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Ordered, Disordered and Partially Synchronized Schools of Fish

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Abstract

We study how an ODE description of schools of fish [1] changes in the presence of a random acceleration. The model can be reduced to one ODE for the direction of the velocity of a generic fish and another one for its speed. These equations contain the mean speed $\bar{v}$ and a Kuramoto order parameter for the phases of the fish velocities, r. We show that their stationary solutions consist of an incoherent unstable solution with $r = \bar{v} = 0$ and a globally stable solution with $r = 1$ and a constant $\bar{v} > 0$. In the latter solution, all fishes move uniformly in the same direction with $\bar{v}$ and the direction of motion determined by the initial configuration of the school. In the second part, the directional headings of the particles are perturbed, in two distinct ways, and the speeds accelerated in order to obtain two distinct classes of non-stationary, complex solutions. We show that the system has similar behavior as the unperturbed one, and derive the resulting constant value of the average speed, verified numerically. Finally, we show that the system exhibits a similar bifurcation to that in [2], between phases of synchronization and disorder. In one case, the variance of the angular noise, which is Brownian, is varied, and in the other case, varying the turning rate causes a similar phase transition.

1   Introduction

The dynamics and structure of a school of fish is a fascinating problem with many applications. The original models for fish interaction were discrete [2] but in [1] an ODE model was derived that could be analyzed by dynamical

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systems methods. Migratory solutions were found in [1] that later became the basis for successful modeling and simulations of the spawning migrations of the Icelandic capelin [3]. The simulations are, however, only an approximation to the motion of the full migrating stock of the capelin that can reach up to 10 billion individuals, a number that still defies direct simulations. Consequently, it is desirable to develop kinetic and hydrodynamic models for the motion of the migrating schools that can be simulated as PDEs and compared with acoustic observations. Interestingly, methods from statistical mechanics can be used for discrete schools with a large number of individuals to gauge the complex structure of the school and the solutions found can be a crucial guide for the development of kinetic and hydrodynamic models. We will use two types of order parameters, both for angle and speed, to implement the analysis in this paper.

We will first study the system of ODEs derived in [1] for a large number of particles, but without perturbations, and show that only two asymptotic states are possible: a disordered state of stationary solutions and an ordered state of migratory solutions. The former in unstable and the latter is stable, in fact it attracts all of phase space except a set of measure zero. This set is the stable manifold of the stationary solutions.

Then we add two types of perturbations modeling realistic situations. The first perturbation is angular noise, a derivative of a Brownian noise with a certain variance. The speed is given a deterministic acceleration in both cases. The result are solutions similar to the unperturbed case. However, now the ordered phase depends on the variance of the noise. When the variance reaches a certain threshold we see a transition to a disordered phase.

The second type of noise are (deterministic) angles randomly chosen from a Lorentz distribution to model a preferred turning rate. This is a behavior that is observed in slow moving schools. In this case we also observe a disordered phase and a partially synchronized phase that is the generalization of the partially synchronized Kuramoto solution to the perturbation of the ODE model in [1]. Now the disordered phase depends on the coupling constant that is magnitude of the inertia term of the speeds. When this inertia is large enough the disordered phase transitions to the partially synchronized one.

Part I

2 Original equations

We analyze the model of ordinary differential equations (ODEs) presented in [1]. We note that the model in [1] did not include noise of any kind and this will be addressed in the analysis below. However, the influence of deterministic perturbations on the system in [1] was investigated in [4]. Let $(x_k(t), y_k(t))$ be the Cartesian coordinates of the kth particle at time t, and let $v_k(t) \geq 0$ and $\phi_k(t)$ be the modulus (speed) and phase of its velocity, respectively, at time t. Let N be the number of particles. We assume that the coupling is a mean field coupling, or that all the particles can sense each other. We first describe the model briefly and then derive the equations of [1] by adding inertia to the system of discrete equations.

