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A MODEL OF SPACETIME EMERGENCE IN THE EARLY UNIVERSE

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Abstract

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This thesis proposes and develops much of the groundwork for a model of emergent physics, posited to describe the initial condition and early evolution of a universe. Two different considerations motivate the model. First, the spacetime manifold underlying general relativity and quantum theory is a complex object with much structure, but its origin is unexplained by the standard picture. Second, it is argued, the usual assumption of the preexistence of this manifold leads to possibly intractable theoretical (not observational) difficulties with the usual cosmological inflation idea. Consistent with both considerations, the assumption of a manifold that precedes a big bang cosmology is dropped; instead, a spacetime manifold with metric, Lorentz symmetry, and manifestation of standard quantum fields propagating on the spacetime all emerge in the model from a simpler, statistically scale invariant underlying structure, driven by an inflation-like process.

The basic structural components of the model are a stochastic (not quantum or classical) scalar field on a general metric space, plus a collection of quantum fields that supply the matter content once spacetime begins to emerge. Importantly, standard quantum fields cannot be defined on the pre-emergent space; this is addressed by assuming quantum theory exists a priori, and then postulating that quantum fields can begin to manifest once an approximate spacetime has emerged. Atypical fluctuations in the scalar field transiently break the statistical scale invariance in a localized region of the general metric space; a very small subset have field configurations of approximate spacetimes which can potentially evolve into an initial condition for a universe. Spacetime structure and geometry then arise from the dependence of propagation speeds and spatial/temporal distances on variations in the scalar field; these variations are seeded by the matter (quantum) fields.

The thesis develops the mathematics of the basic components of the model in some detail, outlines a mechanism whereby scale invariance is broken and dimensionality

is fixed, and develops processes and scenarios wherein variations in the scalar field can lead to spacetime geometry in an inflation-like process. The resulting picture of spacetime is then compared with that of general relativity.

To Anne, Valerie and Isaac,

whose immeasurable patience, sacrifice and support

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Chapter 1

Introduction

1.1 Aims and overview

This thesis will attempt to lay the groundwork for a model of emergent physics, in which a spacetime manifold with metric, as well as standard fields propagating on it, emerge from a simple and general underlying structure. This emergence is viewed both in terms of emergent laws of physics governing a restricted range of length and energy scales, and also as a process of emergence, through an inflation-like phase in the early universe. Indeed, the predictive success and theoretical challenges of inflation form a major motivation for the ideas that will be explored.

The Big Bang cosmology plus inflation, when married with general relativity (GR) and the Standard Model of particle physics, form an extremely successful and well-tested description of laboratory physics as well as the physics of the observable universe and its evolution from very early times. However, there are many open questions in pushing beyond these well-described regimes. One of particular interest here is that of cosmological initial conditions, which as described below appear extremely 'special' even with the inclusion of inflation. This thesis will approach the problem of unlikely initial conditions by treating the observable universe as a part of a larger system, but one that arises via an emergence process that occurs during an inflation-like era.

Because this model will require a significant departure from the typical conceptual and mathematical toolkit of high-energy physics and cosmology, this thesis will pay particular attention to motivation of the ideas, both from perceived deficiencies in current models, as well as the mathematical and physical self-consistency and simplicity

of the proposed alternatives. This introduction gives, in the next section, a review of the inflationary model of early universe cosmology, and also points out open problems in that view. This motivates the emergence scenario, which the next section sketches in broad form. Following that is an outline of the content of the thesis and the basic ideas it puts forth.

1.2 Inflation and its difficulties

Cosmological inflation was invented in the early 1980s to provide a natural explanation for three general characteristics of the observable universe: its homogeneity and isotropy at scales greater than several hundred Mpc, given that the gravitational instability of a Friedmann-Lamaître universe should lead to growing lumpiness of matter at all scales [57]; its spatial flatness; and an apparent absence of heavy monopoles that grand unified theories predict [19]. Guth produced the first complete physical model [26], followed by proposals by Linde [46, 47] and Albrecht and Steinhardt [5] which fixed problems with Guth's original proposal. Computations of expected fluctuations cosmic microwave background were soon computed for the improved model [63, 27], and were extended to $\Omega = 1$ cold dark matter dominated universes with scale invariant adiabatic initial conditions; see Refs. [11, 12]. Not only is the inflation idea elegant and explanatory, it is also in excellent agreement with observations thus far. [44].

As this very brief history of inflation suggests, its observationally verified predictions largely originate from work done in the 1981-1984 time frame. It is these predictions one usually considers when discussing confirmations of inflation: the successes are very real, but as theoretical work has continued it has become apparent the inflation picture is deeper and more cloudy than what the preceding suggests, particularly in its theoretical implications and limitations.

The essential motivation behind cosmological inflation is to give a 'natural' explanation for the origin of what appears to be an extremely unnatural state of the universe today, which state was preceded by a series of even more unnatural states that led to it[1]. It is usual to characterize the improbable condition today in terms of two problems that demand explanation [54]. The 'horizon problem,' or homogeneous and isotropic spatial distribution of energy at scales greater than a few hundred megaparsecs,

[1] **Universe** will always mean the observable universe unless otherwise noted.

is characterized by a distribution of energy density $\rho(\mathbf{x})$, smoothly distributed with a fractional variation $\delta\rho/\rho \lesssim 10^{-4}$ over at least 10^{84} causally disconnected regions. The 'flatness' (or initial velocities) problem arises from the extreme fine tuning of the initial velocities, such that the huge negative gravitational energy of all matter is somehow matched, within one part in 10^{-56}, by a huge positive initial kinetic energy; with less tuning the universe would collapse or become dilute too soon to be consistent with observations.

Inflation addresses the horizon and flatness problems naturally with the help of a (or more than one) specially introduced scalar quantum field, or inflaton, that slowly 'rolls' down a gently sloped plateau of its potential. Consistent with general relativity, a properly crafted inflaton potential can yield an exponential expansion of space by a factor of e^{75} or more, flattening and homogenizing the energy density to an extent needed to obtain the inferred unnatural initial state [54]. Inflation continues until the inflaton reaches a steeply descending region where the inflaton gains kinetic energy and drops into a minimum of its potential; it ends with the inflaton oscillating rapidly in a reheating phase, whereby it seeds the universe with all of its radiation and matter. This elegant idea, however, has an unintended side effect: due to the quantum mechanical nature of the inflaton, potentials that are adequate to obtain 75 or more e-folds of inflation often imply that inflation will not end everywhere at once. We will return to this issue in Subsect. 1.2.2.

1.2.1 Penrose's entropy argument

Inflation is intended to start from a generic initial state of randomly fluctuating energy density and obtain a very flat and homogeneous universe like what we see. In 1989 Penrose [58] offered a qualitative argument based on a statistical mechanical analysis that strongly indicates inflation was a very improbable path through phase space to the present state of the universe, compared to many other paths which could obtain the same present day state without inflation. His argument depends on Hamiltonian evolution in general relativity.

General relativity with matter is a Hamiltonian system (*e.g.*, Wald [71]). The one-particle phase volume is the product of the differential spatial volume and the

differential momentum 'volume' element with lowered indices [54],

$$d^3x\, d^3p = dx^1 dx^2 dx^3 dp_1 dp_2 dp_3\,. \tag{1.1}$$

This is invariant under general coordinate transformations. According to the Liouville theorem, the total phase space volume of a Hamiltonian system (here, of the universe) is invariant under canonical transformations. This is straightforward to prove in flat spacetime, so consider a curved spacetime. At each point on a particle trajectory, it is always possible (by definition of a manifold) to choose a locally inertial coordinate system, so that the Liouville theorem in flat spacetime applies there. Since the volume element (1.1) does not depend on the coordinate system, the phase space volume will be constant along the particle trajectory. Consequently, the Liouville theorem also holds in generally relativistic spacetimes.

Penrose's argument can be briefly summarized as follows. Since (according to the Liouville theorem) the canonical measure is invariant under Hamiltonian evolution, it is easiest to define the measure at late times; the answer should not depend on when the measure is applied. Hence it is sufficient to estimate the phase space volume at late times. As a reference point, the entropy of the CMB in the observable universe today is approximately[2]

$$S_\gamma = \log\Gamma \sim \left(T_\gamma H_0^{-1}\right)^3 \sim 10^{88}\,, \tag{1.2}$$

where H_0^{-1} is the Hubble radius today and the CMB temperature $T_\gamma = 2.7K$.

However, entropy of gravitational clumping dominates the CMB entropy. Assume 10^{80} baryons in the universe, all contained in galaxies of 10^{11} solar masses $M_\odot$, with each galaxy containing a central black hole of mass $10^6 M_\odot$. The Bekenstein-Hawking entropy formula for a black hole of mass M and horizon surface area $A = 16\pi G^2 M^2$ is $S_{\rm bh} = A/4G$ (G is Newton's constant). This gives a total entropy $S_{\rm now} \sim 10^{101}$. Penrose estimated an entropy bound $S_{\rm max} \sim 10^{123}$ for the far future under the assumption of a 'big crunch,' a closed universe scenario where all 10^{80} baryons coalesce into a single black hole. A more plausible scenario is an open universe, but this has an unbounded phase space. Hence, consistent with Penrose's bound, assume $S_{\rm max} \sim 10^{120}$.

Since the phase space volume is constant under Hamiltonian evolution, a probability measure can be readily constructed from phase space volume fractions. If we

[2]We use 'natural' units in which $\hbar = c = k_B = 1$, where $\hbar$ is Planck's constant divided by 2π, c is the speed of light and k_B is Boltzmann's constant.

denote by $\Gamma_{\rm now}$ the number of states (or subvolume of $\Gamma_{\rm max}$) that are consistent with the condition $U_{\rm now}$ of the observable universe today, then the probability of randomly choosing a point in $\Gamma_{\rm max}$ that is consistent with the state of the universe today is simply

$$P(U_{\rm now}) = \frac{e^{S_{\rm now}}}{e^{S_{\rm max}}} \sim \frac{e^{10^{101}}}{e^{10^{120}}} \approx e^{-10^{120}} .$$

Hence the initial conditions at the Big Bang would need to be tuned to one part in $e^{10^{120}}$ to obtain a state as low entropy as our present universe, assuming Hamiltonian evolution. Penrose estimates an initial condition constrained to one part in $e^{10^{60}}$ would generously supply the necessary conditions to create our solar system and all the life on earth from random assembly by particle collisions. Thus, anthropic arguments are inadequate to explain one part in $e^{10^{120}}/e^{10^{60}} \simeq e^{10^{120}}$: Why would a vast, evolving assembly like the observable universe be needed for us to exist?

Inflation was invented to give a natural explanation for such improbable initial conditions at the Big Bang. In addressing the horizon problem, the inflation scenario assumes thermalization of inhomogeneities occurred through interactions prior to inflation; then inflation pushed the thermalized region into 10^{84} causally disconnected regions. But thermalization serves to obtain a more probable, higher entropy state from an inhomogeneous, lower entropy state: the pre-thermalization state must have been even more special than the thermalized pre-inflation state. Gibbons and Turok [21] quantitatively constructed a canonical measure to estimate the probability of N e-folds of inflation in single field, slow-roll inflation models and concluded that $P(N \text{ e-folds}) \sim e^{-3N}$, or e^{-180} for 60 e-folds. There are many more ways to reach the current state of the universe without going through the extraordinarily rare initial state where inflation must have begun, compared to going through inflation. More recently, Steinhardt [64] has presented a Liouville argument to similar effect.

1.2.2 Predictivity problem of eternal inflation

Steinhardt has also argued [64] that eternal inflation compromises the predictivity of inflation models in a crucial sense: it either implies the universe we observe is extremely improbable or it requires crafting a probability measure specifically to allow concluding our universe is typical. This is a different difficulty with inflation than the question of whether or not inflation should occur at all. The following summary gives Steinhardt's argument.

Steinhardt opines that he and others made a significant judgement error in the early days of inflation model building by assuming quantum mechanics would have only perturbative effects on an essentially classical scalar field picture. What was found instead is that quantum mechanics and inflation make, in his words,"a toxic mix." In all inflation models quantum mechanics introduces stochastic perturbations into the otherwise predictable evolution of the scalar field toward the minimum of its potential where inflation ends [54]; *e.g.*, in false vacuum inflation models the tunneling rates are probabilistic. Thus, inflation will not gracefully end everywhere at once. Specifically, 'rogue regions' at the tail of the distribution will, with low but finite probability, delay their exit from inflation. Due to their continued inflationary expansion, such 'procrastinating' regions grow exponentially in overall volume even if their volume fraction decreases. Because inflating regions are never completely killed off, inflation is generically eternal to the future once it begins [25, 1].

Moreover, because inflation is future-eternal, the uncertainties of quantum mechanics ensure different pockets do not look the same. As Guth puts it [24], "Anything that can happen will happen, and it will happen an infinite number of times." This is one way to view the so-called measure problem: when everything is possible and occurs infinitely many times, how do we measure and compare probabilities?

The simplest and most natural measure is the volume measure, where probabilities are given by relative volumes as in Penrose's entropy argument. However, it can be shown [68] that at any given time the total volume occupied by pocket universes much younger than ours is exponentially greater than the total volume occupied by universes as old as ours — universes like ours should be exponentially suppressed according to the volume measure. Anthropic arguments do not help: our universe is more homogeneous, spatially flat, and has a more scale invariant CMB than life requires; much younger (and hence far more typical) universes than ours should also be hospitable to intelligent observers.

Measures other than the volume measure are certainly possible. Such a measure should predict our universe is typical within an eternally inflating Universe. The measure must not appear contrived to avoid disagreement with observations, so the choice of measure should be justified by an underlying physical principle. Steinhardt posits that a 'measure principle' is not likely to be unique; hence, the eternal inflation causes everything to happen that can happen, but all the statistical predictions of the

theory ultimately originate from the (subjective) choice of measure. And if we take that theoretic approach, he argues, we should compare the predictive power of the chosen measure on an eternally inflating Universe with predictions of other cosmological models that rely on a different preferred measure, and this should include cosmological models where there is no inflation at all.

One could claim that, even though eternal inflation is an unintended side effect of the inflation idea, we can use it to side-step the Liouville and entropy arguments. Specifically, once eternal inflation begins, essentially all universes will be pocket universes with inflation in their past according to the volume measure, rendering the Liouville and entropy arguments irrelevant. While this may be true, it trivially requires assuming *a priori* the inflation picture is correct; otherwise eternal inflation could not exist to counterbalance the entropy and Liouville arguments. It does not help decide whether inflation actually occurred. Moreover, the argument is inconsistent — the volume measure on which it depends already predicts universes like ours are exponentially suppressed.

There are obvious difficulties with testing eternal inflation scenarios, given that we have only one universe to observe and anything outside it lies beyond our Hubble horizon. Observational tests proposed thus far are restricted to analyzing the CMB for possible evidence of collisions between pocket universes in the distant past. Such collisions can occur in false vacuum eternal inflation models as follows [3, 2]. In a rogue region the inflaton is trapped in a local minimum of the potential, creating a de Sitter-like phase that maintains inflation. A pocket universe forms when, in a localized region, the inflaton tunnels out of the false vacuum state and then rolls to the state of true vacuum (or at least a lower energy local minimum), ending inflation. This 'bubble nucleation' is a phase transition, so a domain wall separates the pocket universe from the surrounding de Sitter vacuum. Consistent with the scalar field equation of motion the domain wall accelerates outward and the bubble expands into the de Sitter vacuum effectively at the speed of light; the bubble interior, meanwhile, evolves as a Friedmann-Lamaître cosmology like that of our visible universe. Since bubble nucleation is a probabilistic event, there is a finite probability that two (or more) bubbles will nucleate near enough that the accelerating walls of the two bubbles outrun the exponential expansion of space between them, and the bubble walls collide, potentially with observable effects. [4]. A recent study [20] attempted to detect characteristic inhomogeneous signatures of bubble

collisions in the Wilkinson Microwave Anisotropy Probe (WMAP) seven year data, but found no statistically significant evidence for them.

1.3 Assessment

The theoretical difficulty where we now find ourselves can be summarized as follows. We look out at the observable universe and find it is homogeneous, isotropic and spatially flat at large scales. Moreover, the universe is expanding; this and the homogeneity and isotropy mean matter obeys the Hubble law at large scales. Extrapolating backward in time (and also considering other variables, for example the CMB spectrum and the relative densities of hydrogen, helium and other light elements to name just two) we infer the universe began with a Big Bang about 13.7 billion years ago and has expanded since then.

However, Penrose's entropy argument shows the universe we observe is extremely unlikely, having probability of order $\exp(-10^{101})$ or $\exp(-10^{123})$ depending on the choice of upper entropy bound. Since general relativity has a Hamiltonian formulation, then Liouville's theorem applies; this means the evolution is time reversible. Given the improbable current state, we can consider the set of possible trajectories through phase space that could lead to it, and thus infer the universe almost certainly was even more homogeneous and isotropic when it was much younger — much more so, in fact, because otherwise inhomogeneities would have caused much more gravitationally induced clumping of matter than what we observe today. Our extremely unlikely universe practically demands an explanation of how it came about.

The clever idea of cosmological inflation offers a compelling, even 'natural,' explanation for the extreme homogeneity and isotropy we infer must have existed at the time of the Big Bang. It also quantitatively accounts for the observed structure of the CMB. Unfortunately, inflation introduces all the substantial theoretic difficulties outlined in the previous section, among others. These problems make conventional inflation scenarios much less convincing from a theoretical standpoint.

Despite substantial theoretic difficulties, the fact that inflation allows substantive predictions that agree so well with observations strongly suggests there is something correct about the inflation idea, at least in the simple picture where inflation occurs and then ends gracefully. Just one case in point is the spectral index n_s. Linde pointed

out [48] in 2005 that it is very difficult to construct inflation models with an exactly flat spectrum of metric perturbations; most inflation models predicted $n_s = 1 \pm 0.2$ but not $n_s = 1$. Linde remarked it would be interesting to see if observations agreed with the prediction $n_s \neq 1$, which was unknown at the time because observations were consistent with $n_s = 1$. Recent WMAP results [44] find $n_s = 0.968 \pm 0.012$ at the 68 percent confidence level, effectively answering Linde's question. This kind of agreement with substantive predictions argues for considering new early universe models that incorporate some key aspects of inflation but are not subject to its main difficulties. This may mean not relying on general relativity, or even assuming the universe at the time of the Big Bang was a Hamiltonian system, for example.

The crux of the entropy and Liouville arguments is the time-asymmetric evolution of the universe with time-symmetric laws. Physical laws are apparently time symmetric, including quantum mechanics, general relativity, classical mechanics and electromagnetism [22]. Thus, if we reverse the flow of time and evolve the system with time-symmetric laws, we expect from the second law of thermodynamics evolution to a more generic, higher entropy condition. On the other hand, there is widespread acceptance that the time asymmetry or 'arrow of time' embodied in the second law of thermodynamics ultimately has a cosmological origin: low entropy conditions were preceded by even lower entropy conditions, reaching a minimum at the time of the Big Bang (or even earlier for inflation scenarios). This creates the severe problem of explaining the conflict between expectations of time-symmetric evolution and our inferences from observations that a low entropy Big Bang was in our past.

We can thus frame the question as: How can we obtain the extremely improbable initial conditions that are necessary for time symmetric laws to generate the inferred time-asymmetric evolution? However, we have not yet defined what we mean by a law. This thesis will adopt the following working definition: A physical law will mean a dynamical rule that acts consistently within a specified domain of applicability and for all times; its action may be deterministic or statistical. The domain of applicability necessarily includes spacetime characteristics like curvature and smoothness. These are usually implicit in the metric, but one must also consider the distance scale. According to conventional wisdom, spacetime is no longer well defined at the Planck scale, and consequently neither are laws that require a smooth, consistent spacetime. Conventional wisdom does not posit an origin of a special Planck scale; it just assumes

the scale exists and a future nonperturbative theory of quantum gravity will allow us to deal with it.

Spacetime plays a critical role in physical laws. A physical law will not act time symmetrically if the spacetime on which it is defined evolves time-asymmetrically and that evolution is independent of the law. However, the nature of spacetime near the time of the Big Bang is completely opaque to us; it is possible no well defined spacetime existed then at any scale. We can assume general relativity (or a future theory of quantum gravity which obtains general relativity as a low energy limit) will provide a well defined spacetime on which our time-symmetric theories can live, but general relativity and current contenders for quantum gravity all assume prior manifold structure. Nonetheless, other than convenience and theoretical prejudice, neither of which confers any obligation on Nature, we have no real evidence that even spacetime manifold structure existed at the time of the Big Bang.

1.3.1 Motivation for an emergence picture

The main content of Sect. 1.2 can be summarized thusly. Inflation is a clever and explanatory idea that makes generic predictions in excellent agreement with observations, but it does not fulfill its original mandate. At least in its usual form, it does not offer a natural explanation of the extremely improbable state of the observable universe because it depends on preceding, even more improbable initial conditions. Since inflation offers no insight into the origin of the very special initial conditions it requires, its viability depends on extreme fine tuning or predictivity-destroying eternal inflation scenarios, undermining the naturalness of the inflation idea. Ultimately, the problem originates from trying to obtain the improbable initial condition with time symmetric laws.

The problem of initial conditions arguably has two distinct aspects:

- an extremely improbable initial state; and
- the origin and properties of spacetime, quantum fields, and the particular forms of matter interactions we observe.

There is general agreement that the first issue is a genuine, significant problem. Whether one expects an explanation of the second aspect or is content to accept such elements

as *a priori* attributes of Nature is largely a matter of philosophy rather than physics. The perspective adopted in this thesis is that they require explanation.

Addressing the first issue with time symmetric laws acting within a closed system appears unworkable: fluctuations from typical states to improbable states are exponentially suppressed as the entropy argument (Subsect. 1.2.1) shows. Proposals by Carroll and Chen [17] and Greene et al. [23] both regard the observable universe as a subsystem of something larger. For example in Ref. [23] a boundary condition is externally imposed, but if the enclosing system evolves with time symmetric laws in a global phase space it is unclear that the problem is solved; the origin and likelihood of the special boundary condition beg explanation.

It is perhaps remarkable that most proposals for early universe scenarios build on the same physical objects, mathematical structures, and formalisms that are so successful in describing the low-energy physics of the present day universe. Presumably, if general relativity and low energy formulations of quantum mechanics are limiting cases of an ultraviolet-complete theory, then that theory must contain many of the mathematical elements of QM plus GR like Hilbert spaces, manifolds, and spacetime symmetries.

From this perspective, when conceptual difficulties like the entropy and Liouville arguments confront successful ideas like inflation, the logical response is to retain the successful ideas and propose ways to rectify their weaknesses. The more natural, predictive and 'battle tested' the ideas are, the more sense it makes to pursue this approach. It has been the general response to difficulties with inflation thus far.

When developing theories of the very early universe, no direct observational guidance exists so it is necessary to rely on extrapolations from known physics, consistency arguments, and aesthetic considerations, none of which can provide reliable guidance the way observations can. Notably, there is a widespread belief that at the small distance scales that were dominant near the time of the Big Bang, at least some of our theories and usual conceptions of smooth spacetime are untrustworthy. So one can reasonably wonder why many early universe scenarios accept such theories and conceptions as a starting point. Moreover, we have no reliable understanding of the future quantum gravity theory, only ideas that have been worked out with varying degrees of completeness and plausibility. Thus, when conceptual problems arise, we do not have the same mandate for patching up existing early universe ideas as we do for theories

with direct observational support.

In this thesis, the foregoing will be a rationale for considering new scenarios which rest on some different assumptions than do existing early universe ideas. Naturally, interesting new scenarios must skirt existing conceptual problems without introducing even worse ones. Of specific interest will be scenarios where general relativity can be considered an emergent theory, in contradistinction to a low-energy limit of an ultraviolet-complete theory.

Emergence scenarios offer an interesting way to overcome conceptual difficulties because they do not imply explicit connections between the principles and mathematical structures of the emergent and more fundamental theories. If former conceptual difficulties become artifacts of an emergent picture at low energy, they lose their fundamental quality and may even disappear altogether in the regime of interest.

An emergent theory in the sense intended here describes phenomena statistically, analogous to the way pressure and the ideal gas law emerge in a statistical description of a gas of microscopic molecules at sufficiently high temperature. Namely, the more fundamental theory must contain some domain or regime where new phenomena appear statistically after coarse-graining certain degrees of freedom. We can then say the new theory emerges at a characteristic scale given by the scale of coarse graining. Phenomena which are not manifest in the fundamental theory will statistically emerge above the characteristic distance scale, and they will be described by the emergent theory. The coarse-graining scale is not arbitrary — it is the specific scale at which the emergent phenomena first become statistically discernible at some chosen significance level.

1.4 Overview of the Emergence Picture

The global system from which the observable universe is presumed to emerge will be taken to be a stochastic field φ, also endowed with dynamical properties, which lives on a simply connected general metric space $\mathfrak{M}$. The φ field is neither quantum nor classical. Almost everywhere it statistically manifests symmetry in the greatest possible sense: scale invariance, homogeneity, isotropy[3], and without spatial dimension; it

[3]Even in a general metric space a tangent space can be defined, albeit without an associated vector space.

altogether lacks notions of physical time, space, matter and even energy, but nonetheless it can be considered a kind of thermal system in that φ is self interacting and its amplitude obeys a Brownian probability distribution function along every curve in $\mathfrak{M}$. Accordingly, φ field modes k are unbounded, $0 < k < \infty$.

The observable universe will be taken to be a subsystem of this global system, a dynamical phase of finite extent on $\mathfrak{M}$ which consists of a collection of interacting quantum fields living on an emergent spacetime manifold $\mathscr{M}$. The only available origin of a dynamical phase in this picture is a large, extraordinarily improbable fluctuation of the stochastic phase which effects a local phase transition on a subset $\mathfrak{U}$ of $\mathfrak{M}$. Because the new phase has finite extent and has no sharply defined boundary, it can be described by a finite range of φ modes, $[k_{\min}, k_{\max}]$. The ranges $(0, k_{\min})$ and $(k_{\max}, \infty)$ remain in the stochastic phase, and, moreover, modes in $[k_{\min}, k_{\max}]$ introduce perturbative stochastic effects into the dynamical phase. Thus, the two phases coexist within $\mathfrak{U}$.

An elementary particle in flat spacetime is an irreducible representation of the group $P \times G$, where P is the Poincaré group and G is the internal symmetry group [8]. For the Standard Model $G = U(1) \times SU(2) \times SU(3)$. However, Poincaré symmetry is not a symmetry of $\mathfrak{M}$ because $\mathfrak{M}$ has no manifold structure and therefore cannot directly support a spacetime. Hence, elementary particles cannot exist in the scenario unless the fluctuation simultaneously brings about two conditions, which are also conditions for subsequent cosmogenesis:

- A φ field configuration transiently obtains an 'approximate' spacetime with approximate Poincaré symmetry in a neighborhood $\mathfrak{S} \subset \mathfrak{U}$, so that a quantum field can manifest transiently on $\mathfrak{S}$; and
- a quantum fluctuation creates an elementary particle ψ (or particle pair), *e.g.* of the Standard Model, in a neighborhood $\mathfrak{S}' \subset \mathfrak{S}$.

Due to its highly special nature, this kind of fluctuation will occur in only an exceedingly small subset of already rare large fluctuations; almost always the dynamical phase should instead quickly dissipate with no spacetime or matter field transiently appearing.

Given this context, a scenario is now proposed whereby the dynamical phase, obtained by a very special fluctuation of the above sort, can evolve into an inflation-like process and then a Big Bang. The φ field will mediate the emergence of $\mathscr{M}$; then, once $\mathscr{M}$ has emerged, the quantum fields will determine the geometry on $\mathscr{M}$ through their

interactions with φ. In general, the distance $\mathfrak{d}(\mathfrak{x},\mathfrak{y})$ between two points $\mathfrak{x},\mathfrak{y} \in \mathfrak{M}$, where $\mathfrak{d}$ is the distance metric on $\mathfrak{M}$, will be unrelated to the physical distance between $\mathfrak{x}$ and $\mathfrak{y}$ when $\mathfrak{x}$ and $\mathfrak{y}$ are considered points on $\mathscr{M}$.

Assume for illustration purposes that the Standard Model describes the complete particle spectrum for our universe all the way to the Planck scale. Let a fluctuation realize the two conditions above, so that a ψ field transiently exists on $\mathfrak{S}$ with a single ψ quantum. The ψ field will have quantum corrections that involve other fields of the Standard Model, *e.g.* the photon. The initial fluctuation only introduced the ψ field, but quantum corrections to it requires the existence of other quantum fields. In that sense, the ψ field brings into manifestation on $\mathfrak{S}$ those quantum fields with which it interacts. Those fields in turn acquire quantum corrections which require additional quantum fields, and so on: ψ on $\mathfrak{S}$ thereby implies the full Standard Model on $\mathfrak{S}$, at least transiently.

Thus, the approximate nature of the Poincaré symmetry in the scenario should not change the particle spectrum or interactions compared to what they are in a fully emerged spacetime. Instead, the effect of the symmetry being only approximate is presumably to make the quantum fields extremely unstable: energy and momentum are only approximate notions, and quantum corrections are in no sense perturbative. In that regime the distinction between virtual and 'real' processes is unclear (and probably unimportant) because lifetimes are so short and have a very large variance.

The persistence of the nonperturbative quantum 'soup' on $\mathfrak{S}$, *i.e.*, its evolution toward cosmogenesis, depends on making the spacetime progressively more smooth and the Poincaré symmetry more exact. The interactions between the quantum fields and non-quantum φ determine how this occurs, which is via a 'bootstrap' process:

- The approximately localized field quanta act as sources of φ inhomogeneity on $\mathfrak{S}$.
- Given the central role of φ in determining spacetime structure, large φ inhomogeneities will cause localized changes of the (approximate) spacetime structure which can affect virtual processes. For example independent scattering of virtual particles can prevent subsequent absorption (*e.g.*, of virtual photons) or annihilation (*e.g.*, of virtual lepton pairs), effectively creating new free particles which then acquire their own quantum corrections.
- The φ inhomogeneities propagate and disperse, making φ variations more spatially

homogeneous in the interior of $\mathfrak{S}$.

- Greater homogeneity of φ increases the smoothness of the approximate spacetime and improves the approximate Poincaré symmetry on $\mathfrak{S}$.
- Greater Poincaré symmetry increases the stability of the quantum fields, and hence particle lifetimes.

This cycle occurs recursively until spacetime becomes sufficiently smooth and homogeneous on $\mathfrak{S}$.

The components of the above process will be developed later; most of its particulars are not important to this overview. What is important is that the interaction of a field quantum with φ involves an approximately localized transfer of energy from the particle to φ. Since all quantum fields interact with φ they all transfer energy to it. Hence, the characteristic energy scale or mean particle energy will continually decrease on $\mathfrak{S}$ as the energy density of φ increases, consistent with some approximate notion of energy conservation. The process ends with stable particles and a stable characteristic energy scale once spacetime has fully emerged and Poincaré symmetry holds to excellent approximation.

The φ field on $\mathfrak{S}$ interacts with the vacuum at the boundary of $\mathfrak{S}$. The interaction acts dissipatively on the evolving spacetime, via thermalization. Thus, if the spacetime is to persist, the region $\mathfrak{S}$ on which it lives must grow on $\mathfrak{U}$, and eventually $\mathfrak{U}$ must grow on $\mathfrak{M}$. This will occur automatically as long as the assumed φ dynamics allow the approximate spacetime structure near the boundary of $\mathfrak{S}$ to induce approximate spacetime structure in $\mathfrak{U}\backslash\mathfrak{S}$ adjacent to the boundary. Then a stable spacetime can persist in the interior after developing via the bootstrap process — a transition region, comprised of an approximate spacetime in a very early stage of emergence, protects it from thermalization by the vacuum outside $\mathfrak{S}$.

The bootstrap process is inflation-like in two ways. First, consistent with general relativity (but independently of it, since no smooth manifold exists), the increasing energy density of φ will cause the spacetime on $\mathfrak{S}$ to expand relative to any two fixed points $\mathfrak{x}, \mathfrak{y} \in \mathfrak{S}$. This is analogous to placing a ruler between $\mathfrak{x}$ and $\mathfrak{y}$, keeping the distance $\mathfrak{d}(\mathfrak{x}, \mathfrak{y})$ constant, but continually rescaling the ruler so it measures increasingly large physical distances as the process proceeds. As the spacetime becomes progressively

smoother the 'doubling time' for the physical distance between $\mathfrak{x}$ and $\mathfrak{y}$ will increase, but the expansion is clearly exponential in time.

Second, independently of the ruler rescaling above, the self coupling of φ at the boundary of $\mathfrak{S}$ allows φ to mediate a continually growing diameter of $\mathfrak{S}$ on $\mathfrak{M}$: the coupling between the quantum fields and φ just inside the boundary can thereby indirectly induce approximate spacetime structure just outside the boundary. As $\mathfrak{S}$ expands, the bootstrap process will proceed as in the interior, but at a different rate because it is at an earlier stage of the process. Since the boundary region is itself an emerging, exponentially expanding spacetime and the boundary expansion is continual, the total spacetime volume of $\mathfrak{S}$ will continue grow after the inflation-like stage ends in the 'older' interior of $\mathfrak{S}$.

The continual expansion of $\mathfrak{S}$ on $\mathfrak{M}$ will cause the eventual demise of the spacetime and all matter in it. Energy conservation, even when it is only approximate, implies that the characteristic energy scale at the beginning of the bootstrap process in the boundary region will decrease as the diameter of $\mathfrak{S}$ increases. This will cause the expansion of $\mathfrak{S}$ to eventually terminate: since average mode amplitudes in the stochastic phase increase with decreasing k, there will be some diameter of $\mathfrak{S}$ for which the mode amplitudes in the boundary region are similar to those of the stochastic phase. Once that condition occurs, it no longer makes sense to consider the boundary region a separate dynamical phase, and the expansion of $\mathfrak{S}$ thereby ends. Then nothing can stop the inexorable thermalization of the dynamical phase by the stochastic vacuum; the diameter of $\mathfrak{S}$ will gradually decrease until the dynamical regime dissipates altogether. What starts as an extremely improbable fluctuation from a typical condition ends in a typical condition — ultimately, a cosmology is just another way to dissipate a (large) φ fluctuation.

An implicit assumption for the initial fluctuation is that the ψ field on $\mathfrak{S}$ is defined in the same spacetime dimension as the transient, approximate spacetime on $\mathfrak{S}$. If the dimensions are different, the ψ particle will not be in an appropriate representation of G and cannot manifest. By extension, all the other quantum fields brought into manifestation by ψ must also exist in the same spacetime dimension in this picture. However, it is possible in principle that other, non-manifesting quantum fields may be consistent in $n+1$ spacetime dimensions for $n \neq 3$, so that in general an initial fluctuation can bring about the same bootstrap process for any n in which quantum

fields are consistent. Presumably, $n = 3$ is merely the most probable.

Clearly the bootstrap process has no separate 'reheating' phase as conventional inflation scenarios do: particle production and exponential spacetime expansion deterministically end together when spacetime (and hence matter distribution) is sufficiently homogeneous and isotropic. It does not contain an unobserved inflaton field, nor a special potential that ensures both enough inflation and a CMB that is compatible with observations. Hence, the model is strongly constrained and thus subject to falsification: it must reproduce all the successes of conventional inflation but without the 'flexibility' of employing new fields and *ad hoc* potentials that are only relevant at very early times.

Because the bootstrap process automatically terminates locally once the local spacetime and matter distribution are sufficiently homogeneous and isotropic, in principle the scenario fulfills one of the central goals of inflation without implying eternal inflation. The way it addresses the horizon problem, again in principle, is a little more subtle. From the description of the process above, it is apparent that particles are free to interact and thus redistribute energy among themselves even as the characteristic energy scale steadily decreases. While a decreasing energy scale cannot correspond to global equilibrium, if the energy scale decreases at approximately the same rate throughout a region $\mathfrak{S}' \subset \mathfrak{S}$, then a state of relative equilibrium (relative to the characteristic scale) can exist within $\mathfrak{S}'$. Indeed, this should be the case: the degree of inhomogeneity determines the particle production rate and the rate at which spatial distances increase between two fixed points $\mathfrak{x}, \mathfrak{y} \in \mathfrak{M}$, but interactions among nearby particles should make the distribution of inhomogeneities relatively homogeneous within $\mathfrak{S}$ at each stage of the process.

The extraordinarily small probability that a fluctuation can lead to cosmogenesis raises a question: Is the emergence picture vulnerable to the Liouville, entropy or 'Boltzmann brain' arguments that helped motivate it? General considerations indicate it is not. The stochastic phase of φ almost everywhere on $\mathfrak{M}$ is maximally uncorrelated and free of information. Typical, small fluctuations will be insufficient to create information, and will rapidly dissipate by self diffusion. Although the stochastic phase represents an equilibrium-like condition in that φ modes have a thermal spectrum, the nature of the phase precludes defining a set of distinct states or density of states of φ in any region of $\mathfrak{M}$. There are multiple reasons for this:

- Volumes and surfaces are undefined on the general metric space $\mathfrak{M}$ because no

dimensioned space exists; while balls and spheres of finite radius are compact on $\mathfrak{M}$, using them as a finite cover is inadequate because they are no more measurable than the object they cover.

- Scale invariance means no function of φ is uniquely measurable because the choice of measure is intrinsically arbitrary.

- Absence of physically meaningful notions of distance or time precludes conserved dynamical quantities that could usefully define a state.

Moreover, one-dimensional curves are the most general geometric objects on $\mathfrak{M}$, so it is not possible to quantify relative likelihoods of more general φ configurations, *e.g.* over a given neighborhood on $\mathfrak{M}$. Hence, in the stochastic phase there is neither a useful definition of entropy nor a useful measure of probabilities of different φ configurations arising from a fluctuation.

Even if the relative likelihood of different possible fluctuations cannot be made precise, there seems to be a heuristic sense in which some resulting configurations are more likely than others. But how might one frame the issue? Intuition in 3+1 dimensions with well defined measures of spatial and temporal intervals is unhelpful for characterizing the scale-free φ on a space-less and timeless $\mathfrak{M}$. Different choices of measure can change the apparent likelihood of the same φ configuration. So can the (fictitious) assignment of different topologies to the same region of $\mathfrak{M}$ to mimic different choices of emergent manifolds. Thus, it appears that characterizing relative likelihoods of different fluctuations from the stochastic phase is not possible, even heuristically: the idea is too ill-defined to make sense.

The 'Boltzmann brain' argument posits that it is far more probable a fluctuation will obtain a transient sentient observer with false memories than a large universe that produces observers as a by-product; hence, since we see a universe rather than transient 'brains,' the universe was surely not the product of a fluctuation. This argument assumes a prior state space that does not exist in the emergence scenario, but even so the argument can be dispatched on other grounds as follows.

For probabilistic reasons, the emergence picture essentially offers a single way to produce a lump of stable matter, even something as simple as an electron or photon. Specifically, the bootstrap process requires the production of vast numbers of particles via the inflation-like process, described above, to obtain the homogeneity and isotropy

of spacetime needed for stable particles. A cosmology is a (apparently inevitable) consequence of this process rather than a precursor to it. Hence, production of sufficiently stable matter to build an observer is an all-or-none affair: a 'sufficiently large' fluctuation will either lead to rapid dissipation or cosmogenesis. For that reason anthropic arguments can act as a kind of superselection principle, albeit a non-predictive one: existence of intelligent observers implies prior cosmogenesis, although in principle the converse need not be true. Hence, Boltzmann brains should not exist because an emergent spacetime already requires huge quantities of matter to stabilize the spacetime manifold, making it is vastly 'cheaper' to create observers from all that existing matter than to create, via independent fluctuations, the many particles and requisite spacetime for a transient brain. (Even if such a brain could be assembled from many independent fluctuations, a cosmology would ensue regardless, progressing by the same bootstrap process that ordinarily starts from a single particle.)

1.5 Thesis Plan

1.5.1 Thesis outline

Sect. 1.4 describes a bootstrap process wherein spacetime and matter emerge concurrently in an inflation-like phase, the goal of which is to provide a plausible origin of the initial conditions of the Big Bang without the theoretical difficulties of the conventional inflation idea. The main task of Part I of the thesis is to develop basic components that implement this process, working in a general metric space with no prior manifold structure. Part II presents the basic emergence scenario for 3+1 dimensional spacetime and examines the consistency of the construction with some aspects of established physics. Briefly, key questions addressed in each chapter are thus:

1. Why consider an emergence picture; what is it?
2. From what does spacetime emerge?
3. What existing and new mathematical tools provide full mathematical control of the emergence?
4. How do dynamics emerge?
5. Is the emergence picture consistent with QM?

6. How well does the emergence picture reproduce general relativity?

An irreducible scalar field φ, postulated in Chapter 2, will play the central role in this thesis. It manifests both stochastic and dynamical properties. Through its interactions with quantum fields φ mediates the emergence and geometry of spacetime. The picture that will be developed differs substantively from the usual picture of quantum gravity: while quantum fields determine spacetime geometry through its interactions with the mediating field, gravity itself is not a quantum field because (as will be seen), while φ acts as an effectively quantized field in its interactions with 'true' quantum fields, in the regime where it mediates gravity φ is not a quantum field. Although gravity in the emerged spacetime has a geometric interpretation, the emergent manifold is only approximately smooth. From this perspective the classical spacetimes described by general relativity are idealizations that break down at small distance scales because stochasticity becomes important.

Part I: Basic components

Chapter 2 motivates and develops the basic postulates of the non-quantum, non-classical, irreducible, real-valued scalar field φ, as well as the general metric space $\mathfrak{M}$ on which the field lives. $\mathfrak{M}$ is not endowed with manifold structure, nor is it even a product space. The rationale for this choice is philosophical: if a spacetime emerges dynamically, there is no *a priori* reason for preferring a particular number of spatial dimensions (or even for it being an integer), metric signature, local flatness, or differentiability; hence, the elementary manifold and its attributes should emerge dynamically.

Chapter 3 focuses on the mathematical framework needed to work with the intrinsic stochasticity of φ and the lack of prior manifold structure. The intrinsic stochasticity renders φ nondifferentiable because φ has infinite variation along every finite path. This necessitates use of the stochastic calculus, which differs in important respects from the usual calculus of functions of finite variation. Spectral analysis of functions of infinite variation also require special care. While the mathematical theory of stochastic calculus and processes is well developed, the construction of stochastic differential equations is somewhat different in form from partial differential equations that typically occur is physics. Sect. 3.3 addresses this, using the fact that stochastic fluctuations are perturbative in the dynamical phase to define an approximate deriva-

tive. This derivative can then be used anywhere a partial derivative would occur in a physical setting, recognizing its statistical nature implies an uncertainty. The fact that the most general objects of study in a general metric space like $\mathfrak{M}$ are one-dimensional curves requires adaptation of standard stochastic process theory. Specifically, only one dimensional stochastic processes can be defined on $\mathfrak{M}$, but elementary dynamics involve local interactions with the entire neighborhood at each point.

Chapter 4 focuses on the emergence of dynamics of φ, starting from the postulates in Chapter 2. The first part develops the stochastic phase; there, the amplitude of φ along every curve obeys the probability density function for Brownian motion, implying the φ motions are scale invariant in that phase. A dynamical phase occurs in a region $\mathfrak{S}$ of $\mathfrak{M}$ if a very improbable fluctuation sufficiently breaks scale invariance in $\mathfrak{S}$; this preferred scale is characterized by a finite range of Fourier modes $[k_{\min}, k_{\max}]$ whose amplitudes exceed their corresponding stochastic phase amplitudes by some physically motivated factor. The finiteness of the range implies intrinsic hard cutoffs in descriptions of phenomena, both ultraviolet and infrared — spacetime geometry, energy, momentum, and phenomena like particles have only a statistical meaning, a meaning that exists only within (and relative to) the preferred scale. An origin is proposed for product space structure where none previously existed; it is posited that the spacetime dimension is essentially fixed by the initial fluctuation that breaks scale invariance. Since there is no connection between the dynamics on an emergent spacetime and the scale-free dynamics on the general metric space where they are initially defined, it is necessary to introduce postulates to provide that connection. After considering these issues, the field equation for φ is derived from the elementary dynamics in 1+1 dimensions for motions along curves.

Part II: Spacetime and matter emergence

A common viewpoint is that a composite field theory of quantum fields interacting with a non-quantum field is not consistent with quantum mechanics. Chapter 5 is largely devoted to demonstrating that such a field theory can indeed be consistent. The composite theory has two sectors which should evolve somewhat independently of each other: a classical sector in which the quantum fields live, and a classical-stochastic sector which determines the emergence and geometry of spacetime. To help argue that

the composite field theory is consistent with QM, a formalism developed by Hall and Reginatto [32] for consistently describing composite systems of interacting quantum and classical subsystems will be employed. The Hall-Reginatto formalism is summarized, along with the Schrödinger representation of quantum field theory that it assumes. Appendix A reviews Hamilton-Jacobi field theory; this is used to define the dynamics of a non-quantum field component of the composite field theory. The formalism is applied to a model composite field theory of φ plus a collection of quantum fields. Further supporting the consistency of the composite field theory, the φ becomes effectively quantized in the quantum sector; this and its self interaction provides the necessary connection between the quantum fields and the spacetime metric which is developed concretely in Chap. 6. To assist in modeling the effects of the quantum fields on the φ motions, a toy model of a particle is constructed.

Chapter 6 brings together the material of Part I and part of Chapter 5 to concretely show how a manifold emerges. Quantitative arguments demonstrate that a statistically homogeneous energy density of the φ field in the classical-stochastic sector implies boost symmetry; homogeneity and isotropy thus imply emergent Poincaré invariance. Arguments are then presented that the Lorentz invariance thus derived is actually local Lorentz invariance, so that introducing inhomogeneous matter distributions while maintaining the assumptions of the derivation lead to a more general metric like that obtained in GR. Finally the inflation-like scenario already considered in Sect. 1.4 is examined further.

1.5.2 How to read this thesis

This thesis is lengthy and considers many new ideas. Since reading it completely at first exposure may be excessively burdensome, it is useful to tailor the reading to one's goals. What follows hopefully helps in that regard.

A first reading can reasonably consist of this introductory chapter, Sect. 2.2 which introduces the φ field, Sect. 3.3 which defines the approximate derivative, Sect. 4.5 which introduces the φ field equation, and Chapter 6. That should give some basic idea of the emergence picture. Sect. 2.1 includes a brief summary of some standard background from topology, and Sects. 3.1 and 3.2 review some needed definitions and theorems from the mathematical theory of stochastic calculus and stochastic processes; these may be useful if those ideas are unfamiliar. The temptation to skip Sect. 1.6

(Notation and Conventions) should be resisted; notation has been designed to reduce ambiguity, but not knowing it may cause confusion at times.

A more careful first reading would include skimming the background material in Sects. 2.1, 3.1 and 3.2 to find motivations and other comments specific to the emergence picture. One can also skim sections that appear interesting in Chapter 4. If the composite field theory of φ and a collection of quantum fields is of interest, most of Chapter 5 stands alone and may be read without previously reading Part I.

Limited summaries of standard material are occasionally provided. Specifically, these include standard definitions from topology and general metric spaces in Sect. 2.1; stochastic calculus and stochastic processes (Sects. 3.1 and 3.2); the Schrödinger representation of quantum field theory (Subsect. 5.1.2); and a very brief summary of determining the spacetime geometry from the stress-energy tensor in GR (Subsect. 6.2.1). These may serve as a quick review or brief introduction, but may be skipped if the material is already familiar. However, all these sections except the review of the Schrödinger representation of QFT and the GR summary also include thesis-specific discussion or postulates, so they should at least be skimmed for non-standard material and remarks.

1.6 Notation and Conventions

Standard conventions

Unless otherwise noted, spatial components of vectors and tensors will be represented symbolically by Roman letter indices (*e.g.* $i = 1, 2, 3$), and spacetime components by lowercase Greek letters (*e.g.* $\mu = 0, 1, 2, 3$). A three-vector (p^1, p^2, p^3) will be written in non-component form as $\vec{p}$ or $\mathbf{p}$; a four-vector (p^0, p^1, p^2, p^3) will be denoted by p. Tensors of rank greater than one have a boldface font in non-component form.

The metric signature of d-dimensional spacetime will conform to the 'East Coast' convention. For example, Minkowski spacetime with $d = 4$ has the metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\,; \quad \gamma_{ij} = \mathrm{diag}(1, 1, 1)\,; \tag{1.3}$$

i.e., γ_{ij} is the usual Euclidean metric. More generally, the physical metric will be denoted by $\mathbf{g}$, or $g_{\mu\nu}$ in component form.

The Einstein convention will be in effect: summation over a repeated raised and lowered index is implied; *e.g.* $p^\mu x_\mu = \sum_{\mu=0}^{3} p^\mu x_\mu = \sum_{\mu,\nu=0}^{3} p^\mu x^\nu g_{\mu\nu}$ is the contraction of four-vectors p and x.

Nonphysical versus physical spaces

Some objects will be introduced to provide a mathematical framework for defining physical objects and quantities. Notation will be employed to distinguish such objects versus ones with a more direct physical interpretation. For example, two different spaces will be important:

1. **Nonphysical metric space** $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$, where $\mathfrak{d}$ is a distance function, or metric, on a set Σ. This space contains no notion of dimensionality, area or volume; the most general objects of interest are one-dimensional paths and curves. A nonphysical 'space-time' $\mathfrak{M}_\mathfrak{t} \equiv (\Sigma, \mathfrak{d}, \mathfrak{t}$, *i.e.*, $\mathfrak{M}$ augmented with nonphysical time, will find frequent use.

2. **Physical spacetime** $(\mathscr{M}, \mathbf{g})$ consisting of a physical metric $\mathbf{g}$ on a d-dimensional manifold $\mathscr{M}$.

Symbols for nonphysical entities will generally be Latin letters written with a **fraktur** font. For example: $\mathfrak{U} \subset \mathfrak{M}$ denotes a subset of the general metric space, $\mathfrak{t}$ is a nonphysical-time coordinate and $\mathfrak{d}$ is a distance metric on $\mathfrak{M}$. Latin letters in a Roman or script font will label physical quantities and objects with a physical interpretation, *e.g.*, $S \subset \mathscr{M}$ might denote a subset of a spacetime manifold $\mathscr{M}$.

One minor abuse of notation that will occur from time to time is to denote an n-dimensional Euclidean space E^n as $\mathbb{R}^n$. The two are not the same because E^n is an affine space but $\mathbb{R}^n$ is not.

For coordinate names, the font will distinguish the space on which the coordinate is defined.

- Nonphysical space. A fraktur font labels coordinates, *e.g.*, $\mathfrak{x}$, $\mathfrak{t}$ are coordinates on $\mathfrak{M}_\mathfrak{t}$.

- Physical spacetime. Coordinate names on $(\mathscr{M}, \mathbf{g})$ will use an italicized Roman font, *e.g.*, x and t.

Unfortunately, some letters look similar in Roman and fraktur fonts, like $\mathfrak{t}, t$ or $\mathfrak{x}, x$.

Symbols for distances and time durations

Accents below the symbol name will distinguish the underlying space on which it is defined:

- Abstract quantities on $\mathfrak{M}_\mathfrak{t}$ have an **under-bar**, *e.g.* $\mathfrak{T} = \delta\mathfrak{t}$ for a time duration or ℓ for a distance $\mathfrak{d}(x, y)$. However, when a symbol appears as a subscript no under-bar is shown, e.g., an open ball of radius ε centered at $\mathfrak{x}$ would be written $B_\varepsilon(\mathfrak{x})$.

- Physically meaningful quantities on $\mathfrak{M}_\mathfrak{t}$ have an **under-tilde**, *e.g.* a physically interesting interval $\underset{\sim}{\mathcal{T}} = \delta t$ might correspond to a nonphysical time interval $\tau = \delta\mathfrak{t}$; both are defined with respect to $\mathfrak{M}_\mathfrak{t}$.

- Denote physically meaningful quantities on $(\mathscr{M}, \mathbf{g})$ by omitting the above accents. Presumably $\tau \propto \underset{\sim}{\mathcal{T}}$, even though τ and $\underset{\sim}{\mathcal{T}}$ are defined on different spaces.

- *Labels for points* on $\mathfrak{M}$ or $(\mathscr{M}, \mathbf{g})$ are always unaccented. For example, σ could label a point on either $\mathfrak{M}$ or $(\mathscr{M}, \mathbf{g})$; its meaning should be clear from the context.

Fields versus functions of a variable

Notation can help distinguish whether a function argument is a set of points, a variable or a point. This will be especially important in expressions involving φ because φ can occur in so many contexts.

The argument of a function will be enclosed by parentheses or square brackets, and multiple arguments may be separated by commas or semicolons. A spatial parameter can be a point, coordinate range, or subset; a temporal parameter can be a particular time or a time interval.

1. *Single-valued arguments*

   - $\phi(x, t)$: the comma signifies x and t vary independently; alternatively both x and t may be fixed if the context indicates.
   - $\phi(x; t)$: x is fixed and t varies, or vice versa; the context determines which case applies.

2. *Multiple-valued arguments (sets or intervals)*

   - $\phi(x; I_T]$ given an interval $I_T \equiv [t_0, t_0+T]$: the left parenthesis identifies the first argument as single-valued; the semicolon specifies one argument is fixed while the other varies; and the right square bracket makes clear the second argument is a set or interval. The context is that I_T is a fixed interval; hence x must be the varying argument. (If I_T were arbitrary but fixed it could be replaced by δt.)
   - $\varphi[\gamma]$ given $\gamma \subset \mathfrak{M}$: denotes the 'instantaneous' φ field configuration on the subset γ. Square brackets make clear γ is a set.
   - $\varphi[\gamma; \mathfrak{t})$ given $\gamma \subset \mathfrak{M}$: denotes the 'time evolution' of $\varphi[\gamma]$ (context is that γ is a fixed subset; thus $\mathfrak{t}$ is what varies).
   - Point labels may be used instead of coordinates, e.g., $\varphi(\sigma; \mathfrak{t})$ is the time evolution of φ at the fixed point σ.

Arguments of functionals

In some places square brackets will be employed in their more conventional role in functionals. Like the notation above brackets and parentheses can be mixed:

- $F[A, B]$: F is a functional of A and B.
- $F[A, t)$: F is a functional of A and a function of t.
- Notation for functionals and fields may be combined in an expression, *e.g.*, $F[\varphi[\gamma]]$ could denote a functional of the field configuration on $\gamma \subset \mathfrak{M}$.

It should be clear from the context whether square brackets indicate a functional versus a function of a set or interval.

Standard versus new formal results

To distinguish standard mathematical results from definitions and propositions obtained in this thesis, the terms **Def**, **Axiom** and **Theorem** will be employed when summarizing established mathematics, whereas **Def***, **Postulate** and **Proposition** will denote precise statements that apply to the new ideas developed herein. The unfortunate

use of 'Def*' to indicate new definitions reflects my lack of imagination in proposing an equivalent word for 'definition.'

Verbatim quotations

Related to the statements of standard mathematical results above, there is often no advantage in rewording statements of axioms, definitions and theorems by cited authors because they are already clearly and concisely phrased. To identify verbatim restatements of such results, the citation will often immediately appear within the parenthesized definition or theorem. For example, in the title of Def. 3.1.1, the citation immediately follows the title inside the parentheses thusly:

(Convergence properties of random variables. [42]**)**

Part I

Pre-Emergent Space

Chapter 2

Foundation

The introduction described some severe difficulties cosmology faces in explaining the extraordinarily improbable state of the observable universe with time symmetric physical laws, motivating a picture of emergent spacetime and abandonment of a pre-existent manifold. Absent a prior manifold, theory offers very limited guidance, a state of affairs that persists until a manifold has emerged. Hence, philosophical considerations can play an important guiding role in choosing the elementary postulates of the pre-emergent space.

Two general assumptions or guiding principles have been adopted *a priori* in this thesis:

- All physical laws, phenomena and numerical relationships should be comprehensible dynamical consequences of an elementary theory. In turn, the underlying theory should rest on very simple assumptions containing the barest of prior structure.

- The elementary theory should imply a typical equilibrium-like condition and contain a stochastic component that allows a temporary, localized exit from the typical condition. This atypical condition should then dynamically evolve, consistent with the first assumption.

Of course, both of these general assumptions reflect a particular philosophical viewpoint about the comprehensibility of Nature — our knowledge of Nature neither indicates nor contraindicates either of them. The underlying viewpoint is consistent with the general progress of physics thus far, although the continuing incomprehensibility of some

foundational aspects of quantum theory are arguably in tension with it. Furthermore, the first assumption has appeal from the standpoint of theory falsifiability: a theory which conforms to it should be highly constrained, offering little flexibility for fine tuning, *ad hoc* addition of parameters or imposition by *fiat* of new symmetries to evade future empirical disagreement. Having offered this limited rationale, no further attempt will be made to justify these assumptions; they should, however, be remembered for their role as implicit guiding principles.

The first assumption does not imply that a 3+1 dimensional universe is inevitable — it is sufficient that a 3+1 dimensional universe is not unlikely. The assumption does suggest that the full range of possible physical laws and phenomena should be constrained or even fully determined by the spacetime dimension. The second assumption allows chance initial conditions to determine the dimensionality as well as 'environmental' laws that particular distributions of matter may dictate, *e.g.* a kind of 'multiverse.' It also motivates pursuing a picture where spacetime and physical laws are emergent rather than directly derivable.

The influence of the guiding assumptions is greatest in this chapter. They help motivate the definitions and postulates that will form the foundation for the rest of the thesis. Broadly, this foundation can be viewed as a proposal for answering the question, "What is it that spacetime might emerge from?"

Such definitions and postulates will appear below in two primary contexts. The first is the mathematical space on which the elementary physical objects and dynamics will be defined. As Sect. 1.4 indicated, this will be a nonphysical general metric space $\mathfrak{M}$ augmented by a nonphysical time $\mathfrak{t} \in \mathbb{R}^+$. This space will be specified in Sect. 2.1: its particular properties will be motivated by the guiding assumptions, but its definition will rely on established mathematics. The second context is the elementary physical scalar field φ, which will be defined on the augmented space $\mathfrak{M}_\mathfrak{t} \equiv \mathfrak{M} \times \mathbb{R}^+$ in Sect. 2.2. Here also the guiding assumptions will motivate the choice of properties; the relevant standard mathematics is Fourier analysis and the stochastic calculus.

Throughout this chapter, significant effort is expended to give the rationale for each postulate, state relevant definitions, and clearly identify all nontrivial assumptions. At a minimum this should make it easier to verify inferences and determine the origin of conclusions that differ from what one's intuition or implicit theoretical priors might suggest.

2.1 Topological Space $(\Sigma, \mathcal{T}_\Sigma)$

An elementary particle in flat spacetime is an irreducible representation of the group $P \times G$, where P is the Poincaré group and G is the internal symmetry group [8]. As a symmetry of spacetime, Poincaré symmetry cannot manifest if no manifold exists, then the quantum fields whose excitations are the elementary particles cannot manifest either. Hence assume, as already indicated, there exists an irreducible, non-quantum field φ which mediates the emergence of a physical spacetime with Poincaré symmetry. Assign no properties to φ yet; its properties and laws will be proposed in Subsect. 2.2. For now, just assume a theory of initial conditions will depend only on a field theory for φ.

Since φ is a field, it must be a field over a set; denote this set Σ. Σ must contain enough structure to permit construction of a theory, but it is desirable to keep prior structure to a minimum to avoid imposing artificial constraints on the theory. Since a field theory needs notions of neighborhoods, paths and connectedness, promote Σ to a topological space with topology $\mathcal{T}_\Sigma$. Jänich [40] defines a topological space thusly[1]:

Def. 2.1.1 (Topological space. [40]**)** *A topological space is a pair* $(\Sigma, \mathcal{T}_\Sigma)$ *consisting of a set* Σ *and a set* $\mathcal{T}_\Sigma$ *of open subsets of* Σ*, such that the following axioms hold:*

Axiom 1. *Any union of open sets is open.*

Axiom 2. *The intersection of any two open sets is open.*

Axiom 3. $\varnothing$ *and* Σ *are open.*

Following standard convention, we will refer to the space $(\Sigma, \mathcal{T}_\Sigma)$ as Σ unless noted otherwise.

A field theory obviously requires some notion of relative 'closeness' of points. Moreover, there are pragmatic considerations: it is very difficult to imagine making much progress in developing a field theory for φ without tools like elementary calculus, so we need a more precise notion of distance between points. We need Σ to be a metric space.

Def. 2.1.2 (Metric space. [40]**)** *A metric space is a pair* $(\Sigma, \mathfrak{d})$ *consisting of a set* Σ *and a real function* $\mathfrak{d} : \Sigma \times \Sigma \to \mathbb{R}$*, called the 'metric,' such that:*

[1] Most definitions in this section will be taken from Jänich [40].

M1. $\mathfrak{d}(x,y) \geq 0$ *for all* $x, y \in \Sigma$ *and* $\mathfrak{d}(x,y) = 0$ *if and only if* $x = y$.

M2. $\mathfrak{d}(x,y) = \mathfrak{d}(y,x)$ *for all* $x, y \in \Sigma$.

M3. *(Triangle Inequality.)* $\mathfrak{d}(x,z) \leq \mathfrak{d}(x,y) + \mathfrak{d}(y,z)$ *for all* $x, y, z \in \Sigma$.

To simplify notation, henceforth $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$.

