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NEW PERFORMANCE-VESTED STOCK OPTION SCHEMES

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Abstract. In the present paper, we advocate two effective non-traditional performance-based stock option schemes: Parisian and Asian executives' stock option plans. Under a Parisian option scheme, the stock price should have outperformed a certain stock price for a fixed length of time. Under an Asian scheme, the executives' compensation is coupled with the average performance of the stock price. Both schemes make the manipulation through the executives less likely. In the Parisian scheme, it can be achieved by setting the length of excursion sufficiently long and in the Asian scheme, by requiring the average rate of return of the stock to exceed a relatively high fixed rate of return. We focus on the valuation of these new performance-vested stock options and conduct some numerical analyses based on the valuation formulae we obtain.

Keywords: Executive Stock Options, Asian Options, Parisian Options
JEL: G12, G13, G34

1. Introduction

Most of the executives' compensations include some options. The executive stock option (ESO) plans are usually based on the performance of the firm's assets directly. They are plain-vanilla call options granted with a fixed strike price equaling the stock price at the granting date. An on-going discussion about the current executives' stock options is that these options can be easily manipulated by the executives. The manipulation is done by influencing either the timing of granting or the stock price at the maturing date. The executives can set the timing of granting (unscheduledly) to their own benefit. They might grant the options before an anticipated stock increase or after a stock decrease. Lie (2005) provides a discussion on this issue. Since the executives' bonuses are solely dependent on the terminal stock price, the executives will try to increase the terminal value as much as possible. Some suggest that the bonuses shall be coupled with a certain reference portfolio such that the manipulation cannot be done so easily. But it is still a very controversial

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point. Johnson and Tian (2000) introduce a sequence of nontraditional executives' stock option plans, one of which is performance-vested options. These options cannot be exercised unless the stock price hits a prescribed barrier (larger than the stock price at the grant date) during the option life, which could prevent the executives from manipulation. However, the barrier trigger only depends on a single touch of the barrier by the underlying stock price. The executives can still manipulate the exercise of the option quite easily in their favor.

Hambrick and Sanders (2007) study 950 companies (listed on the Standard & Poor's 500, Mid-Cap and Small-Cap indices in 1998) from 1994 to 2000 and find that those whose chiefs get more than half their compensations in stock options are far more likely to take risks in more and bigger acquisitions and to spend heavily on research and equipment. According to their study a higher payment of a CEO in stock options coincides with a more extreme performance of the firm's stock. The reason behind it is that ESOs allow the CEOs to profit from the upside risk but not to suffer from downside risk. However, the more volatile the firm's stock price, the higher the chances there is a big loss. Given these results it is time to reconsider executives'incentives schemes.One suggestion would be to simply reduce the fraction of the executives' payment in stock options. These changes might not be popular with ambitious managers as the executives' profits from a brilliant move would be lower.

In the present paper, we advocate two more effective performance-based stock option schemes that make manipulation less likely: Parisian and Asian executives' stock option plans. In the Parisian option scheme, the stock price should have outperformed a certain stock price which is fixed at the granting date for a fixed length of time d. Apparently, the longer the outperforming time needs to be, the more costly a manipulation will be. The chance of the manipulation becomes consequently smaller. Under an Asian scheme, the executives' compensation is coupled with the average performance of the stock price. If it is required that the average rate of return of the stock exceeds a fixed rate of return, it is very unlikely that the executives can benefit from a one-time manipulation of the stock price. In our analysis, we do not take into consideration all the special characteristics of ESOs and in particular ignore the following features incorporated in these options: early-exercise feature (see. e.g. Sircar and Xiong (2007)), non-tradable restriction (see e.g. Carpenter (2000)), and reloading or resetting feature (see e.g. Dybvig and Loewenstein (2003)).

Parisian options do not have a long history in the literature on exotic options. They are introduced by Chesney et al. (1997) and subsequently developed by Moraux (2002), Anderluh and van der Weide (2004) and Bernard et al. (2005). In a standard Parisian up–and–in option, the contract is knocked in if the underlying asset value remains consecutively above the barrier for longer than some predetermined time d before the maturity date. In the context of with–profit life insurance contracts, Chen and Suchanecki (2007) apply the Parsian barrier option framework to incorporate more realistic bankruptcy procedures (Chapter 11 bankruptcy procedure) in the market valuation of life insurance liabilities.

Asian options own a longer history. An Asian option is a financial option on the value of the arithmetic average of some underlying asset during a prespecified time interval. Even if the underlying asset is assumed to be log-normal, no closed-form solutions can be obtained. A stream of literature has focused on developing numerical methods to achieve approximation results. See e.g. Kemna and Vorst (1990), Turnbull and Wakeman (1991), Vorst (1992), Levy (1992), Curran (1994), Rogers and Shi (1995) and Nielsen and Sandmann (2003), just to quote a few.

The remainder of the article is organized as follows. In Section 2, assumptions about the underlying firm's asset process are made and a base contract specification of ESOs is introduced. Section 3 is dedicated to introducing performance-vested stock options with Parisian feature. The valuation formula is derived and some numerical results (particularly concerning the length of excursion and the riskiness (volatility) of a portfolio) are conducted. Section 4 focuses on the performance-vested stock options with Asian option schemes. Some numerical analyses are carried out as well. Finally, the results of this article and possible further research are summarized in the conclusion.

2. Base contract

This section introduces a simple valuation framework for two contract specifications of ESOs. To specify the underlying assets process, we start immediately with the equivalent martingale (also risk-neutral probability) measure Q. It is assumed that under Q the price process of the firm's assets $\{S(t)\}_{t\in[0,T]}$ follows a geometric Brownian motion

$$dS(t) = S(t)(rdt + \sigma dW_t) \tag{1}$$

in which σ denotes the deterministic volatility of the asset price process $\{S(t)\}_{t\in[0,T]}$. We assume that the firm continuously rebalances its investment portfolios such that the asset return volatility remains the same over time. Furthermore, $\{W_t\}_{t\in[0,T]}$ in (1) is the unique risk–neutral Q–martingale. Solving this differential equation, we obtain

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}.$$

The benchmark contract specification is formulated immediately as a standard European call option:

$$\max\left\{\frac{S(T)}{S(0)} - K, 0\right\} := \left[\frac{S(T)}{S(0)} - K\right]^+ \tag{2}$$

in which the bonus participation is unconstrained. Here a ratio process is used as underlying, unlike in conventional formulation in which the firm's assets serve as underlyings. Note that multiplying the option with the initial asset value leads to the conventional formulation immediately. We have chosen ratio processes because of mathematical tractability.

The arbitrage-free price of the contract payoff in (2) is

$$\begin{aligned} E\left[e^{-rT}\left[\frac{S(T)}{S(0)}-K\right]^{+}\right] =&\Phi(d_1)-Ke^{-rT}\Phi(d_2) \\ d_{1/2} =&\frac{\ln\frac{1}{Ke^{-rT}}\pm\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}} \\ \Phi(t) =& \int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx. \end{aligned} \tag{3}$$

3. Parisian scheme

This section mainly introduces the Parisian-type performance-vested executives' stock options, which are compared with the base contract specification introduced in Section 2.