The model is related to that in [2]. Let us assume that the position of the k-th fish at time $t + \Delta t$ is given by

$$\begin{pmatrix} x_k(t + \Delta t) \\ y_k(t + \Delta t) \end{pmatrix} = \begin{pmatrix} x_k(t) \\ y_k(t) \end{pmatrix} + v_k(t) \begin{pmatrix} \cos(\phi_k(t)) \\ \sin(\phi_k(t)) \end{pmatrix} \Delta t, \tag{1}$$

in terms of its position and its velocity $v_k(t)(\cos(\phi_k(t)), \sin(\phi_k(t)))$ at time t. The unit vector parallel to the fish velocity satisfies the mean-field relation

$$\begin{pmatrix} \cos(\phi_k(t + \Delta t)) \\ \sin(\phi_k(t + \Delta t)) \end{pmatrix} = \frac{1}{N} \sum_{j=1}^{N} \begin{pmatrix} \cos(\phi_j(t)) \\ \sin(\phi_j(t)) \end{pmatrix}. \tag{2}$$

As introduced in [5], at each time step the speed of the k-th fish is the average of the speeds of all the fish in the school calculated at the previous time step:

$$v_k(t + \Delta t) = \frac{1}{N} \sum_{j=1}^{N} v_j(t). \tag{3}$$

Letting $\Delta t \to 0^+$ we obtain the following ODE:

$$\begin{pmatrix} \dot{x}_k(t) \\ \dot{y}_k(t) \end{pmatrix} = \frac{1}{N^2} \sum_{j=1}^{N} v_j(t) \sum_{l=1}^{N} \begin{pmatrix} \cos(\phi_l(t)) \\ \sin(\phi_l(t)) \end{pmatrix}. \tag{4}$$

In polar coordinates,

$$z_k = r_k e^{i\theta_k} \tag{5}$$

$$\dot{z}_k = v_k e^{i\phi_k}, \tag{6}$$

Equation (4) can be rewritten as

$$\dot{z}_k = \frac{1}{N^2} \sum_{l=1}^{N} v_l \sum_{j=1}^{N} e^{i\phi_j}. \tag{7}$$

Now we assume that the school has some inertia $\alpha^{-1}\ddot{z}_k$ which should be added to the left-hand side of the previous equation, thereby obtaining

$$\ddot{z}_k + \alpha\dot{z}_k = \frac{\alpha}{N^2} \sum_{l=1}^{N} v_l \sum_{j=1}^{N} e^{i\phi_j}, \tag{8}$$

where $\alpha > 0$ is the turning rate at which the fish respond to the direction and speed of their neighbors. We now substitute $\dot{z}_k = v_k e^{i\phi_k}$ and $\ddot{z}_k = (\dot{v}_k + iv_k\dot{\phi}_k)e^{i\phi_k}$ into Equation (8), and equate real and imaginary parts. We then obtain the same equations as in [1] for the velocity and direction angle:

$$\dot{v}_k = \frac{\alpha}{N^2} \sum_{i=1}^{N} v_i \sum_{j=1}^{N} \cos(\phi_j - \phi_k) - \alpha v_k, \tag{9}$$

$$v_k\dot{\phi}_k = \frac{\alpha}{N^2} \sum_{i=1}^{N} v_i \sum_{j=1}^{N} \sin(\phi_j - \phi_k). \tag{10}$$

Differentiating (5) and using $\dot{z}_k = v_k e^{i\phi_k}$, we find:

$$\dot{r}_k = v_k \cos(\phi_k - \theta_k), \tag{11}$$

$$r_k\dot{\theta}_k = v_k \sin(\phi_k - \theta_k). \tag{12}$$

2.1 Simplified equations in terms of order parameters

With $\bar{v} := \frac{1}{N} \sum_{i=1}^{N} v_i$, Equations (9) and (10) become:

$$\dot{v}_k = \alpha\bar{v}\frac{1}{N} \sum_{j=1}^{N} \cos(\phi_j - \phi_k) - \alpha v_k \tag{13}$$

$$v_k\dot{\phi}_k = \alpha\bar{v}\frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k). \tag{14}$$

These equations can be rewritten in terms of the Kuramoto order parameter,

$$re^{i\psi} := \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j}, \tag{15}$$

or equivalently,

$$r = \frac{1}{N} \sum_{j=1}^{N} \cos(\phi_j - \psi) \tag{16}$$

and

$$\frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \psi) = 0, \tag{17}$$

where $r(t) \in [0,1]$ measures the coherence of the population and $\psi(t) \in]-\pi, \pi]$ is the average phase. Clearly, the order parameter reaches its maximum of 1 when all the angles $\phi_k = \psi$ are the same.

