which is a Nonstandard strategy with reformulation.

Semi-sequential Nature of Multiplication CMPs

The CMPs presented in Part 1 had minimal overlap in terms of time and structure. They occurred in chronlogical order, and only a small minority of the as-if shared ideas belonged to more than one CMP. In the multiplication strand, by contrast, there was substantial temporal and structural overlap between CMPs. This was due, in part, to the order of topics in the curriculum. Whereas the class made sense of the standard addition and subtraction algorithms on Days 7 and 8, the class did not make sense of the standard multiplication algorithm until Day 14 (and the standard division algorithm on Day 15). Thus, viewed as inclusive of each operation, CMP1 spanned much of the whole-number portion of the course. At the same time, students were encouraged to perform mental multiplication beginning on Day 6, and they used single-digit SMA beginning on that day. So, with respect to multiplication, in particular, CMP1 and CMP6 ran in parallel, both of them spanning Days 6, 7, 8, 13, and part of 14. Figure 59 illustrates the chronological relationship between the multiplication CMPs.

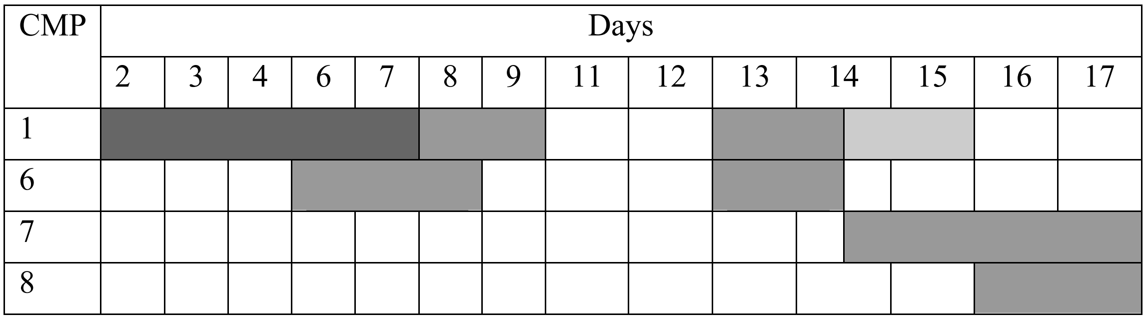


Figure 59. Chronological relationship between CMPs 1, 6, 7, and 8.[21]

CMP7 began when CMP6 ended on Day 14, as students made sense of partial products in the 2-by-2-digit case. On the other hand, these CMPs overlapped structurally, as described in this section. CMP7 preceded CMP8 chronologically. However, these co-existed on Days 16 and 17.

Conclusion

The methodology of Rasmussen and Stephan (2008) afforded a rigorous analysis of collective activity in the mathematics content course: (Phase 1) coding of 208 individual arguments, using Toulmin’s model, led to (Phase 2) the identification of as-if shared ideas, based on the three criteria, and this, in turn, led to (Phase 3) the categorization of these ideas as belonging to more general mathematical activity, i.e., the identification of classroom mathematical practices. In this chapter, I presented two

[21] CMP1 is operation-inclusive. The darkest cells correspond to the period in which the class assumed the authority of all four standard algorithms. The medium-gray cells are specific to multiplication and division. The light gray is limited to the division algorithm.

progressions through CMPs, the first involving a strand of activity related to place value, addition, and subtraction, and the second focused on multiplication. I conceptualize these as two cases of sociogenesis of number sense.

Each sociogenetic case study represents an actual learning route of number sense development on the collective level. I found that these paralleled the envisioned learning route in terms of the progression along the spectrum from Standard to Nonstandard. At the same time, the analyses fleshed out the developmental process for both of the content strands. The analyses reported in this chapter thus addressed prospective elementary teachers' number sense development from the social lens. On the microgenetic level, particular class members made mathematical arguments. However, in looking chronologically across the set of arguments made in whole-class discussion, and considering whether or not these were accepted, the unit of analysis was not a single individual or many individuals; rather, it was the classroom community, taken as one entity. As a result, the analyses with respect to each content strand led to single, coherent accounts of number sense development, specific to that strand.

Taking into account the psychological lens, I recognize that individual students reasoned in different ways. Through the establishment of normative ways of reasoning, a foundation was laid for the class to make sense of novel arguments regarding nonstandard strategies, such as Valerie's argument concerning Shifting the Difference or Amelia's weight argument. The evolution of classroom mathematical practices up to those points set the stage for these arguments, which were founded on previous established as-if shared ideas.

Individual students made crucial contributions in the process of advancing the mathematical activity of the class. Each argument made in whole-class discussion was constructed by an individual or co-constructed by a few individuals. In this sense, the reasoning of those individuals served as a resource for the class in the process of number sense development. In the case of Valerie's argument, reasoning about the difference as a distance between number-locations had previously been established. However, it took an argument such as Valerie's to apply that way of reasoning in the justification of a new and more sophisticated subtraction strategy. Her argument put together pieces of a puzzle in a way that had not occurred before in whole-class discussion.

In fact, there are different stories of number sense development on the individual level. These vary in the extent of improvement. They vary in starting and ending points. But they also vary qualitatively in terms of how individual students reasoned and how their reasoning developed over time. In the next chapter, I shift from the social to the psychological lens and report two cases of ontogenesis of number sense, which correspond to the two content strands according to which this chapter was organized. In Chapter 6, I focus on accounting for change on the individual level in ways that capture the nuances of students' reasoning.

Chapter 6: Results Part 3: Ontogenesis of Number Sense

Chapters 4 and 5 addressed microgenesis and sociogenesis of number sense. This chapter addresses the ontogenesis of number sense. I present two case studies. In the section titled Brandy's Developing Understanding of Addition and Subtraction concerns Brandy's developing understanding of addition and subtraction as she moved from inflexible to flexible. . In the section titled Valerie's Developing Understanding of Multiplication describes Valerie's developing understanding of multiplication. Each case study begins with a detailed description of the participants' initial mathematical reasoning. I account for the development of the participants' reasoning over time by drawing on the participants' written work and coordinating changes in her reasoning with the emergence and establishment of as-if shared ideas. I then present a detailed account of the participants' reasoning in the second interview. I also include additional insights into each case that came from the SST interviews.

In the Conclusion section, I take a step back from these two case studies and consider what they suggest about the ontogenesis of number sense in prospective elementary teachers. I consider how insights gained from these cases may be useful beyond the cases themselves. I also relate these to the local instruction theory. Both cases contribute to my understanding the phenomenon of prospective elementary teachers' number sense development and include findings that represent contributions to the field. In the third section, I summarize and discuss these points.

Brandy's Developing Understanding of Addition and Subtraction

Brandy represents a story of number sense development in the area of whole-number addition and subtraction. She grew from being MASA-bound in mental addition and subtraction to reasoning flexibly about both operations. Her improved flexibility can be accounted for in terms of the interaction between knowledge that she brought with her to the course and ideas that she was exposed to in class. Brandy was able to make sense of Standard and Transition strategies with the help of an enhanced understanding and awareness of place value. She made sense of Nonstandard strategies from class on the basis of ways of reasoning that were familiar to her but applied in new ways. Brandy was aware of changes in her reasoning, and the pre/post contrasts that she reported were corroborated by my analyses.

This section describes how Brandy's reasoning developed over a period of about two months. The richest descriptions come from her interviews. The first interview provides detailed evidence of her initial reasoning about addition and subtraction, as well as some evidence of her relevant understanding of place value. The second interview involved the same tasks, and so it affords a contrast with her first interview responses. In between these, her written responses to homework and test questions provide evidence of her reasoning, especially with regard to particular strategies. The results of the sociogenetic analysis presented in Chapter 5 allow for the coordination of Brandy's developing reasoning with particular classroom events and with the establishment of normative ways of reasoning. Following the second interview, additional homework and exam responses provide additional evidence of how Brandy's reasoning changed. Finally, her SST interview responses provide evidence of Brandy's own reflections on her experience in the course.

Brandy's Initial Reasoning about Addition and Subtraction

This subsection describes Brandy's initial reasoning about addition, subtraction, and place value, on the basis of first interview and NSRS pretest data. Brandy was an 18-year-old freshman Liberal Studies major. She reported having taken algebra, algebra 2, geometry, and math analysis in high school. Brandy scored 21 of 37 on the NSRS pretest, which placed her in the Low group, relative to the class. Brandy performed mental addition and subtraction inflexibly. When invited to perform mental computation in different ways, she entertained Compensation strategies. Brandy was capable of computing sums with Double Compensation and producing valid arguments. This was not the case for subtraction. She considered no alternative strategies. In discussing the standard algorithms, Brandy displayed an unsophisticated understanding of place value, essentially reasoning about numbers as being composed of digits with no place-value meaning. At the same time, outside of the context of the standard algorithms, she was capable of viewing multidigit numbers as being composed of tens and ones.

Brandy's addition and subtraction strategy ranges. In her first interview, Brandy reasoned inflexibly about addition and subtraction. For her, addition and subtraction mapped to the MASA, regardless of the given numbers. Even the computations 38 + 99 and 125 – 49 she solved with the MASAs. She made no apparent choice of strategy, despite the proximity of 99 to 100 and of 49 to 50. As far as her responses to the basic mental computation tasks indicated, Brandy's was blind to any affordances of particular numbers. Only the operation was a consideration. Once she determined that she was going to add or subtract, these operations were simply performed using the MASAs. Figure 60 depicts Brandy's initial strategy ranges for addition and

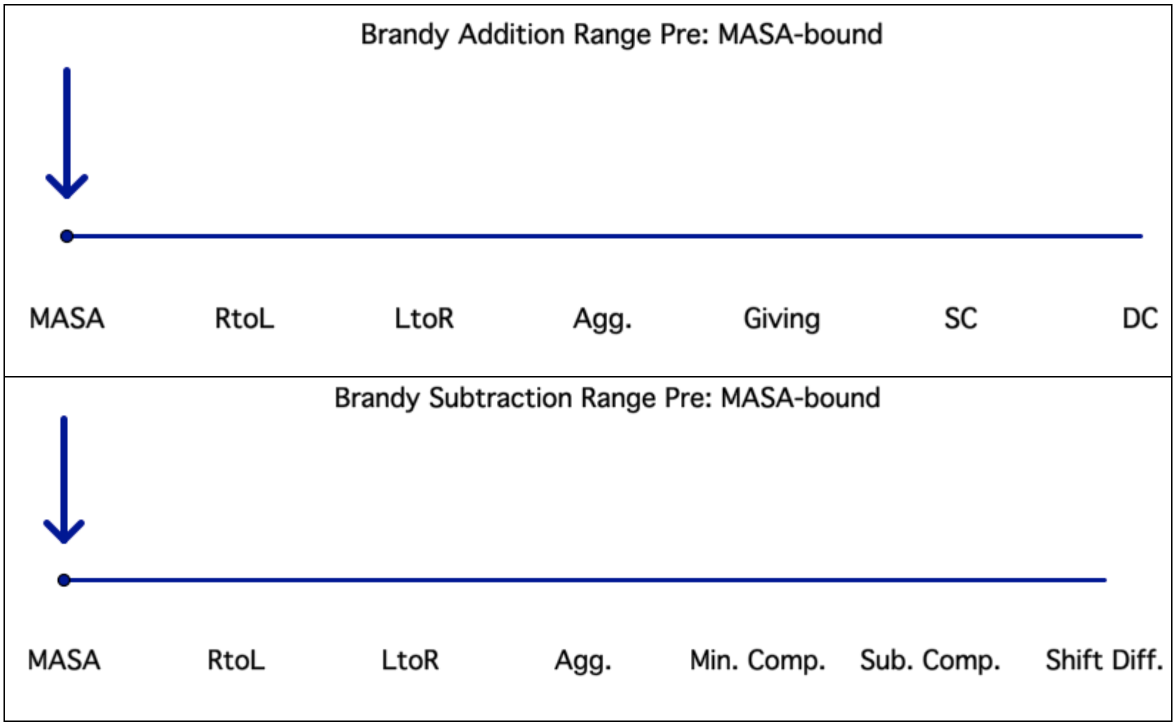


Figure 60. Brandy's initial range of subtraction strategies.

subtraction. Of the seven distinct, valid mental addition strategies that prospective elementary teachers in our research have used, Brandy used one—the MASA. Similarly, of the seven distinct, valid mental subtraction strategies that our research participants have used, Brandy used only the MASA.

Brandy's scaffolded alternatives. In the pre interview, I also investigated Scaffolded Alternatives. As a reminder, after the basic Bobo tasks, participants were given an additional computation for each operation. This time, they were asked to perform the given computations in multiple ways. After one explanation had been completed, the participant was asked if she had another way of solving the problem. Participants were asked this until they said that had no other way or until they had used five different strategies. In attempting to account for Brandy's number sense development, her Scaffolded Alternatives from the pre interview are viewed as resources that she brought with her to the course. These did not belong to her strategy ranges proper because they only arose in the context of the scaffolded tasks. However, these do provide additional evidence concerning Brandy's initial reasoning about addition and subtraction.

In the basic Bobo tasks for addition and subtraction, Brandy used the MASAs exclusively. However, in the Scaffolded Alternatives tasks, she did entertain the possibility of non-MASA approaches. In the case of both addition and subtraction, she perceived the possibility of rounding both numbers to decades before computing. This rounding did not need to be conventional. The numbers could be rounded up or down. She also recognized the need to compensate for these rounding moves. She reasoned about compensation straightforwardly, applying the same reasoning to both addition and subtraction: If she rounded a number up, she had effectively added, and she would need to subtract the same amount to compensate. Likewise, if she rounded a number down, she had effectively subtracted, and she would need had to add the same amount to compensate. This reasoning is valid for addition, and so Brandy had a valid Scaffolded Alternative for mental addition. When it came to subtraction, however, Brandy did not distinguish the role of subtrahend versus minuend in reasoning about compensation. Her Scaffolded Alternative for subtraction was an invalid compensation strategy.

Beyond compensation, Brandy considered no other possibilities for mental addition and subtraction. She said she could not think of any other way of performing the given computations. In terms of Brandy's development from inflexible to flexible, I view her compensation reasoning as belonging to her ZPD with respect to mental addition and subtraction. Although her strategy ranges proper were MASA-bound, her scaffolded strategy ranges were Polarized. She could choose between the MASA and Double

Compensation. (She used essentially the same Double Compensation strategy for both operations. With respect to rounding and compensating, her reasoning was identical.)

Brandy's Scaffolded Strategy Range for addition is represented in Figure 61. In the ZPD, Brandy's addition reasoning was Polarized. She could imagine using either the MASA or her Double Compensation strategy. Her alternative strategy was a legitimate one, and she was able to provide a mathematically valid justification for it. Brandy's subtraction reasoning in the ZPD was also Polarized. However, in this case, the alternative that she considered was not legitimate. She entertained rounding both the minuend and subtrahend and compensating for the balance of the rounding moves. Her reasoning with regard to minuend compensation was correct, but she did not reason any differently about subtrahend compensation. (Because I do not include invalid strategies in participants' strategy ranges, I do not include here a figure depicting Brandy's scaffolded strategy range for subtraction.)

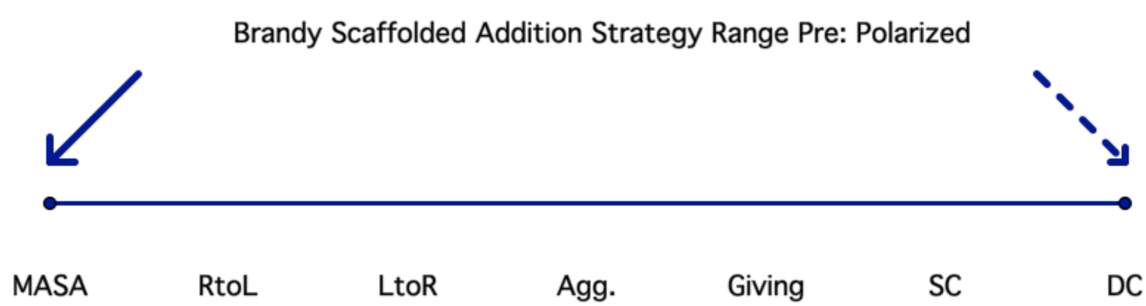


Figure 61. Brandy's scaffolded strategy range for addition in the pre interview.

Brandy's understanding of the standard algorithms. The Operations Tasks from her pre interview provide evidence of how Brandy understood the standard addition and subtraction algorithms at the beginning of the course. Essentially, she thought about the algorithms in terms of digits that inhabited columns. The behavior of these digits in the algorithms was governed by rules, (e.g., “you have to borrow one”), conventions (e.g., where and how to write the borrowed 1), and dogmatic justifications (e.g., “you can’t subtract a bigger number from a smaller number”).

In the Standard Addition Task, Brandy added 9 and 8 in the ones place and obtained a sum of 17. She wrote 7 in the ones place for the sum and carried 1 to the tens place. Her justification for the carrying move was that a two-digit number could not be written in a single column. It was not clear that she saw the carried 1 as representing ten. In fact, she was given so much opportunity to express that idea during the task that the absence of such a statement suggests she did not see the 1 as representing ten. As far as her actions and utterances indiciate, in the context of the algorithm, 17 meant a 1 and a 7 next to each other. Figure 62 depicts Brandy’s argument concerning carrying the 1 in addition. The following is the complete transcript of Brandy’s response to the Standard Addition Task:

> *Interviewer:* I’d like you do to 259 plus 38 in just the normal written way that you would do it.
>
> *Brandy* [writing as she speaks]: So, 9 plus 8 is 17. So, I put the 7 here [pointing] and bring up the 1. So, 5 plus 3 is 8, plus 1 is 9. And bring down the 2. So, the answer is two hundred and 97.
>
> *Interviewer*: Okay. Can you tell me about the little 1 that you put up there?
>
> *Brandy*: Oh, yeah. ‘cause so, 9 plus 18 is 17 [writes 9 + 18 = 17]. So, you – if you could see it [gesturing frantically] it would be like 17 still [pointing at the 1 and the 7] but like if you put the 7 here and bring up the 1, to make it look like that [points at the 17 that she wrote to the side]. And then I would add it [gestures completing the algorithmic work]. Does that make- or?
>
> Interviewer: Yeah. Why do you put the 7 down there and the 1 up there?
>
> *Brandy:* Well, because if you were gonna put the 7 up here and the 1 right there, it would look like it’s 71 when it’s not? I guess.

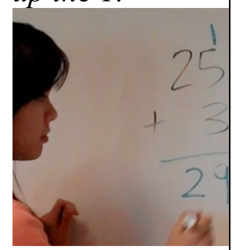
Claim: You have to carry a 1 ("I put the 7 here [pointing] and bring up the 1.")	
Data: 17 is a two-digit number ("Yeah, or 'cause like, you couldn't do 9 plus 8. You couldn't do that [writes 1 next to 7 in the answer so that it reads 2917 instead of 297].")	*"I put the 7 here and bring up the 1."*
Warrant: Only one-digit numbers can be written in a single column ("Yeah, or 'cause like, you couldn't do 9 plus 8. You couldn't do that [writes 1 next to 7 in the answer so that it reads 2917 instead of 297].")	
Backing: Rules/conventions that must be followed ("I just know you just do.")	*"You couldn't do that"*

Figure 62. Brandy's argument about carrying the 1 in the pre interview.

> *Interviewer:* Okay.
>
> *Brandy:* Yeah, or 'cause like, you couldn't do 9 plus 8. You couldn't do that [writes 1 next to 7 in the algorithmic answer so that is reads 2917 instead of 297]. You'd have to – I'm not sure why you bring it up [erases the 1 next to the 7]. I just know you just do. And, um, yeah 'cause it still looks like 17 [pointing]: 1 and 7. You're yeah, bringing it up. (Brandy, personal communication, September 13, 2010)

In the Standard Subtraction Task, Brandy borrowed in the typical fashion. She said that she needed to borrow to make the minuend larger in order to subtract ("you can't subtract 3 from 2"). In circular fashion, Brandy argued that borrowing 1 turned the 2 into twelve because of the fact that the 2 needed be made larger, and 12 satisfied that requirement. She also emphasizes the placement of the "little imaginary one"—that it was written to the left of the 2, as opposed to above the 2 or to its right. Brandy talked as if writing the 1 elsewhere might change its value. Brandy's arguments are depicted in Figures 63a and 63b.

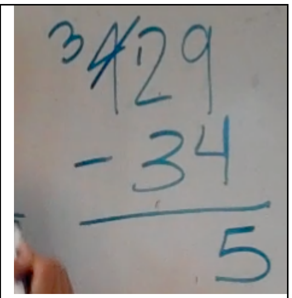
Claim: You have to borrow 1 *("you have to borrow from the next number. So, what you would do is, you'd cross out 4 and put 3 right there. And put this like little imaginary 1 right there.")*	
Data: 2 is less than 3; 12 is more than 3 *(" 'cause 2 is smaller than 3, you have to make it bigger. So, by putting 1 on the left side, makes it 12, which is bigger than 3.")*	
Warrant: You can't subtract a larger number from a smaller number (within a given column); The little 1 together with the 2 becomes 12* *("since 2 is smaller than 3, you have to borrow from the next number"; "you have to make it bigger.")*	

Figure 63a. Brandy's argument that you have to borrow 1.

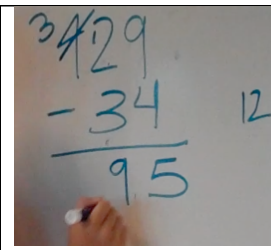
Claim:* The little 1 together with the 2 becomes 12 ("So, by putting 1 on the left side, makes it 12")	
Data: The little 1 is written to the left of the 2; 12 is bigger than 2 ("So, by putting 1 on the left side, makes it 12, which is bigger than 3.")	
Warrant: You need to make the 2 bigger, and turning it into 12 makes it bigger ("have to make it bigger than this one [points to the 3 in the subtrahend]")	

Figure 63b. Brandy's sub-argument concerning borrowing the 1.

Interviewer: I'd like you to do a subtraction one on either side of that [your addition work], whatever you prefer. But go ahead and leave it up for now.

Brandy: Oh. Okay.

Interviewer: And I'd like you to do 429 minus 34.

Brandy: [writing] So, 9 minus 4 is 5. Two minus 3, since 2 is smaller than 3, you have to borrow from the next number. So, what you would do is, you'd cross out 4 and put 3 right there. And put this like little imaginary 1 right there. Because that little imaginary 1 makes it look like – makes it 12 [writes 12 over to the side]. So, 12 minus 3 is 9. Since you borrowed from 4, it makes it 3. So, you just bring down the 3, and the answer would be 3 hundred and 95.

Interviewer: Okay. Um, so, tell me about the little one in this one.

> Brandy: Oh, well, ‘cause like, since like you’re borrowing—you’re borrowing 1—just one number—just one from this one [points to the 4]. So, you [slashing gesture]—it’s kind of like 4 minus 1 equals 3. You have that one left over, so you have to bring it to this 2 to make it 12. Well, you have to make it—you just have—you have to put it on the left side to make it the bigger—‘cause you have to make it bigger than this one [points to the 3 in the subtrahend] ‘cause 2 is smaller than 3, you have to make it bigger. So, by putting 1 on the left side, makes it 12, which is bigger than 3. So, 12 minus 3 would equal 9. (Brandy, personal communication, September 13, 2010)

Finally, Brandy argued that the little 1 carried to the tens place in addition meant the same thing as the little 1 borrowed into the tens place in subtraction. The similarity that she pointed to was that, in both cases, the regrouping move did not change the given numbers: In addition, 1 (in the tens place) and 7 (in the ones place) is still 17. In subtraction, when she borrows 1 from the 4 to make the 2 into 12, Brandy then subtracts 3 from 12 to get 9. The 4 that she borrowed from became a 3, which she brought down, obtaining 39. Brandy noted that 42 – 3 = 39 as well, so that borrowing did not affect the answer. Working columnwise, she “had to borrow.” However, she was capable of viewing the adjacent 4 and 2 as 42. At the same time, there is no evidence that she saw this 42 as representing 42 tens. She referred simply to “numbers” when speaking about borrowing, saying, “you’re borrowing one—just one number.”

> *Interviewer:* Um… [Int. approaches the whiteboard] how come—so, with this one [pointing to the carried 1 in the addition problem] you have a 1 and a 5 and you did, like, 1 plus 5. Here [pointing to the borrowed 1 in the subtraction problem] you have a 1 and a 2, and you don’t read it as 1 plus 2, you read it as 12.
>
> *Brandy*: Yeah.
>
> *Interviewer*: Why is that?
>
> *Brandy*: Um, I’m not really sure. It’s just how they taught me in math. But, like, ‘cause, well, they’re, they’re, they’re different. They call it different things. For instance, when you subtract, you have to borrow so

you can make this one bigger to subtract it from the smaller one. But then, like, when you add, if like – if these two [pointing to the digits in the ones place] are too big, like they exceed more than two – a two-digit number – you have to like put, um, the rest of the number up there. Like, even if it was like 20, you'd put the 0 here and the 2 there 'cause it's more than a one-digit number. 'Cause you could only like – you only like, like one-digit numbers. 'Cause it's like in – it's kind of like in columns like this [draws vertical segments to indicate columns]. So, 17, like 17 [pointing at the separate 1 and 7], then you would add it. You wouldn't borrow because that's what you would do in subtraction. You're just bringing it up to add it together.

Interviewer: Okay. So, would you say that the little 1's, uh, here and here [pointing at the board] mean the same thing or different things?

Brandy: Um, different things. Because this little 1 here [pointing to the subtraction problem] makes this like a really big number compared to this [point to the addition problem]. It's just like one out of – it's like basically 12 versus 1. You just need uh – oh, actually, hold on [steps back from board]… Um, yeah, actually, I think, um, well yeah, liiike, I guess they kinda do mean the same thing because they are both [points to separate digits in each problem] big numbers, except that you're just—oh, yeah, they are the same thing because it's like you're not – you're not losing anything from this because you're – it's still basically 17 still being added into it. Just you wouldn't put it there [pointing to ones place of sum]. You would just add it onto here [pointing to tens place] on the second row – row. While here, you're not losing that extra one unit. You're just adding it onto here to equal – to make it 9. Does that make sense? I don't know.

Interviewer: Okay. Yeah. Can you explain why when you read this one, it's like a one – like one plus five – whereas when you read this one, it makes it into twelve? Instead of being like one plus two?

Brandy: Oh, um.

Interviewer: Does that question make sense?

Brandy: Yeah, it makes sense. It's just hard to explain. Well, 'cause you're – since you're borrowing from this one, making it 3. It's like, uhhh… well, I, uh, it's like, I guess 'cause like there would be really no wayyy. Oh, basically, it's like saying 42 minus - it's like 42 minus 3 is basically 39. It's just a different way of figuring that out. Because you can't subtract a small number from a bigger number. You'd have to make it bigger, of course, by borrowing that one unit, while this one you're just adding it on? Um. It's hard to explain. I don't-

Interviewer: Okay. (Brandy, personal communication, September 13, 2010)

Figure 64 represents Brandy's argument that the little 1's meant the same thing. Brandy's arguments concerning the meanings of the little 1 in addition and subtraction were characterized by interpreting numerals as consisting of concatendated digits, rather than as representing numbers composed of ones, tens, and hundreds (Thanheiser, 2009). She assered the rules and attempted to justify these. Her justifications were grounded in the behavior of digits in columns, following the conventions and dogmatic justifications associated with the standard algorithms. Her argument that the little 1's in addition and subtraction meant "the same thing" was not a particularly clear one. She seemed to argue that in both carrying and borrowing, the numbers were unchanged. For her, it followed that the 1's meant "the same thing." However, she never explicitly stated what the 1's meant to her, i.e., what that "same thing" was. It might be fair to rephrase her claim as something like "the little 1's function similarly."