Def. 2.1.2, while standard, may seem not to hold for Minkowski space. That is not true, however, because the underlying space is $\mathbb{R}^4$; $\mathfrak{d}(x,y)$ is a distance function on $\mathbb{R}^4$, not $\mathbb{R}^{1,3}$. Stated differently, a space must have a prior topology in order to define a manifold $\mathscr{M}$ and a Minkowski metric $\boldsymbol{\eta}$ on $\mathscr{M}$; $\mathfrak{d}$ determines that topology, so it cannot be the same as $\boldsymbol{\eta}$.

Balls and spheres in $\mathfrak{M}$ will find repeated use, so collect their definitions here.

Def. 2.1.3 (Balls and spheres. [40]**)** *Let* $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$ *and* $x, y \in \mathfrak{M}$.

B1. $B_\varepsilon(x)$ *is the* ***open ball*** *of radius* ε *(or* ε*-ball) centered at* x*:*

$$B_\varepsilon(x) \equiv \{y \,|\, \mathfrak{d}(x,y) < \varepsilon\}\,.$$

B2. $\bar{B}_\varepsilon(x)$ *is the* ***closed ball*** *of radius* ε *centered at* x*:*

$$\bar{B}_\varepsilon(x) \equiv \{y \,|\, \mathfrak{d}(x,y) \leq \varepsilon\}\,.$$

B3. $K_\varepsilon(x)$ *is the* ***sphere*** *(or shell) of radius* ε *centered at* x*:*

$$K_\varepsilon(x) \equiv \{y \,|\, \mathfrak{d}(x,y) = \varepsilon\} = \bar{B}_\varepsilon(x) \backslash B_\varepsilon(x)\,.$$

The metric induces a topology on the set Σ as follows.

Def. 2.1.4 (Topology of a metric space. [40]**)** *Let* $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$. $V \subset \Sigma$ *is called open if for every* $x \in V$ *there is an* $\varepsilon > 0$ *such that* $B_\varepsilon(x)$ *is still contained in* V. *The set* $\mathcal{T}_\Sigma(\mathfrak{d})$ *of all open sets of* Σ *is called the* ***topology*** *of* $\mathfrak{M}$.

To make Def. 2.1.1 for a topological space $(\Sigma, \mathcal{T}_\Sigma(\mathfrak{d}))$ consistent with Def. 2.1.4, we need $(\Sigma, \mathcal{T}_\Sigma)$ to be metrizable. The space $\mathfrak{M}$ is metrizable if there is a metric $\mathfrak{d}$ on the set Σ such that $\mathcal{T}_\Sigma(\mathfrak{d}) = \mathcal{T}_\Sigma$.

Elementary calculus requires well defined limiting procedures, so we care about convergence of sequences along paths in $\mathfrak{M}$.

Def. 2.1.5 (Convergence in a topological space. [39]) *A sequence* $\{\sigma_n\}$ *of points in* $\mathfrak{M}$ ***converges*** *to* σ *if for each neighborhood* U *of* σ *there exists an integer* k *such that* $\sigma_n \in U$ *whenever* $n \geq k$. *If* $\{\sigma_n\}$ *converges to* σ *in this sense, then* σ *is a* ***limit point*** *of* $\{\sigma_n\}$, *and we can write* $\sigma_n \to \sigma$. *A sequence* $\{x_i\}$ *such that* $d(x_i, x_j) \to 0$ *as* $i, j \to \infty$ *is called a* ***Cauchy sequence****. Every convergent sequence is a Cauchy sequence.*

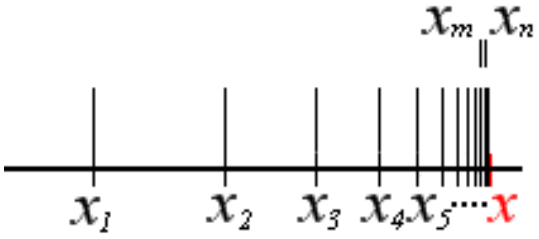


Figure 2.1: Convergent sequence.

Def. 2.1.6 (Completeness of a space. [39]) *If every Cauchy sequence is convergent in a space, the space is called complete.*

Hausdorff separability guarantees uniqueness of limits of sequences in $\mathfrak{M}$. A topological space is Hausdorff separable if every pair of different points $x, y \in \mathfrak{M}$ have disjoint neighborhoods. All metric spaces are Hausdorff separable [40]; for example, if $\mathfrak{d}(x, y) = \varepsilon > 0$, then subsets $P = \{p \,|\, \mathfrak{d}(x, p) < \varepsilon/2\}$ and $Q = \{q \,|\, \mathfrak{d}(x, q) < \varepsilon/2\}$ are disjoint.

If physical spacetime is emergent, it must emerge from something; assume that physical spacetime emerges in consequence of the laws of φ. Since a metric has been imposed on Σ largely for pragmatic reasons, $\mathfrak{d}$ should have no discernible relation to the emergent metric on the emergent physical spacetime. That is, $\mathfrak{d}$ is part of the 'scaffolding' of the theory: it will be used to define and construct its elements, but it should leave no visible trace in the eventual theory. To achieve this ideal, $\mathfrak{M}$ should contain a bare minimum of structure; certainly it should not imply a spacetime manifold. Notions like dimensionality, metric signature, and even product space topology should emerge along with the physical metric, and not in any way be implied by or traceable to $\mathfrak{M}$.

Curves, paths and arcs are primary objects of study in general metric spaces. An **arc** will mean a curve distinguished by some property, for example a geodesic (see Def. 2.1.14).

Def. 2.1.7 (Curve. [16]) *Let $\mathfrak{M}$ be a general metric space. A curve is a continuous map $\mathfrak{c} : I \to \mathfrak{M}$, where $I = [t_a, t_b] \subset \mathbb{R}$ is an interval having a nonempty, connected interior. (For notational convenience this curve may sometimes be denoted $\mathfrak{c}(t_a, t_b)$.)*

Def. 2.1.8 (Path in a metric space. [16]) *A path γ of curve $\mathfrak{c}$ is its image, i.e., the set $\{\, \mathfrak{c}(t) \,|\, t \in I \,\} \subset \mathfrak{M}$, having endpoints $\gamma(t_a) = \sigma_a$ and $\gamma(t_b) = \sigma_b$. (For notational convenience this path may sometimes be denoted $\gamma(\sigma_a, \sigma_b)$.)*

Although not required by their definitions, the following convention will usually be employed: a path will indicate a spatially continuous subset of $\mathfrak{M}$ that is coordinatized as $x : \mathbb{R} \mapsto \gamma$, whereas a curve will refer to a map $\mathfrak{c}(t)$ parameterized by time. With this convention, a curve has an associated speed.

Def. 2.1.9 (Bounding speed of a curve. [16]) *Given a curve $\mathfrak{c} : I \to \mathfrak{M}$ parameterized by an interval $I = [t_a, t_b]$, and a path $\gamma(\sigma_a, \sigma_b)$ which is its image: $\rho \geq 0$ is a **bounding speed** of $\mathfrak{c}$ if $\mathfrak{d}(\mathfrak{c}(s), \mathfrak{c}(t)) \leq \rho|s - t|$ for all $s, t \in I$.*

Def. 2.1.10 (Speed of a curve. [16]) *The **speed**, or metric derivative, of a curve $\mathfrak{c}(t)$ is the infimum of the bounding speed at t, or the limit superior,*

$$|\dot{\mathfrak{c}}(t)| = \limsup_{|t-s|\to 0} \frac{\mathfrak{d}\big(\mathfrak{c}(t) - \mathfrak{c}(s)\big)}{|t - s|} \,. \tag{2.1}$$

Hence distinct curves can have the same image and thus correspond to the same path. For example, a path can be traced at different speeds, or a closed path can be traversed $n \in \mathbb{N}$ times. The speed may vary along the curve.

One topological attribute of $\mathcal{T}_\Sigma$ that could potentially manifest in a physical theory is any non-connectedness of $\mathfrak{M}$. Given that φ is to mediate spacetime emergence, it is sensible to assume φ completely covers $\mathfrak{M}$; otherwise the 'uncovered' part would be inaccessible to a theory. Similarly, it makes sense to assume at least one path exists between each two points $x, y \in \mathfrak{M}$; otherwise $\mathfrak{M}$ would break up into disjoint subsets, and only the subset where we as observers find ourselves would be meaningful in a theory: $\mathfrak{M}$ should be path connected.

Def. 2.1.11 (Connectedness. [40]**)** $\mathfrak{M}$ *is said to be* ***path-connected*** *if every two points* $\sigma_a, \sigma_b \in \mathfrak{M}$ *are connected by a path* γ *such that* $\gamma(t_a) = \sigma_a$ *and* $\gamma(t_b) = \sigma_b$. $\mathfrak{M}$ *is also* ***simply connected*** *if, furthermore, whenever* $\gamma : [t_a, t_b] \to \mathfrak{M}$ *and* $\eta : [t_a, t_b] \to \mathfrak{M}$ *are two paths such that* $\gamma(t_a) \equiv \eta(t_a)$ *and* $\gamma(t_b) \equiv \eta(t_b)$*, then* γ *and* η *are homotopic (informally, each can be 'continuously deformed' to the other).*

Path connectedness alone is insufficient to prevent observational consequences of artifacts of $\mathcal{T}_\Sigma$ like exotic effects from 'holes' in the space. Assume $\mathfrak{M}$ is simply connected to preclude such holes. Furthermore we should require that $\mathfrak{M}$ is a complete metric space (Def. 2.1.6), *i.e.*, a space in which every Cauchy sequence in $\mathfrak{M}$ converges in $\mathfrak{M}$. Requiring completeness precludes exotic features of the metric that might lead to non-converging sequences with the same effect as holes.

The metric must meet additional requirements to serve its pragmatic role in defining properties and dynamics of φ. The metric distance $\mathfrak{d}(x, y)$ should be analogous to the geodesic distance between x and y on a manifold; it is the 'shortest distance' between x and y consistent with the topology of $\mathfrak{M}$. The next three definitions make this precise.

Def. 2.1.12 (Path and Curve Lengths in a Metric Space. [16]**)** *Given a curve* $\mathfrak{c}$ *defined as in Def. 2.1.7, the length* $\mathfrak{L}(\mathfrak{c})$ *is the supremum:*

$$\mathfrak{L}(\mathfrak{c}) = \sup \sum_{i=0}^{n-1} \mathfrak{d}\left(\mathfrak{c}(t_i), \mathfrak{c}(t_{i+1})\right), \tag{2.2}$$

taken over the set of all finite partitions $(t_a = t_0, t_1, \ldots, t_n = t_b)$ *of* $\mathfrak{c}(t_a, t_b)$*. If* $\mathfrak{L}(\mathfrak{c}) < \infty$ *then* $\mathfrak{c}$ *is called* ***rectifiable****; otherwise* $\mathfrak{c}$ *is* ***non-rectifiable****. If a path* $\gamma(\sigma_a, \sigma_b)$ *is the image of* $\mathfrak{c}$*, then* $\mathfrak{L}(\gamma) = \mathfrak{L}(\mathfrak{c})$*;* γ *is rectifiable if* $\mathfrak{L}(\mathfrak{c}) < \infty$*.*

The curve length can also be expressed in terms of its speed (Def. 2.1.10):

$$\mathfrak{L}(\mathfrak{c}) = \int_0^T |\dot{\mathfrak{c}}(t)| \, dt \,. \tag{2.3}$$

Def. 2.1.13 (Intrinsic Metric, and Length Spaces. [40]**)**
Let $\Gamma(x,y) \equiv \{\gamma_\lambda : [x,y] \to \mathfrak{M} \,|\, \gamma_\lambda \subset \mathfrak{M},\ \lambda \in \Lambda\}$ *with* Λ *an indexing set for all rectifiable paths between* x, y*. The* ***intrinsic metric*** *is a function* $\mathfrak{d}_I : \Sigma \times \Sigma \to \mathbb{R}$*, such that* $\mathfrak{d}_I(x,y) = \inf\{\mathfrak{L}(\gamma_{xy}) \,|\, \gamma_{xy} \in \Gamma(x,y)\}$*. If* $\mathfrak{d}(x,y) = \mathfrak{d}_I(x,y)$ *for all* $x, y \in \Sigma$ *then* $\mathfrak{M}$ *is a* ***length space****.*

We will assume $\mathfrak{M}$ is a length space. The meaning of a geodesic, or minimum path between x and y, readily follows from the intrinsic metric $\mathfrak{d}_I$:

Def. 2.1.14 (Geodesic. [40]**)** *A* ***geodesic*** *or* ***arc*** *is a curve* $\mathfrak{c}$ *for which* $\mathfrak{L}(\mathfrak{c}) = \mathfrak{d}_I(x,y)$*.*

Euclidean spaces are especially simple to work with; for example, parallel transport trivially holds. When defining the properties and motions of φ on $\mathfrak{M}$, it will be convenient if $\mathfrak{M}$ is as 'Euclidean-like' as possible. Since $\mathfrak{M}$ is neither one-dimensional nor a product space, this can be by analogy only. The idea is to impose a homogeneous metric which, given a ball $B_\varepsilon(a)$ centered on a and a map f that 'translates' the ball from a to b, both preserves the distance between points in the ball (no deformation) and ensures the distance between each point x and its image $f(x)$ is invariant for all $x \in B_\varepsilon(a)$ (no 'rotation'):

Def*. 2.1.15 (Homogeneous Metric.) *Let* $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$ *be a complete, simply connected metric space and* $f : B_\varepsilon(x) \to B_\varepsilon(y)$ *be a continuous invertible map defined for some* $B_\varepsilon(x), B_\varepsilon(y) \subset \mathfrak{M}$*, such that* $\mathfrak{d}(a, f(a)) = \mathfrak{d}(b, f(b))$ *and* $\mathfrak{d}(a,b) = \mathfrak{d}(f(a), f(b))$ *for all* $a, b \in B_\varepsilon(x)$*. (This is trivially true when* x *and* y *are the same point.) If* f *exists for all choices* $x, y \in \Sigma$*, then* $\mathfrak{d}$ *is a* ***homogeneous metric*** *on* Σ*.*

This allows defining the counterpart of a straight line in Euclidean space:

Def*. 2.1.16 *A geodesic of* $\mathfrak{M}$ *is a* ***straight*** *curve, and its image a straight path, if* $\mathfrak{d}(x,y)$ *is homogeneous in the sense of Def*. 2.1.15.*

For utility, the metric on $\mathfrak{M}$ will be assumed homogeneous in the sense above. Developing equations of dynamics and notions like 'parallel transport' and wavefront propagation along a path, for example, are much easier when there are no local variations of the metric. If we can say that two closed balls $\bar{B}_\varepsilon(x)$ and $\bar{B}_\varepsilon(y)$ for $x \neq y$ 'look alike' in some sense, even when $\mathfrak{M}$ has minimal geometric structure, then we can more

easily characterize how the 'shape' and extent of an object change as it 'moves' on $\mathfrak{M}$, compared to the more general case where 'deformations' are introduced by inhomogeneity of $\mathfrak{d}$.

Imposing homogeneity on $\mathfrak{d}$ is a very strong constraint on $\mathfrak{M}$. Nevertheless, as noted in the discussion following Def. 2.1.6, $\mathfrak{d}$ is a nonphysical metric and thus can have no observable consequences. This requirement holds whether or not $\mathfrak{d}$ is homogeneous. Insensitivity to the form of $\mathfrak{d}$ is the important constraint — that is what imposes a severe restriction on what dynamics can be postulated. In that sense, there is no new loss of generality in postulating $\mathfrak{M}$ is homogeneous. (Indeed, Proposition 4.2.2 will show that the stochastic behavior implied by Postulate 2.2.8 makes it is unnecessary to postulate a homogeneous metric: any choice of metric on Σ that is consistent with Postulate 2.1.27 sans the homogeneity requirement will be suitable; see also Rmk. 4.2.3.)

For future convenience, two related definitions can be given here:

Def. 2.1.17 (Distance between sets. [40]**)** *Let $\mathfrak{M}$ be a metric space. The distance between a point x and a set $A \subset \mathfrak{M}$ is*

$$\mathfrak{d}(x, A) = \inf\{\mathfrak{d}(x, y) \,|\, y \in A\}\,, \tag{2.4}$$

and the distance between two sets $A, B \subset \mathfrak{M}$ is

$$\mathfrak{d}(A, B) = \inf\{\mathfrak{d}(x, y) \,|\, x \in A,\ y \in B\}\,. \tag{2.5}$$

Homogeneity on $\mathfrak{d}$ allows useful generalized notions of angle, inner product and norm. Angles can be defined from a generalization of the cosine law for triangles, $C^2 = A^2 + B^2 - 2AB\cos\theta$.

Def. 2.1.18 (Angle in a metric space, $\tilde{\angle}$. [16]**)** *The angle between two curves $\mathfrak{c}_1$ and $\mathfrak{c}_2 : [0, 1] \to \mathfrak{M}$ with the same origin $\mathfrak{c}_1(0) = \mathfrak{c}_2(0) = y$ is*

$$\tilde{\angle}(\mathfrak{c}_1(0), \mathfrak{c}_2(0)) \equiv \lim_{s,t\to 0+} \tilde{\angle}\mathfrak{c}_1(s)\, y\, \mathfrak{c}_2(t) \tag{2.6}$$

when the limit exists, where $\tilde{\angle}$ is the ***comparison angle****,*

$$\tilde{\angle}xyz \equiv \arccos \frac{\mathfrak{d}(x, y)^2 + \mathfrak{d}(y, z)^2 - \mathfrak{d}(x, z)^2}{2\,\mathfrak{d}(x, y)\,\mathfrak{d}(y, z)}\,. \tag{2.7}$$

Def. 2.1.19 (Metric space inner product. [16]) *Let the* ***norm*** $\|\dot{\mathfrak{c}}\|$ *be the speed of* $\mathfrak{c}(t)$ *at* $t = 0$*,*

$$\|\dot{\mathfrak{c}}\| \equiv \limsup_{t\to 0} \frac{\mathfrak{d}(\mathfrak{c}(t), \mathfrak{c}(0))}{|t|} . \tag{2.8}$$

Then the **inner product** *is*

$$\langle \mathfrak{c}_1, \mathfrak{c}_2 \rangle \equiv \|\dot{\mathfrak{c}}_1\| \|\dot{\mathfrak{c}}_2\| \cos\left[\tilde{\angle}(\mathfrak{c}_1(0), \mathfrak{c}_2(0))\right] . \tag{2.9}$$

Next, consider to what extent properties of $\mathfrak{M}$ can and should determine what might be characterized as intrinsic 'scales.' First, consider boundedness:

Def. 2.1.20 (Boundedness. [14]) *The* ***diameter*** *of a region* U *of* $\mathfrak{M}$ *is* $diam(U) = \sup\{\mathfrak{d}(x, y) \,|\, x, y \in U\}$*; then* U *is* ***bounded*** *if* $diam(U)$ *is finite.*

Remark 2.1.21 (Closed and bounded metric spaces. [14]) *Every metric space is closed [14]. Moreover, if* $\mathfrak{d}$ *is a metric then* $\rho = \mathfrak{d}/(1+\mathfrak{d})$ *is also a metric, so that, given the homeomorphism* $h : \Sigma \to \Sigma$ *defined by* $h(x) = x$*,* $(\Sigma, \mathfrak{d})$ *and* (Σ, ρ) *are topologically equivalent. Thus, a bounded metric can be equivalent to an unbounded metric.*

Def. 2.1.22 (Total boundedness. [14]) *A metric space* $\mathfrak{M}$ *is totally bounded if for every* $\varepsilon > 0$ *there is a finite set* $\{x_1, x_2, \ldots, x_n\} \subset \mathfrak{M}$ *such that* $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_n) = \mathfrak{M}$*.*

Total boundedness can be also be stated in terms of Cauchy sequences (Def. 2.1.5) [14]:

Theorem 2.1.23 *A metric space* $\mathfrak{M}$ *is totally bounded if and only if every sequence has a Cauchy subsequence.*

Compactness is a useful property when it is present.

Def. 2.1.24 (Compactness. [40]) *A topological space is called* ***compact*** *if every open cover possesses a finite subcover. This means that* $\mathfrak{M}$ *is compact if the following holds: If* $\mathfrak{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ *is an arbitrary open cover of* $\mathfrak{M}$*, i.e.,* $U_\lambda \subset \mathfrak{M}$ *open and* $\bigcup_{\lambda\in\Lambda} U_\lambda = \mathfrak{M}$*, then there is a finite number of* $\lambda_1, \ldots, \lambda_r \in \Lambda$ *such that* $U_{\lambda_1} \cup \cdots \cup U_{\lambda_r} = \mathfrak{M}$*.*

The following theorem characterizes compactness in general metric spaces [14]:

Theorem 2.1.25 *A metric space is compact if and only if it is complete and totally bounded.*

The role of $\mathfrak{d}$ will primarily be restricted to defining the dynamics of φ. An important goal is that $\mathfrak{d}$ should have no discernible relation to a dynamically emergent physical metric $\mathbf{g}$. If $\mathfrak{M}$ is bounded then $diam(\mathfrak{M})$ is the maximum possible $\mathfrak{d}$-measured distance scale. In light of Rmk. 2.1.21, however, the distinction between bounded and unbounded is artificial; an equivalent metric can make $\mathfrak{M}$ unbounded and *vice versa*. Hence, to ensure $\mathfrak{d}$ has no effect on the physical distances measured by $\mathbf{g}$, physical distances will be determined by the φ dynamics alone. That is, the dynamics of φ can effect an arbitrary rescaling of $\mathfrak{d}$-measured distances[2], thereby obtaining an unbounded physical space from a bounded metric space. Hence, $\mathfrak{M}$ can be taken to be bounded without loss of generality.

If $\mathfrak{M}$ is bounded and an emergent physical spacetime metric does not depend on the form of $\mathfrak{d}$, then Σ cannot be a discrete set. Discreteness would imply $\inf\{\mathfrak{d}(x,y)\,|\,x,y\in\Sigma\}$ imposes a finite minimum distance on $\mathfrak{M}$. The following theorem guarantees that every neighborhood of $x\in\Sigma$ is an infinite set when $\mathfrak{M}$ is complete (Def. 2.1.6).

Theorem 2.1.26 *(Munroe 1953, p.27 [55]) Given a neighborhood $B_\varepsilon(x)=\{y\,|\,\mathfrak{d}(x,y)<\varepsilon\}$, if x is a limit point of $P\subset\mathfrak{M}$ then for every $\varepsilon>0$, $P\cap B_\varepsilon(x)$ is an infinite set.*

The desired properties of $\mathfrak{M}$ can be summarized in a single postulate.

Postulate 2.1.27 (Metric space $\mathfrak{M}$.) *An irreducible field φ (Sect. 2.2) can be defined on an uncountable set Σ with the following properties and structure:*

T1. *Associated with Σ is a metric $\mathfrak{d}:\Sigma\times\Sigma\to\mathbb{R}$ such that Def. 2.1.2 holds. That is, $\mathfrak{M}\equiv(\Sigma,\mathfrak{d})$ is a metric space, and hence also a topological space $(\Sigma,\mathcal{T}_\Sigma)$ having the topology $\mathcal{T}_\Sigma$ which is induced by $\mathfrak{d}$.*

T2. *$\mathfrak{M}$ is complete (Def. 2.1.6).*

T3. *$\mathfrak{M}$ is simply connected (Def. 2.1.11).*

T4. *$\mathfrak{M}$ is homogeneous in the sense of Def*. 2.1.15. (Rmk. 4.2.3 will explain why this postulate is unnecessary; it is imposed for its utility.)*

T5. *In light of Rmk. 2.1.21, $\mathfrak{M}$ can be taken to be bounded without requiring it to be actually postulated.*

[2]Penrose diagrams represent infinite distances finitely. Here, on the other hand, infinite distances should be obtainable by (dynamically) rescaling finite distances.

2.2 Irreducible Field φ on $(\Sigma, \mathfrak{d})$

Sect. 2.1 proposes a minimal 'canvas,' $\mathfrak{M}$, upon which a theory might be drawn and painted. In this dubious metaphor, the paint is the irreducible φ field that will mediate spacetime emergence and determine its properties. This section motivates and postulates specific attributes of φ.

Energy and momentum are dynamical quantities with very special status in physics. They are fundamental to the Hamiltonian (and hence Lagrangian) formalism on which quantum theory depends, and they determine spacetime geometry in general relativity. In an emergence picture, however, they cannot be taken for granted because they depend on physical space and time which do not exist prior to spacetime emergence:

- The Hamiltonian is the generator of time translations, but the absence of time implies there is no Hamiltonian to generate it;
- linear momentum is the generator of spatial translations, but the absence of a notion of relative position implies momentum is undefined; and
- angular momentum is the generator of rotations, but since a notion of angle is not meaningful neither is angular momentum.

Even if it were possible to define consistent notions of energy and momentum on $\mathfrak{M}$, without a unique map from points on $\mathfrak{M}$ to points on a physical manifold $\mathscr{M}$, *i.e.*, a physical spacetime with $\mathscr{M}$ unique, the exercise would be purely formal. That is why $\mathfrak{M}$ should be considered nonphysical. The physical entity is φ, which requires the mathematical structure of $\mathfrak{M}$ for its definition and manipulation in a theory.

Thus, energy and momentum should be considered emergent notions and not elementary properties of φ. It makes no sense to specify the dynamics of φ by writing down a Lagrangian. Instead it is necessary to specify a more elementary form of dynamics, one that can be defined locally on $\mathfrak{M}$. This will require augmenting $\mathfrak{M}$ with a nonphysical cosmic time $\mathfrak{t}$ to do so. Then the motions of φ should be expressible on curves parameterized by $\mathfrak{t}$; curves are the most general, measurable geometric objects on $\mathfrak{M}$.

Finally, if φ is the only irreducible field that can manifest in the absence of physical spacetime, its properties and dynamics must provide whatever is necessary to

generate an 'initial condition' $\varphi[\mathfrak{U};\mathfrak{t}_0)$ in a region $\mathfrak{U}$ of $\mathfrak{M}$. Otherwise cosmogenesis would not be possible in this picture.

2.2.1 Elementary oscillator field, $\tilde{\omega}[\mathfrak{M}]$

A primary role of φ is to mediate the emergence of spacetime, relying only on the minimal prior structure of Postulate 2.1.27 and local, elementary properties of φ. An elementary property of φ obviously cannot emerge (or else it would not be elementary). The lack of notions like spatial dimension and angle, φ on $\mathfrak{M}$ means φ cannot be vector-, tensor-, or spinor-valued; φ is effectively constrained to be scalar valued. Hence take φ to be a real-valued field as the simplest choice.

What might the elementary scalar property of φ represent? If energy is an emergent notion as the Introduction argues, energy is not a viable candidate. If φ were a quantum field it might represent a probability amplitude, but it is not. No physical quantities are obvious candidates — φ likely has no direct counterpart in existing theories.

In a picture where the initial conditions for the Big Bang are obtained from φ and a collection of quantum fields, a scenario for cosmogenesis must include at least two stages. The quantum fields cannot manifest without at least some form of primitive spacetime, so in the 'primordial' condition the only available dynamics come from φ self interaction. Then the initial stage, the onset of cosmogenesis, is a transition from the primordial condition of $\varphi[\mathfrak{M}]$ to a primitive, predecessor spacetime in a neighborhood $\mathfrak{U} \subset \mathfrak{M}$; it can depend only on properties of $\varphi[\mathfrak{U}]$. The second stage presumably encompasses the evolution of that predecessor into an emergent spacetime in $\mathfrak{U}$. Since a primitive spacetime exists from the first stage, the quantum fields can now also participate.

What this means is the properties of φ, in particular whatever the real-valued quantity at each point of $\mathfrak{M}$ represents along with the φ field equation, must contain the wherewithal to obtain at least the limited form of physical spacetime of the first stage above. Hausdorff separability of the points in $\mathfrak{M}$ means that, in principle, φ at a point σ can vary relatively to φ on the infinitesimal neighborhood $B_\varepsilon(\sigma)$. That and the simple connectedness of $B_\varepsilon(\sigma)$ suggests σ could be a precursor to a position in physical space. The variations of φ at σ, ignoring the variations of φ on its neighborhood $B_\varepsilon(\sigma)\backslash\sigma$, can be taken to be evolution with respect to a precursor of time. Finally, interactions of

$\varphi(\sigma)$ with $\varphi[B_\varepsilon(\sigma)\backslash\sigma]$ could cause propagation of field changes and thereby connect the precursors of space and time to obtain a precursor of spacetime.

Hence assume the elementary property of φ at each point somehow obtains a physical precursor of local time, such that it eventually manifests in local dynamics (*e.g.*, signal propagation), randomness in time, and, indirectly, even an arrow of time. Propose $\delta_{\mathfrak{t}}$ as the minimum interval of this 'primitive' time.

Now consider the propagation of φ inhomogeneities on $\mathfrak{M}$. The 'signal' propagation is constrained by the boundedness of $\mathfrak{M}$, Postulate 2.1.27(T5): Clearly if $\mathfrak{M}$ is bounded, then $\mathfrak{d}$ measures a finite maximum distance $diam(\mathfrak{M})$; see Def. 2.1.20. Since $\mathfrak{M}$ is homogeneous by Postulate 2.1.27(T4), no straight path between two points of $\mathfrak{M}$ will be longer than $diam(\mathfrak{M})$. Hence, in principle, if $\mathfrak{d}$ directly or indirectly measures signal propagation distances, the bounded nature of $\mathfrak{M}$ could be detected in an emergent spacetime by observing signals that were emitted sufficiently distant in the past. This would violate a requirement of Sect. 2.1 that the form of $\mathfrak{d}$ must have no observable effect in an emergent spacetime.

One way to avoid this violation is to assume the primitive time has no minimum interval, *i.e.*, $\delta_{\mathfrak{t}} \to 0$. To accommodate a flow of primitive time, it will be convenient to mathematically abstract the notion by augmenting $\mathfrak{M}$ with 'cosmic time,' a real-valued affine parameter $\mathfrak{t}$. Then by rescaling cosmic time intervals $\Delta\mathfrak{t}$ and the distance metric $\mathfrak{d}$, the effective scale of phenomena can be whatever we wish. As long as the propagation speed of signals scales in the same way as the characteristic scale of emergent phenomena, we can have a finite but arbitrarily large physical spacetime emerge in $\mathfrak{U}$. Such rescaling should allow, in principle, obtaining a scale invariant fundamental theory wherein different, arbitrarily large universes can arise at different cosmic times or different regions of $\mathfrak{M}$ from the same dynamics but different initial conditions. This picture motivates two postulates.

Postulate 2.2.1 (Cosmic time.) *The* cosmic time $\mathfrak{t}$ *is a continuous, dimensionless, monotonically increasing, real-valued affine parameter which orders a sequence of φ configurations on $\mathfrak{M}$. Cosmic time applies globally on $\mathfrak{M}$ by fiat, so that a time-augmented metric space can be defined as $\mathfrak{M}_{\mathfrak{t}} \equiv \mathfrak{M} \times \mathbb{R} \equiv (\Sigma, \mathfrak{d}; \mathfrak{t})$.*

Remark 2.2.2 *Cosmic time has the same mathematical status as $\mathfrak{M}$. Like $\mathfrak{M}$, it is an artificial construction in that it is introduced solely to provide minimal mathemat-*

ical structure for defining the physical elements and their motions. Consequently, an emergent physical spacetime should contain no evidence of cosmic time.

The φ field will be introduced indirectly. Define a field of point-like oscillators, $\tilde{\omega}[\mathfrak{M}]$, of inconstant, arbitrarily large frequency ($\omega(\sigma) \to \infty$) as the elementary substructure of the φ field.

Postulate 2.2.3 (Irreducible oscillator field, $\tilde{\omega}[\mathfrak{M};\mathfrak{t}]$.) *A unique simple harmonic oscillator $\tilde{\omega}(\sigma;\mathfrak{t})$ exists at each $\sigma \in \mathfrak{M}$, whose frequency and evolution are measured with respect to cosmic time.*

Due to a paucity of suggestive symbols, $\omega(\sigma)$ will denote two different things: $\tilde{\omega}(\sigma)$ will denote the oscillator at the position σ, while $\omega(\sigma)$ (no tilde) will denote the frequency of the oscillator $\tilde{\omega}(\sigma)$.

The difference in phase between oscillators will be needed to define φ.

Def*. 2.2.4 (Oscillator position, $\vartheta(\sigma;\mathfrak{t})$.) *For all $\sigma \in \mathfrak{M}$, the* dimensionless position $\vartheta(\sigma;\mathfrak{t}) \in [0, 2\pi)$ *is the difference in position of $\tilde{\omega}(\sigma;\mathfrak{t})$ from $\tilde{\omega}(\sigma_0;\mathfrak{t})$ at the same cosmic time $\mathfrak{t}$, where $\sigma_0 \in \mathfrak{M}$ is a fixed (but arbitrary) reference point.*

Def*. 2.2.5 (Relative oscillator phase, $\phi(\sigma,\sigma';\mathfrak{t})$.) *Let $\mathfrak{c} : [0,1] \to \mathfrak{M}$ be a geodesic with $\mathfrak{c}(0) = \sigma$ and $\mathfrak{c}(1) = \sigma'$, and let T_n be a sequence of partitions $(0{=}t_1, t_2, \ldots, t_n{=}1)$. Then the dimensionless relative phase between $\tilde{\omega}(\sigma)$ and $\tilde{\omega}(\sigma')$ at time $\mathfrak{t}$ is the supremum over partitions of accumulated relative oscillator positions:*

$$\phi(\sigma,\sigma';\mathfrak{t}) \equiv \sup \lim_{n\to\infty} \sum_{i=1}^{n-1} \big[\vartheta(\mathfrak{c}(t_i);\mathfrak{t}) - \vartheta(\mathfrak{c}(t_{i+1});\mathfrak{t})\big]. \tag{2.10}$$

The relative phase between two oscillators is more general than their relative displacement. The former can be any finite number while the latter is restricted to $[0, 2\pi)$.

Postulate 2.2.6 (Elementary oscillator properties.) *The oscillator field of Postulate 2.2.3 has the following properties.*

W1. ***Infinite oscillator frequency*** *in the sense that $\omega(\sigma;\mathfrak{t}) \to \infty$ for all $\mathfrak{t}$. In general $\omega(\sigma;\mathfrak{t})$ is not constant on any time interval.*

W2. ***Continuity*** *of relative oscillator phase: Given* $\tilde{\omega}(\sigma;\mathfrak{t})$ *and its infinitesimal neighborhood,* $\tilde{\omega}[B_\varepsilon(\sigma);\mathfrak{t})$*,*

$$\vartheta(\sigma;\mathfrak{t}) - \vartheta(\sigma';\mathfrak{t}) \to 0 \;\; as \;\; \varepsilon \to 0\,, \quad \forall \sigma' \in B_\varepsilon(\sigma)\,. \tag{2.11}$$

W3. ***Finite but unbounded relative phase*** $\phi(\sigma,\sigma';\mathfrak{t})$ *at any time* $\mathfrak{t}$ *when* $\mathfrak{d}(\sigma,\sigma')$ *is finite.*

The physical field is the oscillator field, $\tilde{\omega}[\mathfrak{M}]$ given by Postulates 2.2.3 and 2.2.6. No new postulate is needed for the elementary scalar field that will be of primary interest in this thesis; it is sufficient to define φ in terms of the oscillator field:

Def*. 2.2.7 (Irreducible scalar field, $\boldsymbol{\varphi[\mathfrak{M}_\mathfrak{t}]}$.) *The dimensionless real scalar field* φ *on* $\mathfrak{M}_\mathfrak{t}$ *is*

$$\varphi(\sigma;\mathfrak{t}) \equiv \phi(\sigma,\sigma_0;\mathfrak{t})\,, \tag{2.12}$$

where $\sigma_0 \in \mathfrak{M}$ *is a fixed reference point for all* $\mathfrak{t}$ *and* $\phi(\sigma,\sigma_0;\mathfrak{t})$ *is the relative phase (Def*. 2.2.5) between* $\tilde{\omega}(\sigma;\mathfrak{t})$ *and* $\tilde{\omega}(\sigma_0;\mathfrak{t})$*.*

Thus $\varphi(\sigma_0;\mathfrak{t}) = 0$ for all $\mathfrak{t}$ by definition. Finiteness of φ is protected by the continuity property, Postulate 2.2.6(W2).

2.2.2 Intrinsic stochasticity of φ

Neither the postulates for φ nor for $\mathfrak{M}_\mathfrak{t}$ on which φ lives provide an origin of initial conditions for cosmogenesis. Clearly, a homogeneous scalar field contains no information and hence no phenomena. One solution is to postulate an initial state $\varphi[\mathfrak{M};\mathfrak{t}=0)$ that happens to be conducive to cosmogenesis, then let φ evolve deterministically as a classical field. That idea feels contrived and non-explanatory — it leaves completely unanswered the question of what produced the initial state. A more natural origin of inhomogeneity is random φ fluctuations, originating from an intrinsic property of the point-like oscillators: that would make φ a stochastic field.

According to Postulate 2.2.3 the oscillator field $\tilde{\omega}[\mathfrak{M}]$ is the underlying irreducible physical field; the φ field is defined in terms of $\tilde{\omega}[\mathfrak{M}]$ through Def*. 2.2.7. Since the oscillators are assumed to be the most elementary entities, no external constraint can

exist to impose a 'natural' or intrinsic mean frequency. Thus a reasonable assumption is that an oscillator's frequency can drift over time relatively to the reference oscillator $\tilde{\omega}(\sigma_0)$. This would be due to the accumulation of small stochastic 'impulses,' with the continuity property keeping the frequency difference $\omega(\sigma;\mathfrak{t}) - \omega(\sigma_0;\mathfrak{t})$ finite. Assuming the probability density function for the resulting fluctuations $\delta\varphi(\sigma;\mathfrak{t})$ is scale invariant, similar behavior should occur at all $\mathfrak{d}$-measured scales. Randomness can thus introduce finite inhomogeneities into $\varphi[\mathfrak{M}_\mathfrak{t}]$ that, in principle, provide the initial conditions for cosmogenesis.

Electronic oscillators offer a useful analogue of this idea. Thermally induced random voltage variations modulate the frequency and manifest as oscillator jitter. During a period T that is short compared to the oscillator period but long compared to the time scale of thermal fluctuations, a large number of thermal fluctuations of different amplitudes superpose to obtain a measurable change, δV. A good model [73] for thermal noise is Gaussian white noise, $\mathcal{N}_\mathrm{w}(t)$, which has the properties:

- Expected value $\langle \mathcal{N}_\mathrm{w}(t) \rangle = 0$, where the mean is an ensemble mean;
- Autocorrelation (autocovariance) function

$$\mathcal{R}_\mathrm{w}(\tau) = \langle \mathcal{N}_\mathrm{w}(t+\tau)\, \mathcal{N}_\mathrm{w}(t) \rangle$$

  which states that the correlation length is zero; and
- Spectral density function which is constant for all ω,

$$\mathcal{S}_\mathrm{w}(\omega) = \mathcal{S}_0 = \mathrm{const}\,.$$

In physical systems these properties are idealizations. For example, no practical noise source has a constant spectrum at all scales; the atomic scale imposes an obvious cut-off. For stochasticity in φ the properties must be taken seriously to preserve the scale invariance of the theory. (Scale invariance is a necessary condition for ensuring the form of the distance metric $\mathfrak{d}$ has no observable consequences, given that $\mathfrak{d}$ measures the 'absolute' scale.) Rmk. 3.1.12 will discuss this further.

Stochastic fluctuations $\delta\varphi(\sigma;\mathfrak{t})$ can be a consequence of white noise as follows. During a sample period τ, continual oscillator jitter means $\mathcal{N}_\mathrm{w}(\mathfrak{t})$ corresponds to a large number of frequency changes $\delta\omega(\sigma)$, which can be thought of as a series of impulses

that modulate $\omega(\sigma)$. Since $\tilde{\omega}(\sigma)$ is an infinite frequency oscillator, $\delta\omega/\omega$ is infinitesimal, and assuming a uniform distribution for the jitter is reasonable. The cumulative effect of many frequency-changing impulses during τ is that the oscillator position deviates, compared to what it would have been in the absence of impulses, by an amount $\delta\vartheta(\sigma) \in \mathbb{R}$ which may be positive or negative.

Formally, Brownian noise is the integral of white noise (see Subsect. 3.1.2), so the cumulative deviations in oscillator position over a time τ will obey a probability density function for Brownian noise (Def. 3.1.2), *i.e.*, Normal distribution with zero mean and variance τ. The correspondence with classical Brownian motion of a particle is straightforward: collisions of the particle with molecules in the surrounding fluid act as impulses that change the particle velocity v, analogously to jitter that changes $\omega(\sigma)$, and the cumulative effect of the impulses is that the displacement of the particle from its starting position obeys the PDF for Brownian motion, like the displacement $\vartheta(\sigma)$ for the oscillator $\tilde{\omega}(\sigma)$. The Brownian nature of $\delta\vartheta(\sigma; I_\tau]$, $I_\tau \equiv (\mathfrak{t}, \mathfrak{t}+\tau]$, can also be viewed as a random walk, where an impulse $\delta\omega(\sigma, \mathfrak{t}_*)$ at time $\mathfrak{t}_*$ causes a step $\delta\vartheta(\sigma, \mathfrak{t}_*)$ of variable size that may be positive or negative.

It will be more useful to work with relative phase changes $\phi(\sigma, \sigma_0; \mathfrak{t})$ than deviations from expected oscillator position $\delta\vartheta(\sigma; \mathfrak{t})$. Since the white noise impulses also change the nonlocally-defined relative phase $\phi(\sigma, \sigma_0; \mathfrak{t})$, the field $\varphi(\sigma; \mathfrak{t}) \equiv \phi(\sigma, \sigma_0; \mathfrak{t})$ will obey the PDF for Brownian noise whenever dynamics can be ignored.

Hence, the net change in $\varphi(\sigma)$ during a time period τ can be expected to have two contributions: intrinsic stochasticity of $\varphi(\sigma)$ as developed above, and a deterministic dynamical response to changes of φ in the neighborhood of σ. The intrinsic stochasticity is specified thusly:

Postulate 2.2.8 (Oscillator stochasticity.) *Let τ be an arbitrary period of cosmic time. During an interval $I_\tau = [\mathfrak{t}_0, \mathfrak{t}_0+\tau]$ with $\mathfrak{t}_0$ arbitrary, the oscillator $\tilde{\omega}(\sigma)$ undergoes a frequency change $\delta\omega(\sigma; I_\tau]$ with a corresponding total change in relative phase $\delta\varphi(\sigma; I_\tau] \equiv \delta\phi(\sigma, \sigma_0; I_\tau]$. This change includes a stochastic contribution $\delta\varphi(\sigma; I_\tau]_{\text{rand}}$ which obeys the probability density function for Brownian noise:*

$$f(\delta\varphi, \tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(\delta\varphi)^2}{2\tau}\right), \qquad \tau = \mathfrak{t} - \mathfrak{t}_0\,, \tag{2.13}$$

i.e., the Normal distribution with zero mean and variance τ (see Def. 3.1.2). The total change $\delta\varphi(\sigma; I_\tau]$ obeys the PDF (2.13) *almost everywhere on $\mathfrak{M}_\mathfrak{t}$.*

The last sentence of the postulate asserts that stochastic fluctuation is the usual condition for $\varphi[\mathfrak{M}_\mathfrak{t}]$, and hence $\tilde{\omega}[\mathfrak{M}]$, at all distance and time scales. Then manifestation of dynamical phenomena of physical interest will only occur in special cases of random origin. The dynamical contribution to $\delta\varphi$, *i.e.*, a coupling term $\delta\varphi(\sigma; I_\tau]_{\text{coupling}}$, will be specified in the next subsection.

A characteristic of Brownian motion is infinite variation along every finite length curve. (**Variation** in this sense is a property of a function, and has little to do with the calculus of variations; see Subsect. 3.1.1.) Infinite variation of $\varphi[\mathfrak{c}]$ is the reason that φ is not differentiable anywhere in the usual sense. It motivates the introduction of an approximate partial derivative to replace the usual partial derivative, *e.g.*,

$$\lim_{\delta\mathfrak{t}\to 0} \frac{\delta\bar{\varphi}(\sigma;\mathfrak{t})}{\delta\mathfrak{t}} = \bar{\partial}_\mathfrak{t}\varphi(\sigma;\mathfrak{t})\,.$$

(The approximate derivative is denoted by $\bar{\partial}_\mathfrak{t}\varphi$ instead of $\partial_\mathfrak{t}\bar{\varphi}$ to conform with the notation introduced in Sect. 3.3.)

Def. 2.2.9 (Variation of a function. [42]**)** *The variation of a real-valued function $\phi(t)$ over an interval $[a,b] \subset \mathbb{R}$ is*

$$\mathfrak{V}_\phi([a,b]) = \sup \sum_{i=1}^{n} \left|\phi(t_i^n) - \phi(t_{i-1}^n)\right|,$$

with the supremum taken over partitions $a = t_0^n < t_1^n < \cdots < t_n^n = b$. Since the sum increases as the partition is made finer, the variation of ϕ also can be written

$$\mathfrak{V}_\phi([a,b]) = \lim_{\delta_n\to 0} \sum_{i=1}^{n} \left|\phi(t_i^n) - \phi(t_{i-1}^n)\right|, \qquad \delta_n = \max_{1\le i\le n} (t_i - t_{i-1})\,. \tag{2.14}$$

Hence the variation of ϕ along a curve $\mathfrak{c}(t)$ is

$$\mathfrak{V}_\phi(\mathfrak{c}) = \sup \lim_{n\to\infty} \sum_{i=1}^{n} \left|\phi(\mathfrak{c}(t_i)) - \phi(\mathfrak{c}(t_{i-1}))\right|. \tag{2.15}$$

2.2.3 Elementary dynamics of φ

Elementary dynamics of the φ field are to be described in terms of the distance metric and cosmic time. Assuming each oscillator is completely characterized by its frequency and relative phase, dynamics require coupling between oscillators. It will be assumed all dynamics are local. That is, consistent with Hausdorff separability

(separability into neighborhoods), an oscillator $\tilde{\omega}(\sigma)$ will only couple to its infinitesimal neighborhood, $\tilde{\omega}[B_\varepsilon(\sigma)\backslash\sigma]$. If local dynamics are what maintain the phase continuity of Postulate 2.2.6(W2), the coupling strength between neighboring oscillators must be infinite in an infinitesimal neighborhood.

Specification of the dynamics must respect the role of $\mathfrak{M}_\mathfrak{t}$ as a mathematical space which has been introduced to allow precise definition of elements of the theory, and not take $\mathfrak{M}_\mathfrak{t}$ to be a primitive form of physical spacetime. That is, there is no intrinsic correspondence between $\mathfrak{M}_\mathfrak{t}$ and an emergent physical spacetime $(\mathscr{M}, \mathbf{g})$; dynamics provide the only connection between them. Encoding any geometric notions into the postulates for dynamics, *e.g.* finite dimensionality or the existence of orthogonal directions, would imply that such geometric notions are ultimately derived from $\mathfrak{M}_\mathfrak{t}$ rather than being emergent. Inevitably, that would leave a trace of $\mathfrak{M}_\mathfrak{t}$ in the description of the physical spacetime, in conflict with the requirement that $\mathfrak{M}_\mathfrak{t}$ has no observable consequences.[3]

Curves are the most general measurable geometric objects on $\mathfrak{M}_\mathfrak{t}$, so they will be the starting point for defining the dynamics. That is, the coupling can be defined between two oscillators $\tilde{\omega}(\sigma)$ and $\tilde{\omega}(\sigma')$, say along a straight path $\gamma(\sigma, \sigma')$ in the limit $\mathfrak{L}(\gamma) \to 0$. Each oscillator $\tilde{\omega}(\sigma)$ couples to its neighborhood $\tilde{\omega}[B_\varepsilon(\sigma)]$, so the net coupling of $\tilde{\omega}(\sigma)$ should just be the average contribution from all straight paths $\gamma(\sigma, \sigma')$ for all $\sigma' \in B_\varepsilon(\sigma)$, in the limit $\varepsilon \to 0$. Hence, by postulating the dynamics in terms of paths or curves and developing a procedure for averaging over neighborhoods (the task of Sect. 3.5), it should be possible to avoid over-specifying the dynamics.

As an irreducible field, φ is locally defined, in contradistinction to reducible fields derived from it which are not locally defined. For example, a reducible field $\Phi(\sigma)$ in n emergent spatial dimensions may be constructed by averaging over over a neighborhood of σ, but such averaging is intrinsically nonlocal because all samples at spatially separated points are taken at the same cosmic time. Thus the dynamics of φ must also be locally defined. In developing the dynamics it will be more convenient to work directly with the relative phase ϕ than with φ, then at the end restate the result in terms of φ.

The idea to be captured by the coupling between oscillators is an analogue

[3]Certainly connectedness of an emergent spacetime is inherited from $\varphi[\mathfrak{M}_\mathfrak{t}]$, so arguably a physical spacetime inherits that from $\mathfrak{M}_\mathfrak{t}$. However, Rmk. 2.2.2 argues connectedness of $\mathfrak{M}_\mathfrak{t}$ is inherited from the oscillators.

of Newton's second law for a mass on a spring, $\ddot{x} = -\frac{k}{m}x$. While suggestive, the mechanical analogy is imperfect because an accelerated mass has inertia and thus can oscillate about its equilibrium position, whereas the oscillators $\tilde{\omega}(\cdot)$ have no inertia.

Consider an oscillator at σ coupled to neighbors at σ_1, σ_2 lying on the same curve. Fig. 2.2 shows the analogous mechanical system: a mass is at rest at σ, connected to two springs having constant k, respectively fixed at distances Δx_1 and Δx_2 from σ and exerting forces $F_1 = -k\Delta x_1$ and $F_2 = -k\Delta x_2$. Each unstretched spring has zero length so its extension is always positive. Then $\ddot{x}(\sigma) = -\frac{k}{m}(\Delta x_2 - \Delta x_1)$. With damping, $F_{\rm d} \propto \dot{x}$ so $\ddot{x}(\sigma) = -2\eta\dot{x} - \frac{k}{m}(\Delta x_2 - \Delta x_1)$.

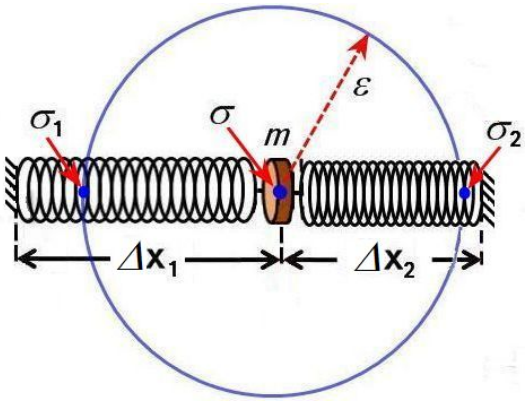


Figure 2.2: Spring analogue.

Now consider the correspondence between oscillator coupling and the spring analogue in Fig. 2.2. A 'displacement' is now a relative phase (Def*. 2.2.5), so the correspondence is $\Delta x_i \to \phi(\sigma_i, \sigma)$:

$$k\,\Delta x_i \;\to\; g\,\phi(\sigma_i,\sigma;\mathfrak{t})\,; \quad m\ddot{x} \;\to\; \rho\frac{\delta^2}{\delta\mathfrak{t}^2}\phi(\sigma_i,\sigma;\mathfrak{t})\,, \tag{2.16}$$

where g is the coupling strength analogous to the spring constant and ρ is a field 'inertia' analogous to the mass. Finite differences have replaced derivatives in the second line because, while derivatives like $\ddot{\phi}$ and $\dot{\phi}$ appear to be consistent with Postulate 2.2.3, the postulated intrinsic stochasticity of $\omega(\cdot)$ makes ϕ nondifferentiable. The finite difference approximation to the second derivative can be computed by a centered difference approximation,

$$\frac{\delta^2\phi(\sigma;\mathfrak{t})}{\delta\mathfrak{t}^2} = \frac{\phi(\sigma;\mathfrak{t}+h) - 2\phi(\sigma;\mathfrak{t}) + \phi(\sigma;\mathfrak{t}-h)}{h^2} + \mathcal{O}(h^2)\,, \tag{2.17}$$

where $h = \tau/2$. Because this finite difference approximation is for local dynamics the error term $\mathcal{O}(h^2)$ will vanish as $\tau \to 0$. To be physically meaningful, τ cannot be smaller than the time needed for field changes to propagate the distance $\mathfrak{d}(\sigma_i, \sigma)$.

Fig. 2.2 shows two springs whose superposed forces determine the acceleration. By analogy, $\rho\,\delta^2/\delta\mathfrak{t}^2[\phi(\sigma_2,\sigma;\mathfrak{t}) - \phi(\sigma,\sigma_1;\mathfrak{t})] = -g\cdot[\phi(\sigma_2,\sigma;\mathfrak{t}) - \phi(\sigma,\sigma_1;\mathfrak{t})]$ will also be obtained by superposition.

The field 'inertia' ρ serves to reduce the rate at which the field $\varphi(\sigma;\mathfrak{t})$ can change in time, but is non-dissipative. It should be present because the field cannot change instantaneously at σ to remove an inhomogeneity, except perhaps in the limit of infinite coupling with its neighborhood. Presumably, stochastic and propagating inhomogeneities continually modify $\varphi(\sigma;\mathfrak{t})$ at all time scales, so that any given inhomogeneity must 'compete' with all the others in its ability to determine the new field value at σ. The property of the field that can serve the role of ρ is the total variation, $\mathfrak{V}_\phi(\mathfrak{c})$, for $\mathfrak{c}$ in the neighborhood of σ. Because the total variation is time dependent, we need $m\ddot{x} \to \frac{\delta}{\delta\mathfrak{t}}[\mathfrak{V}_\phi(\mathfrak{c})\,\delta\varphi/\delta\mathfrak{t}]$. Hence, let $\rho = \rho(\mathfrak{V}_\phi(\mathfrak{c})) = \rho_0\mathfrak{V}_\phi(\mathfrak{c})$, where ρ_0 is a constant.

Next consider the coupling strength g, analogous to k. In the mechanical analogy, the spring constant depends on the material and temperature. The φ field has only one 'material' of interest, the oscillators. The effect of the temperature is to weaken the molecular bonds (reduce Young's modulus) due to thermal motion within the material and thereby reduce the spring constant. The analogous quantity for oscillator coupling is the total variation, Def. 2.2.9.

Remark 2.2.10 (Coupling strength, g(·).) *If $\mathfrak{d}(\sigma,\sigma')$ is kept constant, an increase in the total variation of $\phi(\sigma,\sigma')$ corresponds to an increase in number or magnitude of sign reversals of ϕ along a geodesic $\mathfrak{c} : [0,1] \to \gamma(\sigma,\sigma')$. Propagation of changes in ϕ along $\mathfrak{c}$ thus involves corresponding reversals in direction of propagation, thereby reducing the propagation speed and thus the effective coupling strength. Intrinsic stochasticity of the oscillators, asserted by Postulate 2.2.8 for every point of $\mathfrak{M}$, is presumably the dominant source of variation, but sign changes due to dynamical motions must also be considered.*

Thus the coupling between $\tilde{\omega}(\sigma)$ and $\tilde{\omega}(\sigma')$ should be sensitive to the total variation of ϕ along the geodesic $\mathfrak{c}$. That is, the distance scale $\mathfrak{d}(\sigma,\sigma')$ is fixed, then g decreases as the total variation along curves between σ and σ' increases. In the locality limit, $\mathfrak{d}(\sigma,\sigma') \to 0$,

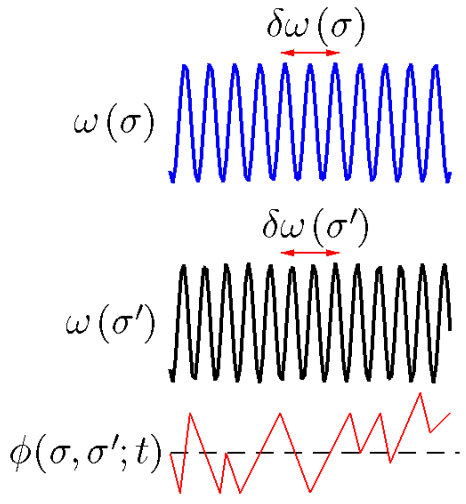


Figure 2.3: Variation of ϕ.

contributions by the geodesic $\mathfrak{c}$ *dominate so that* $g \propto 1/\mathfrak{V}_\phi(\mathfrak{c})$*. From this it follows that* $g \to \infty$ *as* $\mathfrak{d}(\sigma, \sigma') \to 0$*, and the continuity of* ϕ *asserted by Postulate 2.2.6(*W2*) can be taken to be due to infinite coupling in this limit.*

Therefore, rather than assuming the field coupling strength $g(\phi(\sigma_i, \sigma; \mathfrak{t}))$ *depends only on* $\mathfrak{d}(\sigma_i, \sigma)$*, take it to depend on the total variation of* ϕ *along the curve* $\mathfrak{c}$ *between those points. That is, the coupling strength* g *should be a function of the form,*

$$g = g\big(\mathfrak{V}_\phi(\mathfrak{c})\big) = g_0\, \mathfrak{V}_\phi(\mathfrak{c})^{-1}\,, \tag{2.18}$$

where g_0 *is a 'bare' dimensionless coupling constant. To compute* g *over an ensemble,* $\mathfrak{V}_\phi$ *must be computed for each member. The behavior of* g *must be postulated (see Postulate 2.2.11), but the foregoing provides the motivation.*

In general, the form (2.18) is local only if the speed of the curve is the same as the propagation speed of a field change. In all cases, the coupling strength is locally defined in the limit $\mathfrak{L}(\mathfrak{c}) \to 0$.

Since $\mathfrak{M}$ is simply connected and each oscillator couples to all oscillators in its infinitesimal neighborhood, there are uncountably many curves between σ and σ' — the net coupling should depend on contributions from all of them. A path integral is required, but defining a path integral to sum over all curves between σ and σ' is likely to be highly nontrivial in a general metric space.

Because ϕ does not vary smoothly along a curve due to intrinsic stochasticity of the oscillators, the simple relative phase $\phi(\sigma_i, \sigma)$ in Eq. (2.16) is not a correct analogue of a spring displacement Δx_i. In general, given fixed values for ϕ at the endpoints σ_i and σ, *i.e.*, $\phi(\sigma_i, \sigma_0)$ and $\phi(\sigma, \sigma_0)$ fixed, a single sample that compares $\phi(\sigma_i, \sigma)$ with the mean $\langle \phi(\sigma_i, \sigma) \rangle$ along a geodesic $\mathfrak{c}$ almost surely yields $\phi(\sigma_i, \sigma) \neq \langle \phi(\sigma_i, \sigma) \rangle$ due to the stochastic fluctuations along $\mathfrak{c}$. Hence, take the correct analogue of Fig. 2.2 to be

$$\frac{k}{m}\,\Delta x_i \to\ \big\langle g(\mathfrak{V}_\phi(\sigma_i, \sigma; \mathfrak{t})) \cdot \phi(\sigma_i, \sigma; \mathfrak{t}) \big\rangle_{2\tau}\,; \quad m\ddot{x}\ \to\ \frac{\delta}{\delta \mathfrak{t}} \left\langle \rho(\mathfrak{V}_\phi(\mathfrak{c}))\, \frac{\delta}{\delta \mathfrak{t}} \big\langle \phi(\sigma_i, \sigma; \mathfrak{t}) \big\rangle_\tau \right\rangle_\tau , \tag{2.19}$$

where 2τ is the time interval of interest. The averaging interval for the second derivative approximation is chosen to be $h = \tau$ to be consistent with the finite difference approximation (2.17).

The expectation value $\big\langle g(\mathfrak{V}_\phi(\sigma_i, \sigma; \mathfrak{t})) \cdot \phi(\sigma_i, \sigma; \mathfrak{t}) \big\rangle_{2\tau}$ in Eq. (2.19) requires explanation. It can be interpreted as an ensemble average of similarly prepared field configurations along a geodesic between σ_i and σ, where each member of the ensemble is

obtained by a (cosmic) time average with the same starting time $\mathfrak{t}_*$ and duration 2τ. Thus the configuration term obeys a probability density function.

To be consistent with the requirement that the nonphysical metric has no observable consequences, time scales like τ must be determined dynamically in a scale invariant manner. Like the approximation (2.17) for (2.16), if the 'acceleration' of ϕ is to be attributed to the configuration, τ cannot sensibly be smaller than the time needed for field changes to propagate the distance $\mathfrak{d}(\sigma_i, \sigma)$, *i.e.*, along a geodesic. On the other hand, if τ is much longer than the propagation time the error term in the approximation (2.17) may render Eq. (2.19) an inadequate representation of the local dynamics.

Unlike the spring analogy, $\tilde{\omega}(\sigma)$ at time $\mathfrak{t}_*$ couples to an (uncountable) infinitesimal neighborhood, $\tilde{\omega}[B_\varepsilon(\sigma)\backslash\sigma]$ for $\varepsilon \to 0$. In this limit the coupling is completely local. All oscillators on $\bar{B}_\varepsilon(\sigma)$ will contribute equally to the net change of $\omega(\sigma)$ because the distance metric $\mathfrak{d}$ is homogeneous, so without loss of generality it is sufficient to stipulate the coupling between $\tilde{\omega}(\sigma)$ and $\tilde{\omega}$ at a single point $\sigma' \in \bar{B}_\varepsilon(\sigma)\backslash\sigma$. The net coupling, then, is the sum of contributions of the form (2.19) for all $\sigma' \in \bar{B}_\varepsilon(\sigma)\backslash\sigma$.