In order to avoid manipulation of the managers, we assume that the bonus pays out only when the rate of return of the underlying asset has stayed above a certain rate sufficiently long. Mathematically speaking, it corresponds to the feature of a Parisian option[1]. Assume, we are interested in the modelling of a Parisian up–and–in option. With standard Parisian options, the underlying asset value shall stay consecutively above a certain boundary for a time longer than some pre–specified time window d before the maturity date. In this framework, the executives' stock options can only be exercised when the following technical condition is satisfied:

$$T_B^+ = \inf\left\{t>0\middle|\left(t-g_{B,t}^S\right)\mathbf{1}_{\{S(t)>B(t)\}}>d\right\}<T \tag{4}$$

with

$$g_{B,t}^S = \sup\{s\le t|S(s)=B(s)\},$$

where $g_{B,t}^S$ denotes the last time before t at which the value of the assets A_t hits the barrier $B(t)$. T_B^+ gives the first time at which an excursion above B_t lasts more than d units of time. In the following we will assume an *exponential* barrier $B(t)=B(0)e^{gt}$, i.e. the executives shall aim to increase the rate of return of the firm with a constant rate g.

[1]Parisian options distinguish themselves between standard and cumulative Parisian options. Only standard Parisian options are considered here.

Note that

$$\begin{aligned}
g^S_{B,t} =& \sup\{s \le t : S(s) = B(s)\}\\
=& \sup\left\{s \le t : S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)s + \sigma W_s\right\} = B(0)e^{gs}\right\}\\
=& \sup\{s \le t : S(0)\exp\{m\,\sigma\, s + \sigma W_s\} = B(0)\}\\
=& \sup\{s \le t : S(0)\exp\{\sigma Z_s\} = B(0)\}\\
=& \sup\{s \le t : Z_s = b\}\\
=& g_{b,t}.
\end{aligned}$$

with $m := \frac{1}{\sigma}(r - g - \frac{1}{2}\sigma^2)$, $b := \frac{1}{\sigma}\ln\left(\frac{B(0)}{S(0)}\right)$ and $Z_s := W_s + ms$. $\{Z_t\}_{0\le t\le T}$ is a martingale under probability measure P, which is defined by the Radon-Nikodym density

$$\frac{dP}{dQ}\Big|_{\mathcal{F}_T} = \exp\left\{-mZ_T - \frac{m^2}{2}T\right\}.$$

Hence, the condition in (4) is equivalent to

$$T_b^+ := \inf\{t > 0 | (t - g_{b,t})\mathbf{1}_{\{Z_t > b\}} > d\} < T. \tag{5}$$

Hereby we transform the event, the excursion of the stock price below the exponential barrier $B(t) = B(0)e^{gt}$, $B(0) \ge S(0)$, to the event, the excursion of the Brownian motion Z_t below a constant barrier b. This simplifies the entire valuation procedure. Under the new probability measure P the asset price $S(T)$ can be expressed as

$$S(T) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right\} = S(0)\exp\left\{\sigma Z_T\right\}\exp\{gT\}$$

It is well known that the price of a T-contingent claim $\phi(S(T))$ corresponds to the expected discounted payoff under the equivalent martingale measure Q, i.e.,

$$\begin{aligned}
& E\left[e^{-rT}\phi(S(T))\mathbf{1}_{\{T_b^+ < T\}}\right]\\
=& E_P\left[\frac{dQ}{dP}\Big|_{\mathcal{F}_T} e^{-rT}\phi(S(T))\mathbf{1}_{\{T_b^+ < T\}}\right]\\
=& e^{-(r+\frac{1}{2}m^2)T}E_P\left[\phi(S(0)\exp\{\sigma Z_T\}\exp\{gT\})\exp\{mZ_T\}\mathbf{1}_{\{T_b^+ < T\}}\right]
\end{aligned}$$

where the indicator function results from the characteristic of the Parisian option. Only when the excursion above b is longer than d, the issued options do not lose their values. Applying this to our contract, the price of such a contract is determined by:

$$\begin{aligned}
& E\left[e^{-rT}\left[\frac{S(T)}{S(0)} - K\right]^+ \mathbf{1}_{\{T_b^+ < T\}}\right]\\
=& \frac{e^{-(r-g+\frac{1}{2}m^2)T}}{S(0)}E_P\left[\exp\{mZ_T\}\left[S(0)e^{\sigma Z_T} - S(0)K\,e^{-gT}\right]^+ \mathbf{1}_{\{T_b^+ < T\}}\right]
\end{aligned}$$

Two cases are distinguished to price the above mentioned expectations because different relation among the initial stock price, the strike and the barrier leads to different valuation of the parisian options with the use of inverse Laplace transformation method. In Appendix A, the valuation formulae for both cases $K < 1$ and $K > 1$ are given in detail.

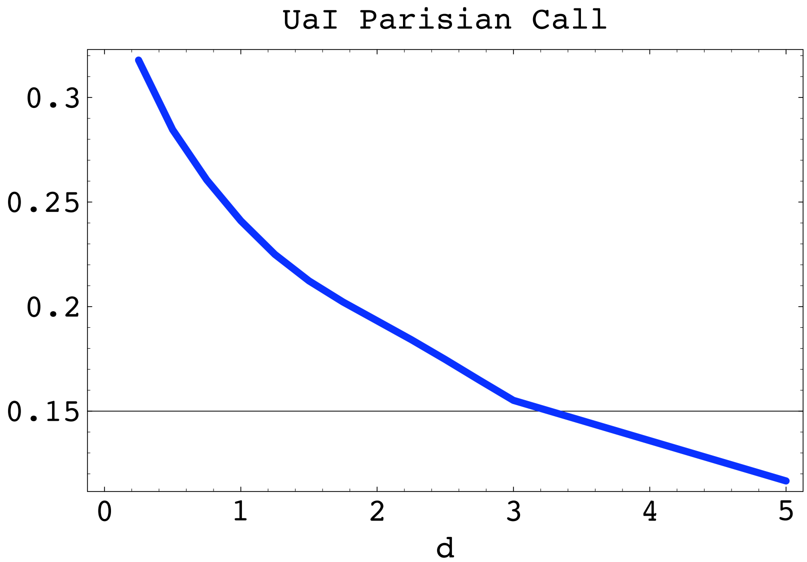

FIGURE 1. Up-and-in Parisian call as a function of d with parameters: $S(0) = 100; B(0) = 110; T = 10; r = 0.05; g = 0.02; K = 1.0; \sigma = 0.15$.

With the incorporation of the Parisian feature, the executive stock options are worth much less to the firm's managers. This effect is substantially strengthened when the length of excursion is set longer. As illustrated in Figure 1, the longer the length of excursion, the lower the value of the Parisian up-and-in call. In the literature, a standard up-and-in barrier feature (c.f. Johnson and Tian (2000)) has been built in the valuation of executive stock options. One big disadvantage of standard barrier options is that the barrier trigger only depends on a single touching of the barrier by the underlying price process and the option value can be easily manipulated. Parisian feature apparently hinder such manipulative behavior, because the trigger event does not depend on a single hit of the barrier but on the time spent beyond the barrier. Thus, the longer this time period is, the more costly a manipulation would be.

Figure 2 illustrates the effect of the volatility on the value of the Parisian up-and-in options for different d values. As a comparison, the value of a plain vanilla call option is plotted as a function of the volatility, too. First, it should be noted that the volatility has a non-monotonic effect on the value of Parisian up-and-in option. But here for the given parameters, the Parisian option value increases in volatility. Second, the incorporation of the excursion dampens the effect of the volatility. When a longer time should be spent above the threshold, the less effect the volatility is going to incur. It is hence less likely to manipulate through the volatility. The value of the plain-vanilla call amounts to 0.6732 (from 0.3946) as the volatility increases from 7.5% to 50%. In case of up-and-in Parisian options, the magnitude of increase in the value is much less substantial, for instance for $d = 0.5$, the value varies from 0.2521 to 0.4674, for $d = 1$ from 0.2109 to 0.3893, and for $d = 3$ from 0.1338 to 0.2691.