Brandy's understanding of number composition. In the context of the standard algorithms, Brandy reasoned in terms of digits. It was not evident from her explanations that she viewed the digits in different places as having different values. However, in the numeration tasks, she did display evidence of some understanding place value. Brandy was capable of viewing two-digit numbers as being composed of ones and tens. She recognized that 63 consisted of 6 tens and 3 ones. She was also capable of viewing 63 as composed of 5 tens and 13 ones, 4 tens and 23 ones, and so on. Thus, she could conceive of one ten as being equivalent to ten ones, which is precisely the understanding necessary to make sense of regrouping between the ones and tens places. Note that this is not

<table>
<tr><td>Claim: The little 1’s mean the same thing
(“I guess they kinda do mean the same thing because they are both [points to separate digits in each problem] big numbers, except that you’re just—oh, yeah, they are the same thing”)</td><td rowspan="3">
“it’s still basically 17 still being added into it. Just you wouldn’t put it there. You would just add it onto here.”</td></tr>
<tr><td>Data: 1 and 7 is still 17; 42 – 3 = 39, (and that’s the same as what you get when you using the borrowing procedure)
(“it’s still basically 17 still being added into it. Just you wouldn’t put it there [pointing to ones place of sum]. You would just add it onto here [pointing to tens place] on the second row – row. While here, you’re not losing that extra one unit. You’re just adding it onto here to equal – to make it 9”)
(“it’s like saying 42 minus - it’s like 42 minus 3 is basically 39. It’s just a different way of figuring that out.”)</td></tr>
<tr><td>Warrant: In both addition and subtraction, you can still read off the original numbers; regrouping didn’t really change anything
(“they are both [points to separate digits in each problem] big numbers”; “you’re not losing anything from this because you’re – it’s still basically 17 still being added into it. Just you wouldn’t put it there [pointing to ones place of sum]. You would just add it onto here [pointing to tens place] on the second row – row. While here, you’re not losing that extra one unit”)</td></tr>
</table>

Figure 64. Brandy’s argument that the little 1’s mean the same thing.

sufficient to make sense of regrouping between the tens and hundreds places, and prospective elementary teachers may be able to meaningfully account for regrouping in the ones/tens but not the tens/hundreds (Thanheiser, 2009). Nonetheless, Brandy had some productive understanding of place value. However, when she operated with and talked about the standard algorithms, whether mentally or in writing, no sense of the value of digits was apparent.

Summary of Brandy’s initial reasoning. Brandy came to the course with knowledge of addition, subtraction, and place value that is relevant to the story of her development. In this analysis, Brandy’s initial ways of reasoning are viewed as resources that (a) enabled her to make sense of new ideas and (b) from which her number sense developed. I summarize those resources here. Brandy knew the standard addition and subtraction algorithms. When it came to reasoning about these algorithms, she displayed a concatenated digits conception of multidigit numbers. However, multidigit numbers meant more to her than just concatenated digits. She was capable of viewing a two-digit numbers as being composed of tens and ones in multiple ways. In her reasoning about the standard subtraction algorithm, she conveyed a meaningful view of subtraction as a take-away process. Finally, when asked to perform the same addition and subtraction computations in multiple ways, Brandy entertained rounding and compensating, and she reasoned straightforwardly about compensation. These aspects of Brandy's reasoning are all relevant to her development from inflexible to flexible in mental addition and subtraction.

Brandy's Development from Inflexible to Flexible

This section describes the development of Brandy's reasoning related to addition and subtraction between the first and second interviews. The presentation is chronological, drawing on homework and test responses. Along the way, I highlight connections to Brandy's initial reasoning. I also point out the timing of Brandy's reasoning with classroom activity and normative ways of reasoning.

Brandy's developing reasoning between the first and second interviews. Brandy participated in classroom activities during the Quantitative Reasoning and Place Value units. On Days 7 and 8, several ideas related to place-value notation and canonical and noncanonical number composition became normative. On Day 8, Trina's addition (Levelling) strategy was introduced. Trina added 95 and 27 by changing the addends to 100 and 22, which made it easy for her to see that the sum was 122. Two other students co-constructed a justification for Trina's strategy. They argued that her strategy involved "just taking 5 from 27 and giving it to 95" and that doing this was valid because "You're still adding all the same numbers, just in different places."

On Day 9, the idea of taking part of one addend and giving it to the other to form a nice number came to function as if shared. The class decided to name Trina's strategy "Borrow to Build." Borrow to Build had a strong influence on Brandy's reasoning. In her initial reflection on Trina's strategy (in a homework assignment that she completed between Days 8 and 9) Brandy made sense of the strategy in terms of number composition and straightforward compensation. She viewed the strategy in terms of forming a nice number and then compensating for doing so: "if you borrow a number of units on one side that you did not have to begin with, you have to take away the same amount of units to the other side so you can have an equal number." Brandy suggested calling Trina's strategy "the equalizer."

In her reflection on Trina's strategy, Brandy used her compensation reasoning to make sense of Levelling. The Double Compensation addition strategy that Brandy had entertained as a Scaffolded Alternative was different from Trina's strategy. Brandy had rounded both of the given numbers independently, added the rounded numbers, and then compensated for her rounding moves. Trina's strategy, by contrast, involved rounding one of the given numbers and compensating immediately by changing the other number in the opposite fashion (she added 5 to 95, so she subtracted 5 from 27). Despite these differences, Brandy interpreted Trina's strategy in terms of compensation and was able to make sense of it on the basis of her straightforward compensation reasoning.

Subtraction Aggregation was introduced to the class on Day 3 when Aaron used it to compute a difference in the Sisters and Brothers Problem. On Day 9, reasoning about subtraction as a cumulative process of decrease came to function as if shared. On Day 11, the class watched a video clip of Connor using a Subtraction Aggregation strategy and discussed his reasoning. Also on Day 11, reasoning about subtraction as a take-away process became normative.

In Journal #5, Brandy compared Connor's subtraction strategy to the standard algorithm, arguing that both involved a take-away view of subtraction. She described the numbers in Connor's subtraction strategy in terms of tens and ones, and she argued that his strategy was legitimate because the same amount was taken away as in the standard algorithm: "We are taking one whole 10 & 7 ones away from 2 whole tens and five ones. Connor's strategy takes away the same amount but breaks it down into 2 different parts." This response also shows Brandy taking place value into account in reasoning about both the standard subtraction algorithm and a nonstandard subtraction strategy.

On Day 11, the class also speculated about how Connor would solve a particular addition problem, given what they knew about his approach to a related subtraction computation. As a result, there was a discussion of Addition Aggregation. As part of Journal #5, Brandy made a drawing to illlustrate Connor's hypothetical addition strategy. Her drawing emphasized number composition in terms of ones, fives, and tens. She suggested the name "Break it add add it" for this strategy. Brandy also related Connor's addition strategy to Borrow to Build. In her view, both of these strategies made use of decomposition of number to make the computation easier.

Brandy also drew a distinction between Connor's addition strategy and Borrow to Build. She said that both strategies involved "breaking numbers into 2 separate parts." However, Connor's strategy involved breaking up one of the numbers, whereas Borrow to Build "breaks 2 numbers into 1 different part." The latter phrase sounds a bit puzzling. My interpretation of Brandy's meaning is that in Borrow to Build, both numbers get changed, and that change involves the *same* amount. It is noteworthy that Brandy was attending to these kinds of distinctions because her initial Scaffolded Strategy Ranges were Polarized. She had considered only one alternative way of performing addition or subtraction. Now she was attending to different ways of making use of decomposition. At the same time, it is not clear that Brandy interpreted Connor's strategy in terms of aggregation. She seemed to view it as compensation of a different sort than Borrow to Build. Thus, compensation seemed to be the basic lens through which she makes sense of nonstandard strategies.

In Journal #6, Brandy reported on a mental computation that she had performed outside of school. Brandy solved a problem concerning how much money she owed in dues to her sorority. She owed $200 for the semester, and she had paid $94 thus far. Brandy rounded $94 to $100. She knew that $200 minus $100 was $100. Then she recognized that she had "borrowed an extra 6" which she had to compensate for by adding $6 to her estimate of $100. So, she found that she owed $106. Brandy related her strategy to Borrow to Build. Although hers was a subtraction strategy and "Borrow to Build" named an addition strategy, she saw a connection in the sense of adding to 94 to form 100 "in order to solve this equation in an easy manner." So, when she referred to borrowing to build in this context, I think she had in mind rounding to a nice number.

In fact, Brandy used Subtrahend Compensation here, which is a sophisticated subtraction strategy that many prospective elementary teachers have difficulty making sense of. In her Scaffolded Alternatives, Brandy had attempted to solve a subtraction problem by rounding the subtrahend and compensating, and she had done so in an invalid way. Both problems had involved money. However, in this case, thinking meaningfully about the context helped her to draw the correct conclusion about how to compensate. Her ten-structured drawing likely supported her reasoning (see Figure 65). Brandy's drawing represented $200 as composed of $100 and $100. She represented each $100, in turn, as composed of ten $10's. Finally, a single $10 was explicitly decomposed into ten ones, grouped as 6 ones and 4 ones.

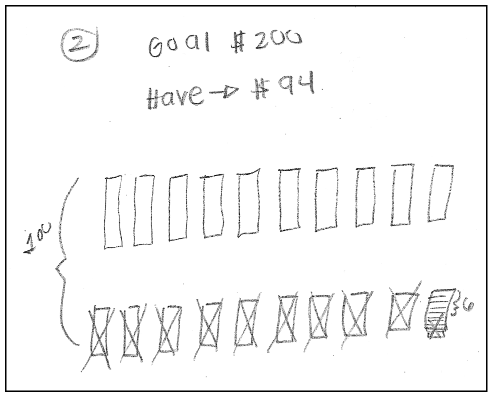


Figure 65. Brandy's drawing related to her subtrahend compensation strategy.

In a homework assignment in Section 3.3, Brandy used Subtrahend Compensation appropriately, this time in a naked-number context. She solved 340 – 49 (numbers of her choosing) by adding 1 to the subtrahend and the compensating appropriately: "First I take the 50 from 340 and that's 290. Then I put the 1 back and it's 291." Her reasoning about compensation was not made explicit here. However, her language, "put the 1 back," suggests reasoning about compensation differently than she had initially. Brandy added 1 to 49 before computing. Following her straightforward compensation reasoning from the first interview, it would follow that she should compensate by subtracting 1. She might instead have said, "I put an extra 1 into the problem, so now I have to take it out." She reasoned differently about compensation here. Her wording implies that she saw adding one to the subtrahend as *taking one more away* from the minuend, and so this extra one that had been taken away had to be *put back*.

Homework in Section 3.3 came after class on Day 12, when reasoning about differences as distances between number-locations came to function as if shared. (Recall Valerie's argument that 364 – 79 = 365 – 80 = 385 – 100 = 285, based on viewing the minuend and subtrahend as locations on a number line and shifting those locations equally.) In the homework from 3.3, Brandy first used Shifting the Difference, following an example of a child's reasoning. However, she used this strategy in a way that was valid but not especially helpful. To solve her subtraction computation, 340 – 49, she added 1 to both numbers to get 341 – 50. Then she added 10 to both numbers to get 361 – 70, and she concluded that the difference was 291. The first shift—adding 1 to both—clearly made for nicer numbers. However, the second shift—adding 10 to both—did not seem to serve that purpose. I find it hard to believe that Brandy could readily see a difference of 291 in 361 – 70 and not in 341 – 50. I think this was a case in which she followed an example of a child's reasoning without understanding the goal of the strategy: making the difference apparent. (For example, shifting to 391 – 100 would have made the difference readily apparent.) It is not clear how Brandy understood Shifting the Difference at this point. She may have understood why the strategy was legitimate (along the lines of Valerie's argument) but not recognized the other important aspect of the strategy, which was to make the difference apparent.

Homework in Section 3.3 also included solving addition and subtraction problems using an empty number line. When working with the empty number line, Brandy exclusively used aggregation strategies. That is, she worked cumulatively from left to right or from right to left. The number-jumps that she made were nicely chosen to take advantage of number composition. For example, she solved 72 – 38 by starting at 72, subtracting 2 to get to 70, then subtracting 30, which landed her at 40, and finally subtracting 6 to end up at 34. (See problem (c) Figure 66).

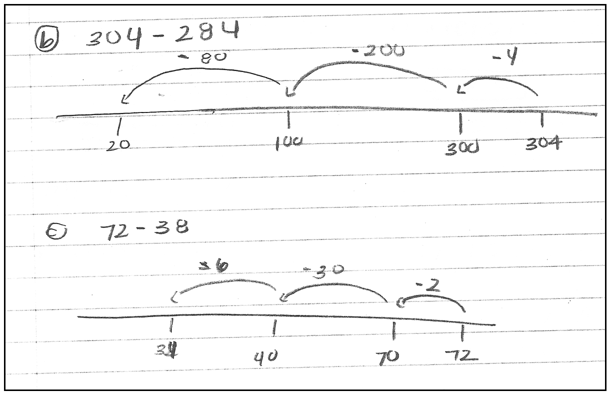


Figure 66. Two of Brandy's empty-number-line solutions.

For the first time, Brandy seemed to be using aggregation strategies, and as best I can tell from her written work, she used this in sensible ways. At the same time, the fact that her solutions were all left-to-right or right-to-left aggregation strategies may indicate that she used the empty number line in a somewhat procedural fashion. That is, she made choices, but these choices were rather constrained. For example, to solve 304 – 284, she might instead have opted to use Shifting the Difference. Since 304 and 284 are very close to one another, viewing both the minuend and subtrahend as number-locations and thinking about the difference as the distance between them would have been more convenient than starting with the minuend and subtracting off the subtrahend, as Brandy did. In each instance of subtraction using the empty number line, Brandy started with the minuend and subtracted in chunks according the subtrahend. She did likewise in addition, solving 62 + 49 by starting with 62, adding 40, and then adding 9. Likely, if she had not been using the empty number line, she would have solved this problem by taking advantage of the proximity of 49 to 50.

In a homework assignment in Section 3.7, Brandy used Single Compensation to solve both 135 + 98 and 114 + 92 in order to detemine which sum was larger. Single Compensation is distinct from Borrow to Build, in that the compensation move is made after computing the sum, rather than before. Single Compensation follows easily from Brandy's ability to perform Double Compensation initially. However, it has the advantage of being simpler and more efficient. Brandy did not use Single Compensation in her first interview, so this specific variant of compensation may have been new to her.

In the same assignment, Brandy correctly compared 46 – 19 with 46 – 17 on the basis of the amount being subtracted (i.e., she appropriately accounted for the effect of changing the subtrahend). Brandy said, "By looking at this equation, we would already know that 46 – 17 would result in a bigger number since we are subtracting less from 46." Again, she distinguished the effect of the subtrahend and accounted for it in her reasoning about differences.

As part of a review assignment for Midterm 1, students reviewed the named strategies from class and generated examples and illustrations for each of these. Brandy wrote appropriate examples of the aggregation strategies Separate-Add-Add and Separate-Subtract-Subtract and created fitting ENL illustrations. For Borrow to Build, she made an illustration involving ten-strips and little squares for ones. Her example for back-and-forth subtraction did not fit the meaning from class. Her example was valid, but it was not a back-and-forth (Compensation) strategy. It was Subtraction Aggregation by tens (400 – 40 as 400 – 10 – 10 – 10 – 10 = 360). Brandy gave a valid example of Shifting the Difference, but the choice of numbers and the choice of shift were not clearly suited to the strategy (500 – 35 = 505 – 40 = 465). She made an ENL illustration that showed that both differences were equal. However, it was again unclear whether she thought in terms of shifting to locations that made the difference easy to recognize.

On Midterm 1, one of the items described someone attempting Subtrahend Compensation but being unsure whether to add or subtract to compensate. Brandy answered correctly and gave a take-away/excess justification. The language of "borrowing" entered in to her explanation: "She subtracted an additional 5 units that was not in the first equation. She needs to add the 5 units to get the correct answer. She borrowed too much."

Summary of Brandy's developing reasoning between the interviews. Making use of number composition and reasoning in terms of compensation were prominent themes in Brandy's written responses. Borrow to Build—both the strategy itself and its name—seemed to profoundly influence Brandy's reasoning. The meaning that she associated with this strategy may have morphed somewhat over time. Initially, it was an addition strategy that she understood in terms of straightforward compensation. However, she later related to a subtraction strategy (for $200 – $94) for which the compensation was not straightforward. In that case, the connection seemed to be in terms of rounding to form a nice number (and compensating in some way). This connection surfaced again in her midterm response, in which she used the term *borrow* rather loosely to refer to rounding the subtrahend up. As best I can tell, Brandy's personal meaning for Borrow to Build included the as-if shared meaning for the classroom community but was broader than that. For her, Borrow to Build referred to a class of addition and subtraction strategies. From my perspective, these are the Compensation strategies. However, Brandy understood Aggregation strategies, at least initially, in terms of compensation.

Brandy's take-away meaning for subtraction helped her to make sense of Subtrahend Compensation and Subtraction Aggregation and to relate the latter to the standard subtraction algorithm. She used the idea of subtraction as a take-away process repeatedly and productively. Brandy also became far more aware of number composition in her reasoning about addition and subtraction. She took advantage of proximity to decade numbers, and her illustrations associated with compensation strategies emphasized ones, tens, and hundreds. Brandy was able to make sense of the compensation aspect of Shifting the Difference but may not have connected that strategy with the general heuristic of making a computation easier. Finally, Brandy was able to use the number line to represent Aggregation strategies, but these did not seem to figure prominently in her reasoning.

Brandy's Reasoning in the Second Interview

In the second interview, Brandy reasoned about addition and subtraction more meaningfully and flexibly than in the first. She could justify the standard algorithms by reasoning in terms of place value. She used multiple mental computation strategies for both operations. These included compensation strategies, for which the antecedents can be identified in her Scaffolded Alternatives from the first interview. She also used strategies that did not appear in her initial Scaffolded Strategy Ranges.

Brandy's reasoning about the standard algorithms. In contrast to her reasoning about the standard algorithms in terms of concatenated digits in the first interview, in the second she reasoned in terms of groups of ones (Thanheiser, 2009). That is, she thought in terms of place value and took account of the amounts represented by digits. For example, she reasoned about the number 259 as consisting of 2 hundreds, 5 tens, and 9 ones. In this way, she was able to justify regrouping in the standard addition algorithm.

Interviewer: Okay. I'd like you to solve an addition problem: 259 plus 38.

[Brandy writes on board]

Brandy: Okay, so I add the ones place first. And I have 17. So, um, that's, like—that's too many ones, because I can't have—I can't put 17 down [B temporarily writes 17 in the ones place and then erases it] So, I'd have to add one ten here, to the tens place [writes carried 1] and then keep the remaining 7 ones [points to the 7] in the ones place. So, now I have 5 tens, 3 tens, and 1 ten [points at digits in tens column] and I have to equal it together. And that's 9. So, I have 9 tens here [writes 9 in tens place of answer]. And now I'm in the hundreds place, so I have to drop it down – I have 2 hundreds, so I can put the 2 down here [writes 2 in hundreds place of answer] 'cause I don't add it with anything here. So, the answer is 2 hundred and 97. (Brandy, personal communication, November 4, 2010)

As the interviewer, I did not feel the need to ask follow-up questions regarding regrouping the addition algorithm because I thought that Brandy's argument was clear from her explanation.

Figure 67 represents Brandy's adding regrouping argument. Brandy's subtraction regrouping arguments also took place value into account:

Interviewer: Could you leave that up?

Brandy: Oh, yeah.

Interviewer: and also do a subtraction problem.

Brandy: Okay.

Claim: You have to carry the 1 *("I'd have to add one ten here, to the tens place [writes carried 1]")*
Data: 8 + 9 = 17; 17 is too many to have in the ones place *("that's too many ones, because I can't have—I can't put 17 down [B temporarily writes 17 in the ones place and then erases it] So, I'd have to add one ten here")*
Warrant: When ten or more ones are in the ones place, a group of ten must be formed and moved to the tens place; The little 1 represents ten *("that's too many ones… So, I'd have to add one ten here, to the tens place [writes carried 1] and then keep the remaining 7 ones [points to the 7] in the ones place.")*

Figure 67. Brandy's addition regrouping argument.

Interviewer: on either side, uh, 429 minus 34.

[Brandy writes problem on board and begins]

Brandy: So, yeah, so then I, um, I subtract the ones place first. So, 9 minus 4 is 5 [writes 5 in ones place of answer] So, I have 5 ones. I have 2 tens and 3 tens [pointing], and I can't sub – since 2 is smaller than 3, I can't subtract that. So – I only have 2 tens and 3 tens – you can't do that. So, I have to, um, borrow from the hundreds place [crosses out the 4 in 429]. So, I have to borrow one hundred [raises one finger] – or just, yeah – one hundred. So, I'm adding – so this one I'm gonna be adding next to it [writes borrowed 1] represents one hundred. So, then this goes back [writes 3 next to the crossed out 4 in hundreds place of minuend]. Since I borrowed, now I have 3 hundreds. So, one hundred plus 2 tens is, um, one twenty? I'm sorry, yeah. One. One hundred, and then I have 2 tens. So – but this [gestures over the 1 and 2 in the tens place of minuend] will look like it's 12. So, 12 minus 3 is 9 [writes 9 in tens place of answer]. So, I have 9 tens. And then I would drop down the 3 in the hundreds place [points to 3 in hundreds place of minuend and writes 3 in hundreds place of answer]. So, it's 395.

Interviewer: Okay. Um, when you were talking about the little 1 just now, you said that the 1 and the 2 would look like 12.

Brandy: Yeah, it looks like 12. Like when people add – when people subtract – we're just – well – people – like people – I mean, it doesn't really mean 12. It means a hundred and twenty. Because this is 2 tens [points to the 2 and then writes 20 above her work] so that's 20, and you're adding the hundred [writes 100 + to the left of the 20]. So, it's 120. Minusss, wait, yeah. One hundred and twenty minus – oh, yeah – one hundred and twenty minus 30 'cause you have three tens – that's what

> you're mult – or, subtracting – so that'd be 90, which is 9 tens. (Brandy, personal communication, November 4, 2010)

Brandy made two related arguments concerning the borrowed 1 in subtraction, the first concerning the need to borrow and the second concerning the resulting amount in the tens place. Her argument concerning the need to borrow was similar to that made in the first interview ("since 2 is smaller than 3, I can't subtract that"). However, in her second interview response, she took into account the meanings of the digits, viewing them as numbers of tens, rather than ones ("I have 2 tens and 3 tens").

Brandy's argument concerning the meaning of the borrowed amount contrasts starkly with the argument that she made in the first interview. She had previously argued that the little 1 made the 2 into 12 because it had to do: She needed to borrow to make 2 into a number bigger than 3, and 12 was bigger was bigger than 3, so the little 1 made the 2 into 12. In the second interview, she made a clear and valid argument (Figure 68) based on the meanings of the digits. Brandy talked about the digits 1 and 2 in the tens place as both "twelve" and "one hundred twenty" and she effectively argued that these digits could be treated as 12 because they represented 120.

Claim: The little 1 together with the 2 can be treated as 12 in the tens place ("this [gestures over the 1 and 2 in the tens place of minuend] will look like it's 12. So, 12 minus 3 is 9")
Data: The 1 represents one hundred; The 2 represents two tens; 100 + 20 = 120 ("it doesn't really mean 12. It means a hundred and twenty. Because this is 2 tens [points to the 2 and then writes 20 above her work] so that's 20, and you're adding the hundred [writes 100 + to the left of the 20]. So, it's 120.")
Warrant: 120 – 30 = 90; 12 – 3 = 9 ("one hundred and twenty minus 30 'cause you have three tens… so that'd be 90, which is 9 tens.")

Figure 68. Brandy's argument concerning the meaning of the little 1 together with the 2.

Brandy also compared the meanings of the little 1's. In contrast to her first interview response, she argued that the 1's meant different things. Also in contrast to that response, she was explicitly about the amounts that the little 1's represented. (See Figure 69.) She also substantiated her claims about the amounts represented by each little 1 on the basis of their origins: The one in the tens place in the addition problem came from ten ones in the ones place, whereas the one in the tens place of the subtraction problem came from the hundreds place. (This piece is a sub-argument for which I do not include an argumentation diagram here.):

> *Interviewer*: Okay. Um, so, in the addition problem and the subtraction problem, you ended up writing a little 1.
>
> *Brandy*: Yeah, which would – yeah, mm-hmm.
>
> *Interviewer*: Do those little 1's mean the same thing or different things?
>
> *Brandy*: Um, different things. The 1 here [pointing to the 1 in the subtraction problem] represents a hundred since I borrowed from the hundreds place beforehand to make it bigger – 'cause you can't subtract a smaller number by a bigger number. And then this one [pointing to the addition problem] I had too many ones, so I had to give my ten to this – to the tens place. Just one ten. And then this represents 9 tens, 7 ones, and 2 hundreds.
>
> *Interviewer*: Okay. (Brandy, personal communication, November 4, 2010)

Claim: The little 1's mean different things ("Um, different things.")
Data: The little 1 in the addition problem represents ten; The little 1 in the subtraction problem represents one hundred ("The 1 here [pointing to the 1 in the subtraction problem] represents a hundred… And then this one [pointing to the addition problem]… Just one ten.")
Warrant (implicit): Ten and one hundred are different

Figure 69. Brandy's argument comparing the meanings of the little 1's.

Brandy's second-interview arguments related to the standard addition and subtraction algorithms were consistent with what Thanheiser (2009) calls the *Groups of Ones* conception of multidigit numbers. She saw digits in the ones place as numbers of ones, digits in the tens place as numbers of groups of ten ones, and digits in the hundreds place as numbers of groups of one hundred ones. The only relevant way of viewing reference units in the context of these tasks that did not arise in Brandy's responses was viewing one hundred as ten tens. She talked about the *12* in the tens place in the subtraction problem as one hundred twenty based on distinct values for the two digits (1 hundred plus 2 tens). She did not seem to view it as 12 tens. The Groups of Ones conception is the second most sophisticated in Thanheiser's framework. The most sophisticated is *Reference Units*. The essential difference between these categories of reasoning is the ability to shift reference units by viewing the *1* in *12* (in the tens place) as both one hundred and as ten tens. Brandy did not make that way of reasoning explicit.

Brandy's addition and subtraction strategy ranges. Brandy moved from Inflexible to Flexible in both addition and subtraction from the first to the second mental computation interview. In terms of her specific strategy ranges, she went from MASA-bound to Transition+ in addition. In the second interview, she used both Transition strategies for addition—Right to Left and Left to Right—as well as one nonstandard strategy—Single Compensation. In subtraction, she moved from MASA-bound to Spread. She did not drop the MASA, but she added Aggregation, Minuend Compensation, and Subtrahend Compensation. Brandy's addition and subtraction strategy ranges are illustrated in Figure 70.

The addition and subtraction strategies that Brandy used in the post interview were all valid mental computation strategies. Furthermore, she was able to produce mathematically valid justifications for each of the non-MASA strategies that she used. As

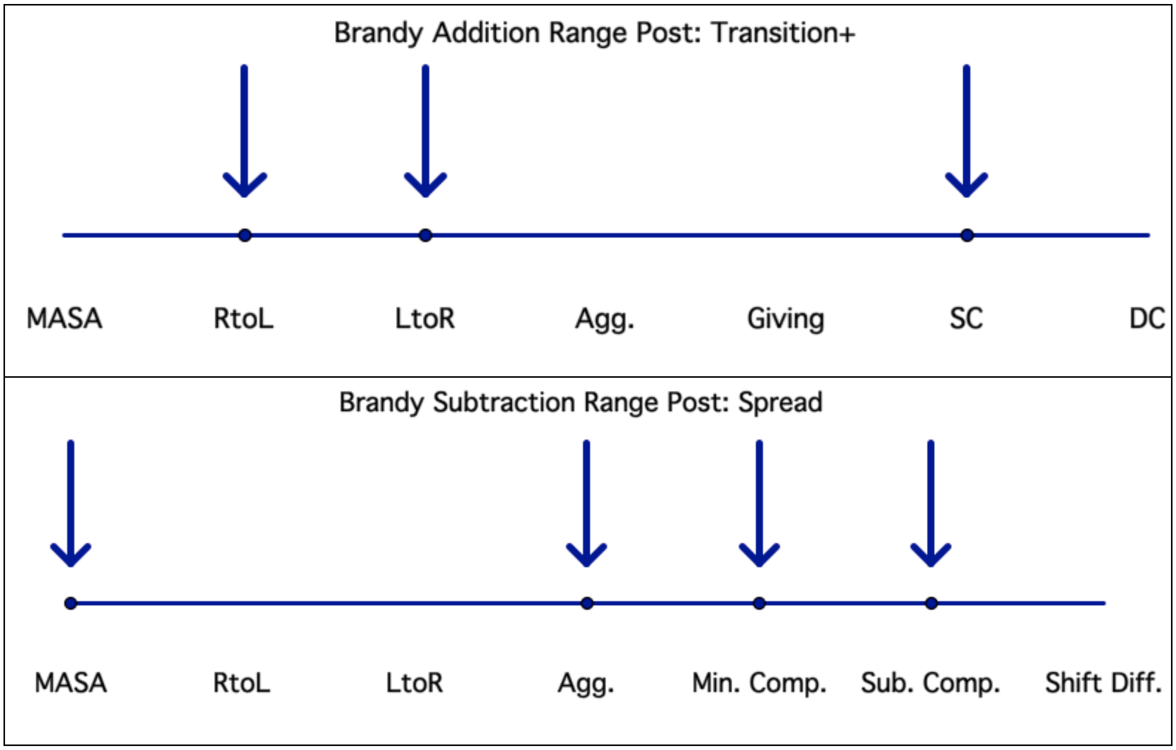


Figure 70. Brandy's second-interview strategy ranges for addition and subtraction.

such, these strategies are regarded as legitimately belonging to her strategy ranges for addition and subtraction.