How to proceed to construct such a sum is not at obvious because notions of volume and area are undefined in a general metric space like $\mathfrak{M}$. That is, while $\bar{B}_\varepsilon$ is compact so that a finite cover of $\bar{B}_\varepsilon$ exists, the volume of each cover element must be determined by its own cover, and so on *ad infinitum*. Thus, measures on balls and spheres in $\mathfrak{M}$ have no quantitative basis like intervals on $\mathbb{R}$ — an exact integral over $\bar{B}_\varepsilon(\sigma)$ does not exist because $\bar{B}_\varepsilon$ is ultimately unmeasurable.

Nonetheless, because ϕ is a scalar, it is possible to define an integral for computing $\langle\phi[\bar{B}_\varepsilon(\sigma)]\rangle$ that converges in probability to the 'true' mean value, and that will be good enough. The idea is to use the statistical and dynamic properties of φ to obtain a meaningful notion of 'similar directions' on a ball in $\mathfrak{M}$, and use that together with the desired accuracy of the approximate mean value to choose an element of 'area' or 'volume' to compute the mean. Sect. 3.5, and Subsect. 3.5.3 in particular, develops this integral. The needed statistical properties of φ will be determined in Chap. 4, especially Proposition 4.2.2. Thus, the postulate for local motion of φ below must restrict consideration to the contributions by ϕ along individual geodesics with endpoints $\sigma' \in \bar{B}_\varepsilon(\sigma)$ and σ, *i.e.*,

$$\frac{\delta}{\delta\mathfrak{t}}\left\langle \rho(\mathfrak{V}_\phi(\mathfrak{c}))\,\frac{\delta}{\delta\mathfrak{t}}\big\langle\phi(\sigma_i,\sigma;\mathfrak{t})\big\rangle_\tau\right\rangle_\tau = -\big\langle g(\mathfrak{V}_\phi(\sigma',\sigma;\mathfrak{t}))\cdot\phi(\sigma',\sigma;\mathfrak{t})\big\rangle_{2\tau} \tag{2.20}$$

(means are taken over over the time period τ), and then imply the general case by postulating that the net motion is just the sum of contributions by such geodesics for all $\sigma' \in \bar{B}_\varepsilon(\sigma)$.

The coupling of an oscillator to its neighborhood, developed above using a spring analogy, can be summarized thusly. The coupling acts to minimize the deviation of an oscillator's phase (and hence its frequency) from that of its neighborhood, such that the restoring 'acceleration' of the relative phase is proportional to the relative phase, ϕ. Continuity of phase is imposed by Postulate 2.2.6(W2), so deviations of phase between neighbors $\tilde{\omega}(\sigma), \tilde{\omega}(\sigma')$ can be at most infinitesimal: the coupling strength $g \to \infty$ as $\mathfrak{d}(\sigma, \sigma') \to 0$, but for finite $\mathfrak{d}(\sigma, \sigma')$ the coupling strength depends on the total variation of ϕ along the geodesic between σ and σ'. The instantaneous relative phase is important; if neighboring oscillators $\tilde{\omega}(\sigma), \tilde{\omega}(\sigma')$ are instantaneously in phase with slightly different frequencies at time $\mathfrak{t}_*$, then the coupling has no effect until a frequency difference leads to a nonzero relative phase. Because the coupling is with each neighbor separately, the coupling of $\tilde{\omega}(\sigma)$ to its entire infinitesimal neighborhood $\tilde{\omega}[\bar{B}_\varepsilon(\sigma)]$ determines the net acceleration of $\omega(\sigma)$ by superposition. Eq. (2.20) describes this superposition.

The relative phase $\phi(\sigma, \sigma'; \mathfrak{t})$ is the physically relevant variable that determines the field acceleration, but it is inconvenient because two points, σ and σ', must be explicitly specified for each field value. It will generally be more convenient to work with a field that explicitly depends only on a single point. The φ field (Def*. 2.2.7), *i.e.*, $\varphi(\sigma; \mathfrak{t}) \equiv \phi(\sigma, \sigma_0; \mathfrak{t})$, is thus a more convenient scalar field than ϕ, even though it is defined nonlocally. For this reason, φ is the irreducible scalar field of primary interest in this thesis.[4]

Postulate 2.2.11 (Local motion of φ.) *Let $\mathfrak{c}$ be a geodesic with image $\gamma(\sigma, \sigma') \subset \bar{B}_\varepsilon(\sigma)$, τ be an arbitrary period of cosmic time, and $I_\tau = [\mathfrak{t}_0, \mathfrak{t}_0 + \tau]$ be a time interval beginning at arbitrary $\mathfrak{t}_0$.*

D1. **Sources of motion.** *During an interval I_τ, the φ field at σ undergoes a net motion*

$$\delta\varphi(\sigma; I_\tau] = \delta\varphi(\sigma; I_\tau]_{\text{coupling}} + \delta\varphi(\sigma; I_\tau]_{\text{rand}}\,, \tag{2.21}$$

where $\delta\varphi(\sigma; I_\tau]_{\text{rand}}$ is given by Postulate 2.2.8, i.e., it obeys the Brownian noise PDF (2.13).

[4] Of course, ϕ and φ represent the same physical field since the relative phase can always be recovered from φ, *i.e.*, $\phi(\mathfrak{x}, \mathfrak{x}'; \mathfrak{t}) = \varphi(\mathfrak{x}; \mathfrak{t}) - \varphi(\mathfrak{x}'; \mathfrak{t})$.

D2. **Dynamic contribution.** *The contribution by* $\delta\varphi[\mathfrak{c}; I_\tau]$ *to* $\delta\varphi(\sigma; I_\tau]_{\text{coupling}}$ *is implicitly defined by*

$$\frac{\delta}{\delta\mathfrak{t}}\left\langle \rho(\mathfrak{V}_\phi(\mathfrak{c}))\frac{\delta}{\delta\mathfrak{t}}\big\langle\phi(\sigma',\sigma;\mathfrak{t})\big\rangle_\tau\right\rangle_\tau = -\big\langle g(\mathfrak{V}_\phi(\mathfrak{c}))\cdot\phi(\sigma',\sigma;\mathfrak{t})\big\rangle_{2\tau} \tag{2.22}$$

in the limit $\varepsilon\to 0$*, where* $\phi(\sigma',\sigma;\mathfrak{t})$ *is the relative phase (Def*. 2.2.5) between* $\tilde\omega(\sigma)$ *and* $\tilde\omega(\sigma')$ *at time* $\mathfrak{t}$*, and* $g(\cdot)$ *is the coupling strength; a subscript on the angle brackets denotes the interval over which the mean is computed. The 'inertia' term* $\rho(\mathfrak{V}_\phi(\mathfrak{c})) = \rho_0\mathfrak{V}_\phi(\mathfrak{c})$*, where* ρ_0 *is a constant.*

D3. **Coupling strength,** $g(\cdot)$**.** *The coupling strength* $g(\cdot)$ *in* (2.22) *has the form* (2.18)*:*

$$g = g\big(\mathfrak{V}_\phi(\mathfrak{c})\big) = g_0\,\mathfrak{V}_\phi(\mathfrak{c})^{-1}\,, \tag{2.23}$$

where g_0 *is the 'bare' dimensionless coupling constant and* $g\to\infty$ *as* $\mathfrak{d}(\sigma,\sigma')\to 0$*.*

D4. **Local propagation speed.** *The instantaneous coupling strength* g *in* (2.23) *is proportional to the instantaneous propagation speed of local changes in* $\phi(\sigma,\sigma';\mathfrak{t})$ *along* $\mathfrak{c}$*.*

D5. **Net motion.** *The net local motion of* $\varphi(\sigma;\mathfrak{t})$ *is obtained by a simple sum of contributions* (2.22) *for all* $\sigma'\in\bar B_\varepsilon(\sigma)$ *(*$\varepsilon\to 0$*), relating* φ *and* ϕ *by Def*. 2.2.7.*

Sect. 3.5 will develop a suitable procedure for computing the average over $\bar B_\varepsilon(\sigma)$ in D5.

Coupling between oscillators implies propagation of field changes, since the coupling allows an oscillator $\tilde\omega(\sigma)$ to effect a change in a neighboring oscillator $\tilde\omega(\sigma')$. However, the notion of propagation speed, or delay in the effect of a change of φ at σ appearing at σ', is nonphysical without an emergent spacetime.

Remark 2.2.12 (Propagation.) *Oscillator coupling will cause propagation of changes in* $\varphi(\sigma)$ *in the following sense. Consider a closed neighborhood of* σ*,* $\bar B_\varepsilon(\sigma) = \{\sigma' \,|\, \mathfrak{d}(\sigma,\sigma')\le\varepsilon\}$*. According to Postulate 2.2.11,* $\tilde\omega(\sigma)$ *couples to oscillators on the sphere* $K_\varepsilon(\sigma)$*. Let* $\bar B_{\varepsilon'}(\sigma)$ *be another closed neighborhood of* σ *such that* $\bar B_\varepsilon(\sigma)\subset B_{\varepsilon'}(\sigma)$*; then* $\delta\varepsilon = \varepsilon'-\varepsilon > 0$*. Taking* $\delta\varepsilon\to 0$*, then for each* $\sigma'\in K_\varepsilon(\sigma)$ *there is a closed ball* $\bar B_{\delta\varepsilon}(\sigma')$ *that contains at least one point in* $\bar B_{\varepsilon'}(\sigma)$*, and Postulate 2.2.11 again applies. In this sense changes of* $\varphi(\sigma)$ *can be said to propagate from* $\bar B_\varepsilon(\sigma)$ *into* $\bar B_{\varepsilon'}(\sigma)\backslash\bar B_\varepsilon(\sigma)$*, and vice versa.*

The explicit dependence of the coupling on the field configuration $\varphi[\mathfrak{c}; I_\tau]$ suggests a crude analogy with point charges q_1 and q_2 at σ and σ', respectively, which are separated by a dielectric. Just as the electrostatic force between q_1 and q_2 depends in part on the particular characteristics of the dielectric, the coupling strength between $\varphi(\sigma; I_\tau]$ and $\varphi(\sigma'; I_\tau]$ may depend on how φ varies between σ and σ during the relevant interval I_τ.

Continuity of φ, asserted by Postulate 2.2.6(W2), implies neighboring oscillators $\tilde{\omega}(\sigma), \tilde{\omega}(\sigma')$ are not truly independent. The following remark argues this does not contradict Postulate 2.2.8.

Remark 2.2.13 *Postulates 2.2.11 and 2.2.8 are mutually consistent. Consider the probability density for Brownian noise, Eq.* (2.13)*:*

$$f(\delta\varphi, \delta\mathfrak{t}) = \frac{1}{\sqrt{2\pi\delta\mathfrak{t}}} \exp\left(-\frac{(\delta\varphi)^2}{2\delta\mathfrak{t}}\right), \qquad \delta\mathfrak{t} = \mathfrak{t} - \mathfrak{t}_0\,.$$

This PDF is both scale invariant and symmetric with respect to the sign of $\delta\varphi$. Moreover it applies everywhere on $\mathfrak{M}$, independent of the time, so Postulate 2.2.8 assumes symmetry of Eq. (2.13) *with respect to 'translation' along curves on $\mathfrak{M}_\mathfrak{t}$. Hence, the field motion postulate 2.2.11 is consistent with Brownian noise of $\delta\varphi(\sigma; I_\tau]_{\text{rand}}$ for all $\sigma \in \mathfrak{M}$ if it maintains all of the symmetries assumed by Postulate 2.2.8.*

Postulate 2.2.11 does not depend on position. Furthermore, it is immediately apparent that the form of the dynamics (2.22) *is scale invariant and time independent. Therefore, the coupling preserves the symmetries of space and time translation and scale invariance. Finally, the coupling depends on the magnitude of $\delta\bar{\varphi}$ but not its sign, so it preserves the symmetry of the probability density* (2.13) *with respect to the sign of $\delta\bar{\varphi}$. Hence, it maintains all the symmetries assumed by Postulate 2.2.8.*

Clearly the coupling has some effect on the behavior of the oscillators; otherwise it could not maintain continuity of φ along every curve $\mathfrak{c} \subset \mathfrak{M}_\mathfrak{t}$. It is apparent the coupling has a rescaling *effect on the time dependence $\delta\mathfrak{t}$ in Eq.* (2.13)*: the stochastic variation at each point proceeds more slowly compared to zero coupling because it is the 'average' fluctuation over the oscillator neighborhood, i.e., $\omega[\bar{B}_\varepsilon(\sigma)]$, and not the fluctuation $\delta\omega(\sigma)$ by itself, that determines the rate at which the stochastic fluctuation of $\omega(\sigma)$ proceeds as a Brownian noise process. Since the cosmic time is nonphysical, there is no difficulty in rescaling it to accommodate the effect of the coupling on the rate of the Brownian process.*

Postulate 2.2.11 fully specifies the local dynamics of φ at a point $\sigma \in \mathfrak{M}$, but it cannot be directly used to describe the dynamics over an extended region S where ε is finite. In this more general case, the dynamical motions of φ, *e.g.* of a wave front, are subject to interactions with φ homogeneities that enter S from the outside. Such interactions will lead to random scattering of wave fronts so that $\mathfrak{c}$ is no longer a geodesic, and moreover they will modify $\mathfrak{V}(\mathfrak{c})$. Thus the postulate only describes the interactions at each point in S but not $\varphi[S]$ overall. The more general case will require path integration; the postulate is the starting point for developing such a procedure.

The contribution (2.22) to the field motion will be the basis for an equation of motion for φ which describes propagation of field changes. While the equation of motion should ultimately be a consequence of Postulate 2.2.11, derivation of the former from the latter is not immediate:

- An equation of motion is a physical statement: it requires well defined notions of physical space, physical time, and spacetime dimensionality. The time-augmented metric space $\mathfrak{M}_\mathfrak{t}$ contains none of these.

- Variational methods, *i.e.*, the Euler-Lagrange equations, are inadequate for determining the equation of motion because notions like energy and momentum require a physical spacetime. Space and time are to acquire meaning as a result of a dynamical emergence process — using energy to define dynamics requires circular reasoning.

- Postulate 2.2.11 is more general than propagation. Coupling can maintain the continuity of φ, for example, even in a general metric space.

2.2.4 Preferred scale

The postulates introduced thus far apply to the abstract realm of φ on $\mathfrak{M}_\mathfrak{t}$. There, the field is scale invariant in its stochastic variation and its dynamics. However, physics as we understand it is not scale-free: particle masses, coupling constants and quantum uncertainty all have associated scales. Dynamics in general are not scale free. Hence, it is necessary to break the symmetry of scale invariance of φ in some region of $\mathfrak{M}$, at least over some range of distance scales, $[\mathfrak{l}_{\min}, \mathfrak{l}_{\max}]$, so that different behavior appears compared to the scale invariant motions of φ. Presumably, breaking scale invariance

is a precursor to, or at least something concurrent with, the appearance of physical distances and time scales on an emergent spacetime manifold $\mathscr{M}$. An descriptive term for this condition is a preferred scale.

Remark 2.2.14 *A* **preferred scale** $[\mathfrak{l}_{\min}, \mathfrak{l}_{\max}]$ *exists in a region* $\mathfrak{U}_{\mathfrak{t}} \subset \mathfrak{M}_{\mathfrak{t}}$ *when dilatation symmetry (scale invariance of* φ*) is broken in* $\mathfrak{U}_{\mathfrak{t}}$*. This can be informally defined in the following way. Take* $\mathfrak{U}$ *to be a ball* $B_R(\sigma)$ *of finite radius* R*, and let* $\Gamma = \{\gamma \,|\, \gamma \subset \mathfrak{U},\ \gamma \text{ straight}, \sigma \in \gamma\}$ *be the diameters of* $\mathfrak{U}$ *that contain* σ*. If for all* $\gamma \in \Gamma$ *a mode decomposition of* $\varphi[\gamma]$ *obtains a continuous range of modes* $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ *whose amplitudes differ from their expected values by* $\delta\mathfrak{k}/\mathfrak{k} \geq \delta_k$*, then at the significance level* δ_k *a preferred scale exists in* $\mathfrak{U}_{\mathfrak{t}}$*, and is characterized by* $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}] = [\mathfrak{l}_{\min}^{-1}, \mathfrak{l}_{\max}^{-1}]$*. The expected values are given by a Brownian motion PDF; see Subsect. 4.2.1.*

The usefulness of this definition requires that spectral analysis is valid for functions of infinite variation. This is the case as Subsect. 3.1.2 will show.

Chapter 3

Stochastic Processes and Calculus of φ

The previous chapter developed a number of definitions and a set of postulates for an irreducible, real-valued scalar field φ which lives on a general metric space $\mathfrak{M}$ (and its time-augmented version $\mathfrak{M}_{\mathfrak{t}}$). That can be viewed as a proposal for answering the question, "What is it that spacetime might emerge from?" The present chapter is devoted to choosing existing mathematical tools and, where needed, limited development of additional tools, to describe how spacetime can actually emerge from the proposed foundation.

Of particular import is the field stochasticity postulate 2.2.8: it introduces intrinsic stochasticity into φ by assuming the change in φ at a point σ during an infinitesimal interval $d\mathfrak{t}$ has two contributions, one dynamical and the other stochastic. The stochastic contribution is to obey the probability density function for Brownian motion. Some of its basic characteristics and stochastic processes more generally will be summarized in this chapter. A property of Brownian motion of special interest is scale invariance. The dynamics, postulated in Postulate 2.2.11 and developed in Chap. 4, also have a scale invariant form. Thus, in general, $\varphi[\mathfrak{M}_{\mathfrak{t}}]$ will be scale invariant except in rare regions where scale invariance is sufficiently broken that a preferred scale $[\mathfrak{l}_{\min}, \mathfrak{l}_{\max}]$ can exist for a finite time.

Two other properties φ will inherit from its Brownian motion PDF are nondifferentiability almost everywhere and infinite variation along every curve of finite length. These mean, respectively, that differentiation and integration methods of ordinary calculus are inadequate. Since it is hard to imagine getting very far without calculus it is necessary to define procedures for integration and differentiation that are appropriate

for random functions of infinite variation. The stochastic calculus provides the needed tools.

Intrinsic stochasticity has been postulated to provide a source of initial conditions for cosmogenesis. While a simple idea, its effects on the emergence picture are substantial. It means the picture must ultimately be probabilistic, even though the dynamics must dominate the randomness in regions that contain a persistent preferred scale. Hence, a summary of some important results from the mathematical theory of stochastic processes, followed by development of appropriate definitions of integration and differentiation for $\varphi[\mathfrak{M}]$, will provide a necessary mathematical foundation for the dynamics and phenomena that Chap. 4 will consider.

The theory of stochastic processes is well developed for spaces with a well defined dimension. Unfortunately, generalization of stochastic processes to a general metric space does not appear to have been developed in the literature. Thus, the need to develop some generalizations is unavoidable, and where needed that has been pursued in very limited form in this chapter.

3.1 Stochastic Processes

This section and the next summarize some results from the mathematical theory of stochastic processes, focusing especially on Brownian motion. The discussion is of a general mathematical nature, and does not refer specifically to the space $\mathfrak{M}_{\mathfrak{t}}$ or the field φ. For that reason, this section and next will not strictly adhere to the notation introduced by Sect. 1.6.

3.1.1 Stochastic processes and Brownian motion[1]

A **stochastic process** or random process [42] is a collection $\{X(t)\}$ of identically distributed random variables $X(t_1), ..., X(t_n)$ with $t_0{=}0 < t_1 < t_2 < \cdots < t_n{=}T$, $1 \leq n < \infty$. A given instance of the process is a **path** $x(t)$, *e.g.* a function of time t $(0 \leq t \leq T)$ which can be represented as the finite sum

$$x(t) \;=\; c_0 I_0(t) + \sum_{i=1}^{n-1} c_i I_{(t_i,\, t_{i+1})}(t) \qquad \big[c_i = x(t_i)\,,\; x(t_i) \in X(t_i)\big], \tag{3.1}$$

[1] Klebaner [42], source of much of the material in this subsection, gives an accessible introductory treatment of stochastic calculus and mathematical theory of Brownian motion.

where I_A is an **indicator** (or characteristic function) of a set A, defined thusly: $I_A(\omega) \equiv 1$ if and only if $\omega \in A$ and zero otherwise. (A path of a process has no relation to Def. 2.1.8 for a path on $\mathfrak{M}$.) Since $x_i \equiv x(t_i)$ is randomly drawn from $X(t_i)$, $x(t)$ is one of an ensemble of possible paths in $\{X(t)\}$.

From Eq. (3.1), it is apparent that performing computations with $\{X(t)\}$ requires knowing the collective properties of all possible paths, which in turn are determined by the probability distribution of $X(t)$. Hence, the process $\{X(t)\}$ is determined by all its finite dimensional probability distributions,

$$P_n(X(t_1) \le x_1,\ X(t_2) \le x_2,\ \ldots,\ X(t_n) \le x_n), \quad 1 \le n < \infty\,, \tag{3.2}$$

for any choice of times $0 \le t_1 < t_2 < \cdots < t_n < T$; $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Let $p_i(x_i, t_i)$ denote the probability density for $X(t_i)$, and write the joint probability density for $\{X(t)\}$ as $p_n(x_1, t_1; \ldots; x_n, t_n)$. Then the probability that a given path drawn from ensemble $\{X(t)\}$ takes a value in the range $[x_1, x_1{+}dx_1)$ at t_1, then a value in $[x_2, x_2{+}dx_2)$ at t_2, and so on, is just

$$P(X(t_1) \ = \ x_1, \ldots, X(t_n){=}x_n) \ = \ p_n(x_1, t_1; \ldots; x_n, t_n)\, dx_1 \cdots dx_n\,. \tag{3.3}$$

The m-point **correlation function** is

$$\langle x(t_1) \cdots x(t_m) \rangle \qquad = \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_m p_m(x_1, t_1; \ldots; x_m, t_m)\, x_1 \cdots x_m\,. \tag{3.4}$$

Convergence properties play a a central role.

Def. 3.1.1 (Convergence properties of random variables. [42]**)** *Given a random variable X, a sequence $\{X_n\}$* ***converges*** *to X*

X1. **in distribution** *if their distribution functions $F_n(x)$ converge to the distribution function $F(x)$ at any point of continuity of F.*

X2. **in probability** *if, for any $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$.*

X3. **almost surely (a.s.)** *if, for any value of ω, outside a set of zero probability $X_n(\omega) \to X(\omega)$ as $n \to \infty$.*

In what follows, all relations involving conditional expectations, such as equalities and inequalities, are to be understood in the 'almost sure' sense, X3 above.

The well developed mathematical theory of Brownian motion, or Wiener process, will be of particular interest in that Postulate 2.2.8 states that the field $\varphi(\mathfrak{t})$ at a point σ obeys the probability density function (PDF) for Brownian motion when oscillator coupling is not too strong. This process, $\{B(t)\}$ where $B(t)$ is a single path, is a model for the cumulative effect of pure (*i.e.*, 'white') noise. For example, if $B(t)$ is the position of a particle) at time t then the pure noise that effects a displacement $B(t+\Delta t) - B(t)$ during Δt might be random molecular collisions.

A derivation of one-dimensional Brownian motion as the continuum limit of a one-dimensional random walk highlights the origin of some of the properties of the Wiener process. The generalization to random walks in $\mathbb{R}^n$ for $n > 1$ is merely a matter of treating each component as an independent 1-D random walk.

There are two general approaches to the derivation. Both start with the notion of a particle moving in discrete steps of length ε at equally spaced discrete times $t_1, \ldots, t_n$ with $\tau = t_{i+1} - t_i$. Represent the position x of the particle at time t_i as $X_i = X(t_i) = x$. The direction of a step at time t_i is a random variable with associated probabilities

$$p_+ \,=\, P((X_{i+1} - X_i) = +\varepsilon)\,, \quad p_- = P((X_{i+1} - X_i) = -\varepsilon) = 1 - p_+\,, \tag{3.5}$$

where p_+ and p_- are the probabilities of steps of length ε in the (+) and (-) directions, respectively. If $p_+ = p_-$ there is no bias in either direction, so that the expected value $\bar{x} = 0$; if $p_+ > p_-$, the bias will cause $\bar{x}$ to increase in time, corresponding to a drift velocity.

The first approach recognizes that the discrete random variable $X(t)$ obeys a binomial distribution, and then uses the central limit theorem in the continuum limit $\varepsilon, \tau \to 0$ to deduce that $X(t)$ (t now continuous) is a normally distributed random variable; that and the other properties of the random walk imply that in the limit of infinitesimal step size the random walk is a Brownian motion process. The other approach is to obtain a difference equation for the probability $P(X_i = x)$ at discrete time t_i, and show it leads to the diffusion equation in the continuum limit, which has a Gaussian solution; that and the other properties of the random walk again correspond to Brownian motion. The second approach is developed below because it better elucidates ideas that will be important in later chapters.

Eq. (3.5) implies the difference equation,

$$P(X_{i+1} = x) = p_+ \cdot P(X_i = x + \varepsilon) + p_- \cdot P(X_i = x - \varepsilon)\,,$$

which after subtracting $P(X_i{=}x)$ from both sides and rearranging becomes

$$P(X_{i+1}{=}x) - P(X_i{=}x) = \tfrac{1}{2}\left[P(X_i = x{+}\varepsilon) - 2P(X_i{=}x) + P(X_i = x{-}\varepsilon)\right] + \frac{p_- - p_+}{2}\left[P(X_i{=}x{+}\varepsilon) - P(X_i{=}x{-}\varepsilon)\right] .$$

After dividing by time step τ, this difference equation can be written in terms of the discrete variables x, t as

$$\frac{P(x,t+\tau) - P(x,t)}{\tau} = \frac{\varepsilon^2}{2\tau}\,\frac{P(x+\varepsilon,t) - 2P(x,t) + P(x-\varepsilon,t)}{\varepsilon^2} + \frac{\varepsilon(p_+ - p_-)}{\tau}\,\frac{P(x+\varepsilon,t) - P(x-\varepsilon,t)}{2\varepsilon} . \quad (3.6)$$

The left hand side is a finite difference, forward derivative approximation to $\partial P(x,t)/\partial t$, the first term on the right side is $\frac{\varepsilon^2}{2\tau}$ multiplying a centered second derivative approximation to $\partial^2 P(x,t)/\partial x^2$, and the last term on the right is $\frac{\varepsilon}{2}(p_+ - p_-)$ multiplying a centered first derivative approximation to $\partial P(x,t)/\partial x$. In the limit $\varepsilon \to 0, \tau \to 0$ where x and t become continuous, we should identify the discrete probability $P(X_k = x) = \int_{x-\varepsilon/2}^{x+\varepsilon/2} P(x', k\tau)dx'$. Then Eq. (3.6) becomes the diffusion equation with finite 'diffusion coefficient' D and 'drift velocity' v:

$$\begin{aligned} \frac{\partial P}{\partial t} &= D\,\frac{\partial^2 P(x,t)}{\partial x^2} + v\,\frac{\partial P(x,t)}{\partial x} ; \\ D &= \lim_{\substack{\varepsilon\to 0,\\ \tau\to 0}} \frac{\varepsilon^2}{2\tau} , \quad v = \lim_{\substack{\varepsilon\to 0,\\ \tau\to 0}} \frac{\varepsilon(p_+ - p_-)}{\tau} . \end{aligned} \quad (3.7)$$

If the particle starts at $x = 0$ at $t = 0$, the probability it is at x at time $t > 0$ is the solution to Eq. (3.7), *i.e.*, the Gaussian

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}}\,\exp\left(-\frac{(x-vt)^2}{4Dt}\right) .$$

Take $D = \frac{1}{2}$ and rescale x, t accordingly to give the Normal probability density with mean vt and variance t:

$$f(x,t) = \frac{1}{\sqrt{2\pi t}}\,\exp\left(-\frac{(x-vt)^2}{2t}\right) . \quad (3.8)$$

The random walk is a discrete stochastic process which becomes the Brownian motion process in the continuum limit of infinitesimal ε, τ. In both cases, the process is a collection of random variables $\{X(t)\}$, and the evolution of the process is determined

by its finite dimensional probability distribution, Eq. (3.2), with each **path** having the form (3.1).

Brownian motion has three defining properties:

Def. 3.1.2 (Brownian motion. [42]**)** *A stochastic process* $\{B(t)\}$ *is Brownian motion if it has these three properties:*

1. **Normal increments.** $B(t) - B(s)$ *has* $N(0, t-s)$ *distribution, i.e., Normal distribution with zero mean and variance* $(t - s)$*, for arbitrary times* $s < t$ *with* $0 \leq t, s < \infty$*.*

2. **Independent increments.** $B(t) - B(s)$ *is independent of* $B(u),\ 0 \leq u < s$*.*

3. **Continuity of paths.** *The* $B(t),\ t \geq 0$ *are continuous functions of* t*.*

Properties 1 and 2 determine all the finite dimensional distributions; it can be proven that all are Gaussian. Property 3 is not obvious when $B(t)$ is expressed in the form of Eq. (3.1), since in general $c_{i+1} - c_i$ will be finite; understanding this requires the concept of versions of a process [42].

Def. 3.1.3 *Two stochastic processes* X, Y *are called* **versions** *of one another if* $P(X(t) = Y(t)) = 1$ *(i.e., they converge in probability) for all* $t,\ 0 \leq t \leq T$*.*

Since a random process is determined by its finite dimensional distributions (3.2), if processes X and Y differ on a nonempty set of probability zero then with probability 1 they can be taken to be the same process. That is, they are different versions of the same process, and we are free to choose whichever is most convenient, generally the most continuous one. It can be proved that if certain conditions hold, there will always be at least one continuous version of a process. Brownian motion satisfies those conditions, so in that sense Brownian motion processes are continuous.

Def. 2.2.9 defined the variation of a real-valued function $g(t)$ over an interval $[a, b] \subset \mathbb{R}$ as

$$\mathfrak{V}_g([a, b]) = \sup \sum_{i=1}^{n} \left| g(t_i^n) - g(t_{i-1}^n) \right|, \tag{3.9}$$

with the supremum taken over partitions $a = t_0^n < t_1^n < \cdots < t_n^n = b$. A theorem gives necessary and sufficient conditions for a function to have finite variation:

Theorem 3.1.4 (Jordan decomposition theorem. [42]**)**
Any function $g(t) : [0, \infty) \to \mathbb{R}$ of finite variation can be expressed as the difference of two increasing functions: $g(t) = a(t) - b(t)$. If g is right-continuous, it can be expressed as the difference of two right-continuous increasing functions.

A sufficient condition for g to have finite variation is [42] :

Theorem 3.1.5 *If g is continuous and g' exists with $\int |g'(t)|\, dt < \infty$, then g is of finite variation.*

From the theory of functions, a function defined on an interval $[a, b]$ can have no more than countably many jumps. Furthermore [42],

Theorem 3.1.6 *A finite variation function can have no more than countably many discontinuities. Moreover, all discontinuities are jumps.*

Scale invariance will be needed below and elsewhere:

Def. 3.1.7 (Scale invariance or dilatation symmetry.) *A function $f(x)$ is **scale invariant**, or has **dilatation symmetry**, if, for some choice of s and all dilatations (or dilations) λ, it is invariant under the rescaling*

$$f(\lambda x) = \lambda^s f(x)\,. \tag{3.10}$$

Brownian motion paths have some (provable) properties, collected here into a theorem[2]:

Theorem 3.1.8 (Brownian motion properties. [42]**)**

Almost every sample path $B(t) \in \{B(t)\}$, $0 \le t \le T$,

B1. *is a continuous function of t;*

B2. *is not monotone increasing or decreasing on* any *interval;*

B3. *is not differentiable at any point;*

B4. *has infinite variation (Def. 2.2.9) on every interval, no matter how small;*

B5. *is scale invariant;*

B6. *has covariance function* $\mathrm{Cov}(B(t), B(s)) = \min(t, s)$*; and*

[2]Theorems in this subsection will usually be stated without proof.

B7. *has quadratic variation on* $[0,t]$ *equal to* t *for any* t. *The* **quadratic variation of Brownian motion** $[B,B](t)$ *is defined as*

$$[B,B](t) \quad = \quad [B,B]([0,t]) \quad = \quad \lim \sum_{i=1}^{n} \left| B(t_i^n) \; - \; B(t_{i-1}^n) \right|^2, \quad (3.11)$$

where for each n, $\{t_i^n\}_{i=1}^n$ *is a partition of* $[0,t]$, *and the limit is taken over all partitions with* $\delta_n = \max_i (t_{i+1}^n - t_i^n) \to 0$ *as* $n \to \infty$. *(Smooth functions have zero quadratic variation, but* $B(t)$ *is smooth nowhere.)*

Proof: **(Partial proof.** [42]**)**

[Part B1**.]** Kolmogorovs condition guarantees the continuity of paths of a random process with continuous time under this condition: If for some $\gamma > 0$ and $\beta > 0$, and for all t, s there is a positive constant C such that

$$\langle |X(t) - X(s)|^\gamma \rangle \leq C^{1+\beta} \,,$$

then paths of $X(t)$ are continuous functions.

A property of Gaussian random variables Y having a zero mean and variance $\sigma^2 = \langle Y^2 \rangle$ is that $\langle Y^4 \rangle = \sigma^4$; hence, for a Brownian motion process, $\langle |X(t) - X(s)|^4 \rangle = 3|t-s|^2 = 3\sigma^4$. Comparing with Kolmogorov's condition, $\gamma = 4$, $\beta = 1$ and $C = 3$ and continuity follows.

[Part B4**.]** Since $B(t)$ is right-continuous, this follows from Def. 3.1.4 and property B2.

[Part B5**.]** From Eq. (3.8), $f(x,t)$ depends only on $(x - vt)^2/t$. Choose a new scale $\ell > 0$ and let $X(t) \to Y(t) = \ell X(t/\ell^2)$. This implies another Brownian motion process $\{B_\ell(t)\}$, which is indistinguishable from $\{B(t)\}$ except for the change of scale. The result immediately follows. ∎

Finally, define the martingale and Markov properties of a stochastic process.

Def. 3.1.9 (Martingale property. [42]**)** *If we know the path* $x(t)$ *of the process* $\{X(t)\}$ *up to time* t', *and* $X(t') = x(t')$, *and the expectation value at any future time is* $x(t')$, *then the process is said to have the martingale property.*

Brownian motion processes have three main martingales [42]:

Theorem 3.1.10 *Let* $B(t)$ *be a Brownian motion path. Then*

M1. $B(t)$ *is a martingale.*

M2. $B(t)^2 - t$ *is a martingale.*

M3. *For any* u, $\exp\left(uB(t) - \frac{u^2}{2}t\right)$ *is a martingale.*

A process with the Markov property has no 'memory' of how it arrived at its current state x; if we know the current state of the process, then the future evolution of the process is independent of its past. More precisely, the process $\{X(t)\}$ has the **Markov property** if the conditional distribution of $X(t+s)$ given $X(t) = x$ does not depend on the earlier values of $x(t)$; it may depend on the present value x, however. Brownian motion has this property [42]:

Theorem 3.1.11 *Brownian motion* $\{B(t)\}$ *possesses the Markov property.*

3.1.2 Spectral analysis of stochastic functions

Spectral analysis for stochastic processes has been thoroughly investigated and its validity established [10]. On every finite interval, almost every path $B(t) \in \{B(t)\}$ of Brownian noise (or motion) has infinite variation, so as the process duration T becomes infinite the spectrum of $B(t)$ will contain modes of all frequencies. The variance of $\{B(t)\}$ is t, so in the general case as $T \to \infty$, the displacement $|B(t)|$ of the path from its origin is unbounded and its Fourier integral will diverge. Nonetheless, $B(t)$ can be developed as a Fourier integral for finite T, and then the limit $T \to \infty$ taken. This will now be done for stationary stochastic processes more generally; $\{B(t)\}$ is but one important example.[3]

Let $B(t)$ be a path of a stationary stochastic process, and assume its mean $\bar{B}(t) = 0$, where the mean is taken over the ensemble $\{B(t)\}$. Consider a process $y(t)$ which is a restriction of $B(t)$ to a long but finite time T, *e.g.*,

$$y(t) = \begin{cases} B(t), & -T/2 < t < T/2\,, \\ 0, & |t| > T/2\,. \end{cases} \tag{3.12}$$

Then its Fourier integral is well defined:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, A(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} df\, A(f)\, e^{i2\pi f t}\,, \\ \tilde{y}(f) &= \int_{-\infty}^{\infty} dt\, A(t)\, e^{-i2\pi f t}\,. \end{aligned} \tag{3.13}$$

[3] Much of this discussion will closely follow Wang and Uhlenbeck [73].

Since $y(t)$is real, $A^*(f) = A(-f)$. From Parseval's theorem,

$$\int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty} |A(f)|^2 df\,, \tag{3.14}$$

where

$$A(f) = \int_{-\infty}^{\infty} y(t)\, e^{i2\pi ft}\, dt = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\, e^{i2\pi ft}\, dt\,. \tag{3.15}$$

Use (3.14) to compute the mean square of $y(t)$. Since $|A(f)|^2$ is an even function,

$$\left\langle y^2(t)\right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} y^2(t)\, dt = \int_0^{\infty} \mathcal{S}_y(f)\, df\,, \tag{3.16}$$

where the factor of two from the evenness of $|A(f)|^2$ is absorbed into $\mathcal{S}_y(f)$,

$$\mathcal{S}_y(f) \quad = \quad \lim_{T\to\infty} \frac{2}{T} |A(f)|^2 \quad = \quad \lim_{T\to\infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} y^2(t)\, e^{i2\pi ft}\, dt \right|\,. \tag{3.17}$$

$\mathcal{S}_y(f)$ is the **spectral density** (or power spectral density) of $y^2(t)$.

Next consider the autocorrelation function $\mathcal{R}_y(\tau)$. Using a similar approach,

$$\mathcal{R}_y(\tau) \;=\; \langle y(t)\, y(t+\tau)\rangle \;=\; \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} y(t)\, y(t+\tau)\, dt\,. \tag{3.18}$$

Insert the Fourier transforms (3.13) and use $\tilde{y}^*(f) = \tilde{y}(-f)$ so the domain becomes only the positive frequencies:

$$\begin{aligned}
\mathcal{R}_y(\tau) &= \lim_{T\to\infty} \frac{1}{T} \iiint_{-\infty}^{\infty} dt\, df\, df'\, \tilde{y}(f)\, \tilde{y}(f')\, e^{-i2\pi f\tau}\, e^{i2\pi(f+f')t} \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} df\, |\tilde{y}(f)|^2 e^{-i2\pi f\tau} \\
&= \lim_{T\to\infty} \frac{2}{T} \int_0^{\infty} df\, |\tilde{y}(f)|^2 \cos(2\pi f\tau) \\
&= \int_0^{\infty} df\, \mathcal{S}_y(f)\, \cos(2\pi f\tau)\,,
\end{aligned} \tag{3.19}$$

where the second step used $\int_{-\infty}^{\infty} \exp(i2\pi(f+f')t) = \delta(f+f')$. Hence Eqs. (3.17) and (3.19) imply the Fourier cosine transform pair,

$$\mathcal{R}_y(\tau) \;=\; \int_0^{\infty} df\, \mathcal{S}_y(f)\, \cos(2\pi f\tau) = \langle y(t)\, y(t+\tau)\rangle\,, \tag{3.20}$$

$$\mathcal{S}_y(f) \;=\; 4\int_0^{\infty} d\tau\, \mathcal{R}_y(\tau)\, \cos(2\pi f\tau)\,. \tag{3.21}$$

This is the content of the Wiener-Khintchine theorem. It can be restated thusly: The correlation function,

$$\rho(\tau) = \frac{\langle y(t)\, y(t+\tau)\rangle}{\langle y^2\rangle} = \frac{\mathcal{R}_y(\tau)}{\sigma_y^2}\,, \tag{3.22}$$

and normalized spectrum,

$$S(f) = \frac{\mathcal{S}_y(f)}{\int_0^\infty df\, \mathcal{S}_y(f)}\,, \tag{3.23}$$

are each uniquely related to the other's Fourier cosine transform.

The foregoing explicitly assumed a zero mean for the process, *i.e.*, $\bar{y} = 0$ obtained for the ensemble over the interval T. If that assumption is dropped so that the ensemble mean $\bar{y}$ computed over the interval T is arbitrary, the spectral density (3.17) should be rewritten,

$$\mathcal{S}_y(f) = \lim_{T\to\infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} \big(y(t) - \bar{y}\big)\, e^{i2\pi f t}\, dt \right|^2 . \tag{3.24}$$

Then Eq. (3.16), which follows from Parseval's theorem, becomes

$$\int_0^\infty \mathcal{S}_y(f)\, df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big(y(t) - \bar{y}\big)^2\, dt \;=\; \sigma_y^2\,. \tag{3.25}$$

This shows the integral of the spectral density is the variance for the process.

Gaussian white noise $\mathcal{N}_{\rm w}(t)$ is a very simple case, but is important because of its relationship to Brownian noise. Specifically [9], Brownian noise is the cumulative effect of Gaussian white noise $\mathcal{N}_{\rm w}(t)$:

$$B(t) - B(0) = \int_0^t dB(t') = \int_0^t \mathcal{N}_{\rm w}(t')\, dt'\,, \tag{3.26}$$

with initial condition $B(0) \equiv 0$. Subsect. 2.2.2 summarized the defining properties of $\mathcal{N}_{\rm w}(t)$ thusly: It has zero mean over an ensemble; its spectral density (or spectral intensity) is flat, *i.e.*, $\mathcal{S}_{\rm w}(\omega) = \text{const}$ for all ω; and its autocorrelation length is zero:

$$\begin{aligned} \mathcal{S}_{\rm w}(\omega) &= \text{const}\,, \\ \mathcal{R}_{\rm w}(\tau) &= \frac{\mathcal{S}_{\rm w}}{2}\,\delta(\tau)\,. \end{aligned} \tag{3.27}$$

Remark 3.1.12 *The properties* (3.27) *of white noise in physical systems are in general idealizations [73]. The spectral density* $\mathcal{S}_{\rm w}(\omega) = \text{const}$ *will never occur in a physical system, although it may be constant over a large range of frequencies. Furthermore, the correlation length will never be truly zero in practice, although it may be effectively zero over the time scales of interest.*

However, a constant spectral density and zero correlation length cannot simply be idealizations when applied to the irreducible field φ*. If* $\mathcal{S}_{\rm w}(\omega)$ *were* not *constant, then*

there would necessarily be nonphysical scales $\mathfrak{l}_{\min}, \mathfrak{l}_{\max}$ *where the spectrum was no longer white. This would set a preferred scale in the theory, destroying its scale invariance — the nonphysical scale* $\mathfrak{l}_{\min}$ *could be deduced in principle by measuring the spectrum. Hence, the assumptions* (3.27) *must be taken more literally for* φ.

The Langevin equation can be used to characterize the spectrum of a Brownian process. First consider the equation of motion for a free particle of mass m and instantaneous velocity v subject to a random force $K(t)$ and friction force fv:

$$m\frac{dv}{dt} + fv = K(t)\,.$$

Assume $K(t)$ has a mean of zero and a very sharp autocorrelation function, so that its spectrum is nearly white; the spectral density of $K(t)$ is $4fk_BT$, where f is the friction coefficient, k_B is Boltzmann's constant and T is the ambient temperature of the medium. Next consider a simple R-L electrical circuit that is subject to a thermal noise source with spectral density $4Rk_BT$, which applies a purely random fluctuating EMF $\mathcal{E}(t)$ to this circuit,

$$L\frac{di}{dt} + Ri = \mathcal{E}(t)\,.$$

A single equation describes both systems:

$$\frac{dy}{dt} + \beta y = \mathcal{N}(t)\,, \tag{3.28}$$

which has the form of a Langevin equation.

Assume that $\mathcal{N}(t)$ is a Gaussian white noise source $\mathcal{N}_{\rm w}(t)$ with constant spectral density $\mathcal{S}_{\rm w}(\omega) = 4D$:

$$\begin{aligned}\langle \mathcal{N}_{\rm w}(t)\rangle &= 0\,,\\ \langle \mathcal{N}_{\rm w}(t_1)\,\mathcal{N}_{\rm w}(t_2)\rangle &= 2D\,\delta(t_1 - t_2)\,.\end{aligned}$$

The second equation follows from Eq. (3.21) after expressing $\mathcal{S}_{\rm w}(t)$ as an integral of a delta function and equating integrands, *i.e.*,

$$\mathcal{S}_{\rm w}(\omega) \;=\; \int_{-\infty}^{\infty} d\tau\, 4D\,\delta(\tau) = \int_{-\infty}^{\infty} d\tau\, 2\cos(\omega\tau)\,\langle \mathcal{N}_{\rm w}(t)\,\mathcal{N}_{\rm w}(t+\tau)\rangle\,.$$

Remark 3.1.13 *The Langevin equation* (3.28) *should provide a reasonable model for the stochastic frequency changes which are intrinsic to the oscillators* $\tilde{\omega}(\cdot)$ *from which* φ

is derived. The stochastic frequency-changing impulses are represented by a white noise source $\mathcal{N}_{\rm w}(\mathfrak{t})$. *For a massive particle undergoing Brownian motion,* $y(t)$ *represents the velocity* $v(t)$ *of the particle; because the particle has inertia the position continues to change as* $v\Delta t$ *in the absence of additional impulses from* $\mathcal{N}_{\rm w}(\mathfrak{t})$.

For φ *motions,* (3.28) *is a more appropriate model of* $\varphi(\sigma;\mathfrak{t})$ *than the 'velocity'* $d\varphi/dt$. *That is, according to Postulate 2.2.8,* $\delta\varphi(\sigma;I_{\mathfrak{T}}] = \delta\varphi(\sigma;I_{\mathfrak{T}}]_{\rm coupling} + \delta\varphi(\sigma;I_{\mathfrak{T}}]_{\rm rand}$ *so that* $\varphi(\sigma;\mathfrak{t})$ *should* not *continue to change according to* (3.28) *without additional impulses from* $\mathcal{N}_{\rm w}(\mathfrak{t})$*; except through* $\delta\varphi(\sigma;I_{\mathfrak{T}}]_{\rm coupling}$. *Then the 'friction' coefficient* β *in* (3.28) *corresponds to the damping effect of the dynamical coupling between* $\tilde{\omega}(\sigma)$ *and its neighborhood, computed as an ensemble mean as discussed in Subsect. 2.2.3.*

Since $\mathcal{N}_{\rm w}(t)$ is Gaussian, $y(t)$ is also a Gaussian random process. Substituting the solution $e^{i\omega t}$ into the characteristic equation $y' + \beta y = 0$ of the Langevin equation (3.28), the system response function $H(\omega) = (i\omega + \beta)^{-1}$. Hence the power spectrum is

$$\mathcal{S}_y(\omega) = |H(\omega|^2\, \mathcal{S}_{\rm w}(\omega) = \frac{4D}{\beta^2 + \omega^2}\,, \tag{3.29}$$

as shown in Fig. 3.1.

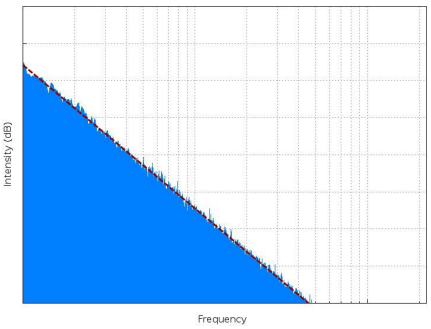


Figure 3.1: Brownian noise spectrum [74]. The spectrum was obtained from 10^4 samples of a Brown noise sound source. The slope of the dashed line is -2, corresponding to mode intensity $I \propto \omega^{-2}$ (or $I \propto k^{-2}$ for spatial modes).

Since $\rho(\tau)$ is the Fourier cosine of the normalized spectrum (3.23),

$$\rho(\tau) = e^{-\beta|\tau|}\,. \tag{3.30}$$

Eq. (3.22) relates this to the autocorrelation function as $\mathcal{R}_y(\tau) = \sigma_y^2\,\rho(\tau)$. Substituting the power spectrum (3.29) and this expression for $\mathcal{R}_y(\tau)$ into Eq. (3.21),

$$\frac{4D}{\beta^2 + \omega^2} = 4\int_0^\infty d\tau\, \sigma_y^2 e^{-\beta|\tau|}\cos(\omega\tau) = \sigma_y^2 \frac{\beta}{\beta^2 + \omega^2}\,. \tag{3.31}$$

Hence, the variance is

$$\sigma_y^2 = \langle y^2\rangle = \frac{D}{\beta}\,. \tag{3.32}$$

The constant D can be interpreted as a diffusion constant.

The Fokker-Planck equation for the Langevin equation (3.28) is [73]

$$\frac{\partial P}{\partial t} = \beta \frac{\partial}{\partial y}(yP) + D\,\frac{\partial^2 P}{\partial y^2}\,. \tag{3.33}$$

Here, $P = P(y(t)\,|\,y_0)$ is the probability distribution for being in state y at time $t = t_0+\tau)$, given the state y_0 at t_0. Note that this equation reduces to the PDE (3.7) for the continuum limit of a random walk when $\beta = 0$, namely when there is no 'friction.' The fundamental solution to (3.33) is

$$P(y,t\,|\,y_0) \qquad = \qquad \left(\frac{\beta}{2\pi D(1-\rho^2)}\right)^{1/2} \exp\left(-\frac{\beta\,(y-y_0\rho)^2}{2D(1-\rho^2)}\right) \tag{3.34}$$

The limit $t \to \infty$ gives the stationary PDF:

$$W_1(y) = \lim_{t\to\infty} P(y,t\,|\,y_0) = \left(\frac{\beta}{2\pi D}\right)^{1/2} \exp\left(-\frac{\beta y^2}{2D}\right) \tag{3.35}$$

consistent with $\sigma_y^2 = D/\beta$ from Eq. (3.32). A fluctuation from this stationary distribution at time t_1 obeys, over a shorter time $\tau = t_2 - t_1$, the PDF,

$$\begin{aligned} W_2(y_1 y_2 t) &= W_1(y_1)\,P(y_2,t\,|\,y_1) \\ &= \frac{\beta}{2\pi D\,(1-\rho^2)^{1/2}}\,\exp\left(-\frac{\beta}{2D(1-\rho^2)}\left(y_1^2+y_2^2-2\rho y_1 y_2\right)\right), \end{aligned} \tag{3.36}$$

where $\rho = \rho(\tau)$ is the correlation function (3.30). Since Brownian motion is a Markov process, subsequent steps will also obey Eq. (3.36).

Remark 3.1.14 *In computing mean values, the average is implicitly assumed to be taken over an ensemble of similarly prepared paths of the process, for example all paths* $\{B(t)\}$ *which start at time* t_0 *and are observed over an interval* T*. Under an assumption of* ***ergodicity****, mean values over an ensemble will be the same as their time averages, i.e., averages computed for a single path* $B(t) \in \{B(t)\}$ *over a period* $T \to \infty$*. In general, an ensemble average and time average will be different when the governing probability density is time dependent.*

For a ***stationary process*** *the probability density is independent of time, and hence the two methods of computing a mean will yield the same result [73]. Since the Brownian motion/noise process* $\{B(t)\}$ *is stationary (and hence so is the white noise process), ensemble and time averages can be used interchangeably. Finally, if the*

parameter t represents a spatial displacement along a curve $\mathfrak{c}(t)$ and $T = \mathfrak{L}(\mathfrak{c})$ is the length of the curve, then the same equivalence of ensemble mean and interval mean should hold true, as long as the process $B(t)$ is stationary along $\mathfrak{c}(t)$.

3.2 Stochastic Calculus

The stochastic calculus is designed to work with nondifferentiable functions like φ, where ordinary calculus fails and ordinary differentials do not exist[4]. It considers only regular stochastic processes, which are either continuous, or else are right-continuous with left limits or left-continuous with right limits. Regularity further requires that the process has at most a countable number of discontinuities, all of them jumps.

3.2.1 Stochastic differentials

A stochastic differential equation (SDE) takes a different form than a partial differential equation normally encountered in physics, made necessary because SDEs involve nondifferentiable functions. To allow an exact treatment, they are expressed in terms of differentials, not derivatives. The following remark summarizes how differentials of nondifferentiable functions are given meaning in the stochastic calculus [42].

Remark 3.2.1 *Although ordinary differentials do not exist for nondifferentiable functions, it is possible to define an integral with respect to such functions, and through an integral representation give meaning to differentials of them. Specifically, once an integral has been defined with respect to a stochastic function w (w possibly has infinite variation), i.e., $\int_a^t f(s)\,dw(s)$, then if the relation holds that*

$$X(t) = X(a) + \int_a^t f(s)\,dw(s) \qquad \text{for all } t > a\,, \tag{3.37}$$

then **by agreement or convention** *this integral relation can be written in differential form as*

$$dX(t) = f(t)\,dw(t)\,.$$

[4]The brief review of some of its highlights in this section will generously borrow from Ref. [42]. Like the previous section, this section will not strictly adhere to the notation introduced by Sect. 1.6.

In this sense X possesses a differential with respect to w. That is, the differential dw of nondifferentiable function w has no independent meaning; it is a shorthand notation derived exclusively from the integral relation (3.37).

This form is not well suited to an equation of motion or other dynamical differential equations. Sect. 3.3 will start with a different construction of a differential and define an approximate derivative that will be employed in this thesis to describe dynamics, implicitly or explicitly. The applicability of the approximate derivative relies on an assumption that stochastic effects are perturbative to the dynamics, an assumption that only holds in what will be called the dynamics-dominated regime (or dynamical regime) of $\varphi[\mathfrak{M}_t]$. Dynamics domination accompanies and maintains a preferred scale (Rmk. 2.2.14); the preferred scale gives unique physical meaning to quantities of distance, time, rates of change, and so on.

Due to its probabilistic origin, a preferred scale should be rare. Almost everywhere φ will be a stochastic process with no meaningful dynamics. This will be called the stochastics-dominated regime (or stochastic regime) of $\varphi[\mathfrak{M}_t]$: here, the assumption does of perturbative stochasticity does not hold and the standard theory of stochastic differential equations must be used instead.

The established mathematical theory of stochastic processes apparently has not been extended to general metric spaces like $\mathfrak{M}$ which are not product spaces. Thus, even the stochastic calculus may not be wholly adequate for describing φ in the stochastic regime without some generalization. No such generalization will be attempted here.

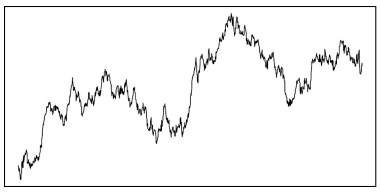
3.2.2 Stochastic integrals

When integrating a stochastic function, the integration measure is obtained from the underlying stochastic process. Recall from Subsect. 3.1.1 that a stochastic process is a collection $\{X_i(t)\}$ of random variables $X_i(t) \equiv X(t_i)$ of identical probability distribution. The collection must have a finite number of elements, corresponding to a finite partition of the parameter t of the process. Even if the continuum limit is of interest, a stochastic process requires a finite dimensional probability distribution, Eq. (3.2):

$$P(X(t_1) \leq x_1,\ X(t_2) \leq x_2,\ \ldots,\ X(t_n) \leq x_n), \qquad 1 \leq n < \infty,$$

for any choice of times $0 \leq t_1 < t_2 < \cdots < t_n < T$, with $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Since n can be arbitrarily large, an arbitrarily precise description of a path $X(t)$ of the process can be obtained by making the partition of the interval $0 \le t \le T$ sufficiently fine. For example, consider one path $B(t)$ for Brownian motion in one spatial dimension:



This could represent the displacement of a particle as a function of time. If $B(t)$ corresponds to the amplitude $\varphi(t)$ along a coordinatized path $\gamma \subset \mathfrak{M}$, then t_i would represent a displacement along γ.

The generalization of a process $\{X(t)\}$ to n-dimensional Euclidean space in $n+1$ spacetime dimensions is straightforward. If $\mathbf{B}(t)$ represents the path of a particle undergoing Brownian motion in three spatial dimensions, then $\mathbf{B}(t)$ can be thought of as a 1-D path of a point-like object through a 3+1 dimensional embedding spacetime, with $\mathbf{B}(t_i)$ one point on that curve.

Denote $dX(t)$ the integration measure with respect to the process $\{X(t)\}$, and $\int_0^T Y(t)\, dX(t)$ the integral of a function $Y(t)$ with respect to $\{X(t)\}$ over an interval $[0, T]$. The integral should have the property that, when $Y(t) = c = \text{const}$,

$$\int_0^T Y(t)\, dX(t) = c \cdot \int_0^T dX(t) = c\,(X(T) - X(0)) \ . \tag{3.38}$$

A **simple stochastic process** is one where $Y(t)$ is constant during each subinterval $(0, t_1], \ldots, (t_i, t_{i+1}], \ldots, (t_{n-1}, T]$ with n finite. Consider $n = 2$ with $Y(t) = c_0$ on $[0, t_1]$ and $Y(t) = c_2$ on $(t_1, T]$, and require that $\int_0^T Y(t)\, dX(t)$ has the additional property $\int_0^T Y(t)\, dX(t) = \int_0^{t_1} Y(t)\, dX(t) + \int_{t_1}^T Y(t)\, dX(t)$; both integrals evaluate as (3.38).

Integration of more general functions with respect to $\{X(t)\}$ proceeds by writing $Y(t)$ in terms of indicator functions $I_{(t_i, t_{i+1}]}(t)$ as in Eq. (3.1),

$$\int_0^T Y(t)\, dt = c_0 I_0(t) + \sum_{i=1}^{n-1} c_i I_{(t_i,\, t_{i+1}]}(t)\,,$$

where $I_{(t_i, t_{i+1}]}(t) \equiv 1$ if $t \in (t_i, t_{i+1}]$ and zero otherwise. Hence, taking the limit $\max_{1 \le i < n}(t_{i+1} - t_i) \to 0$ with n finite but arbitrarily large, then $X(t_{i+1}) - X(t_i) \to dX(t_i)$ with probability 1, and $\int_0^T Y(t)\, dX(t)$ becomes a sum of integrals each of the form (3.38).

First consider integration with respect to a process $\{A(t)\}$ of finite variation, *i.e.*, where the variation $V_A([a, b]) = \sup \sum_{i=1}^n |A(t_i^n) - A(t_{i-1}^n)| < \infty$. The Jordan

decomposition (Thm. 3.1.4) says that any function of finite variation is a difference of two monotone functions, so an integral defined with respect to a monotone function is sufficient.

Given an interval $[a,b]$ and a partition $a = t_0^n < t_1^n < \cdots < t_n^n = b$ with $\delta = \max_{1\le i\le n}(t_i^n - t_{i-1}^n)$ and $t_{i-1}^n \le \xi_i^n \le t_i^n$, define the **Stieltjes integral** of f with respect to a monotone function g on interval $(a,b]$ thusly [42]:

$$\int_a^b f\,dg \quad = \quad \int_a^b f(t)\,dg(t) \quad = \quad \lim_{\delta\to 0}\sum_{i=1}^{n} f\big(\xi_i^n\big)\Big(g\big(t_i^n\big) \;-\; g\big(t_{i-1}^n\big)\Big)\,. \tag{3.39}$$

Now consider integration with respect to a process of infinite variation, Brownian motion being the most common of such processes. Specializing to the Brownian motion process, $X(t) \equiv B(t)$, leads to the **Itô integral** for a simple stochastic process [42],

$$\int_0^T Y(t)\,dB(t) = \sum_{i=0}^{n-1} c_i\big(B(t_{i+1}) - B(t)\big)\,. \tag{3.40}$$

If $Y(t)$ is a deterministic function that contains no random variables, *e.g.*, $Y(t) = t^2$, then stochasticity only enters through the measure $dX(t)$. More generally, $Y(t)$ can be a stochastic process: $Y(t) = \{Y(t_i)\,|\,1 \le i \le n\text{–}1\}$. For example, $\int_0^T B(t)\,dB(t)$ is the integral of Brownian motion with respect to Brownian motion on $[0,T]$. This is straightforwardly accomplished by replacing each constant c_i in (3.40) with a value $b(t_i)$ drawn from the random variable $B(t_i)$.

Integrals having the form of the Itô integral (3.40) are not the most general stochastic integrals. Most generally, the underlying process is a semimartingale.

Def. 3.2.2 (Semimartingale. [42]**)** *A regular right-continuous with left limits adapted process is a semimartingale if it can be represented as a sum of two processes: a local martingale* $\{M(t)\}$ *and a process* $\{A(t)\}$ *of finite variation,*

$$S(t) = S(0) + M(t) + A(t)\,, \qquad \textit{with } M(0) = A(0) = 0\,. \tag{3.41}$$

Thus, the most general stochastic integral, $\int_0^T H(t)\,dS(t)$, breaks up into a sum of two integrals,

$$\int_0^T H(t)\,dS(t) = \int_0^T H(t)\,dM(t) + \int_0^T H(t)\,dA(t)\,.$$

Since A is a function of finite variation, the second integral on the right hand side can be done path by path with a Stieltjes integral (3.39), provided H is integrable with respect to A, *i.e.*, $\int_0^T |H(t)|\,dV_A(t) < \infty$.

3.3 Differentiation of φ

Postulate 2.2.11 is the starting point for obtaining the dynamics of $\varphi[\mathfrak{M}_\mathfrak{t}]$. It asserts the coupling of $\varphi(\mathfrak{x})$ to its neighborhood $B_\varepsilon(\mathfrak{x})$ depends on differences in field values, not the field values themselves. Namely, a constant scalar value can be added to φ at each point with no net effect: $\varphi(\mathfrak{x}) \to \varphi'(\mathfrak{x}) = k + \varphi(\mathfrak{x})$ for all $\mathfrak{x} \in \mathfrak{M}$ leaves φ and φ' indistinguishable in their behavior. Hence, the dynamics should depend on field derivatives and variations, but not absolute field magnitudes.