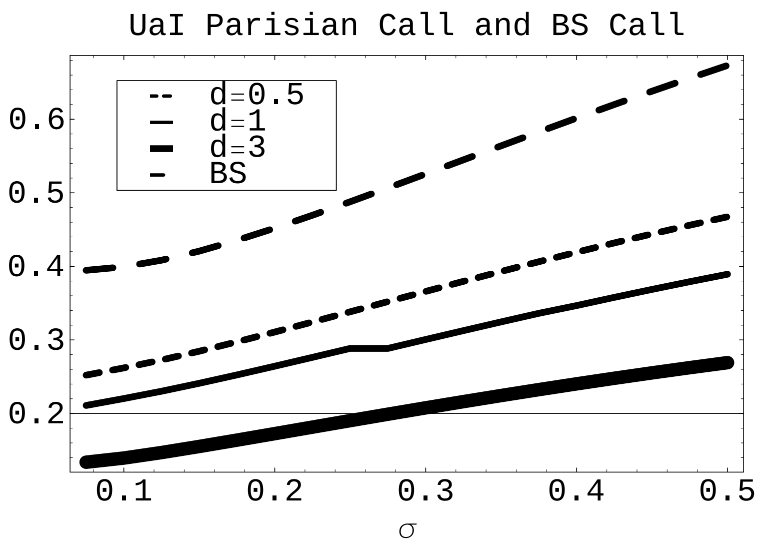

FIGURE 2. Up-and-in Parisian call and plain vanilla call as a function of σ for different d values with parameters: $S(0) = 100; B(0) = 110; T = 10; r = 0.05; g = 0.02; K = 1.0$.

4. Asian scheme

4.1. **Contract specification 3.** Here the bonus indicator depends on the periodic rates of return:

$$\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K(N)\right]^+ \quad \text{with} \quad K(N) \geq N \tag{6}$$

$K(N)$ can be eventually formulated as $K(N) = N(1+\beta)$. It can also be extended to spreads. The basic idea is that the bonus is only paid out when the average return is larger than a certain value. For example the average rate of return has to grow at least linearly with a certain rate. Mathematically this payoff scheme corresponds to an Asian-style call option. Unfortunately, a closed-form solution for the price of arithmetic Asian options is not available. Using the methods developed in Nielsen and Sandmann (2003) we derive lower and upper bounds for the price. As shown in their paper these bounds perform very well.

First note that the today's arbitrage-free price equals the expected discounted payoff under the risk-neutral measure Q:

$$C_A(K,N) = E\left[e^{-rT}\left[\sum_{i=0}^{N-1}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(t_{i+1}-t_i)+\sigma(W(t_{i+1})-W(t_i))\right)-K(N)\right]^+\right]$$

with $T=t_N$ and $t_0=0$.

4.1.1. *Lower bound.* A simple lower bound is derived by using the fact that the geometric average is no greater than the arithmetic average.

$$\frac{1}{N}\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S_{t_i}} \geq \left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(ti)}\right)^{\frac{1}{N}}. \tag{7}$$

Thus, we can bound the price from below by a geometric Asian-Call option.

$$\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1}}{S(t_i)}-K\right]^+ \geq \left[N\left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{1}{N}}-K\right]^+.$$

The geometric average takes the simple form

$$\left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{1}{N}} = \left(\frac{S(T)}{S(t_0)}\right)^{\frac{1}{N}} = \exp\left(\frac{1}{N}\left((r-\frac{1}{2}\sigma^2)T+\sigma W_T\right)\right).$$

Hence, the lower bound $C_G(K,N)$ is given by the price of a European type call option:

$$C_G(K,N) = N\exp\left(\left(\frac{1}{N}-1\right)rT+\frac{1}{2}\frac{1}{N}\sigma^2T\left(\frac{1}{N}-1\right)\right)\Phi(d_G)-K\exp(-rT)\Phi\left(d_G-\frac{\sigma}{N}\sqrt{T}\right)$$

where

$$d_G = \frac{\log\left(\frac{N}{K}\right)+\frac{1}{N}(r-\frac{1}{2}\sigma^2+\frac{1}{2}\frac{\sigma^2}{N})T}{\frac{\sigma}{N}\sqrt{T}}.$$

A better bound can be achieved by the conditioning approach presented in Rogers and Shi (1995) and Nielsen and Sandmann (2003). The starting point is the inequality:

$$\begin{aligned} E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\right] &= E\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle| Z\right] \\ &\geq E\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle| Z\right]^+\right] \\ &:= C_{l,Z}(K,T)e^{rT} \end{aligned}$$

where Z is an $\mathcal{F}_T$-measurable Gaussian random variable. Hereby we have used the property of iterated expectation in the first step. The basic idea of the second step is that the N stochastic variables in the arithmetic average are replaced by the projection on a single random variable Z. Therefore the resulting expected value will be solvable in closed form. The higher the correlation between the arithmetic average and Z the lower the resulting pricing error. A natural choice for Z would be the geometric average in (7). In the following we will set Z equal to the logarithm of the geometric average as this random variable contains the same information as the geometric average but is easier to handle.

We show in Appendix B that for a general sequence of times $\{t_i\}_{i=1}^N$ the price $C_{l,Z}(K,T)$ equals the weighted average of European style call options with different volatilities and strike prices. Here we will present the result for the special case of equidistant time periods, i.e. $t_{i+1} - t_i = \frac{T}{N}$.

PROPOSITION 4.1. *Given equidistant time periods the lower bound $C_{l,Z}(K,T)$ obtained by the conditioning approach is*

$$C_{l,Z}(K,T) = Ne^{r(\frac{1}{N}-1)T}\Phi(d_1) - e^{-rT}K\Phi(d_2).$$

with

$$d_1 = \frac{N\log(N/K)}{\sigma\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} + \frac{1}{2}\frac{\sqrt{T}\sigma}{N} \qquad d_2 = d_1 - \frac{\sigma\sqrt{T}}{N}$$

PROOF: Let $Z = \frac{W_T}{\sqrt{T}}$ be a standard Gaussian random variable under Q. We obtain

$$\begin{aligned}
C_{l,Z}(K,N) =& e^{-rT}E\left[E\left[\sum_{i=0}^{N-1}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(t_{i+1}-t_i) + \sigma(W(t_{i+1})-W(t_i))\right) - K\middle|\frac{W_T}{\sqrt{T}} = z\right]^+\right] \\
=& e^{-rT}E\left[\left[\sum_{i=0}^{N-1}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)\left(\frac{T}{N}\right) + \sigma(W(t_{i+1})-W(t_i))\right) - K\middle|\frac{W_T}{\sqrt{T}} = z\right]^+\right] \\
=& e^{-rT}E\left[\left[\sum_{i=0}^{N-1}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)\left(\frac{T}{N}\right) + \sigma\frac{\sqrt{T}}{N}z + \frac{1}{2}\sigma^2\left(\frac{T}{N}-\frac{T}{N^2}\right)\right) - K\right]^+\right] \\
=& e^{-rT}E\left[\left[N\exp\left(r\left(\frac{T}{N}\right) - \frac{1}{2}\sigma^2\frac{T}{N^2} + \sigma\frac{\sqrt{T}}{N}z\right) - K\right]^+\right].
\end{aligned}$$

From Step 2 to 3, we have used $E[W(t_{i+1})-W(t_i)|Z] = \frac{\sqrt{T}}{N}Z$ and $Var[W(t_{i+1})-W(t_i)|Z] = \left(\frac{T}{N} - \frac{T}{N^2}\right)$.