The following relations for the order parameter are easily obtained:

$$\frac{1}{N} \sum_{j=1}^{N} \cos(\phi_j - \phi_k) = r \cos(\psi - \phi_k) \tag{18}$$

and a corresponding one for sine instead of cosine, which combined with Equations (13) and (14) give

$$\dot{v}_k = \alpha \bar{v} r \cos(\psi - \phi_k) - \alpha v_k \tag{19}$$

$$v_k \dot{\phi}_k = \alpha \bar{v} r \sin(\psi - \phi_k), \tag{20}$$

where the mean-field behavior of the model is apparent. From these equations, we obtain a simple description of the average velocity,

$$Re^{i\mu} := \frac{1}{N} \sum_{k=1}^{N} v_k e^{i\phi_k}. \tag{21}$$

In fact, substituting (19) and (20) in the result of time-differentiating (21), we get

$$\left(\dot{R} + iR\dot{\mu}\right) e^{i\mu} = \frac{1}{N} \sum_{k=1}^{N} (\dot{v}_k + iv_k\dot{\phi}_k) e^{i\phi_k} = \alpha \bar{v} r e^{i(\psi - \mu)} - \alpha R e^{i\mu}.$$

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Dividing by $e^{i\mu}$ and separating real and imaginary parts, we get

$$\dot{R} = \alpha\bar{v}r\cos(\psi - \mu) - \alpha R \tag{22}$$

and

$$R\dot{\mu} = \alpha\bar{v}r\sin(\psi - \mu). \tag{23}$$

3 Dynamics of the order parameters

3.1 The Kuramoto order parameter

We now discuss the dynamics of the order parameters r and ψ. We let $r_\infty = \lim_{t\to\infty} r(t)$ and $\psi_\infty = \lim_{t\to\infty} \psi(t)$ denote equilibrium values of the order parameters, whenever the limits exist. By differentiating Equation (15) and multiplying the result by $e^{-i\psi}$, we obtain:

$$\dot{r} + ri\dot{\psi} = \frac{1}{N}\sum_{k=1}^{N} i\dot{\phi}_k e^{i(\phi_k - \psi)}. \tag{24}$$

Assuming now that all fish are moving, $v_k > 0$ for all k, we can obtain $\dot{\phi}_k$ from Equation (20) and insert it into Equation (24), which then becomes:

$$\dot{r} + ri\dot{\psi} = \frac{1}{N}\sum_{k=1}^{N} i\alpha\bar{v}r\frac{1}{v_k}\sin(\psi - \phi_k)e^{i(\phi_k - \psi)}. \tag{25}$$

Equating real parts we obtain the following equation for the dynamics of r:

$$\dot{r} = \alpha\bar{v}r\frac{1}{N}\sum_{k=1}^{N}\frac{1}{v_k}\sin^2(\psi - \phi_k) \tag{26}$$

and for ψ:

$$r\dot{\psi} = \alpha\bar{v}r\frac{1}{N}\sum_{k=1}^{N}\frac{1}{v_k}\sin(\psi - \phi_k)\cos(\psi - \phi_k) \tag{27}$$

which can be simplified to

$$\dot{\psi} = \alpha\bar{v}\frac{1}{N}\sum_{k=1}^{N}\frac{1}{2v_k}\sin(2(\psi - \phi_k)) \tag{28}$$

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if $r \neq 0$.

Now, from Equation (26) we see that $\dot{r} \geq 0$; with $\dot{r} = 0$ only if $r = 0$ (randomly oriented fish velocities), or $r = 1$ and $\phi_k = \psi$, for all k (completely ordered fish velocities). A fish school with randomly oriented velocities, $r = 0$, is an unstable state, as r increases with any small deviation with $r > 0$. The fish school should eventually end up in the state of completely ordered velocities $r_\infty = 1$ and $\phi_k = \psi_\infty$, for all k.

3.2 Long term behavior of the average velocity and the directional angles

Now, summing Equation (19) over k, dividing by N, and using Equation (16) we obtain an equation for the derivative of $\bar{v}$:

$$
\begin{aligned}
\dot{\bar{v}} &= \alpha \bar{v} \left(r \frac{1}{N} \sum_{k=1}^{N} \cos(\psi - \phi_k) - 1 \right) \\
&= \alpha \bar{v} (r^2 - 1).
\end{aligned}
\tag{29}
$$

Similarly, Equation (20) turns into

$$
\frac{1}{N} \sum_{k=1}^{N} v_k \dot{\phi}_k = 0.
\tag{30}
$$

Equation (29) shows that the average speed goes to zero unless $r = 1$ which would mean that the fish velocities are all directed along the same vector. In the latter case, the average speed may take on an arbitrary real value.