As suggested in Subsect. 3.2.1, the stochastic calculus is inadequate once dynamics become important because it provides no counterpart to ordinary and partial derivatives that occur throughout physics. This is not a deficiency of the stochastic calculus itself: functions of infinite variation like φ are nowhere differentiable on any finite interval, and there is nothing the stochastic calculus or any other consistent mathematical theory can do to change that fact.

The theory of stochastic processes is a mathematically precise model of the stochastic regime, but in the dynamics-dominated regime where partial differential equations of physics typically apply, fluctuations are perturbative. This behavior can be captured by an approximate derivative: the mean (or 'classical') derivative $\bar{\partial}\varphi$ is obtained by smoothing over small distance scales, whereas the standard deviation from this mean captures the effects of the stochastic perturbations. The smoothing scale for $\bar{\partial}\varphi$ is presumably the minimum physically resolvable distance and time scales ε and τ, respectively. At scales smaller than ε and τ the fluctuation amplitudes are too small to contribute to the classical motions.

However, the coarse graining introduces a physically relevant uncertainty into the derivative, so that the PDE describes a distribution whose mean is the classical case. That is, a PDE constructed from approximate derivatives should be interpreted in an ensemble sense. (Thus an approximate derivative may also be called a statistical derivative.) The ensemble is the set of field configurations that are compatible with the temporal and spatial boundary conditions.

Of course, the same PDE governs the individual members of the ensemble; each member corresponds to a particular, actual instance. Studying the evolution of a single configuration may provide information about detailed processes that is lost by averaging over the ensemble. For example, if an unstable particle is taken to be a

dynamical field configuration which is described by a PDE, then the ensemble can give insight into decay rates and decay paths but probably not details of the decay process itself. Study of individual members of the ensemble is needed for insight into the details of the process, *e.g.*, a growing instability induced by an atypical fluctuation.

Continuous derivatives can exist only for functions of bounded variation, so the first step is to define a procedure for obtaining a version of φ that has bounded variation. Since Fourier modes of arbitrarily high frequencies are responsible for the infinite variation of φ, the obvious way to bound the variation is to introduce a cutoff scale using scale-free criteria. Specifically, let $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ be a preferred scale (Rmk. 2.2.14), and $\varepsilon^{-1} \in [\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ and $\tau^{-1} \in [\omega_{\min}, \omega_{\max}]$ respectively be the spatial and temporal cutoff scales for approximate derivatives along curves on $\mathfrak{M}_\mathfrak{t}$. Implementing the cutoff is straightforward: a smoothed $\bar{\varphi}(x)$ is obtained by averaging over an interval of length ε or τ, centered on x; then the derivative is computed as $\lim_{\Delta x \to 0}[\bar{\varphi}(x+\Delta x) - \bar{\varphi}(x)]/\Delta x$. Thus the approximation is introduced by the smoothing, not by the limiting process.

The cutoff scales ε, τ are not arbitrary. Presumably, random spatial variations on $\mathfrak{d}$-measured scales smaller than ε and random temporal variations on cosmic time scales smaller than τ cannot discernibly affect phenomena at the preferred scale. Namely, the corresponding physical values $\underset{\sim}{\varepsilon}, \underset{\sim}{\tau}$ are physical resolution limits; *i.e.*, $\underset{\sim}{\varepsilon}$ is the minimum distance below which physical changes cannot be reliably discerned in principle by any experiment, and similarly $\underset{\sim}{\tau}$ is the minimum time interval between two events below which the two events cannot be distinguished by any conceivable experiment. This physical origin of cutoff scales will be considered in Subsect. 4.4.4.

In the remainder of the chapter, the nonphysical ε, τ, and curves on $\mathfrak{M}_\mathfrak{t}$ will be used rather than their physical counterparts $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\tau}$. This is because the approximate derivatives will be defined on $\mathfrak{M}_\mathfrak{t}$. Postulate 4.4.5 connects $(\underset{\sim}{\varepsilon}, \underset{\sim}{\tau})$ with (ε, τ).

For spatial derivatives, the goal is a useful directional derivative $\bar{\partial}_\mathfrak{x}\varphi$ along an arbitrary path γ; $\mathfrak{x}$ is a coordinatization of γ. It is necessary to work with paths, and curves more generally, because they are the basic objects of study in a general metric space like $\mathfrak{M}$.

The first step is to smooth φ without requiring that γ is straight (Def. 2.1.16). To do this, obtain the mean $\bar{\varphi}$ over a distance ε using stochastic integration, then assign $\bar{\varphi}$ to the midpoint. Distances along γ are readily computed with the distance metric $\mathfrak{d}(\mathfrak{x}, \mathfrak{y})$; see Def. 2.1.12.

Assume that φ in the stochastics-dominated regime is a Brownian motion process, $\{B(\mathfrak{t})\}$. According to Thm. 3.1.8(B1) there is always a continuous version of a given path $B(\mathfrak{t})$ of the process. Hence use the Itô integral (3.40) at fixed time $\mathfrak{t}$ to compute

$$\bar{\varphi}(\mathfrak{x};\mathfrak{t}) = \frac{1}{\varepsilon}\int_{\mathfrak{x}-\varepsilon/2}^{\mathfrak{x}+\varepsilon/2} \varphi(s;\mathfrak{t})\, dB(s)\,. \tag{3.42}$$

This smoothing procedure allows defining an approximate derivative that can be used in place of a partial derivative with the recognition that the smoothing scale ε introduces an uncertainty into the computed derivative.

Def*. 3.3.1 (Approximate derivative, $\bar{\partial}\varphi$.) *Given a path $\gamma : [0,1] \to \mathfrak{M}$, compute the approximate derivative $\bar{\partial}_{\mathfrak{x}}\varphi$ along γ thusly:*

- *Coordinatize γ with a smooth invertible map $\mathfrak{x} : \mathbb{R} \mapsto \gamma$.*
- *Smooth $\varphi(\mathfrak{x})$ over characteristic scale ε by applying Eq.* (3.42) *for each $\mathfrak{x} \in \gamma$. This coarse-graining of φ is consistent with the assumption that phenomena at the ε scale are substantially independent of scales less than ε.*
- *Compute the* approximate derivative *of $\varphi(\mathfrak{x};\mathfrak{t})$ at fixed $\mathfrak{t}$ with*

$$\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x};\mathfrak{t}) = \lim_{\Delta\mathfrak{x}\to 0} \frac{\Delta\bar{\varphi}(\mathfrak{x};\mathfrak{t})}{\Delta\mathfrak{x}}\,, \quad \textit{with } \Delta\mathfrak{x} \textit{ on } \gamma\,. \tag{3.43}$$

A completely analogous approximate derivative $\bar{\partial}_t\varphi(\mathfrak{x};\mathfrak{t})$ can be obtained by fixing $\mathfrak{x}$ and smoothing $\varphi(\mathfrak{x};\mathfrak{t})$ over an interval of length τ in Eq. (3.42), then using a limiting procedure like (3.43). Note that the smoothing (3.42) is only first order, so $\bar{\partial}\varphi(\mathfrak{x},\mathfrak{t})$ has infinite variation.

Remark 3.3.2 (Second and higher-order derivatives.) *We also need to define the second derivative of $\varphi(\mathfrak{x},\mathfrak{t})$. If $\bar{\partial}_{\mathfrak{x}}\varphi$ were a smooth function, we could differentiate it along γ in the traditional way as*

$$\partial^{\mathfrak{x}}\big(\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})\big) = \lim_{h\to 0 \text{ on } \gamma} \frac{\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}+h;\mathfrak{t}) - \bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x};\mathfrak{t})}{h}\,. \tag{3.44}$$

Recall $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})$ was obtained by smoothing $\varphi(\mathfrak{x}',\mathfrak{t})$ over a fixed interval $[\mathfrak{x}'-\frac{\varepsilon}{2},\ \mathfrak{x}'+\frac{\varepsilon}{2}]$, for all $\mathfrak{x}' \in [\mathfrak{x}-h,\ \mathfrak{x}+h]$. That is, statistics for computing $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})$ are obtained by sampling φ over the interval $[\mathfrak{x}-\frac{\varepsilon}{2}-h,\ \mathfrak{x}+\frac{\varepsilon}{2}+h]$, which has length $\varepsilon+2h$. Thus computing

the second derivative according to (3.44) *requires sampling over a total interval length* $\varepsilon+4h$*. Although* h *is vanishingly small it is still finite — taking the second derivative using* (3.44) *requires differentiating an infinitely varying function of* φ *over a finite distance* $2h$*, which is not possible.*

Hence, taking a second derivative $\bar{\partial}^2_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})$ *requires smoothing* $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})$ *according to Eq.* (3.43)*, substituting* $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x},\mathfrak{t})$ *for* $\varphi(\mathfrak{x},\mathfrak{t})$*. This procedure can be extended to higher order derivatives with an accompanying increase of the sampling interval by* $2h$ *required for each successive derivative. Note these are the same intervals used to compute approximate derivatives in terms of finite differences, given discrete time steps* τ *and spatial steps* ε*. However, space and time are* not *discrete here, by assumption.*

3.4 Tangent Spaces on $(\Sigma, \mathfrak{d})$

Postulate 2.2.11 implies that all dynamics are localized. Even propagation of field changes (Rmk. 2.2.12) should be a succession of local changes on $\mathfrak{M}_{\mathfrak{t}}$. Since an uncountable number of unique paths can intersect at a point $\mathfrak{x}$, field evolution at $\mathfrak{x}$ can occur in each of the 'directions' that correspond to those uncountable paths. (Path uniqueness in this context means that, given any two straight paths $\gamma, \gamma' \subset \mathfrak{M}$, each containing $\mathfrak{x}$, then $\gamma \not\subseteq \gamma'$ and $\gamma' \not\subset \gamma$.) There are uncountably many relevant, independent field derivatives at $\mathfrak{x}$, and the evolution of $\varphi(\mathfrak{x},\mathfrak{t})$ should depend on the collective effect of all of them.

Although $\mathfrak{M}$ is a metric space, it is not a product space, and hence there is no tangent space at a point in the usual sense, *i.e.*, a vector space with an orthogonal basis. First generalize the notion of tangency on $\mathbb{R}^n$ or any normed linear space to a general metric space:

Def. 3.4.1 (Tangent paths. [16]) *Two paths* $\gamma_i : I_i \to \mathfrak{M}$ *for* $i = 1, 2$ *(*I_i *an interval) are* **tangent** *at* $\mathfrak{x} \in I_1 \cap I_2$ *if*

$$\lim_{h\to 0} \frac{\mathfrak{d}(\gamma_1(\mathfrak{x}+h), \gamma_2(\mathfrak{x}+h)}{h} = 0\,. \tag{3.45}$$

Tangency can be used to define the notion of a path direction on $\mathfrak{M}$:

Def*. 3.4.2 (Direction of a path.) *Given a straight path* η *(Def. 2.1.16) tangent at* $\mathfrak{x}$ *to another path* γ*, then* γ *has the* ***η-direction*** *at* $\mathfrak{x}$*. If both paths are straight,* γ *and* η *have the same direction if and only if* $\eta \subseteq \gamma$ *or* $\gamma \subset \eta$*.*

This allows generalizing the directional derivative to $\varphi[\gamma]$:

Def*. 3.4.3 (Directional derivative.) *Given an arbitrary path γ and a straight path η tangent to γ at $\mathfrak{x}$, the* ***directional derivative*** *at $\mathfrak{x}$ is the approximate derivative $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})$, and has the η-direction.*

Given a notion of direction, we can construct a more general form of tangent space at $\mathfrak{x}$ that is appropriate for $\mathfrak{M}$. First consider how tangent spaces are constructed on a manifold $\mathscr{M}$. The tangent space $T_x\mathscr{M}$ at $x \in \mathscr{M}$ consists of all equivalence classes of differentiable functions $u : [0,1) \to \mathscr{M}$ with $u(0) = x$. Two functions $u, v : [0,1) \to \mathscr{M}$ are in the same equivalence class if they are tangent at zero, *i.e.*,

$$\lim_{h\to 0^+} \frac{\|u(h) - v(h)\|}{h} = 0\,. \tag{3.46}$$

This idea can be carried over to a general complete metric space [67]. Denote $C^0([0,1),\mathfrak{M})$ the space of all functions $u : [0,1) \to \mathfrak{M}$, continuous at zero. With the help of Def. 3.4.1 of tangency, Eq. (3.46) can be generalized.

Def. 3.4.4 (Equivalence classes.) *Let $u, v \in C^0([0,1),\mathfrak{M})$ with $u(0) = v(0) \equiv \mathfrak{x}$. Then u and v are in the same equivalence class if they are tangent at zero, that is if*

$$\lim_{h\to 0^+} \frac{\mathfrak{d}(u(h), v(h))}{h} = 0\,. \tag{3.47}$$

Denote the **tangent space** *$T_{\mathfrak{x}}\mathfrak{M}$ to mean the set of all equivalence classes of functions $u \in C^0([0,1),\mathfrak{M})$ with $u(0) = \mathfrak{x}$, and the* **tangent bundle** *$T\mathfrak{M}$ to mean the disjoint sum of all $T_{\mathfrak{x}}\mathfrak{M}$ for all $\mathfrak{x} \in \mathfrak{M}$.*

While Def. 3.4.4 is general, it will not be the preferred definition for tangent spaces and tangent bundles on $\mathfrak{M}$. The simple connectedness and homogeneity of $\mathfrak{M}$ (Postulates 2.1.27 (T3, T4)) allow a particularly simple construction of a tangent space at $\mathfrak{x}$. Specifically, identify each equivalence class with a straight path from $\mathfrak{x}$ to a unique point on the sphere $K_{\varepsilon/2}(\mathfrak{x})$. The choice $diam(K_{\varepsilon/2}(\mathfrak{x})) = \varepsilon$ is the distance over which a directional derivative $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})$ should be computed along each diameter γ of $B_{\varepsilon/2}(\mathfrak{x})$.

Def*. 3.4.5 *Given a 'homogeneous' metric $\mathfrak{d}$, the* ***tangent space*** *$T_{\mathfrak{x}}\mathfrak{M}$ is the set $\{\eta\}$ of all straight paths from $\mathfrak{x}$ to points on the sphere $K_{\varepsilon/2}(\mathfrak{x})$,*

$$T_{\mathfrak{x}}\mathfrak{M} \equiv \left\{\, \eta : [0,1] \to \mathfrak{M},\ \eta(0) = \mathfrak{x},\ \eta(1) = \mathfrak{y} \mid \mathfrak{y} \in K_{\varepsilon/2}(\mathfrak{x}),\ \mathfrak{d}(\mathfrak{x},\mathfrak{y}) = \varepsilon/2 \,\right\}. \tag{3.48}$$

The ***tangent bundle*** *$T\mathfrak{M}$ is the disjoint sum of all $T_{\mathfrak{x}}\mathfrak{M}$ for all $\mathfrak{x} \in \mathfrak{M}$.*

Note $T_{\mathfrak{x}}\mathfrak{M} \cong K_{\varepsilon/2}(\mathfrak{x})$, an isomorphism implied by (3.48).

The homogeneity of $\mathfrak{d}$ permits defining a kind of 'transport' or translation of an ε-ball along a path. This requires some way to relate the directions in $B_\varepsilon(a)$ to those in $B_\varepsilon(b)$ when a and b are nearby. On a manifold, parallel transport is defined via a connection, so we would like to generalize this idea to $\mathfrak{M}$.

Def*. 3.4.6 (Connection.) *Given a complete, simply connected metric space $\mathfrak{M} \equiv (\Sigma, \mathfrak{d})$ where $\mathfrak{d}$ is homogeneous in the sense of Def*. 2.1.15, define a connection on $\mathfrak{M}$ to be a continuous invertible map*

$$\Theta : T_a\mathfrak{M} \to T_b\mathfrak{M} \quad (\textit{for all } a \in \mathfrak{M} \textit{ with } \mathfrak{d}(a,b) \to 0)\,, \tag{3.49}$$

such that $\mathfrak{d}(y_a, z_a) = \mathfrak{d}(y_b, z_b)$ for all $y_a, z_a \in \bar{B}_\varepsilon(a)$ and all $y_b, z_b \in \bar{B}_\varepsilon(b)$.

Since ε is arbitrary in Def*. 3.4.6, it holds everywhere inside a ball $\bar{B}_{\varepsilon'}(x)$, *i.e.*, for any ε such that $0 < \varepsilon \leq \varepsilon'$. Hence the connection can be used to transport an arbitrary ball along any finite-length path by successive applications of (3.49).

Def*. 3.4.7 (Transport.) *Transport of an ε-ball along a path γ is the translation of $\bar{B}_\varepsilon(\mathfrak{x})$ along γ, consistent with a connection* (3.49) *on $\mathfrak{M}$, such that $\bar{B}_\varepsilon(b)$ is homeomorphic to $\bar{B}_\varepsilon(a)$ for all $a, b \in \gamma$.*

The homogeneity of $\mathfrak{d}$ means this holds for all paths in $\mathfrak{M}$: all $\bar{B}_\varepsilon(\cdot) \subset \mathfrak{M}$ are homeomorphic. Hence the tangent bundle on $\mathfrak{M}$ is trivial: $T_{\mathfrak{x}}\mathfrak{M}$ is invariant for all $\mathfrak{x} \in \mathfrak{M}$. This generalizes the idea that the metric $\mathfrak{d}$ is flat, *i.e.*, that $\mathfrak{M}$ is the generalization of a flat, homogeneous space. (Since $\mathfrak{d}$ is not a physical metric, homogeneity of $\mathfrak{d}$ clearly has no bearing on whether an emergent spacetime $(\mathscr{M}, \mathbf{g})$ is flat.)

3.5 'Surface' and 'Volume' Integrals on $(\Sigma, \mathfrak{d})$

It will be necessary to integrate in more general contexts than simple paths. Difficulties arise when trying to construct integrals of φ over surfaces and volumes in the emergence picture. A tangent space of Sect. 3.4 does not correspond to a tangent space on a manifold. Indeed, these tangent spaces have no intrinsic finite dimensional basis, nor even a countable one: surfaces and volumes are undefined on $\mathfrak{M}$. Moreover, even in $\mathbb{R}^n$, the φ field has infinite variation so that ordinary integrals of φ over surfaces and volumes will not converge.

However, these and other difficulties that arise when trying to integrate in more general contexts than one-dimensional paths appear to be fully contained. Derivatives are not meaningful in the scale invariant, stochastics-dominated regime because scale invariance means there is no preferred choice of smoothing distance ε. In that regime the statistics of φ are determined by a homogeneous and scale invariant probability density function so that phenomena that could distinguish one direction or distance from another do not exist. This saves the consistency of the picture: The same decisive breaking of scale invariance that leads to a locally preferred scale also implies some notion of dimensionality. That is, the insurmountable difficulties only arise where differential operators and general integrals have no meaning, but once they become meaningful those difficulties become surmountable. This issue will be explored in Subsect. 3.5.1 and again in Chap. 4

The preceding argument notwithstanding, it is important that the transition between the stochastic and dynamic regimes remains under full mathematical control to ensure self consistency. A transition between the two regimes takes two forms: spatial (or temporal), because boundaries of the region where scale invariance is broken are not sharply defined; and distance scale dependence, because scale invariance is broken over only a finite range of $\mathfrak{d}$-measured distance scales. In transition regions or scales, both stochastics and dynamics are relevant. These considerations point to a need for being careful.

3.5.1 Fundamental difficulties of integrals over regions

The first step is to clarify the difficulties of constructing consistent integrals over regions more general than simple paths. Then general strategies for overcoming each one can be considered. Some of the problems are specific to the stochastics-dominated regime, and thus can be side-stepped. Others require more work.

As suggested by this section's introduction, generalizing differential elements of area (da) and volume (dV) on $\mathbb{R}^n$ to their analogues on $\mathfrak{M}$ encounters the obvious difficulty that $\mathfrak{M}$ is a general metric space with no intrinsic notion of spatial dimension. Hence there is no way to meaningfully define areas or volumes. One idea might be to define a 'standard ball' with some fixed radius, and then determine the maximum number of them that could be packed into a given region $\mathfrak{U} \subset \mathfrak{M}$; such a 'packing measure' would provide a way to define a primitive notion of the volume of $\mathfrak{U}$. This

would work if $\mathfrak{U}$ could be mapped to $\mathbb{R}^n$, but with n undefined how could such a packing be specified?. Nonetheless, in the special case where the mean value of φ is to be computed over a sphere or ball, an approximate integral can be usefully defined over a region of $\mathfrak{M}$. Subsect. 3.5.3 will outline such a computation.

For the most part, however, this problem can be side-stepped. There is no quantity of physical interest that requires a volume or surface integral in the stochastics-dominated regime. The scale invariance of that regime, in particular, precludes interesting phenomena — physical phenomena require a preferred scale. This makes the impossibility of computing surface and volume integrals irrelevant. Once scale invariance is broken the problem has a very elegant solution, even during the transition to a well-defined physical spacetime. This solution will be developed in Chap. 4.

Now assume scale invariance is broken so that surfaces and volumes can be defined. Three significant problems arise when integrating φ or its derivatives over a surface or volume.

First, the stochastic properties of φ must be respected even in a dynamics-dominated regime. Integrating φ over surfaces and volumes requires extending the Itô integral (3.40) to $n > 1$ dimensions. Infinite variation means exact differentials $\varphi(d\mathfrak{x}) = d\varphi(\mathfrak{x})$ do not exist (see Rmk. 3.2.1); clearly, differentials $\varphi(dV)$ and $\varphi(da)$ are undefined for the same reason. This difficulty is fundamental, but can be overcome by constructing an **approximate differential**, such that the resulting integral converges in probability (Def. 3.1.1) to the 'true' value under appropriate conditions. It requires a suitable smoothing procedure to handle the infinite variation analogously to Eq. (3.43) for approximate derivatives.

The second and third problems are consequences of the lack of any intrinsic relationship between directions on $\mathfrak{M}$ and directions on an emergent physical spacetime $(\mathscr{M}, \mathbf{g})$. At an arbitrary point $\mathfrak{x} \in \mathfrak{M}$ an approximate derivative can be computed for each direction $\eta \in T_{\mathfrak{x}}\mathfrak{M}$. Unlike a physical metric $\mathbf{g}$, $\mathfrak{d}(\mathfrak{x}, \mathfrak{y})$ is just a scalar distance function that implies no relationship between directions in $T_{\mathfrak{x}}\mathfrak{M}$. There is no intrinsic notion of orthogonal directions on $\mathfrak{M}$ that would allow expressing $\bar{\partial}_\eta\varphi$ in terms of a finite dimensional basis at $\mathfrak{x}$. That is, correctly representing changes in φ at $\mathfrak{x}$ using the postulated structure of $\mathfrak{M}$ requires knowing all elements of the uncountable set $\mathcal{D}_{\mathfrak{x}} = \{\, \bar{\partial}_\eta\varphi \,|\, \eta \in T_{\mathfrak{x}}\mathfrak{M} \,\}$. On the other hand, summing differential elements $\mathcal{D}(d\mathbf{a})$ and $\mathcal{D}(dV)$ with an integral requires a finite dimensional basis. Two difficulties arise:

1. $\mathfrak{M}$ contains no intrinsic properties that suggest a way to obtain a finite-dimensional basis.

2. Assuming a finite dimensional basis exists, it is still necessary to compute $\mathcal{D}(da)$ and $\mathcal{D}(dV)$ from $\mathcal{D}_{\mathfrak{x}}$ for all $\mathfrak{x} \in da$ or $\mathfrak{x} \in dV$. Assume some function relates the $\mathcal{D}_{\mathfrak{x}}$ over an entire differential element. Then obtaining the differential elements $\mathcal{D}(da)$ or $\mathcal{D}(dV)$ requires summing the contributions of all elements of $\mathcal{D}_{\mathfrak{x}}$ at each $\mathfrak{x}$, *i.e.*, $\bar{\partial}_\eta \varphi(\mathfrak{x})$ for all $\eta \in T_{\mathfrak{x}}\mathfrak{M}$. Constructing an integral over all directions in $\mathcal{D}_{\mathfrak{x}}$ requires that $\mathcal{D}_{\mathfrak{x}}$ is countable, but $\mathcal{D}_{\mathfrak{x}}$ is uncountable because $T_{\mathfrak{x}}\mathfrak{M}$ is uncountable; the sum cannot be performed.

The first difficulty cannot be solved by relying on properties of $\mathfrak{M}$. Reduction to a finite dimensional basis must be accomplished dynamically — orthogonality of spatial directions must be an emergent notion of the dynamics-dominated regime.

The second problem can be solved if we can construct an approximate integral that obtains the desired sum over directions to whatever precision we require. This requires reducing $\mathcal{D}_{\mathfrak{x}}$ to a finite set. The approximate integrals will be satisfactory provided they converge in probability to their 'true' values.

3.5.2 Obtaining finite covers of balls and spheres on $(\Sigma, \mathfrak{d})$

The ability to obtain a finite cover of a ball or sphere in $\mathfrak{M}$ will be important to the remainder of this section. The postulated properties of $\mathfrak{M}$ provide sufficient conditions for spheres and closed balls to be compact, which is what we need.

Specifically, given the metric homogeneity imposed by Postulate 2.1.27(T4), Theorems 2.1.23 and 2.1.25 imply finite subcovers of radius-ℓ spheres K_ℓ and closed balls $\bar{B}_\ell$ exist:

> **Thm. 2.1.25:** A metric space is compact if and only if it is complete and totally bounded.

> **Thm. 2.1.23:** A metric space is totally bounded if and only if every sequence has a Cauchy subsequence.

Postulate 2.1.27(T2) already asserts $\mathfrak{M}$ is complete. In light of Rmk. 2.1.21, $\mathfrak{M}$ can be taken to be bounded, and bounded metric spaces are closed [14]. This, the

simple connectedness of $\mathfrak{M}$ and the assumed homogeneity of $\mathfrak{M}$ imply every sequence in $\mathfrak{M}$ has a limit point. Given completeness, every sequence has a Cauchy subsequence so that the conditions for $\mathfrak{M}$ compact are met. Finally, every compact set has a finite subcover; balls of radius $\frac{1}{n}$ with $n \in \mathbb{N}$, centered on a finite set of points $\sigma_1, \sigma_2, \ldots, \sigma_{N(n)}$ in $K_\ell(\sigma)$ or $\bar{B}_\ell(\sigma)$, give the required collection.

3.5.3 Integrating over uncountably many directions

Consider a 'surface' integral of φ over a sphere $K_\ell(\sigma)$ in $\mathfrak{M}$. Integration requires a measure on sets (*i.e.*, neighborhoods in K_ℓ, not points in K_ℓ), so $T_\sigma\mathfrak{M}$ must be mapped to a countable partition of $K_\ell(\sigma)$. Fig. 3.2 illustrates the construction of a finite subcover of $K_\ell(\sigma)$ from N 'surface' elements da_i:

$$\bigcup_{i=1}^{N} da_i = \bigcup_{i=1}^{N} \left(B_\varepsilon(\sigma_i') \cap K_\ell(\sigma) \right), \qquad \sigma_i' \in K_\ell(\sigma)\,. \tag{3.50}$$

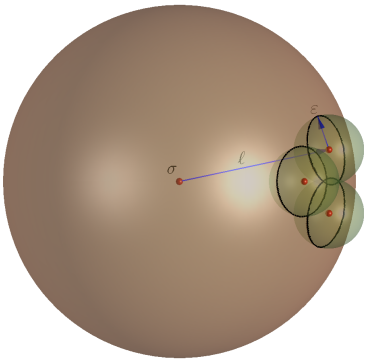


Figure 3.2: Cover of $K_\ell(\sigma)$ by N balls of radius $\varepsilon < \ell$.

The idea is to compute the mean of φ over $K_\ell(\sigma)$ by approximating the mean value $\bar{\varphi}$ on each element da_i, then average all N mean values to obtain an approximation for $\bar{\varphi}[K_\ell(\sigma]$. That is, construct a sequence of approximations,

$$S_N = \frac{1}{N} \sum_{i}^{N} \bar{\varphi}[da_i]\,, \tag{3.51}$$

over finer and finer finite partitions, or subcovers, of $K_\ell(\sigma)$. If this sequence converges in probability (Def. 3.1.1) to the 'true' average $S_\infty = \bar{\varphi}[K_\ell(\sigma)]$ as $N \to \infty$ (so that $\mathrm{diam}(da_i)$ becomes arbitrarily small), *i.e.*, if for any desired $\epsilon > 0$

$$\lim_{N\to\infty} P\left(\left|S_N - S_\infty\right| > \epsilon\right) \to 0\,, \tag{3.52}$$

then the sum (3.51) is a valid approximate integral for computing the mean value $\bar{\varphi}[K_\varepsilon(\sigma)]$. In order to interpret this mean as an ensemble mean, the expectation values $\bar{\varphi}[da_i]$ must be computed over an ensemble of similarly prepared field configurations for each $da_i \subset K_\ell(\sigma)$.

The construction (3.51) clearly requires the existence of a finite subcover of $K_\ell(\sigma)$ with balls of radius $\varepsilon < \ell$. Subsect 3.5.2 shows this can always be obtained for spheres (and closed balls) in $\mathfrak{M}$. In approximating the (ensemble) mean $\bar{\varphi}[da_i]$, the idea is to use something like the intermediate value theorem of real analysis: there is a point $\sigma'' \in B_\varepsilon(\sigma')$ such that $\varphi(\sigma'') = \bar{\varphi}[B_\varepsilon(\sigma')]$. Presumably this can be proven in a general metric space with a 'homogeneous' metric, like $\mathfrak{M}$, by generalizing from the provable observation that there is a point x on each straight path $\gamma(\sigma', \sigma'')$, $\sigma'' \in K_\varepsilon(\sigma')$, such that $\varphi(x) = \bar{\varphi}[\gamma]$. Thus, if we can choose a point in $B_\varepsilon(\sigma') \cap K_\ell(\sigma)$, *i.e.*, a point from each element of the cover, such that $\varphi(\sigma')$ is in some sense 'close enough' to the true mean value $\bar{\varphi}[B_\varepsilon(\sigma') \cap K_\ell(\sigma)]$ at arbitrary time $\mathfrak{t}$, then we can define an approximate integral by taking the average of φ over all elements of the cover.

To make this idea more precise, choose an acceptable uncertainty with which the mean value of φ must be known on an element of the cover, and denote it by δ_φ. Also choose a relative frequency ϵ, such that it is phenomenologically acceptable (*i.e.*, experiments cannot detect violations of physical laws) if $\varphi(\sigma')$ differs from the true mean value by more than δ_φ a fraction ϵ of the time. This criterion of 'phenomenological acceptability' is quite important — as ϵ becomes smaller, implying higher experimental precision or larger statistical sample sizes, inexactness in the mean value computation will show up in higher order terms of a model of a phenomenon, and thus become observable.

Hence, for all $\sigma'' \in B_\varepsilon(\sigma')$, the PDF which governs stochastic fluctuations (*i.e.*, Eq. (2.13) for $\delta\varphi(\sigma')_{\text{rand}}$), together with the dynamical response to $\delta\varphi_{\text{rand}}$ due to interactions with the field on $B_\varepsilon(\sigma')\backslash\sigma'$, imply that

$$P\big(|\varphi(\sigma''; \mathfrak{t}) - \varphi(\sigma'; \mathfrak{t})| > \delta_\varphi\big) < \epsilon\,, \qquad 0 < \epsilon < 1, \tag{3.53}$$

for all $\mathfrak{t}$. That is, $\varphi(\sigma')$ can be taken as the average value of φ on the cover element $B_\varepsilon(\sigma') \cap K_\ell(\sigma)$ with probability $1-\epsilon$, within the specified uncertainty δ_φ.

Together, δ_φ and ϵ imply the value of ε and thus N, the number of elements needed to cover $K_\ell(\sigma)$. For example, decreasing ε while keeping δ_φ fixed implies a

smaller relative frequency ϵ of erroneously computed mean values. Or decreasing ε while keeping ϵ fixed decreases the uncertainty of the computed mean value. Nonetheless, δ_φ cannot be made arbitrarily small because broken scale invariance implies a minimum length scale exists. Hence, is probably best to regard δ_φ as a 'constant of Nature' and take ε (and thus N) as implied by the choice of ϵ and *vice versa*. Note the minimum length scale does not imply a minimum value of ε because ε is an effective smoothing scale for φ — its effect is to coarse grain φ and thus smooth out φ variations contributed by the stochastic regime at distance scales less than ε, but the scale invariance of the stochastic regime allows arbitrarily small values of ε.

Hence the value $\varphi(\sigma')$ will converge in probability to $\bar{\varphi}[B_\varepsilon(\sigma') \cap K_\ell(\sigma)] \pm \delta_\varphi$ as $N \to \infty$, where N is the number of elements in the cover of $K_\ell(\sigma)$. It follows that the average of the N such values will also converge in probability to the true average $\bar{\varphi}[K_\ell(\sigma)]$ in that limit. Hence define the approximate average of φ on $K_\ell(\sigma)$, convergent to the true value $\bar{\varphi}[K_\ell(\sigma)] \pm \delta_\varphi$ in the limit $N \to \infty$, as

$$\bar{\varphi}[K_\ell(\sigma)] \simeq \frac{1}{N} \sum_i^N \varphi(\sigma_i') \,, \tag{3.54}$$

where σ_i' is the center of the i^{th} element of the cover.

An analogous 'volume' average $\bar{\varphi}[\bar{B}_\ell(\sigma)]$ can be constructed as (3.54), given a finite cover of $\bar{B}_\ell(\sigma)$ by balls $B_\varepsilon(\sigma')$ with $\sigma' \in \bar{B}_\ell(\sigma)$. Taking the limit $N \to \infty$ as in Eq. (3.54), the average will converge in probability to the true average $\bar{\varphi}[\bar{B}_\ell(\sigma)] \pm \delta_\varphi$ by reasoning similar to the sphere case.

3.5.4 Volumes and packing measures in $\mathfrak{M}$

Integration can be used to construct differential operators in $\mathbb{R}^n$, for example [6],

$$\begin{aligned} \boldsymbol{\nabla}\varphi(x) &= \lim_{d\tau \to 0} \frac{\int \varphi \, d\mathbf{a}}{\int d\tau} \,, \\ \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}(x) &= \lim_{d\tau \to 0} \frac{\int \boldsymbol{\Phi}(x) \cdot d\mathbf{a}}{\int d\tau} \,, \end{aligned} \tag{3.55}$$

where $\int d\tau = dV$ is the differential volume element and $d\mathbf{a}$ is the (oriented) area of a surface of dV. Because $\int d\tau$ in (3.55) is a spatial volume and not a φ related quantity, it can be evaluated in $\mathbb{R}^n$ but not $\mathfrak{M}$. As Subsect. 3.5.1 explained, this is not a conceptual

problem because notions of volume and area are only meaningful at a preferred scale. If there is a preferred scale, there are maps from regions of $\mathfrak{M}$ to $\mathbb{R}^n$, in principle; $\int d\tau$ can be evaluated indirectly, subject to uncertainties discussed in Subsect. 3.5.5.

In the emergence picture, the spatial dimension n is determined dynamically, subject to the initial conditions of the particular instance of the preferred scale. Thus the physical volumes $\int d\tau$ will depend on those dynamics.

Let $\mathfrak{U}$ be a simply connected, compact region of $\mathfrak{M}$. Because $\mathfrak{U}$ has a finite subcover (Subsect. 3.5.2), it should also be possible to define a packing of balls in $\mathfrak{U}$ without reference to spatial dimension. This indicates a packing measure can be defined on $\mathfrak{U}$. Edgar [18] discusses packing measures in general metric spaces.

3.5.5 Uncertainty in φ on surfaces and volumes

By assumption, in the dynamics-dominated regime the φ field within a region U of $\mathbb{R}^n$ is primarily determined by the φ field equation — stochasticity is perturbative. The ability to construct useful integrals of φ over surfaces and volumes to describe dynamics completely depends on this behavior. The reason is straightforward.

In general, stochastic contributions will introduce diffusion into the evolution; see, *e.g.*, Eqs. (3.7) and (3.32).[5] The variance in a diffusion process is proportional to the time. Thus, if diffusion is significant, summing the infinitesimal field changes over a finite surface or volume can obtain a value that deviates significantly from the classical expectation inferred from the dynamics alone. If, however, stochasticity is perturbative, *i.e.*, the dynamics effectively dominate the φ evolution and thereby limit the diffusion, the variance introduced by diffusion will be small. Then the integral should obtain the classical expectation within a small uncertainty.

Hence, if stochasticity is perturbative, it is reasonable and convenient to work only with smooth functions of φ on U, and separately compute the uncertainty in terms of the variance over an ensemble. Nonetheless, it is important to specify a formal integration procedure that considers the perturbative stochasticity explicitly in order to demonstrate mathematical consistency. Determining the components of that procedure has been the main task of this section.

[5] Brownian motion is the governing stochastic process for diffusion.

3.6 Differential Operators on Infinitely Varying φ on $\mathfrak{M}$

Although the tangent space of Sect. 3.4 has no physical correspondence to a tangent space on an emergent spacetime manifold, it will serve a necessary role in generalizing vector operators like the gradient and divergence of φ to $\mathfrak{M}$. Derivatives are not meaningful in the scale invariant stochastics-dominated regime because there is no preferred choice of smoothing distance ε. Derivatives become relevant only after a preferred scale emerges, *i.e.*, after decisively breaking scale invariance. Dynamical evolution that characterizes interesting physical phenomena can only exist at a preferred scale; such evolution is best described with respect to a physical metric $\mathbf{g}$ very different from the nonphysical metric $\mathfrak{d}$. Thus the need for generalized vector operators which rely on $\mathfrak{d}$ but not $\mathbf{g}$ may not be obvious.

The dynamics of φ on an emergent spacetime $(\mathscr{M}, \mathbf{g})$ must have representations on both $(\mathscr{M}, \mathbf{g})$ and $\mathfrak{M}_\mathfrak{t}$, because both spaces contain the same physical situation but embody different viewpoints. Spacetime manifolds are locally $\mathbb{R}^d$: given the role of φ in determining physical spacetime, there must be a physically motivated map from φ on $B_\varepsilon \subset \mathfrak{M}_\mathfrak{t}$ to φ on $B_{\underset{\sim}{\varepsilon}} \subset \mathbb{R}^d$.

Moreover, the dynamical process governing evolution from the initially broken scale invariance to the establishment of emergent spacetime cannot lead to a sharp boundary between the emergent $(\mathscr{M}, \mathbf{g})$ and the stochastics-dominated $\mathfrak{M}_\mathfrak{t}$. There will be a finite range of $\mathfrak{d}$-measured Fourier modes (or distance scales) which manifest a statistically significant departure from their scale invariant probability density; this range characterizes the preferred scale where interesting physics presumably occurs. Distance scales far below the preferred scale will remain dominated by stochastics. Hence there should be transition regions of Fourier space at the extremes of the preferred scale where both stochastics and dynamics are relevant. To maintain full mathematical control over the transition region, and over the emergence process more generally, operators which are defined with respect to $\mathfrak{d}$ are needed.

Assume a finite cover of N elements on a sphere, $K_\varepsilon \subset \mathfrak{M}$. By identifying each element of the tangent space at σ with a point on $K_\varepsilon(\sigma)$, the cover implies a basis of $\frac{N}{2}$ elements can be constructed for $T_\mathfrak{r}\mathfrak{M}$ that can be mapped to a sphere $K_{\underset{\sim}{\varepsilon}}(x)$ on $\mathbb{R}^n$. Fig. 3.3 illustrates the basic idea. With no notion of spatial dimension on $\mathfrak{M}$, in general n is undefined, and thus arbitrary. Hence, the notion of a gradient, or 'net direction'

for $\bar{\partial}\varphi(\mathfrak{x})$ is ill-defined. If, however, there is some *a priori* choice for n, then it may be possible to define a meaningful map from the $\frac{N}{2}$ 'basis elements' of $T_{\mathfrak{x}}\mathfrak{M}$ to a basis for $T_x\mathbb{R}^n$.

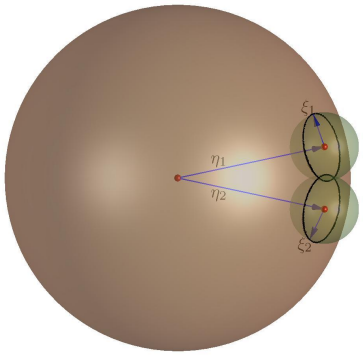


Figure 3.3: Different emergent directions.

3.6.1 The derived field Φ

The derived field, $\mathbf{\Phi}(\mathfrak{x})$, is the starting point for defining spatial differential operators on a region $\mathfrak{U}_{\mathfrak{t}} \subset \mathfrak{M}_{\mathfrak{t}}$. The presumption is that dynamical contributions to $\varphi[\mathfrak{U}_{\mathfrak{t}}]$ are small compared to a homogeneously thermalized φ which acts as an (effective or actual) stochastic regime, and that the perturbative dynamics determine both the smoothing interval ε and effective spatial dimension n.

Because $\mathfrak{M}$ is not a product space, it has no intrinsic spatial dimension n that determines the tangent space $T_{\mathfrak{x}}\mathfrak{M}_{\mathfrak{t}}$ has n basis vectors. Whether $T_{\mathfrak{x}}\mathfrak{M}_{\mathfrak{t}}$ or $T_x\mathscr{M}$, a tangent space has uncountably many elements of course, but $\mathfrak{M}_{\mathfrak{t}}$ is not an inner product space. Thus, considering only the intrinsic structure of $\mathfrak{M}_{\mathfrak{t}}$, relationships among the uncountable directions in $T_{\mathfrak{x}}\mathfrak{M}_{\mathfrak{t}}$ are undefined at best. The generalization to $\mathfrak{M}$ of a gradient is uncountably many field derivatives $\bar{\partial}_\eta\varphi$ at each $\mathfrak{x} \in \mathfrak{M}$, not an economical, n-component representation like $\boldsymbol{\nabla}\varphi$ on $\mathbb{R}^n$.

Hence, it is useful to define a derived field $\mathbf{\Phi}(\mathfrak{x})$ as the collection of directional derivatives along the straight paths which comprise $T_{\mathfrak{x}}\mathfrak{M}$, computed over distances ε. An isomorphism between the tangent space at $\mathfrak{x}$ and a sphere centered on $\mathfrak{x}$ (Sect. 3.4) suggests a convenient representation of $\mathbf{\Phi}(\mathfrak{x})$. Given a sphere $K_{\varepsilon/2}(\mathfrak{x})$ define two sets of

straight paths,

$$\{\, \eta_{\mathfrak{y}}(\mathfrak{x},\mathfrak{y}) \mid \mathfrak{y} \in K_{\varepsilon/2}(\mathfrak{x}) \,\},$$
$$\{\, \gamma_{\mathfrak{y}}(a,b) \mid \gamma_{\mathfrak{y}} = \eta_a \cup \eta_b \text{ is a diameter of } K_{\varepsilon/2}(\mathfrak{x}) \,\},$$

where each $\eta_{\mathfrak{y}}$ is isomorphic to a element of $T_{\mathfrak{x}}\mathfrak{M}$. From this definition $\mathfrak{L}(\gamma) = \varepsilon$. Assign each approximate derivative $\bar{\partial}_{\eta_a}\varphi(\mathfrak{x})$ to a point $a \in K_{\varepsilon/2}(\mathfrak{x})$, such that the derivative is computed along γ in the direction $\mathfrak{x} \to a$. Since $\mathbf{\Phi}(\mathfrak{x})$ is defined thusly for all $\mathfrak{x} \in \mathfrak{M}$, $\mathbf{\Phi}[\mathfrak{M}]$ is a real-valued instantaneous field on $\mathfrak{M} \times K_{\varepsilon/2}$; $\mathbf{\Phi}[\mathfrak{M}_{\mathfrak{t}}]$ is its time dependent generalization.

Denote this real-valued field of directional derivatives on $K_{\varepsilon/2}(\mathfrak{x})$ by $\Upsilon_{\mathfrak{x}}[K_{\varepsilon/2}(\mathfrak{x})]$. Let $\{a,b\} = \gamma \cap K_{\varepsilon/2}(\mathfrak{x})$, and note two elements $\xi_a, \xi_b \in T_{\mathfrak{x}}\mathfrak{M}$ correspond to each choice of $\gamma = \eta_a \cup \eta_b$: ξ_a for $\mathfrak{x}$ to a and ξ_b for $\mathfrak{x}$ to b. Moreover, η_a and η_b correspond to opposite directions on γ: derivatives along η_a are computed $\mathfrak{x} \to a$ while derivatives along η_b are computed $\mathfrak{x} \to b$. In this sense, the elements of $T_{\mathfrak{x}}\mathfrak{M}$ which correspond to η_a and η_b can be said to have **opposite directions**. Accordingly, require $\Upsilon_{\mathfrak{x}}(a) = -\Upsilon_{\mathfrak{x}}(b)$, and impose the convention that $\Upsilon_{\mathfrak{x}}(a) = \bar{\partial}_{\gamma}\varphi(\mathfrak{x})$ if the coordinatization of γ assigns $\mathfrak{x}$ a smaller coordinate value than a; otherwise $\Upsilon_{\mathfrak{x}}(a) = -\bar{\partial}_{\gamma}\varphi(\mathfrak{x})$.

Def*. 3.6.1 (Derived field $\mathbf{\Phi}$.) *At a scale ε, the derived field $\mathbf{\Phi}(\mathfrak{x})$ is a real-valued field $\Upsilon_{\mathfrak{x}}(\cdot)$ over a sphere $K_{\varepsilon/2}(\mathfrak{x})$, i.e., the collection of all directional approximate derivatives at $\mathfrak{x}$. For all $\mathfrak{x} \in \mathfrak{M}$ and all $\xi_a, \xi_b \in T_{\mathfrak{x}}\mathfrak{M}$,*

$$\mathbf{\Phi}(\mathfrak{x}) = \Big\{ \Phi_{\gamma}(\mathfrak{x}) \equiv \bar{\partial}_{\gamma}\varphi(\mathfrak{x}) = \Upsilon_{\mathfrak{x}}(a) = -\Upsilon_{\mathfrak{x}}(b) \;\Big|\; \{a,b\} = \gamma \cap K_{\varepsilon/2}(\mathfrak{x}),\, \gamma = \eta_a \cup \eta_b;\, \eta_a, \eta_b \cong \xi_a, \xi_b \Big\}, \quad (3.56)$$

*where each approximate derivative $\bar{\partial}_{\gamma}\varphi(x)$ is computed over the distance $\mathfrak{L}(\gamma) = \varepsilon$. By convention, $\Upsilon_{\mathfrak{x}}(a) = \bar{\partial}_{\gamma}\varphi(\mathfrak{x})$ when the coordinatization of γ assigns $\mathfrak{x}$ a smaller coordinate value than a, and $\Upsilon_{\mathfrak{x}}(a) = -\bar{\partial}_{\gamma}\varphi(\mathfrak{x})$ otherwise. Denote the **γ-component** of $\mathbf{\Phi}(\mathfrak{x})$ by $\Phi_{\gamma}(\mathfrak{x})$.*

A computation of $\mathbf{\Phi}(\mathfrak{x})$ could be similarly performed if $\mathfrak{M}$ were a smooth n-dimensional manifold, replacing approximate derivatives with ordinary partial derivatives and obtaining the uncountable set of possible directions from $T_x\mathscr{M}$ instead of $T_{\mathfrak{x}}\mathfrak{M}$. A set of n basis vectors can always be obtained at each $x \in \mathscr{M}$, so that a gradient

$\boldsymbol{\nabla}\varphi(x)$ is fully specified by n components expressed in the chosen basis at x. The most general analogue on $\mathfrak{M}$ of the n-component $\boldsymbol{\nabla}\varphi(x)$ on $\mathscr{M}$ is a restricted version of $\boldsymbol{\Phi}(x)$, defined in the next subsection.

3.6.2 Gradient of $\varphi(\mathfrak{x})$ on $\mathfrak{M}$

Let $\bigcup_i^{2N} B_\varepsilon(\mathfrak{y}_i)$ be a finite subcover of $K_{\varepsilon/2}(\mathfrak{x})$ obtained as in Subsect. 3.5.2 with $\ell \to \varepsilon/2$. Let $\mathfrak{Y}^N = \{\mathfrak{y}_1, \ldots, \mathfrak{y}_N\}$ be the set of unique directions obtained from this cover: Given the $2N$ elements of the cover centered on $\{\mathfrak{y}_1, \ldots, \mathfrak{y}_{2N}\}$, omit $\mathfrak{y}_j$ from $\mathfrak{Y}^N$ if $\mathfrak{y}_i$ and $\mathfrak{y}_{j\neq i}$ lie on the same diameter γ and the angle (Def. 2.1.18) $\tilde{\angle}(\eta_N(\mathfrak{x}, \mathfrak{y}_N), \eta_i(\mathfrak{x}, \mathfrak{y}_i)) < \tilde{\angle}(\eta_N(\mathfrak{x}, \mathfrak{y}_N), \eta_j(\mathfrak{x}, \mathfrak{y}_j))$ (*i.e.*, η_N is the reference direction).

Define the **N component gradient** $\boldsymbol{\Phi}^N(\mathfrak{x})$:

$$\boldsymbol{\Phi}^N(\mathfrak{x}) = \left\{ \Phi_{\gamma_i}(\mathfrak{x}) = \Upsilon_{\mathfrak{x}}(\mathfrak{y}_i) \mid \mathfrak{y}_i \in \mathfrak{Y}^N \right\}, \tag{3.57}$$

where $\Upsilon_{\mathfrak{x}}(\mathfrak{y}_i)$ is computed as in Subsect. 3.6.1. To simplify notation, $\Phi_{\gamma_i}(\mathfrak{x})$ may also be denoted by $\Phi_i(\mathfrak{x})$.

The connection (3.49) allows transporting $K_{\varepsilon/2}$ along any path γ, mapping the directions on $K_{\varepsilon/2}(\mathfrak{x})$ to those on nearby $K_{\varepsilon/2}(\mathfrak{x}' \neq \mathfrak{x})$, and hence relating directional derivatives on $\boldsymbol{\Phi}(\mathfrak{x})$ and $\boldsymbol{\Phi}(\mathfrak{x}')$. This allows constructing a finite dimensional (N component) 'gradient' over any finite distance at the chosen characteristic scale ε.

3.6.3 Divergence of $\Phi(\mathfrak{x})$ on $\mathfrak{M}$

Because $\boldsymbol{\Phi}(\mathfrak{x})$ already contains one derivative, the divergence $\mathbf{div}\,\boldsymbol{\Phi}(\mathfrak{x})$ involves second derivatives. Hence the considerations of Rmk. 3.3.2 for higher order derivatives apply. It is now necessary to employ a second sphere $K_\varepsilon(\mathfrak{x})$ of twice the radius used for $\boldsymbol{\Phi}(\mathfrak{x})$ because second derivatives must be computed over a total distance 2ε.

Consider two opposite directions $\xi_a, \xi_b \in T_{\mathfrak{x}}\mathfrak{M}$ and denote their corresponding straight paths by $\eta_a(\mathfrak{x}, a), \eta_b(\mathfrak{x}, b) \subset \bar{B}_{\varepsilon/2}(\mathfrak{x})$ as in the previous subsection; $\gamma_\varepsilon(a, b) = \eta_a \cup \eta_b$ is a diameter of $K_{\varepsilon/2}(\mathfrak{x})$ as before. The second derivatives along a diameter $\gamma_{2\varepsilon} \supset \gamma_\varepsilon$ are now computed thusly:

- Define $\chi_a(\mathfrak{x}, \mathfrak{y}_1) \supset \eta_a$ and $\chi_b(\mathfrak{x}, \mathfrak{y}_2) \supset \eta_b(\mathfrak{x}, b)$ with $\mathfrak{y}_1, \mathfrak{y}_2 \in K_\varepsilon(\mathfrak{x})$ and $\chi_a \cup \chi_b = \gamma_{2\varepsilon}$.
- Compute a continuous succession of approximate first derivatives along $\gamma_{2\varepsilon}$ over distances ε in the usual way (3.43), where the initial derivative is along χ_a, the last

along χ_b, and at $\mathfrak{x}$ it is just $\Upsilon_{\mathfrak{x}}(a)$ for $\Phi_\gamma(\mathfrak{x})$. See Fig. 4.3. This set of approximate first derivatives defines a new scalar field φ' along γ_ε.

- Smooth $\varphi'[\gamma_\varepsilon]$ over a distance ε and compute its approximate derivative in the same way as for first derivatives.

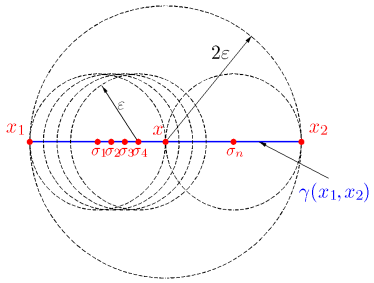


Figure 3.4: Computing second derivatives.

The resulting second derivatives for all $\mathfrak{y} \in K_\varepsilon(\mathfrak{x})$ can now be assigned to points on $K_\varepsilon(\mathfrak{x})$ to obtain a field ${}^{(2)}\mathbf{\Phi}(\mathfrak{x})$ on $K_\varepsilon(\mathfrak{x})$. Like the derived field $\mathbf{\Phi}$, ${}^{(2)}\mathbf{\Phi}(\mathfrak{x})$ is defined for all $\mathfrak{x} \in \mathfrak{M}$ and hence it is a real-valued instantaneous field on $\mathfrak{M} \times K_{\varepsilon/2}$. Its time dependent generalization to $\mathfrak{M}_\mathfrak{t}$ is immediate.

Working in a finite 'basis' allows working with separate components of ${}^{(2)}\mathbf{\Phi}(\mathfrak{x})$. The minimum number of basis elements (call this number $2N$ for consistency with the discussion of gradients) must now be determined by convergence in probability for approximate first derivatives of φ, not φ itself. Like the computation of $\mathbf{\Phi}(\mathfrak{x})$, where $\Upsilon_{\mathfrak{x}}(a) = -\Upsilon_{\mathfrak{x}}(b)$, an analogous situation exists for ${}^{(2)}\mathbf{\Phi}(\mathfrak{x})$. Thus, given a cover of $2N$ elements, only N second derivatives must be retained as components of ${}^{(2)}\mathbf{\Phi}^N(\mathfrak{x})$.

Hence assume ${}^{(2)}\mathbf{\Phi}(\mathfrak{x})$ contains N components, where the set $\mathfrak{Y}$ that identifies these components has been constructed according to the prescription of Subsect. 3.6.2 for N-component gradients. Define the **N-divergence** of $\mathbf{\Phi}(\mathfrak{x})$ as

$$\mathbf{div}\, \mathbf{\Phi}^N(\mathfrak{x}) = \sum_{i=1}^{N} {}^{(2)}\Phi_i^N(\mathfrak{x})\,. \tag{3.58}$$

Chapter 4

Emergence of Dynamics

Enough structure is now in place to take steps toward obtaining physics from the postulated foundation of Chap. 2, utilizing the mathematical tools and structures examined or developed in Chap. 3. That material has been developed in some detail with special attention to ensuring self-consistency. The present chapter will consider how dynamics can emerge from that foundation.

A significant part of Chap. 3 was devoted to careful treatment of the intrinsic stochasticity of φ. A common way of accommodating stochasticity in physics is to treat it as a contribution to the equation of motion. For example, the Langevin equation includes a time dependent random force, $\mathcal{F}(t)$, which introduces randomness so that the motion can only be determined in a probabilistic sense:

$$m\frac{dv}{dt} = \sum_i F_i + \mathcal{F}(t) - \alpha v\,, \tag{4.1}$$

where v is the velocity, the F_i are the deterministic forces and αv is a dissipative force which pushes the system toward equilibrium. However, the Langevin equation is only relevant if dynamics are well defined, and this is not the case prior to manifold emergence. Until then, φ is essentially a random process — the stochastic calculus is the appropriate tool for describing its motions in that regime.

Moreover, a spacetime requires a preferred scale (Rmk. 2.2.14), a continuous range of modes, $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ where dynamics and phenomena are not scale invariant, but a preferred scale only exists over a finite range of modes. This means there is a transition region of Fourier space near $\mathfrak{k}_{\min}$ and $\mathfrak{k}_{\max}$ where the Langevin equation is inapplicable. To maintain full mathematical control from the stochastic regime through

the transition region, it is again necessary to rely on the stochastic calculus to ensure self consistency at all scales.

Dynamics emerge from the stochastic regime when the motions of φ are mostly governed by the coupling equation (2.22) and intrinsic stochasticity is relegated to a perturbation of it. Then the stochastic calculus can be relegated to a formal role in the definition of the approximate derivative (Def*. 3.3.1) which is implicitly assumed whenever a partial derivative occurs; this formally accommodates infinite variation of φ due to modes $\mathfrak{k} > \mathfrak{k}_{\max}$ and contributes a small uncertainty to the derivatives. The primary task of this chapter is to examine the emergent dynamics.

4.1 The Two Regimes of φ on $\mathfrak{M}_\mathfrak{t}$

Chap. 2 assumed scale invariance is a very desirable property of $\varphi[\mathfrak{M}_\mathfrak{t}]$. The dynamics and stochastic behavior were carefully specified to be scale invariant with respect to the distance metric $\mathfrak{d}$. Hence $\mathfrak{d}$ on $\mathfrak{M}$ must be regarded as arbitrary, even though Postulate 2.1.27(T4) specifies it to be homogeneous. The rationale is there is no intrinsic means for selecting an *a priori* absolute scale which every cosmology must respect, so if the preferred scale is absolute a fundamental theory must arbitrarily impose it by *fiat*. On the other hand, experience shows observable phenomena generally have a (fixed) characteristic scale: if $\varphi[\mathfrak{M}_\mathfrak{t}]$ cannot exhibit phenomena at some preferred scale the program begun in the preceding chapters is doomed. From that standpoint, scale invariance is potentially an undesirable property.

The tension between the desirability of scale invariance and the need for preferred-scale phenomena is minimized if φ can inhabit two somewhat stable or persistent regimes:

- **Stochastics domination.** This is characterized by a scale invariant probability density function (PDF). It is the expected regime for φ globally.
- **Dynamics domination.** This is characterized by 'broken' scale invariance in some localized neighborhood $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$ of arbitrary diameter, arising in accordance with the PDF for φ, such that the broken symmetry condition is maintained (and evolved) via dynamics on $\mathfrak{U}_\mathfrak{t}$ — stochastics play a perturbative role. Hence this regime governs a persistent preferred scale (Rmk. 2.2.14) required for cosmology and physics in general.

To realize this program, it is necessary that φ obeys dynamical laws so that a dynamic regime can persist once it arises in $\mathfrak{U}_\mathfrak{t}$, but those dynamics are mostly suppressed in $\mathfrak{M}_\mathfrak{t}\backslash\mathfrak{U}_\mathfrak{t}$ so that φ acts as a random process along almost all curves outside $\mathfrak{U}_\mathfrak{t}$. The characteristic scale of dynamics in $\mathfrak{U}_\mathfrak{t}$ is arbitrary because $diam(\mathfrak{U})$ is arbitrary; hence the laws governing the φ dynamics must have a scale invariant form.[1]

The two regimes have a concrete origin in the oscillator stochasticity postulate (2.2.8) and the field motion postulate (2.2.11). The field motion postulate attributes the change in φ at σ during a time interval I_τ to two contributions:

$$\delta\varphi(\sigma; I_\tau] = \delta\varphi(\sigma; I_\tau]_{\text{coupling}} + \delta\varphi(\sigma; I_\tau]_{\text{rand}}\,, \tag{4.2}$$

where the stochastic contribution $\delta\varphi(\sigma; I_\tau]_{\text{rand}}$ is a Brownian noise, or Brownian motion process of the φ field at σ. Looking ahead to Proposition 4.2.2, Eq. (4.2) implies that $\delta\varphi[\mathfrak{c}]_{\text{rand}}$ is also a Brownian motion process, where $\mathfrak{c}$ is an arbitrary curve in $\mathfrak{U}_\mathfrak{t}$.

The contribution $\delta\varphi[\mathfrak{c}]_{\text{coupling}}$ qualitatively plays several roles. It explains the continuity of $\varphi[\mathfrak{c}]$ implied by Postulate 2.2.6(W2). Since the forms of both the stochastics and dynamics of φ are scale invariant, $\delta\varphi[\mathfrak{c}]_{\text{coupling}}$ must maintain continuity at all scales. Second, $\delta\varphi[\mathfrak{c}]_{\text{coupling}}$ should cause dispersion of localized fluctuations because such fluctuations represent (usually very small) deviations from an 'equilibrium' condition.

When $\delta\varphi[\mathfrak{c}]_{\text{coupling}}$ plays only the two roles above, variations of φ along any curve $\mathfrak{c}$ in $\mathfrak{U}_\mathfrak{t}$ should remain consistent with the probability densities in time (4.3) and space (4.5) over most of their range. Then $\delta\varphi[\mathfrak{c}(\delta\mathfrak{t})] = \delta\varphi[\mathfrak{c}; I_\mathfrak{t}]_{\text{rand}} + \delta\varphi[\mathfrak{c}; I_\mathfrak{t}]_{\text{coupling}}$ characterizes field variations in the **stochastic regime**.