Letting $\bar{\sigma} = \frac{\sigma}{N}$ and $\bar{K} = \frac{K}{N}\exp\left(-rT\left(\frac{1}{N}-1\right)\right)$, we have

$$\begin{aligned}
C_{l,Z}(K,N) =& e^{-rT}E\left[\left[N\exp\left(r\left(\frac{T}{N}\right)-\frac{1}{2}\bar{\sigma}^2T+\bar{\sigma}\sqrt{T}z\right)-K\right]^+\right]\\
=& \left(Ne^{rT\left(\frac{1}{N}-1\right)}\right)e^{-rT}E\left[\left[\exp\left(rT-\frac{1}{2}\bar{\sigma}^2T+\bar{\sigma}\sqrt{T}z\right)-\bar{K}\right]^+\right]\\
=& Ne^{rT\left(\frac{1}{N}-1\right)}\mathrm{Call}(T,\bar{K},\bar{\sigma}).
\end{aligned}$$

■

4.1.2. *Upper bound.* The next step is developing an upper bound. The simplest approach is based on the geometric average exploiting the fact that

$$E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+ \leq E\left[N\left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{1}{N}}-K\right]^+ + E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-N\left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{1}{N}}\right].$$

Thus, the upper bound takes the form:

$$NC_G\left(\frac{K}{N},N\right)+e^{-rT}\left(\sum_{i=0}^{N-1}e^{r\Delta t_i}-N\exp\left\{r\frac{T}{N}+\frac{1}{2}\frac{\sigma^2}{N}T\left(\frac{1}{N}-1\right)\right\}\right)$$

where $C_G(\frac{K}{N},N)$ denotes the price of a geometric Asian Call option with strike $\frac{K}{N}$. Applying the conditioning method we can derive a sharper upper bound. Denoting the pricing error made when applying the conditioning method by ϵ, the upper bound may be written as

$$C_{u,Z}(K,N) = C_{l,Z}(K,N)+\epsilon \tag{8}$$

As before we set $Z = \frac{W_T}{\sqrt{T}}$. The pricing error at time T takes the form

$$e^{rT}\epsilon = E\left[E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle|Z\right]-\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle|Z\right]\right]^+\right].$$

In Appendix C we solve the problem for a general sequence of time periods, but here we will only consider the simple case of constant differences between the time periods, i.e. $t_{i+1}-t_i = T/N = \Delta$.

Proposition 4.2. *If the time periods are equidistant, the pricing error* ϵ *is given by*

$$\begin{aligned}
\epsilon =& \frac{e^{-rT}}{2}\Phi(d)^{\frac{1}{2}}\left[\left(\exp\left(2\Delta r+\sigma^2\frac{\Delta^2}{T}\right)\Phi(d-2\sigma\Delta)\right)\left((N^2-N)\left(\exp\left(-\frac{\sigma^2\Delta}{T}\right)-1\right)\right.\right.\\
&\left.\left.+N\left(\exp\left(\sigma^2\Delta-\frac{\sigma^2\Delta^2}{T}\right)-1\right)\right)\right]^{\frac{1}{2}}
\end{aligned}$$

with

$$d = \frac{N}{\sigma\sqrt{T}}\left(\log\frac{K}{N}-\frac{1}{N}\left(r-\frac{1}{2}\sigma^2\right)T\right).$$

PROOF: We will show Appendix C that an upper bound on ϵ satisfies

$$\epsilon = e^{-rT}\frac{1}{2}\left(E\left[Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle| Z=z\right]\mathbf{1}_{z<d}\right]\right)^{\frac{1}{2}}\left(E\left[\mathbf{1}_{z<d}\right]\right)^{\frac{1}{2}}$$

where d is the value of $z=\frac{W_T}{\sqrt{T}}$ for which $N\left(\prod_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{1}{N}}=K$. First, we derive an expression for the conditional variance of the sum of returns

$$Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle| Z\right]=E\left[\left(\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^2\middle| Z\right]-E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle| Z\right]^2.$$

We introduce the notation $\Delta W_i=(W(t_{i+1})-W(t_i))$ and will use the fact that $Cov[\Delta W_i,\Delta W_j|Z]=\Delta-\frac{\Delta^2}{T}$ if $i=j$ and $=-\frac{\Delta^2}{T}$ otherwise. The first term of the variance can be written as

$$\begin{aligned}
&E\left[\left(\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^2\middle| Z\right]\\
=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}E\left[\exp\left(\Delta(r-\frac{1}{2}\sigma^2)+\sigma\Delta W_i\right)\exp\left(\Delta(r-\frac{1}{2}\sigma^2)+\sigma\Delta W_j\right)|Z\right]\\
=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\exp\left(2\Delta(r-\frac{1}{2}\sigma^2)+2\sigma\frac{\Delta}{\sqrt{T}}Z+\frac{1}{2}\sigma^2 Var(\Delta W_i+\Delta W_j|Z)\right)\\
=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\exp\left(2\Delta(r-\frac{1}{2}\sigma^2)+2\sigma\frac{\Delta}{\sqrt{T}}Z+\frac{1}{2}\sigma^2(2\Delta-2\frac{\Delta^2}{T}+2\Delta\mathbf{1}_{\{i=j\}}-2\frac{\Delta^2}{T})\right)\\
=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\exp\left(2\Delta r+2\sigma\frac{\Delta}{\sqrt{T}}Z+\sigma^2\mathbf{1}_{\{i=j\}}\Delta-\frac{1}{2}\sigma^2\left(4\frac{\Delta^2}{T}\right)\right)\\
=&(N^2-N)\exp\left(2\Delta r+2\sigma\frac{\Delta}{\sqrt{T}}Z-2\sigma^2\frac{\Delta^2}{T}\right)+N\exp\left(2\Delta r+2\sigma\frac{\Delta}{\sqrt{T}}Z+\sigma^2\Delta-2\sigma^2\frac{\Delta^2}{T}\right)
\end{aligned}$$

The second term takes the following form:

$$\begin{aligned}
E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle| Z\right]^2=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}E\left[\frac{S(t_{i+1})}{S(t_i)}\middle| Z\right]E\left[\frac{S(t_{j+1})}{S(t_j)}\middle| Z\right]\\
=&\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left(\exp\left(\Delta\left(r-\frac{1}{2}\sigma^2\right)+\sigma\frac{\Delta}{\sqrt{T}}Z+\frac{1}{2}\sigma^2\left(\Delta-\frac{\Delta^2}{T}\right)\right)\right.\\
&\left.\exp\left(\Delta\left(r-\frac{1}{2}\sigma^2\right)+\sigma\frac{\Delta}{\sqrt{T}}Z+\frac{1}{2}\sigma^2\left(\Delta-\frac{\Delta^2}{T}\right)\right)\right)\\
=&N^2\exp\left(2\Delta r+2\sigma\left(\frac{\Delta}{\sqrt{T}}\right)Z-\sigma^2\frac{\Delta^2}{T}\right).
\end{aligned}$$