In both cases, Brandy adopted two ways of reasoning that she used readily in the Bobo tasks. For both addition and subtraction, Compensation was one of these. So, we can partially account for the change in Brandy's mental computation activity based on the antecedents seen in her Scaffolded Alternatives. Specifically, valid addition compensation was a possibility for her coming into the class. We know that she gained additional experience using various strategies, and she reported becoming more confident in her mathematical abilities. This alone may account for her adoption of addition compensation. It does not account for her adoption of Transition strategies.

In terms of subtraction, Brandy perceived rounding as a possibility in the pre interview, but she did not understand how to compensate appropriately. In the post interview, Brandy used valid minuend and subtrahend compensation. She did not round both the minuend and subtrahend within a given problem. She was able to provide a mathematically valid justification for compensation in both cases. So, with respect to subtraction compensation, it seems that Brandy learned to distinguish the effects of rounding the subtrahend versus the minuend. She gained this understanding during her experience in the course, and it enabled her to use subtraction compensation strategies. As with addition, Brandy adopted another way of reasoning that is not accounted for by her Scaffolded Alternatives, namely Aggregation.

Brandy's scaffolded alternatives. I also investigated scaffolded alternatives in the second interview. For a single addition computation, Brandy considered using the MASA and variants of both Compensation and Aggregation. To compute 47 + 88, she found the sum of 135 in several different ways. She used Double Compensation by rounding both numbers up and then compensating for the net effect of rounding. She also used Double Compensation by rounding both numbers down and then compensation for the net effect of rounding. The latter strategy could be considered Left to Right Separation. However, Brandy described it in terms of Compensation (subtracting from the addends, finding the sum of the rounded numbers, then adding to compensate).

Brandy also used versions of Aggregation as Scaffolded Alternatives. She started with 88 and added 40 to get 128, and then added 7 more to get 135. She also described counting up from 88 by tens, rather than making a single jump of 40. She also used this strategy starting from 47 instead and counting by tens all the way to 127, then adding the remaining 8 to get 135. Although these latter strategies were less efficient, they were distinct, and they involved reasoning about addition in a way that did not show up in Brandy's addition strategy range for the basic Bobo tasks.

To solve a subtraction computation, 153 – 78, Brandy also used a variety of strategies. She performed the MASA, correctly obtaining a difference of 75. She also used a rare Compensation strategy that involved doubling. She knew that 78 plus 78 equaled 156, which was only 3 more than 153. So, she subtracted 3 from 78 to get an answer of 75. When using this strategy, Brandy seemed to be thinking in terms of how much she needed to add to 78 to get 153. Her initial estimate of 78 was too high, so she corrected for it. Doubling 78 to get 156 was evidently an easy computation for her.[22]

Brandy made an attempt at Double Compensation. She rounded 153 and 78 to 10 and 80, respectively. Rather than reasoning separately about her two rounding moves (which we know from the basic Bobo tasks she could have done), she wanted to make a single compensation move to account for both. She knew that adding 3 and 8 together would give her 11, and this would not be helpful, and she got stuck at that point. Because she seemed a little frustrated, I gave her the option of letting that strategy go and moving on, which she did. As a result, it is not clear whether she could have sorted out Double Compensation for subtraction. We know that she had valid arguments for both Minuend Compensation and Subtrahend Compensation, but coordinating the two still presented a

---

[22] Children in the United States often use known doubles to derive less familiar arithmetic facts (Fuson et al., 1997). Brandy's strategy seems to be a more sophisticated version of this approach. I expect that 78 + 78 = 156 was not a known fact for her, but was one that she was able to compute readily.

challenge. In contrast to her first interview response, Brandy recognized that her attempt at Double Compensation was flawed.

Brandy also used an Aggregation strategy that involved counting up from 78 to 153. She counted by tens from 78 to 148. The she knew she would have to add 5 more to get from 148 to 153. She reported that she had added a total of 75.

Summary of Brandy's reasoning in the second interview. In the second interview, Brandy was able to justify the standard algorithms by reasoning in terms of place value. She displayed a more sophisticated conception of multidigit numbers in the context of reasoning about the algorithms. Brandy used Transition and Nonstandard strategies as part of her strategy ranges proper. The new strategies that she adopted (relative to her first interview ranges) included both strategies similar to those that arose in her first-interview Scaffolded Alternatives and strategies that she had not entertained in the first interview but had come up in class.

Brandy's Scaffolded Strategy Ranges in the second interview also included strategies that went beyond her second-interview strategy ranges proper. She used variants of addition aggregation. She also used a Subtrahend Compensation strategy of a different sort than she had used previously, by beginning with an estimate of the amount she would have to add to the subtahend to reach the minuend. Although Brandy had been able to make sense of Shifting the Difference when presented with it in homework and test items, she did not use this strategy spontaneously. When she considered rounding both minuend and subtrahend and compensating simultaneously, she did not think in terms of a distance between number-locations but fell back on her straightforward compensation reasoning. Brandy's Scaffolded Strategy Ranges in the second interview went beyond their analogoues in the first interview. She exhibited growth, and at the same time her reasoning in the ZPD showed room for further development.

Brandy's Reflections on Her Experience in the Course

In the SST interview, I had the opportunity to ask participants about their experience in the course. Brandy's reflections on her experience are the focus of this piece of her case study.[23] When asked about her mathematical background, Brandy described her high school math classes as having been procedurally focused. She gave the example of learning a song to memorize the quadratic formula in an algebra class. Brandy contrasted her previous experiences in math classes with her experience in the content course in terms of the emphasis on understanding why procedures worked and the emphasis on become familiar with different strategies.

Brandy reported not having done much mental computation before she took the content course: "I never really thought about using mental math before. I always just relied on a calculator." She also reported noticing that her reasoning had changed:

> Like, before I would—it took—like, I felt like it took a longer time trying to think about it, and I could only think about it like standard algorithm way. So, in my head, I tried to like cross everything out, and it would just, in my head it got kind of like—I wasn't sure, really. And then when I—the second interview—I used some of the strategies I learned from class to like figure out the problems, the answers.

She also said that she was using mental computation more outside of school: "Whenever I go shopping too, I just, when something's 30% off, I just calculate it in my head and

[23] I also asked targeted questions concerning particular standout strategies and tools from class. In Brandy's case, the data that add most to this account of her development are simply her reflections on her experience in the course. Because the empty number line was not very influential in Brandy's reasoning, the tasks and questions related to it do not connect well with the themes in the story of her development. In terms of standout strategies from class, Brandy's responses in the SST interview were as expected based on her reasoning in the second interview.

subtract by that. I never—usually, I would just use my calculator, but then I can do it in my head now" (Brandy, personal communication, November 9, 2010).

Brandy reported valuing the use of different strategies, "Because it can help you, like out in the world. You can't always rely on a calculator." Brandy associated using mental computation strategies other than the MASA with thinking meaningfully about mathematics, and she expressed valuing that meaningful thinking. In talking about using different strategies, she said, "It's a different way of thinking, basically. It makes you think, like, more logically and everything. So then, instead of like crossing it all out 'cause when you use paper and pencil, you don't really think about it. You're just writing it down" (Brandy, personal communication, November 9, 2010). Brandy gave the example of computing 1000 – 3, pointing out how much "crossing it out" would be required to perform that computation with the standard algorithm, whereas she could just think about it and determine that the answer was 997.

She described valuing understanding of the standard algorithms: "I think that teachers should like focus on teaching students why we do things the way we do, like why we carry, why we borrow when we add—I mean, subtract—and stuff like that. We should focus on that so kids can apply it to, like, to higher level math." She also noted that children can come up with their own strategies, and that this had been a revelation for her: "Children come up with their own strategies, and I never really thought—I always thought like teachers would teach them a certain way, and kids would go by that. But kids can also form their own like ways of thinking in how they solve their problems" (Brandy, personal communication, November 9, 2010).

Brandy's reflections on her experience suggested that there had been positive changes in her beliefs about mathematics, teaching, and learning.[24] Not all of these changes are owing to Brandy's experience in the content course alone. She was concurrently enrolled in a 1-unit children's mathematical thinking (CMT) course, which was designed to be a companion to the content course. Some of the statements that Brandy made about children's mathematical thinking seemed to be related to her experiences in the CMT course. Philipp et al. (2007) studied the experiences of prospective elementary teachers in the same mathematics content course that was the setting for this research. They found that students who concurrently studied children's mathematical thinking while taking the content course developed more sophisticated beliefs about mathematics, teaching, and learning, and improved their mathematics content knowledge more than those who did not. Based on my study of Brandy, it seems to me that the development of her mathematical reasoning can be accounted for on the basis of her experience in the course, in terms of the interaction between the knowledge that she brought with her and ideas encountered in class. However, her experience in the CMT course likely had a powerful influence on her beliefs. This, in turn, may have led to greater motivation to learn the content.

[24] As beliefs were not a focus of this study, I did not use a pre-instruction measure of the interview participants' beliefs. However, in her reflections on her experience in the course, Brandy drew contrasts with her previous mathematical experiences and she repeatedly referenced what she had learned and how she had changed during the course thus far.

Brandy's Continued Development after the Interviews[25]

There is not much data concerning Brandy's reasoning about whole-number addition and subtraction after the SST interview. However, there is evidence worthy of mention. This comes from the second midterm, an additional homework assignment, and the NSRS posttest.

Brandy's additional exam and homework responses. On Midterm 2, Brandy made sense of a hypothetical student's subtraction aggregation strategy. She illustrated it with an ENL diagram. Her explanation emphasized number composition and the idea of cumulative increase. Brandy also evaluated an example of Shifting the Difference (presented numerically with numbers aligned) as legitimate. She said, "Adding each number by the same amount of units will result in the same answer." She generated an example of this strategy. This time, her choice of numbers and shift size was fitting (762 – 295, then add 5 to each). She seleced a subtrahend of 295, which is close to the benchmark number 300, and she chose to shift minuend and subtrahend such that the subtrahend would become 300. Thus, for the first time, Brandy showed evidence of both understanding why Shifting the Difference worked *and* understanding the goal of making the difference readily recognizable. Given additional occasions for mental subtraction, Brandy may have adopted this strategy at some point.

In Journal #11, Brandy gave an example of a mental subtraction strategy that she had performed in the context of buying a movie ticket. She gave her cousin $20 to pay for

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[25] The second mental computation interview took place at the end of the whole-number portion of the course. The SST interview took place the following week. Several weeks of instruction focused on rational number remained. During this portion of the course, there were a few tasks related to mental computation that provided additional data for this analysis.

a \$9.50 movie ticket. Brandy reasoned that \$9.50 + \$0.50 equaled \$10, and that \$10 + \$10 equaled \$20, so her cousin owed for \$10.50. She again saw her strategy as an example of Borrow to Build, saying, “I borrowed the 50 cents to build \$10 and added from there to reach \$20. I then knew my cousin owed me \$10.50.” She illustrated her reasoning in aggregation-fashion in an ENL diagram. In this instance, Brandy’s reasoning was similar to that in computations she had performed in the past. However, here, she used this way of reasoning in a rational-number context. This may have been related to her experience in the rational-number portion of the content course.

Change in Brandy’s NSRS responses. Brandy scored 29 of 37 on the NSRS posttest, which was a substantial improvement over her pretest score of 21. Few questions on the NSRS relate directly to whole-number addition and subtraction. However, one does stand out as relevant to this analysis. Item #16 asked about the number of digits in the sum of two 3-digit numbers (Figure 71). On the pretest, Brandy incorrectly answered (d), that the sum could have three, four, or five digits. On the posttest, she correctly answered (c), that the sum could have three or four digits. On its own, this item does not provide much insight into Brandy’s reasoning. However, the result is consistent with other changes in her reasoning about place value and addition. I would conjecture that the change in Brandy’s response to this item was either related to the meaning that she associated with regrouping in the standard addition algorithm or that she shifted from thinking about this item in terms of the standard algorithm to thinking about it in terms of a nonstandard strategy like Borrow to Build. In any case, this item provides evidence from a very different sort of task than those used in the interviews that

16. The sum of a 3-digit number and a 3-digit number:

a) must be three digits
b) must be four digits
c) can be three digits or four digits
d) can be three digits, four digits, or five digits

Figure 71. NSRS Item #16.

Brandy grew to make more sense of the relationship between place-value notation, number magnitude, and the addition operation.

Summary of Brandy's Number Sense Development

Brandy developed from inflexible to flexible in mental addition and subtraction. Specifically, she represents a case of change from MASA-bound to Transition+ in addition and from MASA-bound to Spread in subtraction. In the first interview, Brandy was dependent on the standard addition and subtraction algorithms. She did not understand why these algorithms worked, but she knew that they did. This situation used to puzzle me. I thought of inflexible prospective elementary teachers being dependent on the standard algorithms *despite* not understanding them. Cases like Brandy's have led me to think differently: she was dependent on the standard algorithms *because* she did not understand them.

At the beginning of the course, Brandy had knowledge of place value that she could have used to make sense of the standard algorithms. However, when performing and reasoning about those algorithms, she did not draw upon that knowledge. In using the algorithms, she reasoned about numerals as consisting of digits. In the course of her experience in the place value unit, she came to reasoning about the amounts represented by digits according to their place values and to think of the carrying and borrowing procedures in terms of regrouping. As a result, she became able to make sense of the values of the little 1's and, hence, to understand the details of the algorithms.

The reasoning involved in making sense of the standard addition and subtraction algorithms overlaps with the reasoning involved in making sense of Transition strategies. In fact, there is in some cases a fine line between Right to Left Addition and the MASA. The distinction is precisely thinking about the numbers in place-value terms, especially concerning regrouping. Brandy reasoned in terms of groups of ones, tens, and hundreds in some of her written homework responses. She also used this reasoning in second-interview tasks concerning the standard algorithms, as well as the Transition strategies that she used in mental computation. In the basic mental computation tasks in the second interview, Brandy's addition strategy range did not include the MASA. Instead, she used Transition for 3 of 4 addition computations. Thus, the fact that she made sense of both the standard addition algorithm and Transition strategies for addition led to her no longer using the MASAs and using Transition strategies instead.

Brandy also came to the course with the idea of rounding, computing, and compensating as an alternative to the MASAs. Compensation strategies were not part of her strategy ranges proper, but they arose in her Scaffolded Alternatives. Brandy's straightforward compensation reasoning was sufficient to support valid Addition Compensation. It is no great surprise then that she adopted Compensation strategies, being that reasoning in terms of compensation occurred in her Scaffolded Alternatives. However, Brandy's case suggests that Scaffolded Alternatives may provide valuable insights into students' reasoning. Not only did Compensation move from Brandy's scaffolded strategy ranges to her strategy ranges proper, but Compensation was the lens through which Brandy interpreted many non-MASA strategies, including some that I would categorize as Transition and Aggregation strategies.

Borrow to Build (Levelling) was especially influential in Brandy's story of number sense development. She understood this strategy in terms of straightforward compensation and in terms of forming nice numbers for ease of computation. Brandy carried these ideas further, seeing "borrowing to build" in many strategies that I would categorize differently. The fact that the Levelling strategy was given the name *Borrow to Build* in the class seemed to be significant for Brandy. She used the language repeatedly in her written responses. She also mentioned the strategy by name in her second interview and in her SST interview.

Brandy also came to the course with the idea of subtraction as a take-away process. This is an idea that probably every prospective elementary teacher brought to the course, and it is one that I never saw as particularly desirable or productive. However, Brandy used this idea, together with what for her was the essence of Borrow to Build—rounding to form a nice number and then compensating appropriately—to construct valid Subtrahend Compensation on her own, before this strategy had been introduced in class. It may be that Brandy never explicitly recognized that the minuend and subtrahend in subtraction play distinct roles and affect the difference in opposite ways. This would explain why she was still unable to coordinate minuend and subtrahend compensation in a single computation when she attempted to do so in the second interview. However, she did construct valid Minuend Compensation and Subtrahend Compensation arguments for separate computations by reasoning appropriately about how to compensate on the basis of the effects of her rounding moves.

Brandy did not adopt Shifting the Difference, at least not in the span of the first and second interviews. This may have been due to the fact that she was late to recognize a key aspect of that strategy. She made sense of the legitimacy of shifting the minuend and subtrahend, but she was late to catch on the idea of shifting these to nice locations that would make the difference readily apparent.

Brandy adopted Aggregation strategies to some extent. Addition Aggregation did not show up in her proper addition strategy range in the second interview, but it did show up in her Scaffolded Alternatives. She did use a Subtraction Aggregation strategy in the second interview, and she used another variant as a Scaffolded Alternative. Aggregation was not part of Brandy's Scaffolded Strategy Range in the first interview, for either addition or subtraction. So, as best I can tell, it was a new idea for her, and she was in the process of incorporating it into her repertoire at the time of the second interview. I interpret her use of the less sophisticated Aggregation by Tens as an indicator of the newness of this kind of strategy for her. Even though counting by tens was cumbersome, Brandy may have been more comfortable doing so because she felt more sure of her the validity of her approach when computing that way.

Finally, Brandy was congizant of a contrast between her previous experiences in mathematics classes and her experience in the content course. She expressed that she valued making sense in mathematics and that she saw various strategies as sensible and useful. Brandy also developed ideas about children's mathematical thinking and about how she thought children should be taught. Some of these changes are likely attributable to her experience in the children's mathematical thinking course. I conjecture that taking that course concurrently positively influenced Brandy's beliefs, and her beliefs about mathematics, teaching, and learning may have motivated her to learn and thus contributed to the improvement in her number sense.

Valerie's Developing Understanding of Multiplication

Valerie's case represents a story of number sense development in whole-number multiplication. She grew from being reliant on an invalid mental multiplication strategy to being able to make sense of partial products and using a variety of strategies. As in the case of Brandy, Valerie's development can be accounted for in terms of the interaction between knowledge that she brought with her to the course and ideas that she was exposed to in class. On the other hand, this story proceeds differently. It is specific to one operation, multiplication, and it is focused even more specifically on Valerie's reasoning about partial products. Her reasoning about these changes from the first to the second interview, and this change is related to a poignant classroom event. At the same time, when it comes to the second and third interviews, Valerie is not entirely clear and consistent in her reasoning about partial products. Her ability to account for these depends on the task at hand and on the tools that she uses to help organize her thinking.

This section describes how Valerie's reasoning developed over a period of about two months. The richest descriptions come from her interviews. Data from the first interview affords a detailed description of her initial reasoning about multiplication. Valerie's second-interview responses afford a similarly detailed description and contrasts with her first interview responses. In between the interviews, Valerie's written responses to homework and test questions provide important evidence of her reasoning. She also made a notable contribution to whole-class discussion that figures into her story. Valerie's SST interview responses not only provide evidence of Valerie's reflections on her experience in the course, they provide additional valuable data concerning her reasoning about multiplication.

Valerie's Initial Reasoning about Multiplication

This subsection describes Valerie's initial reasoning about multiplication on the basis of first interview and NSRS pretest data. Valerie was an 18-year-old freshman Liberal Studies major. She reported having taken a remedial algebra class in her freshman year of high school, geometry as a sophomore, algebra 2 as a junior, and precalculus as a senior. Valerie scored 29 of 37 on the NSRS pretest, which placed her in the High group, relative to the class.

Valerie performed mental multiplication using a go-to strategy that was invalid. When invited to perform mental multiplication in different ways, she considered only the MASA. Valerie's understanding of the standard multiplication algorithm could be characterized as procedural in nature. Much like Brandy's initial reasoning about the addition and subtraction algorithms, Valerie reasoned in terms of the behavior of digits in columns. Valerie's reasoning about partial products in mental multiplication seemed to be unrelated to her procedural knowledge of the standard algorithm.

Valerie's mental multiplication strategy.[26] In her first interview, Valerie used only one strategy for each of the basic Bobo multiplication tasks. She multiplied tens by tens and ones by ones, and then added these two partial products together. For example, she reasoned as follows about multiplying 24 and 15:

[26] Valerie did not have a multiplication strategy range to speak of. She made no choices based on the given numbers. Valerie used only one strategy, and it was invalid.

> *Interviewer*: Bobo's offering a package of 15 oboes for 24 dollars per oboe, but you have to buy the whole package. So, what would the whole package cost?
>
> *Valerie*: [pause] Two hundred and twenty.
>
> *Interviewer*: And how did you get that?
>
> *Valerie*: Well, I broke it up, into segments. So, I took—I made it 20 times 5 [sic], or 20 times 10, rather.
>
> *Interviewer*: Okay.
>
> *Valerie*: And 20 times 10 is 200.
>
> *Interviewer*: Uh-huh.
>
> *Valerie*: And then I took 5 and 4 and multiplied it, and that's 20. And I added those two together to get 220.
>
> *Interviewer*: Gotcha. Um, so kind of a similar question to one I asked before: how do you know that that's it—that that's gonna give you the right answer?
>
> *Valerie*: Because, as long—so, if you add, if you take it away [positions her hands apart from one another and perched on the table] so that it's 20 and 10, you're still left with 5 and 4. So, you need to multiply those together to get how much those would equal. And if you add both of them together, you'll get the answer of what it would be if you combined them all together [brings her hands together and clasps them]. It's just breaking them into smaller parts, but as long as they all come together after you've multiplied both sets and add 'em together, it's the right answer.
>
> *Interviewer*: Okay. (Valerie, personal communication, September 14, 2010)

Valerie went on to use the same strategy for each multiplication task. To find the cost of a package of 19 oboes at $21 per oboe, she computed 10 x 20 = 200, 1 x 9 = 9, and 200 + 9 = 209. This time, she gave a shorthand explanation, "I broke it up again," acknowledging that she used the same strategy and referencing her previous justification. Proceeding similarly, she reasoned that 25 x 16 equaled 230 and that 99 x 15 equaled 945 (Valerie, personal communication, September 14, 2010).

Valerie confirmed that this strategy of multiplying the tens by the tens and the ones by the ones was her go-to strategy for mental multiplication. Her response also provides some insight into why she used this strategy:

> *Interviewer*: So, I noticed that for each of these last four questions that you did it the same way. Do you agree?
>
> *Valerie*: Mm-hm [nods]
>
> *Interviewer*: Is that like your normal way of doing multiplication?
>
> *Valerie*: Well, if I don't have a calculator, yeah [laughs]
>
> *Interviewer*: Okay.
>
> *Valerie*: Sometimes, like if I have it in front of me and I have to write it down, I'll actually like go through each of the numbers and, like, make the two columns,
>
> *Interviewer*: Uh-huh
>
> *Valerie*: but if I can do it in my head, I can't do that [the standard algorithm] without mixing the numbers up. So, I have to make it simpler, and that's the simplest way I know how to do it.
>
> *Interviewer*: Okay. (Valerie, personal communication, September 14, 2010)

Valerie reported that the strategy she had used for the basic Bobo multiplication tasks was the go-to mental multiplication strategy that she normally used. She also explained that doing the MASA for multipication was difficult. She preferred her strategy because of its simplicity.

Valerie reported doing mental multiplication, and she was aware of the particular strategy that she used. Because two of the four partial products were ignored, Valerie's answers were consistently less than the correct answer. Some answers were so far off that one would have an expectation that they did not seem reasonable. For example, for 21 times 19, Valerie got an answer of 209. If asked to estimate this product, people typically say that it is close to 20 x 20, which is 400. Valerie's answer was about half that, and yet she did not question it. Alternatively, one might notice that 21 x 10 = 210, which is greater than 209. It seemed that Valerie was not in the habit of assessing the reasonableness of her answers and/or she was so confident in her multiplication strategy that she saw no reason to question her answers.

Valerie's (lack of) scaffolded alternatives. Solving a set of multiplication problems, using a single strategy for each, made it possible for Valerie to solve each problem incorrectly and not notice that anything was wrong. When it came to her Scaffolded Alternatives for multiplication, Valerie was asked to perform one computation in multiple ways. I expected that Valerie would obtain inconsistent answers, and I was curious how she would react to that.

To perform the computation 45 x 12, Valerie first used her go-to strategy. She multiplied 40 by 10 and 5 by 2, and added these results together to get an answer of 410. When asked for an alternative, Valerie performed the MASA. This led to a different answer than 410.

> *Interviewer*: Do you have a different way of doing this?
>
> *Valerie:* Um. [pause] Well, the way I would do it sometimes is that I'll do it in my head but like I was doing it on paper. So, I would line them up. And 2 times 5 is 10, so you leave the 0 and carry the 1. So, 2 times 4 is 8, plus 1 is 9. So, you have 9 on the first column—or the first row.
>
> *Interviewer*: Okay.
>
> *Valerie:* And then in multiplication you have to bring down the zero. So, it's 0 underneath the other 0. And 1 times 5 is 5, and 1 times 4 is 4. [finger arithmetic] So, you're left with [pause, voice goes soft] 450? Yeah, 450 plus—that's not right. [pause] Hm. I did that wrong. [laughs] It's harder to do it unless I have paper in front of me.

Interviewer: Yeah.

Valerie: But I know the answer's 410. It's just, I did it wrong.

Interviewer: Okay.

Valerie: That's all. I don't have any other ways! [laughs]

Interviewer: Alright.

Valerie: I would have to write it out, or I would have to do that [I think "that" refers to her go-to strategy] in my head. (Valerie, personal communication, September 14, 2010)

Valerie did arrive at inconsistent answers. However, rather than question her strategy, she concluded that she must have made a mistake when performing the MASA. Due to the relative complexity of the MASA, Valerie was comfortable assuming she had made a mistake when using it—that she had somehow mixed the numbers up. She was sure that her first answer was correct. She considered no other way of performing the computation, nor did she attempt to assess which answer was more reasonable. It seemed that cases of disagreement only reinforced her faith in her strategy and her aversion to the MASA.

Valerie's understanding of partial products. To better understand how Valerie thought about partial products in double-digit multiplication, consider the details of her argument (Figure 72). She claimed that 24 times 15 equaled to 220. As data, she described her computational steps. She computed the products of 20 and 10 and of 4 and 5, obtaining 200 and 20, respectively. Then she added those together to get her answer of 220. The warrant in her argument was the idea of breaking the numbers up into smaller parts, multiplying those smaller parts, and then adding the results together.

I describe the backing that Valerie offered as *accounting for all of each factor*. My understanding of her reasoning is as follows: By multiplying 20 by 10, she had

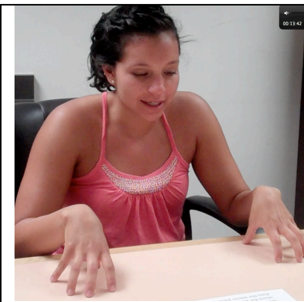
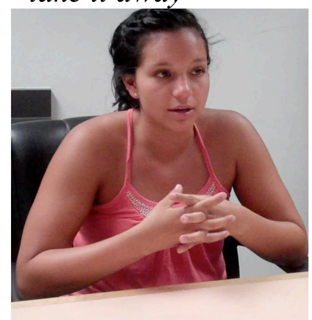
<table>
<tr><td>Claim: 24 x 15 = 220
(Valerie: Two hundred and twenty.)</td><td rowspan="4">
“take it away”

“combined them all together”</td></tr>
<tr><td>Data: 20 x 10 = 200, 4 x 5 = 20, 200 + 20 = 220
(Valerie: 20 times 10 is 200… And then I took 5 and 4 and multiplied it, and that’s 20. And I added those two together to get 220.)</td></tr>
<tr><td>Warrant: Breaking it up, multiplying, and adding it back together[27]
(“I broke it up, into segments.”)
(“It’s just breaking them into smaller parts, but as long as they all come together after you’ve multiplied both sets and add ‘em together, it’s the right answer.”)</td></tr>
<tr><td>Backing: Accounting for all of each factor
(“Because, as long—so, if you add, if you take it away so that it’s 20 and 10, you’re still left with 5 and 4. So, you need to multiply those together to get how much those would equal. And if you add both of them together, you’ll get the answer of what it would be if you combined them all together.”)</td></tr>
</table>

Figure 72. Valerie’s invalid partial products argument.

accounted for most of what needed to be multiplied. Rather than dealing with the whole 24 and the whole 15, she had dealt first with 20 of the 24 and with 10 of the 15. Once she multiplied the 20 and 10 together, she was finished with those parts. That left only the 4 and 5. She accounted for the 4 and 5 by multiplying those together. At that point, there was no multiplication left to do because she had dealt with the entirety of both factors. She had accounted for the 20 and the 10, obtaining 200, and she accounted for the 4 and the 5, obtaining 20. She added 200 and 20 together for her final answer of 220. She asserted that she would have obtained the same answer if she had not broken up the 24 and 15 but had dealt with the whole amounts at once.