The third role of $\delta\varphi[\mathfrak{c}]_{\text{coupling}}$ is strictly dynamical: it is responsible for the **dynamic regime** in which an equation of motion, derived from the field motion postulate (2.2.11), exists and has deterministic (classical) solutions; stochasticity merely perturbs and adds small uncertainties to the solutions. Given that the dynamic regime departs from primarily stochastic behavior, scale invariance is clearly broken. The dynamic regime is the regime of interesting physics.

[1] In this chapter, scales are assumed to be distances measured with respect to $\mathfrak{d}$ or time intervals measured with respect to cosmic time.

4.2 The Stochastic Regime of φ

The stochastic regime is the answer to the question, What does spacetime emerge from? This section studies some of its properties.

4.2.1 Characterizing the stochastic regime

The field stochasticity postulate (2.2.8) asserts that, during a time interval $I_\mathfrak{t} \equiv [\mathfrak{t}_0, \mathfrak{t}_0 + \delta\mathfrak{t}]$ and for each $\sigma \in \mathfrak{M}$, $\varphi(\sigma; I_t]$ undergoes the net variation (4.2). The stochastic contribution $\delta\varphi_{\text{rand}}$ is a Brownian noise process $\{B(\mathfrak{t})\}$ and therefore the paths of the process manifest the three defining properties of Brownian motion given by Def. 3.1.2:

1. **(Normal distributed increments.)** $B(t) - B(s)$ has distribution $N(0, |t-s|)$, *i.e.*, the Normal distribution with zero mean and variance $(t-s)$, for arbitrary times $s < t$ with $0 \leq (t,s) < \infty$.

2. **(Independent increments.)** $B(t) - B(s)$ is independent of $B(u),\ 0 \leq u < s$.

3. **(Continuity of paths.)** The $B(t),\ t \geq 0$ are continuous functions of t.

The probability density for Brownian noise without dissipation is given by Eq. (3.8) with $v = 0$ and $x \to \delta\varphi(\sigma; \delta t] = \varphi(\sigma, t_0 + \delta t) - \varphi(\sigma, t_0)$:

$$f(\delta\varphi, \delta t) = \frac{1}{\sqrt{2\pi\delta t}} \exp\left(-\frac{(\delta\varphi)^2}{2\delta t}\right); \qquad \delta t = t - t_0\,. \tag{4.3}$$

A new proposition will now be stated and proved. It generalizes the field stochasticity postulate to general continuous curves $\mathfrak{c} \subset \mathfrak{M}_\mathfrak{t}$, suggests a way to characterize the stochastic behavior of $\varphi[B_\ell(\sigma); \mathfrak{t})$. It is also important in its own right.

Two definitions will be needed. Let $\mathfrak{l} = \mathfrak{L}(\mathfrak{c})$ be the $\mathfrak{d}$-measurable length of a curve $\mathfrak{c}(t)$. In general the parameter t is either a temporal or spatial displacement from the start of the curve.

Def*. 4.2.1 *Let $\mathfrak{c} \subset \mathfrak{M}_\mathfrak{t}$ be parameterized by t and let γ be its image on $\mathfrak{M}$. Then $\mathfrak{c}(t)$ is **purely space-like** if t is a displacement along γ, $0 \leq t \leq \tau$, at some time $\mathfrak{t}_*$ and total spatial displacement $\tau = \mathfrak{L}(\mathfrak{c})$; and it is **purely time-like** if t is the time, $\mathfrak{t}_0 \leq t \leq \mathfrak{t}_0 + \tau$, at some point σ and total temporal duration τ.*

Proposition 4.2.2 *Let γ be a path on $\mathfrak{M}$, and $\mathfrak{C}[\gamma]$ be a family of continuously parameterized curves,*

$$\mathfrak{C}[\gamma] = \{\mathfrak{c}^a(t) \mid \mathfrak{c}^a : \mathbb{R}^+ \to \gamma,\ 0 < t \leq \tau^a\}\,,$$

such that the maps from $\mathfrak{c}$ to γ are invertible and τ^a is the duration of $\mathfrak{c}^a$. For each $\mathfrak{c}^a \in \mathfrak{C}$, denote the stochastic contribution to $\{\varphi[\mathfrak{c}^a]\}$ along $\mathfrak{c}^a$ by $\{\varphi[\mathfrak{c}^a]_{\rm rand}\}$. Then

i. In the stochastic regime, i.e., when $\varphi[\mathfrak{c}^a]_{\rm coupling}$ is at most perturbative in the decomposition (4.2)*, $\{\varphi[\mathfrak{c}^a]\}$ is a Brownian noise process along $\mathfrak{c}^a$, having the PDF* (4.3)*.*

*ii. For each $\gamma(\mathfrak{x}) = \mathfrak{c}^a(\mathfrak{t})$ in which $\mathfrak{L}(\gamma) > 0$ there exists a time-invariant **cosmic time rescaling** at $\gamma(\mathfrak{x})$,*

$$\delta\mathfrak{t} \to s^a(\gamma(\mathfrak{x}))\cdot\delta\mathfrak{t}\,, \qquad s^a[\gamma] = \{\, s^a(\sigma) \,|\, \sigma \in \gamma\,\}\,, \tag{4.4}$$

whereby $\{\varphi[\mathfrak{c}^a]_{\rm rand}\}$ and $\{\varphi[\gamma;\mathfrak{t}{=}{\rm const})_{\rm rand}\}$ have the same probability density.

iii. For all $\mathfrak{c}^{b\neq a} \in \mathfrak{C}[\gamma]$ in which $\mathfrak{L}(\gamma) > 0$, $s^b[\gamma] \neq s^a[\gamma]$.

Proof: For Part i it must be shown that the three defining properties of Brownian noise hold for $\{\,\varphi[\mathfrak{c}(t)]_{\rm rand}\,\}$, for all $\mathfrak{c} \in \mathfrak{C}$. Then $\{\varphi(\mathfrak{x};\mathfrak{t})\,\}$ is a Brownian noise process in the stochastic regime because the increments $\delta\varphi \to \delta\varphi_{\rm rand}$ in that regime. In what follows take $\delta\varphi = \delta\varphi_{\rm rand}$.

The case $\mathfrak{L}(\gamma) = 0$ (*i.e.*, $\gamma(\sigma,\sigma) = \sigma$ with $\tau^a > 0$) is an automatic consequence of the field stochasticity postulate 2.2.8. Next consider the purely space-like process $\{B(\mathfrak{x})\,\}$, $0 < \mathfrak{x} \leq \mathfrak{L}(\gamma)$, where each sample path of the process is just a single member of an ensemble, $B(\mathfrak{x}) = \varphi[\gamma;\mathfrak{t})$ with $\mathfrak{t}$ fixed. Continuity of the process is ensured by Postulate 2.2.6(W2) for a purely space-like curve: $d\varphi = |\varphi(\mathfrak{x},\mathfrak{t}) - \varphi(\mathfrak{x}',\mathfrak{t})| \to 0$ as $|\mathfrak{x}-\mathfrak{x}'| \to 0$. According to the field stochasticity postulate, $\{\varphi(\mathfrak{x};\mathfrak{t})\,\}$ and $\{\varphi(\mathfrak{x}';\mathfrak{t})\,\}$ are independent Brownian noise processes and therefore have independent increments and obey the PDF (4.3). Since the sum of Brownian processes is also a Brownian process, $\{B(\mathfrak{t})\,\} = \{\varphi(\mathfrak{x};\mathfrak{t}) - \varphi(\mathfrak{x}';\mathfrak{t})\,\} = \{\varphi(\mathfrak{x};\mathfrak{t})\,\} - \{\varphi(\mathfrak{x}';\mathfrak{t})\,\}$ is a Brownian process (where $\mathfrak{x},\mathfrak{x}'$ are fixed and $\mathfrak{t} \in \mathbb{R}^+$), but so is $\{B(\mathfrak{x})\,\} = \{\varphi(\mathfrak{x};\mathfrak{t}) - \varphi(0;\mathfrak{t})\,\}$ ($\mathfrak{t}$ fixed and $\mathfrak{x} \in (0,\mathfrak{L}(\gamma)]$).

The argument above assumes only the points on γ contribute, but Postulate 2.2.11 implies all paths in the tangent space of $\mathfrak{x} \in \gamma$ contribute to the dynamics

and thus to $\{\varphi(\mathfrak{x};\mathfrak{t})\}$. However, in the stochastic regime the dynamics maintain continuity of φ but do not modify the form of the PDF (4.3), leaving the argument intact. Hence $\{\varphi[\gamma;\mathfrak{t})\}$ is a space-like Brownian process in the stochastic regime, and obeys the PDF

$$f(\delta\varphi,\delta\ell)=\frac{1}{\sqrt{2\pi\delta\ell}}\,\exp\left(-\frac{(\delta\varphi)^2}{2\,\delta\ell}\right),\qquad 0<\delta\ell\leq\mathfrak{L}(\gamma)\,. \tag{4.5}$$

It remains to generalize the purely space-like case to all $\mathfrak{c}^a\in\mathfrak{C}[\gamma]$. Partition $\gamma(0,\mathfrak{L}(\gamma))$ as $\gamma_n=(0=\mathfrak{x}_1<\mathfrak{x}_2<\cdots<\mathfrak{x}_{n-1}<\mathfrak{x}_n=\mathfrak{L}(\gamma))$; likewise partition $[\mathfrak{t}_0,\mathfrak{t}_0+\tau^a]$ as τ^a_m.

Clearly we can choose $n=m$ so that the partitions of γ_n and τ^a_m have the same number of elements. Then for each curve $\mathfrak{c}^a$, each partition $\tau^a_{m=n}$ implies a unique partition γ_n (since $\mathfrak{x}_i=\mathfrak{c}(\mathfrak{t}_i)$), and the partitions τ^a_n and γ_n imply a unique partition of $\mathfrak{c}^a$ also with n elements. Since every curve can be approximated by a set of constant-speed curves, without loss of generality we can consider the case $\dot{\mathfrak{c}}^a(\mathfrak{t})=\delta\ell/\delta\mathfrak{t}=\mathfrak{L}(\gamma)/\tau^a_n=$ const. Examining the probability densities (4.3) and (4.5), it is clear they are identical for all $\mathfrak{L}(\gamma)>0$ if we rescale $\delta\mathfrak{t}$ in Eq. (4.3) as $\delta\mathfrak{t}\to s^a[\gamma]\cdot\delta\mathfrak{t}$, where $s^a[\gamma]=\mathfrak{L}(\gamma)/\tau^a_n$ is a time-independent constant function. This demonstrates Part ii. (After the rescaling, $\delta\mathfrak{t}$ in Eq. (4.3) is reinterpreted as a time interval an 'observer' moving along $\mathfrak{c}^a$ would 'measure.') A different constant speed curve $\mathfrak{c}^b(t)$ with $\dot{\mathfrak{c}}^b(t)=\mathfrak{L}(\gamma)/\tau^{b\neq a}_n$ will have a rescaling function $s^b[\gamma]\neq s^a[\gamma]$, so $s[\gamma]$ depends on the speed of $\mathfrak{c}$, which is Part iii. Since every curve can be approximated by a sequence of constant speed curves, the result generalizes to all $\mathfrak{c}\in\mathfrak{C}[\gamma]$. ∎

Thus $\{\delta\varphi[\mathfrak{c}]_{\rm rand}\}$ is a one-dimensional Brownian noise process described by the PDF (4.3), where t parameterizes the curve $\mathfrak{c}$; $\mathfrak{c}(t)$ need not be either purely time-like or purely space-like.

Proposition 4.2.2 implies that for each $\gamma\subset\mathfrak{M}$ there is a unique family of rescaling functions $S[\gamma]$ such that there is one and only one $s\in S[\gamma]$ for each $\mathfrak{c}\in\mathfrak{C}[\gamma]$. Hence, after rescaling $\delta\mathfrak{t}$ using $S[\gamma]$, there is no way in principle to determine $\mathfrak{c}(t)$ by examining $\{\,\varphi[\mathfrak{c}]_{\rm rand}\,\}$. This is a physical implication, even though it has been derived in terms of a nonphysical metric $\mathfrak{d}(x,y)$ and nonphysical cosmic time $\mathfrak{t}$. In the stochastic regime where $\{\varphi[\mathfrak{c}]\}=\{\,\varphi[\mathfrak{c}]_{\rm rand}\,\}$ this means there is no way to physically distinguish motion from non-motion by observing the process $\{\varphi[\mathfrak{c}]\}$.

Remark 4.2.3 *Proposition 4.2.2 justifies a previous claim that the particular choice of* $\mathfrak{d}(\mathfrak{x},\mathfrak{y})$ *has no physical consequence. Consider Postulate 2.1.27*(T4)*, which states the nonphysical metric is homogeneous. Homogeneity implies all straight paths* $\gamma \subset \mathfrak{M}$ *of the same length* ℓ *have equivalent* $S[\gamma]$*. If the postulate is eliminated so that* $\mathfrak{d}$ *is* not *homogeneous, it will still always be possible to find some* $S[\gamma]$ *such that* $\{\varphi[\mathfrak{c}^a]_{\text{rand}}\}$ *is again invariant for all choices of* $\mathfrak{c}^{\mathfrak{a}} \in \mathfrak{C}[\gamma]$*, although now* $\{\gamma^a \,|\, \mathfrak{L}(\gamma^a) = \ell,\ \gamma^a$ *straight*$\}$ *will not, in general, have equivalent* $S[\gamma]$*. The (time independent) rescaling needed to obtain effective homogeneity of* $\mathfrak{d}$ *is possible because it is essentially a mathematical choice or convention; it is* not dynamical *in origin and has no dynamical consequences. This means Postulate 2.1.27*(T4) *is superfluous:* any *metric that is consistent with the other requirements of Postulate 2.1.27 is suitable; a homogeneous metric is simply a convenience.*

Given the flexibility to rescale the time so that $\{\varphi[\mathfrak{c}^a]\}$ and $\{\varphi[\mathfrak{c}^b]\}$ are indistinguishable for all $\mathfrak{c}^a, c^{b\neq a} \in \mathfrak{C}[\gamma]$, $\{\varphi[\mathfrak{c}]\}$, not $\{\varphi[\gamma;\mathfrak{t}_*)\}$ or $\{\varphi(\sigma;\mathfrak{t})\}$, should be considered the fundamental process. The time-like process $\{\varphi(\sigma;\mathfrak{t})\}$ restricted to a single point σ is a special case, where $\gamma = \gamma(\sigma,\sigma) = \sigma$ so that $\mathfrak{C}[\sigma]$ contains exactly one element, $\mathfrak{c}(t) = \mathfrak{t}$; similarly $S[\sigma]$ contains a single element, $s(\sigma) = 1$. The space-like process $\{\varphi[\gamma;\mathfrak{t}_* = \text{const})\}$ is also a special case: it can be evaluated like any other curve in $\mathfrak{C}[\gamma]$, with $s[\gamma] = \mathfrak{L}(\gamma)/\delta\mathfrak{t}$ in the limit $\delta\mathfrak{t} \to 0$.

Remark 4.2.4 (Inapplicability of equation of motion to stochastic regime.) *The equation of motion holds in the dynamic regime and in the transition between the stochastic and dynamic regimes, but does not hold in the stochastic regime. This can be seen as follows.*

Together, Proposition 4.2.2 and the scale invariance of Brownian motion mean that, in the stochastic regime, the probability density (4.3) *is the same for all rescaled times* $\mathfrak{t}$ *and along all curves* $\mathfrak{c} \in C[\gamma]$ *for all* $\gamma \subset \mathfrak{M}$*. Since* $\{\varphi(\sigma;\mathfrak{t})\}$ *is a stationary process, there is no useful way to distinguish different times. Furthermore,* $\{\varphi(\sigma;\mathfrak{t})\}$ *'looks the same' at all points in* $\mathfrak{M}$*, so there is no meaningful way to distinguish different points nor measure physical distances. That is, just as there is no way to determine the state of motion by observing the process* $\{\varphi[\mathfrak{c}(\mathfrak{t})]\}$*, there is no way to distinguish spatial intervals from time intervals. The stochastic regime admits no notions of physical distance, position, time, motion, or scale; it has the greatest possible symmetry. Absent*

notions of distance, time or motion, there is no sense in which the equation of motion can apply to the stochastic regime; it only applies if the stochastic regime does not hold.

Subsect. 3.1.2 discussed spectral analysis of time dependent stochastic processes, which included procedures suitable for determining the spectral density $\mathcal{S}(\omega)$ and time correlations $\langle y(\mathfrak{t})\, y(\mathfrak{t}+\tau)\rangle$ in terms of the cosmic time at a single point. Spectral analysis of spatially extended processes in terms of wavenumber modes $\mathfrak{k}$ will also be necessary. The methods of Subsect. 3.1.2 can be used by fixing the time and taking samples over an ensemble of paths or over very long spatial distances to determine the spectral density (3.17), $\mathcal{S}(\mathfrak{k})$, and correlation function (3.22), for example $\langle y(\mathfrak{x})\, y(\mathfrak{x}+\ell)\rangle$. Of special interest are modes below the cutoff $\mathfrak{k}_{\max}$ (and similarly $\omega_{\max}$) which arise when scale invariance is broken. As indicated by the intensity-frequency relationship for a Brownian noise source in Fig. 3.1, coefficients $A(\mathfrak{k}) \to 0$ as $\mathfrak{k} \to \infty$ and similarly for $A(\omega)$. Thus a mode decomposition of $\varphi[\mathfrak{c}]$ can be obtained to any desired accuracy by choosing an appropriate but finite range of frequencies.

What is an appropriate interpretation of a spectrum obtained along curves $\mathfrak{c}(\mathfrak{t})$? According to Proposition 4.2.2, $\varphi[\mathfrak{c}]$ has the spectrum of Brownian noise. Because $\mathfrak{c}(\mathfrak{t})$ maps to a single point $\mathfrak{x}$ in the image of $\mathfrak{c}$, the spectrum $\mathcal{S}(\omega)$ implies $\mathcal{S}(\mathfrak{k})$ and vice versa — the two spectra contain identical information. This is a manifestation of the indistinguishability of space and time in the stochastic regime.

4.2.2 Statistics of neighborhoods

To fully characterize the stochastic regime, it is necessary to generalize the statistical behavior of φ for points and curves to φ over neighborhoods, $B_\ell(\sigma)$. The simplest approach is to use the property of scale invariance. Consider the stochastic behavior of the process, $\{\varphi(\sigma,\mathfrak{t})\}$, as the limit

$$\lim_{\ell\to 0} \{\varphi[B_\ell(\sigma);\mathfrak{t})\,\} = \{\varphi(\sigma;\mathfrak{t})\,\}\,.$$

To maintain scale invariant behavior of φ on $\mathfrak{M}_\mathfrak{t}$ in the stochastic regime, the statistics of the mean field value $\bar{\varphi}[B_\ell(\sigma)]$ at a time $\mathfrak{t}$ must obey the PDF (4.5) for all finite ℓ.

The Brownian nature of neighborhoods in the stochastic regime can also be seen straightforwardly by considering φ on the sphere $K_\ell(\sigma)$. According to the field stochasticity postulate, $\varphi(\sigma'; I_\mathfrak{t}]$ is a Brownian noise process for all $\sigma' \in K_\ell(\sigma)$. Sums

of Brownian noise processes are Brownian noise processes, so if a finite sum can be constructed on $K_\ell(\sigma)$ the Brownian nature of $\varphi[K_\ell(\sigma); I_\mathfrak{t}]$ will follow. Subsect. 3.5.3 gives the procedure for constructing the desired sum.

The martingale property of Brownian noise gives a third way to see the Brownian behavior of $\bar{\varphi}[K_\ell(\sigma); \mathfrak{t})$. The field $\varphi(\sigma; \mathfrak{t})$ in the stochastic regime is a Brownian noise process in time. The martingale property (Def. 3.1.9) implies that if the path $\varphi(\sigma; \mathfrak{t})$ of the process $\{\varphi(\sigma; \mathfrak{t})\}$ up to time $\mathfrak{t}'$ is known, then the expected value at σ at any future time is $\varphi(\sigma; \mathfrak{t}')$. Likewise, if the path of the process $\{\varphi(\mathfrak{c}(t); \mathfrak{t}_*)\}$ along a curve $\mathfrak{c}$ is known up to the point $\sigma' = \mathfrak{c}(t')$, where $\mathfrak{t}_*$ is fixed and σ' varies as t increases with $\mathfrak{c}(0) = \sigma$, then the expected value at any point $\mathfrak{c}(t'') > \mathfrak{c}(t')$ is $\varphi(\mathfrak{c}(t'); \mathfrak{t}_*)$. Taking the latter point of view, and letting $\mathfrak{c} \equiv \gamma$ be a radius of $K_\ell(\sigma)$, we have $\bar{\varphi}[K_\ell(\sigma); \mathfrak{t}_*) = \varphi(\sigma; \mathfrak{t}_*)$ at $t = \mathfrak{L}(\gamma) = \ell$. As before, since $\varphi(\sigma; \mathfrak{t})$ is a Brownian noise process in time, $\bar{\varphi}[K_\ell(\sigma); \mathfrak{t})$ will also be a Brownian process in time. By similar reasoning, if K_ℓ is transported along a curve $\mathfrak{c}(t)$, $\bar{\varphi}[K_\ell(\mathfrak{c}(t))$ is a Brownian noise process as it moves along $\mathfrak{c}$.

4.3 Breaking Scale Invariance

In the emergence picture, stochastics domination is the natural or 'equilibrium' condition for φ, presumably holding almost everywhere on $\mathfrak{M}_\mathfrak{t}$. In that regime, φ is statistically homogeneous and scale invariant — it contains no information at all, since there is no attribute of the stochastic regime that allows distinguishing two points $\sigma, \sigma' \in \mathfrak{M}$, nor two cosmic times $\mathfrak{t}$ and $\mathfrak{t}'$, nor one $\mathfrak{d}$-measured scale from another. In that regime $\varphi[\mathfrak{M}_\mathfrak{t}]$ has maximal symmetry in the most general sense possible.

Moreover, φ is the physical entity, not $\mathfrak{M}_\mathfrak{t} \equiv \mathfrak{M} \times \mathbb{R}$. Consequently, despite the superficial appearance of cosmic time as an absolute, globally applicable Newtonian time coordinate, there is no physically meaningful global 'rest frame.' The scale invariance of φ and its dynamics is the reason: Scale invariance with respect to both cosmic time and $\mathfrak{d}$-measured neighborhood diameter leaves no way in principle to distinguish the state of motion on $\mathfrak{M}_\mathfrak{t}$ by any observation of φ configurations. In fact, this is true even if scale invariance is broken in some region S because, again, the scale invariance of φ outside S makes it impossible to determine a state of motion of S on $\mathfrak{M}$. Hence there is no preferred sequence of spacelike slices $\varphi[\mathfrak{M}]$.

To simplify the discussion below, all curves will have constant speed, $\dot{\mathfrak{c}}(t) =$

const, and each map $\mathfrak{c} : \mathbb{R}^+ \to \gamma \subset \mathfrak{M}$ is continuous and invertible. For all $t \geq 0$ (and $t \leq \tau$ if the curve has finite duration τ), $\mathfrak{c}(t)$ is a point in the image of $\mathfrak{c}$. Except for the special case of purely time-like $\mathfrak{c}$ where γ is a point, γ is a continuous path on $\mathfrak{M}$ and t directly corresponds to a displacement $\ell = \dot{\mathfrak{c}}t$ from the start of the path. For the purely time-like case, $\delta\mathfrak{t} = \dot{\mathfrak{c}}t$. Fourier transforms are valid on Brownian paths [41], so the spectra of φ modes along curves will be the starting point.

4.3.1 Transient preferred scale

Consider what it means to break scale invariance in a neighborhood $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$, ignoring for now the process which leads to it. The stochastic regime is scale invariant because φ is a random process along all curves in $\mathfrak{M}_\mathfrak{t}$, and its PDF is scale invariant. The PDF is Eq. (4.3), denoted below by $f_0(\delta\varphi, \delta t)$, where t parameterizes the constant-speed curve of interest.

That is, continuously taking samples while moving at constant speed v along a curve $\mathfrak{c}(t)$ in the stochastic regime, the spectral density over an ensemble will be Eq. (3.29),

$$\mathcal{S}_0(\omega) = \alpha/\omega^2 \tag{4.6}$$

for some constant α; this is illustrated by Fig. 3.1. In terms of spatial displacements along the image of $\mathfrak{c}$, (4.6) corresponds to

$$\mathcal{S}_0(\mathfrak{k}) = \alpha'/\mathfrak{k}^2 \tag{4.7}$$

where $\alpha' = \alpha/v$ since $\omega = v\mathfrak{k}$ and v is assumed constant. To break scale invariance, the spectral density must deviate from $\mathcal{S}_0$ over some range of modes $[\mathfrak{k}_{\text{min}}, \mathfrak{k}_{\text{max}}]$, where $\mathfrak{k}_{\text{max}}^{-1} \simeq \text{diam}(\mathfrak{U})$. These modes are deterministic solutions of an equation of motion, so their increased amplitudes compared to $\mathcal{S}_0$ must be sufficiently large to lead to distinguishable phenomena. The stochastic motions of all modes ($\mathfrak{k} > 0$) in $\mathfrak{U}_\mathfrak{t}$ are perturbations of those solutions.

The development of a proposal for dilatation symmetry (scale invariance) breaking will start with an unsuitable model in order to assess what does not work. Fig. 4.1 is a cartoon of the first attempt; compare with Fig. 3.1.

Two interpretations of Fig. 4.1 will be entertained. The first is that broken scale invariance in $\mathfrak{U}_\mathfrak{t}$ is characterized by a different probability density $f_1(\delta\varphi, \delta t)$ for

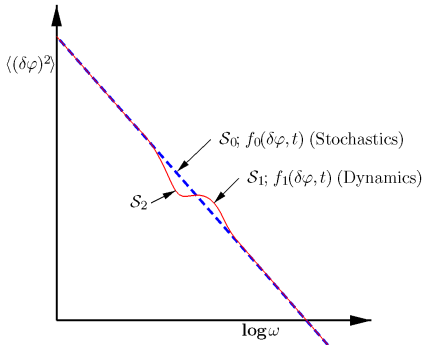


Figure 4.1: Hypothetical breaking of scale invariance. The frequency $\omega \sim 1/\delta\mathfrak{t}$, where $\delta\mathfrak{t}$ is the sampling interval; then $\delta\varphi(\sigma)$ is the departure from $\langle\varphi(\sigma)\rangle \equiv 0$ after $\delta\mathfrak{t}$.

the same interval δt. Then PDF in $\mathfrak{U}_\mathfrak{t}$ after breaking scale invariance would look like

$$f(\delta\varphi,\delta t) \;\to\; \begin{cases} f_0(\delta\varphi,\delta t)\,, & 0 \quad \leq t < t_{\min} \\ f_1(\delta\varphi,\delta t)\,, & t_{\min} \leq t \leq t_{\max} \\ f_0(\delta\varphi,\delta t)\,, & t_{\max} < \mathfrak{t} < \infty\,. \end{cases} \tag{4.8}$$

A problem with Eq. (4.8) is that it treats f_1 as a probability density in the same sense as f_0, even though dynamics alone are clearly responsible for the change $f_0 \to f_1$. That is, f_0 is stationary and time- and space-translation invariant, whereas f_1 cannot be either — dynamics imply non-random field evolution in both space and time.

Another possibility, also suggested by Fig.4.1, is that the spectral density changes due to a shift of 'power' from one range of modes to another, *i.e.*, the change from $\mathcal{S}_0 \to \mathcal{S}_1$ is accompanied by an opposite change $\mathcal{S}_0 \to \mathcal{S}_2$. Such a simple picture is suggested by conservation of energy, but there is no notion of energy, at least until some form of manifold exists. However, there are other difficulties with this idea.

First, it is not clear how a shift of 'power' from one part of the spectrum to another could be maintained over time without additional postulates, given that the form of the coupling (Postulate 2.2.11) should cause two different frequency ranges, $[\omega_{\min},\omega_{\max}]$ for $\mathcal{S}_1$ and $[\omega'_{\min},\omega'_{\max}]$, for $\mathcal{S}_2$, to shift toward each other to reduce their frequency difference. This is because both ranges coexist in the same spatial region $\mathfrak{U}$ at the same time. Since such a shift should not affect the phenomena governed by $\mathcal{S}_1$ because the dynamics have a scale invariant form, such shift is consistent with an equation of motion. As the two ranges shift toward each other, the modes comprising the range with spectral density $\mathcal{S}_1$ will no longer have the needed excess amplitudes — a dynamic regime could not persist in the scenario of Fig. 4.1.

Second, the part labeled $\mathcal{S}_0$ in Fig. 4.1 is isotropic in the stochastic regime,

where isotropy at a point $\mathfrak{x} \in \mathfrak{U}_\mathfrak{t}$ means the statistics of φ motions are the same in all directions given by the tangent space $T_\mathfrak{x}\mathfrak{U}_\mathfrak{t}$; see Sect. 3.4. Although isotropy with respect to $T_\mathfrak{x}\mathfrak{U}_\mathfrak{t}$ is reasonable for $\mathcal{S}_0$, it is less reasonable for $\mathcal{S}_1$. Phenomena are solutions to an equation of motion, but phenomena of physical interest live in a space with some notion of spatial dimension. The equation of motion must exist in some finite number n of spatial dimensions. Thus, since $\mathcal{S}_1$ deviates from $\mathcal{S}_0$ due to the equation of motion, $\mathcal{S}_1$ can at best be isotropic in n spatial dimensions. That means the 'shape' of $\mathcal{S}_0$ and $\mathcal{S}_1$ can only be compared in the n independent spatial directions represented in the equation of motion.

The second problem in particular indicates the property of the stochastic regime that must be modified: isotropy of the spectral density with respect to $T_\mathfrak{x}\mathfrak{U}_\mathfrak{t}$ along all curves that meets the assumptions of Proposition 4.2.2. If there were a range of modes $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ in $\mathfrak{U}_\mathfrak{t}$ for which the spectral density depends on the path the curve takes, that would mean scale invariance is broken in $\mathfrak{U}_\mathfrak{t}$ — invariance of the spectral density along all candidate curves would be absent.

For interesting dynamics to emerge, it is necessary that a set of special curves emerge, so that the spectral density is no longer the same for all curves in $\mathfrak{U}_\mathfrak{t}$. Denote this set of special curves at a point $\mathfrak{x} \in U_t$ by $\mathfrak{G}_\mathfrak{x}$. Since Postulate 2.22 indicates the equation of motion will be some kind of wave equation, the spectral density should be different along curves where waves propagate than along arbitrary curves which are only sometimes tangent to the propagation curves. Thus, $\mathfrak{G}_\mathfrak{x}$ acts like a set of geodesics, not of $\mathfrak{M}$ but of a different space $\mathscr{M}$ that is in the process of emerging.

Because the spectrum along the curves of $\mathfrak{G}_\mathfrak{x}$ includes solutions to an equation of motion, the spectral density along the propagation paths of the waves will be different than $\mathcal{S}_0$. Denote this spectral density by $\mathcal{S}_1$, although it should be noted that $\mathcal{S}_1$ will depend on both the position and the emergent geodesic in the general case.

If in some small neighborhood of $\mathfrak{x}$ the field evolution along all curves in $\mathfrak{G}_\mathfrak{x}$ can be decomposed as a linear combination of field evolutions along a subset of n members of $\mathfrak{G}_\mathfrak{x}$, where each such 'component' separately solves the equation of motion, then those n members act like a basis of $T_\mathfrak{x}\mathscr{M}$ in the emerging space. In this way a product space can arise at $\mathfrak{x}$ where none previously existed.

It is not necessary that only one curve in $\mathfrak{U}_\mathfrak{t}$ maps to one curve in $\mathfrak{G}_\mathfrak{x}$. To retain generality, it should be possible that symmetries exist so that a curve in $\mathfrak{G}_\mathfrak{x}$ corresponds

to an equivalence class of curves in $\mathfrak{U}_\mathfrak{t}$.

The discussion above motivates using the spectral density to define what it means to break scale invariance, or dilation symmetry. Because Postulate 2.2.8 asserts there are only two sources of field change, $\delta\varphi(\sigma;\delta\mathfrak{t}]_{\text{rand}}$ which has spectral density $\mathcal{S}_0$, and $\delta\varphi(\sigma;\delta\mathfrak{t}]_{\text{coupling}}$ which leads to an equation of motion, the deviation of $\mathcal{S}_1$ from $\mathcal{S}_0$ in the definition below is ultimately due to the action of the equation of motion.

Def*. 4.3.1 (Dilatation symmetry breaking.) *Let $\mathcal{S}_0$ be the spectral density* (4.7) *of the stochastic regime, $0 < \mathfrak{k} < \infty$, and let all spectral densities $\mathcal{S}_1(\mathfrak{c})$ in the following be computed over an ensemble of similarly prepared field configurations in $\mathfrak{U}_\mathfrak{t}$. For all $x \in \mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$, if there exists a set $\mathfrak{G}_\mathfrak{x}$ of curves containing $\mathfrak{x}$, such that the deviation $\delta_\mathfrak{k}$ in spectral densities $\mathcal{S}_1(\mathfrak{c})$ (for all $\mathfrak{c} \in \mathfrak{G}_\mathfrak{x}$) and $\mathcal{S}_0$ for all $\mathfrak{k} \in [\mathfrak{k}_{\text{min}}, \mathfrak{k}_{\text{max}}]$ exceeds a threshold ϵ, then*

- *Dilatation symmetry is broken in $\mathfrak{U}_\mathfrak{t}$ at a significance level ϵ, where $\mathfrak{k}_{\text{min}}^{-1} \simeq \text{diam}(\mathfrak{U}_\mathfrak{t})$.*
- *The range of modes $[\mathfrak{k}_{\text{min}}, \mathfrak{k}_{\text{max}}]$ gives the preferred scale (Rmk. 2.2.14).*

Fig. 4.2 illustrates the general idea. No attempt will be made at a specific quantification of ϵ in this thesis.

If $\varphi[\mathfrak{U}_\mathfrak{t}]$ meets additional requirements, then $\mathfrak{U}_\mathfrak{t}$ also contains a pre-emergent space in this picture:

Def*. 4.3.2 (Pre-emergent space, $\mathscr{M}$.) *Let $\varphi[\mathfrak{U}_\mathfrak{t}]$ satisfy Def*. 4.3.1. If $S_\mathfrak{t}$ is a simply connected subregion of $\mathfrak{U}_\mathfrak{t}$ such that, for each $\mathfrak{x} \in S_\mathfrak{t}$, there is a special set of curves $\mathfrak{G}_\mathfrak{x}$ along which the field evolution can be decomposed as a linear combination of field evolutions along n members of $\mathfrak{G}_\mathfrak{x}$ (where each such 'component' separately solves the equation of motion), then*

- *$S_\mathfrak{t}$ contains a pre-emergent space, $\mathscr{M}$.*

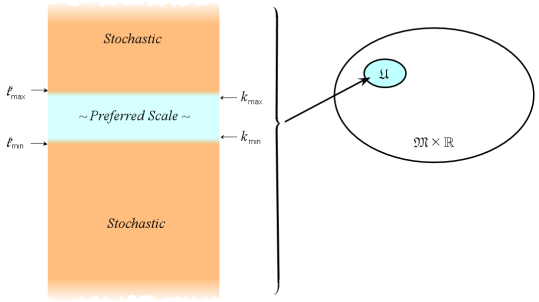


Figure 4.2: A preferred scale is bounded by $[\mathfrak{k}_{\text{min}}, \mathfrak{k}_{\text{max}}]$. Modes $\{\mathfrak{k}\}$ are measured by $\mathfrak{d}$.

- *The n members of $\mathfrak{G}_\mathfrak{x}$ are a basis of $T_\mathfrak{x}\mathscr{M}$.*
- *$\mathscr{M}$ has spatial dimension n.*

A region $\mathfrak{U}_\mathfrak{t}$ that meets the requirements of Def*. 4.3.1 may contain multiple disjoint subregions that meet the requirements of Def*. 4.3.2. There is no requirement that all such subregions have the same dimension n. It is also possible that the spatial dimension of a subregion of $\mathfrak{U}_\mathfrak{t}$ may change during evolution toward an emergent spacetime, if such an evolution occurs.

4.3.2 Fluctuations leading to a preferred scale

A good probabilistic model that describes how fluctuations of the stochastic regime can lead to a transient preferred scale with n emergent directions as developed in Subsect. 4.3.1 is presently absent. A transient preferred scale is a necessary precursor of cosmogenesis, needed to generate a persistent preferred scale. Simultaneous fluctuations according to the PDF (4.8) over a finite $\mathfrak{d}$-measured neighborhood is an implausible way to get an n-dimensional preferred scale due to the very large transitions $\delta\varphi$ needed for dynamics to manifest and the exponential suppression of such dynamics. Moreover, the fluctuations would need to respect the continuity of φ along all curves.

A more likely scenario starts from a very low-amplitude preferred scale, not necessarily having the same number of emergent directions as the 'final' preferred scale, and then grows into its final form by a combination of dynamics and many additional, highly unlikely fluctuations of the Brownian type. That is, it presumably appears as the outcome of a process rather than a single event. It should be possible to enforce continuity at all stages, since continuity is necessary for both the dynamic and stochastic regimes.

A good model which describes the most probable process is not required to proceed. For present purposes it is sufficient to note that it must occur in order to get dynamical phenomena from $\varphi[\mathfrak{M}]$, whatever the reason.

4.4 Physical Observability of φ Motions

Although it should be possible to develop an elementary theory on the base space $\mathfrak{M}_\mathfrak{t} \equiv \mathfrak{M} \times \mathbb{R}$, at the end there obviously needs to be a way to relate it to ob-

servations by physical observers in a physical spacetime $(\mathscr{M}, \mathbf{g})$ if the theory is to be a physical one. In the strong emergence picture in this thesis, connecting fields on the nonphysical $\mathfrak{M}_\mathfrak{t}$ to fields on $(\mathscr{M}, \mathbf{g})$ is neither immediate nor trivial. The main problem is that a physical spacetime is not implied by φ on $\mathfrak{M}_\mathfrak{t}$; that and the intrinsic stochasticity of φ suggest it is impossible in principle for any observer to obtain an unambiguous map from φ on $\mathfrak{M}_\mathfrak{t}$ to φ on $(\mathscr{M}, \mathbf{g})$, or vice versa. The best we can do is demonstrate that φ on $(\mathscr{M}, \mathbf{g})$ is compatible with φ on $\mathfrak{M}_\mathfrak{t}$, then find relationships on $\mathfrak{M}_\mathfrak{t}$ that should also hold on $(\mathscr{M}, \mathbf{g})$ to obtain concrete connections between the two realms. That is the task of this section.

From the form of the local dynamics in Postulate 2.2.11(D2), or, alternatively, anticipating the field equation that will be developed in Sect. 4.5, it is reasonable to expect the field equation that governs dynamics of φ will be a wave equation. Thus, it will be assumed throughout this section that a preferred scale $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$ exists in a region $\mathfrak{U}_\mathfrak{t}$ of $\mathfrak{M}_\mathfrak{t}$, the dynamic regime completely dominates the stochastic regime in $\mathfrak{U}_\mathfrak{t}$ so that perturbative stochastic influences can be ignored, and φ inhomogeneities propagate as waves.

4.4.1 Wave propagation speed

Rmk. 2.2.10 argued that an increase in the total variation of $\phi(\mathfrak{x}, \mathfrak{x}')$ corresponds to an increase in number or magnitude of sign reversals of ϕ along the geodesic $\mathfrak{c}$ whose image is $\gamma(\mathfrak{x}, \mathfrak{x}')$. Thus, propagation of changes in ϕ along $\mathfrak{c}$ involves corresponding reversals in direction of propagation, thereby reducing the propagation speed and the effective coupling strength. The reduction in coupling strength as $\mathfrak{V}_\phi(\cdot)$ increases is analogous to reducing a spring constant by increasing the total length of the wire that comprises the spring.

Postulate 2.2.11 codified this idea for local dynamics in parts (D3) and (D4). Since the field equation should inherit at least most of its properties from the local dynamics, these two postulates will provide the foundation for the propagation speed over longer distances.

The model curve for what follows has duration T (*i.e.*, $0 \leq t \leq T$) and image $\gamma(\mathfrak{x}, \mathfrak{x}')$ whose length $\mathfrak{L}(\gamma)$ is finite:

$$\mathfrak{c} : [0, T] \to \mathfrak{M}\,; \quad T \text{ and } \mathfrak{L}(\mathfrak{c}) \text{ are finite}\,.$$

It will be assumed propagation occurs along $\mathfrak{c}$, and the total propagation time T from $\mathfrak{x} \to \mathfrak{x}'$ will be allowed to vary from one trial to the next. The instantaneous position of the wavefront is $\mathfrak{c}(t)$, and the instantaneous propagation speed is $v = d\mathfrak{c}(t)/dt = \dot{\mathfrak{c}}$.

Postulate 2.2.11(D4) and Eq. (2.23) relate the instantaneous propagation speed of local changes in $\phi(\mathfrak{x}, \mathfrak{x}'; \mathfrak{t})$ along $\mathfrak{c}$ to the instantaneous coupling strength g. Since $v \propto g$, and noting $\mathfrak{V}_\varphi(\mathfrak{c}) = \mathfrak{V}_\phi(\mathfrak{c})$,

$$v \;=\; v_0\, \mathfrak{V}_\varphi(\mathfrak{c})^{-1}\,. \tag{4.9}$$

Because the total variation $\mathfrak{V}_\varphi(\mathfrak{c})$ is an additive function and v_0 is a constant, Eq. (4.35) also holds for arbitrary T and $\mathfrak{L}(\mathfrak{c})$ assumed for the model curve.

Postulates 2.2.11(D3) and (D4) also imply $v \to \infty$ as $\mathfrak{d}(\mathfrak{x}, \mathfrak{x}') \to 0$. This implication can be checked for self consistency as follows. Assume as in Rmk. 2.2.10 that a wavefront can be scattered by φ inhomogeneities; this is the proposed explanation for why increasing the total variation slows the net propagation speed. If we let the propagation speed of a wavefront correspond to the finite diffusion speed of a particle undergoing Brownian motion, no generality will be lost because diffusion is generally slower than propagation. However, the instantaneous speed is nonetheless infinite:

Theorem 4.4.1 *The instantaneous speed of a particle undergoing Brownian motion is infinite.*

Proof: Eq. (3.7) for a random walk in the continuum limit says that

$$\frac{\partial P}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2} + v\,\frac{\partial P(x,t)}{\partial x}\,; \qquad D = \lim_{\substack{\varepsilon\to 0,\\ \tau\to 0}} \frac{\varepsilon^2}{\tau}\,, \quad v = \lim_{\substack{\varepsilon\to 0,\\ \tau\to 0}} \frac{\varepsilon}{\tau}(p_+ - p_-)\,, \tag{4.10}$$

where D is the diffusion coefficient and $v = \frac{D}{\varepsilon}(p_+ - p_-)$ is the particle drift velocity. Since D is finite, clearly the instantaneous particle velocity $\varepsilon/\tau \to \infty$ as $\tau \to 0$. The Fokker-Planck equation (3.33) for Brownian motion with friction is more general than Eq. (3.7), but the same conclusion holds. ∎

Physical particles cannot have infinite instantaneous velocities, so the theorem is obviously inapplicable to real particles[2], but that is unimportant to the present argument. The theorem adheres to assumptions already postulated for φ. Because φ in

[2]Thm. 4.4.1 is a consequence of infinitesimal step size. An infinitesimal step size involving physical particles would require that collisions occur in zero time, that intermolecular distances are infinitesimal everywhere, and that position and momentum can be simultaneously determined to arbitrary accuracy in violation of the uncertainty principle.

the stochastic regime is a Brownian process both temporally and spatially, it inherits the properties of Brownian motion including infinite variation and scale invariance. An infinite instantaneous propagation speed is therefore acceptable in the limit $\mathfrak{L}(\mathfrak{c}) \to 0$.

Returning to Eq. (4.35), the absolute propagation speed v along the model curve is ill defined because $\mathfrak{V}_\varphi(\mathfrak{c})$ is infinite and v_0 is unspecified (v_0 might be finite or infinite). The absolute speed is also nonphysical because it is defined with respect to the distance metric $\mathfrak{d}(\mathfrak{x}, \mathfrak{x}')$ and cosmic time, both of which are nonphysical.

Consider a ratio of nonphysical speeds:

$$v = \frac{v_0}{\mathfrak{n}(\varphi[\mathfrak{c}(t)])}; \quad \mathfrak{n}(\varphi[\mathfrak{c}]) \equiv \frac{\mathfrak{V}_\varphi(\mathfrak{c})}{\mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}})}, \tag{4.11}$$

where $\mathfrak{c}_{\text{ref}}$ is some reference curve which has the same duration T as the model curve $\mathfrak{c}$. It is not obvious whether this is workable because $\mathfrak{n}$ is a ratio of two infinite quantities. The usefulness of (4.11) depends on finding constraints on $\varphi[\mathfrak{c}]$ and $\varphi[\mathfrak{c}_{\text{ref}}]$ that make $\mathfrak{n}$ well defined and finite.

First consider a regime where φ is a Brownian noise process, a reasonable approximation when motions of φ are very small. This is the stochastic regime, implying $\mathfrak{c}$ and $\mathfrak{c}_{\text{ref}}$ lie in a region of $\mathfrak{M}_\mathfrak{t}$ where φ is scale invariant. As Brownian processes, $\varphi[\mathfrak{c}]$ and $\varphi[\mathfrak{c}_{\text{ref}}]$ can diverge arbitrarily — the finiteness of (4.11) cannot be ascertained.

Try instead taking the mean $\langle \mathfrak{n}(\varphi[\mathfrak{c}]) \rangle$ as the relevant ratio. Again working in the stochastic regime,

$$\begin{aligned}
\langle \mathfrak{n} \rangle &= \left\langle \frac{\mathfrak{V}_\varphi(\mathfrak{c})}{\mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}})} \right\rangle \\
&= \left\langle \left(\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle + \delta\mathfrak{V}_\varphi(\mathfrak{c}) \right) \left(\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle + \delta\mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \right)^{-1} \right\rangle \\
&\simeq \left\langle \frac{\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle}{\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle} \left(1 + \frac{\delta\mathfrak{V}_\varphi(\mathfrak{c})}{\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle} \right) \left(1 - \frac{\delta\mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}})}{\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle} \right) \right\rangle \\
&= \left[\frac{\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle}{\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle} \left(1 - \frac{\cancelto{0}{\left\langle \delta\mathfrak{V}_\varphi(\mathfrak{c})\, \delta\mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \right\rangle}}{\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle \langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle} \right) \right] \\
&= \left(\frac{\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle}{\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\text{ref}}) \rangle} \right),
\end{aligned}$$

where the correlation vanishes in the second to last step because fluctuations at space-like separated points are independent for Brownian processes. Directly using the definition 2.2.9 of $\mathfrak{V}_\varphi(\mathfrak{c})$, compute the mean value

$$\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle = \left\langle \sup \sum_{i=1}^{n} \left| \varphi(\mathfrak{c}(t_i^n)) - \varphi(\mathfrak{c}(t_{i-1}^n)) \right| \right\rangle = \sup \sum_{i=1}^{n} \left| \left\langle \varphi(\mathfrak{c}(t_i^n)) \right\rangle - \left\langle \varphi(\mathfrak{c}(t_{i-1}^n)) \right\rangle \right|.$$

By assumption, $\varphi(\mathfrak{c}(t_i^n))$ and $\varphi(\mathfrak{c}(t_{i-1}^n))$ are Brownian noise processes. The ensemble average $\langle \varphi(\mathfrak{c}(t_i^n)) \rangle$ should be the same as the time average due to the stationarity of the scale invariant vacuum: the ensemble can be generated by shifting the entire curve by different values δt. This and the martingale property of Brownian motion (see Thm. 3.1.10 and Def. 3.1.9) implies $\langle \varphi(\mathfrak{c}(t_i^n)) \rangle = \varphi(\mathfrak{c}([t+\delta t]_i^n))$, and similarly for $\varphi(\mathfrak{c}(t_{i-1}^n))$. Thus, $\langle \mathfrak{V}_\varphi(\mathfrak{c}) \rangle \big|_{t'} = \mathfrak{V}_\varphi(\mathfrak{c}) \big|_t$. The same reasoning holds for $\langle \mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref}) \rangle$. No advantage is accrued by considering the average $\langle \mathfrak{n} \rangle$ compared to considering a single sample $\mathfrak{n}$.

Hence $\mathfrak{n}$ is undefined when either curve lies in a region of $\mathfrak{M}_\mathfrak{t}$ where φ is scale invariant. While this conclusion is useful, it does not imply $\mathfrak{n}$, as it is defined in Eq. (4.11), is finite when scale invariance is broken.

To examine that case, consider a 'boundary case' where $\mathfrak{c}$ and $\mathfrak{c}_{\rm ref}$ are the same curve, lying within a region of $\mathfrak{M}_\mathfrak{t}$ where a mode decomposition of φ deviates from the scale invariant spectrum. Let $\varphi[\mathfrak{c}]_{\rm broken}$ be the deviation of φ from scale invariance along $\mathfrak{c}$, *i.e.*,

$$\varphi[\mathfrak{c}_{\rm ref}] = \varphi'[\mathfrak{c}_{\rm ref}] + \varphi_{\rm broken}[\mathfrak{c}_{\rm ref}], \tag{4.12}$$

where $\varphi'[\mathfrak{c}_{\rm ref}]$ is scale invariant. In general $\mathfrak{V}_\varphi(\mathfrak{c})_{\rm broken}$ is finite while $\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})$ is not, so $\mathfrak{V}_\varphi(\mathfrak{c})_{\rm broken}/\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})$ will be nonvanishing only on a set of field configurations of measure zero, *i.e.*, those configurations of $\varphi[\mathfrak{c}_{\rm ref}]$ which have finite variation because they contain only finitely many jumps. Then

$$\mathfrak{n}(\varphi[\mathfrak{c}_{\rm ref}]) \;=\; \left(\frac{\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})}{\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})}\right) = 1 = \left(\frac{\mathfrak{V}_{\varphi'}(\mathfrak{c}_{\rm ref})}{\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})} + \cancelto{0}{\frac{\mathfrak{V}_\varphi(\mathfrak{c})_{\rm broken}}{\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})}}\right).$$

This means that $\mathfrak{n}$ in Eq. (4.11) is completely determined by the scale invariant contribution, and the conclusion that Eq. (4.11) is not well defined, reached when $\mathfrak{c}$ or $\mathfrak{c}_{\rm ref}$ are in a scale invariant region, generalizes to everywhere on $\mathfrak{M}_\mathfrak{t}$.

Apparently there are no constraints on $\varphi[\mathfrak{c}]$ and $\varphi[\mathfrak{c}_{\rm ref}]$ that can make (4.11) unambiguous and thus finite. The problem must lie in the proposed definition (4.11) for $\mathfrak{n}$. Nevertheless, Eq. (4.12) suggests a modification of (4.11) that can make $\mathfrak{n}$ finite. The idea is to work with the finite variations $\mathfrak{V}_\varphi(\mathfrak{c})_{\rm broken}$ and $\mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})_{\rm broken}$, and then determine what constraint(s) are necessary to ensure their ratio remains finite even if $\mathfrak{V}_\varphi(\mathfrak{c}) - \mathfrak{V}_\varphi(\mathfrak{c}_{\rm ref})$ diverges.

A finite variation $\mathfrak{V}_\varphi(\mathfrak{c})_{\text{broken}}$ implies a finite **effective cutoff** scale $\mathfrak{l}_{\min}$ (and vice versa), the smallest distance scale where there is a deviation from scale invariance. That is, given a mode decomposition of $\varphi[\mathfrak{c}_{\text{ref}}]$ (see Subsect. 3.1.2), modes $\mathfrak{k} > \mathfrak{l}_{\min}^{-1}$ can be ignored without loss of consequential information because the emergent physics will be a consequence of deviations from scale invariance. Scale invariance means there is no intrinsically preferred scale, so the cutoff scale $\mathfrak{l}_{\min}$ can occur at any $\mathfrak{d}$-measured scale where a chance fluctuation breaks scale invariance sufficiently. (In this thesis a 'physically interesting situation' will usually mean a cosmology.) Sect. 4.3 considers the physical origin of $\mathfrak{l}_{\min}$ and shows natural cutoffs will inevitably accompany the initial conditions of a cosmology; its introduction here is not a new assumption.

The finite variation $\mathfrak{V}_\varphi(\mathfrak{c})_{\text{broken}}$ can be defined more generally as a residual variation:

Def*. 4.4.2 (Residual variation.) *Given a cutoff mode $\mathfrak{k}_{\max} = \mathfrak{l}_{\min}^{-1}$ and a curve $\mathfrak{c}(t) \subset \mathfrak{M}_\mathfrak{t}$, $t_0 \leq t \leq t_0+\tau$ for some parameter t and interval length τ: let $\mathfrak{V}_{\bar{\varphi}}\big|_{\mathfrak{k}<\mathfrak{k}_{\max}}$ be the variation of $\varphi(\mathfrak{c}(t))$ when φ has been smoothed over intervals of length $\mathfrak{k}_{\max}^{-1}$. Then the **residual variation** of φ along $\mathfrak{c}$ is*

$$V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c}) = \mathfrak{V}_\varphi(\mathfrak{c}) - \mathfrak{V}_{\bar{\varphi}}(\mathfrak{c})\big|_{\mathfrak{k}>\mathfrak{k}_{\max}}; \tag{4.13}$$

$\mathfrak{V}_\varphi(\mathfrak{c})$ is defined by Def. 2.2.9.

A straightforward way to compute $\bar{\varphi}(\mathfrak{c}(t))$ is the 'smoothing' procedure summarized in Eq. (3.42), where $\varepsilon = \mathfrak{k}_{\max}^{-1}$ is the smoothing interval. This procedure is also used to compute $\bar{\varphi}(\mathfrak{c}(\mathfrak{t}), \mathfrak{t})$ in Eq. (2.22), although the choice of ε may be different there. The smoothing forces a sharp cutoff at $\mathfrak{k}_{\max}$ so that $V_\varphi(\mathfrak{c})$ ignores modes greater than $\mathfrak{k}_{\max}$. The procedure requires extending $\mathfrak{c}$ at each end by a distance $\varepsilon/2$ to allow computing $\bar{\varphi}(\mathfrak{c}(\mathfrak{t}), \mathfrak{t})$ at the ends, but otherwise it is as expected. (The curve can be extended by geodesics of $\mathfrak{d}$ which are tangent to $\mathfrak{c}$; Def. 3.4.1 defines tangent paths.)

Def*. 4.4.2 is the basis for defining $\mathfrak{n}$ as a relative variation, superseding Eq. (4.11). The relative variation will be useful because it is a scalar quantity that characterizes different φ configurations, one that is independent of the nonphysical attributes of curves on $\mathfrak{M}_\mathfrak{t}$ when $\mathfrak{c}$ and $\mathfrak{c}_{\text{ref}}$ have the same length and duration.

Def*. 4.4.3 (Relative variation.) *Let $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$ be a region throughout which φ deviates from scale invariance over a continuous range of modes $[\mathfrak{k}_{\min}, \mathfrak{k}_{\max}]$. Given a*

curve $\mathfrak{c} \subset \mathfrak{U}_\mathfrak{t}$ *and a reference curve* $\mathfrak{c}_{\text{ref}} \subset \mathfrak{U}_\mathfrak{t}$ *of equal length and temporal duration, the* ***relative variation*** *with respect to* $\mathfrak{c}_{\text{ref}}$ *is*

$$\mathfrak{n}(\varphi[\mathfrak{c}]) \equiv \left(\frac{V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c})}{V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c}_{\text{ref}})} \right). \tag{4.14}$$

Because $\mathfrak{k}_{\max}$ is finite, the infinite variation along each curve is due to modes greater than $\mathfrak{k}_{\max}$; hence both the numerator and denominator of (4.14) are nonzero and finite, implying $\mathfrak{n}$ is finite. While finiteness of (4.14) is necessary for its usefulness, it is not sufficient. Persistence of a deviation of $\varphi[\mathfrak{U}_\mathfrak{t}]$ from scale invariance is the most important auxiliary condition. If scale-specific configurations do not persist, then neither will physical phenomena that originate from those deviations from scale invariance; physical laws would not manifest the invariance that makes them useful.

As a ratio, Eq. (4.14) gives $\mathfrak{n}$ a scale invariant definition, as required. That is, $\mathfrak{k}_{\max}$ will be determined by a scale invariant random process so quantities defined relative to it have a scale-free meaning. Although the variations $V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c})$ and $V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c}_{\text{ref}})$ are defined with respect to the nonphysical distance metric $\mathfrak{d}$ and cosmic time $\mathfrak{t}$, the ratio (4.14) hides all references to these nonphysical objects.

Write the propagation speed along a curve $\mathfrak{c}$ of duration T, in a scale-free form that is useful on either $\mathfrak{M}_\mathfrak{t}$ or a physical spacetime:

- Define a reference curve $\mathfrak{c}_{\text{ref}}$ which also has duration T;
- Take $v_0 \equiv \text{const}$ to be the invariant propagation speed along the chosen $\mathfrak{c}_{\text{ref}}$ (can choose a 'natural' set of units in which $v_0 \equiv 1$ if desired);
- Compute $\mathfrak{n}(\varphi[\mathfrak{c}])$ according to Def*. 4.4.3; and
- Relate the (non-constant) propagation speed along $\mathfrak{c}$ to v_0 by

$$v = \frac{v_0}{\mathfrak{n}(\varphi[\mathfrak{c}])}\,. \tag{4.15}$$

The choice of the symbol $\mathfrak{n}$ in (4.14) is intended to suggest an analogy with the refractive index n in geometric optics. Eq. (4.15) is not uniquely implied by Postulate 2.2.11, so it must be postulated; see Postulate 4.4.5(S1).

4.4.2 Physical lengths and times

Let $\mathfrak{c}$ be a geodesic of duration τ along which a wave propagates, and let $\mathfrak{L}(\mathfrak{c}) = \ell$. Denote the reference curve, implicitly required to determine the relative variation $\mathfrak{n}$ in Eq. (4.15), by $\mathfrak{c}_{\text{ref}}$. Using (4.15) for the propagation speed along $\mathfrak{c}_{\text{ref}}$,

$$v = \frac{v_0}{\mathfrak{n}(\varphi[\mathfrak{c}_{\text{ref}}])} = v_0 = \frac{\ell_0}{\tau}\,,$$

which is clearly constant because v_0 is defined to be a universal constant. Along $\mathfrak{c}$, $v = v_0/\mathfrak{n}(\varphi[\mathfrak{c}])$ is not constant in general. However, the propagation speed can be taken to be v_0 (and hence constant) by rewriting v:

$$v_0 = \mathfrak{n}v = \frac{\ell\mathfrak{n}}{\tau} = \frac{\ell\mathfrak{n}^{1-p}}{\tau\mathfrak{n}^{-p}} = \frac{\ell'}{\tau'}\,.$$

Taking $p = \frac{1}{2}$,

$$v_0 = \frac{\ell\mathfrak{n}^{1/2}}{\tau\mathfrak{n}^{-1/2}} \quad \Longrightarrow \quad \ell' = \ell\,\mathfrak{n}^{1/2}\,, \quad \tau' = \tau\,\mathfrak{n}^{-1/2}\,. \tag{4.16}$$

This is just a rewrite of Eq. (4.15) in terms of ℓ and τ, but it shows how the propagation speed can be taken to be a universal constant with lengths and times being what change with $\mathfrak{n}$. The choice $p = \frac{1}{2}$ is completely unmotivated for now, but the rationale for the choice will become clear in Chap. 6.

Taking the transformations (4.16) of lengths and times to be correct, Def*. 4.4.3 can be used to define a scale-free relationship between physical and nonphysical distances along an arbitrary curve $\mathfrak{c}(t)$:

Def*. 4.4.4 (Variational length $L_\varepsilon(\mathfrak{c})$.) *The variational length $L_\varepsilon(\mathfrak{c})$, measured with respect to a specified cutoff scale τ, is*

$$L_\tau(\mathfrak{c}) = \sup \lim_{n\to\infty} \sum_{i=1}^{n} \Big|\mathfrak{c}(s_i) - \mathfrak{c}(s_{i-1})\Big|\,, \tag{4.17}$$

where

$$s_i = \begin{cases} t_i\,\mathfrak{n}^{1/2}(\varphi[\mathfrak{c}(t_i)]) & \text{for purely space-like } \mathfrak{c} \\ t_i\,\mathfrak{n}^{-1/2}(\varphi[\mathfrak{c}(t_i)]) & \text{otherwise.} \end{cases} \tag{4.18}$$

Here $\mathfrak{n}(\varphi[\mathfrak{c}])$ is the relative variation defined by Def. 4.4.3, and $\mathfrak{n}(\varphi[\mathfrak{c}(t_i)])$ means that $\mathfrak{n}(\cdot)$ is computed over the interval $[\frac{1}{2}(t_i + t_{i-1}),\ \frac{1}{2}(t_i + t_{i+1})]$ for $1 \le i < n$; for $i = 0$ and $i = n$, respectively, $\mathfrak{c}$ is extended by geodesics (tangent to the curve) of length $\frac{1}{2}(t_{i+1} - t_i)$ and $\frac{1}{2}(t_i - t_{i-1})$, and the values computed with the fictitious t_{-1} or t_{n+1}.*

4.4.3 Concretely connecting $\varphi[\mathfrak{M}_\mathfrak{t}]$ to physical spacetime

The complete absence of reference to the background structure $\mathfrak{M}_\mathfrak{t}$ in the relative variation in Eqs. (4.14) and (4.15) means it is appropriate to use $\mathfrak{n}$ in expressions for quantities whose effects are to be observed within a physical spacetime. The relative variation only refers to field variations that can be inferred, in principle, within that spacetime. The inference requires a model of interactions between φ and quantum fields whereby the there is a theoretical link between observable distributions of energy and the variations of φ, but that is not a problem of principle. Chap. 5 will take steps toward such a link.