Hence, the conditional variance equals

$$Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z\right] = (N^2-N)\left(\exp\left(2\Delta r+2\sigma\frac{\Delta}{\sqrt{T}}Z-\sigma^2\frac{\Delta^2}{T}\right)\left(\exp\left(-\sigma^2\frac{\Delta^2}{T}\right)-1\right)\right)$$
$$+N\left(\exp\left(2\Delta r+2\sigma\frac{\Delta}{\sqrt{T}}Z-\sigma^2\frac{\Delta^2}{T}\right)\left(\exp\left(\sigma^2\left(\Delta-\frac{\Delta^2}{T}\right)\right)-1\right)\right).$$

In conclusion, the pricing error is

$$\epsilon = \frac{e^{-rT}}{2}\Bigg[\left(\exp\left(2\Delta r+\sigma^2\frac{\Delta^2}{T}\right)\Phi(d-2\sigma\Delta)\right)\left((N^2-N)\left(\exp\left(-\frac{\sigma^2\Delta}{T}\right)-1\right)\right.$$
$$\left.+N\left(\exp\left(\sigma^2\Delta-\frac{\sigma^2\Delta^2}{T}\right)-1\right)\right)\Bigg]^{\frac{1}{2}}\Phi(d)^{\frac{1}{2}}. \qquad \blacksquare$$

4.1.3. *Comparative Statics.* With the incorporation of the arithmetic weighting feature, the value of the ESO option (expected rate of return of the manager) decreases substantially. As illustrated in figure 3, the more time periods are considered, the lower the value of the Asian Call.

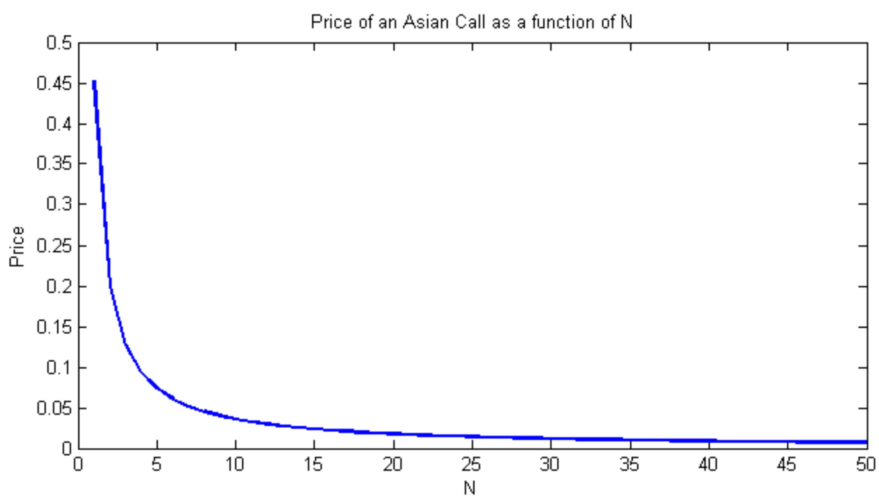

FIGURE 3. Price of $\frac{1}{N}$ Asian calls as a function of N with parameters $\sigma = 0.2$, $T = 10$, $r = 0.05$ and $K = N$.

It is seen in figure 4, that the magnitude of the increase in the value caused by the volatility is much smaller. The price of the Asian call is much less influenced by changes in σ than the vanilla call.

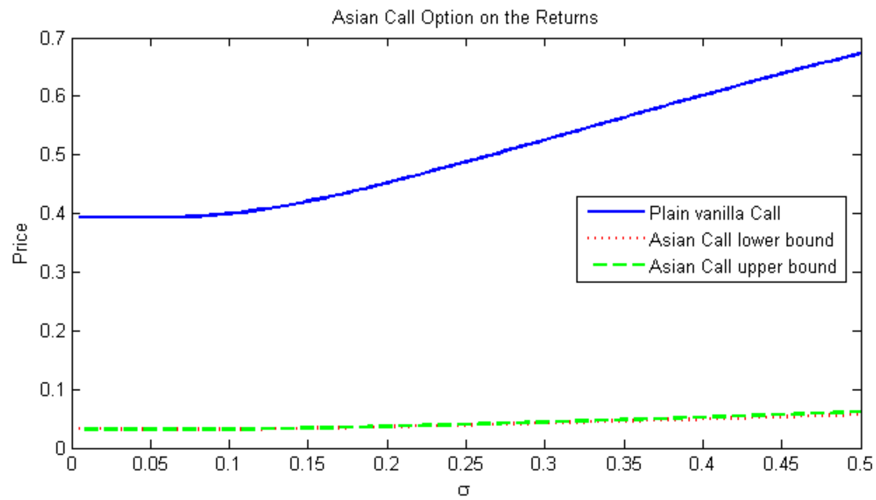

FIGURE 4. $\frac{1}{N}$ Asian calls and a plain vanilla call as a function of σ with parameters $N = 20$, $T = 10$, $r = 0.05$ and $K = N$ for the Asian call and $K = 1$ for the plain vanilla call.

Next, we consider the functional relationship between the price of an Asian call and a vanilla call with respect to the strike price K. We observe that the value of the Asian call decreases faster than the value of the corresponding plain vanilla call.

The error made by using the lower bound as an approximation for the price of an Asian Call seems to be negligibly small. Figure 5 plots the maximal error made by using the lower bound. In absolute terms the error ϵ is quite small compared to the price of the contract. In figure 6 we use a Monte Carlo simulation to approximate the true value and compare it with the bounds. Obviously, the lower bound performs better than the upper bound.

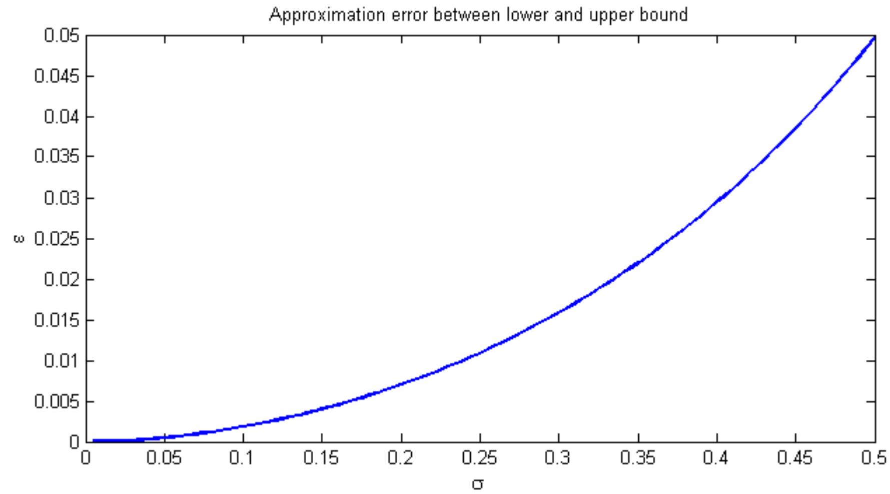

FIGURE 5. Difference between lower and upper bound of $\frac{1}{N}$ Asian calls as a function of σ with parameters $N = 20$, $T = 10$, $r = 0.05$ and $K = N$.