[27] This idea is similar to separating, multiplying, and adding (albeit applied in an invalid way). In this analysis, I am focused on Valerie’s particular reasoning at the beginning of the course. As such, I view the idea at a finer grain size, and I use language that matches hers.

No meaning for the multiplication operation itself was apparent in her reasoning. Reasoning about multiplication in terms of repeated addition did not arise in Valerie's argument. Valerie's strategy enabled her to simplify double-digit multiplication by reducing it to computing decades times decades (e.g., 20 x 10) and units times units (e.g., 4 x 5). Even though the context for these problems had a repeated-addition structure (finding the total cost of a package of oboes, given the number of oboes and the price per oboe), Valerie's reasoning about partial products did not seem to be informed by this structure. She recognized the story problems as calling for multiplication, and then she reasoned in naked-number terms. In reasoning about 24 and 15 as naked numbers, she did not distinguish the roles of multiplier and multiplicand. The numbers seemed to have equal status and to function in the same way in the computation: they consisted of two parts, the tens and the ones, and these were to be multiplied. Multiplication itself (when it came to partial products) seemed to be a black-box process, involving recalled facts. I would conjecture that single-digit products were all recalled facts for Valerie and that she multiplied decades by a combination of recall and annexing zeroes. This conjecture is supported in part by Valerie's reasoning about the standard multiplication algorithm.

Valerie's understanding of the standard multiplication algorithm. Valerie preferred not to use the MASA for multiplication. However, she was familiar with the standard algorithm, and she reported that she sometimes used it when performing multiplication on paper. The multiplication Operations Task from her pre interview provides evidence of how Valerie understood the standard multiplication algorithm at the beginning of the course. Valerie solved 27 x 13 by the standard algorithm. She described her steps, and she made arguments concerning three nontrivial aspects of the algorithm.

Figure 73 shows Valerie’s primary written work. She did additional written work in support of her arguments. The full transcript of her response is included below. Figure 74 shows Valerie’s supporting written work.

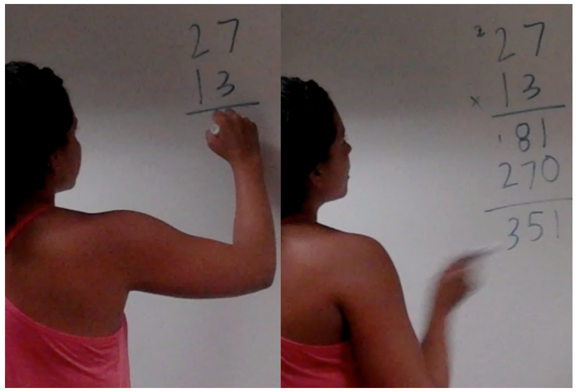


Figure 73. Valerie’s primary written work for 27 x 13.

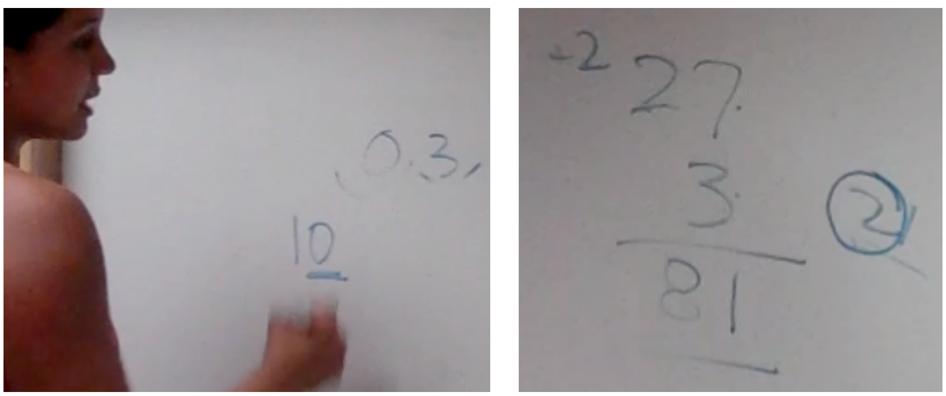


Figure 74. Valerie’s supporting written work: (left) Valerie’s work to justify writing a 0 in the second row; (right) Valerie’s work to justify carrying the 2.

Valerie: K. So, 7 times 3 is 21. So, you put the 1 down here and take up the 2. So, when you do multiplication, this number is being added. So, it’s like addition, where you’re adding this number. It doesn’t change the place of this number [pointing to the 2 in 27]. So, 3 times 2—so, you multiply the same. You multiply these two. So, 3 times 2 is 6. You add 2 to 6 to get 8.

Interviewer: Okay.

Valerie: K. Then when you go on to the next column, you’re not multiplying by this number anymore [underlines the 3 in 13]. You’ve moved over to this number [draws an arrow from 3 to 1], which means that this number got multiplied by an increment of ten [writes “x 10” off to the side]. So, that means you move the answer over to write here [points to 7 in second row of her work]. So, imagine that this isn’t even here [erases the 0 from the second row]. So, you would move the answer from right here [underlines ones place in second row] to over here [underlines tens place] because it’s moving up. So, if you had that [writes 0.3] and you multiplied by ten, you take the zero and you count one [draws an arc from the right of the 3 to the left of the three]—or. One. [draws an arc from the decimal point to the left of the zero]. Multiply by ten. So, it would be [traces over the 0 and 3]. Anyway.

[She demonstrated that incorrectly. She showed 0.3 x 10 = 0.03.]

Valerie: so, you have the zero here [rewrites 0 in second row]. So, you bring down the zero. And I always just thinking of it as, because you’re multiplying by ten, you just put the zero from the ten.

Interviewer: Okay.

Valerie: Then you start with this. So, you multiply 1 and 7, which is 7, and 1 and 2, which is 2. And then, from here, you add that together [writes +]. 1 and 0 is 1. 8 and 7 is 15. Carry the 1, which is being added to write here [draws an arrow from the carried 1 to the 2 below it] and that’s 3. So, 351.

Interviewer: Okay. You made a point about this [points to the “+2” written to the left of the 2 in 27] being added, like in addition. How come it’s added?

Valerie: Because you’re not taking anything away from another number to add on to this number. This is simply being carried over from an—think of this as—so, if you have [writes standard algorithm setup for 27 x 3 off to the side], this is a 1 [writes 1 in ones place]. Think of it as like 21. Think of the 2 as like overflow. You can’t fit the overflow in the answer because it’ll change. A way to think about it is, if you have three numbers you’re multiplying with [points to the three digits 2, 7, and 3] most likely your answer is gonna be three numbers. So, you can’t add this one [referring to just writing down 21 in the answer space] because you’ll have four numbers rather than three.

Interviewer: Okay.

Valerie: So, you take—you have 21. You have an overflow. When you have an overflow in addition, you take the overflow number [circles the 2 in 21, written off to the side] and put it toward the other side. So, that’s

> why you put the 2 up there, and add that to whatever's being multiplied, which will be 8. (Valerie, personal communication, September 14, 2010)

In her response to the task, Valerie made three arguments of interest:

1. You have to carry the 2
2. You have to add the carried 2
3. You have to write a 0 in the ones place of the second row

The details of each of her arguments are described below.

In Arg VM1.1 (Figure 75), Valerie argued that the 2 had to be carried based on the idea of "overflow." This argument was similar to the one that Brandy made about carrying in the addition algorithm in her first interview. She reasoned about 21 as consisting of two digits and argued that only one of those digits could be placed in the righthand column of the answer. Otherwise, the number of digits in the answer would increase, and it would be incorrect as a result. Valerie made an interesting generalization regarding numbers of digits when she inserted that having three digits in the problem would likely result in three digits in the answer. This is not true in general. In fact, it was not true in her example of 27 x 3 = 81. Moreover, in lieu of some meaningful justification, it suggests reason about the multiplication algorithm in terms of patterns of behavior of digits, rather than in terms of place value and partial products.

In Arg VM1.2 (Figure 76), Valerie argued that the carried 2 (discussed in VM1.1) should be added. That is, once she multiplied 3 by 2, obtaining 6, she argued it was appropriate to add the carried 2 to 6 and obtain 8, which should be written below. She gave less attention to the addition step than to the carrying step. Nonetheless, she did make an identifiable argument.

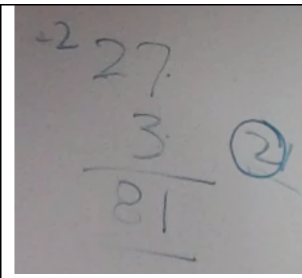
Claim: You have to carry the 2	
Data: "You have an overflow." 7 x 3 = 21, and 21 will not fit in the ones column of the answer.	
Warrant: When you have an overflow, you have to carry the overflow digit When you have an overflow in addition, you take the overflow number [circles the 2 in 21, written off to the side] and put it toward the other side.	
Backing: If you don't carry the overflow digit, you'll get the wrong answer You can't fit the overflow in the answer because it'll change. A way to think about it is, if you have three numbers you're multiplying with [points to the three digits 2, 7, and 3] most likely your answer is gonna be three numbers. So, you can't add this one [referring to just writing down 21 in the answer space] because you'll have four numbers rather than three.	

Figure 75. Arg VM1.1: Valerie's argument concerning carrying in multiplication.

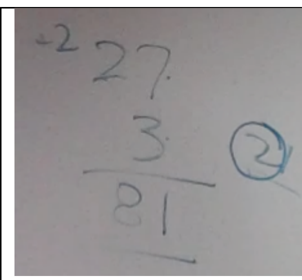
Claim: You have to add the carried 2	
Data: The 2 was carried, not borrowed ("Because you're not taking anything away from another number to add on to this number. This is simply being carried over…")	
Warrant: Carried numbers get added in multiplication, as in addition ("it's like addition, where you're adding this number.")	

Figure 76. Arg VM1.2: Valerie's argument concerning adding the carried digit in multiplication.

Figure 77 represents Valerie's argument concerning writing the zero. Valerie demonstrated moving the decimal point incorrectly. According to her work, 0.3 x 10 = 0.03. However, the focus of our discussion is on her reasoning about multiplication. Whether she moved the decimal point in the appropriate direction or not, she was attempting to justify the need to write a 0 in the second row by appealing to a rule for multiplying by ten. Even though she identified the 1 in 13 as representing ten, she did not

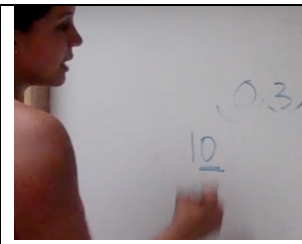
<table>
<tr><td>Claim: You have to write a 0 in the ones place of the second row</td><td rowspan="3">
“Multiply by ten.”</td></tr>
<tr><td>Data: When you multiply by the 1 in 13, you’re really multiplying by 10
(“when you go on to the next column, you’re not multiplying by this number anymore [underlines the 3 in 13]. You’ve moved over to this number [draws an arrow from 3 to 1], which means that this number got multiplied by an increment of ten”)</td></tr>
<tr><td>Warrant: When you multiply by 10, you move the decimal point
(“So, if you had that [writes 0.3] and you multiplied by ten, you take the zero and you count one [draws an arc from the right of the 3 to the left of the three]—or. One. [draws an arc from the decimal point to the left of the zero]. Multiply by ten. So, it would be [traces over the 0 and 3].”)</td></tr>
</table>

Figure 77. Arg VM1.3. Valerie’s argument concerning writing a 0 in the second row in multiplication.

talk about the associated partial products as numbers of tens (7 tens and 20 tens). Rather, she used one procedure to justify another.

It may be difficult to see in Valerie’s initial reasoning much in the way of productive resources. However, Valerie’s knowledge of the standard algorithm turns out to be an important resource that supported her ability to make sense of partial products.

Valerie’s Developing Reasoning between the First and Second Interviews

This section describes the development of Valerie’s reasoning related to multiplication between the first and second interviews. The presentation is chronological, drawing on homework and test responses, as well as a significant contribution that Valerie made to whole-class discussion on Day 14. Valerie attended class faithfully and was an active participant in classroom activities, including whole-class discussions. It appears there was little development to speak of prior to Day 14. However, Valerie’s written responses provide additional information concerning her reasoning about multiplication and, in particular, her interpretations of strategies discussed in class. The analysis of the time between interviews is separated into the following landmarks: prior to Day 14, Day 14 and Valerie's subsequent reflections, and reasoning about weights (Days 16 and 17).

Valerie's reasoning before Day 14. On Day 4, Aaron presented the strategy that he had used to compute 600 ÷ 25 mentally. Aaron thought in terms of finding out how many 25's there were in 600. He knew that there were four 25's in 100 and six 100's in 600, so he multiplied 4 by 6 to get his answer of 24. In Journal #2, students were asked to reflect on Aaron's strategy. Valerie described his strategy as follows:

> [Aaron] broke down the problem, much like could be done in addition or subtraction. For example, if asked to add 88 and 22, you could add 80 +20 = 100. Then with the remaining 8 and 2 omitted in the first equation, you are left with 10. In order to get the correct answer, you need to add the parts, being 100 and 10, to get the whole/total answer of 110. [Aaron]
>
> started with an equation he knew, 25 x 4 = 100 (like 4 quarters = 1 dollar). Now he needed to get that 100 to 600, by multiplying by 6. Because he used multiplication, the 4 and 6 need to be multiplied rather than added to reach the final answer. 6 x 4 = 24

The addition example that Valerie gave was a Left-to-Right Separation strategy, which bears little resemblance to Aaron's division strategy. However, the comparison that she drew to an addition problem is reminiscent of her argument for her invalid multiplication strategy. She referred to breaking the problem down into tens and ones, computing, and then combining. The latter leveraged the associative property of multiplication, and this aspect was not apparent in Valerie's explanation. Also, with respect to Aaron's argument, she argued that Aaron needed to multiply 4 by 6 "because he used multiplication." It is not clear that Valerie could have provided any valid backing for this vague warrant. I imagine that she could have made sense of this strategy by reasoning about quarters and dollars. Instead, she reasoned vaguely that Aaron "broke down the problem."

In the second part of Journal #2, students were asked to apply Aaron's strategy to another division problem, 1200 ÷ 50. These numbers were chosen because they lent themselves to Aaron's strategy. However, Valerie's written work was strikingly out of sync with the example of Aaron's strategy from class. She computed 1000 ÷ 50 and 200÷ 50 separately, which was not consistent with using associativity. Neither did she use Aaron's strategy to compute the partial quotients. She focused on the number of zeroes in the dividends and divisors. (See Figure 78.)

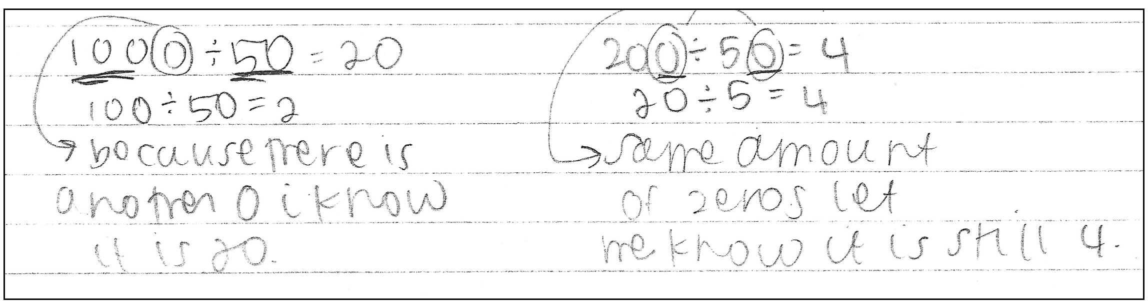


Figure 78. Valerie's attempt to apply Aaron's strategy.

Valerie's computation for 1000 ÷ 50 could have been an instance of Aaron's strategy (e.g., by recognizing that there were two 50's in 100 and reasoning that there would be ten times as many 50's in 1000). However, her explanation suggests thinking in terms of annexing zeroes. Valerie's observation that 200 ÷ 50 = 20 ÷ 5 = 4 is true, but hernjustification reads as an attempt to appeal to a procedure, and that procedure has nothing to do with Aaron's strategy. She seemed to view the strategy vaguely as involving breaking numbers down, computing, and putting them back together. This enabled her to see his strategy as similar to Addition Separation, to offer an explanation that sounded much like her argument for her invalid multiplication strategy, and to misapply his strategy, treating it as involving distributivity and annexing zeroes, rather than associativity.

As described in Chapter 5, multiplication computations arose during the place value unit. On Day 6, two students presented their strategy for computing the product of 64 and 8: 60 x 8 = 480, 4 x 8 = 32, and 480 + 32 = 512. This was an instance of separating, multiplying, and adding with a single-digit multiplier. In Journal #3, students were asked to reflect on this strategy and to consider how it related to the standard algorithm. They were asked how the ones and tens were recorded in the standard algorithm. They were also asked how many whole tens were in the answer of 512 and where those tens came from. In Journal #3, Valerie's response had no relationship to the standard algorithm. She said that 512 would need 8 more ones in order to make 52 tens.

In class on Day 7, Zelda separated, multiplied, and added to compute the product of 49 and 7 as 40 x 7 + 9 x 7. In a follow-up homework assignment, students were asked to suggest a name for this strategy and to make a diagram that would capture the "key idea" of the strategy. Valerie suggested the name "Break-and-Bake cookie" strategy. She explained the metaphorical relationship that she saw between cookie baking and this multiplication strategy:

> First you are presented with a big slab of cookie dough (or a large multiplication problem). You break the cookie dough down into individual cookies (break down numbers). You arrange the cookies on a tray to bake (solve the respective sets of problems) and finally count the finished cookies that were once connected (add solutions together to find total)

Valerie's explanation related separating, multiplying, and adding to the process of baking cookies in a very explicit way. Valerie also drew a picture (Figure 79) to depict the cookie-baking process in relation to the multiplication strategy.

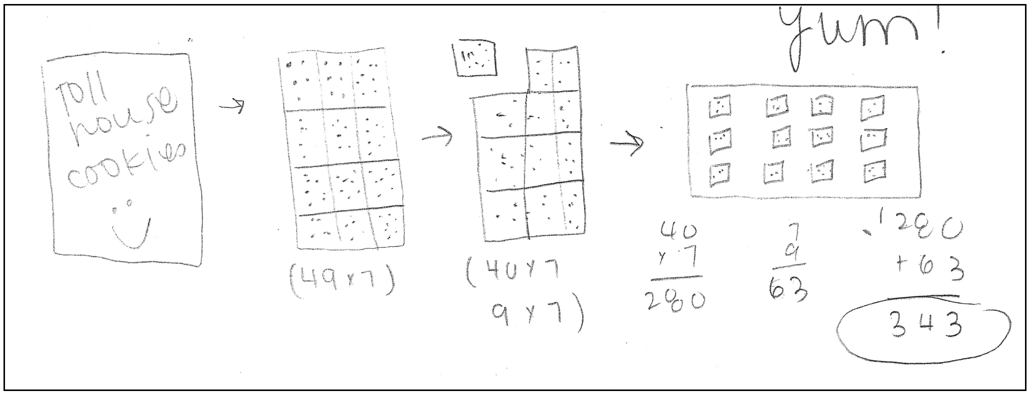


Figure 79. Valerie's cookie-baking drawing.

Unlike many other students, Valerie was specific about how each step in the cookie-baking process might relate to the steps in the strategy. While the strategy involved breaking numbers up and putting them back together, the nature of the entities that were broken apart was different than those put together. In this case, the number 49 was broken into 40 and 9. It was not then added back together as 40 + 9. Rather, each part was multiplied by 7, and then the partial products were added together. The only meaning that I can see for multiplying by 7 in Valerie's analogy is putting the cookies in the oven and baking them. She ignored the significance of tens and ones, and her drawing in no way took dimension into account.

I interpret Valerie's characterization of Zelda's strategy as related to her interpretation of Aaron's strategy, as well as to her reasoning about 24 x 15 in her first interview. Although the details of these strategies were different, they involved breaking numbers up, computing, and putting them back together. Thinking about nonstandard strategies in these general terms would not enable Valerie to make necessary distinctions between valid and invalid strategies. Through the lens of Valerie's reasoning, it seems like a wide range of strategies, both valid and invalid, would have appeared equally reasonable.

Change in Valerie's reasoning about partial products. On Day 14, students worked on the Carpet Problem. Initially, all students agreed that the area of the 23-by-23-foot room was 409 square feet. Valerie explained how she had arrived at that answer. Valerie's primary argument (Figure 80) involved reasoning about partial products in terms of pairing up tens and ones. (Her ancillary argument involved annexing zeroes and is omitted here.)

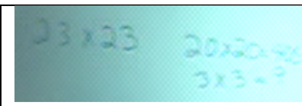
Claim: A room measuring 23ft by 23ft has an area of 409 square feet (*Valerie*: [I] got 409.)	
Data: 20 x 20 = 400; 3 x 3 = 9; 400 + 9 = 409 (*Valerie*: Okay, so I made both 23's into 20… so it's 400. And then I did 3 times 3 is 9 and so I added 400 and 9 and got 409.)	
Warrant: (Implicit) Reasoning about partial products in terms of pairing up tens and ones	

Figure 80. Valerie's primary argument concerning the product of 23 and 23.

Initially, the class agreed with Valerie's solution. However, students found that the standard multiplication algorithm gave a different answer, of 529, so Valerie's solution was rejected. Students went on to make sense of why Valerie's strategy was invalid, to construct a valid strategy involving two partial products, and finally to account for all four partial products by relating them to steps in the standard algorithm.

In a journal assigned after Day 14, students were asked to reflect on the invalid strategy that they had agreed with initially. Students were asked whether they were surprised that computing 20 x 20 + 3 x 3 turned out to be wrong and to describe why it had seemed to make sense. In her response, Valerie said,

> This kind of strategy only works with addition. If you were to add 23 and 23, separating the numbers into 20 + 20 and 3 + 3 would make sense (separate-add-add). When I first saw the problem, I thought of it in terms of separate-add-add. I did not take into consideration that multiplication would be different, seeing that the addition strategy was so fresh in my mind. (recency effect in psychology)

Valerie attributed her use of an invalid multiplication strategy to the application of an idea that was true for addition, and she claimed that this was the result of the fact that she had been thinking about addition recently ("recency effect in psychology"). However, we know from Valerie's first interview that the invalid multiplication strategy that she described in class was actually her go-to strategy.

Prior to day 14, Valerie seemed to have reasoned similarly about all strategies that involved breaking numbers down. In the above response, she again gave an addition example, but she *contrasted* it with multiplication. She was now asserting that what worked for one operation would not necessarily work for another.

In the second part of the same journal assignment, students were asked whether they could identify a particular realization that they had concerning this strategy. They were asked how they now understood the fact that 20 x 20 + 3 x 3 was *not* equal to 23 x 23. In her response, Valerie addressed the role of partial products. She said,

> When you make 23 into 20, you do not simply take away 3 in a singular sense. You take away 3 groups of 23. If you just multiplied 3 x 3, you have 9, whereas 23 x 3 = 69. That accounts for the 3 groups of 23 taken away to make 20.

This response is important and provides some insight into Valerie's reasoning. As in her first interview responses, Valerie had reasoned in class that the contribution of the 3's to the product was just 3 x 3 = 9. She now articulated a view of the 3 in 23 as being multiplied by the other factor of 23, so that its contribution to the product was 3 x 23 = 69 while recognizing it as a "key realization." She had not expressed this idea in any of the data up to this point.

It is not clear from the above response whether Valerie could have accounted for the contributions of *both* 3's. Her explanation was in line with that of students in Vignette 7.3, who described the product of 23 and 23 as consisting of 23 x 20 and 23 x 3. Given the data, it appears that Valerie's thinking progressed to the point where she could appropriately treat double-digit multiplication as involving two partial products, but it is unclear whether she could account for all four partial products separately.

Valerie's reasoning about Equal Area Shifting and weight. On Day 17, the strategy that involved halving one factor and doubling the other was discussed and was given the name "Equal Area Shifting." In Journal #8, students were asked to apply this strategy to a novel computation in a story problem context:

> In an attempt to save some money, Julie decided not to buy a parking permit for fall semester. Julie got 5 parking tickets over the course of the semester, each for $36. How much money did she end up spending on parking?

Valerie correctly applied Equal Area Shifting. She explained her reasoning with a drawing, number sentences, and in words. (See Figure 81.)

Unlike when she attempted to apply Aaron's strategy, Valerie's drawing and explanation were closely related. She used rectangular area and explicitly addressed halving and doubling the dimensions of a 5-by-36 rectangle to create a 10-by-18 rectangle. This response was correct and her explanation was clear. It is possible that her

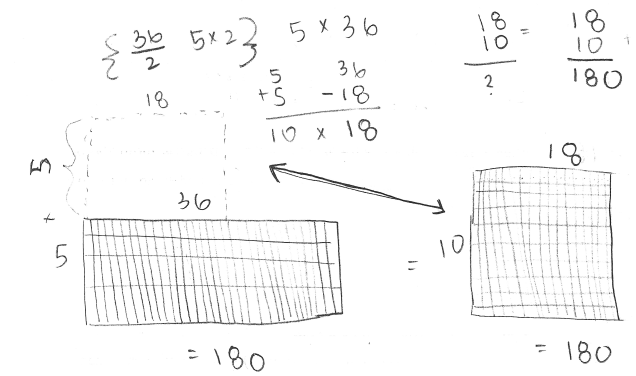
	Julie spent $180 on parking in the fall semester. Equal area shifting involves taking half of the larger number and doubling the smaller number. If you were to cut off half of 36 → the longest part, you attach it to the 5, which would make the dimensions 10 x 18 → 180 36 x 5 = 180

Figure 81. Valerie's application of Equal Area Shifting: (left) Valerie's drawing and number sentences; (right) her written explanation.

response involved following an example from class rather closely (albeit applied to different numbers). On the other hand, all of the details seemed to make sense and were consistent with the strategy, which was not the case when Valerie attempted to apply Aaron's strategy to a different problem.

Valerie's invented notation was interesting. Although she performed the computations correctly, she used an additive notation to record the reformulation of 5 x 36 into 10 x 18. Just above the two-way diagonal arrow in the middle of her diagram, Valerie wrote 5 and 36, then +5 and -18 beneath these, a horizontal bar, and finally 10 x 18. Thus, she recorded the reformulation as if 5 was added to the factor of 5 and 18 was subtracted from the factor of 36. On the other hand, in her explanation, she described these steps in terms of halving and doubling the numbers and related these to the dimensions of the rectangles. Thus, she seemed to sensibly coordinate the additive idea of increasing one factor by 5 and decreasing the other by 18 with the multiplicative ideas of doubling and halving to appropriately account for the weight of the refomulation steps. As Valerie had said in her journal after Day 14, these additive steps did not affect the product "in a singular sense." A change to a factor had a certain weight, which derived from the corresponding partial product.

As part of a homework assignment following Day 17, students solved problems that involved computing and estimating products. One problem asked students to estimate 0.76 x 62. Valerie rounded 0.76 to 0.75 and then computed 0.75 x 62 by halving and doubling. She reformulated the computation to 1.50 x 31, estimated this as 30 x 1.50, and arrived at an estimate of 45.0. In this response, Valerie spontaneously applied the halving and doubling strategy in a problem involving a decimal number and in the context of estimating a product. The context in which this strategy had been introduced was quite different, and this suggests that Valerie was not merely following an example from class. She seemed to recognize an affordance in the given numbers. In her explanation, Valerie said that she knew 0.75 x 2 was 1.50, and there were 31 "groups of 2" in 62. In this way, she made the associative aspect of halving and doubling explicit. This explanation contrasts markedly with Valerie's interpretation of Aaron's strategy from Day 4, in which she seemed oblivious to the associative aspect of his approach.

Another problem in the same homework assignment was a repeat of a task from class that had to do with weight. It asked the following:

> Consider 150.68 x 5.34. In estimating the product by calculating 150 x 5, one has ignored the decimal part of each number. In refining the estimate, which decimal part (0.68 or 0.34) should be the focus? Explain.