In order to avoid referring to $\mathfrak{M}_\mathfrak{t}$, the relative variation assumes a reference curve, $\mathfrak{c}_{\rm ref}$. Field motions along $\mathfrak{c}_{\rm ref}$ act as a standard against which φ motions along all other curves can be quantitatively compared. Thus far, $\mathfrak{c}_{\rm ref}$ has only be used for quantifying propagation speed, but physical lengths and times can also be defined with respect to it. The reference curve will be the basis for quantifying the geometry of physical spacetime.

Given the central importance of $\mathfrak{c}_{\rm ref}$, an unambiguous prescription for obtaining it is needed in order to concretely connect φ motions on $\mathfrak{M}_\mathfrak{t}$ to φ motions on a spacetime $(\mathscr{M}, \mathbf{g})$. It will require a new postulate.

Postulate 4.4.5 (Relationship between $\mathfrak{M}_\mathfrak{t}$ and $(\mathscr{M}, \mathbf{g})$.) *Let $[\mathfrak{t}_{\rm min}, \mathfrak{t}_{\rm max}]$ be a preferred scale in a region $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$, and $\mathcal{O} \in \mathfrak{U}_\mathfrak{t}$ be a privileged but arbitrary reference point that acts as the origin of an observational reference frame. Let $\mathfrak{c}_{\rm ref}(t)$ be an abstract curve of image length L, duration T, and endpoint $\mathcal{O}$, which becomes concrete once L, T, and the base space for $\mathfrak{c}_{\rm ref}$ are all specified.*

S1. *The instantaneous propagation speed (and thus average propagation speed) of φ waves is* defined *to be a universal constant v_0 at all points along $\mathfrak{c}_{\rm ref}(t)$, for all t $(0 \leq t \leq T)$. The propagation speed along other curves is related to v_0 by Eq.* (4.15)*: $v = v_0/\mathfrak{n}(\varphi[\mathfrak{c}])$.*

S2. *If the base space is $\mathfrak{M}_\mathfrak{t}$, $\mathfrak{c}_{\rm ref}(t)$ is a geodesic of the distance metric $\mathfrak{d}$ (hence $\mathcal{O} \in \mathfrak{U}_\mathfrak{t}$); L is measured by $\mathfrak{d}$, T is measured by the cosmic time, the parameter t is the cosmic time, and $v_0 = \dot{\mathfrak{c}}(t)$.*

S3. *If the base space is an emergent spacetime $(\mathscr{M}, \mathbf{g})$, $\mathfrak{c}_{\rm ref}(t)$ is a geodesic of $\mathbf{g}$ (hence*

$\mathcal{O} \in \mathscr{M}$); L and T are measured by $\mathbf{g}$, the parameter t is the proper time in the reference frame where $\mathcal{O}$ is at rest, and $v_0 = c$ is the speed of light.

S4. *The reference curve is associated with the point $\mathcal{O}$, and is fixed on $\mathfrak{M}_\mathfrak{t}$ or $\mathscr{M}$ only if $\mathcal{O}$ is fixed on either.*

S5. *Both the quantum fields and the φ field move in the same (emergent) spacetime and transform according to the same spacetime symmetries.*

S6. *A geodesic of $\mathfrak{M}_\mathfrak{t}$ may not be a curve on $\mathscr{M}$, or vice versa; an unambiguous map between $\mathfrak{M}_\mathfrak{t}$ and $(\mathscr{M}, \mathbf{g})$ may not exist.*

By convention, the fixed reference point $\sigma_0 \in \mathfrak{M}$, with respect to which φ is defined everywhere on $\mathfrak{M}$ by Def*. 2.2.7, will be taken to be the point $\mathcal{O}$ which is in the image of $\mathfrak{c}_{\text{ref}}$.

Part (S5) is needed because the equations of motion and symmetries of the quantum fields are defined outside the emergence picture. It ensures that once φ has mediated an emergent spacetime, the quantum fields will also be subject to that spacetime. Moreover, the quantum fields will not physically manifest where the spacetime does not exist.

Part (S6) implies the full preferred-scale vacuum structure of an emerged spacetime is inaccessible to an observer. The variation of φ must be computed in terms of the emergent metric, so only comparisons between the vacuum structure along $\mathfrak{c}_{\text{ref}}(t)$ and other curves are possible. This restriction is compatible with computing the relative variation (4.14).

Generally each observer will be associated with a privileged point $\mathcal{O}$. There is no requirement that $\mathcal{O}$ be the same point for every observer; consistency only requires that all computations a given observer makes are performed with respect to the same $\mathcal{O}$. Hence, an observer A who computes quantities with respect to $\mathcal{O}_A$ may obtain different values than an observer B who computes quantities with respect to $\mathcal{O}_B$. However, because observer A computes all values with respect to the same privileged point $\mathcal{O}_A$ and similarly for B and $\mathcal{O}_B$, A can immediately obtain the values computed by B using a transformation that maps computed values at $\mathcal{O}_B$ to values at $\mathcal{O}_A$ (and vice versa).

4.4.4 Physical basis for the smoothing intervals ε, τ

An equation of motion (a restricted version of the general field equation; see Footnote 3) can hold in the dynamic regime and the transition between the stochastic and dynamic regimes, but cannot apply to the stochastic regime where position and time have no physical meaning. Hence the full field equation is useful only when space and time are well defined; by extension spacetime position and smoothing intervals ε and τ for approximate derivatives can also be defined. The equation of motion should thus be understood to apply usefully when stochasticity is perturbative, *i.e.*, in the dynamic regime or strong coupling limit, $\delta\varphi(\mathfrak{x}; I_\tau]_{\text{rand}} \simeq 0$ in Eq. (2.21). This limit will be assumed implicitly.

Approximate derivatives of the nowhere-differentiable φ correspond to ordinary partial derivatives that appear in a PDE for a classical smooth field, in this sense: in both cases derivatives track changes in the field at the highest resolution that is physically meaningful. Smooth fields are differentiable everywhere, so the smallest physically interesting distance is infinitesimal. On the other hand, descriptions of deterministic phenomena of φ in the dynamic regime would not benefit by incorporating scales smaller than the minimum physically resolvable distance and time scales — such small scales would probe the stochastic regime where fluctuations are essentially uncorrelated with the phenomena of interest. Hence, the minimum physically resolvable distance $\tilde{\varepsilon}$ and minimum physically resolvable time $\tilde{\tau}$ should play the same role for $\varphi[\mathfrak{M}_\mathfrak{t}]$ as their infinitesimal counterparts dx and dt do for smooth fields on a smooth manifold $\mathscr{M}$.

Hence, when computing approximate derivatives for the field equation, the formally defined averaging intervals ε and τ respectively used to compute $\bar{\partial}_x$ and $\bar{\partial}_t$ on $\mathfrak{M}_\mathfrak{t}$ should be identified with the minimum physically resolvable intervals $\tilde{\varepsilon}$ and $\tilde{\tau}$. Assuming this identification, the accents above ε and τ will be omitted henceforth. Provided that ε and τ are much smaller than the minimum distances and times that experiments can probe, approximate derivatives of a nowhere-differentiable φ can be viewed as partial derivatives of a smooth φ. The only limitations of this viewpoint are that position and time uncertainties inevitably exist in the solutions due to the underlying stochasticity.

The foregoing makes it clear the finite intervals τ and ε explicitly and implicitly appearing in Eq. (2.21) have physical meaning and thus should not be freely chosen.

Without a physical motivation for a particular choice for τ, there is no reason to expect an arbitrary τ will yield a physically correct field equation. Clearly τ is related to how quickly the oscillator at $\mathfrak{x}$ dynamically responds to a change in oscillator neighborhood of $\mathfrak{x}$. It is desirable that the field equation predicts a slower response due to rapid random motions, but if τ in the field equation is chosen too large, the field equation will 'filter out' very rapid but non-random changes within the oscillator neighborhood, and thus incorrectly model the actual dynamics.

Thus, τ is the minimum physically resolvable interval of time in another sense. Taking τ to be larger than that minimum value can, in principle, lead to incorrect predictions when probing the smallest distance scales of the preferred scale. On the other hand, taking τ to be too much smaller than the minimum physically resolvable interval could produce a less clear picture of dynamics at the smallest scales because stochastic contributions are over-represented.

We will now propose physically motivated values of τ and ε. Computations will be in terms of $\mathfrak{d}$-measured distances and cosmic time intervals, but using the results of Subsects. 4.4.1 and 4.4.2 they can be reexpressed as quantities in a physical spacetime.

Given a preferred scale, $[\omega_{\min}, \omega_{\max}]$, an obvious absolute lower bound for τ is $\omega_{\min}^{-1}$, but that choice would almost certainly be too small. A reasonable choice would start with a theoretically motivated spectral density $\mathcal{S}(\omega)$ for φ modes in the preferred scale vacuum, then compute the variance σ_φ^2 of the 'process' $\varphi(\mathfrak{x};\mathfrak{t})$ from Eq. (3.25):

$$\int_0^\infty \mathcal{S}_\varphi(f)\, df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big(\varphi(\mathfrak{x};t) - \bar{\varphi}(\mathfrak{x})\big)^2\, dt \;=\; \sigma_\varphi^2\,. \tag{4.19}$$

Now introduce a cutoff frequency $f_{\rm co}$ such that $\mathcal{S}_\varphi(f) \equiv 0$ for all $f > f_{\rm co}$. Determine $f_{\rm co}$ to be the frequency which gives $\sigma_\varphi = 1$ in (4.19). With the choice $\tau = f_{\rm co}^{-1}$, predictions and 'measurements' of $\varphi(\mathfrak{x},\mathfrak{t})$ should agree within one standard deviation at the highest frequencies of the preferred scale.

Determination of the minimum physically resolvable distance, ε, can be done in exactly the same way, replacing f by the inverse wavelength $k = \lambda^{-1}$.

$$\int_0^\infty \mathcal{S}_\varphi(k)\, dk = \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} \big(\varphi(\mathfrak{x};t) - \bar{\varphi}(\mathfrak{x})\big)^2\, dk \;=\; \sigma_\varphi^2\,. \tag{4.20}$$

Introduce a cutoff frequency $k_{\rm co}$ such that $\mathcal{S}_\varphi(k) \equiv 0$ for all $k > k_{\rm co}$, and determine $k_{\rm co}$ to be the frequency which gives $\sigma_\varphi = 1$ in (4.20). Finally, choose $\varepsilon = k_{\rm co}^{-1}$.

Modes in the dynamic regime are propagating modes, $e^{i(\mathfrak{k}\mathfrak{x}-\omega\mathfrak{t})}$, so it is reasonable to expect that the smoothing intervals ε and τ at any given point are related to each other in a particular way. Specifically, they are presumably related by the propagation speed:

$$v = \frac{\varepsilon}{\tau}\,. \tag{4.21}$$

This relationship will be assumed when deriving the field equation.

Remark 4.4.6 *The limits of a preferred scale, $(\mathfrak{k}_{\min}, \mathfrak{k}_{\max})$ and $(\omega_{\min}, \omega_{\max})$ have corresponding physical limits $(k_{\min}, k_{\max})$ and $(\varpi_{\min}, \varpi_{\max})$ which must be defined relative to the lower bound $(\mathfrak{k}_{\min}, \omega_{\min})$ of the preferred scale if they are to have a scale-free definition. The physical limits can be estimated but not directly measured by the physical metric: presumably, the metric itself depends on the motions of the modes between the physical limits, which limits its resolution. Hence, the physical bounding modes are hard cutoffs. That is, there is no sense in which any time or distance scale can have physical meaning outside $[\varpi_{\min}^{-1}, \varpi_{\max}^{-1}]$ or $[k_{\min}^{-1}, k_{\max}^{-1}]$. For example, it is clearly impossible to construct an approximate Fourier integral for a function which varies on distance scales smaller than $k_{\max}^{-1}$ because the modes that are needed to capture the variations do not exist. These physical cutoffs act as natural ultraviolet and infrared cutoffs for quantum field theory computations.*

4.5 Field Equation for φ

At the level of elementary dynamics, *i.e.*, prior to emergence of spacetime and manifestation of matter fields, the only mathematical space in which to work is $\mathfrak{M}$ or its time-supplemented version $\mathfrak{M}_{\mathfrak{t}}$. The field self-coupling is only indirectly observable through its consequences, for example the equation of motion.[3] Hence, in principle it is possible to fully define and work with the field equation on $\mathfrak{M}_{\mathfrak{t}}$, then transform to physical variables when deriving physical consequences.

[3]Conventionally, 'field equation' and 'equation of motion' are interchangeable terms, but in this section the field equation will denote the most general dynamical equation derived from postulates 2.2.11; the equation of motion will mean the field equation restricted to the dynamic regime where stochasticity is perturbative.

4.5.1 Generalizing from local dynamics

A brief review of the local dynamics developed in Subsect. 2.2.3 will get things started. The physical entity is the field of infinite frequency, point-like oscillators, $\tilde{\omega}[\mathfrak{M}]$. A cosmic time, whose mathematical status as an *a priori* nonphysical measure of intervals is the same as the distance metric $\mathfrak{d}$, is introduced to allow measuring the position of individual oscillators. Together, $\mathfrak{M}$ and cosmic time provide a minimal mathematical structure that allows defining neighborhoods, describing interactions between oscillators, and describing the evolutions of spatially extended configurations of oscillators, in a mathematically precise way.[4]

The basis for all dynamics is the coupling of an oscillator $\tilde{\omega}(\mathfrak{x};\mathfrak{t})$ to its infinitesimal neighborhood $\tilde{\omega}[B_\varepsilon(\mathfrak{x});\mathfrak{t})$, which can be expressed as a sum over pairwise couplings of $\tilde{\omega}(\mathfrak{x};\mathfrak{t})$ to $\tilde{\omega}(\mathfrak{x}';\mathfrak{t})$ for all $\mathfrak{x}' \in B_\epsilon(\mathfrak{x})$. The pairwise coupling when $\varepsilon \to 0$ is implicitly defined by Postulate 2.2.11:

$$\frac{\delta}{\delta \mathfrak{t}} \left\langle \rho(\mathfrak{V}_\phi(\mathfrak{c})) \frac{\delta}{\delta \mathfrak{t}} \big\langle \phi(\mathfrak{x},\mathfrak{x}';\mathfrak{t}) \big\rangle_\tau \right\rangle_\tau = -\big\langle g(\mathfrak{V}_\phi(\mathfrak{c})) \cdot \phi(\mathfrak{x},\mathfrak{x}';\mathfrak{t}) \big\rangle_{2\tau} \tag{4.22}$$

where $\phi(\mathfrak{x},\mathfrak{x}';\mathfrak{t})$ is the relative phase (Def*. 2.2.5) between $\tilde{\omega}(\mathfrak{x})$ and $\tilde{\omega}(\mathfrak{x}')$ at time $\mathfrak{t}$, and $\mathfrak{V}_\phi(\mathfrak{c})$ is the total variation of ϕ (Def. 2.2.9), computed along $\mathfrak{c}$ by Eq. (2.15). A subscript on the angle brackets denotes the interval over which the mean is computed. The 'inertia' $\rho(\cdot)$ has the form

$$\rho = \rho\big(\mathfrak{V}_\phi(\mathfrak{c})\big) = \rho_0\, \mathfrak{V}_\phi(\mathfrak{c})\,, \tag{4.23}$$

where ρ_0 is a constant. The coupling strength $g(\cdot)$ has the form

$$g = g\big(\mathfrak{V}_\phi(\mathfrak{c})\big) = g_0\, \mathfrak{V}_\phi(\mathfrak{c})^{-1}\,, \tag{4.24}$$

where $\mathfrak{c}$ is the geodesic between $\mathfrak{x},\mathfrak{x}'$.

Since all dynamics of φ are local, Eqs. (4.22) and (4.24) are the starting point for developing a field equation that describes dynamics over spatially extended regions. The field equation will only be relevant to the dynamic regime — Rmk. 4.2.4 has already argued why it has no effect in the stochastic regime.

In generalizing from local dynamics to a field equation, working with dynamics along curves appears to be the appropriate approach. A curve is the most general

[4] To reiterate, the assumption, indeed requirement if a scale free elementary theory is to be obtained, is that the description of an emergent physical spacetime is completely free of any traces of the particular definitions of $\mathfrak{d}$ and cosmic time.

measurable geometric object in a general metric space. By its definition 2.1.7, a curve $\mathfrak{c} : [0,1] \to \mathfrak{M}$ has a static image, *i.e.*, a path on $\mathfrak{M}$, and a real valued parameter t, $0 \leq t \leq 1$. Def. 2.1.10 allows computing a 'speed' $\dot{\mathfrak{c}}(t)$, but the definitions do not imply t should be interpreted as an elapsed time. Nevertheless, it would be very convenient for present purposes to interpret a general curve $\mathfrak{c}(t)$ as a time-dependent physical displacement from $\mathfrak{c}(t=0)$.

Curves have a fixed image on $\mathfrak{M}$ and t has a fixed range (*e.g.*, $[0,1]$), so the most general analogue of $\mathfrak{c}(t)$ in a physical spacetime is a specific world line. A field equation in 1+1 dimensions should describe field evolution along any world line, so the model 'base space' on which the field equation can be defined should be an abstract curve, one whose image and range of t are unspecified. If t is taken to be the cosmic time it ceases to be a parameter, and the requirement that it has an endpoint at $t = 0$ or any other fixed value must be dropped.

In taking t to be the cosmic time the curve is defined on $\mathfrak{M} \times \mathbb{R}^+$, not $\mathfrak{M}$. The analogy between a curve on $\mathfrak{M} \times \mathbb{R}^+$ and world line on a physical spacetime is clear, but it is only an analogy. Specifically, $\mathfrak{M} \times \mathbb{R}^+$ is a nonphysical space with no intrinsic relationship to a physical spacetime, and there is no necessity (nor is it even desirable) that there is a direct correspondence between curves on $\mathfrak{M}_\mathfrak{t}$ and world lines on $(\mathscr{M}, \mathbf{g})$. A fixed point on $(\mathscr{M}, \mathbf{g})$ might be a point, a curve, or a more general region on $\mathfrak{M}_\mathfrak{t}$, for example.

The local motion postulate 2.2.11 attributes both a dynamical and stochastic contribution to the local motion of $\varphi(\mathfrak{x};\mathfrak{t})$ for $\mathfrak{x} \in \mathfrak{M}$:

$$\delta\varphi(\mathfrak{x}; I_\tau] = \delta\varphi(\mathfrak{x}; I_\tau]_{\text{coupling}} + \delta\varphi(\mathfrak{x}; I_\tau]_{\text{rand}} \,, \tag{4.25}$$

where τ is a chosen sampling period that also appears in the pairwise coupling (4.22). The total $\varphi[\mathfrak{x}; I_\tau]$ is what couples to its neighborhood, not just $\varphi[\mathfrak{x}; I_\tau]_{\text{coupling}}$, so a randomly fluctuating term should rightfully be included in the field equation. Denote this perturbation by $F(\mathfrak{t})$.

The averaging period τ, left unspecified by Postulate 2.2.11, will play an important role in the field equation. Its interpretation becomes straightforward by comparing the argument inside the outer angle brackets on the left side of Eq. (4.22) with the definition of an approximate derivative, Eq. (3.43): It can be identified with $\bar{\partial}_\mathfrak{t}\phi(\mathfrak{x}, \mathfrak{x}'; \mathfrak{t})$ computed over a smoothing period τ. Note the actual smoothing interval in this case

is $[\mathfrak{t}-\tau/2,\ \mathfrak{t}+\tau/2]$.

From Rmk. 3.3.2, higher order approximate derivatives require increasing the sampling interval to obtain the same smoothing. That means $\tau \to 2\tau$ for $\bar{\partial}_{\mathfrak{t}}^2\varphi$ in this case. Fig. 4.3 illustrates the basic idea, although for a spatial approximate second derivative along a path $\gamma(\mathfrak{x}_1, \mathfrak{x}_2)$. Hence the full left side of (4.22) looks like $\bar{\partial}_{\mathfrak{t}}^2\phi(\mathfrak{x}, \mathfrak{x}'; \mathfrak{t})$ (ignoring averaging and the total variation), and 2τ is the total time to compute a second approximate time derivative.

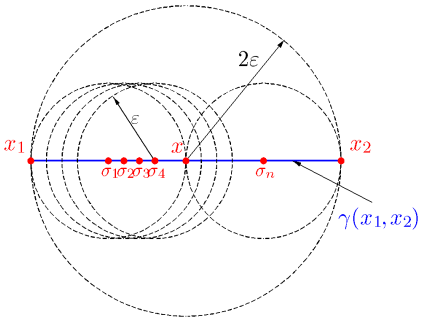


Figure 4.3: Spatial second derivative at $\mathfrak{x}$ in the $\mathfrak{x}_1 \to \mathfrak{x}_2$ direction. First compute first derivatives at $\sigma_1, \sigma_2, \ldots, \sigma_n$ by smoothing over a smaller distance ε. Average these over the interval $[\sigma_1, \sigma_n]$, then compute the approximate derivative of the smoothed first derivative to get the approximate second derivative, $\bar{\partial}_{\mathfrak{x}}^2\varphi(\mathfrak{x})$. See Eq. (3.43).

After rewriting the left side of Eq. (4.22) in terms of approximate derivatives, the starting point for getting a 1+1 dimensional field equation is

$$\bar{\partial}_{\mathfrak{t}} \left\langle \rho(\mathfrak{V}_\phi(\mathfrak{c}))\, \bar{\partial}_{\mathfrak{t}} \left\langle \phi(\mathfrak{x}, \mathfrak{x}'; \mathfrak{t}) \right\rangle_\tau \right\rangle_\tau = \left\langle g(\mathfrak{V}_\phi(\mathfrak{c})) \cdot \phi(\mathfrak{x}', \mathfrak{x}; \mathfrak{t}) \right\rangle_{2\tau} + F(\mathfrak{t})\,. \tag{4.26}$$

4.5.2 Field equation in 1+1 dimensions

The form of Eq. (4.26) suggests the field equation will be a wave equation, so consider how to write the right side of (4.26) in terms of a spatially extended field configuration. Since there are no external boundaries, waves should propagate; call their propagation speed v, not necessarily constant. Propagation of φ inhomogeneities is presumably the primary means by which finitely separated oscillators affect each other.

Subsect. 4.4.4 developed a physically motivated smoothing time τ (Eq. (4.19)) for approximate time derivatives, and distance ε (Eq. (4.20)) for approximate space derivatives. Thus $\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}; \mathfrak{t})$ can be computed along the propagation path by the usual prescription (3.43):

$$\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}; \mathfrak{t}) = \lim_{\delta\mathfrak{x}\to 0} \frac{\Delta\bar{\varphi}(\mathfrak{x}; \mathfrak{t})}{\delta\mathfrak{x}}\,,$$

where $\bar{\varphi}(\mathfrak{x};\mathfrak{t})$ is the average field obtained by smoothing over a distance ε, symmetric about $\mathfrak{x}$.

The choice of ε also acts as a cutoff for the rapidly changing random function $F(\mathfrak{t})$. Since $F(\mathfrak{t})$ represents stochastic motions of φ contributed by modes outside the preferred scale, it is reasonable to ignore it when dynamics strongly dominate. This will typically be the case when an equation of motion is relevant. Hence take $F(\mathfrak{t}) \simeq 0$ in (4.26) and drop it for now.

Let $\bar{B}_\varepsilon(\mathfrak{x}) \subset \mathfrak{M}$ be a closed neighborhood of x, and let ξ_1, ξ_2 be radii of $\bar{B}_\varepsilon(\mathfrak{x})$ such that $\gamma(\mathfrak{x}_1,\mathfrak{x}_2) = \xi_1(\mathfrak{x},\mathfrak{x}_1) \cup \xi_2(\mathfrak{x},\mathfrak{x}_2)$ is a diameter of $\bar{B}_\varepsilon(\mathfrak{x})$. See Fig. 4.4.

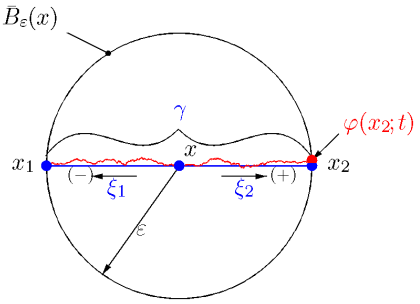


Figure 4.4: In the mechanical oscillator Fig. 2.2, the net force on m vanishes when $\delta x_1 = \delta x_2$. An analogous result holds when $\delta\varphi_1 = \delta\varphi_2$. Since ξ_1 and ξ_2 have directions given by elements of $T_{\mathfrak{x}}\mathfrak{M}$, ξ_1 and ξ_2 have opposite orientation. The high frequency noise on $\varphi[\gamma]$ is modeled by $F(\mathfrak{t})$.

Postulate 2.2.11 obtains the net motion of $\varphi(\mathfrak{x};\mathfrak{t})$ over the interval I_τ by summing contributions (4.22) by all neighbors of σ over the period τ. This means the coupling strength g, Eq. (4.24), is an additive function: $g(a+b) = g(a) + g(b)$. In 1+1 dimensions the sum over an infinitesimal neighborhood contains two terms, one for each nearest neighbor of $\mathfrak{x}$. If $\mathfrak{c}$ has image γ and $\mathfrak{c}_1$ and $\mathfrak{c}_2$ respectively have images ξ_1 and ξ_2, after dropping $F(\mathfrak{t})$ the right side of Eq. (4.26) can be written

$$\begin{aligned}\big\langle g\big(\mathfrak{V}_\phi(\mathfrak{c}_1)\big)\cdot\phi(\mathfrak{x},\mathfrak{x}_1;\mathfrak{t}) + g\big(\mathfrak{V}_\phi(\mathfrak{c}_2)\big)\cdot\phi(\mathfrak{x}_2,\mathfrak{x};\mathfrak{t})\big\rangle_{2\tau}\\ = \big\langle g\big(\mathfrak{V}_\phi(\mathfrak{c}_1)\big)\cdot\phi(\mathfrak{x},\mathfrak{x}_1;\mathfrak{t}) - g\big(\mathfrak{V}_\phi(\mathfrak{c}_2)\big)\cdot\phi(\mathfrak{x},\mathfrak{x}_2;\mathfrak{t})\big\rangle_{2\tau}\end{aligned} \tag{4.27}$$

where ξ_1 and ξ_2 are the images of curves $\mathfrak{c}_1(t)$ and $\mathfrak{c}_2(t)$, respectively, with $\mathfrak{t}_0 \le t \le \mathfrak{t}_0+\delta\mathfrak{t}$.

Essentially all the contribution to the total variation $\mathfrak{V}_\phi(\cdot)$ is due to stochasticity at distance scales smaller than the preferred scale. The stochastic contributions are perturbative in the dynamic regime, and will be smoothed out after averaging over time τ. What remains is essentially the same as the residual variation (4.13), $V_\varphi^{(\mathfrak{t}_{\max})}(\mathfrak{c})$.

Over a time scale τ or distance scale ε, *i.e.*, the smoothing scales for approximate derivatives, the residual variation can be taken to be constant throughout $\bar{B}_\varepsilon(\mathfrak{x})$

over a time scale τ; indeed, if that were not so it would mean ε and τ were not correctly chosen in the neighborhood of $\mathfrak{x}$. Hence, take $\mathfrak{V}_\phi(\mathfrak{c}_2) = \mathfrak{V}_\phi(\mathfrak{c}_1) = \mathfrak{V}_\phi(\mathfrak{c}_0)$ in (4.27), where the image of $\mathfrak{c}_0$ is an arbitrary radius of $\bar{B}_\varepsilon(\mathfrak{x})$; after smoothing the constancy assumption should be met. Also note $\phi(\mathfrak{x},\mathfrak{x}_i;\mathfrak{t}) = \varphi(\mathfrak{x};\mathfrak{t}) - \varphi(\mathfrak{x}_i;\mathfrak{t})$. Then the argument inside the outer angle brackets in (4.27) becomes

$$\begin{aligned} &g\big(\mathfrak{V}_\phi(\mathfrak{c}_1)\big)\cdot\phi(\mathfrak{x}_1,\mathfrak{x};\mathfrak{t}) - g\big(\mathfrak{V}_\phi(\mathfrak{c}_2)\big)\cdot\phi(\mathfrak{x},\mathfrak{x}_2;\mathfrak{t}) \\ &\qquad = g(\cdot)\cdot\big\{[\varphi(\mathfrak{x}_2;\mathfrak{t}) - \varphi(\mathfrak{x};\mathfrak{t})] - [\varphi(\mathfrak{x};\mathfrak{t}) - \varphi(\mathfrak{x}_1;\mathfrak{t})]\big\}\,, \end{aligned} \tag{4.28}$$

where $g(\cdot) = g(\mathfrak{V}_\phi(\cdot)) = g(\mathfrak{V}_\varphi(\cdot))$ since $\mathfrak{V}_\varphi(\cdot) = \mathfrak{V}_\phi(\cdot)$.

The temporal and spatial dependence of the 'tension' and 'density' can be modeled in two ways. The first is to take measures of time and distance to be invariant along $\mathfrak{c}_0$ and let the tension and density vary. The other is to take the tension and density to be constant along $\mathfrak{c}_0$ but the measures of time and distance vary. This second viewpoint is illustrated in Subsect. 4.4.2 (see Eq. (4.16)) for wave propagation speeds, and will be adopted here.

Since $\mathfrak{V}_\varphi$ rescales distances in the adopted viewpoint, it is convenient (and suggestive) to define a matrix $g_{\mu\nu}$:

$$g_{\mathfrak{tt}} = 1/\mathfrak{V}_\varphi^{(\mathfrak{t})} = h_\mathfrak{t}^2\,, \quad g_{\mathfrak{xx}} = \mathfrak{V}_\varphi^{(\mathfrak{x})} = h_\mathfrak{x}^2\,, \quad g_{\mathfrak{xt}} = g_{\mathfrak{tx}} = 0 \tag{4.29}$$

with $g^{\mu\nu} = 1/g_{\mu\nu}$. Call $h_\mathfrak{x}$ the position- and time-dependent 'scale factor' which tells how distances vary in the $\mathfrak{x}$ direction in the neighborhood of $\mathfrak{x}$; similarly $h_\mathfrak{t}$ is the scale factor for the time direction. That is, $h_\mathfrak{x}\varepsilon$ and $h_\mathfrak{t}\tau$ are the rescaled smoothing distance and time, respectively. The particular expressions assigned to $g_{\mathfrak{tt}}$ and $g_{\mathfrak{xx}}$ above will be assumed without justification below; they will be obtained analytically in Chap. 6.

To be clear, the matrix defined by Eq. (4.29) should be viewed as a **naming convention** intended to make the field equation have a more familiar appearance. There is no assumption it is a valid spacetime metric, that it generalizes to more than 1+1 dimensions, or even that it is a tensor. Those issues cannot be considered until Chap. 6.

Since $\mathfrak{x}_1 = \mathfrak{x} - \varepsilon$ and $\mathfrak{x}_2 = \mathfrak{x} + \varepsilon$, (4.28) can be written in terms of central

differences (time coordinate omitted):

$$\begin{aligned} g(\cdot)\cdot\big\{[\varphi(\mathfrak{x}_2)-\varphi(\mathfrak{x})][\varphi(\mathfrak{x}_2)-\varphi(\mathfrak{x})]-[\varphi(\mathfrak{x})-\varphi(\mathfrak{x}_1)]\big\} \\ &= g(\cdot)\cdot h_{\mathfrak{x}}\varepsilon\Big\{\frac{\varphi(\mathfrak{x}+\varepsilon)-\varphi(\mathfrak{x})}{h_{\mathfrak{x}}\varepsilon}-\frac{\varphi(\mathfrak{x})-\varphi(\mathfrak{x}-\varepsilon)}{h_{\mathfrak{x}}\varepsilon}\Big\} \\ &\simeq g(\cdot)\cdot h_{\mathfrak{x}}\varepsilon\big\{h_{\mathfrak{x}}^{-1}\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}+\tfrac{\varepsilon}{2})-h_{\mathfrak{x}}^{-1}\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}-\tfrac{\varepsilon}{2})\big\} \\ &= g(\cdot)\cdot h_{\mathfrak{x}}\varepsilon^2\,\frac{\zeta(\mathfrak{x}+\frac{\varepsilon}{2})-\zeta(\mathfrak{x}-\frac{\varepsilon}{2})}{2(\varepsilon/2)}\,, \end{aligned} \tag{4.30}$$

where $\zeta(\mathfrak{x}\pm\frac{\varepsilon}{2})=h_{\mathfrak{x}}^{-1}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x}\pm\frac{\varepsilon}{2})$, using Eq. (4.29). The field was implicitly smoothed over a distance ε between the second and third steps above. After smoothing the approximate first derivatives in the numerator of the last step of (4.30), again over a distance ε,

$$g(\cdot)\cdot h_{\mathfrak{x}}\varepsilon^2\,\bar{\partial}_{\mathfrak{x}}\zeta(\mathfrak{x}) \;=\frac{g_0}{\mathfrak{V}_\varphi^{(\mathfrak{x})}}\,h_{\mathfrak{x}}h_{\mathfrak{t}}\varepsilon^2\,\bar{\partial}_{\mathfrak{x}}\left(\frac{1}{h_{\mathfrak{x}}h_{\mathfrak{t}}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right). \tag{4.31}$$

This can be straightforwardly put into the traditional form. Since $\sqrt{-g}=h_{\mathfrak{x}}h_{\mathfrak{t}}$ and $g^{\mathfrak{x}\mathfrak{x}}=1/g_{\mathfrak{x}\mathfrak{x}}$, then $\sqrt{-g}g^{\mathfrak{x}\mathfrak{x}}=1/h_{\mathfrak{x}}h_{\mathfrak{t}}$ so that, using Eq. (4.29),

$$\begin{aligned} \frac{g_0}{\mathfrak{V}_\varphi^{(\mathfrak{x})}}\,h_{\mathfrak{x}}\varepsilon^2\,\bar{\partial}_{\mathfrak{x}}\left(\sqrt{-g}g^{\mathfrak{x}\mathfrak{x}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right) &= \frac{g_0}{\mathfrak{V}_\varphi^{(\mathfrak{t})}}\,\varepsilon^2\,\frac{h_{\mathfrak{x}}h_{\mathfrak{t}}}{h_{\mathfrak{x}}^2h_{\mathfrak{t}}^2}\,\bar{\partial}_{\mathfrak{x}}\left(\sqrt{-g}g^{\mathfrak{x}\mathfrak{x}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right) \\ &= \frac{g_0}{\mathfrak{V}_\varphi^{(\mathfrak{t})}}\,\varepsilon^2\,\frac{1}{\sqrt{-g}}\,\bar{\partial}_{\mathfrak{x}}\left(\sqrt{-g}g^{\mathfrak{x}\mathfrak{x}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right), \end{aligned} \tag{4.32}$$

since $\mathfrak{V}_\varphi^{(\mathfrak{t})}/\mathfrak{V}_\varphi^{(\mathfrak{x})}=1/h_{\mathfrak{t}}^2h_{\mathfrak{x}}^2$.

The left side can be put into a similar form, repeating much of the reasoning above for difference equations in time. Using Eq. (4.23) to write $\rho=\rho_0\mathfrak{V}_\varphi^{(\mathfrak{t})}$, inserting the scale factors where appropriate, and ignoring the angle brackets, the left side of Eq. (4.26) can be written

$$\rho(\cdot)\cdot h_{\mathfrak{t}}\tau^2\,\bar{\partial}_{\mathfrak{t}}\left(\frac{1}{h_{\mathfrak{t}}}\bar{\partial}_{\mathfrak{t}}\varphi(\mathfrak{x};\mathfrak{t})\right) \;=\rho_0\,\mathfrak{V}_\varphi^{(\mathfrak{x})}\,\tau^2\,\frac{1}{\sqrt{-g}}\,\bar{\partial}_{\mathfrak{t}}\left(\sqrt{-g}g^{\mathfrak{t}\mathfrak{t}}\,\bar{\partial}_{\mathfrak{t}}\varphi(\mathfrak{x};\mathfrak{t})\right). \tag{4.33}$$

The remaining issue is the how to handle the time-averaging brackets that appear on both sides of Eq. (4.26). The time averaging was implicitly performed when taking approximate time derivatives in Eq. (4.33), but what about the time average on the right side of (4.26)? In analyzing the right side of the field equation, interactions were taken to be along the curves $\mathfrak{c}_1$ and $\mathfrak{c}_2$ because they are due to φ inhomogeneities propagating along those curves. Hence, the distinction between time averaging and space averaging is artificial in deriving the field equation. Because modes in general

propagate in the dynamic regime, it is reasonable to expect the fixed relationship between ε and τ given by Eq. (4.21). Hence, the space averaging performed in computing the approximate derivatives in Eq. (4.32) has presumably already performed the time averaging on the right side of Eq. (4.26). Then the averaging brackets can be dropped there as well.

Equating Eqs. (4.32) and (4.33) and slightly rearranging gives a homogeneous wave equation,

$$\begin{aligned}\frac{1}{\sqrt{-g}}\,\bar{\partial}_{\mathfrak{t}}\left(\sqrt{-g}g^{\mathfrak{tt}}\,\bar{\partial}_{\mathfrak{t}}\varphi(\mathfrak{x};\mathfrak{t})\right) &= \left[\frac{g_0}{\rho_0}\,\frac{1}{\mathfrak{V}_\varphi^{(\mathfrak{x})}\mathfrak{V}_\varphi^{(\mathfrak{t})}}\,\frac{\varepsilon^2}{\tau^2}\right]\frac{1}{\sqrt{-g}}\,\bar{\partial}_{\mathfrak{x}}\left(\sqrt{-g}g^{\mathfrak{xx}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right)\\ &= v^2\frac{1}{\sqrt{-g}}\,\bar{\partial}_{\mathfrak{x}}\left(\sqrt{-g}g^{\mathfrak{xx}}\,\bar{\partial}_{\mathfrak{x}}\varphi(\mathfrak{x})\right),\end{aligned} \tag{4.34}$$

where

$$v \equiv \left(\frac{g_0}{\rho_0}\,\frac{1}{\mathfrak{V}_\varphi^{(\mathfrak{x})}\mathfrak{V}_\varphi^{(\mathfrak{t})}}\right)^{\frac{1}{2}}\frac{\varepsilon}{\tau}\,. \tag{4.35}$$

Evidently, v is the wave propagation speed, measured with respect to the distance metric $\mathfrak{d}$ and cosmic time. If v is constant, and moreover if $g_{\mu\nu}$ is a tensor on a manifold with Lorentzian signature, the equation can also be expressed in the traditional form:

$$\frac{1}{\sqrt{-g}}\,\bar{\partial}_\mu\left(\sqrt{-g}g^{\mu\nu}\,\bar{\partial}_\nu\varphi(\mathfrak{x})\right) = 0\,. \tag{4.36}$$

Chap. 6 will consider whether $g_{\mu\nu}$ in (4.34) meets these requirements.

Subsect. 4.4.1 obtained a scale-free expression for the propagation speed, Eq. (4.15):

$$v = \frac{v_0}{\mathfrak{n}(\varphi[\mathfrak{c}])}\,, \tag{4.37}$$

where $\mathfrak{n}(\varphi[\mathfrak{c}])$ is the relative variation (Def*. 4.4.3) along $\mathfrak{c}$; v_0 is the speed along a reference curve $\mathfrak{c}_{\text{ref}}$ (used to compute the relative variation), defined to be a universal constant. This propagation speed is consistent with the one obtained in the field equation (4.34) because both depend on the field variation in the same way. Their relationship is even more straightforward if

$$v_0 \sim \sqrt{\frac{g_0}{\rho_0}}\,\frac{\varepsilon}{\tau}\,,$$

which is similar in form to the propagation speed for waves on a string.

If the relative variation $\mathfrak{n}$ is used instead of the total variation $\mathfrak{V}_\varphi$, the field equation (4.34) becomes scale-free. It is only necessary to use the propagation speed (4.37).

As it is written, the field equation is a homogeneous wave equation. This is not particularly suitable because it contains no way to describe interactions and self interactions involving φ. That means it cannot be complete.

At least part of the problem, and maybe all of it, is that all the analysis leading to it assumed interactions only occurred along geodesics. This is certainly wrong. If, instead, the field equation (4.34) describes interactions due to a single path, then it can be applied to all possible paths between the points of interest, *i.e.*, it can be the basis for path integration. The next subsection will explore this issue further.

Interestingly, once path integration is introduced, the stochasticity of φ leads to something like annealed randomness.[5] This is important because annealed randomness can introduce effective interactions into φ, *e.g.* a $\lambda\varphi^4$ interaction [7]. Thus, it is possible that the necessary self interactions will occur without introducing them in an explicit postulate.

Regardless, the φ field equation must have one or more interaction terms if it is to underlie a physical theory. Hence, take the general form of the field equation with the random perturbation $F(\mathfrak{t})$ restored to be

$$v^2 \frac{1}{\sqrt{-g}} \bar{\partial}_{\mathfrak{r}} \left(\sqrt{-g} g^{\mathfrak{r}\mathfrak{r}} \bar{\partial}_{\mathfrak{r}} \varphi(\mathfrak{r}) \right) - \frac{1}{\sqrt{-g}} \bar{\partial}_{\mathfrak{t}} \left(\sqrt{-g} g^{\mathfrak{t}\mathfrak{t}} \bar{\partial}_{\mathfrak{t}} \varphi(\mathfrak{r};\mathfrak{t}) \right) = \frac{d\mathcal{V}(\varphi)}{d\varphi} + F(\mathfrak{t}) \,, \tag{4.38}$$

where v is the wave propagation speed and $\mathcal{V}(\varphi)$ is the effective potential. The effective potential may include an effective mass.

4.5.3 Generalization to $n+1$ dimensions

Mathematically, the field equation (4.38) immediately generalizes from $n = 1$ to arbitrary n. It is only necessary to define new scale factors $g_{ij} = \delta_i^j h_i^2$, where δ_i^j is the Kronecker delta, add additional terms of the same form as the first term of of (4.38), and take v to have n components $v_1, v_2, \ldots, v_n$ with $v^2 = \sum_{i=1}^n v_i^2$.

Physically, the generalization obviously only makes sense where there are $n > 1$ independent directions. The meaning of 'independent directions' in the emergence picture is described in Subsect. 4.3.1 and codified in Def.*4.3.2. Generalization to $n > 1$ dimensions does not require (or imply) Lorentz symmetry of the space or a tensorial

[5] Thanks to Tom Banks for bringing annealed randomness to my attention.

nature of $g_{\mu\nu}$ in Eq. (4.38). Applicability of (4.38) to a physical spacetime does impose such requirements, along with others like smoothness of space and time in some sense.

4.5.4 The need for path integration

The local dynamics of Subsect. 2.2.3 take the simple form of Postulate 2.2.11 because the coupling between neighboring points σ and σ' is just a nearest neighbor interaction. Hence it is only necessary to consider the geodesic between σ and σ' in Eqs. (4.22) and (4.24) to completely describe the interaction between neighboring oscillators $\tilde{\omega}(\sigma;\mathfrak{t})$ and $\tilde{\omega}(\sigma';\mathfrak{t})$.

If $\ell = \mathfrak{d}(\sigma,\sigma')$ is finite, it should still be possible to specify an interaction between $\tilde{\omega}(\sigma;\mathfrak{t})$ and $\tilde{\omega}(\sigma';\mathfrak{t})$, but now it will be an effective interaction. That is, the relative phase $\phi(\sigma,\sigma';\mathfrak{t})$ can only influence the evolution of $\omega(\sigma;\mathfrak{t})$ by first propagating to the infinitesimal neighborhood $B_\varepsilon(\sigma)$, after which the local dynamics determine the effect on $\omega(\sigma;\mathfrak{t})$. Since ℓ is finite, $\phi(\sigma,\sigma';\mathfrak{t})$ can propagate to $B_\varepsilon(\sigma)$ along uncountably many paths $\gamma(\sigma,\sigma')$, *e.g.* the uncountable set of geodesic paths $\{\gamma(\sigma',\sigma'') \,|\, \sigma'' \in B_\varepsilon(\sigma)\, \mathfrak{L}(\gamma) = \ell\}$. Moreover, uncountably many curves have the same image $\gamma(\sigma',\sigma'')$, and there is no *a priori* justification for excluding any of them in computing the coupling.

By similar reasoning, since ℓ is finite there are uncountably many intermediate points $\{\xi_1,\xi_2,\ldots\}$ along the geodesic $\gamma(\sigma,\sigma')$ which should be analyzed similarly to σ. Considering only the contribution by γ, before $\phi(\sigma,\sigma';\mathfrak{t})$ can influence $\omega(\sigma;\mathfrak{t})$ it must influence $\omega(\xi_i;\mathfrak{t})$ due to $\phi(\sigma',\xi_i;\mathfrak{t})$, which it does by the same local dynamics at ξ_i as it does for σ. As before, there are uncountably many contributions to change in $\omega(\xi_i;\mathfrak{t})$ due to $\phi(\sigma',\xi_i;\mathfrak{t})$ because ξ_i has an uncountable neighborhood.

The reasoning above makes it clear that the interactions over finite distances, being indirect and thus effective, are vastly more complicated than the direct, fully local interactions where the contribution from the geodesic between two neighbors completely dominates contributions from other paths between the same points. Path integration is the usual tool for dealing with such situations. However, path integration methods have been developed for use in established spacetimes, not general metric spaces.

Present interest in path integrals in general metric spaces is primarily restricted to regions of $\mathfrak{M}_\mathfrak{t}$ where scale invariance has been broken in the sense of Def*. 4.3.1 but which lack a well defined dynamic regime. One important example is presumably the earliest stage of cosmogenesis. Yet it is doubtful path integration would be appropriate

in that case even if it could be rigorously defined. The quantum fields are almost certainly in a non-perturbative regime during that era due to the same lack of an established spacetime, so it is unclear what quantity could act as a small parameter that would allow expanding a path integral as a perturbation series. The initial stage of cosmogenesis is likely to require a nonperturbative approach.

The other primary example where path integration in a general metric space might be desirable is the netherworld between the stochastic and dynamic regimes at the boundaries of a preferred scale. Fluctuations of φ would likely be large in that case; like the initial stage of cosmogenesis, it is not clear what could act as an expansion parameter.

Hence it appears a reasonably well established spacetime should exist whenever path integration could be a useful computation tool. As will be argued in Chap. 6, a small mean square fluctuation of φ is necessary for an established spacetime, and that can provide a small parameter for perturbation theory.

Alas, while the last point lessens the practical necessity of path integration methods in a general metric space it does not argue against the theoretical desirability of them. The chief reason is that the field equation must be defined on $\mathfrak{M}_\mathfrak{t}$ first because that is the postulated base mathematical space, and path integration should be at least formally a part of that definition. Only after appropriate relationships have been established between $\mathfrak{M}_\mathfrak{t}$ and an emergent physical spacetime $(\mathscr{M}, \mathbf{g})$ can the full field equation be promoted to a physical spacetime.

Part II

Emergence and Post-Emergent Spacetime

Chapter 5

Composite Field Theory

One of the premises of this thesis is that quantum fields do not manifest if no manifold or other basic elements of spacetime exist. Hence, if spacetime emerges from a condition where the rudiments of spacetime do not exist *a priori*, then it is reasonable to consider the possibility that quantum mechanics also emerges from a condition where its rudiments are absent. If so, then quantum theory may be an effective description that hides important microscopic degrees of freedom. The problem of emergent quantum mechanics is not readily addressed, if indeed it is emergent. Hence the following 'provisional' approach will be adopted instead:

- Assume a collection $\mathcal{Q}$ of quantum fields exists *a priori*, but their manifestation is contingent on the emergence of a physical manifold.
- Assume standard QM is valid once spacetime emerges, at least when gravitational effects are not important.
- Restrict the role of the quantum fields in gravity to interactions with φ; these induce motions of φ which lead to spacetime emergence, spacetime persistence and gravity.
- Sidestep foundational questions of QM that would require attention if QM were actually emergent. (To do otherwise would lead the discussion well beyond the scope of this thesis.)

The present chapter will assume a spacetime manifold $\mathscr{M}$ has already emerged. This allows focusing on the interactions between the quantum fields and the φ field, especially

in providing the necessary conditions for φ to act as a mediator between the stress-energy tensor $T_{\mu\nu}$ and spacetime geometry. A specific candidate model that explains how φ can actually play the mediating role in manifold emergence and spacetime geometry will be postponed until Chap. 6.

A common viewpoint is that a composite field theory of quantum fields interacting with a non-quantum field is not consistent with quantum mechanics in a fundamental theory. The first section of this chapter will argue this view is not correct, at least when the role of the non-quantum field is to mediate gravity as does φ. It will review a formalism for mixed classical and quantum ensembles wherein quantum and classical fields can coexist on equal footing while remaining consistent with quantum theory. Although φ is not a classical field, it can play the role of the classical subsystem in the formalism.

5.1 Composite Theory in 3+1 Dimensions

Given the assumption that emergence of a spacetime manifold $\mathscr{M}$ is mediated by the semi-stochastic scalar field φ, it is necessary to ascertain the constraints on models wherein $\mathscr{M}$ dynamically emerges on $\mathfrak{M}_{\mathfrak{t}}$ via φ coupled to $\mathcal{Q}$, the collection of model quantum fields. In the emergence picture, the members of $\mathcal{Q}$ depend on φ because they live on the spacetime $(\mathscr{M}, \mathbf{g})$ whose structure is generated by φ; in turn, φ depends on the members of $\mathcal{Q}$ through their backreaction on φ. Hence, coupling between φ and the members of $\mathcal{Q}$ remains important after emergence of $\mathscr{M}$ is complete.

In determining constraints on a composite field theory of $\{\varphi, \mathcal{Q}\}$, it is obviously helpful to start from an established body of empirical and theoretical knowledge and then work backwards. Specifically, if a consistent picture can be constructed in 3+1 dimensions, then restriction of the dynamics to the time-augmented general metric space $\mathfrak{M}_{\mathfrak{t}} \equiv \mathfrak{M} \times \mathbb{R}$ can offer a productive way to obtain elements of a more elementary theory that is consistent with emergent 3+1 dimensional spacetime.

In 3+1 dimensions a very strong and thus extremely useful constraint on models clearly exists: consistency with general relativity and quantum theory, at least in the regime where those theories are empirically established. The substantial mathematical framework that already exists for analyzing quantum and gravitational theory gives another compelling reason to first focus on the 3+1 dimensional case.

This section starts by introducing two sectors of the composite field theory: quantum and classical-stochastic. The φ field participates in both, but in different ways. In the quantum sector φ couples to the quantum fields, and in a limited sense it thereby acts as a gravitational source like the quantum fields. In the CS sector φ can be treated as an **effectively classical** field, but now its role is a mediating field which 'implements' the 3+1 dimensional spacetime, only acting as a gravitational source in a passive (albeit nonlinear) way. These two sectors coexist because φ participates in both, so it is crucial that the coupling between the 'classical' φ in the CS sector and the quantum fields in quantum sector is consistent with quantum mechanics.

The Hall-Reginatto configuration ensemble formalism, introduced in Subsect. 5.1.3, provides a framework to ensure a consistent composite field theory. Due to the weakness of gravity it should be unnecessary to work within the formalism to do actual field theory calculations in most cases, a claim considered in Subsect. 5.1.6. The formalism imposes consistency constraints nevertheless.

5.1.1 The quantum and classical-stochastic sectors

Consider an interacting field theory of a collection of $\mathcal{N}_\mathcal{Q}$ quantum fields:

$$\mathcal{Q} = \{\, \xi_j \,|\, 1 \leq j \leq \mathcal{N}_\mathcal{Q} \,\} \tag{5.1}$$

plus a non-quantum scalar field $\varphi \notin \mathcal{Q}$. Assume for simplicity they live on a spacetime with Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,; \quad \gamma_{ij} = \mathrm{diag}(1,1,1)\,. \tag{5.2}$$

Let Σ be a space-like hypersurface in some reference frame; it has Euclidean metric γ_{ij}.

The configuration space for the theory is the product space ($\mathcal{A}_\varphi$ is the field configuration space for φ)

$$\mathcal{A} = \mathcal{A}_\varphi \times \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_{\mathcal{N}_\mathcal{Q}}\,. \tag{5.3}$$

Since this is a field theory, $\mathcal{A}_j$ is the infinite-dimensional space of all possible field configurations on Σ for the field ξ_j. For each $\xi_j \in \mathcal{Q}$, assume a basis $\{\,|A_j\rangle\}$ on $\mathcal{A}_j$ for which the field operator is diagonal. For the 'classical' φ, let the basis $\{\,|\varphi\rangle\}$ be a complete set $\{\varphi_k \mid 0 < k < \infty\}$ of mode functions on Σ. Then the 'coordinate' basis on the space (5.3) is a complete set of kets,

$$\{\,|A\rangle\} = \{\,|\varphi\rangle\} \otimes \bigotimes_{j=1}^{\mathcal{N}_\mathcal{Q}} \{\,|A_j\rangle\}\,.$$

Denote the (infinite dimensional) Hilbert space for $|A_j\rangle$ by $\mathscr{H}_j$ so that the Hilbert space for a state ket $|\Psi\rangle = |\varphi\, A_1 A_2 \cdots A_{\mathcal{N}_\mathcal{Q}}\rangle$ is the product space

$$\mathscr{H} = \mathscr{H}_\varphi \otimes \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_{\mathcal{N}_\mathcal{Q}}\,. \tag{5.4}$$

Let the field theory have the Lagrange density

$$\mathcal{L} \;=\; \sum_{j=1}^{\mathcal{N}_\mathcal{Q}} \mathcal{L}_{\rm free}(\xi_j, \partial_\mu \xi_j, \varphi) \;+\; \mathcal{L}_\varphi(\varphi, \partial_\mu \varphi) \;+\; \sum_{\substack{i=1,\\ j=i}}^{\mathcal{N}_\mathcal{Q}} \mathcal{L}_{\rm int}(\xi_i, \partial_\mu \xi_i, \xi_j, \partial_\mu \xi_j, \varphi)\,, \tag{5.5}$$

where φ has the Lagrange density

$$\mathcal{L}_\varphi(\varphi, \partial_\mu \varphi) = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm rand} = \;\tfrac{1}{2}\left[-c^{-2}\dot\varphi^2 + (\nabla\varphi)^2 - m^2\varphi^2\right] - \mathcal{V}_{\rm int}(\varphi) + \mathcal{L}_{\rm rand}\,. \tag{5.6}$$

Thus $\mathcal{L}_\varphi$ defines an interacting scalar field augmented by a stochastic term $\mathcal{L}_{\rm rand}(\varphi, \partial_\mu\varphi)$ which remains unspecified for now. ($\mathcal{L}_{\rm rand}$ will play a significant role in the early stages of spacetime emergence.) The propagation speed c is a function $c(\mathbf{x}, t) = v_0/\mathfrak{n}(\mathbf{x}, t)$, where $\mathfrak{n}(\mathbf{x}, t)$ is the relative variation of φ (Def*. 4.4.3); see Eq. (4.37). The effective mass m will be considered in Sect. 5.4.

Much of this chapter will assume the viewpoint that, to the extent that gravity is very weak compared to other interactions, the composite field theory can usually be partitioned into two sectors or subsystems: quantum and classical/stochastic. The φ field mediates the interactions between these two sectors.

- **Quantum sector.** This contains the matter fields and observables of the quantum mechanical world, *i.e.*, essentially all non-gravitational phenomena. The quantum sector describes whatever goes into the stress-energy tensor $T_{\mu\nu}$ that determines the spacetime geometry.

- **CS sector.** Interactions with the quantum fields in the quantum sector induce motions in φ which transfer energy to the CS sector. These motions then implement the structure and geometry of spacetime within which the quantum sector lives. The zero point motions of the quantum fields are also considered part of this sector, and hence are not included in $T_{\mu\nu}$.

Hence, gravitation is a phenomenon that is to be described within the CS sector. That is, gravity is to be mediated by the non-quantum manifestation of the φ field.[1]

To demonstrate consistency of this picture with quantum theory, a formalism developed by Hall and Reginatto [32] will be employed. It works with ensembles of field configurations on configuration space. Except for the zero-point motions of the quantum fields, quantum theory is described by the quantum sector. To usefully employ the Hall-Reginatto formalism, it is necessary to consider an effectively classical φ coupled to the quantum fields. This can be most easily accomplished while maintaining the distinction between the quantum and CS sectors by demoting the 'quantum φ' to a classical field and then considering only the quantum sector. This modification will only be necessary to establish consistency of the composite field theory; once this has been done, the distinction between the quantum φ in the quantum sector and the classical/stochastic φ in the CS sector can be restored.

The formalism is most natural in the Schrödinger representation of quantum field theory. After briefly reviewing selected aspects of this representation, the remainder of the section will summarize the formalism and use it to demonstrate consistency between quantum theory and a composite field theory of φ and a collection $\mathcal{Q}$ of quantum fields. The argument is not restricted to three spatial dimensions.

5.1.2 Schrödinger representation of field theory

The Schrödinger representation is the natural representation for quantum field theory on configuration space. Symanzik showed [65, 49] the Schrödinger representation exists for all renormalizable quantum field theories to all orders of a perturbation expansion, and with suitable regularization the Hamilton operator also is well-defined. Thus the Schrödinger representation is suitable for all fields in $\mathcal{Q}$.

Start with the Schrödinger equation,

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right)\psi(\mathbf{x},t) = 0 \tag{5.7}$$

(with $\hbar$ explicitly shown), can be interpreted as a matter field equation on Σ for ψ [60]. The (infinitely many) degrees of freedom are the values of $\psi(\mathbf{x},t)$ for all $\mathbf{x} \in \Sigma$ at a given time t. Assume Hamilton's principle applies to ψ. Then variation of the action

[1] In Sect. 5.2, φ will be promoted to an effectively-quantum field in the quantum sector, so it can contribute to the renormalization flow of the 'true' quantum fields.

at each point $\mathbf{x}$,

$$0 = \delta \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L}\left(\psi, \nabla\psi, \dot{\psi}, t\right) ,$$

obtains the (infinite set of) Euler-Lagrange equations for ψ, one equation for each $\mathbf{x} \in \Sigma$. If

$$\mathcal{L} = i\hbar\psi^*\psi - \frac{\hbar^2}{2m}(\nabla\psi^* \cdot \nabla\psi) - V\psi^*\psi \,, \tag{5.8}$$

then the Euler-Lagrange equations applied to (5.8) yield the Schrödinger equation (5.7) for the conjugate field ψ^*. (Varying (5.8) with respect to ψ^* gives (5.7) itself.) Since

$$\pi(\mathbf{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(\mathbf{x}, t)} = i\hbar\psi^*(\mathbf{x}, t)$$

is the momentum field conjugate to ψ, the Hamiltonian

$$\mathcal{H}(\psi, \psi^*) = \pi\,\dot{\psi} - \mathcal{L} = \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi + V\psi^*\psi \,, \tag{5.9}$$

The reformulation (5.9) in terms of matter fields (ψ, ψ^*) is purely formal. To obtain a quantum field theory, the fields are 'second quantized' in the usual way. The canonical variables $\psi(\mathbf{x})$ and $\pi(\mathbf{x})$ become operators on the elements Ψ of some Hilbert space $\mathscr{H}_\Sigma$. The infinite dimensional function space of all smooth field configurations $\mathcal{A} = \{A(\mathbf{x})\}$ on Σ comprises the configuration space of the system. The (time dependent) state ket $|\Psi\rangle$ is a complex-valued wave functional $\Psi[A, t] = \langle A|\Psi\rangle$, where the argument A runs over all elements of $\mathcal{A}$. In the Schrödinger representation of QFT, one works in a 'coordinate' basis $\{\,|A\rangle\}$ on Fock space in which the field operator $\psi(\mathbf{x})$ is diagonal [36], so that

$$\psi(\mathbf{x})\,\Psi[A] = A(\mathbf{x})\,\Psi[A] \;\Leftrightarrow\; \psi(\mathbf{x})\,|A\rangle = A(\mathbf{x})\,|A\rangle \,. \tag{5.10}$$

Hence, $\Psi[A]$ can be interpreted thusly [49]: $|\Psi[A]|^2$ is proportional to the probability (in the Heisenberg picture) that the quantum field $\psi(\mathbf{x}, t)$ has the value $A(\mathbf{x})$ at time $t = 0$. Note Ψ is not a point function of $\mathbf{x}$; it depends on the function ψ for all $\mathbf{x}$.

The commutation relations are

$$\begin{aligned} &[\psi(\mathbf{x}),\, \psi(\mathbf{y})] = [\pi(\mathbf{x}),\, \pi(\mathbf{y})] = 0, \\ &[\psi(\mathbf{x}),\, \pi(\mathbf{y})] = i\hbar\,\delta(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{5.11}$$

which requires that

$$\pi(\mathbf{x})\,\Psi[A] = \frac{\hbar}{i}\,\frac{\delta}{\delta A(\mathbf{x})}\Psi[A] \,. \tag{5.12}$$

Here $\delta/\delta A(\mathbf{x})$ denotes functional differentiation with respect to $A(\mathbf{x})$.

The Schrödinger equation for Ψ is the functional differential equation[2],

$$i\hbar \frac{\partial}{\partial t}\Psi[A,t] = H\Psi[A,t]\,. \tag{5.13}$$

The Hamiltonian operator H is obtained via Eqs. (5.10) and (5.12).