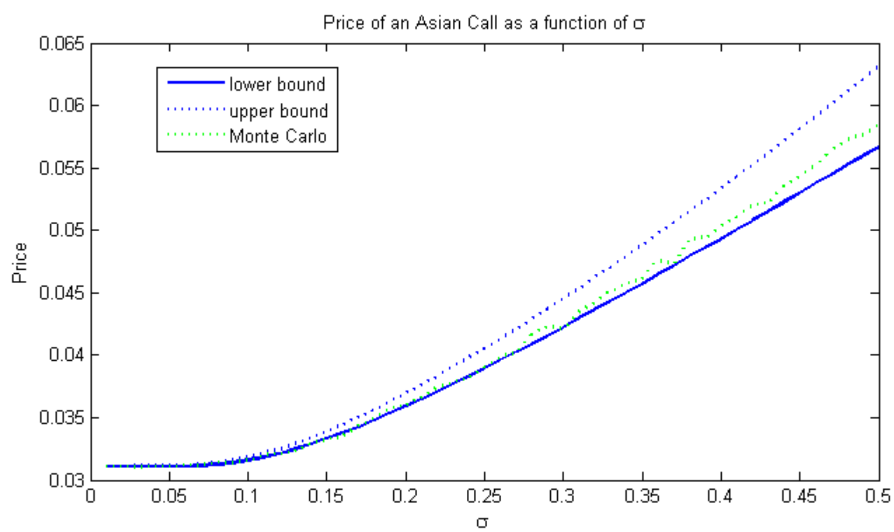

FIGURE 6. Price of $\frac{1}{N}$ Asian call as a function of σ with parameters $N = 10$, $T = 10$, $r = 0.05$ and $K = N$. We use 20000 Monte Carlo simulations.

Contract 3 can be extended easily. For instance, it can be extended by additionally requiring that the terminal rate of return shall be larger than 1. The economic intuition behind this contract is that the manager should make no losses in the last period. As before there does not exist a closed-form solution for the price of this contract but upper and lower bounds can be derived. We label this new payoff scheme as contract 4.

By the assumption that the contract is traded (or can at least be perfectly hedged by traded securities) the price equals the expected discounted payoff under the equivalent martingale measure Q:

$$C_{A2}(K,N) = e^{-rT} E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K\right]^+ \mathbf{1}_{\left\{\frac{S(T)}{S(t_{N-1})}\geq 1\right\}}\right]$$

with $T = t_N$.

In figure 7 we plot the upper and lower bounds for contracts 3 and 4. For a better comparison we also calculate the price for the two contracts with a Monte Carlo simulation. As expected introducing the additional restriction of contract 4 lowers the price. Surprisingly, the price of the new contract is not monotonically increasing in σ any more. For low values of σ an increase in the volatility can actually decrease the price. In contract 3 the payoff was increasing in σ as only the positive effect of a higher risk was taken into consideration. On the contrary a higher σ in contract 4 has two effects. On the one hand there is the same positive effect on the payoff as in contract 3. On the other hand for a certain range of σ a higher volatility makes it also more likely that the final return is below 1 and hence the price becomes lower.

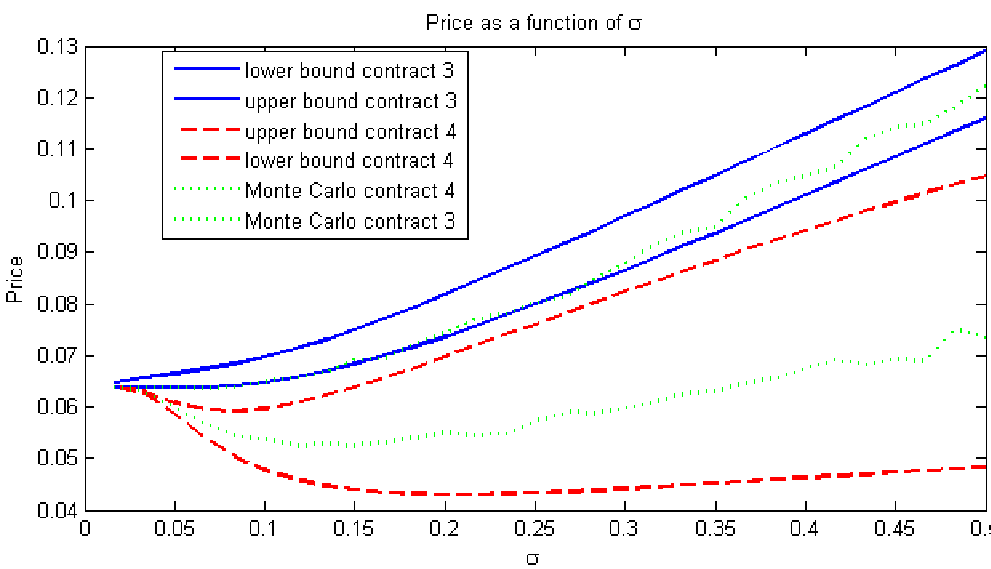

FIGURE 7. Price of $\frac{1}{N}$ Asian call of contract 3 and contract 4 as a function of σ with parameters $N = 5$, $T = 10$, $r = 0.05$ and $K = N$. We use 20000 Monte Carlo simulations.

5. Conclusion

We develop two new performance-vested executives' stock option schemes: Parisian and Asian scheme. Both schemes are able to effectively prevent a firm's executives from taking on too large risks. Furthermore, these schemes make it also less likely for the executives to manipulate the executives' stock options in their favor. In order to make ESO compensation effective, Parisian scheme requires the executives to ensure that the rate of return of the firm's asset remains above a fixed level for at least d length of time, whereas Asian scheme requires them to achieve a relatively high average rate of return. In the Parisian case, we achieve closed-form valuation formulae, while in the Asian case, lower and upper valuation bounds are developed. Under both schemes, we observe that an increase in the risk level (volatility) does not increase the value of ESOs substantially, hence the incentives of the executives to take large risks are effectively reduced.

Our analysis has focussed on the valuation of the ESO schemes. It would be interesting to analyze the welfare implications in a utility based framework. Compared to the current payoff schemes like American style and plain vanilla ESOs, can the shareholders benefit from the new schemes?

Appendix A. Valuation formulae of ESO under the Parisian scheme

As mentioned in the main text, different relation among the initial stock price, the strike and the barrier leads to different valuation of the parisian options with the use of the inverse Laplace transformation method: a) $K < 1$ (the option is in-the-money at the initial time); b) $K \geq 1$ (the option is at- or out-of-the money at the initial time). In both cases, the value of Parisian up-and-in options can be calculated by using inverse Laplace transform (c.f. Chesney et. al. (1997)). In case a), the value of Parisian up-and-in option is given by

$$\begin{aligned}
&\frac{e^{-(r-g+\frac{1}{2}m^2)T}}{S(0)} E_P\left[\exp\{mZ_T\}\left[S(0)e^{\sigma Z_T} - S(0)K\,e^{-gT}\right]^+ \mathbf{1}_{\{T_b^+ < T\}}\right] \\
=&\frac{e^{-(r-g+\frac{1}{2}m^2)T}}{S(0)} \int_k^\infty e^{my}(S(0)e^{\sigma y} - S(0)Ke^{-gT})h_2(T,y)dy \\
=&e^{-(r-g+\frac{1}{2}m^2)T} \int_k^\infty e^{my}(e^{\sigma y} - Ke^{-gT})h_2(T,y)dy
\end{aligned}$$

where $k = \frac{1}{\sigma}\ln(K)$ and $h_2(T,y)$ is described by inverting the corresponding Laplace transform which is given by