In her response (Figure 82), Valerie drew on ideas from class, which may have been in her notes or simply remembered. Trina had argued for that the 0.34 in 5.34 weighed more that the 0.68 in 150.68 by estimating the decimal parts as the fractions 1/3

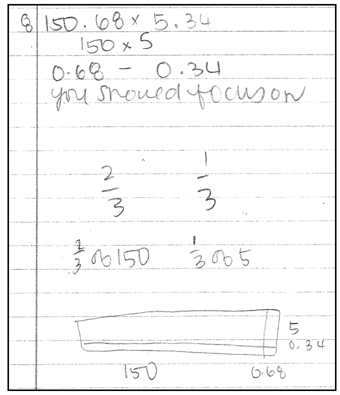


Figure 82. Valerie's response to a homework task concerning weight.

and 2/3 and reasoning in terms of partial products. The Instructor had shared a rectangular area diagram and used it to argue that the 0.34 weighed more. Valerie's written response included elements of both of these arguments. However, Valerie did not explicitly answer the question.

Valerie's area drawing seems appropriate, but there is no evidence that she made any substantive use of it. Her use of the fractions 2/3 and 1/3 appears incorrect. She wrote "2/3 of 150" and "1/3 of 5." If by this she meant to refer to partial products 2/3 x 150 and 1/3 x 5, then her response suggests either thoughtlessness or a misundertanding of the relationship between parts of a factor and partial products. It could also be that she meant the word *of* in an informal, colloquial sense, so that she was referring to the 2/3 belonging to the 150 and the 1/3 belonging to the 5. However, in that case, she would not have presented sufficient data to answer the question. (Also, in light of Valerie's second and third interview responses, this seems unlikely.)

Valerie had become aware that her old go-to strategy was invalid, and she had begun to attempt to account for partial products correctly. However, she was inconsistent in her ability to do this. I believe that this inconsistency was not a fluke but was due to how she organized her thinking in attempting to account for partial products. This theme will be explored further in the subsections that follow.

Summary of Valerie's developing reasoning between the interviews. In the period prior to Day 14, Valerie's reasoned vaguely about nonstandard as involving breaking numbers down, computing, and putting numbers back together. Reasoning at this grain size, nonstandard strategies made the same kind of sense, both within operations and across operations, and whether those strategies were valid or invalid. On Day 14, Valerie used her go-to multiplication strategy in class and explained her reasoning. Although the other students agreed with her initially, it turned out that Valerie's strategy was invalid. The focus of activity on Day 14 was making sense of why that strategy was invalid, how it could be modified to create a valid strategy, and how it related to the standard algorithm.

I believe that Valerie understood some of the arguments that were made on Day 14. It is safe to say that she came away from class knowing that her strategy was not right. Following Day 14, there is evidence that she began to make distinctions between nonstandard strategies and consider more carefully why they would work or not work. In her response concerning her invalid strategy, she first expressed reasoning about partial products in terms of part of one factor being multiplied by *all* of the other factor.

Whereas Valerie had not understood the associative aspect of Aaron's strategy from Day 4, she did make sense of Equal Area Shifting after it was introduced in class on Day 17. She even spontaneously applied the idea of halving and doubling to estimate the product of decimal numbers. In these responses, Valerie appropriately accounted for the weights of parts of factors in terms of partial products. However, when it came to a direct question concerning weight in estimating products—the very same question that had been discussed in class—Valerie gave a sloppy response in which she did not coordinate partial products correctly.

Valerie's Reasoning in the Second and Third Interviews

Valerie's second (post) and third (SST) interviews took place only one week apart. I see no evidence of development in Valerie's reasoning in between these two interviews, and data from both contribute to my understanding of her reasoning after the end of the whole-number unit. So, I include data from both interviews in the description of her reasoning in this section. Consistent with the chronological presentation, though, the second-interview data is presented before the third-interview data.

Overview. In the second interview, Valerie reasoned about multiplication more meaningfully and flexibly than in the first. She could justify the details of the standard algorithm by reasoning about digits in terms of their place values. For the basic Bobo tasks, she used three valid strategies. Her scaffolded alternatives included additional, valid strategies.

Despite all of this evidence of improvement, Valerie continued to have difficulty accounting for partial products. For the first product that she computed mentally, she started to use her old invalid strategy, but she quickly recognized that it was wrong. She corrected herself to a degree by identifying one additional partial product. However, she was unable to account for all four partial products. She went on to solve the other basic Bobo multiplication tasks correctly, using valid strategies, but she avoided accounting for all four partial products explicitly.

In the Jessica Task, Valerie was asked directly to reason about the invalid strategy that she had relied on in the first interview. In this context, she argued that the strategy was not right, and she was eventually able to construct a coherent argument to that effect. This was not without difficulty, however. She was not readily able to account for all partial products appropriately. It was only when she used the standard algorithm to help organize her thinking that she was able to sort these out.

In the third interview, Valerie reasoned about weight in the context of estimating a product. She did this quickly and correctly without any difficulty. She drew sensible pictures and clearly explained the relationship between partial products and partial rectangles. In this context, rectangular area clearly seemed to support her reasoning about partial products. Valerie was also able to justify the relationship between multiplication and rectangular area. Valerie also reflected on her experience in the course in the third interview, and she talked about how her thinking about mental computation had changed.

Valerie's multiplication strategies. In her second interview, Valerie reasoned differently about computing products than she had two months before in three important ways. First, she thought in different ways about partial products and how to account for them. Second, she distinguished between the roles of multiplier and multiplicand using the language of "groups of." Third, she recognized affordances in some numbers. Her responses to the four basic Bobo multiplication tasks are each of interest because her reasoning varied across the set of tasks and illustrated aspects of this developing understanding.

Recall that in the first interview, Valerie computed 24 x 15 by multiplying the tens by the tens and the ones by the ones, obtaining an answer of 220. In the second interview, she started off similarly but soon questioned her approach.

> *Interviewer*: So, if you recall, Bobo started selling oboes in packages. So, he's offering a package of 15 oboes for 24 dollars per oboe. What would that whole package cost?
>
> *Valerie:* Um, [pause] twooo-twenty? That's not right. [pause] I get 220, but I don't think that's right.
>
> *Interviewer*: How did you get 220?
>
> *Valerie:* Well, I just did [shakes her head] but that's not right. It's more than that. Well, what I did is I did, like, 20 times 10 is 200, and then I did 4 times 5. But that's not right.
>
> *[Valerie thinks longer and briefly considers whether her answer could be right, but she decides that it is not. She says she would have to write this down to solve it.]*
>
> *Valerie:* I would have to write it down. I can't do it in my head.
>
> *Interviewer*: K. So, you said 220
>
> *Valerie [interrupting]:* But I don't think that's right
>
> *Interviewer*: and then you said it's not right. Why do you say it's not right?
>
> *Valerie:* Because it's not—it would be a different story if, like, if it was like 24 times five. Because then it would make sense to do 20 times 5 and then 4 times 5, because it's only one place. But now that it's 15, I would have to multiply [positions her hand out in front of her and points down at the table] like, I can't just multiply, um, 10 by 20 and then just 4 and 5 [seems to trace out the corresponding digits in the standard algorithm, tapping her finger as she says each number, and moving vertically (relative to a view looking down on the table) from 10 to 20 and from 4 to 5]. I have to do, like, 24 times 10 [moves her finger horizontally and then vertically] and then like 4 times 5 [moves her finger diagonally] and like, I have to write it [makes a sweeping gesture, as if to erase her imaginary written work]—write it down. It doesn't—it's like too much multiplication for me to do in my head without writing it down [laughs]

Interviewer: K. Is there any way you can figure it out without writing it down?

Valerie: [pause] Okay, well, if I did 20 times 10, is 200. So, that takes care of these two places. So then 20 times 5 [pause] is another hundred. So, that's 300. And then 4 times 5 is 20. So, that's 320 [shakes her head]. Well, I guess that makes more sense, actually, so it'd be 320. Because if you take and you make it 24—no, 15 times 20 [pauses, mumbles, long pause]. Okay, well, 15 times 2 is 30. So, if you made it 15 times 20, that's 300. And then you would make it 5 times 4, and that's 20, so 320.

Interviewer: So, you're doing [gestures with pen in the air, similarly to what Valerie was doing earlier with her finger in the table]

Valerie: [laughs]

Interviewer: a few different multiplications

Valerie: yeah

Interviewer: depending on how you're doing it

Valerie: mm-hm

Interviewer: How do you know, like, which ones to do?

Valerie: Well, you have to, like, make sure that all of the places are accounted for. So, like I said, if it was like 24 times 5, then it would make sense to do, like, 20 times 5 and 4 times 5. Like, that makes more sense. But because there's an extra tens place in there, you have to account for all the places. So, you can't just be like oh, it's 24 times 10 [repeats gesturing on table but with her whole hand] and then whatever, whatever. You have to, like, account for all of them, if that makes sense.

Interviewer: Okay.

Valerie: I don't [shakes her head] I still don't even know if that's right [laughs]

[I interpret "that" here as referring to her revised answer of 320.] (Valerie, personal communication, November 11, 2010)

In this response, Valerie clearly conveyed that she knew her old go-to strategy was invalid. She articulated an understanding of appropriate partial products in the 1-digit-times-2-digit case. She knew that the 2-by-2 case was more complicated, and she expressed a concern for accounting for all of the partial products. In some obvious ways, this response contrasts with her first interview response: she knew that just multiplying 20 by 10 and 4 by 5 was incorrect; she ended up computing three partial products, rather than two; as a result, she arrived at a different answer, 320 rather than 220; although this answer was still incorrect, it was closer to correct; she accounted for three of the four partial products, and her answer of 320 was not far off from the correct answer of 360.

Valerie's reasoning in this case is also similar to her reasoning in the first interview in the sense that she was again playing a kind of digit-matching game. The differences were in the details of her approach. Whereas in the first interview she thought in terms of *accounting for all of each factor*, in this response she thought in terms of *accounting for all partial products*. Yet it was not clear to her exactly how to do that. There was not an evident principle guiding her matching decisions. She stopped at three partial products and a total of 320 not because she could justify that she had exhausted all pairings but, as far as I can tell, simply because she did not see or could not think of another pairing. When reasoning in terms of accounting for all of each factor in the first interview, it had been easy for Valerie to know when she was finished. She used the 20 and the 10, then she used the 4 and the 5, and there was nothing left, so she was done. However, accounting for all partial products was more complicated. Parts of factors needed to be used more than once and paired up in just the right ways.

The story of Valerie's second-interview responses does not end there. She solved each problem differently, and her reasoning changed as she moved through the problems. Valerie solved the next problem (19 x 21) by using the MASA. She solved the problem correctly and without any apparent difficulty. To solve 25 x 16, Valerie dealt again with partial products and again experienced some difficulty. However, her reasoning about this problem was quite different than her reasoning about the first. It involved distinguishing between the role of multiplier and multiplicand:

> *Interviewer*: What if Bobo offers a package of 25 oboes for 16 dollars per oboe? What would that whole package cost?
>
> *Valerie:* Ummm [pause] 250
>
> *Interviewer*: How did you get 250?
>
> *Valerie:* Wait [pause] yeah. Because I did 10 times 25—oh—which is 250. Okay, so it's three hundred [cringes]. Because I did 10 times 25, which is 250. And then I thought of the 25's in terms of like quarters. So, there's four quarters in a dollar. So, it'd be 250, and then another four quarters is a dollar, so that's 350. So, it's 400 because then you have two extra quarters left over, which is 50 cents. So, it's not 250 [shakes her head and waves], I
>
> *Interviewer*: Okay. So, how did you get from 250 to 400?
>
> *Valerie:* Okay. So, I did 10 times 25, which is 250. And then you have to do 6 times 25.
>
> *Interviewer*: Oh, okay.
>
> *Valerie:* and so, if you do four of 'em, it's a hundred, and then you have two left over, which is 50 cents. So, it comes out to be 400.
>
> *Interviewer*: Gotcha. (Valerie, personal communication, November 11, 2010)

Valerie reasoned about 16 times 25 in terms of finding the sum of sixteen 25's. She computed ten 25's, which gave her 250. Then four more 25's gave her another 100, for a subtotal of 350. Finally, the remaining two 25's brought the total product to 400. Valerie's strategy could be represented as $16 \times 25 = (10 + 6) \times 25 = 10 \times 25 + 6 \times 25 = 250 + (4 + 2) \times 25 = 250 + 4 \times 25 + 2 \times 25 = 250 + 100 + 2 \times 25 = 350 + 50 = 400$.

After she had computed the product that way, Valerie realized that she could have made it simpler by reasoning in terms of quarters all along. As I was making my notes about her strategy above, she spontaneously suggested another way:

> *Valerie:* Or I guess you could just do four groups of 25. Ha [laughs] Because if you have 16 quarters, it's four dollars. So, 400.
>
> *Interviewer*: Okay. (Valerie, personal communication, November 11, 2010)

I did not follow up on this spontaneous alternative because the interview protocol did not include discussion of alternative strategies during the basic Bobo tasks. I view Valerie's Quarters strategy as belonging to her Scaffolded Strategy Range since it only occurred to her after she had solved the problem by distributivity. Nonetheless, this strategy is noteworthy since it is distinct from the first. Valerie recognized that four 25's made 100, and there were four such groups that could be formed from sixteen 25's. Formally, this strategy involves the associative property of multiplication: 16 x 25 = (4 x 4) x 25 = 4 x (4 x 25) = 4 x 100 = 400. This approach is related to Aaron's division strategy from Day 4, which Valerie did not understand at the time that it was introduced.

To compute the cost of 99 oboes at $15 per oboe, Valerie used a different strategy, which took advantage of the proximity of 99 to 100:

> *Interviewer*: What if Bobo offers a package of 99 oboes for 15 dollars per oboe. What would that whole package cost?
>
> *Valerie:* Oh, okay. So, if you made it a hundred oboes, um, a hundred times 15 is 1500. So, you would subtract 15 because we need to subtract one group of 15, which is 1485.
>
> *Interviewer*: Why do you subtract one group of 15?
>
> *Valerie:* Because, when you make 99 into 100, you're not adding just one; you're adding a group of 15. So, because you're adding, like, one group of

> 15, you have to subtract 15—like, the actual number 15—from your final answer.
>
> *Interviewer*: K. Why is it that you're not adding one, you're adding a group of 15?
>
> *Valerie:* Um, because it's [pause, mumbles] So, each group—so, like 99 represents the number of groups of 15 that you have [both hands out and perched on table]. So, like when you multiply—so, like one times 15, which would be one group times 15 units per group is equal to 15, like that's how much is in each group [brings hands together with fingertips touching]. So, if you have 99 groups of 15, when you add a group [holds left hand in fixed position, makes fist with right hand and jumps fist horizontally in an iterative manner], you have to add the entire 15 to make it a solid group.
>
> *Interviewer*: Thank you. (Valerie, personal communication, November 11, 2010)

Valerie computed the product of 99 and 15 by rounding 99 to 100, multiplying, and then compensating appropriately for rounding. Her argument hinged on reasoning about multiplication in terms of repeated addition. She explicitly distinguished the roles of multiplier (99) and multiplicand (15) as a number of groups and a number of units per group, respectively. Valerie's argument is represented by Figure 83.

In her first-interview response, Valerie had reasoned about 99 and 15 (and factors in general) as being of equal status. In this response, by contrast, she clearly distinguished their roles as a number of groups and a number per group, and doing so was fundamental to her justification for her compensation strategy.

Valerie's set of second-interview responses to the basic Bobo multiplication tasks was unusual among the interview participants in the sense that her thinking seemed to evolve from one problem to the next. In her response to the first problem, she did not seem to reason meaningfully about multiplication, and she had difficulty accounting for partial products. In the third problem, she was able to account for partial products

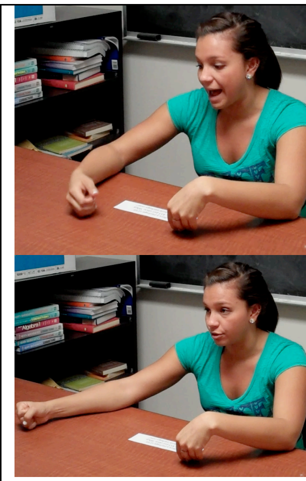
Claim: 99 x 15 = 1485	
Data: 99 + 1 = 100, 100 x 15 = 1500, 1500 – 15 = 1485 "So, if you made it a hundred oboes, um, a hundred times 15 is 1500. So, you would subract 15 because we need to subtract one group of 15, which is 1485." "when you make 99 into 100, you're not adding just one"	
Warrant: Compensating for rounding: You added a group of 15, so you have to subtract 15 "Because, when you make 99 into 100, you're not adding just one; you're adding a group of 15. So, because you're adding, like, one group of 15, you have to subtract 15—like, the actual number 15—from your final answer."	
Backing: Reasoning about multiplication in terms of repeated addition (a number *of* groups times a number *per* group) "So, each group—so, like 99 represents the number of groups of 15 that you have. So, like when you multiply—so, like one times 15, which would be one group times 15 units per group is equal to 15, like that's how much is in each group. So, if you have 99 groups of 15, you have to add the entire 15 to make it a solid group."	*"when you add a group" [holds left hand in fixed position, makes fist with right hand and jumps fist horizontally in an iterative manner]*

Figure 83. Valerie's second interview argument concerning 99 x 15.

appropriately by treating 16 as a double-digit multiplier and 25 as a multiplicand that was to be iterated without decomposition. By the fourth problem, reasoning about multiplication in terms of repeated addition fundamentally informed her approach. She answered quickly and confidently, and she provided a detailed and valid mathematical justification for a sophisticated nonstandard strategy.

As discussed in Chapter 3, the pairs of numbers used for the basic Bobo tasks were ordered so as not to scaffold students' reasoning. Essentially, the problems were ordered from most to least difficult, as best I could anticipate how students might reason. That way, strategies participants used that might have been unique to benchmark numbers would not inform their thinking about problems that were less special. Due to the ordering of problems, it may be that Valerie's reasoning did not progress as she moved from one problem to the next, but rather her approaches were problem-specific.

Valerie's reasoning about 16 x 25 and about 99 x 15 involved reasoning about multiplication in terms of repeated addition, which she also used in her journal assignment after Day 14. I would attribute the trend in her responses, in part, to the special affordances of the numbers 25 and 99, which enabled Valerie to reason appropriately about partial products on the basis of repeated addition. In neither case did she explicitly account for four partial products. In the cases represented by 24 x 15 and 19 x 21, Valerie did not perceive any special affordances. She thought that she needed to account for all partial products separately (in the digit-pairing sense). She was unsure whether she did this successfully in the first problem. She abandoned that approach and solved the second problem by falling back on the MASA, which she knew would enable her to account for partial products correctly.

It seems apparent from her responses that, at the time of the second interview, Valerie's reasoning about partial products in multiplication was still developing. She was using new ideas that were less comfortable and about which she had less confidence. This uncertainty made sense. Valerie had previously been confident of a familiar strategy that had turned out to be invalid. She seemed to proceed more cautiously in the second interview. Nonetheless, the contrast between the two interviews is drastic. In the first, she used the same invalid strategy for each computation, and the only alternative she considered was the MASA. In the second interview, she used different strategies for each problem, and three of these were valid. She knew that her old strategy was invalid, she could explain why this was the case, and she was trying to find a way to reliably account for partial products.

Valerie's scaffolded alternatives. I also investigated scaffolded alternatives in the second interview. These provide further evidence of Valerie taking advantage of (1) number affordances, (2) meanings for multiplication, and (3) awareness of the need to account for partial products. Valerie's responses to those tasks contrasted with her first-interview responses more so than in any other part of the interview. She articulated four distinct, valid mental multiplication strategies, not including the MASA, and produced clear mathematical arguments for each of them. Valerie had also come up with the Quarters strategy as a spontaneous alternative in her reasoning about 16 x 25, and that too is considered to be part of her scaffolded strategy range.

To compute 45 x 12, Valerie first took a valid approach involving two partial products. She decomposed 12 into 10 and 2. She computed 10 x 45 = 450 and 2 x 45 = 90. Then she added 450 and 90 to get her answer of 540. She explained her strategy: "because 10 times 45 accounts for the tens place. So, it accounts for that whole ten, so ten groups of 45. And then I still need to fulfill the two groups of 45. So, that's why I did 2 times 45." For her first strategy, Valerie used a valid partial products approach that involved decomposing one of the two factors. She justified her approach by reasoning about multiplication in terms of repeated addition. Specifically, she thought in terms of 45 x 12 in terms of twelve groups of 45.

As an alternative strategy, Valerie used doubling and halving. She explained:

> *Valerie*: Well, I know that 2 times 45 is 90. So, that means that I could do 90 times 6. And 9 times 6 is 54, so that means that 90 times 6 would be 540.

> *Interviewer*: Okay. Um, so you did 2 times 45 is 90, and then 90 times 6 is 540. So, how come you can do it that way?
>
> *Valerie*: Because I'm doubling what, how many groups there are, which means that I'd have to compensate by taking half of how much it costs, because there's more groups, so I'm like splitting it.
>
> *Interviewer*: Oh, okay [writing notes]. Say that again. You're doubling what?
>
> *Valerie*: So, it'd be kind of the same thing, like, I have like, for every 45 oboes, there's gonna be 12 dollars. So, if I doubled the amount of how many oboes, I would have to like decrease the price by half to keep the proportion the same. (Valerie, personal communication, November 11, 2010)

Valerie took advantage of the affordances of 45 and 12 by doubling 45 and halving 12. By doubling 45, she got 90, which is an easy number to work with. Specifically, it is a decade number that affords annexing zeroes, which she used as an ancillary strategy. The number 12 is even, and so it afforded halving. Valerie reduced the problem to 9 x 6, which was a known multiplication fact for her. She was able to justify her doubling and halving strategy. She also used annexing zeroes appropriately. She went on to describe it in terms of multiplying 6 by 90, rather than just 9, so that the answer was 540, not 54, because she was "taking groups of ten."

As a third strategy for computing 45 x 12, Valerie used Subtractive Distribution. She rounded 45 to 50, computed 12 x 50, and then compensating correctly for her rounding move:

> *Interviewer*: What about a different way of doing this one?
>
> *Valerie*: Okay, well, let's see. Fiftyyy. Yeah, this makes sense. So, I'm gonna do it like in terms of 50 cents. So, if I made 45 into 50, so I would have 12 times 50, which if I did that in terms of like cents, it'd be 6 dollars
>
> *Interviewer*: Okay.

> *Valerie*: because half of 12 is 6, so it would be 6 dollars. So, it's 600. Um, so, because I made 45 into 50, that's 5 away. So, that means I added on 12 groups of 5. So, I would have to subtract those 12 groups of 5 to get the answer, and 12 times 5 is 60. So, 600 minus 60 is 540. (Valerie, personal communication, November 11, 2010)

Valerie's primary strategy in this case was Subtractive Distribution. She had also used this strategy to compute 99 x 15 in the basic Bobo tasks. She used it here in a less specialized case. The proximity of 45 to 50 afforded the strategy, but 45 is not as close to 50 as 99 is to 100. In selecting numbers for these tasks, I considered 45 to be in the possible benchmark category, whereas 99 is in the obvious benchmark category.

In the above response, Valerie also used doubling and halving as an ancillary strategy. Thinking in terms of money, she reasoned that 12 times 50 cents would be 6 dollars because "half of 12 is 6." My educated guess would be that she thought: 50 cents times 2 is $1, and then $1 times 6 is $6 (similar to her previous strategy). Another possibility is that she thought: 50 cents is half of a dollar, so 12 times 50 cents is half of $12. Either way, she again used the associative property with a factor of 2. The fact that she thought in terms of dollars and cents also lends credence to considering Quarters to be legimitately in her scaffolded strategy range.

As her final strategy[28] for computing this product, Valerie computed all four partial products explicitly. Specifically, she first split the product into three partial products in an appropriate way. Then she clarified that she could split it further into four:

Interviewer: Can you think of a different way of doing this one?

---

[28] According to the interview protocol, since Valerie had described four strategies for solving 45 x 12, I should have asked for one more. I did not, and I do not recall a reason why. Perhaps I thought Valerie had already described five strategies for computing that product. It is possible that she could have come up with one more strategy if asked to do so.

> *Valerie*: [pause] Yeah. So, if I did 40 times 10, is 400, and then [pause] that means that [whispers 'Oh, this is gonna be complicated'] Okay, so 40 times 10 is 400. And then you would have to do 2 times 40 is 80. And then [pause] and then 12 times 5 is 60. So, that's where, and then if you add all those together, it's 540.
>
> *Interviewer*: Okay. So, you did 40 times 10, 2 times 40, and 12 times 5. Is that right?
>
> *Valerie*: Right.
>
> *Interviewer*: and I kind of want to ask about this again: How do you know that that's accounting for everything?
>
> *Valerie*: Because when I do [Valerie positions her hands on the slip of paper] 40 times 10, that's accounting for both of the tens places [points to digits 4 and 1 on the paper] like 40 times 10. And then if I did—I would have to multiply this ones place [pointing to the 2 in 12] by the entirety of this [runs her finger back and forth over the 45]. So, this ones place times 40, that's 80 [points to the 2 and then the 4]. And then if I switch it [pulls her hand to the side of the paper touching all of her fingertips together], I would have to account for this [points to the 5 in 45] with the entirety of this [runs her finger back and forth across the 12]. And then you could break the—you could break this up into 10 times 5, which is 50, and then 2 times 5, which is 10, and it would still be 60. I just kinda like cut the corner on that one 'cause I knew what the answer was. So, that accounts for every single place value. (Valerie, personal communication, November 11, 2010)

Valerie correctly accounted for partial products throughout her explanation. Initially, she had computed the products as 40 x 10 + 2 x 40 = 80 + 12 x 5 = 60. Then she pointed out that she could decompose 12 x 5 further into 10 x 5 + 2 x 5.

As Valerie carefully accounted for all partial products, the way that she used her fingers was interesting. (See Figure 84.) She took advantage of the fact that the numerals were printed on the paper in front of her. Although these appeared within a sentence, they happened to be more or less aligned vertically. When referring to one-digit-by-two-digit partial products, Valerie positioned her finger on a single digit in one factor and then ran her finger back and forth across the other factor. She did this first to indicate 2 x 45,

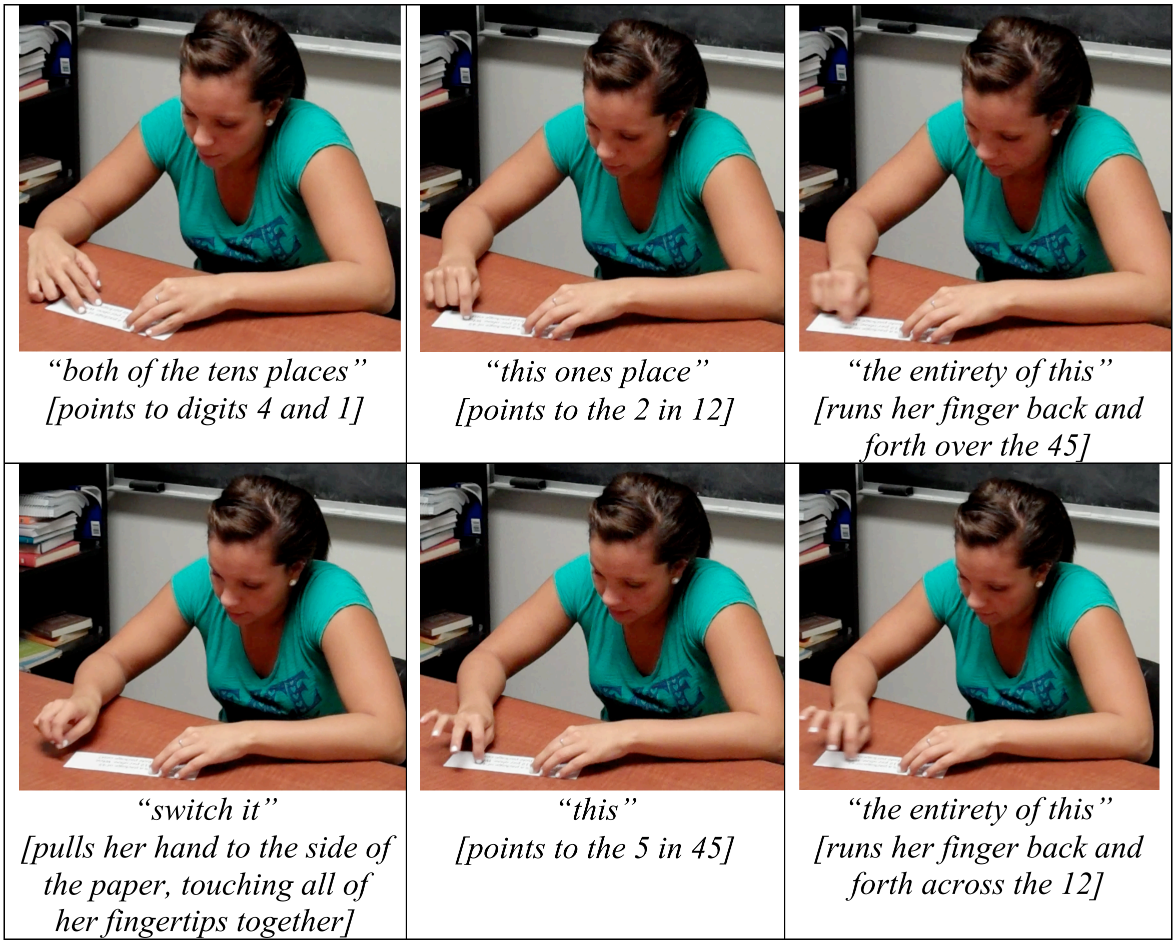
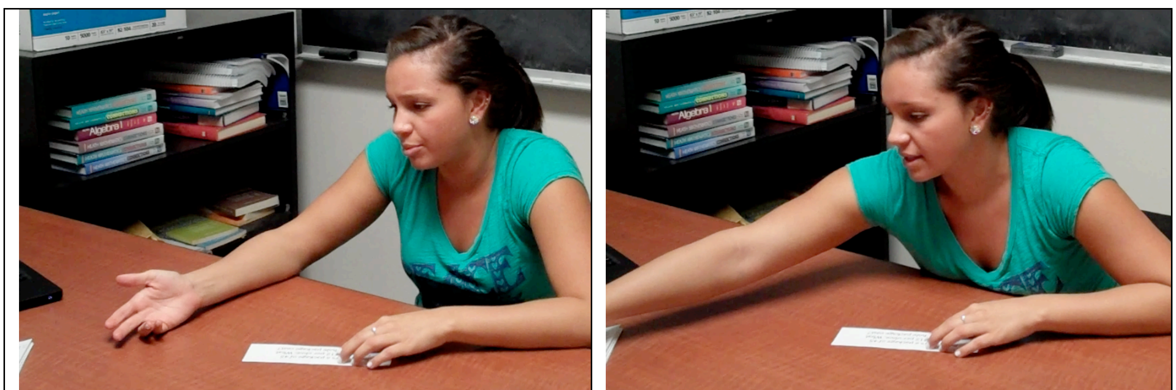


Figure 84. Valerie's finger accounting.

taking the 2 in 12 as the single-digit multiplier. Then she pulled her hand away from the paper as she said, "switch it," and proceeded to refer to 5 x 12, taking the 5 from 45 as the single-digit multiplier. Thus, "switch it" coincided with switching the roles of multiplier and multiplicand.