We therefore see that the system has two characteristic behaviors; on one hand, if there is perfect disorder in the distribution of directional angles the system slows down, approaching the stationary solution $v_k = 0$ for all k, and on the other hand we reach perfect alignment with $r_\infty = 1$, and Equation (19) turns into

$$
\dot{v}_k = \alpha(\bar{v} - v_k),
\tag{31}
$$

and thus the system eventually moves at constant speed with $\dot{\bar{v}} = 0$. As we showed in the previous subsection, the perfectly disordered solution $r = 0$ is unstable and the system eventually reaches a consensus, as shown in Figure 1. This means that, unless the system is perfectly disordered, all the speeds evolve towards a common positive constant $\bar{v}_\infty$. Its value depends on α and

on the initial distribution of velocities (speeds and directional angles). If the initial distribution of the direction angles ϕ_k is perfectly disordered, the sums in Equations (13) and (14) are zero. Then $\dot{v}_k = -\alpha v_k$, the speeds all decay to zero, and $\dot{\phi}_k$ is zero for all t (the phases remain in their initial random configuration). For any other initial state, we have shown before that the system reaches a consensus with $r_\infty = 1$ and $\bar{v}_\infty > 0$.

The behavior of the average velocity (21) follows from Equations (22) and (23). The average phase of the velocity distribution, μ tends towards the average phase ψ according to (23). In turn, Equation (22) implies that R evolves towards a stationary state R_∞ with $\dot{R} = 0$:

$$R_\infty = \bar{v} r_\infty \cos(\psi - \mu). \tag{32}$$

Since μ becomes ψ, we find $R_\infty = \bar{v} r$. Now the stationary values of r are either the unstable state $r = 0$ (random velocity orientations with zero average speed) or the stable state $r = 1$ (completely aligned velocities with $\bar{v}_\infty > 0$). In the latter case, we obtain $R_\infty = \bar{v}_\infty$ which is consistent with the definitions of the Kuramoto and average velocity order parameters. This behavior is shown in Figure 1.

Equation (20) divided by v_k (assuming that $v_k > 0$) is the governing equation of Kuramoto's model for phase oscillators [6] with zero natural frequencies and coupling constant $\alpha \bar{v}/v_k$. For this simple Kuramoto model, we have shown that the long term behavior of the system has two stationary states, (i) $r = 0$, $\bar{v} = 0$ and randomly oriented initial velocity phases, and (ii) $r = 1$, $\bar{v} > 0$ and equal velocity phases. According to Equations (11) and (12) for the positions, state (i) is stationary in space and is unstable, whereas state (ii) is stable, it is stationary in the direction angles ($\theta_k = \phi_k = \psi$) and the fish move uniformly with speed $\bar{v}$. The basin of attraction of state (ii) has full measure so it attracts almost every point in phase space. These results are indeed the same as those of [7]. The model in that reference is very similar to that in [2] and the corresponding synchronized state is also an absorbing state.
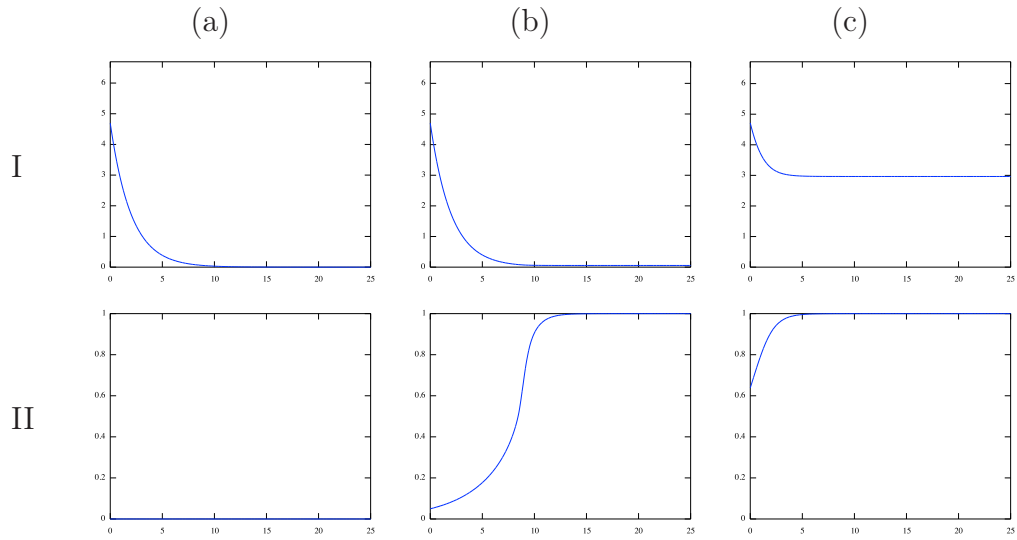
Figure 1: Evolution of the average speed (I) and of the order parameter r (II) according to Equations (9) and (10). In all cases the inertia is $\alpha = 0.5$, and the number of particles is $N = 2000$. The plots differ only in the initial distribution of the directional angles: (a) roots of unity, giving $\bar{v}_\infty = 0$ and $r_\infty = 0$, (b) uniformly randomized, giving $\bar{v}_\infty \simeq 0.05$ and $r_\infty = 1$ (c) uniform of width π, giving $\bar{v}_\infty \simeq 2.96$ and $r_\infty = 1$.