$$\begin{aligned}
\hat{h}_2(\lambda,y) =& \frac{e^{-y\sqrt{2\lambda}}}{\sqrt{2\lambda}\psi(\sqrt{2\lambda d})} + \frac{\sqrt{2\lambda d}e^{\lambda d}}{\psi(\sqrt{2\lambda d})}\left(e^{-y\sqrt{2\lambda}}\left(N\left(-\sqrt{-2\lambda d} + \frac{y-b}{\sqrt{d}}\right)\right.\right. \\
&\left.\left. - N(-\sqrt{-2\lambda d})\right) - e^{(y-2b)\sqrt{2\lambda}} N\left(-\sqrt{2\lambda d} - \frac{y-b}{\sqrt{d}}\right)\right)
\end{aligned}$$

with λ denoting parameter of Laplace transform. In case b), the option is initially at- or out-of-the money. The price of the Parisian up-and-in call can be determined as follows:

$$\begin{aligned}
&\frac{e^{-(r-g+\frac{1}{2}m^2)T}}{S(0)} E_P\left[\exp\{mZ_T\}\left[S(0)e^{\sigma Z_T} - S(0)K\,e^{-gT}\right]^+ \mathbf{1}_{\{T_b^+ < T\}}\right] \\
=&e^{-(r-g+\frac{1}{2}m^2)T}\left(\int_k^b e^{my}(e^{\sigma y} - Ke^{-gT})h_1(T,y)dy + \int_b^\infty e^{my}(e^{\sigma y} - Ke^{-gT})h_2(T,y)dy\right)
\end{aligned}$$

Here $h_1(T,y)$ is also described by inverting the corresponding Laplace transform:

$$\hat{h}_1(\lambda,y) = \frac{e^{(y-2b)\sqrt{2\lambda}}\psi(-\sqrt{2\lambda d})}{\sqrt{2\lambda}\psi(\sqrt{2\lambda d})}.$$

Appendix B. Contract 3: lower bound

A very good lower bound is achieved by the conditioning approach presented in Rogers and Shi (1995) and Nielsen and Sandmann (2003). Remember the following inequality:

$$\begin{aligned}
E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\right] =& E\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle|Z\right]\\
\geq& E\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle|Z\right]^+\right]\\
=& C_{l,Z}(K,T)e^{rT}
\end{aligned}$$

where Z is an $\mathcal{F}_T$-measurable Gaussian random variable.

We show that the prize $C_{l,Z}(K,T)$ equals the weighted average of European style call options with different volatilities and strike prices. Let $Z=\frac{W_T}{\sqrt{T}}$ be a standard Gaussian random variable. We obtain

$$C_{l,Z}(K,N) = e^{-rT}E\left[E\left[\sum_{i=0}^{N-1}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t_i+\sigma(W(t_{i+1})-W(t_i))\right)-K\middle|\frac{W_T}{\sqrt{T}}\right]^+\right]$$

where $\Delta t_i = t_{i+1}-t_i$. Remember that for X being another standard Gaussian variable it holds that

$$E[X|Z]=E[X]+E[XZ]/\sigma_Z^2[Z-E[Z]]$$

Denote $m_i=E[Z(W(t_{i+1})-W(t_i))]=\frac{t_{i+1}-t_i}{\sqrt{T}}$. Thus, using well known results for conditional expectations we obtain

$$\begin{aligned}
E[W(t_{i+1})-W(t_i)|Z] =& \frac{t_{i+1}-t_i}{T}W(T)=m_iZ\\
Cov[(W(t_{i+1})-W(t_i)),(W(t_{j+1})-W(t_j)]|Z) =& \begin{cases}\Delta t_i - m_i^2 & \text{if } i=j\\ -m_im_j & \text{otherwise}\end{cases}
\end{aligned}$$

Hence, the lower bound is given by

$$\begin{aligned}
C_{l,Z}(K,N) =& e^{-rT}E\left[E\left[\sum_{i=0}^{N-1}\exp\left((r-\frac{1}{2}\sigma^2)\Delta t_i+\sigma(W(t_{i+1})-W(t_i))\right)-K\middle|\frac{W_T}{\sqrt{T}}=z\right]^+\right]\\
=& e^{-rT}E\left[\left[\sum_{i=0}^{N-1}\exp\left((r-\frac{1}{2}\sigma^2)\Delta t_i+\sigma m_i z+\frac{1}{2}\sigma^2(\Delta t_i-m_i^2)\right)-K\right]^+\right]\\
=& e^{-rT}E\left[\left[\sum_{i=0}^{N-1}\exp\left(r\Delta t_i-\frac{1}{2}m_i^2\sigma^2+\sigma m_i z\right)-K\right]^+\right]\\
=& e^{-rT}E\left[\left[\sum_{i=0}^{N-1}w_i\exp\left(rT-\frac{1}{2}m_i^2\sigma^2+\sigma m_i z\right)-K\right]^+\right]
\end{aligned}$$

with $w_i = \frac{\exp(r\Delta t_i)}{\exp(rT)}$. As $m_i = \frac{\Delta t_i}{\sqrt{T}} > 0$ the function $w_i \exp(rT - \frac{1}{2}m_i^2\sigma^2 + \sigma m_i z)$ is strictly increasing and convex in z, taking values from 0 to $+\infty$. Thus

$$z^* = \left\{ z \middle| \sum_{i=0}^{N-1} w_i \exp(rT - \frac{1}{2}m_i^2\sigma^2 + \sigma m_i z) = K \right\}$$

is unique. Define $\bar{\sigma}_i = \frac{\sigma m_i}{\sqrt{T}}$ and $K_i^* = \exp\{rT - \frac{1}{2}m_i^2\sigma^2 + \sigma m_i z^*\}$. The expression for the lower bound turns into

$$\begin{aligned}
C_{l,Z}(K,N) =& e^{-rT} \sum_{i=0}^{N-1} E\left[w_i \left(\exp\left(rT - \frac{1}{2}m_i^2\sigma^2 + \sigma m_i z \right) \right) \mathbf{1}_{\{z \geq z^*\}} - w_i K_i^* \mathbf{1}_{\{z \geq z^*\}} \right] \\
=& e^{-rT} \sum_{i=0}^{N-1} w_i E\left[\exp\left(rT - \frac{1}{2}m_i^2\sigma^2 + \sigma m_i z \right) \mathbf{1}_{\{z \geq z^*\}} - K_i^* \mathbf{1}_{\{z \geq z^*\}} \right] \\
=& e^{-rT} \sum_{i=0}^{N-1} w_i E\left[\exp\left(rT - \frac{1}{2}\bar{\sigma}_i{}^2 T + \bar{\sigma}_i W_T \right) \mathbf{1}_{\{z \geq z^*\}} - K_i^* \mathbf{1}_{\{z \geq z^*\}} \right] \\
=& \sum_{i=0}^{N-1} w_i \mathrm{Call}(T, K_i^*, \bar{\sigma}_i).
\end{aligned}$$

Appendix C. Contract 3: upper bound

Applying the conditioning method we can derive a relatively sharp upper bound. Denoting the pricing error made when applying the conditioning method by ϵ, the upper bound may be written as

$$C_{u,Z}(K,N) = C_{l,Z}(K,N) + \epsilon.$$

Denote the rescaled geometric average by $G = N \left(\prod_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} \right)^{\frac{1}{N}} = N \left(\frac{S(T)}{S(0)} \right)^{\frac{1}{N}}$. As the geometric average is no smaller than the arithmetic average it holds that

$$\begin{aligned}
& E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K \right]^+ \right] \\
=& E\left[E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K \right]^+ \middle| G \right] \mathbf{1}_{\{G<K\}} + E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K \right]^+ \middle| G \right] \mathbf{1}_{\{G \geq K\}} \right] \\
=& E\left[E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K \right]^+ \middle| G \right] \mathbf{1}_{\{G<K\}} + E\left[\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} - K \right] \middle| G \right] \mathbf{1}_{\{G \geq K\}} \right].
\end{aligned}$$