Although Valerie's finger accounting took place in response to my request for justification ("How do you know that that's accounting for everything?"), she did not seem to use this gesturing in a way that was intended for me to follow. She was not presenting this digit-pairing approach to me as a form of justification. (She was looking down at the paper, which was oriented for her to read, not for me.) Rather, holding her fingers on digits and acting out the correspondences seemed to help her to organize her thinking. This activity seemed to lead Valerie to the realization that she could split up 12 x 5 into 10 x 5 and 2 x 5, thereby explicitly accounting for all four partial products.

Valerie's answer did not change when she decomposed 12 x 5 into the canonical partial products. There was nothing whatsoever incorrect about the explanation that she had given. Nevertheless, she seemed to feel it important to point out that she could decompose 12 x 5 into 10 x 5 and 2 x 5, so much so that she did something very unusual in the social context of the interview. She reached all the way across the table to point to a specific detail in my notes (Figure 85). Rather than just saying that she could break up the 12, she said, "you could break this up" while pointing to a 12 that I had written down.



"you could break the"	*"you could break this up"*

Figure 85. Valerie reaches across the table.

I learned a lot about Valerie's multiplication reasoning from her responses to the Scaffolded Alternatives tasks. I regard this reasoning as occurring in the ZPD. Valerie had solved four different multiplication problems prior to 45 x 12, and this had given her the opportunity to reason about partial products prior to this task. She first solved 45 x 12 by using a strategy with which she was comfortable. She had used the same primary strategy to solve 16 x 25 by reasoning about it as 10 x 25 + 6 x 25 (although there were differences in the details). Then she had an answer, 540.

When asked for an alternative strategy, Valerie quickly recognized that she could also solve this problem by doubling and halving, which was a strategy she had not used in the basic Bobo tasks but had used sensibly in homework responses. She used Subtractive Distribution to compute 45 x 12 by taking advantage of the proximity of 45 to 50. Finally, she accounted explicitly for all four partial products. She did this by computing the tens-by-tens partial product. Then she accounted for the contributions of the ones by applying reasoning about multiplication in terms of repeated addition from two different perspectives, switching the roles of multiplier and multiplicand to see the partial products 2 times 45 *and* 5 times 12. Finally, Valerie took her accounting for partial products all the way and dealt with the ones-times-ones product separately. She had computed 2 x 40 (ones-by-tens), not 2 x 45. So, she had correctly used the 2 x 5 product just once. She decomposed 12 x 5 into 10 x 5 and 2 x 5, accounting for the tens-by-ones and the ones-by-ones.

Valerie's scaffolded strategy range for double-digit multiplication included the MASA, Valid Partial Products, Subtractive Distribution, Halving and Doubling, and Quarters. Actually, given the triumph represented by Valerie's last strategy above, her case brings to light a grain size issue concerning Valid Partial Products. It was far more challenging for her to account for all four partial products separately than it was to correctly treat a product as consisting of two partial products. A finger-grained scheme would distinguish these strategies. In any event, the pre/post contrast between Valerie's scaffolded strategy ranges is compelling. She moved from having only the MASA and

Invalid Partial Products to being able to use five distinct, valid strategies (at the grain size of my coding scheme).

Valerie's reasoning about the standard algorithm. Valerie's first-interview arguments concerning the standard multiplication algorithm were procedural in nature. In the second interview, she explained the same details of the algorithm, but she did so by reasoning about digits in terms of their place values. Figure 86 shows Valerie's primary written work. The transcript follows:

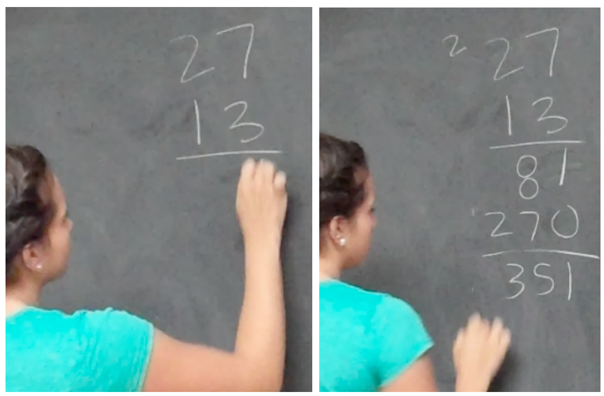


Figure 86. Valerie's algorithmic work.

Interviewer: I'm gonna ask you to do a multiplication problem now: 27 times 13.

[Valerie performs the algorithm quickly and correctly, obtaining the answer of 351]

Interviewer: Alright. Could you walk me through what you did?

Valerie: Yeah. So, 7 times 3 is 21. So, you keep the, uh [writes over the 1 in the first row of work]—so, you have twenty-*one* [writes 21 off to the side] which is two tens and one one. So, you leave the 1 here, and you take the group of two tens up here [pointing to the carried 2]. So, you do 3 times 2, which is 6, and then you add the 2. So, 6, 7, 8. So, it's 8 [writes over the 8 in the first row of work]. And then you need to put a 0 right here [writes over the 0 in the second row of work] to indicate that you're no longer in the ones place. You've moved to the tens place. So, there's that 0. So, 7 times 1 is 7, and 1 times 2 is 2. And then you add these together and get 351.

Interviewer: Okay. How come you add the 2 that you carried?

Valerie: [pause] Okay. So, this is the ones. You're multiplying the ones place and the tens place [points at the 3 in 13 and then the 2 in 27]. So, it's like saying 20—so, if you had this be a 0 [pointing at the 7 in 27]—so, it's like 20 times 3. You're doing like the whole value [underlines the 2 and 7] by 3. So, because you're adding essentially, you're adding 7 groups of 3 to your 20 times 3, you need to—it'd be like saying, okay, well, 20 times 3 is 60, but you have to add—account—for these groups of 3, which would be 21. So, that's why you'd have to add the 20 because you're not just accounting for a 2; you're accounting for the group of 21. So, you have to—you already dealt with the 1, so you have to add a group of 20 [circles the carried 2 with her finger]. So, by adding the 2, it's increasing this number [points to the 2 in 27], which is increasing the tens place; it's not just increasing the ones place.

Interviewer: Okay. And what about the 0 that you put down on the second row?

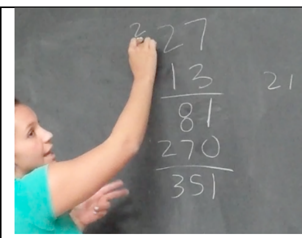
Valerie: So, that's there because—so, if I just did like 7 times 3 [writes standard algorithm setup for 7 x 3 off to the side] or even 7 times whatever, what is it? 27 [writes 2 next to 7, making her example 27 x 3]. Okay, so, I'm only dealing with the ones place right here [circles the 3]. So, that would a 1 [writes 1 in ones place of answer] and I'd carry the 2, and it'd be 81 [writes 8 in tens place of answer]. There's no need for me to worry about the tens place. But because there's a 1 here [points to the 1 in 13 in the original problem] that means that I'm multiplying this 27 by 10 [circles the 13 with her finger] rather than just 1. So, because I'm multiplying it by 10, I have to make it—make the ones say, okay, there's no ones because it's has enough to make a group of 10. (Valerie, personal communication, November 11, 2010)

Valerie was asked to do a multiplication problem: 27 times 13. As in her first-interview response, Valerie made three arguments of interest. The claims are the same as in the first interview. :

1. You have to carry the 2
2. You have to add the carried 2
3. You have to write a 0 in the ones place of the second row

However, Valerie justified these claims differently than she had in the first interview. I detail each of her arguments below.

Figure 87 represents Valerie's argument that you have to carry the 2 from 7 x 3 = 21. Essentially, Valerie argued that the partial product 21 consisted of tens and ones, and the ones belonged in the ones place and the tens belonged in the tens place. This argument contrasts with Valerie's first-interview argument, in which she talked about two-digit numbers not being able to fit in the ones place and said that the 2 was the "overflow."

<table>
<tr><td>Claim: You have to carry the 2</td><td rowspan="3">

"the group of two tens"</td></tr>
<tr><td>Data: 7 x 3 = 21, which consists of two tens and one one
(So, 7 times 3 is 21… so, you have twenty-one, which is two tens and one one.)</td></tr>
<tr><td>Warrant: (Implicit) You write the ones in the ones place and the tens in the tens place
(So, you leave the 1 here, and you take the group of two tens up here [pointing to the carried 2].)</td></tr>
</table>

Figure 87. Valerie's argument that you have to carry the 2.

Figure 88 represents Valerie's argument that you have to add the 2 that was carried. Valerie argued that the 2 represented a number of tens. Specifically, it represented 20. It was carried into the tens place and then added to the partial product 60, which was also a number of tens. The addition step of 6 + 2 = 8 actually represented 60 + 20. By contrast, in the first interview, Valerie had argued that the 2 needed to be added because it had been carried, and carried numbers should be added, as in addition.

Figure 89 represents Valerie's argument that you have to write a 0 in the ones place when you move down to the second row. Valerie argued that this was because you were multiplying by ten, and therefore the partial product would be at least ten. This argument was less clear than the others. Valerie did not explicitly state that the result of

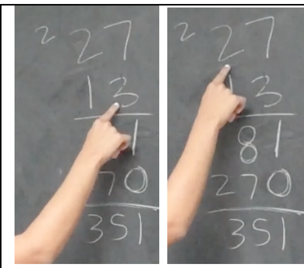
Claim: You have to add the carried 2	
Data: You're really adding 20 to 60 ("it'd be like saying, okay, well, 20 times 3 is 60, but you have to add—account—for these groups of 3, which would be 21. So, that's why you'd have to add the 20")	
Warrant: Reasoning about digits in multiplication in terms of their place values ("So, by adding the 2, it's increasing this number [points to the 2 in 27], which is increasing the tens place; it's not just increasing the ones place.")	*"You're multiplying the ones place and the tens place"*

Figure 88. Valerie's argument that you have to add carried the 2.

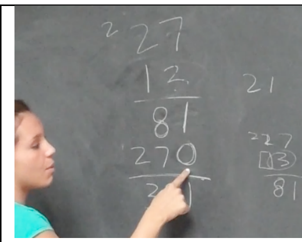
Claim: You have to write a 0 in the ones place of the second row	
Data: Multiplying by ten (I'm multiplying this 27 by 10 [circles the 13 with her finger] rather than just 1.)	
Warrant: The partial product will be at least 10 (So, because I'm multiplying it by 10, I have to make it—make the ones say, okay, there's no ones because it's has enough to make a group of 10.)	*"okay, there's no ones"*

Figure 89. Valerie's argument that you have to write 0 in the ones place of the second row.

multiplying by ten would be *a number of* tens. She only said that it would have enough ones to make ten. This seems like a case of a less than ideal choice of words on Valerie's part. In any event, the argument as stated contrasts with her first-interview argument, which appealed to a procedure for moving the decimal point when multiplying by ten.

Each of Valerie's second-interview arguments concerning the details of the standard multiplication algorithm contrasted with the analogous arguments from her first interview. These contrasts were all of a similar nature. In the first interview, Valerie reasoned in terms of digits, columns, and rules. In the second interview, she reasoned in terms of place value, i.e., the amounts that those digits represented.

Valerie's reasoning about invalid partial products. In the second interview, following the task concerning the standard algorithm, Valerie was posed the Jessica Task:

> My friend Jessica wanted to do 37 times 58 mentally. And, to make it easier, she said that she would do 30 times 50 and 7 times 8 and add them together. Does Jessica's strategy make sense? Why or why not?

Valerie started to respond to this question by comparing answers. She found that Jessica would get an answer of 1500 + 56 = 1556. She was going to perform the standard algorithm, but I stopped her and asked whether she could answer the question without checking Jessica's answer. Valerie immediately responded that Jessica's strategy was "wrong." She wrote the setup for the standard algorithm and said:

> She basically just separated it in half like this [draws a line segment between the tens and ones place]. So, she said, 'Okay, well, I'll make this thirty and this fifty, and that's 1500. Then if I just add these [sic] or multiply these [pointing to 7 and 8] and add them, I'll have the answer. But that's not right. (Valerie, personal communication, November 11, 2010)

In further discussion, Valerie seemed to interpret Jessica's strategy as a compensation strategy: Jessica rounded 37 and 58 down to 30 and 50. She found the product of those, and then she compensated for rounding by adding in the product of 7 and 8. Valerie explained that this was not right because the 7 and 8 actually represented "groups." Initially, Valerie represented these groups incorrectly as "7 groups 30" and "8 groups 50" (written on the board). She knew that the 7 and 8 represented groups, but in this context she initially seemed to have difficulty figuring out what they were groups of. She asked herself, "Would it be 7 groups of 30 or would it be 7 groups of 58?"

Valerie progressed in her thinking about these groups. She said, "Well, maybe if I talk about it'll make more sense." She erased what she had written and proceeded to think in terms of partial products. She said, "To do this [pointing to the standard algorithm setup] you need to multiply—so, she made it 30 and 50, but that means I'd still need to do 8 times 37 and 5 times [sic] um, and 7 times 58." Valerie tapped the digits and made jumps from one to another as she reasoned about pairing them up in partial products. After doing this, she had answered her own question. She concluded, "It would be 7 groups of 58 and 8 groups of 37, is what she still needs to add to it."

Valerie wrote 37 x 8 + 58 x 7 on the board. Taken separately, both of these partial products were sensible. However, taken together, the 7 x 8 product was being double-counted, and I wondered whether Valerie was attuned to this point. I asked how the 37 x 8 and 58 x 7 related to the 7 times 8 that Jessica had done. I said, "She did 30 times 50 and 7 times 8. Does she also need to do those two that you wrote down?" Valerie responded, "If she did these [pointing to 37 x 8 + 58 x 7] this [pointing to 7 x 8] wouldn't matter." She continued thinking and said:

> So, I mean, essentially, she could use it. So, it would have to be [writing on board] like 30 times 8, 50 times 7, and then it would have to be 8 times 7. So, like, if you break it down. So, this [points to the 3 and 8 in 37 x 8] would be 30 times 8. This [points to 7 and 5 in 58 x 7] would be 7 times 50. And you'd still have to do 7 times 8, or 8 times 7, to get the 7 groups of 8 that you took away. (Valerie, personal communication, November 11, 2010)

Figure 90 represents Valerie's written work on the Jessica Task. In Valerie's case, the Jessica Task was not merely a hypothetical situation. This had been Valerie's go-to strategy for multiplication in the first interview and up until Day 14 of class. I took the Jessica task as an opportunity to ask Valerie about that directly. I said, "So, I remember that in your first interview, you used this strategy a few times. You multiplied the tens by

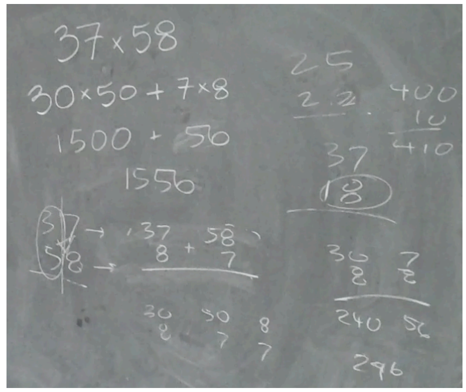


Figure 90. The final record of Valerie's thinking about the Jessica Task.

the tens and the ones by the ones…" Valerie responded, "Yeah, well, I thought it was right, but it's not." I asked her to say more about that.

Valerie used the example of 37 times 8, She said,

> Okay, say that was what you were asking me to solve [writes 37 x 8 on board]. I would just go about it, I'd say, okay, I can just do 30 times 8 and then 7 times 8, and that'll give me the answer, which it will. Because that's 240, 56, so the answer is 296. That works. That's what I was going for last time. Because in this problem, you're only asking me to account for the ones place. So, you're saying you need 37 groups of 8. Like how many do you have? So, that's fine. I can do 30 times 8 is 240, and then I can do these places [points to 7 and 8] and then I just add 'em together. That works. But if you add another place right here [writes 1 in front of the 8, making it 18] then I have to be, okay, I need 10 groups of 37 plus 8 groups of 37. So, because it's like this [steps to the side and points back at the original problem of 37 x 58] I can't just say that, okay, I'll make it 30 and 50 because that takes care of the tens place, and then this [taps the 7 x 8] takes care of the ones place. Because it doesn't account for all the groups. (Valerie, personal communication, November 11, 2010)

Here, by building up from the one-by-two digit case to the two-by-two digit case, Valerie very clearly explained the appropriate partial products as "groups of" and justified why Jessica's strategy was invalid. When I asked Valerie how her thinking about this had changed,

Interviewer: So, what changed? How did you figure this out?

> *Valerie*: [laughs] Well, 'cause in class, I did the wrong thing and got the wrong answer. (Valerie, personal communication, November 11, 2010)

Valerie did not remember the particular instance from class, but she remembered that she had used this strategy and gotten the wrong answer.

> *Interviewer*: So, you did this in class, and you got it wrong, and you remember getting it wrong. Is there a difference in something that you understand?
>
> *Valerie*: Yeah. I think before, I was just like, because we had learned like this [pointing to her 37 x 8 example] like, 'oh, break it up,' I was like, I didn't think about [pointing to 37 x 18] 'oh, well, now there's a different place value.' I didn't think about that. I just thought about, 'oh, well, I can just break it up and add it together. It's easy.' But it doesn't work that way because once you move to a different, um, place value, instead of, if you're—because essentially if you're in the ones place, you think about it like everything is multiplied by one. So, it doesn't change, like, it's value; it just stays the same. So, that means, I can, you know, do [taps back and forth from 8 to the 3 in 37 and from 8 to the 7 in 37] just break it up, and it's the same. But once you move into the tens place, I have to say, 'well, the value of one is no longer one. It's a value of ten. So, every time I think that I'm just multiplying by one, I'm really multiplying by ten. So, that's why there's more groups. (Valerie, personal communication, November 11, 2010)

I understand Valerie's explanation in terms of relating place value, partial products, and groups of. The fact that a digit in the tens place represented a number of tens, not ones, was not in itself a new idea for Valerie. However, in the above explanation, she related this idea to the number of groups of the multiplicand that were represented by a product. In other words, when she talked about the 1 in 18 x 37 as representing ten, I do not think she meant that in the sense that, when she multiplied, she would have to put a zero at the end of a numeral. Rather, she was articulating an interpretation of that ten as representing ten *groups* of 37. She was explaining that the product consisted of 10 groups of 37 and 8 groups of 37, and that this was "more groups" than the invalid strategy could account for.

Valerie's reasoning about weight. The SST interview provided the opportunity to explore Valerie's multiplication reasoning further. I posed a task that involved comparing estimates and that was similar to tasks discussed in class on Days 16 and 17:

> *Interviewer*: Let's say that you want to estimate the product of 21 and 73.
>
> *[Valerie writes standard algorithm setup for 21 x 73 on board]*
>
> *Interviewer*: The question is: which would give the closer estimate
>
> *Valerie*: [whispers] Oh, no!
>
> *Interviewer*: [laughs] 20 times 73 or 21 times 70?
>
> *[Valerie writes down standard algorithm setups for 20 x 73 and 21 x 70]*
>
> *Interviewer*: Which would give the closer estimate?
>
> *Valerie*: Okay, let's see if I remember how to draw this
>
> *[Valerie draws a rectangle with dimensions labeled 20 and 73. She then adds a skinny rectangle to it, by increasing the side labeled 20 by 1 unit]*
>
> *Valerie*: [mumbling] so, that's 1 times 73 [writes 1 x 73 below her drawing and then writes 73 inside the skinny rectangle]
>
> *Valerie*: and then
>
> *[Valerie draws another rectangle, which she partitions into two smaller rectangles. She labels the unpartitioned side 21 and the two parts of the partitioned side 70 and 3. She writes 63 inside the smaller partial rectangle.]*
>
> *Valerie*: so this would give you [writes "closer estimate" next to the drawing corresponding to 21 x 70]
>
> *Interviewer*: Okay. Can you explain what you drew?
>
> *Valerie*: This took me a really long time to get, by the way [laughs]
>
> *Interviewer*: [laughs]
>
> *Valerie*: In class, I didn't understand how to do it. Um, okay, so this is what you have [circles 21 x 73] and a lot of times you think, okay, I can just estimate, like when you add. So, you're saying, okay, I estimated, say,

> 20 to 73. That's easier 'cause there's a zero, so it won't take as long. And this one's the same thing [pointing at 21 x 70] I could estimate it, and there'd still be a zero. So, you need to look at it in terms of this [points to drawings]. You draw, if you draw a picture, and say you have 20 and 73 [underlines dimension labels on her first drawing]. So, the original length of this would be 21, but you're taking away one unit. So, it's gonna be a unit that's 1 by 73 'cause this hasn't changed [points to 73 label on top of rectangle]. So, you do 1 times 73, which is 73. And then this one [points to her second drawing] you're taking, going from 73 to 70. So, you're taking off 3, which still has a height, or width or whatever, of 21. So, you need to do 21 times 3, which is 63. So, that's less. [steps to the side] So, if you do this [points to 21 x 70] it's gonna have, it's gonna be 63 away from the actual answer. So, it's gonna be closer [points to partial rectangles in each drawing] because there's a smaller gap. (Valerie, personal communication, December 1, 2010)

In the above response, Valerie's use of rectangular area (Figure 91) seemed to help her organize her thinking about partial products in the context of comparing weights to determine the closer estimate. I used probes and follow-up questions to find out more about the difficulties that she had alluded to with understanding "this" as well as her understanding of the relationship between multiplication and rectangular area and the role of her rectangular area drawings in her reasoning.

> *Interviewer*: Okay. You said that it took you a long time to get this in class. Could you say more about that? Like what was difficult about it?
>
> *Valerie*: I just didn't understand it 'cause I, like I—I don't know, like I'd never thought about it. 'Cause whenever I was asked to estimate, it was never 'which one is closer?' It was always just 'estimate the answer.' So, I mean, I would just do 20 times 70, and get however much that is, right now, I think it's 1400. Like that's what I was always asked to do. I was never asked which one's closer. So, when they asked, 'well, which ones closer,' I always just thought, well, this one's obviously closer [points to 20 x 73] because 73, and then you're adding like a smaller one [points at 20]. So, this one's obviously bigger because it looks bigger, you know?
>
> *Interviewer*: Because 3 is more than 1?
>
> *Valerie*: Yeah, 73 looks bigger, so it looks like there's gonna be a bigger answer.

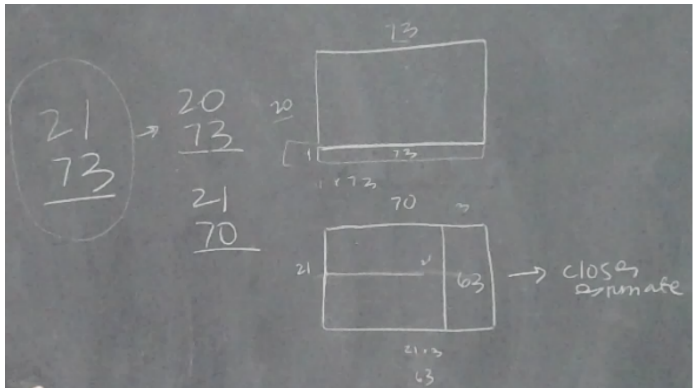


Figure 91. Valerie's work to compare estimates of 21 x 73.

Interviewer: Okay.

Valerie: when it turns out that it's not. And then when she introduced the drawing, I did not understand the drawings 'cause I didn't understand this, so how was I gonna understand this and understand the drawings? It just didn't make sense.

Interviewer: [laughs]

Valerie: So, but now that I like know how to do it, it makes sense because [leans into first drawing] if you're taking away one unit, you're taking away 1 unit by 73, and then [points to second drawing] 3 units by 21. So, it would make sense to multiply them together.

Interviewer: Okay. So, what does it mean when you write 73 inside of that skinny rectangle there?

Valerie: It just means that you're taking away 73 pieces of this whole.

Interviewer: So, you wrote 1 times 73 underneath. Could you just explain that to me, like why is it 1 times 73?

Valerie: [draws a new rectangle off to the side] Because you're finding the area. (Valerie, personal communication, December 1, 2010)

Valerie gave the example of finding the area of a 5-foot-by-5-foot lawn (Figure 92). She divided the rectangle into rows and columns and explained that you multiply 5 and 5 to get the number of "pieces" because there were 5 groups of 5: "If I have 5 groups of 5 pieces, that means that I'm gonna have to multiply the number pieces in a group by

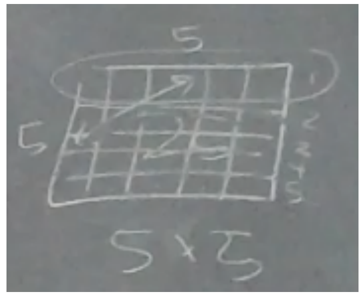


Figure 92. Valerie's drawing to explain the connection between multiplication and area.

the number of groups to get how many single pieces there are" (Valerie, personal communication, December 1, 2010).

> *Interviewer*: Okay. So, for this estimation question, you went right away to making these drawings. Um, is that part of your thinking about this question or is it just like something you're supposed to do 'cause it's what you did in class?
>
> *Valerie*: No, 'cause like, if you just told me this [circles 21 x 70 and 20 x 73] I couldn't figure out what the answer was. I mean, for me, personally. Like if I draw it out, it makes more sense. I mean, I'm sure that I could look at this and say, 'okay, well, I took away one group of 73 [circles the 0 in 20 x 73] and in this one I took away three groups of 21 [circles the 0 in 21 x 70]. I mean, it's the same thing. But if I draw it out, I know I'm right. (Valerie, personal communication, December 1, 2010)

Valerie's responses indicated to me that the initial difficulty she had mentioned concerning closer-estimate questions had to do with the novelty of the idea of accounting for the error in estimates (as opposed to the more familiar task of making rough estimates and not considering how far off they might be). I was also convinced by her responses that she had a sensible understanding of the relationship between multiplication and rectangular area. I believe that Valerie used rectangular area drawings in her work on the estimation task because the form of the task was familiar and she associated such drawings with this type of task. I am also convinced that these drawings made sense to her and helped her to organize her thinking in accounting for the removed amounts by reasoning about partial products in term of partial rectangles.