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Part II

4   Influence of disordered and noisy accelerations

The fish school described by the equations in Part I had two possible long term behaviors; random orientations of fish velocities and zero average speed, and constant average speed and the same direction for the fish velocities. These simple states are similar to the incoherent and totally synchronized states in the Kuramoto model. In this part, we will see how angular noise and accelerated speeds modify these simple behaviors and find a partially synchronized non-stationary state.

4.1   Perturbed equation for $\dot{\phi}_k$

We note that from Equation (20) it is clear that all the direction angles are driven towards the average phase ψ. The long term behavior of the Equations (9) and (10) is therefore a completely aligned school of particles traveling at constant speed. This solution can be viewed as a migratory solution for a school of fish. However, it is quite unlikely that the fish within a school are perfectly aligned. Multiple factors can introduce random elements; such as currents (turbulent or not), physical structure of the fish (fins, tails etc.), influence of food or some environmental factor and so on.

We therefore want to introduce two types of noise to the system of equations. The first type will be a quenched deterministic noise, which will correspond to the disorder in the natural frequencies of the phase oscillators in Kuramoto's well known paper [6]. A possible biological interpretation could be that there is an intrinsic disorder in the rates at which the fish reorient their velocities to follow the average values marked by the school. This would indicate the imperfect ability of the fish to determine the orientation angle. Another possibility is that changing environmental factors alter instantaneously the fish's ability to find the orientation angle. We model this inability by adding a white noise to the right hand side of Equation (20). Below, we investigate the system's behavior when modified by both types of noise.

4.2 Driven speeds

One long term behavior of the system in Equations (9) and (10) is a completely disordered school, which slows down such that the average speed decreases exponentially. We know that such a solution is unstable, as mentioned in the first part of this paper. However, adding any kinds of noise to the directional headings will effectively decrease the coherence of the school of particles, thus decreasing the order parameter r from Equation (15). From Equation (29) we see that a lower value of r will result in the school of particles slowing down at a faster rate. We therefore require the speed to be driven in order for the system to have some interesting dynamics and the schools to move. For now, we assume that the acceleration is a constant $\nu > 0$, so that Equation (19) becomes

$$\dot{v}_k = \alpha \bar{v} r \cos(\psi - \phi_k) - \alpha v_k + \nu. \tag{33}$$

It is furthermore clear that if the system is completely disordered (giving $r = 0$), we find that the average speed tends towards $\bar{v}_\infty = \nu/\alpha$. This limiting value could be achieved in the absence of noise, but the solution with $r = 0$ is unstable. We therefore see that with noise, the order parameter r tends away from zero and reaches a non-zero limiting value $r_\infty < 1$.

If, on the other hand, we have $0 < r_\infty < 1$ induced by noise, the equilibrium velocity will become

$$\bar{v}_\infty = \frac{\nu}{\alpha} \frac{1}{1 - r_\infty^2}, \tag{34}$$

which simulations confirm.

4.2.1 Deterministic noise in $\dot{\phi}_k$

We now add random frequencies ω_k to the right hand side of Equation (20) divided by v_k (we assume that all $v_k > 0$). The ω_k are random i.i.d. variables whose probability density is a zero-mean Lorentzian distribution of width $\gamma = 0.5$. Related results can be found in the review [8] and in [9]. The resulting equation for the phase of the fish velocity is

$$\dot{\phi}_k = \alpha \frac{\bar{v}}{v_k} r \sin(\psi - \phi_k) + \omega_k, \tag{35}$$