For example, an interacting scalar field ϕ with classical Hamilton density $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$ has the corresponding Hamiltonian, now a functional differential operator [49]:

$$H = \int d^3x \left[\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \frac{\hbar^2}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{5.14}$$

Hatfield [36] has computed the quantum corrections to the mass for this QFT in some detail using Rayleigh-Schrödinger perturbation theory. Following Hatfield, the general idea of Rayleigh-Schrödinger perturbation theory is as follows.

The mass m in (5.14) is the physical mass of a ϕ particle. Rewrite it: $m \to m+\delta_m$, where m is now the 'bare' (non-renormalized) mass. Write (5.14) as $H = H_0 + \alpha H_{\text{int}}$, where H_0 is the free-field Hamiltonian and $H_{\text{int}} = \frac{1}{2}\delta_m^2\phi^2 + \frac{\lambda}{4!}\phi^4$ gives the interaction. Expand the wave functional $\Psi_n[\varphi]$ and energy eigenvalue E_n as a power series in α:

$$\begin{aligned} \Psi_n &= \Psi_n^{(0)} + \alpha\Psi_n^{(1)} + \alpha^2\Psi_n^{(2)} + \cdots\,, \\ E_n &= E_n^{(0)} + \alpha E_n^{(1)} + \alpha^2 E_n^{(2)} + \cdots\,. \end{aligned} \tag{5.15}$$

$\Psi_n^{(0)}[\phi]$ satisfies the Schrödinger equation for the free field: $H_0\Psi_n^{(0)} = E_n^{(0)}\Psi_n^{(0)}$. Insert the expansions (5.15) into the Schrödinger equation, $H\Psi_n = E_n\Psi_n$, and collect terms for each order in α. Then take inner products with $\Psi_m[\phi]$ to get

$$\Psi_n^{(1)} = \sum_{n\neq m} \frac{\langle\Psi_m|H_{\text{int}}|\Psi_n\rangle}{E_n - E_m}\Psi_n^{(0)}\,, \tag{5.16}$$

$$E_n^{(1)} = \langle\Psi_m|H_{\text{int}}|\Psi_n\rangle\,, \tag{5.17}$$

$$E_n^{(2)} = \sum_{n\neq m} \frac{|\langle\Psi_m|H_{\text{int}}|\Psi_n\rangle|^2}{E_n - E_m}\Psi_n^{(0)}\,. \tag{5.18}$$

All expectation values are computed as functional integrals. Hatfield [36] gives a detailed explanation of perturbation theory in the Schrödinger representation, and of computations in that representation more generally.

[2]The notation $\Psi[A,t)$ denotes that Ψ is a functional of A and a function of t.

5.1.3 Hall-Reginatto ensemble formalism for fields

Hall and Reginatto (HR) have developed a formalism [32] that is well-suited to examining a composite field theory in an established spacetime. It works with ensembles on configuration space to describe interactions between classical and quantum systems or fields. Use of ensembles allows it to avoid the problems earlier attempts have encountered in coupling quantum mechanics and classical fields. Ref. [32] summarizes various earlier proposals, giving some details and references.

The HR formalism facilitates several tasks in 3+1 dimensional spacetime:

- constructing 'in principle' arguments that a composite field theory with interacting quantum and classical fields, required for classical scalar field-mediated spacetime emergence, is self-consistent;
- indicating how observable consequences can be interpreted statistically; and
- casting the theory into a form that allows using perturbation theory to perform computations.

The brief overview of selected aspects of the HR formalism below draws from Refs. [28, 29, 30, 31, 32].

Let $\{\,|A\rangle\}$ be a complete set of kets that spans the configuration space $\mathcal{A}$, where each point in the function space $\mathcal{A}$ specifies a particular spatial field configuration (now not necessarily scalar) that $A(\mathbf{x})$ can visit. (Recall $A(\mathbf{x})$ is an eigenstate of the field operator; see, *e.g.*, Eq. (5.10).) Then the field functional[3] $\Psi[A,t)$ can be interpreted as an ensemble of wave functions $\Psi(A,t)$, with one member of the ensemble for each possible field configuration $A(\mathbf{x})$.

Assuming a quantum field on configuration space has an intrinsically statistical nature, each possible field configuration $A(\mathbf{x})$ has an associated probability $P(A)$, normalized as $\int \mathcal{D}A\, P(A) = 1$ with $\mathcal{D}A$ the functional measure on the configuration space $\mathcal{A}$. In general $P(A)$ will be time dependent under Hamiltonian evolution. Thus, the ensemble wave functional $\Psi[A,t)$ has an associated probability density functional $P[A,t)$ which associates a probability $P(A,t) \in P[A,t)$ with (*i.e.*, assigns a weight to) each member $\Psi(A,t)$ of the ensemble $\Psi[A,t)$.

[3]Recall the notation (Sect. 1.6) that $F[A,t)$ indicates F is a functional of A and a function of t.

For present purposes, a chief benefit of the Schrödinger representation of QFT is that it works within a Hamiltonian formulation in configuration space. Hence we can perform a canonical transformation from (Ψ, Ψ^*) to a different pair of conjugate functionals that are better suited to the problem. Because the transformation is canonical, the equations of motion will have the same content in the new representation as in the old (Ψ, Ψ^*) representation. As suggested by the previous paragraph, a useful choice is $P[A, t)$ for one of the functionals; call its canonically conjugate functional $S[A, t)$. Thus, a key step in developing the HR formalism is to canonically transform the $(\Psi[A, t), \Psi^*[A, t))$ representation to the $(P[A, t), S[A, t))$ representation.

In classical Hamilton-Jacobi theory, S is the classical action, or Hamilton's principal function. With an appropriate decomposition analogous to the factoring of a wave functional, the Hamilton-Jacobi equation is the classical limit of the Schrödinger equation for a system (see, *e.g.*, Laudau [45], §17). Consider the classical one particle system (A.2). Insert the decomposition $\Psi = P^{1/2}\, e^{iS/\hbar}$ into the Schrödinger equation (5.7). After differentiating and separately equating the real and imaginary parts to zero, we get two differential equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\nabla^2 P^{1/2}}{P^{1/2}} = 0\,, \tag{5.19}$$

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m} \right) = 0\,. \tag{5.20}$$

In the classical limit $\hbar \to 0$, the first equation is equivalent to the Hamilton-Jacobi equation (A.3). The second equation is the continuity equation for $P = |\Psi^* \Psi|$, *i.e.*, P is a probability density which is conserved as the system evolves according to H. Appendix A further reviews classical Hamilton-Jacobi field theory.

Hence define the real-valued functionals (P, S) by the polar decomposition:

$$\langle A|\Psi\rangle = \Psi[A, t) = P^{1/2}\, e^{iS/\hbar}\,, \tag{5.21}$$

so that the Hamiltonian $\tilde{H}$ is a functional of functionals,

$$\tilde{H}[P, S] = \langle \Psi\, |\hat{H}|\Psi\rangle = \int \mathcal{D}A\, P[A, t)\, \hat{H}[P, S]\,. \tag{5.22}$$

Note that the transformation of representation from (Ψ, Ψ^*) to (P, S) also transforms the Schrödinger equation to a nonlinear differential equation. For example, Hamiltonian (5.9) becomes, in the (P, S) representation,

$$\mathcal{H} = P(\mathbf{x}) \left(\frac{(\nabla S)^2}{2m} + V(\mathbf{x}) - \frac{\hbar^2}{4m} \frac{(\nabla P)^2}{P^2} \right)\,. \tag{5.23}$$

The transformation is canonical in the sense of the Hamiltonian formalism, but it is not linear and hence is not unitary.

The interpretation of Eq. (5.22) is that $\tilde{H}$ is the Hamiltonian for an ensemble in the $|A\rangle$ basis. The evolution of $\Psi[A,t] \equiv P^{1/2}\, e^{iS/\hbar}$ is determined by the ensemble Schrödinger equation,

$$i\hbar\, \partial_t \Psi[A,t] = \tilde{H}\Psi[A,t] \quad \Leftrightarrow \quad i\hbar\, \partial_t |\Psi\rangle = \tilde{H}|\Psi\rangle\,. \tag{5.24}$$

Given Hamilton's principle $\delta\alpha = \delta \int L\, dt = 0$ with

$$\alpha = \int dt \left[-\tilde{H} + \int S\, \partial_t P \right] = -\int dt \left[\tilde{H} + \int P\, \partial_t S \right]$$

(up to a total time derivative $\frac{d}{dt}(PS)$ which leaves the equations of motion invariant) the canonical equations of motion for P and S follow for the ensemble as[4]

$$\frac{\partial P}{\partial t} = \frac{\delta \tilde{H}}{\delta S}\,, \qquad \frac{\partial S}{\partial t} = -\frac{\delta \tilde{H}}{\delta P}\,. \tag{5.25}$$

Hence, once $\tilde{H}$ is specified, $S[A,t)$ is determined by integrating the equation of motion for S.

Now consider a system of two interacting fields $\xi_{\mathcal{A}}, \xi_{\mathcal{B}}$, respectively living on configuration spaces $\mathcal{A}, \mathcal{B}$ and governed by ensemble Hamiltonians $\tilde{H}[P_A, S_A], \tilde{H}[P_B, S_B]$. The new degrees of freedom of the second field are handled by adding new coordinates to the P and S functions, just as would be done if the system were purely quantum or purely classical. For example, if $\xi_{\mathcal{A}}$ is a quantum field with P and S functionals $P[A,t)$ and $S[A,t)$, and a classical field $\xi_{\mathcal{B}}$ is added, we have $P[A,t) \to P[AB,t)$ and $S[A,t) \to S[AB,t)$.

Hence, following HR [32], the composite ensemble requires an enlarged configuration space $\mathcal{A} \times \mathcal{B}$, a joint probability density functional $P[AB,t)$, a conjugate functional $S[AB,t)$, and a composite ensemble Hamiltonian $\tilde{H}_{AB}[P,S]$ given by

$$\tilde{H}_{AB}[P,S] = \int d^3x\, P\, \Big(\tilde{H}[P_A, S_A] + \tilde{H}[P_B, S_B] + \tilde{H}_{\rm int}[P,S] \Big)\,. \tag{5.26}$$

Here $\tilde{H}_{\rm int}[P,S]$ describes the energy of interaction between the two fields; see also Eqs. (5.21) and (5.22).

Expectation values represent observables in the formalism [29]. Let q label the quantum configuration, most generally a complete set of kets $\{\,|q\rangle\}$ spanning the

[4]Note that if $F[A] = \int dx\, G(A, \nabla A)$, the functional derivative $\delta F/\delta A = \partial G/\partial A - \nabla \cdot (\partial G/\partial \nabla A)$.

configuration space $\mathcal{A}$, and x label a continuous set of coordinates on the classical configuration space $\mathcal{B}$. For a classical observable $f(x,k)$, the expectation value is given by the functional integral

$$F = \int \mathcal{D}A\,\mathcal{D}B\,P[A,B,t)\,f\big(x,\nabla_x S[A,B,t)\big)\,,$$

whereas a quantum observable $\hat{g}$ is computed as

$$G = \int \mathcal{D}B\,\langle\Psi(x)\,|\hat{g}|\Psi(x)\rangle$$

with the inner expectation value obtained similarly to the ensemble Hamiltonian (5.22).

If the interaction is turned off so that $\tilde{H}_{\rm int}[P,S]=0$, then $\xi_{\mathcal{A}}$ and $\xi_{\mathcal{B}}$ respectively live on their own subspaces of $\mathcal{A}\times\mathcal{B}$. The probability density factors as

$$\begin{aligned} P[AB,t) &= P[A,t)\cdot P[B,t)\,;\ \text{and furthermore,} \\ S[AB,t) &= S[A,t)+S[B,t)\,. \end{aligned} \tag{5.27}$$

With interactions, the system visits off-diagonal elements of $\mathcal{A}\times\mathcal{B}$ so that neither P nor S separates as in (5.27).

The following point by HR [32] clarifies the interpretation of S:

> ... for the configuration ensemble formalism to maintain full generality across the classical and quantum spectrum, no limiting interpretation should be assigned to S [...]. Thus, S will be regarded here simply as the canonical conjugate of the probability density P, with its existence being an immediate consequence of the requirement of an action principle for P.
>
> In particular, for an ensemble of classical particles [...] it will *not* be assumed that the *velocity* of a member of the ensemble at position x is a physically well-defined quantity given by $\nabla S/m$, contrary to the usual trajectory interpretation of the Hamilton-Jacobi equation. This avoids forcing a similar trajectory interpretation in the quantum and quantum-classical cases (although this remains as an option if desired, albeit attended by the types of difficulties associated with the de Broglie-Bohm approach [56]). Such an assumption is in fact *unnecessary* for classical ensembles: if an ensemble is well localized about some position x_0 at time $t=0$ (*e.g.*, via a position measurement at time $t=0$), such that $\nabla S\approx mv$ over the support of P for some constant vector $\mathbf{v}$, then the ensemble will be well localized about x_0+vt a sufficiently short time t later [...].

5.1.4 The composite free field theory with ensembles

The Hall-Reginatto formalism will now be applied to a composite field theory of the φ field (the 'classical' subsystem) coupled to a single quantum field, ψ. Generalization to any number of quantum fields is straightforward, although complicated by back-reactions on φ from quantum-quantum interactions.

To usefully employ the Hall-Reginatto formalism to the composite field theory of φ coupled to a set of interacting quantum fields:

- The stochasticity of φ must be highly suppressed so that classical Hamilton-Jacobi field theory appropriately describes a non-quantum φ. This condition should be met in an established spacetime, since dynamics dominate.

- The quantum and classical-stochastic sectors must effectively decouple from each other, so that the formalism can be restricted to the quantum sector only where spacetime can be modeled classically. Presumably this would not be true inside a black hole or near the time of the Big Bang.

The remainder of this section will focus on applying the HR formalism to the quantum sector where the requirements above are met.

To apply the HR formalism to a subsystem in the quantum sector, φ and the quantum fields must be on the same footing in a given computation. The choice of representation may vary with the problem. For example, Hall, Kumar and Reginatto [30] work in the (P, S) representation for a bosonic field calculation; on the other hand, a Schrödinger representation has already been worked out for fermionic fields[5] so that fermionic calculations may be more expedient with (Ψ, Ψ^*). As long as a canonical transformation effects the change from one Hamiltonian formulation to another, the transformation will preserve the equation of motion.

Working in the Schrödinger representation, the classical φ field functional will be denoted by $\Phi[\varphi, t)$ to easily distinguish it from the wave functional $\Psi[\psi, t)$.

First obtain (P, S) and (Φ, Φ^*) representations for the φ free field Lagrangian. From (5.6),

$$\mathcal{L}_\varphi(\varphi, \partial_\mu\varphi) = \tfrac{1}{2}\left[-\mathfrak{c}^{-2}\pi^2 + (\nabla\varphi)^2 - m^2\varphi^2\right], \tag{5.28}$$

[5] See Hatfield [36] for a computation oriented discussion of fermionic fields in the Schrödinger representation.

where $\pi = \dot{\varphi} = \partial\mathcal{L}/\partial\varphi$. The Hamiltonian density $\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}$ so that, in units where $\mathfrak{c} = 1$,

$$H = \int d^3x\, \mathcal{H}(\varphi, \pi) = \frac{1}{2}\int d^3x\, \left[\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right]. \tag{5.29}$$

The conjugate field for φ is $\pi = \delta S[\varphi, t)/\delta\varphi$, where $S = \int d^4x\, \mathcal{L}$ is Hamilton's principal function. The classical Hamilton-Jacobi equation for the φ field is $\partial S/\partial t + H = 0$:

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\int d^3x \left[\left(\frac{\delta S}{\delta\varphi}\right)^2 + (\nabla\varphi)^2 + m^2\varphi^2\right]. \tag{5.30}$$

The probability functional P is given by the continuity equation,

$$\frac{\partial P}{\partial t} = -\int d^3x\, \frac{\delta}{\delta\varphi}\left(P\frac{\delta S}{\delta\varphi}\right), \tag{5.31}$$

which is normalized as

$$\int \mathcal{D}\varphi\, P = \int \mathcal{D}\varphi\, \Phi^*\Phi = 1\,. \tag{5.32}$$

Eqs. (5.30) and (5.31) together with the normalization (5.32) define the real-valued (P, S) representation for φ. We will adopt the same interpretation of the probability functional as Holland [38] for quantum fields (with minor notation changes):

> ... at time t, $P\mathcal{D}\varphi$ is the probability for the [classical] field to lie in an element of 'volume' $\mathcal{D}\varphi$ about the configuration $\varphi(\mathbf{x})$ for all $\mathbf{x}$. The notation $\mathcal{D}\varphi$ means the infinite product $\Pi_x \mathrm{d}\varphi$ of field volume elements $\mathcal{D}\varphi$ for each value of $\mathbf{x}$.

Now obtain a single functional differential equation for complex-valued $\Phi[\varphi, t)$. Give Φ the polar decomposition (5.21), *i.e.*, $\Phi = Re^{iS}$. Partially differentiate this with respect to time and multiply by i:

$$i\,\frac{\partial\Phi}{\partial t}\Phi = \left[i\,\frac{1}{R}\frac{\partial R}{\partial t} - \frac{\partial S}{\partial t}\right]\Phi\,. \tag{5.33}$$

Obtain an expression for the first term on the right by rewriting (5.31) in terms of $R^2 \equiv P$, performing the indicated differentiations and rearranging:

$$\begin{aligned} i\,\frac{1}{R}\frac{\partial R}{\partial t}\,\Phi &= -\int d^3x \left[\frac{i}{R}\frac{\delta R}{\delta\varphi} + \frac{i}{2}\frac{\delta^2 S}{\delta\varphi^2}\right]\Phi \\ &= \frac{1}{2}\int d^3x \left[-\frac{\delta^2}{\delta\varphi^2} - \left(\frac{\delta S}{\delta\varphi}\right)^2 + \frac{1}{R}\frac{\delta^2 R}{\delta\varphi^2}\right]\Phi\,, \end{aligned} \tag{5.34}$$

where the second line follows from functionally differentiating $\delta^2\Phi/\delta\varphi^2 = \delta^2(Re^iS)/\delta\varphi^2$ and substituting for the integrand of the first line. Since $R = P^{1/2} = (\Phi^*\Phi)^{1/2}$ from

(5.32), then from Eqs. (5.33) and (5.34) we get the following functional differential equation for the (Φ, Φ^*) representation:

$$i\,\frac{\partial\Phi}{\partial t} \quad = \quad \frac{1}{2}\int d^3x\left[-\frac{\delta^2}{\delta\varphi^2}+(\nabla\varphi)^2+m^2\varphi^2+(\Phi^*\Phi)^{-1/2}\,\frac{\delta^2}{\delta\varphi^2}\,(\Phi^*\Phi)^{1/2}\right]\Phi\,. \quad (5.35)$$

Identifying $\pi^2\Phi = -\delta^2\Phi/\delta\varphi^2$, the momentum operator is

$$\pi(\mathbf{x}) = -i\frac{\delta}{\delta\varphi(\mathbf{x})}\,. \quad (5.36)$$

For a quantum field, $\pi(\mathbf{x}) = -i\hbar\,\delta/\delta\varphi(\mathbf{x})$.

This is a classical, relativistic field theory. Functionally differentiating the Hamilton-Jacobi equation (5.30) with respect to φ and using $\dot\varphi = \pi = \delta S/\delta\varphi$ leads to the massive Klein-Gordon equation for φ, as desired [38]. Note (5.35) has a form similar to the Schrödinger equation, but with an extra term:[6]

$$i\,\frac{\partial\Phi}{\partial t} = H\Phi + \left((\Phi^*\Phi)^{-1/2}\,\frac{\delta^2}{\delta\varphi^2}\,(\Phi^*\Phi)^{1/2}\right)\Phi\,. \quad (5.37)$$

5.1.5 Interactions between quantum fields and classical φ

Superficially at least, the procedure above puts the quantum fields and φ on equal footing. However, the composite theory above is uninteresting because it does not include interactions. Adding interactions among the quantum fields is straightforward, as is adding self-interaction to the classical φ. Without interactions between φ and the quantum fields, the quantum sector of the theory looks like any other QFT and the classical sector looks like a classical field theory.

Returning to the two field theory of the previous subsection, the composite state $\Psi[\psi,\varphi,t)$ factors as $|\,\Psi[\psi,t)\Phi[\varphi,t)\,\rangle$, where $\Phi(\mathbf{x})$ evolves classically to obtain the φ configuration, whereas the field operator $\psi(\mathbf{x})$ acts on Ψ but not Φ. However, this too is an uninteresting composite field theory because ψ and φ do not mix. Hence, introduce an interaction Hamiltonian for (perturbative) interactions between the φ and ψ fields, as well as a φ self interaction.

An an example, let the quantum/classical interaction be of the Yukawa form and the φ self interact via a φ^4 potential:

$$H_{\rm int} = \int d^3x\,\left(g\bar\psi\psi\varphi + \frac{\lambda}{4!}\varphi^4\right)\,, \quad (5.38)$$

[6] In de Broglie-Bohm field theory [38] this term is missing from the Schrödinger equation $i\,\frac{\partial\Phi}{\partial t} = H\Phi$ ($\hbar = 1$), but appears in the Hamilton-Jacobi equation (5.30) as the 'quantum potential'.

where λ and g are dimensionless coupling constants. The interesting question is what happens when the interaction (5.38) is treated with perturbation theory.

Unlike quantum fields the perturbation expansion for classical fields contains no loops [66]. Fig. 5.1 shows an example diagram for the self interaction of the classical φ field.

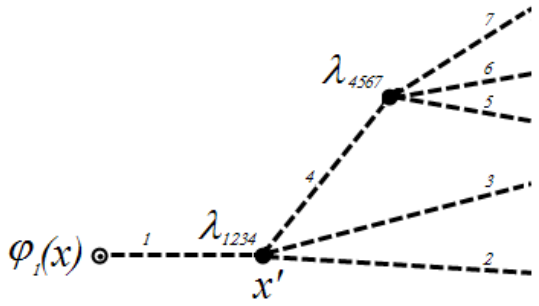


Figure 5.1: Classical $\lambda\varphi^4$ interaction.

The Yukawa interaction in (5.38) contains a leg for classical φ, allowing scattering analogous to Møller scattering; see the left tree-level diagram in Fig. 5.2. Because the interaction is mediated by the classical φ, the interactions cannot include loops like the right diagram in Fig. 5.2 — classical φ cannot contribute to the renormalization flow of the quantum fields.

One potential inconsistency introduced by interactions must be addressed. Salcedo [62] and Hall *et al.* [33] have identified an empirical inconsistency of the HR formalism that can arise if an interacting quantum/classical theory is taken to be fundamental, as it is here. Once the quantum and classical subsystems interact their configuration probability density $P[AB, t)$ no longer factorizes as in Eq. (5.27), *i.e.*, the quantum and classical subsystems become entangled. The inconsistency arises if the interaction is subsequently turned off. One would naturally demand that once the quantum and classical subsystems no longer interact, they should evolve independently. However, the nonlinear term in Eq. (5.23), proportional to $(\nabla P)^2/P^2$, causes them to remain entangled so that a change in one system can cause a change in the other, even though there is no interaction. This is obviously unacceptable if it can occur in the composite theory under consideration.

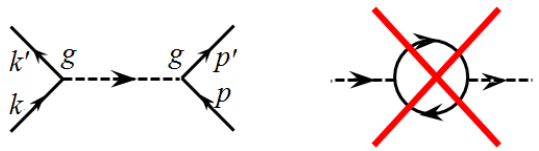


Figure 5.2: Classical Yukawa interaction, $g\bar{\psi}\psi\varphi$. Loops are not allowed.

What saves the consistency of the composite field theory in the HR formalism is that the role of φ is to mediate gravity, but gravity cannot be turned off. Hence there is no way to have $H_{\rm int} = 0$ and thereby bring about the possibility of nonlocal signalling.

For that reason, Hall *et al.* suggest that a classical gravitational field interacting with a quantum system is the only case where their formalism in its present form can apply to a fundamental theory. That is good enough for present purposes.

Gravitational radiation presents a potential challenge to the claim $H_{\rm int}$ cannot become zero after an interaction. A gravitational wave becomes causally disconnected from its source, so clearly its interaction with the quantum system vanishes. Nonetheless, this is not actually a problem. The HR formalism is being used to analyze the quantum sector of the composite field theory, but gravitational radiation rightfully is part of the classical-stochastic sector because it is a perturbation of the metric, consistent with general relativity. It represents a transfer of energy from $T_{\mu\nu}$ (and hence the quantum sector) to the CS sector, so that the associated φ motions now lie outside the domain of the HR formalism.

Hence, introducing quantum/classical interactions has not caused inconsistencies in the composite field theory. It appears justified to conclude a composite field theory of φ and a collection of quantum fields is consistent with quantum theory, as desired.

5.1.6 Role of the Hall-Reginatto formalism

The chief objective of this section, which is to argue that a composite field theory of φ and a collection $\mathcal{Q}$ of quantum fields can be consistent with quantum mechanics in 3+1 dimensional spacetime, is essentially complete. To work within the HR formalism for ensembles in configuration space, φ can be given a Schrödinger representation such that φ and the quantum fields comprising $\mathcal{Q}$ are on the same footing. Since all renormalizable quantum field theories have a Schrödinger representation, this demonstration is general.

The HR formalism imposes constraints on a composite field theory which must be respected to maintain consistency with quantum theory. In particular, measurements of observables must be attributed to an ensemble of similarly prepared quantum-classical field configurations so that quantum uncertainty transfers to φ configurations. The deterministic evolution of a particular configuration $\varphi(\mathbf{x}, t)$ contributes to the ensemble according to the probability density functional $P[\varphi, t]$ in the HR formalism.

Although the HR formalism serves an important purpose in allowing a demonstration that a composite field theory is consistent with quantum theory, it will not be

necessary to actually work within it in this thesis. This is a benefit of the decoupling of the quantum and CS sectors. In the CS sector φ can be treated as a scalar field with both classical and stochastic attributes, ignoring coupling with the quantum fields. The next section will develop a proposal whereby the particles of the quantum fields can be taken to be effectively localized in a hidden-variable sense, which will allow promoting φ to a quantum field in the quantum sector. Hence, in the quantum sector, the only region where the φ field contributes nontrivially to the stress-energy tensor will be in the vicinity of a particle, and in that region the field can be taken to be effectively quantized. Thus, assuming the necessary decoupling of the quantum and CS sectors occurs, φ and the quantum fields can be analyzed in the framework of standard QFT.

5.2 Effective Quantization of φ

The previous section treated φ as a classical field in the quantum sector of the composite field theory. Doing so allowed application of the Hall-Reginatto configuration ensemble formalism to show that a non-quantum field φ interacting with a collection of quantum fields in the quantum sector is consistent with quantum theory, even in a fundamental theory. Having demonstrated the desired consistency, it is appropriate to specify the role of φ in the quantum sector more carefully.

Some troubling issues exist with the naive picture of a classical φ field interacting with quantum fields, insofar as φ mediates gravity in the emergence picture. While there appear to be no internal inconsistencies with the hybrid picture presented thus far, ir is inelegant and has a certain arbitrariness.

1. The inability of a classical field to affect the renormalization flow of the quantum fields goes against the spirit of quantum field theory. Experience and intuition with quantum theory indicates fluctuations of of φ should occur in the quantum sector, and these should induce quantum corrections, at least in principle.

2. If φ acts classically in both the quantum and CS sectors, there is no clearly identifiable behavior that indicates which φ dynamics should be regarded as gravity-inducing behavior that belongs in $T_{\mu\nu}$ and thus lies in the quantum sector, versus implementing the gravitational field within the CS sector.

3. If φ acts classically in both sectors, there is no natural indication of how to accom-

modate energy exchange between them. Gravitational radiation is an example of such an exchange.

These theoretical issues will be taken seriously in light of a central premise of the thesis that Nature is ultimately comprehensible and elegant, not arbitrary. Specifically, the goal is to align the φ field more closely with the quantum fields, at least in the quantum sector where they necessarily interact. It appears possible to do this.

5.2.1 Origin of effective quantization

The key observation that suggests a better approach is as follows. If a particle of a ψ field can be localized to a volume S, then the coupling between the φ field and the particle will also be localized to S. As a quantum of the ψ field, the particle energy is quantized. If the φ field in S has an energy density $\mathcal{E}_{\varphi,\psi}[S]$, due to its coupling with ψ in S, is proportional to the energy of the ψ particle, then φ in S has become **effectively quantized**.[7] That is, φ effectively becomes a quantum field in S through its coupling to the quantized ψ field, but not outside S.

This observation is the seed of a program that will be partially developed in the remainder of this section and Sect. 5.4. The energy $\mathcal{E}_{\varphi,\psi}[S]$ comes from the ψ particle, so the interaction involves an energy transfer from the ψ field to φ. In its effectively quantized form, $\varphi[S]$ is part of the quantum sector because its energy can be considered a contribution from the energy of the ψ field in S. However, since the φ field is also self interacting, its motions in S due to coupling to ψ will couple nonlinearly to φ outside S, leading to an energy transfer to the surrounding vacuum where it becomes part of the CS sector. This energy transfer is both explanatory and problematic: It is explanatory in that it provides the needed mechanism for a long range gravitational field to result from local coupling to a quantum field, but it is problematic because there is no obvious way to stop the energy transfer from dissipating all energy from the ψ particle. Sect. 5.4 will propose a non-obvious resolution of the crucial issue of dissipation, one that also gives a mass to φ in S and thereby avoids introducing a new φ-mediated long range interaction (besides gravity) into the quantum sector.

Localizable particles are also indicated by classical gravitation theory. In Newtonian gravity, the gravitational potential is attributable to precisely localized masses so

[7] 'Effective' in this context means 'for all practical purposes.'

that the gravitational field can be determined exactly everywhere. In general relativity, the stress-energy tensor replaces mass as the field source, but GR is a classical field theory which in principle can be solved exactly once the $T_{\mu\nu}$ is specified everywhere on the spacetime manifold. Thus, both theories share a common assumption that the gravitating sources can be precisely localized, at least in principle.

Quantum uncertainty clearly conflicts with the notion of precisely localizable gravitating sources. For quantum fields, the problem is especially significant because there appears to be no direct analogue of the position operator of nonrelativistic QM. Various no-go theorems indicate true localization of particles is not possible in QFT, although a notion of **effective localization**[8] can make sense. Subsect. 5.2.3 discusses the theoretical difficulties; for the present it is sufficient to assume effective localization has meaning.

Effective localization is only a useful notion if the localization occurs continuously from the time a particle is created until it annihilates. The consequent continuous trajectory of the particle should be viewed as a hidden variable that is subject to all the constraints of quantum mechanics, including the fact that particles of the same species are indistinguishable. For example, particle trajectories are not classical paths that can be precisely predicted if one 'only knows' the complete state of the universe at some instant. Instead, they can be predicted in an ensemble sense according to the rules of QM. Moreover, after detection, one can say the particle followed one trajectory from an ensemble, but QM limits the ability to determine which was the actual path. The quotation at the end of Subsect. 5.1.3 is relevant here.

The notion of effective localization has some empirical support. In a relatively recent experiment [43], particles in a two-slit interferometer were partly localized by weak measurements prior to detection, where 'weak' means partial measurements that perturb the system too little to destroy quantum interference. The authors noted this implies one can operationally define a set of trajectories for an ensemble of trials.

Similarly, the interaction of a particle with the φ field can act as a form of continuous weak position measurement. The weakness of gravity means the 'measurement' will provide very little information about the trajectory of a body unless it is very massive.

Even though the precise trajectory of an effectively localized particle is un-

[8] Again, 'effective' means 'for all practical purposes.'

knowable, the assumption that every particle follows some continuous trajectory over its lifetime is important from a theoretical standpoint. It allows associated a localized energy density of φ with every particle, thereby allowing φ to be treated as a quantum field in the quantum sector with its energy density included in $T_{\mu\nu}$. A new postulate makes the assumption explicit.

Postulate 5.2.1 *Every excitation of a quantum field is an effectively localized, indistinguishable particle which, over its lifetime, follows a continuous (approximate) trajectory between interactions. The definiteness with which the trajectory can be observed or predicted is subject to the rules of quantum mechanics.*

In providing the conditions needed to promote φ to a quantum field, the postulate provides a way to address the first two theoretical difficulties of the composite field theory, mentioned at the beginning of the section.

5.2.2 φ as a quantum field

The φ field will be taken to be a scalar quantum field in the quantum sector of the composite field theory. The justification was given in the previous subsection: given a particle of a quantum field ψ, effectively localized to a region S, and continuous coupling between φ and ψ, $\varphi[S]$ inherits quantization from the ψ particle. This means φ should be promoted to a field operator on the field state functional $\Phi[\varphi, t)$ in S.

Explicitly, let the φ operator and its conjugate π have the same Fourier representation as field operators of a Klein-Gordon field. In Minkowski spacetime [59],

$$\varphi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\Big(a(\mathbf{k})\,e^{-ik\cdot x} + a^\dagger(\mathbf{k})\,e^{ik\cdot x}\Big)$$

$$\pi(\mathbf{x},t) = -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\Big(a(\mathbf{k})\,e^{-ik\cdot x} + a^\dagger(\mathbf{k})\,e^{ik\cdot x}\Big) \tag{5.39}$$

where $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$. In the i^{th} direction, $a(k^i)\,|\Phi\rangle$ obtains the amplitude of positive mode k^i of the φ field in the quantum sector; $a^\dagger(k^i)\,|\Phi\rangle$ acts similarly for negative k^i. Like any other quantum field, its eigenvalues are quantized.

Once the full Lagrangian for φ is defined, including its interactions with all the quantum fields, everything in the quantum sector can be analyzed in the QFT framework. Gravity, which is to be fully described within the CS sector, will not employ the QFT framework.

The energy density of φ that lies within the CS sector can be considered a kind of 'ground state' energy. For a true quantum field, the mean energy per mode ω is $\hbar\omega/2$. In this, φ differs from the 'pure' quantum fields — there is no reason to expect its per-mode energy is that of a quantized harmonic oscillator because φ acts mostly classical in the CS sector. This is not a problem with effective quantization of φ because the ground state energy is not an observable quantity.

To be consistent with other quantum fields, the φ field must manifest quantum fluctuations. This is straightforwardly achieved because φ participates in both the quantum and CS sectors. Chaotic motions of φ in the CS sector in the vicinity of a particle can induce transient changes in the 'quantum φ motions; this can be interpreted as a fluctuation of either positive or negative energy, where a positive energy fluctuation corresponds to a transient transfer of energy from the CS sector (φ 'ground state') to the quantum sector, and a negative energy fluctuation is the converse. Sects. 5.3 and 5.4 will examine energy exchange between the two sectors in more detail.

Energy exchange between the two sectors allows φ to run in loops in perturbation theory like other quantum fields, and thereby contribute to the renormalization flow of the other fields. Its energy can be take to be quantized as $\hbar\omega_{\text{eff}}$, with ω_{eff} defined to be $E_\varphi/\hbar$. Likewise, the effective momentum can be defined by $|\mathbf{p}_{\text{eff}}| = \sqrt{E_{\text{eff}}^2 - m^2}$. The weakness of gravity suggests that the coupling between φ and the quantum fields is weak, so quantum corrections from φ will be correspondingly small, at least at currently accessible energy scales.

A final consideration in loop-level calculations is the momentum cutoff. In the emergence picture, a physical spacetime exists at a preferred scale, *i.e.*, within an 'island' of modes, $[\mathfrak{k}_{\text{min}}, \mathfrak{k}_{\text{max}}]$, of the scale invariant stochastic regime. The $\mathfrak{k}_{\text{max}}$ mode provides an ultraviolet cutoff that cannot be breached by phenomena within the preferred scale — doing so would imply that $\mathfrak{k}_{\text{max}}$ does not really bound the preferred scale. Nonetheless, the actual cutoff should occur at a somewhat longer distance scale where spacetime is at least approximately defined. By similar reasoning, $\mathfrak{k}_{\text{min}}$ acts as a hard infrared cutoff but the actual cutoff mode is presumably greater than $\mathfrak{k}_{\text{min}}$. Since, by assumption, quantum fields only manifest within a preferred scale, emergent spacetime physically imposes momentum cutoffs that are conventionally employed by QFT regularization procedures. This is also true for the φ component of the composite field theory.

The entire discussion has assumed a Minkowski background spacetime. If

spacetime curvature is nontrivial at the assumed particle localization scale R_S, then surely quantum/classical interactions like (5.38) can no longer be approximated as point-like; the value of R_S depends on the curvature scale, and the details of the deterministic φ motions in the localization region S should be considered. Thus, in highly curved spacetimes, the quantum and CS sectors presumably will not decouple, making it necessary to generalize the functional differential equation (5.35) to curved spaces. While needed, a generalization of the composite field theory to curved spacetime will not be pursued in this thesis.

5.2.3 Localization of field quanta

It is important to ascertain whether Postulate 5.2.1, which asserts excitations of quantum fields are effectively localized particles, is consistent with quantum theory. Quantum fields are presumed to be the gravitating sources in this thesis, but classical gravitation theory assumes gravitating sources are classical particles and classical fields.

Localization of particles in QFTs is problematic. Various 'no-go' theorems show that non-interacting quantum field theories do not admit localization of quanta at all. While all realistic QFT are interacting, the issues raised are important enough to consider, especially given the necessity for particle localization.

In Einstein's equations for general relativity,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{5.40}$$

where $G_{\mu\nu}$ is the Einstein tensor and G is Newton's constant, the metric is a classical tensor field. Hence $G_{\mu\nu}$, being constructed as it is from $g_{\mu\nu}$, is a classical tensor field. If the left side of (5.40) is classical then presumably the right side is too. Hence we expect $T_{\mu\nu}$ describes a classical energy density: Particles should be localizable and have well defined momenta and energy everywhere. As Bondi puts it [13]:

> In relativity a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found.

This is in obvious tension with quantum mechanics.

One possible resolution of the apparent conflict is to adopt Rosenfeld's suggestion [61] that the right side of Einstein's equations represents the expectation value of

the quantum mechanical stress-energy tensor operator $\hat{T}_{\mu\nu}$, *i.e.*, Eq. (5.40) becomes

$$G_{\mu\nu} = 8\pi G \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle \,, \tag{5.41}$$

where $|\Psi\rangle$ denotes the complete state of the quantum fields.

Eq. (5.41) is the starting point for the program of semi-classical gravity. Working with the expectation value might eliminate the need for localizable particles with well defined momenta. Be that as it may, the program leads to significant theoretical difficulties. To quote Mattingly [51]:

> - The expectation value $\langle T_{\mu\nu} \rangle$ needs to be regularised to avoid divergences. Wald has done this, but there remains an ambiguity in its definition. Since his regularisation procedure is not scale invariant, there is a problem determining two conserved local curvature terms. The presence of a natural length scale for the theory would resolve this ambiguity, but it is not clear how to determine this scale.
> - Some solutions of the semi-classical Einstein equations are unstable. Small changes in initial conditions produce dramatically different solutions. Some solutions have runaway behavior. Thus we need a way to distinguish physically acceptable solutions from those that are not.
> - There is trouble with choosing the quantum state. "In addition," observe Butterfield and Isham, "if $|\psi_1\rangle$ and $|\psi_2\rangle$ are associated with a pair of solutions γ_1 and γ_2 to $[G_{\mu\nu} = k\langle T_{\mu\nu}\rangle]$, there is no obvious connection between γ_1 and γ_2 and any solution associated with a linear combination of $|\psi_1\rangle$ and $|\psi_2\rangle$. Thus the quantum sector of the theory has curious non-linear features, and these generate many new problems of both a technical and a conceptual nature."

Woodard [76] argues inflation implies a severe difficulty with semi-classical gravity. As he phrases it, one of the tenets of primordial inflation is:

> The structures of today's universe derived from 13.7 billion years of gravitational collapse into the tiny (one part in about 10^5) inhomogeneities provided by quantum fluctuations of the stress-energy tensor near the end of inflation.

He argues this eliminates the option of taking the source of a classical metric to be the expectation value of stress-energy tensor operator in some state, because

> If inflation is correct then the expectation value of the stress-energy tensor at the end of inflation cannot retain inhomogeneities of more than about one part in 10^{78}, otherwise they would have been so big at the beginning of inflation that gravitational collapse would have ensued.

Given these (and other) reasons, working with the expectation value of the stress-energy density evidently does not eliminate the need for localized gravitational field sources. This observation can be taken as evidence that the gravitational field must be quantized, *i.e.*, that both sides of Einstein's equations (5.40) are quantum mechanical. That viewpoint is assumed by all major quantum gravity programs, so it needs no justification here. This thesis takes a different tack: the metric and the manifold on which it lives are emergent; since the quantum fields are presumed unable to manifest without a spacetime manifold, some other field that is not quantum must mediate the manifold emergence. The non-quantum field can be identified as the φ field.

The emergence scenario requires that the field excitations interact with φ as though they are effectively localized. They need not be classically localizable, however. Hence it is important to examine why quanta of free fields are non-localizable and consider the extent to which the issue causes difficulties for interacting fields. The outcome will be that, for all practical purposes, the difficulties with localization are more of philosophical than practical interest — effects of the tails of the position distribution for a particle will very likely be swamped by randomness in $g_{\mu\nu}$ due to φ fluctuations.

After 'second quantization' (Subsect. 5.1.2) the quanta of the free fields in Minkowski spacetime are plane wave solutions of the equation of motion. This is evident, for example, in the Klein-Gordon field operator (5.39). Plane waves are in no sense localized.

As Teller and Halvorson point out [70, 34], the problem of localization is already present in nonrelativistic QM: operators with continuous spectra (like position) have no eigenvectors, so that no vector in the Hilbert space strictly localizes a particle. Although an extension of the Hilbert space (*e.g.*, a rigged Hilbert space) can allow mathematical representation of exact, continuous values, assigning a physical meaning to point-like localization of a particle is highly questionable — its momentum would be completely undetermined, implying an infinite expectation value of kinetic energy [70]. However, it can make sense to localize particles within some region, *e.g.*, of a screen after they pass through a double slit.

Some 'no-go' theorems for localizability of particles indicate there is no relativistic quantum theory of particles which could serve as a 'middle ground' between nonrelativistic particle QM and relativistic QFT:

1. *Malament [50, 35].* Under assumptions of translation covariance, energy bounded

from below, particle localizability to bounded space-like regions of Minkowski spacetime, and locality: the particle cannot be found in any bounded region. This *reductio ad absurdum* indicates conflicting assumptions, presumably localizability and causality.

2. *Hegerfeldt [37].* Under conditions of energy positivity (*i.e.*, the Hamiltonian operator is a hermitian operator which is positive or bounded from below): a particle or system, if initially completely localized to a bounded region S, will immediately develop infinite tails of its spatial distribution so that it is no longer localized to S, except in the special case where it remains in S for all times. The latter result might be a bound state, but would presumably require external potentials.

3. *Busch [15].* Under assumptions similar to #1 above, the conclusions of Malament can be generalized to 'unsharp' localization. (A sharp measurement is like a projection onto an eigenstate, as is assumed for #1 and #2; unsharp measurements involve Hilbert space 'effects' which are not necessarily orthogonal[9].) Halvorson and Clifton [35] subsequently closed a loophole in Busch's theorem.

4. *Halvorson and Clifton [35].* They introduce the assumption of a system of local number operators on bounded space-like regions, which is the minimum requirement to have a localizable particle interpretation of a quantum theory. Under this and assumptions similar to the no-go theorems above, but restricted to non-interacting quantum fields: the particle number is zero everywhere. This *reductio ad absurdum* extends the 'no particles' conclusion to (uninteresting) QFTs.

The usual conclusion is that these no-go theorems show a relativistic quantum theory cannot be a theory of localizable particles. However, as Busch notes this conclusion is not clear [15]:

> [...] the concept of a localization observable–whether sharp or unsharp–involves global elements, namely, the totality of all bounded spatial subsets of S as well as the defining requirement of translation covariance. It is thus quite conceivable that from their very operational definition, sharp or unsharp localization effects cannot be regarded as locally measurable quantities. As local measurability is a premise of the weak Einstein causality postulate (and of the objectivity requirement), this postulate and, along with it, proposition 2 would in this situation become inapplicable. Thereby

[9] See App. A of Ref. [15] for a brief introduction to unsharp measurements

> the validity of weak causality would not be affected by a violation of the commutativity condition; indeed, this condition would lose its intended meaning indicated by the phrase 'local commutativity'.
>
> [...] Up to now we have assumed that bounded spacetime regions–in which local physical operations are to be carried out–can be defined operationally solely by classical physical means. [...] sharp localization may not be an operationally meaningful concept; given that the (quantum) devices used to define spacetime regions are themselves only unsharply localizable, the concept of a local measurement–and with it that of a local observable algebra–would have to be reformulated. This would render weak and strong Einstein locality inapplicable and would call for an operationally significant notion of causality, possibly in the form of a probabilistic concept and involving reference to appropriate levels of detector sensitivities.

In the emergence picture, classical notions of smooth spacetime and (exactly) local operators are untenable. They break down entirely near the small-distance cutoff. (Even the notion of a spacetime point becomes meaningless below the emergence scale, as the manifold itself is no longer evident.)

Regardless of the 'true' status of localizable particles, we can be satisfied with a 'For All Practical Purposes' argument. We need localization and trajectories for consistency with classical gravity, but almost localized particles will do the job. Hegerfeldt's theorem, *i.e.*, instantaneous propagation of tails to infinity, remains valid even if sharp localization is not meaningful, but even so, it probably cannot be exploited for superluminal signaling [15]. From Halvorson and Clifton [35]:

> Our experience shows us that objects (particles) occupy finite regions of space. But the reply to this argument is just as simple: These experiences are illusory! Although no object is strictly localized in a bounded region of space, an object can be well-enough localized to give the appearance to us (finite observers) that it is strictly localized.

After critiquing the approach of Halvorson and Clifton, Wallace [72] also concludes a solution, where 'For All Practical Purposes' particles can be localized, is appropriate:

> From the perspective of this paper, the problem with this approach [of Halvorson and Clifton] is its *a priori* assumption that what we measure are always exactly localised operators. This is, of course, an interpretive axiom of AQFT as it is often presented, but it effectively assumes the presence of outside observers whose measurements cannot be treated within the ordinary dynamics of the QFT. We shall instead construct an account which treats observers as part of the internal dynamics of the system (although,

apart from that difference of emphasis, the solution below will be rather similar in character to that of Halvorson and Clifton).

As a general conclusion it appears taking particles to be precisely localized is problematic, but assuming them to be effectively localized appears consistent with quantum mechanics. Hence Postulate 5.2.1 is restricted to asserting effective localization — it is good enough for the purposes of this thesis.

It must be emphasized that the investigation above assumes non-interacting field theories, which do not exist in Nature. For interacting theories, QFT models the interactions as point-like, indicating localization during the interaction: Plane waves are inappropriate models of particles during interactions. For gravity, the interaction with the gravitational field is continuous, although very weak, so it is reasonable that the interaction is capable of localizing particles at least approximately. That is, a wave packet appears to be a more realistic model of a field excitation than a plane wave.

5.3 Origin of Quantum Fluctuations of φ

Sect. 5.1 assumed the composite field picture can be partitioned into quantum and classical-stochastic sectors, then focused on the quantum sector. This is a viable approach because it is assumed throughout this chapter that gravity is weak and metric fluctuations are small.

Partitioning the total system into two decoupled sectors allows analyzing each with the most appropriate or convenient set of tools, for example QFT for the quantum sector and a combination of classical and stochastic field theory for the CS sector. Each sector has its own energy budget for describing the motions of φ. This is because decoupling means the phenomena of one sector can be analyzed in that sector without regard for phenomena of the other sector; otherwise the two sectors cannot be treated as essentially independent. When a more accurate description is needed, the motions of φ in one sector can be treated as perturbations of the φ motions in the other sector.

Subsect. 5.1.1 qualitatively indicated the roles of the two sectors. By assumption or definition, the quantum sector includes essentially everything that is conventionally represented by the stress-energy tensor $T_{\mu\nu}$, *i.e.*, the contributions from the quantum fields. Thus it corresponds to the right side of Einstein's equations (5.40). The CS sector implements the structure and geometry of spacetime; the hope is this

will reproduce the left side of Einstein's equations. The unobservable zero-point motions of the quantum fields, assumed chaotic like the φ motions in the CS sector, contribute to the CS sector through their coupling to φ. The 'stochastic' part of 'classical-stochastic' comes from the stochastic regime described in Sect. 4.2, but that is of little interest here because it only manifests outside the preferred scale and has no notion of energy.

While each sector has its own energy budget, φ is single valued at every point so the physical field configuration in a given region is the same for both. That presents no obvious problem to the idea of decoupling, due to the following rationale. The phenomena of interest in the CS sector are primarily gravitational, and these, as will be considered later, relate to establishing and maintaining local equilibrium among the modes of the φ field; these modes are in chaotic motion, uncorrelated with the motions of the quantum fields in the quantum sector. (The ground state motions of the quantum fields are also assumed to be chaotic and in a kind of equilibrium.) The quantum sector contains phenomena with more discernible, persistent structure like atoms and the particle spectrum of the Standard Model, *i.e.*, non-gravitational wave-like phenomena that evolve according to quantum mechanics. Thus the evolution of the quantum sector can ignore the CS motions by simply averaging them out; their instantaneous amplitudes are small and should not substantively affect the quantum evolution. The rationale for this claim will be more clear in Subsect. 6.1.2. The evolution of the CS sector can ignore quantum phenomena by regarding them as being superposed on top of the CS motions but not interacting with them, a claim that is reasonable only because gravity interacts so weakly with matter.

The boundary between the two sectors is virtual and thus porous. Since φ is single-valued, if a fluctuation has positive energy in the CS sector, the same fluctuation will occur in the quantum sector with negative energy, *i.e.*, it corresponds to a negative frequency. In order to respect energy conservation of the total system, there must be an accounting of energy transfers across that virtual boundary. The accounting should maintain the illusion of two independent sectors.

The remainder of this section considers a model for transient energy fluctuations in the quantum sector due to the chaotic motions of the CS sector. However, the model will not be suitable for the more deterministic energy transfers to or from the CS sector which are due to the evolution of the quantum fields, like emission of gravitational radiation. That will require a toy model of a particle that can deterministically induce

or respond to φ motions in the CS sector; Sect. 5.4 will propose such a model.

Because the φ field is self interacting, its modes should reach thermal equilibrium in the CS sector, at least locally. For now, assume such an equilibrium exists in a large volume V, as would be expected for a flat spacetime. This suggests the CS sector in V can act as a heat reservoir with which the quantum sector in V can exchange energy. The reservoir temperature is interpreted as the mean kinetic energy density of the random motions of the φ field in the CS sector. Since the (classical) φ field admits a mode decomposition in V, the kinetic energy of φ in the CS sector can be attributed to the sum of the energies of the (classical) φ modes.

It is natural to consider the reservoir, *i.e.*, the CS sector in V, to be one subsystem and the quantum sector in V to be another subsystem; label the two subsystems R and S, respectively. Note that R and S have been defined to account for energy exchanges for only the φ field in V.[10] The reason why R acts as a reservoir but S does not, even though both R and S occupy the same physical volume, has to do with the relatively large energy density of φ in the CS sector that is needed to have a stable spacetime. This claim will be considered in Chap. 6; for now it will be sufficient to assume R can be treated as a reservoir so that energy exchanges with S are at most perturbative in R. Hence, if the total energy of the φ modes is constant in V then the total energy of the φ modes is effectively constant in R, although not necessarily effectively constant in S.

The porousness of the boundary between the two sectors corresponds to thermal contact between R and S. The chaotic, uncorrelated motions of the φ modes in R will naturally lead to effectively random fluctuations of energy within subvolumes of V. As noted above, a fluctuation in the CS sector implies a fluctuation of the opposite sign in the quantum sector, corresponding to a transient transfer of energy between R and S. The lack of correlation between the motions of φ in R and S allows considering the fluctuation to be random in S.

Assume the total energy of the φ modes is constant in V, and moreover assume the total number N of available modes in V is constant. Since R and S are in thermal contact while the 'particle number' N and system volume V are fixed, the setup adequately meets the conditions assumed by the canonical ensemble. Hence, the

[10] The quantum fields also fluctuate due to motions of the unexcited (ground state) modes, but that is not being modeled here, and in fact lies outside the scope of this thesis.

φ fluctuations in S can be analyzed by the machinery of standard statistical mechanics.

Of particular interest is what happens if a fluctuation occurs in a subvolume V' of V which contains an (effectively localized) ψ particle, where ψ here is a generic symbol for some 'true' quantum field. The coupling between φ and ψ causes effective quantization of φ in V', but the fluctuation in V' has its own mode decomposition which contributes to the mode decomposition for the 'quantum φ' and acts as a perturbation of it. The combined φ decomposition should therefore correspond to a different momentum than the unperturbed quantum φ. The momentum change depends completely on the mode decomposition of the fluctuation, both spatial and temporal. The transient momentum change of φ in V' will therefore obey a probability density function. The PDF is determined by the canonical ensemble.

Thus, a fluctuation can manifest as a disturbance of the quantized φ field. In this way, fluctuations in the CS sector can induce quantum corrections of the φ field in the quantum sector. In terms of a Feynman diagram the perturbation in S corresponds to momentum running inside a loop, *e.g.* like Fig. 5.3 if φ has a $\lambda\varphi^4$ interaction.

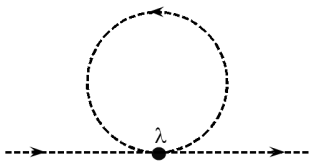


Figure 5.3

5.4 Coupling between φ and Quantum Fields

A primary role of the φ field is mediation between the quantum fields and the structure and geometry of an emergent spacetime $(\mathscr{M}, \mathbf{g})$. The φ field can only play that role if the localized particles of the quantum fields induce localized classical motions in φ. Thus, it is necessary to model the interaction between effectively localized particles of the quantum fields and the effectively quantized φ with a process that contains at least some classical elements.

The setting will be that of the rest of the chapter. That is, it is assumed a $3+1$ dimensional spacetime has already emerged, and the quantum and classical-stochastic sectors are decoupled. The field coupling will be considered in this section and a process wherein the motions of φ in the CS sector generate a spacetime will be proposed in the next chapter. In the spirit of the rest of the thesis, and given the well known problems [76] of mixing general relativity and quantum mechanics, a goal

is to minimize *a priori* constraints on candidate processes. That means nonstandard interpretations of a quantum phenomenon can be entertained if it is explanatory and does not imply conflict with observations.

Coupling between the φ and 'true' quantum fields has two primary consequences. First, it provides the means by which quantum fields induce the motions in φ in the CS sector which determine spacetime geometry. Second, it is responsible for energy transfer from the quantum sector to the CS sector. Sect. 5.3 has already considered energy transfers from the CS to quantum sectors; a goal here is to model energy transfer in the other direction.

The quantum fields, through their coupling with φ, are the sources of inhomogeneity in φ that provide the 'boundary conditions' which drive the dynamical evolution of $\varphi(\mathbf{x}, t)$. We wish to create simple models of these sources and let them act classically (deterministically) on $\varphi(\mathbf{x}, t)$. Provided the φ subsystem is analyzed separately from the quantum subsystem, this can be done by adding terms to the φ Lagrangian (5.6).

5.4.1 The model interaction

As a representative quantum field, consider a quantized Dirac field ψ in the momentum representation in Minkowski spacetime. The ψ field operator which acts on the vacuum state $|\, \Psi[\psi, t)\Phi[\varphi, t)\,\rangle$ is [59]

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{\frac{ip\cdot x}{\hbar}} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{\frac{-ip\cdot x}{\hbar}} \right), \tag{5.42}$$

where the summations are over spins, $u(p)$ and $v(p)$ are smooth functions, $E_{\mathbf{p}}$ is the energy corresponding to 3-momentum $\mathbf{p}$, and $(a, a^\dagger)$ and $(b, b^\dagger)$ are, respectively, (annihilation, creation) operators for the species of Dirac (particle, anti-particle). This is a plane wave solution of the Dirac equation, which in turn follows from the ψ Lagrangian,

$$\mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \,. \tag{5.43}$$

The modes (plane waves for a free field) $e^{\pm ip\cdot x}$ can be interpreted as completely non-localized excitations of definite momentum, but this is not an acceptable interpretation for gravitating sources, for reasons already discussed in Subsects. 5.2.1 and 5.2.3. Instead, consistent with Postulate 5.2.1, excitations of each quantum field should be considered effectively localized, indistinguishable particles with continuous

trajectories between interactions, represented as wave packets constructed as superpositions of plane wave solutions. The consistency with QFT of this interpretation was argued in Subsect. 5.2.3, although the discussion only considered Minkowski spacetime.

It is not necessary to assign a definite trajectory to point-like particles associated with wave packets; indeed, doing so would introduce difficulties that the de Broglie-Bohm approach encounters [56]). Hence, assume that trajectories are meaningful in a probabilistic sense, consistent with the Hall-Reginatto configuration ensemble formalism; see the quotation at the end of Subsect. 5.1.3.

Analogously, wave packets should continuously couple to φ along their trajectories. Given the very weak coupling strength, one can imagine this serves to perform very weak, continuous 'position measurements' on the wave packet as it propagates. Moreover, assuming the motions of φ in the CS sector determine the metric, in principle those measurements will leave an objective gravitational signature while being too weak to collapse the wave function.

The model interaction will be between φ and an effectively localized particle of a Dirac field ψ. Only massive fields will be considered here. Assume a coupling of the Yukawa form,

$$\mathcal{L}_{\text{int},\varphi} = \lambda_\psi \overline{\psi} \varphi \psi \tag{5.44}$$

with very small coupling strength λ_ψ. This coupling will be given two interpretations: quantum and classical. The φ field plays a different role in each.

5.4.2 The quantum picture

In the first interpretation, a ψ particle acts as both a source and absorber of excitations of the φ field. As a source, the ψ particle continuously induces scalar (longitudinal) waves in φ, as represented by the left diagram of Fig. 5.4; it thus continuously transfers some of its energy to φ by radiation. The right diagram of Fig. 5.4 corresponds to a ψ particle continuously absorbing energy from incoming φ radiation, the inverse process. Since both emission and absorption of radiation are continuous, in an equilibrium condition the expectation value of the ψ particle momentum will remain constant, although small fluctuations can occur instantaneously.

The φ radiation is a propagating wave solution, obtained by the φ field operator (5.39). From the ψ field operator (5.42), the ψ particle also has wave solutions. The frequency of the φ waves will be assumed equal to the frequency ω_ψ of the ψ particle that radiates them. Under the assumption of particle localization, ω_ψ actually represents a set of frequencies for a wave packet; to simplify the discussion take ω_ψ to be the central frequency of the wave packet. Furthermore, it will be assumed the outgoing radiation is coherent with the ψ particle. Requiring coherence is contrary to the usual assumption that the phase has no physical meaning, but that will not be true in this case.

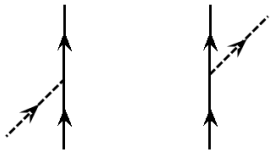


Figure 5.4

The φ radiation will not have energy $\hbar\omega_\psi$, where ω_ψ is the frequency of the ψ particle. Otherwise, the ψ particle would quickly lose all its energy to radiation. Instead, the coupling strength λ_ψ will determine the radiation energy. Since this is taken to be very small in Eq. (5.44), the radiation will contain only a small amount of energy. That is, the φ radiation inherits quantization from ψ through the coupling (5.44), but the quantization is not intrinsic to φ; hence, the energy/frequency relationship for the radiation will be determined by the coupling (5.44) which is the origin of the effective quantization.

The derivation of the φ field equation (4.38) in Sect. 4.5 indicates the φ field is not intrinsically massive. A massless scalar quantum field would have an associated long range interaction. Although gravity is a long range interaction, gravity in the emergence picture is presumed to be mediated by φ in the CS sector, not the quantum sector. That indicates a long range interaction mediated by the quantum φ, however weak, is undesirable. Hence, the quantum φ field should somehow acquire a mass in the vicinity of a particle. This issue is considered further in the next subsection.

5.4.3 The classical picture

The second interpretation of the coupling (5.44) is more classical. The effectively quantized φ must interact with the φ modes which comprise the classical-stochastic sector in order for the quantum fields to have gravitational consequences. Viewed classically, this is no problem because the propagating modes of the quantum φ field are no different than propagating modes of the classical φ field, because ultimately

both are the same field manifesting in two different ways. Quantum mechanics does not offer a similar viewpoint because it has no 'awareness' of classical fields, so it appears the classical view will be most productive. However, the classical interactions are to be considered in an ensemble sense, consistent with the probabilistic nature of QM, so it might be more correct to regard it as a quasi-classical view.

In the quantum picture, a ψ particle of 3-momentum $\mathbf{p}$ in the x_1 direction has a plane wave (packet) solution, where the wave also propagates in the x_1 direction. Likewise, the quantum φ which couples to it can also be given a plane wave solution in the x_1 direction. However, treating φ modes as plane wave solutions in the x_1 direction inadequately represents the physical situation in the non-quantum sector because the particle must gravitate in all directions, not just the x_1 direction. Plane wave solutions for the quantum φ modes would indicate that, after their partial thermalization with the φ vacuum, there is an asymmetric gravitational field of the particle that is incompatible with predictions of Newtonian gravity or GR. Likewise, the ψ particle must absorb φ modes, as in the right diagram in Fig. 5.4, preferentially in the x_1 direction; this would require a completely unmotivated postulate to obtain the behavior.