Valerie's reflections on her experience in the course. At the beginning of the SST interview, I asked Valerie about her previous experiences with math in general and mental math in particular. I then asked about the content course and how her experience in that class compared to her previous mathematical experiences. When asked to talk about her mathematical experiences, Valerie said bluntly, "I don't like math." She described difficulty and confusion in her geometry and precalculus courses in high school. Of her experience with mental math prior to the course, she said, "I'd use it all the time, I guess, if I ever have to do something that's easy. I always used to do it with change when I was little" (Valerie, personal communication, December 1, 2010).

Valerie described how the content course compared to her high school math classes:

> Well, I mean, in this class I'm learn how—it's more of like, normally you don't think about like adding fractions, or you don't think about like how you get to an answer because you just punch it into a calculator. 'Cause like my calculator was my best friend in high school, like I used it all the time. So, I mean, you don't really think about like how do you multiply two-digit numbers, or how do you know like which parts are being added or how you can add them. You don't think about that because you just add it and the calculator does it for you. (Valerie, personal communication, December 1, 2010)

She mentioned multiplication again when talking about what the course was like and that it was challenging:

> I like it because it's not just like a conventional math class, where in calculus or statistics, like people are taking now, that's pretty much the same stuff we went over in high school, just continued. This kind of gives you a new, like, outlook on math, I guess. And it makes it that you think you knew everything there was to do in math, but you can't figure out how

> to multiply two-digit numbers or, like, use different strategies. (Valerie, personal communication, December 1, 2010)

Clearly, double-digit multiplication was a topic that stood out to Valerie when she thought about her experience in the course.

I also asked Valerie whether her use of mental math outside of class hand changed:

> *Interviewer*: So, you said that you did quite a bit of mental math before, since you were a kid. Would you say that your thinking or your habits in mental math have changed over the course of the semester?
>
> *Valerie*: Um [smiles] I, like, find myself thinking about which strategies I use now [laughs]. Like, my friend and I, whenever we like figure something out, or if we ask like a question, like 'oh, there's ten of them' or whatever, whatever. We'll be like, well, what did you use to solve it? Like in all seriousness, we'll be like, well, what did you do? (Valerie, personal communication, December 1, 2010)

She gave an example of adding up the number of remaining meals on her meal plan and explaining her strategy to a friend. As a final reflection, Valerie said, "I guess I still do the same amount of mental math that I did before. It's just that now I realize that I'm doing it, and I realize how I got the answer, and how like it makes sense" (Valerie, personal communication, December 1, 2010).

Change in Valerie's NSRS responses. Valerie scored 32 of 37 possible points (86%) on the NSRS posttest, compared to 29 of 32 points (78%) on the pretest. This was modest improvement overall, and it was on par with the mean gain of 2.92 for students in the High-scoring pretest group. Surprisingly, though, Valerie's responses changed from incorrect to correct on four items related to multiplication.[29] Below, I describe each of

[29] This is surprising since Valerie's score only increased by three points. Yet she gained four points on these four multiplication items. It follows that at least one of Valerie's other responses changed from correct to incorrect. My analysis focused only on Valerie's multiplication reasoning.

these items and Valerie’s pre/post responses. Since the NSRS is a multiple-choice measure with no written work, these responses do not afford access to how Valerie reasoned in selecting them. I offer some conjectures that relate these items to Valerie’s reasoning in her homework and interview responses. However, these are only conjectures. I do not claim to know how Valerie arrived at any of her NSRS responses. I include the data here because there was a conspicuous improvement in Valerie’s answers to multiplication items. This provides corroborating evidence that Valerie’s reasoning about multiplication changed over the course of the semester.

Item #9 asked the following: “93 x 134 = 12462. How much greater than 12462 is the product of 93 and 135?” On the pretest, Valerie selected (d) *I can’t tell without calculating*, which counted as incorrect. On the posttest, she correctly selected (a) 93. This item relates directly to partial products. It seems plausible that Valerie was able to correctly answer it in the posttest by reasoning about multiplication in terms of repeated addition, i.e., thinking of 93 x 134 as 134 groups of 93, and comparing that to 135 groups of 93. That would have been consistent with reasoning that she displayed in the second and third interviews.

Item #17 asked students to select the greatest product from amongst the options (a) 18 x 17, (b) 16 x 18, (c) 17 x 19, and (d) 19 x 15. On the pretest, Valerie incorrectly answered (a). On the posttest, she correctly answered (c). It is difficult to say how Valerie might have arrived at her pretest answer of 18 x 17. In the posttest, she might have compared 18 x 17 with 17 x 19 by reasoning about multiplication in terms of repeated addition (i.e., 19 groups of 17 is more than 18 groups of 17).

Item #3 asked students to select the answer that was “the same as 0.5 x 840.” On the pretest, Valerie incorrectly answered (c) 840 ÷ 5. On the posttest, she correctly answered (a) 840 ÷ 2. This item related multiplication by 0.5 to division by 2. The improvement in Valerie’s answer may have been related to her understanding of doubling and halving. It may also have been related to her understanding of decimal numbers.

Item #15 asked, “How many digits are in the product of a 2-digit number and a 2-digit number?” On the pretest, Valerie incorrect answered (a) *must be three digits*. On the posttest, she correctly answered (c) *can be three digits or four digits*. It is difficult to say how exactly Valerie’s reasoning about this item might have changed. Based on the pre/post contrast in her reasoning about the standard algorithm, I would conjecture that the changed involved thinking in terms of place value, rather than in terms of the rule-based behavior of digits.

Valerie’s answer to Item #23 did not change from correct tocounted as incorrect both on the pre and the post. However, her specific response did change, so I include the data here for completeness. Item #23 asked, “Compare 521 x 5 and 520 + 521 + 522 + 523 + 524. Which is greater?” On the pretest, Valerie incorrectly answered (b) *the result of 520 + 521 + 522 + 523 + 524 is greater*. On the posttest, she left the item blank, which counted as incorrect.

The changes in Valerie’s NSRS responses to multiplication items provide corroborating evidence to the analysis presented here. These results indicate that Valerie’s reasoning about multiplication changed in significant ways during the semester.

Summary of Valerie's reasoning in the second and third interviews. After the end of the whole-number portion of the course, Valerie was interviewed twice about whole-number ideas and about her experience in the course. In these interviews, Valerie reasoned about multiplication more meaningfully and flexibly than in the first interview. She used a variety of valid mental multiplication strategies, and she could meaningfully justify why her old go-to strategy was invalid. Her justifications for the details of the standard algorithm changed from being procedural and digit-focused to involving reasoning about the amounts represented by those digits.

At the same time, Valerie still found reasoning about partial products to be challenging. On several occasions, she struggled to correctly account for partial products. In the Jessica task, Valerie successfully accounted for all partial products by using the pairing of digits in the standard algorithm to help organize her thinking. When she reasoned about weight in the third interview, Valerie used rectangular area in a similar manner, creating drawings that helped her to organize her thinking about partial products and to pair them up appropriately.

Conclusions from the Case of Valerie's Number Sense Development

In this section, I conclude the presentation of this case study. I summarize the development of Valerie's multiplication reasoning. I also discuss the role of particular tools in Valerie's accounting for partial products.

Summary. In her first interview, Valerie was dependent on an invalid mental multiplication strategy.[30] She was confident with this strategy and trusted it more than the

[30] She might have been able to compute one-by-two digit products correctly. I cannot say this for sure because there was no such task in the interview. We know that Valerie's go-to strategy for computing two-by-two digit products was invalid.

MASA. She preferred her strategy due to its simplicity. She found it easy to manage mentally, whereas the MASA seemed too complicated. Valerie could come up with no other mental multiplication strategies. Valerie reasoned about the standard written multiplication algorithm in terms of the rule-based behavior of digits in columns. Her reasoning with her go-to strategy was not connected to her procedural knowledge of the algorithm.

During the semester, Valerie started out reasoning vaguely about nonstandard strategies as involving breaking numbers down, computing, and putting numbers back together. Reasoning at this grain size, it seemed that she was blind to important distinctions between strategies. On Day 14, Valerie used her go-to multiplication strategy in class and learned that it was invalid. From that point on, she was more attuned to distinctions between strategies. In her homework reflection on Day 14, Valerie was able to argue that her old go-to strategy was invalid by reasoning about partial products in terms of part of one factor being multiplied by *all* of the other factor, using the idea of multiplication as repeated addition.

Valerie also encountered strategies that leveraged the associative property of multiplication. When she first saw an example of this, in Aaron's division strategy from Day 4, she did not recognize the associative aspect of it, probably because she simply saw it as another nonstandard strategy that involved breaking numbers down. However, on Day 17, Valerie did make sense of Equal Area Shifting, and in her homework she spontaneously applied the idea of halving and doubling to estimate the product of decimal numbers. Weight was discussed on Day 16 and 17. Valerie gave an unclear homework response that suggested she did not understand how to coordinate partial products correctly.

In her second interview, Valerie reasoned flexibly about computing products, exhibiting the ability to use a variety of valid strategies. Her scaffolded strategy range included halving and doubling, which seemed to be a new strategy for her. Valerie knew that her old strategy was invalid and could justify this fact meaningfully. She could also justify the standard algorithm by reasoning about digits in terms of their place values. Valerie knew that she needed to account for partial products, and she thought carefully about how to do this, but she still found it difficult. Both the standard algorithm and rectangular area served as tool that helped her to organize her thinking.

Valerie reported that she had become more aware of her mental computation activity outside of school and of the strategies that she used. The fact that her old go-to had turned out to be invalid made an impression on Valerie, and she thought more carefully about whether her strategies made sense.

The role of tools and models. Zandieh and Rasmussen (2010) define *models* as "student-generated ways of organizing their activity with observable and mental tools" (p. 58). In this sense, I view rectangular area drawings and the standard multiplication algorithm as observable tools that Valerie used to organize her activity when reasoning about partial products. Specifically, Valerie's rectangular area drawings enabled her to account for the contribution of units digits removed from a product in rounding by relating the given factors to the dimensions of a rectangle and relating the dimensions of the removed part of that rectangle to a partial product. *Similarly*, Valerie's reenactments of the setup and digit-pairing in the standard written multiplication algorithm enabled her to correctly account for partial products in two-by-two-digit multiplication. Without this aid (as in her response to 15 x 24 in the second interview) she was unable to exhaust all pairings and account for the whole product. With the setup of the algorithm as a guide, she was able to successfully *and sensibly* account for partial products by relating them to "groups of."

In designing instruction for this course, the instructor and I had anticipated the importance of rectangular area. Indeed, we have previously written about how we saw it supporting the development of more sophisticated reasoning about products. The role of the standard algorithm, on the other hand, was unanticipated. Valerie came to the content course with this piece of procedural knowledge. She knew the standard multiplication algorithm. When it came to mental multiplication, she preferred her strategy over the MASA. However, in the class, she became aware that her strategy was invalid and she came to understand why. Valerie's previous knowledge of the standard multiplication algorithm was integral to the reasoning that enabled her to understand that her old strategy was invalid *and* enabled her to appropriately account for partial products.

Conclusion

I conclude this chapter by summarizing the results in terms of themes that emerged from the two case studies. I also discuss the issue of comparability of case studies, and I describe specific ways in which the cases of Brandy and Valerie may inform the work of other researchers.

Summary of Insights Gained from the Case Studies

In the paragraphs below, I briefly describe insights into prospective elementary teachers' number sense development that I gained from each case study.

Insights gained from the study of Brandy. At the beginning of the course, Brandy was dependent on the standard addition and subtraction algorithms. It was not the case that Brandy was dependent on these algorithms *despite* not understanding them. Rather, she was dependent on the standard algorithms *because* she did not understand them.

Brandy had knowledge of place value that she could have used to make sense of the standard algorithms. However, when performing and reasoning about those algorithms, she thought in terms of the rule-based behavior of digits in columns, rather than reasoning about digits in terms of their place values.

The reasoning that enabled Brandy to make sense of the standard addition and subtraction algorithms overlapped with the reasoning involved in making sense of Transition strategies. As a result of using her knowledge of place value to understand the standard algorithms and Transition strategies, Brandy came to use the MASAs less often and Transition strategies more often. Essentially, there is a fine line between using Transition strategies and reasoning meaningfully about the standard algorithms.

Compensation was part of Brandy's initial scaffolded strategy range. In the story of her developing reasoning, Compensation strategies not only became part of her strategy ranges proper. Rather, compensation was the lens through which Brandy interpreted many nonstandard strategies. The strategies that occur in prospective elementary teachers' scaffolded strategy ranges reflect particular ways of reasoning. These ways of reasoning may influence how prospective elementary teachers interpret the nonstandard strategies that they encounter.

Brandy latched on to a particular nonstandard strategy from class, Borrow to Build. She seemed to overgeneralize the meaning of this strategy, seeing it in instances that I would categorize as distinct strategies. Yet, even when doing this, she reasoned sensibly and produced valid mathematical arguments. She overgeneralized the application of a strategy name, but she did not overgeneralize mathematical ideas.

Reasoning about subtraction as a take-away process was a commonplace idea that Brandy brought with her to the content course. This idea and others served as productive resources for her learning. For example, Brandy used the idea of subtraction as a take-away process to construct valid Subtrahend Compensation on her own.

My analysis emphasized the purely mathematical ideas in students' arguments. However, Brandy's reasoning about Shifting the Difference suggests that ideas such as making a computation easier (in a particular way) can also be important factors in prospective elementary teachers' understanding and adoption of nonstandard strategies.

Insights gained from the study of Valerie. At the beginning of the course, Valerie was dependent on an invalid strategy. This was possible because she prioritized simplicity when it came to mental computation. Her strategy was easy to manage mentally, whereas the MASA seemed too complicated. Thus, Valerie trusted answers obtained by her strategy. In a case of discrepant answers, she imagined that she must have made an error using the MASA.

Valerie initially reasoned about the standard written multiplication algorithm in terms of the rule-based behavior of digits in columns. Her reasoning with her invalid strategy was not related to her procedural knowledge of the algorithm. Rather, she thought in terms of accounting for all of each factor. In this way, her invalid strategy made sense to her.

Valerie was initially insensitive to distinctions between nonstandard strategies. They all seemed to involve breaking numbers down, computing, and adding things together. As a result, both valid and invalid strategies seemed equally sensible. It was only when Valerie became aware that her go-to strategy was invalid that she began to make more careful distinctions between nonstandard strategies.

Valerie's reasoning about partial products changed, and she became more able to account for them correctly. Reasoning about multiplication in terms of repeated addition was a productive idea in this process. Viewing one factor or the other as the multiplier and appropriately treating double-digit products as consisting of two partial products was more accessible for Valerie than accounting for all four partial products explicitly. She was able to do this by switching repeated-addition perspectives, viewing one factor as the multiplier and then the other. Valerie's ability to account for partial products was also supported by the standard algorithm and rectangular area, which served as tools that helped her to organize her thinking.

How the Case Studies Inform Theory

This chapter has presented two case studies of prospective elementary teachers developing improved number sense. In case study research, it is important to consider how insights gained from individual cases might contribute to an understanding of the phenomenon of interest. In other words, in what can be learned from these analyses?

I would remind the reader that this is fundamentally qualitative research, and the questions of interest are process, rather than variance questions (Maxwell, 2005). The class was selected as a setting for the research because evidence suggested that it was *not* a typical mathematics content course. The prospective elementary teachers enrolled in this course experience improved number sense, and the literature suggests that this kind of change does not typically occur. In cases in which prospective elementary teachers do develop improved number sense, we are seeking to better understand what supported that development.

Within the setting of this special content course, two participants were selected for case studies from among the seven students who voluntarily participated in interviews. Brandy was selected as a case of number sense development with a specific focus on her reasoning about place value, addition, and subtraction. Valerie was selected as a case of number sense development with a focus on multiplication. I do not claim that these two individuals may be regarded as representative of the set of interview participants, let alone the entire class. Rather, the cases were selected on the basis of their potential to illuminate our understanding of certain aspects of the phenomenon of prospective elementary teachers' number sense development.

The cases of Brandy and Valerie contribute to the understanding prospective elementary teachers' number sense development by informing revisions and elaboration to the local instruction theory. This process involves a reflexive relationship between actual learning trajectories and hypothetical learning trajectories, and between hypothetical learning trajectories and the local instruction theory. For example, both case studies led to insights relating to the roles of the standard algorithms in the prospective elementary teachers' number sense development.

We have conceptualized the LIT in terms of prospective elementary teachers moving from dependence on the standard algorithms to reasoning flexibly about the operations. Insights gained from the case studies inform our thinking about these standard algorithms. In both cases, knowledge of the standard algorithm played an important role in the case students' developing reasoning. For Brandy, coming to understand the standard algorithms helped her to move away from dependence on them. In Valerie's case, her knowledge of the standard multiplication algorithm helped her to account for partial products.

These case studies not only inform the research of Nickerson and myself. The rich descriptions also afford their comparability by enabling other researchers to judge for themselves how the findings apply based on the ways in which they overlap with other cases. It is the more general theory not the details of the case that may be applied to future work involving number sense development and prospective elementary teachers.

Chapter 7: Conclusion

In order to support children's learning of elementary mathematics meaningfully, elementary teachers need to understand elementary mathematics meaningfully and flexibly (Ball, 1990; Ma, 1999). However, researchers have found that prospective elementary teachers tend to be reliant on standard algorithms, rather than reasoning meaningfully and flexibly about numbers and operations (Ma, 1999; Newton, 2008; Yang, 2007). Previous research had focused on single or pre/post snapshots of prospective teachers' number sense, as opposed to analyzing the development of number sense. In this study, I set about to analyze the ways in which prospective elementary teachers' number sense develops.

In previous research, we found that prospective elementary teachers enrolled in a mathematics content course informed by a local instruction theory developed improved number sense over the course of a semester-long teaching experiment (Whitacre & Nickerson, 2006). Similarly, in this study, students scored higher on measures of number sense. A subset of the students were interviewed pre and post. They transitioned from being generally dependent on standard algorithms to reasoning more flexibly in mental computation by using more nonstandard strategies.

In this study, I investigated prospective elementary teachers' number sense development during a first mathematics content course that was designed with number sense development as a goal. I duplicated analyses from a previous study and found similar results. I also moved beyond the previous study in several ways, investigating number sense development as a microgenetic, sociogenetic, and ontogenetic process (Saxe & Esmonde, 2005). I asked the following research questions: As prospective elementary teachers participate in a mathematics content course designed to support their development of number sense,

1. How does the number sense of individuals evolve?
2. What ideas come to function as if shared? What classroom mathematical practices emerge and become established?

The ontogenetic strand concerned development over time in individual students' reasoning. This analysis addressed Research Question 1. The sociogenetic strand concerned development over time in the collective activity that occurred in the classroom. This analysis addressed Research Question 2. The microgenetic strand concerned instances of mathematical activity, both in class and in interviews. These analyses contributed to both the ontogenetic and sociogenetic analyses, and thus helped to answer both research questions.

I approached this study from a situated perspective (Cobb & Bowers, 1999). The emergent perspective informed my approach to the research in terms of taking both social and individual lenses to the analysis of number sense development, as well as taking an interest in the relationship between these (Cobb & Yackel, 1996).

I used a variety of methods to perform analyses of the three strands of development. The primary means of assessing change in students' reasoning was via pre/post interviews with seven participants. These microgenetic analyses of participants' strategies and arguments enabled me to make pre/post comparisons of characteristics of their reasoning. I focused especially on flexibility and the distribution of strategies along the spectrum from Standard to Nonstandard.

I analyzed sociogenesis of number sense by using the methodology of Rasmussen and Stephan (2008). I coded arguments made in whole-class discussion using Toulmin's (1969) model. I applied established criteria for identifying ideas that functioned as if shared (Cole, et al., 2011; Rasmussen & Stephan, 2008). I then organized these as-if shared ideas according to more general mathematical activities.

I analyzed ontogenesis of number sense by means of two detailed qualitative case studies (Yin, 1994) of individual participants' reasoning. I analyzed Brandy's developing reasoning about place value, addition, and subtraction, as well Valerie's developing reasoning about multiplication with a particular focus on partial products. I did this through fine-grained analysis and thick description of their reasoning in the interviews, together with analysis of written work, and coordinated with classroom events.

The results of the analyses of the microgenetic, sociogenetic, and ontogenetic strands all contribute to answering my research questions. In the remainder of this chapter, I summarize the study results. I discuss analytic and theoretical contributions of the research. I discuss implications for teaching. Finally, I describe likely directions for future research.

Summary of Results

In this section, I summarize the results presented in Chapters 4, 5, and 6. I frame these results in terms of how they speak to the research questions.

Improvement in Participants' Number Sense

Several types of evidence point to improvement in the study participants' number sense. An established instrument, the Number Sense Rating Scale, was used as a quantitative measure of number sense. This test was administered to 34 students pre and post, and the mean score increased by more than one standard deviation, from 65% to 79%. Furthermore, the mean gain score was greater than that seen in the previous semester for students in all the groups—low, middle, and high. The same 34 students took the Student Preference Survey pre and post. These results indicate that students came to see multiplication and division as less difficult than they had at the beginning of the semester. Students reported increased willingness to perform computations mentally, especially in the case of multiplication. The inclination to actually apply one's understanding of numbers and operations is an aspect of number sense. Thus, these results provide additional evidence of improved number sense.

We have seen that the interview participants became more flexible in whole-number mental computation. The interview participants came to use a wider variety of strategies for a given operation. In addition to becoming more flexible, the interview participants came to use nonstandard strategies far more often. That is, the distribution of strategies used shifted decidedly towards the nonstandard end of the spectrum. Nonstandard strategies are associated with number sense because they are often performed ad-hoc, rather than algorthmically, and thus require students to recruit their understanding of numbers and operations in reasoning about how to perform the computation.

The above results provided evidence *that* the participants' number sense improved. These results amount to a duplication of those of the previous study. Repeating studies is worthwhile, and it is not often done in mathematics education research. Five years after the previous study, with a new group of students and a different instructor teaching the course, comparable results were obtained in a course in which instruction was informed by the local instruction theory. Prospective elementary teachers are developing improved number sense in the course, especially when the teaching is informed by the local instruction theory for number sense development. This noteworthy improvement in the number sense of prospective elementary teachers merits further investigation since the literature suggests that it does not typically occur.

The results presented in Chapter 4 established the premises of the research questions. The prospective elementary teachers' number sense improved during the course. Thus, the classroom teaching experiment provided an appropriate setting to ask how study participants' number sense improved. The research questions were directly addressed by the results presented in Chapters 5 and 6.

Sociogenesis of Number Sense

The results presented in Chapter 5 concerned the sociogenetic strand of prospective elementary teachers' number sense development. I presented evidence that indicated social norms such that the methodology of Rasmussen and Stephan (2008) was appropriate to the analysis of collective activity in the content course. I then told the stories of the classroom math practices that emerged and became established in two whole-number content strands.

The methodology of Rasmussen and Stephan (2008) assumes classroom social norms involving argumentation that are characteristic of inquiry-oriented instruction. The section titled Social Norms presented evidence that the mathematics content course as enacted was characterized by students' inquiry into elementary mathematics. Most of the arguments were made by students, many were co-constructed by the Instructor and one or more students, and few were made by the Instructor alone. Also, expanded arguments occurred approximately as frequently in the content course as they did in an inquiry-oriented linear algebra course (Wawro, 2011).

In the section titled Classroom Mathematical Practices: Place Value, Addition, & Subtraction, I documented a case of sociogenesis of number sense. In the strand of activity involving ideas of place value, addition, and subtraction, I categorized sets of as-if shared ideas into five classroom mathematical practices. In terms of collective activity, the class progressed from (CMP1) assuming the authority of the standard algorithms to (CMP2) making sense of place value to (CMP3) using an understanding of place value to make sense of the standard addition and subtraction algorithms, as well as Transition strategies, to (CMP4) reasoning flexibly about addition and (CMP5) reasoning flexibly about subtraction.

Viewing the progression through CMPs in terms of the Standard-to-Nonstandard framework, CMP1 corresponded to the Standard category (Figure 93). In this initial stage of collective activity, the class relied on standard algorithms. CMP2 and CMP3 correspond to the Transition category. By making sense of place value and relating that understanding to the standard algorithms, the class transitioned from using the MASAs for addition and subtraction to using Right to Left and Left to Right strategies. CMP4 and CMP5 correspond to the Nonstandard categories (with and without reformulation). The class reasoned flexibly about addition and subtraction by making sense of and using various nonstandard strategies.

In the section titled Classroom Mathematical Practices: Multiplication, I documented a second case of sociogenesis of number sense. In the strand of activity related to multiplication, I identified three additional classroom mathematical practices.

Actual learning route	CMP1	CMP2 & CMP3	CMP4 & CMP5	
Envisioned learning route	Standard	Transition	Nonstandard	Nonstandard w/Reformulation

Figure 93. Correspondence between CMPs and envisioned learning route.

The class progressed from (CMP1) assuming the authority of the standard algorithm to (CMP6) separating, multiplying, and adding with single-digit multipliers to (CMP7) separating, multiplying, and adding with double-digit multipliers to (CMP8) reasoning flexibly in computing and estimating products.

In the multiplication strand, we again see a correspondence between CMPs and the Standard-to-Nonstandard framework (Figure 94). In the initial stage of collective activity, the class assumed the authority of the standard multiplication algorithm. Single-digit SMA served as a Transition strategy in students' progression from dependence on the standard algorithm to reasoning flexibly about multiplication. Through separating, multiplying, and adding, students came to reason about multiplication in terms of partial products, which helped to lay the groundwork for making sense of multidigit multiplication. Students used previously established ideas concerning partial products, together with reasoning about products in terms of rectangular area, to make sense of multidigit multiplication as involving four partial products. This enabled them to both justify the standard algorithm and move beyond it by correctly applying SMA to the case of a double-digit multiplier. In the last CMP, students reasoned flexibly about computing products by using multiple nonstandard strategies.

Looking more broadly at the results of both strands of analysis of collective activity, a few points stand out as surprising and noteworthy. The standard algorithms for

Actual learning route	CMP1	CMP6	CMP7	CMP8
Envisioned learning route	Standard	Transition	Nonstandard	Nonstandard w/Reformulation

Figure 94. Correspondence between CMPs and envisioned learning route.

whole-number arithmetic initially functioned authoritatively, and their authoritative status was leveraged productively. It led to a classroom culture in which students made mathematical arguments for the validity of nonstandard strategies. Eventually, students were also able to justify the validity of the standard algorithms.

Once the standard algorithms had been justified, a variety of ideas related to nonstandard strategies came to function as-if shared. Also, established as-if shared ideas came to be used in more sophisticated ways as students reasoned about nonstandard strategies. Even commonplace ideas such as reasoning about subtraction as a take-away process were used productively in students' arguments for the validity of nonstandard strategies. Students' reasoning flexibly about operations did not occur haphazardly. It was the culmination of a set of foundational as-if shared ideas, together with the freedom from the standard algorithms that seemed to result from making sense of them.

Ontogenesis of Number Sense

I presented two case studies of ontogenesis of number sense, which corresponded to the content strands that were studied in the sociogenetic analysis. I described Brandy's developing reasoning about place value, addition, and subtraction in the first section, and I described Valerie's developing reasoning about multiplication in second section. In the Conclusion section, I identified insights gained from the case studies and discussed the issue of generalizing from the case studies to the broader phenomenon of interest.

Brandy's developing reasoning about place value, addition, and subtraction. Brandy developed from Inflexible to Flexible in her reasoning about computing both sums and differences. She did this by applying her understanding of place value to make sense of both standard algorithms and Transition strategies. Knowledge that she brought with her to the course served as a resource in her learning. Her scaffolded alternatives, particularly Compensation strategies influenced how she interpreted the nonstandard strategies that she encountered in class. A particular addition strategy from class, Borrow to Build, was influential in Brandy's developing reasoning about both addition and subtraction. Brandy used the commonplace idea of subtraction as a take-away process to construct valid Subtrahend Compensation, a nonstandard strategy that is counterintuitive for many prospective elementary teachers. Finally, although Brandy could justify the validity of Shifting the Difference, she did not seem to understand how to apply it to make computations easier, and this may explain why she did not adopt the strategy.

Viewing Brandy's developing number sense through the lens of the environment metaphor, she developed increased attunement to constraints and affordances in the domain of numbers and quantities. Specifically, she recognized the affordances of benchmark numbers, such as 49, and selected special strategies on the basis of the given numbers. She also came to distinguish between the minuend and subtrahend in subtraction, which enabled her to makes sense of how to compensate for rounding the subtrahend. Brandy developed a broad heuristic, which she identified as borrowing to build, in which she sought to form nice numbers and compensate appropriately for doing so. She identified analogous ways of rounding, computing, and compensating across different operations and with regard for the particular properties of those operations.