which is very similar to Kuramoto's model for coupled oscillators, with an effective coupling constant $\alpha \frac{\bar{v}}{v_k}$. We note that the effective coupling constant
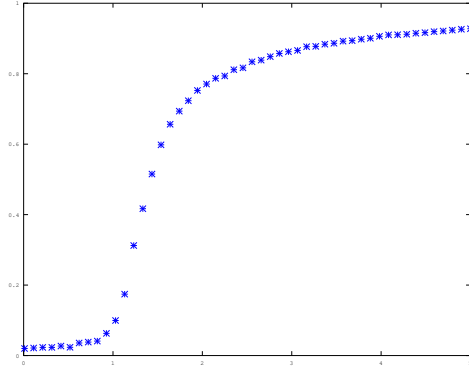
Figure 2: Value of r_∞ as a function of α see Equations (35) and (33). The ω's were drawn from a Lorentzian distribution of width $\gamma = 0.5$. Here, $\nu = 0.2$ and the number of particles is $N = 2000$.

varies until the speeds reach an equilibrium, which the system achieves for both types of angular noise below.

As in Kuramoto's model, in the limit of infinitely many fish we obtain a bifurcation of the equilibrium value of the order parameter r_∞, as the value of α is varied beyond a certain threshold, as shown in Figure 2. For finitely many fish, the transition is not sharp but adding more fish in our simulations we approach the sharp phase transition.

As for the stability of the system, we expect it to show richness in behavior, similar to the Kuramoto model. In [10], it was shown that the incoherent state of the Kuramoto model is unstable above threshold, but neutrally stable below the threshold. This means that the decay of modes to incoherence is similar to Landau damping in plasmas [11] and that there might be rich dynamics with $r = 0$. See [8] and references cited therein. The partially synchronized state in the Kuramoto model is also known to be neutrally stable [9]. We should expect to see a similar behavior in the system above.

Below, we discuss the distribution of the directional angles in the partially synchronized state, and compare to that of the Kuramoto model, shown in Figure 5.

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4.2.2 Brownian noise in $\dot{\phi}_k$

Now, assuming that $v_k \neq 0$ we divide through Equation (20) and add N-dimensional Brownian noise, B_t:

$$\dot{\phi}_k = \alpha \frac{\bar{v}}{v_k} r \sin(\psi - \phi_k) + \dot{B}_t^{(k)}, \tag{36}$$

which we rewrite as

$$\mathrm{d}\phi_k = \left(\alpha \frac{\bar{v}}{v_k} r \sin(\psi - \phi_k) \right) \mathrm{d}t + \mathrm{d}B_t^{(k)}. \tag{37}$$

We use methods from [12] to solve Equation (37).

We note that we expect the behavior of the order parameter to be determined by the variance of the noise, σ^2, similar to the model in [2]. Furthermore, we expect the system to reach a statistically stationary state, where r_∞ and $\bar{v}_\infty$ can be defined. This is indeed the case, which is shown in Figure 3. By varying σ, we obtain a a bifurcation in the order parameter r_∞, which can be seen in Figure 4. This is discussed in more detail below.

4.3 Behavior of the perturbed system

When we add noise to the directional angles as in Equation (36), the dynamics of the order parameter r in Equation (26) can be shown to change according to

$$\dot{r} = \alpha \bar{v} r \frac{1}{N} \sum_{k=1}^{N} \frac{1}{v_k} \sin^2(\psi - \phi_k) + \frac{1}{N} \sum_{k=1}^{N} \dot{B}_t^{(k)} \sin(\psi - \phi_k), \tag{38}$$

and Equation (27) becomes

$$r\dot{\psi} = \alpha \bar{v} r \frac{1}{N} \sum_{k=1}^{N} \frac{1}{v_k} \frac{1}{2} \sin\left(2(\psi - \phi_k)\right) + \frac{1}{N} \sum_{k=1}^{N} \dot{B}_t^{(k)} \cos(\psi - \phi_k). \tag{39}$$

We do not expect the dynamics of the order parameter r of the perturbed system to differ drastically from that of the unperturbed one. Indeed, it turns out that the system reaches a statistically stable state, where r_∞ is defined. From the above equations it is clear that σ is the determining factor for the value of r_∞. We therefore compute that value for a range of values for

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σ, shown in Figure 4. Similar to [2], we obtain a bifurcation of the order parameter when the variance of the noise exceeds a certain threshold. We see that the value of r_∞ reaches 1 when $\sigma \to 0$. At high variance, we expect the system to not be able to synchronize, as is the case, with r fluctuating close to zero. However, below a certain threshold σ_c, the system is able to synchronize, and the transition is very sharp, exhibiting a bifurcation between phases of the system. With $0 < \sigma < \sigma_c$ we have $0 < r_\infty < 1$ and so the system is partially synchronized.