Hence it appears necessary to posit that a particle must radiate φ waves in all directions, not just the direction of momentum. However, to correctly represent the momentum density the radiated φ waves in a given direction should have a frequency given by the relativistic Doppler shift of the Compton frequency of the particle at rest. Symmetrically, the frequency of φ radiation absorbed by a particle from each direction is modified by the relativistic Doppler shift.

This proposal is compatible with the quantum φ having plane wave solutions like the ψ particle to which it couples. The 4-momentum is determined by $\hat{p} = -i\partial_\mu$ operating on the φ field, so the operator will pick out the Doppler shift in the field; the frequencies in the $+x_1$ and $-x_1$ directions will be shifted in opposite directions while the other spatial components are shifted differently, so the momentum operator can extract the 4-momentum information from the φ field. This is a somewhat classical interpretation of the momentum operator, but it shows that the quantum φ field can retain its quantum character while also producing radiation compatibly with it acting as a gravitational source.

The φ radiation from the ψ particle must interact with the φ vacuum in order to 'telegraph' the information about its position and 4-momentum to the φ vacuum, thus

determining its effect on the gravitational field. As emphasized already, the interaction must be very weak to be consistent with the weakness of gravity. A very weak Yukawa coupling λ_ψ in (5.38) constrains the rate of energy transfer between the quantum and CS sectors over short intervals. Notice this constraint applies equally to radiation and absorption of φ waves. Unlike transient random fluctuations, this energy transfer is essentially deterministic.

Now return to the issue raised in the previous subsection, that the φ field must acquire an effective mass in the vicinity of a particle to avoid the introduction of a new and unwanted long range interaction. To see the effect of an acquired mass, consider a classical analogue of the massive Klein-Gordon field. This is a flexible string with additional stiffness provided by a 'bracing' medium, *e.g.* by embedding the string in something like a thin sheet of rubber [53] (p.138). The restoring force due to the string tension is augmented by a restoring force from the bracing medium on each part of the string. If K is a constant that depends on the elastic properties of the bracing medium, T is the string tension, and ρ is the linear density of the string, then the equation of motion of the string is

$$\frac{1}{c^2}\partial_t^2\psi = \partial_x^2 - \mu^2\psi\,; \quad c^2 = \frac{T}{\rho}\,, \quad \mu^2 = \frac{K}{T}\,.$$

Its Green function is

$$G(x,\xi) = \begin{cases} \frac{1}{2\mu}\, e^{\mu(x-\xi)}\,; x < \xi \\ \frac{1}{2\mu}\, e^{\mu(\xi-x)}\,; x > \xi\,, \end{cases} \tag{5.45}$$

where the unit transverse force is steadily applied to the string at $x = \xi$. Thus, the effective mass μ causes the wave motion to decay exponentially. (The bracing medium also has the effect of increasing the natural frequencies of the wave motion and making the propagation speed frequency-dependent so that the medium becomes dispersive.)

Eq. (5.45) illustrates by analogy how even a classical field can suppress long range interactions if it has mass. While at least one promising candidate process exists whereby φ in the Lagrangian (5.6) can acquire a mass in the neighborhood of a particle, none is sufficiently developed at present to allow proposing it here. Mass acquisition requires further study.

5.4.4 Toy model for coupling to φ

Some key properties of a toy model of quantum-classical coupling for massive quantum fields, motivated in the previous two subsections, can be summarized thusly:

- The 'quantum φ' field couples to the other quantum fields via a Yukawa interaction (5.44).
- Through the coupling, the effectively localized particles of the 'true' quantum fields radiate and absorb scalar radiation in the quantum φ field continuously as they moves along their trajectories. The radiation and absorption rates are equal when the particle momentum is constant.
- The φ waves are radiated and absorbed by the particle in all directions, with the frequency in each direction given by its direction-dependent Doppler shifted Compton frequency.
- The scalar radiation interacts and partially thermalizes with the φ vacuum in an extended neighborhood; this conveys particle position and momentum information to the gravitational field.
- The 'quantum φ' field acquires a mass in the vicinity of a massive particle by some as-yet unspecified process that is at least partly classical.

Clearly there needs to be an analogous model for coupling between φ and massless particles of the gauge fields. The propagating virtual field quanta of the gauge fields, *i.e.*, off-shell disturbances in the quantum fields, provide the means for 'potential energy' to couple to φ and thus have gravitational consequences consistent with general relativity. Coupling of virtual quanta can be handled straightforwardly by assuming the disturbance represented by a virtual particle propagates along a trajectory, consistent with its Green function, and couples to φ along the trajectory as would an on-shell particle.

Chapter 6

Emergent Spacetime and Gravity

In this chapter, the assumption of Chap. 5 that a 3+1 dimensional manifold $\mathscr{M}$ has already emerged is dropped. The new objective is to obtain $\mathscr{M}$ and a metric $\mathbf{g}$ on $\mathscr{M}$ via an emergence process.

Some sections will be mostly descriptive rather than quantitative. The intent is to offer some general ideas to help complete the conceptual picture, even though the ideas have not been developed much beyond general speculations. A limited quantitative treatment may also reflect an intent to summarize in words results from earlier in the thesis or obtained by others in order to draw attention to a relevant point.

6.1 Manifold emergence

6.1.1 Origin of n spatial dimensions

The basic field equation (4.38) is a wave equation in 1+1 dimensions, but it readily generalizes to $n > 1$ spatial dimensions, as Subsect. 4.38 discussed. The primary criterion for $n > 1$ is that the spectral density of modes has directional dependence, in the sense that the spectral density in any given direction can be inferred by knowing the spectral density in no more than n different directions while at least n different directions are necessary to determine the spectral density everywhere. That is, a basis of n directions is necessary and sufficient to describe the φ motions everywhere in the space. See the discussion in Subsect. 4.3.1 and Def*. 4.3.2 in particular.

Since the φ field equation is a wave equation, its basic solutions are propagating modes (waves) of the form $e^{i(kx-\omega t)}$. There are no plane wave sources in Nature because

that would require planar sources of infinite extent. Thus, the waves will necessarily attenuate as they propagate in n spatial dimensions, ultimately having an amplitude comparable to that of the uncorrelated φ fluctuations of the vacuum. Unless there are sources of new propagating waves, the thermalized modes will eventually thermalize with the scale invariant vacuum of the stochastic regime and the preferred scale will dissipate.

Chap. 5 posited that the quantum fields are the ultimate source of propagating φ modes, through their coupling to φ. The coupling made φ an effectively-quantum field in the quantum-classical sector of the composite field theory. The propagating φ modes have a limited range due to an effective mass φ acquires in the quantum-classical sector, but φ self interaction leads to energy transfer to the classical-stochastic sector wherein the φ modes thermalize and determine the metric. Postulate 4.4.5(S1) is especially relevant in connecting the variations of φ with the physical spacetime structure.

If the assumption that the quantum fields are the only non-stochastic source of propagating φ modes is correct, then there must be a direct correspondence between the spatial dimension n and the number of independent directions in which the quantum fields couple to φ. (The value of n might come from the number of spacetime dimensions needed for internal symmetries to manifest, for example.) That is because the subsequent interactions among the propagating modes cannot introduce additional spatial degrees of freedom; φ is not an independent source of non-stochastic fluctuations.

While this last observation may be obvious, it is important because it makes the following claim more plausible. Once the initial fluctuation sets the dimension n of the precursor of spacetime in a region $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M}_\mathfrak{t}$, if that primitive initial spacetime has an approximate Poincaré symmetry that is 'good enough' that a fluctuation of a quantum field in $\mathfrak{U}_\mathfrak{t}$ can immediately manifest also in n dimensions, then if cosmogenesis subsequently occurs it will maintain the same spatial dimension n throughout the evolution. That is because the quantum field and the other quantum fields to which it couples should couple to φ in the same n dimensions, 'reinforcing' the initial n dimensional primitive spacetime of the initial fluctuation; this in turn should lead to additional quantum fluctuations in n dimensions, and so on in a recursive, self-bootstrapping fashion.

Thus, the scenario presented in the paragraph above is a proposal for the origin of spacetime dimension in the emergence picture. Fortunately, it is not necessary to postulate the scenario to make progress. The only needed ingredients are that the

quantum fields couple to φ in n independent directions, and that the quantum fields are the sole generating source of dynamically generated motions. Then the resulting thermalization of φ modes will obtain an emergent $n+1$ dimensional spacetime, at least in those cases where an initial fluctuation somehow proceeds into a cosmology rather than quickly dissipating.

6.1.2 Equipartition in the φ vacuum

The 1+1 dimensional equation of motion for φ that was derived in Sect. 4.5 is Eq. (4.38):

$$v^2 \frac{1}{\sqrt{-g}} \bar{\partial}_{\mathfrak{x}} \left(\sqrt{-g} g^{xx} \bar{\partial}_{\mathfrak{x}} \varphi(\mathfrak{x}, \mathfrak{t}) \right) - \frac{1}{\sqrt{-g}} \bar{\partial}_{\mathfrak{t}} \left(\sqrt{-g} g^{tt} \bar{\partial}_{\mathfrak{t}} \varphi(\mathfrak{x}, \mathfrak{t}) \right) = \frac{d\mathcal{V}(\varphi)}{d\varphi} + \mathcal{F}(\mathfrak{t}) \,, \quad (6.1)$$

where v is the wave propagation speed, $\mathcal{F}(\mathfrak{t})$ is a stochastic perturbation and $\mathcal{V}(\varphi)$ is the effective potential. In an established $n+1$ dimensional spacetime, $g_{\mu\nu}$ can be interpreted as the metric and the equation of motion takes the standard form,

$$\frac{1}{\sqrt{-g}} \bar{\partial}_{\mu} \left(\sqrt{-g} g^{\mu\nu} \bar{\partial}_{\nu} \varphi \right) = \frac{d\mathcal{V}(\varphi)}{d\varphi} + \mathcal{F}(t) \,. \quad (6.2)$$

The effective potential was left unspecified, but it was argued that the stochasticity of φ together with the necessity for path integration could give rise to an effective self interaction through annealed randomness, *e.g.* $\lambda\varphi^4$. Regardless, self interactions are necessary for φ to play the role as the mediator of spacetime, so if they do not arise through something like annealed randomness they must be imposed by postulate. Hence, $\mathcal{V}(\varphi)$ must include a self interaction that causes modes to mix and exchange energy (or the counterpart of energy in a pre-emergent spacetime). For the present discussion, let the effective potential have the massless form,

$$\mathcal{V}(\varphi) = \frac{\lambda}{4!}\varphi^4 \,.$$

Let U be a region in which spacetime has fully emerged. To keep the picture simple, assume an established Minkowski spacetime in U with $n = 3$ spatial dimensions. Then the classical Hamiltonian that corresponds to Eq. (6.2) in this simple scenario is

$$H = \int_V \tfrac{1}{2} \left(\pi^2 + (\nabla\varphi)^2 + \frac{\lambda}{4!}\varphi^4 \right) , \quad (6.3)$$

where interactions with the quantum fields are omitted because they are not consequential to equipartition of modes of the φ field in U.

Associate a preferred scale with U, having a range of modes $[k_{\min}, k_{\max}]$ with $\mathrm{diam}(U) = k_{\max}^{-1}$. The $\lambda\varphi^4/4!$ potential is nonlinear, ensuring interactions among all modes in $[k_{\min}, k_{\max}]$. Thus, taking the φ modes of the vacuum in U to be at equilibrium, then, since the dynamics (6.3) are classical, classical equipartition of energy will occur for the kinetic and gradient energies in U. Since $n = 3$ by assumption, each frequency $k \in [k_{\min}, k_{\max}]$ corresponds to four independent degrees of freedom (ω, k_x, k_y, k_z); each will have average energy $k_B T$. The temperature, $T = \langle \frac{1}{2}\pi^2(\omega) \rangle$, of the φ vacuum is undetermined.

The Rayleigh-Jeans law for φ in U gives the energy density in ω-space of the infinitesimal shell, radius ω and thickness $d\omega$:

$$\mathcal{E}(\omega) = \rho(\omega) = k_B T \, \frac{\omega^2}{\pi^2}$$

where k_B is Boltzmann's constant. Integrating over the entire preferred scale range $[\omega_{\min}, \omega_{\max}]$ and taking $\omega_{\min} \simeq 0$ gives the 'ground state' energy per unit volume:

$$\mathcal{E} \simeq \frac{k_B T}{3\pi^2} \, \omega_{\max}^3 = \frac{k_B T}{3\pi^2} \, k_{\max}^3 \, . \tag{6.4}$$

This is presumably a large number. Naively taking $T_{\mu\nu} \sim \mathcal{E}$ in the right side of Einstein's equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, GR predicts a large spatial curvature. However, the naive calculation is not correct in the emergence picture due to the mediating role of φ. Essentially, the φ configuration is the metric through Postulate 4.4.5(S1), and hence it is implicit in the $G_{\mu\nu}$. It is incorrect to include $\mathcal{E}$ in $T_{\mu\nu}$ because $\mathcal{E}$ is already implicit in $G_{\mu\nu}$. This means large $\mathcal{E}$ is entirely compatible with spatial flatness.

The finiteness of $\mathcal{E}$ in Eq. (6.4) has been preserved because the hard cutoffs $\omega_{\max}$, $k_{\max}$ place an upper bound on the modes which can participate in equipartition. Without the cutoff, essentially all the energy would be dumped into the high frequency end of the spectrum, and all modes would contain essentially zero energy. Although φ avoids this so-called 'ultraviolet catastrophe' due to the cutoffs, the high frequency end of the spectrum still contains most of the energy.

Now drop the assumption of an established spacetime in U and consider instead the early era of a cosmology, where a preferred scale exists and a spacetime is in the process of emerging via an inflation-like process. The quantum fields play a crucial role during this era, in that they are the sources of φ modes that interact among each other and thereby drive the exponential expansion of spacetime. Strictly speaking, the

effective potential $\mathfrak{V}(\varphi)$ in the field equation (6.1) should now include interactions with the quantum fields since those interactions are crucial to the evolution of φ.

Nonetheless, interactions between φ and the quantum fields should lead to an effective quantization of φ; see Subsect. 5.2. The subsequent thermalization of the φ modes that originate from the 'quantum φ' is what generates the emerging spacetime, not the non-thermalized modes of the effectively quantized φ. So even in the very early inflation-like era it appears reasonable to continue to ignore the interactions between the quantum fields and φ when considering equipartition of φ modes.

There is another problem with energy equipartition during the inflation-like era: the spacetime has only partly emerged, so a notion of energy is only approximate at best, and describing φ by a Hamiltonian like Eq. (6.3) is presumably invalid. Moreover, during the emergence era there is no sense in which the φ modes are in equilibrium in U, so the even if a valid Hamiltonian existed energy equipartition would not logically follow.

This problem can be addressed as follows. First, the motions of the φ field can be described entirely on the time-augmented metric space $\mathfrak{M}_\mathfrak{t}$, even in an established spacetime, since that is the context of the postulates of the field and its properties. Postulate 4.4.5 allows connecting the description on $\mathfrak{M}_\mathfrak{t}$ to a physical spacetime. Since the physics of φ cannot depend on the choice of space used to describe the field dynamics, any physical conclusions drawn on a physical spacetime $(\mathscr{M}, \mathbf{g})$ must also hold on $\mathfrak{M}_\mathfrak{t}$, and vice versa. That means that if energy equipartition occurs among modes on $(\mathscr{M}, \mathbf{g})$, an equipartition of an analogous quantity must occur on $\mathfrak{M}_\mathfrak{t}$ (but that quantity is not energy because no notion of energy exists on $\mathfrak{M}_\mathfrak{t}$). Hence, what is important is that there is something which undergoes equipartition on $\mathfrak{M}_\mathfrak{t}$ if the necessary preconditions exist, such as an equilibrium-like condition. This suggests the spectral density should be the important quantity to consider, since it can be defined with respect to either $\mathfrak{M}_\mathfrak{t}$ or $(\mathscr{M}, \mathbf{g})$. (The spectral density for the same field configuration, expressed in terms of measures on $\mathfrak{M}_\mathfrak{t}$ and $(\mathscr{M}, \mathbf{g})$, may not be equal, but they should be related by a transformation between the two spaces.)

Second, the lack of anything remotely like equilibrium means equipartition will not occur, at least during the early stages of the inflation-like era. Nonetheless, φ remains a self interacting field, so 'energy' can still be exchanged between modes and thermalization can still proceed in a limited sense. That is good enough. Presumably, as

the emergence process proceeds, the ongoing redistribution of 'energy' will more closely approximate true thermalization, causing a gradual approach to equilibrium and true equipartition.

6.1.3 Lorentz symmetry

Let $U \subset \mathfrak{M}_\mathfrak{t}$ be a region in which spacetime has fully emerged, such that the φ vacuum is completely thermalized, homogeneous, and the spectral density is the same in all directions (isotropy). The task is to show that if the residual variation of φ (Def*.4.4.2) as assumed, Lorentz symmetry will exist in U. The goal is not to derive special relativity from a different set of postulates than is conventional; it is to show that the Lorentz symmetry follows automatically from the emergence picture without being put in 'by hand.'

The analysis will be with respect to the distance metric $\mathfrak{d}$ and cosmic time. Due to the assumed homogeneity and isotropy in U, rotational invariance in the emergent space in the interior of U is automatically satisfied. Thus, obtaining Lorentz symmetry is a matter of demonstrating boost invariance; the 2+1 dimensional case will be sufficient.

Start with the residual variation (Def*. 4.4.2) along a curve $\mathfrak{c} \subset U$ in position space,

$$V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c}) = \mathfrak{V}_\varphi(\mathfrak{c}) - \mathfrak{V}_{\bar{\varphi}}(\mathfrak{c})\big|_{\mathfrak{k}>\mathfrak{k}_{\max}} . \tag{6.5}$$

This is finite because it is attributable solely to the finite range of preferred scale modes, $\mathfrak{k}_{\min} \leq \mathfrak{k} \leq \mathfrak{k}_{\max}$; contributions by $\mathfrak{k} > \mathfrak{k}_{\max}$ which are the source of the infinite variation of φ have been subtracted. Let $\mathfrak{c}$ be a geodesic of the emergent spacetime which spans U. Assume $\mathfrak{c}$ is also a geodesic of $\mathfrak{d}$ because the φ vacuum is homogeneous and isotropic in U; to simplify the setting without losing generality, intrinsic uncertainty in the path (image) of the curve on $\mathfrak{M}$ will be ignored.

Then $V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c})$ and $V_\varphi^{(\omega_{\max})}(\mathfrak{c})$ can be represented by a sum of Fourier modes,

$$\begin{aligned} V_\varphi^{(\mathfrak{k}_{\max})}(\mathfrak{c}) &= \int_{-k_{\max}}^{k_{\max}} \frac{dk}{2\pi} \left[\tilde{a}(k)\, e^{i(kx-\omega t)} + \text{c.c.}\right], \\ V_\varphi^{(\omega_{\max})}(\mathfrak{c}) &= \int_{-\underset{\sim}{\omega}_{\max}}^{\underset{\sim}{\omega}_{\max}} \frac{d\omega}{2\pi} \left[\tilde{a}(\omega)\, e^{i(kx-\omega t)} + \text{c.c.}\right], \end{aligned} \tag{6.6}$$

where $k_{\max}$ and $\underset{\sim}{\omega}_{\max}$ are hard, physically motivated effective cutoffs. They are effective cutoffs in that $k_{\max} < \mathfrak{k}_{\max}$, and similarly $\underset{\sim}{\omega}_{\max} < \omega_{\max}$, because the short-distance (and

long-distance) transition between the dynamic and stochastic regimes is not sharp. The infrared cutoffs $k_{\min} = \mathrm{diam}(U)^{-1}$ and $\underset{\sim}{\omega}_{\min}$ are taken to be zero to simplify the integrals. Having distinguished ω from $\underset{\sim}{\omega}$, henceforth the accent on ω will be dropped.

The modes appearing in the integrals are propagating modes which are solutions of the φ equation of motion. This contrasts with the stochastic regime where modes do not propagate because the equation of motion does not apply.

From Subsect. 6.1.2, equipartition will occur among all modes in $[k_{\min}, k_{\max}]$ and $[\omega_{\min}, \omega_{\max}]$ in U. The average energy per mode is $k_B T = \text{const}$; hence the average energy density per mode is also constant. Call this constant a_0^2; it represents the average kinetic energy density $\frac{1}{2}\langle \pi^2 \rangle$ of each mode ω and the average gradient energy density $\frac{1}{2}\langle (\nabla\varphi)^2 \rangle$ of each component i of each mode k.

From Postulate 4.4.5(S1), the propagation speed c of waves at a preferred scale varies inversely with the residual variation, *i.e.*,

$$c = \frac{v_0'}{V_\varphi^{(\mathfrak{t}_{\max})}(\mathfrak{c})}\,, \qquad v_0' \propto v_0\,. \tag{6.7}$$

(The postulate is stated in terms of the relative variation $\mathfrak{n}$, which is why c is proportional to $v_0/V_\varphi^{(\mathfrak{t}_{\max})}(\mathfrak{c})$ rather than equal to it; the residual and relative variations are related by Def*. 4.4.3.) For the φ vacuum, at equilibrium in U, the residual variation should be the same effective constant along all curves $\mathfrak{c} \subset U$ so that c is constant in U.

Within this setup, a box will be given a relative velocity $\mathbf{v}$ within a 'lab' frame as in Fig. 6.1. The idea is to deduce the change in residual variation of the vacuum which an observer in the box would experience due to Doppler shifting of propagating modes $\sin(k_\pm x - \omega_\pm t)$ entering the box. The non-relativistic form of the Doppler shift must be used because special relativistic effects on dy and dt are to be determined, not assumed. The nonrelativistic form assumes waves propagate through a medium, where the wave source, observer or both may be in motion relative to the medium. The medium in the present context is the relative variation due to the sum of all modes in $[k_{\min}, k_{\max}]$ and $[\omega_{\min}, \omega_{\max}]$, since that determines the propagation speed. Given the dependence of c on $V_\varphi^{(\mathfrak{t}_{\max})}$ the change in measures of distance, dx and dy, and time, dt, can be determined.

A sample computation will only consider the effect of the Doppler shift on the spatial modes, $k_\pm$, but similar reasoning applies to temporal frequencies $\omega_\pm$. The computation depends crucially on whether the observer is taken to be at rest in the

medium or moving relatively to it, so it is necessary to be careful in correctly choosing the setting. There is no single frame which can be declared to be 'at rest' with respect to the medium because there is no frame which acts as the source for all modes — all modes interact and undergo equipartition independently of any notion of a reference frame. That is because, ultimately, all propagating modes live on top of the stochastic regime at distance scales $\mathfrak{k}^{-1} < \mathfrak{k}_{\rm max}^{-1}$, but the stochastic regime contains no notions of either space or time.

Hence, there is no natural choice of rest frame for the medium; the notion of such a frame does not even make sense. Nevertheless, Doppler shift of propagating modes clearly occurs because a particular mode k_- in the lab frame that enters the top of the box in Fig. 6.1 clearly has a different frequency with respect to the box when $v > 0$ than it does when $v = 0$, computed within the lab frame.

One option for the medium's rest frame assumes the lab frame and the box are both moving relatively to the medium. That is, the relative velocity between the box and lab frame is known but their speeds with respect to the medium are undefined. However, this will not work because a Doppler shift computation requires that motion with respect to the medium is at least a meaningful notion, even if the particular relative speed is unknown. The only other option is to declare that the lab frame and frame of the box are both at rest with respect to the medium. Then it does not matter which frame is chosen to do the calculations as long as the entire calculation respects that choice. Since this option is apparently the only possibility that makes sense, it will be assumed henceforth.

Given the observer is stationary in the vacuum, take $k_\pm$ to be generated by sources moving at speed v with respect to the vacuum. Let k_+, k_- in the lab frame be such that their Doppler shift makes them the same k in the observer's frame:

$$k = \frac{k_\pm}{1 \pm v/c}\,.$$

The shift is in the y direction only because v has no x component.

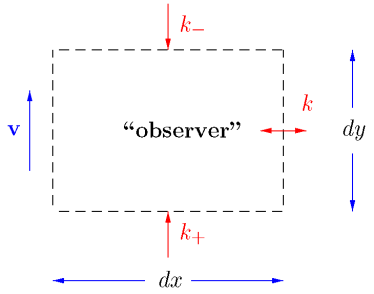


Figure 6.1

In the lab frame, the average amplitude (or weight) of mode k after equipar-

tition is $\tilde{a}(k) = \langle |a(k)|^2 \rangle^{1/2} = a_0/k$ and $k\tilde{a}(k) = k_\pm \tilde{a}(k_\pm)$. For $v > 0$, $k_\pm \to k$ due to Doppler shift (all frequencies are measured by 'lab' frame ruler), *e.g.* for k_+ :

$$k\tilde{a}(k) = k_+\tilde{a}(k_+) \quad \xrightarrow{\text{Doppler shift}} \quad k\,\tilde{a}(k_+) = \frac{k_+}{1+v/c}\tilde{a}(k_+) = \frac{a_0}{1+v/c}\,. \tag{6.8}$$

In the lab frame, mode k leaves the top weighted as $\tilde{a}(k)$ while k_-, weighted as $\tilde{a}(k_-)$, enters through the top. For the bottom, k leaves with weight $\tilde{a}(k)$ while k_+ enters with weight $\tilde{a}(k_+)$. Average weight $\tilde{a}'(k)$ (entering) replaces average weight $\tilde{a}(k)$ (leaving):

$$k\,\tilde{a}'(k) = \tfrac{1}{2}\,a_0\left[(1+v/c)^{-1} + (1-v/c)^{-1}\right] = a_0(1-v^2/c^2)^{-1}\,. \tag{6.9}$$

Thus, from the viewpoint of the lab frame, the entire spectrum is shifted to higher frequencies by a factor $\gamma^2 = (1-v^2/c^2)^{-1}$. In terms of a sum of modes, $V_\varphi^{(k_{\max})}(\mathfrak{c})$ is

$$\int_{-k_{\max}}^{k_{\max}} \frac{dk}{2\pi}\left[\tilde{a}(k)\,e^{i(kx-\omega t)} + \text{c.c.}\right] \quad \xrightarrow{\text{Doppler shift}} \quad \gamma^2 \int_{-k_{\max}}^{k_{\max}} \frac{dk}{2\pi}\left[\tilde{a}(k)\,e^{i(kx-\omega t)} + \text{c.c.}\right]. \tag{6.10}$$

From (6.7), $c \propto 1/V_\varphi^{(k_{\max})}(\cdot)$. The conclusion within the lab frame is that processes in the box are subject to a propagation speed $c' = c/\gamma^2$, necessarily constant because v and c are constant in the box as measured within the lab frame. Hence measures of length and time for the moving (primed) frame are

$$dy' = \alpha dy,\ dt' = \beta dt \text{ such that } \frac{dy'}{dt'} = c' = \frac{\alpha}{\beta}c$$
$$\therefore\ \frac{\alpha}{\beta} = \gamma^{-2}\,. \tag{6.11}$$

Now look at the transverse direction where $V_\varphi^{(k_{\max})}(\cdot)$ is unchanged, so $dx' = dx$.

In the moving frame $v' = 0$. Because the observer's reference frame in the moving box is taken to at rest in the medium, the speed of light is isotropic in that frame, just as it is in the lab frame. Thus, from Fig. 6.2, $\sqrt{c'^2 - v'^2} = c'$ and

$$dx' = c'dt' = dx = \sqrt{c^2 - v^2}\,dt\,,$$

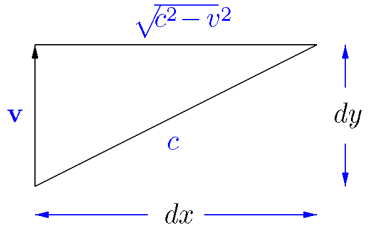


Figure 6.2

where dt is the time required to traverse dx, measured in the lab frame. Hence

$$\frac{dx'}{dx} = 1 = \frac{c'}{c\sqrt{1 - v^2/c^2}} \frac{dt'}{dt} = \frac{dt'}{dt} \gamma^{-1}$$

which, using (6.11), leads to the expected transformation equations for boosts:

$$\begin{cases} dt' = \beta dt = \gamma dt \\ dx' = dx \\ dy' = \alpha dy = (\beta\gamma^{-2}) dy = \gamma^{-1} dy \,. \end{cases} \tag{6.12}$$

From these and isotropy, the Lorentz transformations and other properties of Lorentz invariance in physical systems follow. Little would be gained repeating here what is already available in standard texts on special relativity.

The transformed measures (6.12) are **non-dispersive**. That is, the transformations are insensitive to the mode frequency. This indicates there should be no scale-dependent violation of Lorentz invariance over the entire range of distance and time scales in U, $[k_{\rm max}^{-1}, k_{\rm min}^{-1}]$ and $[\omega_{\rm max}^{-1}, \omega_{\rm min}^{-1}]$ respectively, provided equipartition holds over the entire range of scales.

The result (6.12) will generalize to the number of independent directions, or spatial dimensions, in which equipartition occurs. That and homogeneity and isotropy are the properties of the vacuum that are needed for boost symmetry in the reasoning above.

6.1.4 Metric signature

Having proposed that the product topology of spacetime has a dynamical origin, the next problem is to explore what determines the spacetime dimension, $d = n+m$ where m is the number of time dimensions. The number of spatial dimensions was the subject of Subsect. 4.3.1; assume n is an integer. Here, the task is to determine the metric signature for an emergent spacetime, *i.e.*, m.

Tegmark [69] argues for $m = 1$ (and also $n = 3$) on 'anthropic' grounds. When $m > 1$, namely the ultra-hyperbolic case for Eq. (6.13), there exists no spacelike hypersurface on which initial condition data can be specified, so the initial value problem is ill-posed; specifying the initial conditions on a small subset of the spacetime determines the solution throughout spacetime, so that consistency with observation requires

specification of that data with infinite precision. Fig. 6.3 summarizes this anthropic argument for preferring $(n, m) = (3, 1)$.

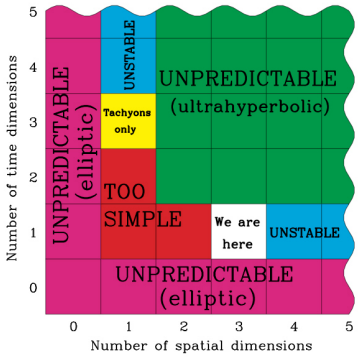


Figure 6.3: Properties of $n + m$-dimensional spacetimes [75] (reproduced under Creative Commons license).

An anthropic argument is of limited use, however, because it is phenomenological: it can indicate what the metric signature should be to allow intelligent observers to exist, but says nothing about the physical origin of that signature. On the other hand, given the premise that spacetime emerges dynamically, it should be possible to infer the metric signature from the φ dynamics.

Consider a general, linear second order partial differential equation in $\mathbb{R}^d$:

$$\left(A_{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} + B_\mu \frac{\partial}{\partial x_\mu} + C \right) \varphi(x) = 0 \,, \tag{6.13}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ and the constant C are all differentiable with respect to the d coordinates. In a fully emerged spacetime where (6.13) describes the covariant partial differential equations of physics, *e.g.* the wave equation, $\Box\varphi = 0$, or diffusion equation, $\left(\nabla^2 - \partial_t\right) \varphi = 0$, $\mathbf{A}$ will have the same eigenvalues and hence the same signature as the metric tensor $\mathbf{g}$ [69].

The dynamics of φ on an emergent spacetime $(\mathscr{M}, \mathbf{g})$ must have representations on both $(\mathscr{M}, \mathbf{g})$ and $\mathfrak{M}_\mathfrak{t}$ because both spaces contain the same physical situation but embody different viewpoints. Consistency requires the same number of time dimensions in both representations, so consider the field equation for φ on $\mathfrak{M}_\mathfrak{t}$. This is well defined along curves, *i.e.*, in 1+1 dimensions. Rewrite the left side of Eq. (6.2), taking $\mathcal{V}'(\varphi) + \mathcal{F}(t) = 0$ and $\sqrt{-g} = 1$ without loss of generality:

$$0 \;=\; \left(A^{\mu\mu}\, \bar{\partial}_\mu^2 + B_\mu\, \bar{\partial}_\mu + C(\varphi) \right) \varphi(\mathfrak{x}, \mathfrak{t}) = \left(A_1\, \bar{\partial}_\mathfrak{x}^2 - A_0\, \bar{\partial}_\mathfrak{t}^2 \right) \varphi(\mathfrak{x}, \mathfrak{t}) \,, \tag{6.14}$$

where $\mu \in \{0, 1\}$. Comparing Eqs. (6.13) and (6.14), $\mathbf{A} \sim \mathrm{diag}(-1, 1)$.

Now consider the same field equation in the $\mathbb{R}^d$ representation. The sign of the eigenvalues of $\mathbf{A}$ must be the same because the physical situation is the same, although their magnitudes will in general be different. Since the eigenvalues of $\mathbf{A}$ and $\mathbf{g}$ are the same, the metric signature is $\mathrm{diag}(-1, 1)$. In more than 1+1 spacetime dimensions, n can change without affecting the content of (6.14), but m cannot. Hence, $m = 1$ so that the metric signature[1] that is consistent with the φ field equation is, in $\mathbb{R}^d$,

$$g_{\mu\nu} = \mathrm{diag}(g_{00}, g_{11}, \ldots, g_{nn}) \;\sim\; (-+\cdots+)\,. \tag{6.15}$$

This reasoning shows $\mathbf{g}$ inherits its signature from the φ equation of motion, which in turn is derived from the elementary dynamics of Postulate 2.2.11(D2).

6.2 Reproducing General Relativity

This section considers the extent to which the emergence picture can reproduce general relativity. It starts with a very brief overview of how the metric is determined in general relativity, followed by a discussion of how the emergence picture reproduces the correspondence between energy density and spacetime geometry.

6.2.1 Computing $g_{\mu\nu}$ in general relativity

In general relativity spacetime is modeled by a pseudo-Riemannian manifold $(\mathscr{M}, \mathbf{g})$, specifically a Lorentzian manifold which, in the convention of this thesis, has signature $(-++\cdots+)$. One starts with Einstein's equations on the pre-existent smooth manifold $\mathscr{M}$ in $n+1$ spatial dimensions:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}\,, \tag{6.16}$$

where G is Newton's gravitational constant and the cosmological constant is omitted for simplicity. The Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}\,, \tag{6.17}$$

where $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ is the Ricci tensor, $R = R^{\mu}{}_{\mu}$ is the Ricci scalar, and the Riemann curvature tensor $R^{\rho}_{\sigma\mu\nu}$ is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{6.18}$$

[1] This assumes the sign convention (6.15) specified in the Introduction.

The Levi-Civita connection $\Gamma^{\sigma}_{\mu\nu}$ is given in terms of the metric by

$$\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu} \right) . \tag{6.19}$$

The energy/momentum density in $T_{\mu\nu}$ determines the metric:

$$R_{\mu\nu} - \tfrac{1}{2} R \, g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

Correctly specifying the energy-momentum density via $T_{\mu\nu}$ everywhere on $\mathscr{M}$ implicitly requires a metric. If the metric is not known, perturbation theory can be used in many cases, using the Minkowski metric to initially specify $T_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is the perturbation. Finally, one solves (6.16) for the (up to) ten independent components of the dynamical tensor field $g_{\mu\nu}$, which is the spacetime metric on $\mathscr{M}$.

6.2.2 More general emergent metrics than $\eta_{\mu\nu}$

The derivation of Lorentz invariance in Subsect. 6.1.3 assumes the φ vacuum is completely thermalized, homogeneous, and has an isotropic spectral density which results from equipartition. In terms of a metric with n spatial dimensions, the transformations (6.12) plus isotropy imply a Minkowski metric, $\eta_{\mu\nu}$.

Importantly, the derivation is purely a local one, *i.e.*, the assumed conditions only need to hold in the vicinity of the infinitesimal box in Fig. 6.1. This means the Lorentz invariance can be taken to hold locally in a neighborhood where the necessary conditions are present. If the spectral density for φ waves is inhomogeneous or anisotropic over an extended region S, local Lorentz invariance will still be obtained provided the φ modes have thermalized and the required conditions of homogeneity and isotropy hold locally within S. The relative variation $\mathfrak{n}(x,t)$, defined by Eq. (4.14) in terms of the residual variation $V_{\varphi}^{(k_{\max})}(\mathfrak{x},\mathfrak{t})$, quantifies local departures from homogeneity in S.

It is helpful to frame this idea in terms of the ideas of the previous chapter. Effectively localized particles of quantum fields couple locally to φ in the quantum-classical sector, and as a result φ effectively becomes a quantum field in that sector. The modes of the quantum φ arise due to the coupling between φ and the particles, not from the internal motions of φ itself; this is clear from the toy model for the coupling,

summarized in Subsect. 5.4.4. However, the φ thermal 'vacuum' is essentially classical, not effectively quantized, and the thermal vacuum is what generates the physical metric. Since φ is a single, self interacting field which acquires quantum or non-quantum characteristics depending on the setting, its self interactions inevitably lead to partial thermalization of the quantum φ modes in the vicinity of the particle, and that leads to local inhomogeneities of the φ vacuum. Moreover, the energy density of the quantum φ modes is proportional to the energy of the particle of the quantum field which is acting as the source, so that after partial thermalization the energy density of the thermal vacuum in the vicinity of the particle increases with the energy of the source (particle). Since the relative variation in the vicinity of the particle, after partial thermalization, also increases in proportion to the energy (characteristic frequency) of the particle, spacetime curvature in the vicinity of the particle should increase with increasing particle energy.

Thus, as long as a local equilibrium is achieved between the quantum φ and the thermal φ in the vicinity of a particle, the thermalization and equipartition will occur locally in the same way as it does for the large-scale vacuum initially assumed in Subsect. 6.1.3. The required equilibrium is such that energy exchange between the quantum φ and thermal φ can continually occur, but the net transfer from one sector to the other is zero. The relative variation will then vary in the same way as the energy density, so that an inhomogeneous energy distribution will obtain an inhomogeneous spacetime geometry.

To make this explicit, rewrite the measures (6.12) for flat space in terms of the relative variation, $\mathfrak{n}(x,t)$, taking the reference curve $\mathfrak{c}_{\text{ref}}$ to be fixed in the lab frame. Since $c' = c/\mathfrak{n}(x,t)$, then from (6.12),

$$\begin{cases} dt' = dt \cdot \mathfrak{n}^{1/2} \\ dx' = dx \\ dy' = dy \cdot \mathfrak{n}^{-1/2}\,. \end{cases} \tag{6.20}$$

The picture not only reproduces the basic idea that matter introduces curvature into spacetime, it also reproduces the crucial aspect of GR that gravity itself acts as a source of gravitation. That is because the propagating φ modes which comprise the φ vacuum have an associated propagation speed that is determined by the relative variation in the same way as the propagating modes of the quantum fields. In a region S where the relative variation increases from the ambient, φ modes propagating in S will

slow, and this in turn will cause the relative variation to increase above what it would be if the modes passed through S without slowing. The increased relative variation then causes further slowing of modes in S until the nonlinear process reaches a 'fixed point' where the relative variation in S achieves an equilibrium value. Idealizing the propagating modes as plane waves, the spatial inhomogeneity of the relative variation in S will lead to deflection of the modes, analogous to how light passing through a material with a varying index of refraction will deflect. This can lead to mixing of different spatial components in the resulting metric.

This picture is completely consistent with the assertion by Einstein's equations that the stress-energy tensor determines the spacetime geometry. Moreover, the picture offers an explanation for why the energy density determines geometry — it indicates an underlying process by which the geometry is determined.

6.2.3 Computing $g_{\mu\nu}$ in the emergence picture

In the emergence picture that has been developed in this thesis, the process of determining the metric is conceptually different than solving Einstein's equations. The corresponding procedure, in which an $n+1$ dimensional manifold $\mathscr{M}$ has already emerged in a region $\mathfrak{U}_\mathfrak{t} \subset \mathfrak{M} \times \mathbb{R}$, can use either of two approaches. Each of the two approaches must obtain the same final result because the physical situation is the same.

The first approach starts by specifying the energy-momentum density everywhere on $\mathfrak{U}_\mathfrak{t}$, and then computing the residual variation in terms of a homogeneous metric $\mathfrak{d}$ (Def*. 2.1.15) and cosmic time, then solving the φ equation of motion (6.2) and taking ensemble averages to arrive at the residual variation. This approach is fine for conceptually developing in-principle arguments, and it has the advantage of not requiring an emergent manifold, but it is probably impractical unless there is a very high degree of symmetry that allows reducing the problem to 1+1 dimensions.

The other, more practical approach starts by specifying the energy-momentum density on a $\mathbb{R}^{n,1}$ background space, then uses the φ equation of motion in $n+1$ dimensions to solve for the relative variation (Def*. 4.4.3). The speed of light, measured with respect to Euclidean measures of space and time, now can vary everywhere according to the relative variation; see Eq. (4.37). Where the relative variation is large, nonlinearity will exist: The relative variation in a region S depends on the speed at which modes propagate in and out of S, and if the relative variation increases in S then the

mean mode amplitudes will also increase in S, leading to a further increase in relative variation in S, and so on. Because the matter content is completely specified on a flat space and the relative variation is then computed on the same space, the field may evolve nonlinearly but it should be well defined at all times, even for arbitrarily large energy-momentum densities.

In the second approach, once the relative variation is known everywhere, the propagation speed (6.7) is also known everywhere in terms of Euclidean measures of space and time. By changing the point of view so that the propagation speed is constant everywhere but the measures of space and time are what vary, the metric can be computed everywhere from the relative variation on $\mathbb{R}^{n,1}$.

Finally, once the metric is known everywhere the Einstein tensor can be computed directly from Eqs. (6.19), (6.18) and (6.17). This indicates a direct correspondence between the spacetime geometry from Einstein's equations and the geometry computed from the relative variation in the emergence picture. However, that does not guarantee the metric computed from the same $T_{\mu\nu}$ in the two pictures will be proportional to each other. That is an important question that requires additional study.

6.2.4 Manifestation of matter fields

General relativity establishes a close connection between matter, or more precisely energy and momentum density, and spacetime geometry. However, GR requires a preexistent pseudo-Riemannian manifold on which the physical spacetime metric $\mathbf{g}$ and stress-energy tensor $\mathbf{T}$ can be defined. A smooth manifold contains somewhat more structure than the time-supplemented metric space $\mathfrak{M}_\mathfrak{t} \equiv \mathfrak{M} \times \mathbb{R}$. Namely, lacking a notion of spatial dimension, $\mathfrak{M}_\mathfrak{t}$ is not a manifold, much less an inner product space. Moreover, GR assumes a well defined local energy density, whereas energy and momentum density are emergent notions in the emergence picture.

In one significant respect, GR is incompatible with the emergence picture in this thesis: Stable solutions of Einstein's equations exist for universes that contain little or no matter at all, such as a Kasner metric [52]. Matter-free solutions are highly problematic in the emergence picture because, while gravitational energy alone could induce curvature as it can in GR (see the latter part of Subsect. 6.2.2), without matter there appears to be no means for generating the spacetime manifold on which a spacetime metric could live. By asserting a prior manifold, GR 'surreptitiously' implies the

existence of such vacuum solutions and skirts the burden of justifying their existence or physical sensibility.

While GR describes how the energy density determines spacetime geometry, its description of how spacetime geometry affects matter is restricted to determining the motions of matter. GR has nothing to say about whether spacetime geometry affects the existence or stability of elementary particles.

6.3 Cosmogenesis and Inflation-like Era

Sufficient groundwork has been laid that an inflation-like process, wherein a spacetime manifold and metric emerge and the quantum fields manifest, can be outlined in a little more detail than in Sect. 1.4. It appears that any spacetime emergence scenario that relies only the elements that have been developed in this thesis will contain the basic features of the process. The discussion will be somewhat general.

In the emergence picture, the equation of motion (6.1) indicates that the physical metric $\mathbf{g}$ should be describable in terms of the relative variation $\mathfrak{n}(\mathfrak{x}, \mathfrak{t})$, Eq. (4.14). Local curvature should correspond to local variations in $\mathfrak{n}$. Obtaining a realistic manifold on which $\mathbf{g}$ lives is the harder part: The relatively small number of postulates and the very limited prior structure they assume significantly limits the possible processes for manifold emergence.[2] Subsect. 4.3.1 took some initial steps by proposing in broad terms how a fluctuation could transiently obtain a product space where none previously existed. A transient product space is a necessary but insufficient condition for cosmogenesis, however.

6.3.1 Initial fluctuation

Subsects. 4.3.1 and 6.1.1 discuss the idea of an initial fluctuation that produces a transient preferred scale. If a fluctuation is to lead to cosmogenesis, there must be mode sources, *i.e.*, manifestation of quantum fields, to amplify the initial fluctuation and lead to its growth. Hence consider a highly improbable fluctuation that produces a φ field configuration with approximate Lorentz symmetry in n spatial dimensions in some finite region $\mathfrak{U}_\mathfrak{t}$ of $\mathfrak{M}_\mathfrak{t}$.

[2] An optimist may regard the paucity of candidate processes as a benefit because it reduces the range of possible models that must be considered.

A field configuration with approximate Lorentz symmetry can come about through a combination of dynamics and randomness, not just randomness. The discussion of Subsect. 6.1.3 shows that Lorentz symmetry corresponds to equipartition in a fully thermalized condition. Since complete thermalization is an equilibrium condition and hence corresponds to maximum entropy in an established spacetime, presumably it is far more likely that a random fluctuation will produce a configuration with approximate Lorentz symmetry than one like Fig. 6.4 which contains a small number of modes. That is, a fluctuation will almost certainly be a wave packet.

As a wave equation, the equation of motion will act on each mode of the fluctuation. Letting Fig. 6.4 represent just two constituent modes of a packet, it is clear that each mode is a solution to a wave equation, and thus will propagate. For a fluctuation in n spatial dimensions, the waves will thermalize in all n dimensions due to the self interaction potential in the equation of motion, so it is reasonable that until the fluctuation dissipates there can be approximate equipartition sufficiently deep in the interior of $\mathfrak{U}_t$. However, the lifetime of the fluctuation is presumably very short: the exterior of $\mathfrak{U}_t$ is governed by the stochastic regime, so as modes propagate outside $\mathfrak{U}_t$ they do so in a region with undefined spatial dimension, and the usual dynamics will no longer apply. Presumably attenuation will be rapid; once amplitudes decrease to those that characterize the stochastic regime, complete thermalization with those modes will occur and dynamics will cease.

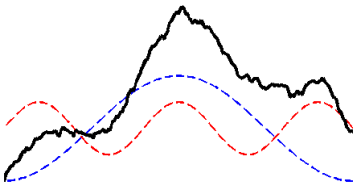


Figure 6.4

The expected rapid dissipation of a fluctuation can be slowed if there are sources of modes that replace those which propagate outside $\mathfrak{U}_t$. Working within the set of assumptions of this thesis, the quantum fields are the primary candidates for mode sources. Being quantized, their particles are in some sense persistent — if they were not persistent in time or space there would be no sense in which they could be considered quantized. Moreover, as asserted by Postulate 5.2.1, the particles are effectively localized which allows them to exist in the interior of $\mathfrak{U}_t$ without also being at the boundary of $\mathfrak{U}_t$ where the requisite symmetry is tenuous at best. This justifies the assumption that effectively localized particles of the quantum fields are the mode sources needed to slow

the dissipation of the preferred scale in $\mathfrak{U}_\mathfrak{t}$.

6.3.2 Initial manifestation of quantum fields

In a relativistically correct quantum field theory in Minkowski spacetime, particles are unitary irreducible representations of the Poincaré group. Within $\mathfrak{U}_\mathfrak{t}$ Lorentz symmetry is only approximate, and almost certainly translation invariance is even more tenuous. Nonetheless, obtaining the 'right' fluctuation by random processes only requires waiting long enough, so let approximate Poincaré symmetry exist over some restricted region $\mathfrak{S}$ in the interior of $\mathfrak{U}_\mathfrak{t}$.

Poincaré symmetry is at best approximate near the time of the initial fluctuation largely because notions of physical time, distance and angle are subject to significant time-dependent statistical variation. But even in an established spacetime measures of distance, time and angle are subject to uncertainty because they are derived from φ, and φ is ultimately a stochastic field.

Hence, in the emergence picture, spacetime symmetries have an intrinsically statistical nature. The origin of Lorentz symmetry developed in Subsect. 6.1.3 indicates how to quantify a deviation from perfect symmetry: it corresponds to the root mean square fluctuation of φ, such that the symmetry becomes classical in the limit $\sigma_\varphi \to 0$. Eq. (3.25) allows computing the variance $\sigma_\varphi^2 = \langle(\delta\varphi)^2\rangle$ from the spectral density, as was done in Eq. (4.19):

$$\int_0^\infty \mathcal{S}_\varphi(f)\, df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big(\varphi(\mathfrak{x};t) - \bar{\varphi}(\mathfrak{x})\big)^2\, dt \;=\; \sigma_\varphi^2\,.$$

Note that σ_φ is derived from the dynamical fluctuations of φ at the preferred scale, not the stochastic fluctuations of the stochastic regime. Nonetheless, in an established spacetime σ_φ will be a property of the classical/stochastic sector, not the quantum sector. This suggests the possibility that quantum uncertainty is ultimately derived from uncertainty in the spacetime itself rather than an intrinsic property of the quantum fields.[3]

In light of the above, the ability of quantum fields to manifest in the absence of exact Poincaré symmetry will be attributed to the variance σ_φ^2:

[3] In an established spacetime, it is reasonable to expect there will be a quantitative relationship between Planck's constant and σ_φ.

Postulate 6.3.1 (Poincaré symmetry and stable particles [provisional].) *Let variance σ_φ^2 be the mean square fluctuation of φ in the dynamic regime. The quantum fields cannot manifest at all when $\sigma_\varphi^2 \to \infty$, and manifest as they do in an established spacetime when $\sigma_\varphi = \sigma_{\varphi_s}$ for some empirically established value $\sigma_{\varphi_s}^2$. For $\sigma_\varphi \in (\sigma_{\varphi_s}, \infty)$, a particle's rescaled lifetime τ/τ_0 (τ_0 is its lifetime in established Minkowski spacetime) is inversely related to $\sigma_{\varphi_s}/\sigma_\varphi$.*

Determination of the particular quantitative inverse relationship between particle lifetime and variance is beyond the scope of this thesis. Given that stable particles must become unstable somehow when σ_φ increases beyond σ_{φ_s}, it cannot be a linear relationship for all particles, and it may depend on the particular field. It is plausible the relationship will be different for different values of spatial dimension n. Nonetheless, the postulate is useful in relating the variance to particle stability, thereby asserting that quantum fields do not manifest in the stochastic regime and become progressively more stable as the variance decreases. The postulate is considered provisional in that it may be quantitatively derivable in a future theory.

Now assume some quantum field ψ fluctuates in $\mathfrak{S}$ before the approximate Poincaré symmetry dissipates, producing a virtual particle or particle pair in $\mathfrak{S}$. This brings the ψ field into manifestation in $\mathfrak{U}_\mathfrak{t}$. Because the Poincaré symmetry is only approximate and the diameter of $\mathfrak{U}_\mathfrak{t}$ is finite, time translation and space translation symmetries are at best approximate. Thus, the ψ particle is virtual.

Assuming the ψ field in some sense has observable, non-gravitational effects in an established spacetime, it has an interaction Lagrangian which describes its coupling with other quantum fields.[4] The approximate nature of the Poincaré symmetry should not change the particle spectrum or interactions, compared to what they are in an established spacetime. Those other fields introduce quantum corrections into the coupling constants, and also the mass if ψ is a massive field. In this sense, the ψ field brings into manifestation the other quantum fields appearing in its Lagrangian. Each field brought into manifestation by ψ has its own interaction Lagrangian, thereby causing those fields to manifest.

Thus, the spacetime dimension $d = n+1$ and eventual particle spectrum are essentially determined completely by the initial φ fluctuation and subsequent quantum

[4] There is no notion of energy at this stage, but the Lagrangian in an established spacetime nonetheless determines what fields couple to ψ, even in an emerging spacetime.

fluctuation, *i.e.*, the number of independent spatial directions in which the ψ field couples to φ and possibly other properties of the ψ field. The subsequent evolution will either end in dissipation (presumably the usual case) or evolve dynamically according to the d-dimensional version of the φ equation of motion, interacting in d spacetime dimensions with the quantum fields. Hence, quantum corrections will cause the full set of quantum fields that can appear in an established $n+1$ dimensional spacetime to also manifest at the time of the initial fluctuation.

Lacking additional constraints on what fields can manifest for a given choice of n, in principle it is possible that there can be disjoint collections of quantum fields $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$, such that only one collection will manifest in a given occurrence of an $n+1$ dimensional spacetime. If $\psi_i \in \mathcal{Q}_i$ appears in an initial fluctuation, then only the fields of $\mathcal{Q}_i$ will manifest if the fluctuation evolves into an established spacetime. For example, in $n = 3$ the fields of the Standard Model would define one such collection $\mathcal{Q}_i$. There is no reason to expect such disjoint collections of fields exist for a given n, but the possibility can only be precluded by new constraints. No such constraints will be considered here; for present purposes they are of peripheral interest anyway.

It is clear that in an approximate spacetime where the field quanta are all virtual there is no sense in which interactions are perturbative. The same holds true for interactions between the 'true' quantum fields and the effectively quantized φ — the approximate spacetime provided by the φ fluctuation will be significantly affected by the self interaction between it and the induced φ modes of the 'quantum φ' field. Thus, after the initial quantum fluctuation, the collection of quantum fields and φ must be treated as a strongly coupled system. Moreover, the composite quantum/φ field theory cannot be separated into quantum and classical/stochastic sectors as described in Subsect. 5.1.1 because there is no equilibrium in an approximate spacetime.

Postulate 6.3.1 asserts the quantum fields are increasingly unstable as Poincaré symmetry becomes less exact. Presumably lifetimes of different particles of the same field have a large variance also. Energy and momentum are at best only approximate notions in this regime.

As long as Poincaré symmetry is approximate and energy and momentum are somewhat ill defined, transitions from one particle type to another which are allowed by internal symmetries (but are broken at lower energy in an established spacetime) can presumably occur rapidly and somewhat freely. The usual dictum of quantum theory

applies: whatever is not prohibited will occur.

6.3.3 Exponential expansion

The spacetime emergence scenario outlined below essentially proceeds by successively reducing the mean square fluctuation in the measures of distance and time until it is sufficiently small, where 'sufficiently small' presumably means the spacetime can support persistent phenomena with a law-like character. The basic process at work is straightforward.

From the previous subsection, when the variance σ_φ^2 is large, particle lifetimes are small and the quantum fields are strongly coupled. Moreover, the φ modes induced by the quantum fields are far out of equilibrium, both in terms of lack of equipartition and with respect to balance between radiation and absorption by the quantum fields. In terms of the model discussed in Subsect. 5.4.2, the lack of equilibrium corresponds to the virtual particles radiating much more energy as φ waves than they absorb. That is, absorption, represented by the left diagram of Fig. 5.4 (repeated in Fig. 6.5) is suppressed compared to radiation, represented in the right diagram. Thus there is a continual transfer of energy from the quantum fields into propagating φ modes.

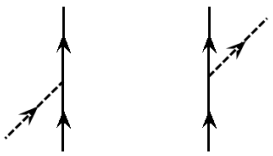


Figure 6.5

The asymmetry between radiation and absorption of φ waves drives a continuous, net transfer of energy (in its approximately defined sense) from the quantum fields to the φ field. The growing radiation field gradually thermalizes. This has some important consequences.

First, the total approximate energy content of the quantum fields decreases with time, even if, in some sense, the total energy of φ plus the quantum fields remains constant. Taking the energy content of the 'non-quantum φ' field to be gravitational energy, consider the correspondence with GR. In GR, the energy of the gravitational field is negative and the energy of the matter fields is positive. This contrasts with the situation above, where the energy of both the matter and gravitational fields is positive. The two viewpoints can be reconciled by taking the energy of the 'non-quantum φ' field to be negative by adding an appropriate negative energy offset.

Second, the net energy transfer increases the φ mode amplitudes, increasing

the relative variation. (The variation along the reference curve $\mathfrak{c}_{\text{ref}}$ used to compute the relative variation needs to be fixed at some early time for this statement to be physically relevant; otherwise the variation along $\mathfrak{c}_{\text{ref}}$ would change as the energy of the φ field increases.) The increased relative variation changes the measures of space and time; see Eq. (6.20). Hence an infinitesimal volume element $dx\,dy\,dz$ in $\mathfrak{S}$ scales as $\mathfrak{n}^{-3/2}$ so that the total volume of $\mathfrak{S}$ (measured, *e.g.*, by the distance metric $\mathfrak{d}$) scales as $\mathfrak{n}^{3/2}$.

As the volume of $\mathfrak{S}$ increases due to net radiation by the virtual particles, the particles will propagate in the added volume just as they did in the prior volume. The φ radiation is still far out of equilibrium and the quantum fields are still strongly coupled, so the net transfer of energy to the φ field with the consequent increase in $\mathfrak{n}$ continues as above, recursively. Thus, the volume of $\mathfrak{S}$ expands exponentially.

Third, and also due to the increase in relative variation, the mean square fluctuation of φ decreases in $\mathfrak{S}$ as the relative variation increases. This is analogous to the mean square pressure fluctuation in a container of ideal gas: as the number of particles increases the mean square fluctuation in pressure decreases, as do the fluctuations of other intensive variables. The decreasing mean square fluctuation is what moderates the exponential expansion and eventually causes it to end. The smoothness of the emergent manifold is also determined by the mean square fluctuation.

Quantitatively modeling the interplay of all these factors is likely to be somewhat complex due to the nonlinearity of the process. Additionally, the relative variation cannot be assumed proportional to the particle content, in that there is additional nonlinearity because the propagation speed of the φ modes is also determined by $\mathfrak{n}$; see the discussion below Eq. (6.20). Moreover, the volume of $\mathfrak{S}$ will increase as the φ field on $\mathfrak{S}$ interacts with the vacuum at the boundary of $\mathfrak{S}$. Clearly the foregoing is a very rough sketch.

Appendix A

Hamilton-Jacobi Field Theory

As a prelude to classical field theory, consider the Lagrangian for a classical system consisting of single particle in a time-independent potential V. That is, $L(q, \dot{q}, t) = \frac{1}{2}m\dot{q}^2 - V(q)$, where $q(t)$ is the particle's trajectory and $\dot{q} \equiv dq/dt$. The Hamilton-Jacobi function, defined as the classical action, is, on a fixed time interval $[t_i, t_f]$,

$$S(q,t) = \int_{t_i}^{t_f} L(q, \dot{q}, t)\, dt\,. \tag{A.1}$$

Hamilton's principle $\delta S = 0$ implies the Euler-Lagrange equation,

$$\frac{dp}{dt} - \frac{\partial L}{\partial q} = 0\,, \qquad p \equiv \frac{\partial L}{\partial \dot{q}}\,. \tag{A.2}$$

The equation of motion immediately follows from (A.2); we could integrate it to obtain $q(t)$ once $V(\mathbf{x})$ is specified. However, it is more useful for present purposes to identify S in (A.1) with Hamilton's principal function, and solve for $S(q,t)$ via the Hamilton-Jacobi equation:

$$\frac{\partial S(q,t)}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0\,; \qquad H(q,p) = p\dot{q} - L\,. \tag{A.3}$$

The momentum p has been rewritten as $\partial S/\partial q$, which follows from integrating the second equality in (A.2) and using (A.1).

The Hamilton-Jacobi equation for a classical scalar field $\phi(\mathbf{x}, t)$ involves a straightforward generalization. The field configuration $\phi(\mathbf{x})$ corresponds to the coordinate q of the single particle system; then $q(t) \sim \phi(\mathbf{x}, t)$. Thus, the Hamilton-Jacobi function (*i.e.*, classical action) is a functional of the field configuration and a function of the time.

The Lagrange density for ϕ in Minkowski spacetime is

$$\mathcal{L} = \tfrac{1}{2}\left(-\dot{\phi}^2 + (\nabla_i\phi)^2 - m^2\phi^2\right) - \mathcal{V}(\phi)\,, \tag{A.4}$$

where $\mathcal{V}(\phi)$ is the potential. The H-J function is

$$S = \int_{t_i}^{t_f} dt \int dx^3\, \mathcal{L}(\phi, \partial_\mu\phi, t)\,;$$

this governs the field evolution from time t_i to t_f. (In a more general spacetime, the action would have explicit time dependence due to factors involving the metric in the covariant volume element $\sqrt{-g}d^3x$ and covariant derivatives.)

Introduce functional derivatives with respect to the coordinate volume, *e.g.*,

$$\delta S = \int d^3x\, \frac{\delta S}{\delta\phi}\,\delta\phi\,.$$

Fix the boundary conditions at time t_i, and vary the action while remaining consistent with the equations of motion to get the functional analogue of Eq. (A.3):

$$\frac{\delta S}{\delta\phi} \;=\; \frac{\partial\mathcal{L}}{\partial\phi} \;=\; \pi(\mathbf{x}, t)\,. \tag{A.5}$$

Thus, the H-J equation for a scalar field theory, corresponding to the single particle case (A.3), is

$$H\left(\phi, \frac{\delta S}{\delta\phi}, t\right) + \frac{\partial S(\phi, t)}{\partial t} \;=\; 0\,, \tag{A.6}$$

where

$$H(\phi, \pi, t) = \int d^3x\left(\pi\dot{\phi} - \mathcal{L}\right)\,. \tag{A.7}$$

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