At the beginning of the course, Valerie was dependent on an invalid strategy, which she preferred over the MASA due to its simplicity. Her reasoning with her invalid strategy was not related to her procedural knowledge of the algorithm. Rather, she thought in terms of accounting for all of each factor. Valerie was initially insensitive to distinctions between nonstandard strategies. It was only when Valerie became aware that her go-to strategy was invalid that she began to make more careful distinctions between nonstandard strategies.

She was able to make sense of the standard multiplication algorithm by reasoning the digits in terms of their place values. Valerie's reasoning about partial products changed, and she became more able to account for them correctly. The commonplace idea of reasoning about multiplication in terms of repeated addition was a productive idea in this process. Viewing one factor or the other as the multiplier and appropriately treating double-digit products as consisting of two partial products was more accessible for Valerie than accounting for all four partial products explicitly. She was able to do this by switching repeated-addition perspectives, viewing one factor as the multiplier and then the other. Valerie's ability to account for partial products was also supported by the standard algorithm and rectangular area, which served as tools that helped her to organize her thinking.

Viewing Valerie's case through the lens of the environment metaphor, she became increasingly attuned to distinctions between nonstandard strategies. In so doing, she became aware of constraints that made certain strategies invalid. Initially all nonstandard strategies seemed to involve breaking numbers up, computing, and then putting numbers back together. She came to view operations as having particular constraints and affordances. For example, a strategy that worked for addition would not work for multiplication. Once Valerie became aware of the need to account for partial products, she also needed to find useful tools for the purpose. Rectangular area and the standard multiplication algorithm both enabled her to organize her accounting activities.

It was not the case that Brandy was dependent on the standard algorithms *despite* not understanding them. Rather, she was dependent on the standard algorithms *because* she did not understand them. Essentially, there is a fine line between using Transition strategies and reasoning meaningfully about the standard algorithms. The strategies that occur in prospective elementary teachers' scaffolded strategy ranges reflect particular ways of reasoning. These ways of reasoning may influence how prospective elementary teachers interpret the nonstandard strategies that they encounter. Reasoning about subtraction as a take-away process was a commonplace idea that Brandy brought with her to the content course. This idea and others served as productive resources for her learning. In addition to mathematical justification, understanding practical aspects of strategy application, such as making a computation easier in a particular way can also be important factors in prospective elementary teachers' understanding and adoption of nonstandard strategies.

Conclusion

Chapters 4–6 presented results of analyses of the microgenesis, sociogenesis, and ontogenesis of number sense. The results presented in the first section of Chapter 4 provided evidence of substantial improvement in the number sense of the study participants. These results established that the setting was appropriate to investigate number sense development. The second section of Chapter 4 presented results of two new analyses, introducing the constructs of strategy ranges and strategy-arguments. These afforded more fine-grained descriptions of the interview participants' mental computation activity. Chapter 4 began to address Research Question 1 and set the stage for the rest of the study results.

Chapter 5 told the story of sociogenesis of number sense in two distinct content strands. These accounts spanned grain sizes, from specific vignettes, to as-if shared ideas, to classroom mathematical practices. The fine-grained descriptions elucidated the details of collective activity and the nuances of students' reasoning. The coarse grain size of progressions through CMPs conveyed a big picture and enabled these results to be related to the local instruction theory. The results presented in Chapter 5 answered Research Question 2.

Chapter 6 presented two case studies of the ontogenesis of number sense. Brandy's case was one of exceptional number sense improvement that spanned two operations. Valerie's case was specific to multiplication and afforded a more fine-grained analysis of her developing reasoning about partial products. Both analyses led to insights concerning the cases of number sense development. The results presented in Chapter 6 answered Research Question 1.

Discussion of Analytic and Theoretical Contributions

In this section, I highlight the innovative aspects of the study. These represent both methodological and empirical contributions to the field.

Categories of Flexibility and Strategy Ranges

My analysis of interview participants' mental computation activity benefited from two extant frameworks. Markovits and Sowder's (1994) framework describes the extent to which mental computation strategies depart from the standard algorithms and, hence, are associated with number sense. Heirdsfield and Cooper's (2004) framework characterizes accurate mental calculators as either inflexible or flexible and describes the distinct mental processes corresponding to these two categories. My analysis of participants' strategy ranges and flexibility built upon and integrated these two frameworks.

The analysis of flexibility made use of the categories Inflexible and Flexible. I also found it useful to introduce an in-between category called *Semiflexible* in order to distinguish individuals who choose between only two possible strategies from those who select from a repertoire of three or more strategies. In the analysis presented earlier, the connection between these categories and strategy ranges was left implicit. Here, I lay out that relationship explicitly.

An individual's strategy range for mental computation was defined in terms of a specific operation as the set of valid strategies that the person used to perform that operation mentally. I conceptualize this set of strategies as ordered along the spectrum from Standard to Nonstandard. There are various ways in which participants' strategies were distributed along that spectrum, and these gave rise to the categories MASA-bound, Polarized, Transitional, Spread, Transition+, and Independent. These strategy ranges relate naturally to the categories of flexibility. For example, an individual whose strategy range is Polarized is necessarily Semiflexible, and an individual who is MASA-bound is necessarily Inflexible.[31] Table 41 illustrates the coordination of the categories of flexibility and strategy ranges.

Table 41. Coordination of Strategy Ranges and Flexibility

Inflexible	Semiflexible	Flexible
MASA-bound	Polarized Transitional	Spread Transition+ Independent

In this coordinated framework, the two dimensions describe distinct aspects of participants' mental computation activity. Flexibility refers to a process related to making choices. Inflexible participants do not make a choice of strategy. Semiflexible participants choose between two possible strategies. Flexible participants choose from amongst three or more strategies. Whereas these categories of flexibility describe the number of distinct strategies that participants use, strategy-range profiles describe the types of strategies that they use. These are described in terms of the Standard-to-Nonstandard spectrum.

Strategy ranges offer a more fine-grained way of characterizing mental computation reasoning. There is also the potential for strategy ranges to provide a useful tool to describe students' reasoning in other domains, such as comparing fractions.

[31] Note that Inflexible need not imply MASA-bound. In fact, Valerie is an exception. Apart from her case, however, the Inflexible participants invariably were MASA-bound. Likewise, Heirdsfield and Cooper (2004) described accurate, inflexible mental calculators as relying on the MASAs specifically. It is not clear how to integrate invalid strategies into the picture of strategy ranges. This matter will require further attention.

Strategy-Arguments

The strategy-argument construct itself represents a contribution to the field. In interview settings, researchers often depend on students' verbal descriptions in order to understand and identify their strategies. These descriptions serve to justify the student's solution to a given task. They are mathematical arguments made by a student in a social context. Viewing them as such acknowledges the human aspects of the research setting. It also can help the researcher to distinguish the mathematical ideas that students use to justify their strategies.

I identified 17 particular strategy-arguments articulated by prospective elementary teachers, including the ideas that students used as warrants and backings. These results represent a contribution to the literature concerning prospective elementary teachers' mathematics content knowledge, as well as to the literature concerning number sense and mental computation. More broadly, the literature concerning mental computation has not emphasized those conceptions that support students' reasoning about the nonstandard aspects of nonstandard strategies, such as which way to compensate. The ideas that the study participants used to justify their strategies, both standard and nonstandard, provide insights into aspects of their reasoning that are important in supporting number sense development.

Documenting Collective Activity

The methodology of Rasmussen and Stephan grew out of their previous research (Stephan & Rasmussen, 2002) was introduced to the field formally in 2008. This methodology was developed in the analysis of data from an inquiry-oriented differential equations class. It has since been used to document collective activity in inquiry-oriented linear algebra (e.g., Wawro, 2011), as well as in a physical chemistry (Cole et al., 2011). To the best of my knowledge, the present study represents the first instance to date of the methodology being used to document collective activity in a mathematics content course for prospective teachers.

Using the methodology to analyze data from a mathematics content course presented challenges due to the layers of activity involved in such a course. As in any mathematics class, the students engaged in mathematical activity. However, these students were prospective teachers. This being the case, the pedagogical implications of the course content were often discussed as well. Furthermore, students' activity was often not purely mathematical but involved reasoning about children's mathematical thinking. I focused my analysis on the mathematical ideas at play in these discussions since my interest lied in number sense development. However, the potential exists for a different analysis, focused instead on the development of pedaogical content knowledge.

Implications for Teaching

This study investigated a course designed to promote prospective elementary teachers' number sense development. I used pre/post surveys and interviews to analyze whether and how participants' number sense improved. Having found evidence of substantial improvement in their number sense, I further investigated the developmental process by documenting both collective activity and individual case studies. The results of this study answer the question of how prospective elementary teachers' number sense can improve substantially during a single-semester mathematics content course.

Given that preservice elementary teachers who have already taken their college mathematics courses are known to have poor number sense (Ball, 1990; Thanheiser, 2010; Tsao, 2005), I expect that many elementary mathematics content courses could benefit by (a) incorporating aspects of the instructional approach that was used in our classroom teaching experiment and (b) taking into account some of the findings of this study. For example, the findings concerning the role of the standard algorithms in students' number sense development stand out to me as noteworthy and useful. These will be incorporated into the local instruction theory and shared with colleagues in mathematics teacher education.

This study led to the development of new analytic tools in the analysis of number sense. I also documented collective activity in a mathematics content course in which prospective elementary teachers developed improved number sense. The study results will also enable me to think about how the NSRS or similar instruments can be revised based on what was learned about prospective elementary teachers' mathematical thinking.

Findings will be shared with the mathematics teacher education community through conversations with colleagues, conference presentations, publications, and curricular materials. I have already reported on the analysis of collective activity at a recent conference (Whitacre, 2012), and several publications related to this work are planned. Dr. Nickerson is one of the authors of the textbook that is used by instructors of the mathematics content course that was studied. This textbook is used at many other institutions across the country. Dr. Nickerson and I plan revisions to the textbook that make use of findings from this study. We have been working on prospective elementary teachers' number sense development for several years now, and the field is beginning to take notice. Recently, a colleague from another university contacted us to ask for suggestions and materials that he could use in a workshop for inservice teachers that will focus on number sense development.

The role of the standard algorithms stands out as a subject of surprising findings from both the sociogenetic and ontogenetic analyses. I never anticipated a classroom math practice characterized by assuming the authority of the standard algorithms, nor did I imagine that such a practice could be leveraged productively and play an important role in the progression toward reasoning flexibly about the operations. On the individual level, I had not anticipated how a student's knowledge of a standard algorithm might actually serve as a productive resource in that she could use that algorithm as a tool to organize her activity.

I had underestimated the importance of place value in the process by which students came to reason flexibly about the operations. Upon reflection, I think this is because I was familiar with the kinds of justifications that students gave for nonstandard strategies, and place value ideas are often slip below the surface of these arguments. In my current conceptualization, students' understanding and awareness of place value stands out as important to their number sense development in two ways.

First, by relating place value ideas to the operations, prospective elementary teachers can make sense of standard algorithms and Transition strategies. Doing so opens the door to a range of other possible ways of performing the operations. For many of these students, who reason inflexibly about the operations at the beginning of the course, I believe that there is a radical transition that occurs when they move from trusting the standard algorithms as ways of performing operations on the basis of their authoritative status to judging for themselves the validity of a given strategy or algorithm on the basis of their own mathematical reasoning.

Second, prospective elementary teachers bring knowledge of place value with them to the course. This is not to say that students do not develop new mathematical understanding during the place value unit of the course. However, what seems to me most significant about their experience in the place value unit is that students become more aware of place value. They come to reason about digits in terms of place value when performing operations, both by the standard algorithms and otherwise, and this is a key to reasoning flexibly about the operations.

Prior to conducting this study, I had taught a total of nine sections of the mathematics content course. I had conducted a previous teaching experiment in the course. I had also supervised instructors of the course for two years. Having been so involved with this course, I find it surprising that I learned so much from this study. I began the study with an interest in certain aspects of the classroom activity, particularly the naming of strategies and the role of particular models in students' reasoning about mental computation. These phenomena certainly figured into both the sociogenetic and ontogenetic strands. The ontogenetic analysis was especially illuminating with regard to those aspects since I had never before studied individual students' thinking in the class in such a detailed way.

Directions for Future Research

Directions for future research related to this work involve both untapped data from the study and future data collection. In this section, I discuss both of these avenues.

Untapped Data

I deliberately collected considerably more data than would be used for the dissertation study itself. This untapped data includes interview data and written work, data related to the rational-number portion of the course, and video of small-group activity. The case studies were selected after all data had been collected. For this reason, sufficient data was collected to afford case studies of all interview participants and for each operation. As discussed in Conclusion section of Chapter 6, additional case studies are of interest and will be useful to account for different extents of c and various developmental trajectories.

We also collected data related to the rational-number portion of the course. This included all data analogous to that collected for the present study: pre/post surveys, fraction comparison interviews, SST interviews, classroom video, and written work. This data will afford analyses analogous to those reported here. The basic interview data has already been analyzed and the results shared with the research community (Whitacre & Nickerson, 2011b). We have also reported on the instructional approach to the rational-number unit, which involved extending the local instruction theory to a new content domain (Whitacre & Nickerson, 2011a). Further analyses will include documenting collective activity during the rational-number unit and analyzing at least one case study, concerning Maricela's developing reasoning about fraction magnitude.

Finally, the analysis of classroom data in this study was limited to whole-class discussions. However, video cameras also recorded activity during small-group work. This activity was certainly integral to students' learning in the course. Very often, students worked in groups on a task and created a shared representation of their work on the group's whiteboard. Students then presented these ideas in whole-class discussion. There is a great deal of opportunity to better understand students' reasoning by analyzing group discussions.

Taking small groups as a unit of analysis would bridge the gap between whole-class discussion and individual students' reasoning. For the most part, the classroom data did not afford access to Brandy or Valerie's reasoning during class. I used homework and test responses to trace the development of their mathematical thinking. However, in small-group discussions, students' thinking is on display during classroom activity. Therefore, I plan to conduct some pilot analyses of small-group activity in order to more carefully consider how to study that activity. There is some precedent for this in the literature (e.g., Scherr & Hammer, 2009; Southerland, Kittleson, Settlage, & Lanier, 2005), but it is an area that is relatively uncharted.

Future Research

There are many ways in which this work might be extended. In the near future, I will begin teaching an elementary mathematics methods course and a course focused on children's mathematical thinking. These courses represent the counterpart to the mathematics content course that I taught in the past and that was the setting for the dissertation study. I anticipate opportunities in these new setting to study aspects of prospective elementary teachers' mathematical preparation that extend beyond content knowledge. There is the possibility of documenting collective activity in discussions of children's mathematical thinking, rather than discussions of the mathematics itself.

Other possible directions include extending the work to other content domains or to other student populations. It is interesting to consider how envisioned learning routes for number sense development change if working with different student populations. Clearly, elementary children who have not yet been taught those algorithms follow different learning routes. The development of their arithemetic reasoning has been well studied (e.g., Carpenter et al., 1999). However, in the number sense literature, middle school students and preservice elementary teachers have been talked about similarly. Authors have identified both of these groups as generally exhibiting poor number sense and relying heavily on standard algorithms and procedures (e.g., Reys & Yang, 1998; Yang, 2007). Thus, it natural to ask whether and how the local instruction theory for number sense development would apply to middle school mathematics instruction. This task would make for an interesting challenge since the occasions for engagement in computational reasoning and discussions of strategies would look different than in an elementary mathematics content course.

Another way of extending the research is to move into different content domains. I mentioned in the previous section the work that Dr. Nickerson and I have done around fractions in the content course. The domain of integers represents another extension of the natural numbers, and this may be a topic worth exploring. Recently, Lamb and colleagues have researched how K-12 students think about integers and integer arithmetic (Lamb et al., 2012). Lamb's research group is working on characterizing integer sense as a form of number sense. Their efforts will likely lead to classroom-based research focused on integer sense development, which could be informed by findings from this study of number sense development.

Appendix 1: Number Sense Rating Scale, adapted from Hsu, Yang, and Li (2001).

Mathematical Thinking Survey

This is a multiple-choice survey intended to investigate aspects of your mathematical thinking. The questions are not meant to be solved by written work, so please do not do any writing, except to indicate your answers. Simply read each question, consider the answer options, and choose the best answer based on your knowledge, reasoning, and/or mental math. Don’t worry if you’re unsure about some of the responses. You can make an educated guess or just indicate that you’re unsure. Please just select the response that best reflects your mental reasoning.

This survey is confidential. Your instructor will not be informed of your responses, and your responses will not affect your course grade. Please just do your best.

Thanks!

1. How many fractions are between $\frac{4}{7}$ and $\frac{5}{7}$?

a) none b) one c) ten d) infinitely many

2. Which fraction below is between $\frac{4}{5}$ and 1 ?

a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{5}{6}$ d) $\frac{5}{5}$

3. Which answer below is the same as 0.5 x 840 ?

a) 840 ÷ 2 b) 5 x 840 c) 840 ÷ 5 d) 0.50 x 84

4. A pizza is cut into 8 equal pieces. Then each piece is cut into 3 equal pieces.

How many equal pieces does the pizza have now?

a) 3 b) 8 c) 12 d) 24

5. How many decimal numbers are between 9.43 and 9.44 ?

a) none b) one c) ten d) infinitely many

6. Compare $\frac{7}{11}$ and $\frac{7}{10}$. Which one is greater?

a) $\frac{7}{11}$ b) $\frac{7}{10}$ c) they are equal d) I'm not sure

7. Compare 7.2 and 7.1987. Which one is greater?

a) 7.2 b) 7.1987 c) they are equal d) I'm not sure

8. Compare 3.111 and 3.1099. Which one is greater?

a) 3.111 b) 3.1099 c) they are equal d) I'm not sure

9. 93 x 134 = 12462. How much greater than 12462 is the product of 93 and 135?

a) 93 b) 134 c) 135 d) I can't tell without calculating

10. Which description of $6\frac{2}{5} \div \frac{15}{16}$ is correct?

   a) greater than $6\frac{2}{5}$

   b) smaller than $6\frac{2}{5}$

   c) equal to $6\frac{2}{5}$

   d) I can't tell without calculating

11. Which description of 87 x 0.09 is correct?

   a) much smaller than 87

   b) a little bit smaller than 87

   c) much greater than 87

   d) a little bit greater than 87

12. Which description of 245 x 0.98 is correct?

   a) greater than 245

   b) smaller than 245

   c) equal to 245

   d) I can't tell without calculating

13. Which description of 0.997 x 0.9 is correct?

   a) greater than 0.9

   b) smaller than 0.9

   c) equal to 0.9

   d) I can't tell without calculating

14. Which description of 487 ÷ 0.99 is correct?

a) greater than 487

b) smaller than 487

c) equal to 487

d) I can't tell without calculating

15. How many digits are in the product of a 2-digit number and a 2-digit number?

a) must be three digits

b) must be four digits

c) can be three digits or four digits

d) can be three digits, four digits, or five digits

16. The sum of a 3-digit number and a 3-digit number:

e) must be three digits

f) must be four digits

g) can be three digits or four digits

h) can be three digits, four digits, or five digits

17. Which product below is the greatest?

a) 18 x 17 b) 16 x 18 c) 17 x 19 d) 19 x 15

18. Which description of 145 x 4 below is correct?

a) greater than 450

b) smaller than 450

c) equal to 450

d) I can't tell without calculating

19. Which answer below is greater than 1?

a) $\frac{2}{5} + \frac{3}{7}$ b) $\frac{1}{2} + \frac{4}{9}$ c) $\frac{3}{8} + \frac{2}{11}$ d) $\frac{4}{7} + \frac{1}{2}$

20. 103 x 236 = 24308. So, 103 x 235 = ?

a) 24307 b) 24205 c) 24335 d) 24544

21. Which description below is correct for the result of $\frac{3}{7} + 1.5$?

a) greater than 2

b) smaller than 2

c) equal to 2

d) Fractions and decimals cannot be added together

22. Which description below is correct for 9 x 99.99?

a) greater than 900

b) smaller than 900

c) equal to 900

d) I can't tell without calculating

23. Compare 521 x 5 and 520 + 521 + 522 + 523 + 524. Which is greater?

a) the result of 521 x 5 is greater

b) the result of 520 + 521 + 522 + 523 + 524 is greater

c) they are equal

d) I can't tell without calculating

24. What is the sum of $\frac{14}{15}$ and $\frac{7}{8}$ *approximately*?

a) 1 b) 2 c) 21 d) can't tell without calculating

25. Which answer is the product of 18 and 19 closest to?

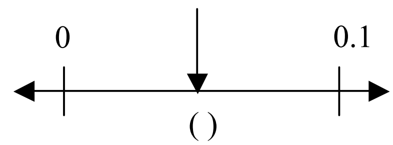
a) 250 b) 350 c) 450 d) 550

26. A cat eats 600 grams of fish every four days. How many grams of fish does this cat eat in six days?

a) 400 b) 600 c) 800 d) 900

27. Which value below is the best estimate of "( )" ?

a) 0.01 b) 0.5 c) 0.05 d) 0.005



28. Mary took a trip. She spent 5 hours traveling to her destination, with an average speed of 80km/hr. Mary's return trip took only 4 hours. What was her average speed on the return trip?

a) 60 km/hr b) 70 km/hr c) 90 km/hr d) 100 km/hr

29. $\frac{1}{2}$ x ( ) = $\frac{4}{8}$, which number below can we put into ( ) ?

a) $\frac{3}{4}$ b) $\frac{3}{6}$ c) 1 d) 4

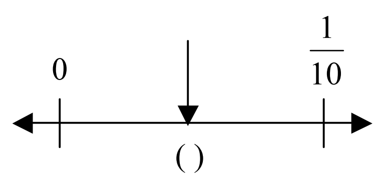
30. Which fraction below is the best estimate of "( )" ?

a) $\frac{5}{10}$

b) $\frac{5}{100}$

c) $\frac{1}{100}$

d) $\frac{5}{1000}$



31. Which fraction below is the best estimate of "( )" ? (Refer to the figure above.)

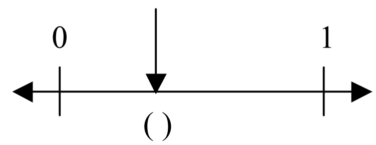
a) $\frac{2}{10}$

b) $\frac{5}{10}$

c) $\frac{1}{20}$

d) $\frac{2}{100}$

32. Which fraction below is the best estimate of "( )" ?



a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{1}{5}$

33. Which is the most typical weight of a male who is 67 inches tall?

a) 22 lbs b) 55 lbs c) 154 lbs d) 431 lbs

34. Which description below is correct for $\frac{2}{5}$?

a) greater than $\frac{1}{2}$

b) equal to 2.5

c) equal to 0.4

d) smaller than $\frac{1}{4}$

35. Which value below is equal to $1\frac{1}{4}$?

a) 1.14 b) 1.41 c) 1.25 d) 1.0

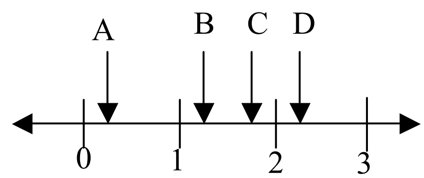
36. Which point in the figure can represent 2.19 best?

a) A

b) B

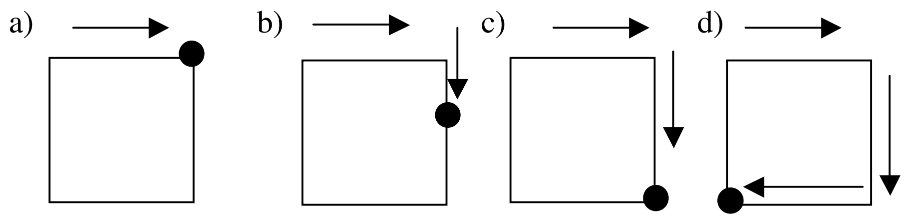
c) C

d) D



37. Sam starts off at the top left corner of a square and moves (clockwise) along it.

Which figure shows the point at which Sam has walked $\frac{1}{3}$ of the way around?



Appendix 2: Student Preference Survey, modified from an instrument developed by McIntosh et al., (1995).

Student Preference Survey

Computations and number comparisons can be performed by calculator, written work, or mentally. Please look at each computation and fraction comparison problem below and indicate whether or not you would do it mentally. Circle Yes or No to indicate your response. It is not necessary to actually perform the computations or comparisons. Please just answer honestly.

Would you compute these mentally?		
1. 37 + 52	Yes	No
2. 78 – 34	Yes	No
3. 15 x 24	Yes	No
4. 420 ÷ 14	Yes	No
5. 64 + 87	Yes	No
6. 178 – 52	Yes	No
7. 19 x 21	Yes	No
8. 570 ÷ 30	Yes	No
9. 96 + 157	Yes	No
10. 82 – 45	Yes	No
11. 25 x 16	Yes	No
12. 275 ÷ 25	Yes	No
13. 38 + 99	Yes	No
14. 125 – 49	Yes	No
15. 99 x 15	Yes	No
16. 900 ÷ 45	Yes	No

Appendix 3: Mental computation interview tasks, used in the first and second interviews.

Instructions: For the following tasks, please find an exact answer *mentally* and explain your thinking. Each task involves a story about Bobo, who sells oboes.

Whole Number Addition

Story: Bobo's oboe business is booming! However, he could use some help keeping track of his sales.

Task	**Notes**
A1. Bobo sells 37 oboes one month and 52 the next month. How many oboes did he sell in those two months?	
A2. Bobo sells 64 oboes one month and 87 the next month. How many oboes did he sell in those two months?	
A3. Bobo sells 96 oboes one month and 157 the next month. How many oboes did he sell in those two months?	
A4. Bobo sells 38 oboes one month and 99 the next month. How many oboes did he sell in those two months?	

Whole Number Subtraction

As everyone knows, oboes don’t grow on trees. Bobo has to spend money to make money.

Task	Notes
S1. If Bobo buys an oboe for $34 and then sells it for $78, how much money does he make?	
S2. If Bobo buys an oboe for $52 and then sells it for $178, how much money does he make?	
S3. If Bobo buys an oboe for $45 and then sells it for $82, how much money does he make?	
S4. If Bobo buys an oboe for $49 and then sells it for $125, how much money does he make?	

Whole Number Multiplication

Story: Bobo is now selling oboes in bunches.

Task	Notes
M1. Bobo offers a package of 15 oboes for $24 per oboe. How much does that package cost?	
M2. Bobo offers a package of 19 oboes for $21 per oboe. How much does that package cost?	
M3. Bobo offers a package of 25 oboes for $16 per oboe. How much does that package cost?	
M4. Bobo offers a package of 99 oboes for $15 per oboe. How much does that package cost?	

Whole Number Division

Task	Notes
D1. If Bobo charges $420 for a package of 14 oboes, what is the price per oboe?	
D2. If Bobo charges $570 for a package of 30 oboes, what is the price per oboe?	
D3. If Bobo charges $275 for a package of 25 oboes, what is the price per oboe?	
D4. If Bobo charges $900 for a package of 45 oboes, what is the price per oboe?	

Appendix 4: Numeration Tasks, used in the first and second interviews.

The number 63 is made up of ___ tens and ___ ones.

Are there other possible answers?

If you have $535 in the bank, how many $10 bills could you withdraw?

How many $100 bills?

How many $5 bills?

A bag of 30 marbles can be put into ___ groups of ___ marbles.

Are there other possible answers?

How many different ways can the marbles be grouped?

Suppose you have 50 groups of marbles, with 18 marbles in each group. If you rearranged the marbles into 100 equal-sized groups, how many marbles would be in each of those groups?

Appendix 5: Operation Tasks, used in the first and second interviews.

Please solve 259 + 38 by the normal, written method.

Follow up: What is the meaning of the little 1 that you wrote above the 5?

Please solve 429 – 34 by the normal, written method.

Follow up: What is the meaning of the little 1 that you wrote above the 2?

Follow up: Does the little 1 from the addition problem mean the same thing as the little 1 in the subtraction problem? Why or why not?

Please solve 27 x 13 by the normal, written method.

Follow up: Why did you shift over (or write a zero) when you moved down to the second line?

Jessica says that 37 x 58 can be solved by taking 30 x 50 + 7 x 8. Does Jessica's method make sense to you? Explain.

Please solve 528 ÷ 8 by the normal, written method.

Follow up: Do the two 6's in your answer mean the same thing? Why do you bring down the 8? Why did the 4 become a 40? Was it 40 before you brought the 8 down? Can't 8 go into 40?

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