It is worth pointing out that the it is not completely clear whether the bifurcation in Figure 4 is continuous or not, i.e. whether the bifurcation is supercritical or subcritical. The transition does seem to be supercritical, but theoretical analysis is required to fully determine the nature of the bifurcation. It is quite possible that the nature of the bifurcation depends on the distribution of the natural frequencies [13]. However, for the Czirok-Viksek model [2, 14] it has been speculated that the nature of the phase transition is determined by how the noise is entered into the angular velocities. In our case, we add a so called "angular noise", which perturbs a perfectly calculated average direction, resulting in a seemingly continuous transition, in accordance with the Czirok-Vicsek model. On the other hand, in [15, 16], it is argued that adding "vectorial noise" to the angular velocities, i.e. by adding noise terms to each of the interactions, results in a subcritical onset of collective motion. But it is important to note that vectorial noise is very different from the original Czirok-Vicsek model, and it is not surprising that the system turns out to be more sensitive to the amplitude of the noise. Indeed, the results of [2] were confirmed in [17].

Finally, we note the difference between the distribution of the directional angles at equilibrium, see Figure 5, with noise added as in Equation (35), and as in Equation (36), even though the value of the order parameter r_∞ is similar. In the latter, the histogram resembles that of a Gaussian distribution. In the former, we obtain a more heavy tail of the distribution, which is a result of the Lorentzian noise. However, it is interesting to note that the former case differs from that of the Kuramoto model (Figure 5(c)), although they should be very similar at equilibrium. Further research will hopefully shed light on this discrepancy. The distributions in Figure 5 were obtained with $N = 32000$ in order to obtain smoother distributions, but the resulting values of the order parameters do not depend on the value of N.
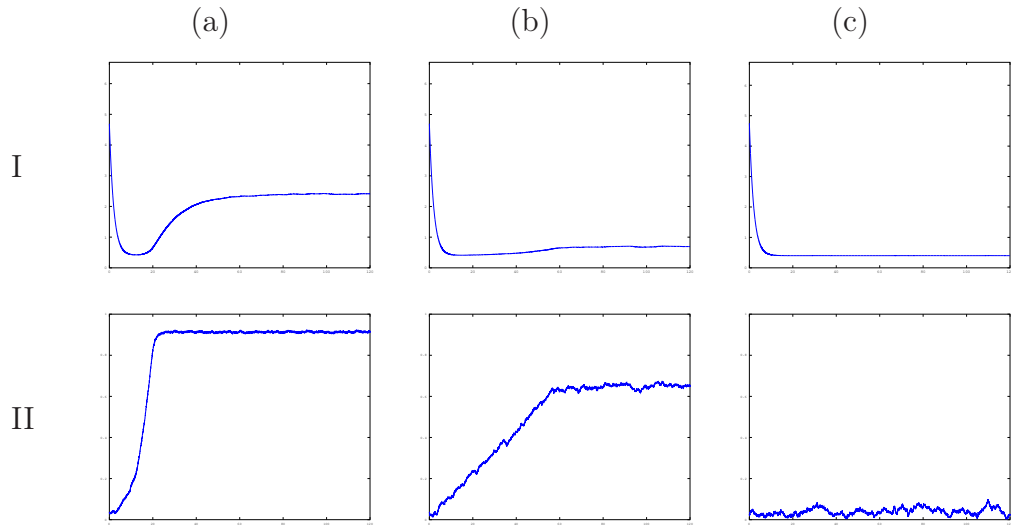
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Figure 3: Evolution of the average speed (I) and of the order parameter r (II) according to Equations (36) and (33). In all cases the inertia is $\alpha = 0.5$, acceleration is $\nu = 0.2$, and the number of particles is $N = 2000$. The initial distribution of the directional angles was uniform in all cases. The plots differ only in the value of σ: (a) $\sigma = 0.40$, (b) $\sigma = 0.68$, (c) $\sigma = 0.82$. In all cases we see that r_∞ is defined, and r fluctuates around that value. A bifurcation in r_∞ occurs as the value of σ is varied, see Figure 4.
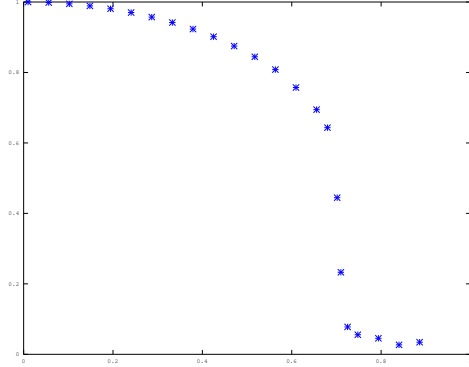
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Figure 4: Value of r_∞ as a function of σ, where σ^2 is the variance of the noise in the directional angles, see Equations (36) and (33). In all cases we have $\alpha = 0.5$, $\nu = 0.2$ and number of particles $N = 2000$. We obtain a bifurcation at $\sigma_c \approx 0.72$.
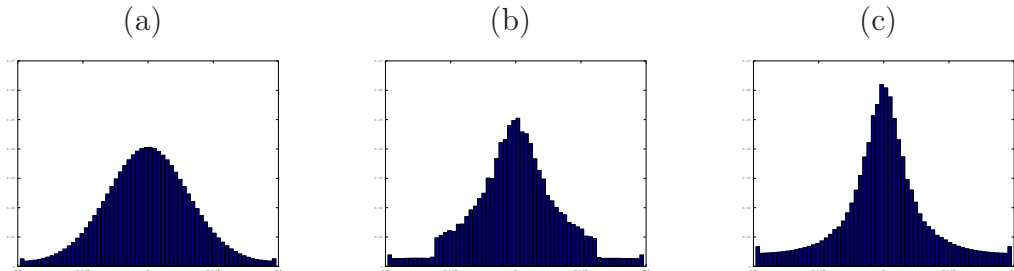

(a)   (b)   (c)

Figure 5: Histogram of the directional angles $\{\phi_k\}_{k=1}^N$ at equilibrium after a long initial transient, averaged over the last 5000 iterations. The figures correspond to (a) Equation (36) with $\alpha = 0.5$, $\nu = 0.2$, and $\sigma = 0.7$, (b) Equation (35) with $\alpha = 1.5$, $\nu = 0.2$ and (c) the Kuramoto model [6] with $K = 1.5$. In cases (b) and (c), the intrinsic turning rates $\{\omega_k\}_{k=1}^N$ were drawn from a Cauchy distribution with $\gamma = 0.5$. In all cases the number of particles is $N = 32000$. The resulting order parameters were (a) $r_\infty \simeq 0.58$ and $\bar{v}_\infty \simeq 0.60$, (b) $r_\infty \simeq 0.59$ and $\bar{v}_\infty \simeq 0.21$, and (c) $r_\infty \simeq 0.58$.

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5   Discussion

We have shown that the system in [1] tends towards a stable traveling solution ($\bar{v}_\infty$ constant) with perfect synchronization $r_\infty = 1$. Perhaps more biologically accurate, we perturb the directional angles, and note that for the system not to come to a halt we have to accelerate the particles. We find that the system reaches an equilibrium and derived the resulting average speed as a function of the model parameters and the order parameters, see Equation (34).

By inspecting Equation (38) we expect the value of σ_c to depend on α, and possibly also ν. We would like to conduct a full investigation of the parameter space, and obtain a relationship between σ_c and α. We speculated that the nature of the bifurcation was supercritical, but require further analysis to determine the true nature.

5.1   Synchronized stationary solutions

We plan to investigate the possible structure of the disordered state in the case of the deterministic perturbation above. It is possible that there exists a sequence of partially synchronized schools of particles, reminiscent of Kuramoto's partially synchronized solutions based on the symmetries of the stationary solutions in [1]. The complexity of these solutions can be studied using order parameters for both the direction headings and the speeds. Analogous to the perturbations of the stationary solutions of a finite number of particles, in [4], the infinite sequence of Kuramoto stationary solutions, if it exists, could perturb to metastable solutions, consisting of complex circling schools. Eventually, all of these schools would tend to the migratory Kuromoto solution consisting of one complex partially synchronized school. It is not clear if this complex sequence survives the perturbations.

5.2   Local interactions

Finally, we note that it would be interesting to explore the behavior of the same model equations, but with local interactions. A broad literature is available on such models, and it is certainly biologically more accurate to limit the visual and sensual range of the particles [18, 19, 20, 21, 22]. We might start with only the zone of orientation. However, we would also be interested in investigating the behavior of the system with other zones of

interaction implemented, e.g. a zone of repulsion. We would like to introduce the zone of repulsion in order to obtain a model which was investigated in [4]. The spatial structure of the models mentioned above has not been dealt with, and it would be very interesting to see whether we can obtain the solutions from [4], and investigate their stability.

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