As before we set $Z = \frac{W_T}{\sqrt{T}}$ and denote by $\phi(.)$ the standard normal density function of Z. Let d be the value of Z for which $G = K$, i.e.

$$d = \frac{N}{\sigma\sqrt{T}} \left(\log \frac{K}{N} - \frac{1}{N} \left(r - \frac{1}{2}\sigma^2 \right) T \right).$$

Therefore, the pricing error at time T takes the form

$$
\begin{aligned}
e^{rT}\epsilon =&E\left[E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle|Z\right]-\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle|Z\right]\right]^+\right]\\
=&E\Bigg[E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle|Z=z\right]\mathbf{1}_{\{z<d\}}+E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]\middle|Z=z\right]\mathbf{1}_{\{z\geq d\}}\\
&-\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle|Z\right]\right]^+\Bigg]\\
=&\int_{-\infty}^{d}\left(E\left[\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\right]^+\middle|Z=z\right]-\left[E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}-K\middle|Z=z\right]\right]^+\right)\phi(z)dz\\
\leq&\frac{1}{2}\int_{-\infty}^{d}\left(Var\left(\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z=z\right)\right)^{\frac{1}{2}}\phi(z)dz\\
=&\frac{1}{2}E\left[\left(Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z=z\right]\mathbf{1}_{\{z<d\}}\right)^{\frac{1}{2}}\left(\mathbf{1}_{\{z<d\}}\right)^{\frac{1}{2}}\right]\\
\leq&\frac{1}{2}\left(E\left[Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z=z\right]\mathbf{1}_{\{z<d\}}\right]\right)^{\frac{1}{2}}\left(E\left[\mathbf{1}_{\{z<d\}}\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

In the last step we have applied Hölder's inequality while in the third step we made use of the fact that for any random variable U it holds that

$$
\begin{aligned}
0\leq&E\left[U^+\right]-E\left[U\right]^+\\
=&\frac{1}{2}\left(E[|U|]-|E[U]|\right)\\
\leq&\frac{1}{2}E\left[|U-E[U]|\right]\\
\leq&\frac{1}{2}Var[U]^{\frac{1}{2}}.
\end{aligned}
$$

Hence, the pricing error ϵ is given by

$$
\epsilon=e^{-rT}\frac{1}{2}\left(E\left[Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z=z\right]\mathbf{1}_{\{z<d\}}\right]\right)^{\frac{1}{2}}\left(E\left[\mathbf{1}_{\{z<d\}}\right]\right)^{\frac{1}{2}}.
$$

Now, we derive an expression for the conditional variance of the sum of returns

$$
Var\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z\right]=E\left[\left(\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\right)^2\middle|Z\right]-E\left[\sum_{i=0}^{N-1}\frac{S(t_{i+1})}{S(t_i)}\middle|Z\right]^2.
$$

We introduce the notation $\Delta W_i = (W(t_{i+1}) - W(t_i))$. The first term of the variance can be written as

$$
\begin{aligned}
& E\left[\left(\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)}\right)^2 \middle| Z\right] \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} E\left[\exp\left(\Delta t_i(r - \frac{1}{2}\sigma^2) + \sigma\Delta W_i\right)\exp\left(\Delta t_j(r - \frac{1}{2}\sigma^2) + \sigma\Delta W_j\right)|Z\right] \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \exp\left((\Delta t_i + \Delta t_j)(r - \frac{1}{2}\sigma^2) + \sigma(m_i + m_j)Z + \frac{1}{2}\sigma^2 Var((\Delta W_i + \Delta W_j)|Z)\right) \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \exp\Bigg((\Delta t_i + \Delta t_j)(r - \frac{1}{2}\sigma^2) + \sigma(m_i + m_j)Z \\
& + \frac{1}{2}\sigma^2(\Delta t_i + \Delta t_j - m_i^2 - m_j^2 + 2\Delta t_i \mathbf{1}_{\{\Delta t_i = \Delta t_j\}} - 2m_i m_j)\Bigg) \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \exp\left((\Delta t_i + \Delta t_j)r + \sigma(m_i + m_j)Z + \sigma^2 \mathbf{1}_{\{\Delta t_i = \Delta t_j\}}\Delta t_i - \frac{1}{2}\sigma^2(m_i^2 + m_j^2 + 2m_i m_j)\right).
\end{aligned}
$$

The second term takes the following form:

$$
\begin{aligned}
E\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} \middle| Z\right]^2 =& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \Bigg(\exp\left((\Delta t_i(r - \frac{1}{2}\sigma^2) + \sigma m_i Z + \frac{1}{2}\sigma^2(\Delta t_i - m_i^2)\right) \\
& \exp\left((\Delta t_j(r - \frac{1}{2}\sigma^2) + \sigma m_j Z + \frac{1}{2}\sigma^2(\Delta t_j - m_j^2)\right)\Bigg) \\
=& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \exp\left((\Delta t_i + \Delta t_j)r + \sigma(m_i + m_j)Z - \frac{1}{2}\sigma^2(m_i^2 + m_j^2)\right).
\end{aligned}
$$

In conclusion the conditional variance equals:

$$
\begin{aligned}
Var\left[\sum_{i=0}^{N-1} \frac{S(t_{i+1})}{S(t_i)} \middle| Z\right] =& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \Bigg(\exp\left((\Delta t_i + \Delta t_j)r + \sigma Z(m_i + m_j) - \frac{1}{2}\sigma^2(m_i^2 + m_j^2)\right) \\
& (\exp\left(\sigma^2(\mathbf{1}_{\{\Delta t_i = \Delta t_j\}}\Delta t_i - m_i m_j)\right) - 1)\Bigg).
\end{aligned}
$$

Finally, the pricing error is

$$
\begin{aligned}
\epsilon =& \frac{1}{2}e^{-rT}\Phi(d)^{\frac{1}{2}}\Bigg(\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\Bigg(\exp\left((\Delta t_i + \Delta t_j)r + \sigma^2 m_i m_j\right)\Phi(d - \sigma(m_i + m_j)) \\
& (\exp\left(\sigma^2(\mathbf{1}_{\{\Delta t_i = \Delta t_j\}}\Delta t_i - m_i m_j)\right) - 1)\Bigg)\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Assume that the differences between time periods is constant, i.e. $\Delta = t_{i+1} - t_i = T/N$ for all $i \in \{0, N-1\}$. Under this condition the following expressions simplify to:

$$
\begin{aligned}
m_i =& \frac{\Delta}{\sqrt{T}} \\
Cov(\Delta W_i, \Delta W_j | Z) =& \begin{cases} \Delta - \frac{\Delta^2}{T} & \text{if } i = j \\ -\frac{\Delta^2}{T} & \text{otherwise} \end{cases} \\
z^* =& \frac{N \log(K/N)}{\sigma \sqrt{T}} - r \frac{\sqrt{T}}{\sigma} + \frac{1}{2} \frac{\sqrt{T} \sigma}{N} \\
C_{l,Z} =& N e^{rT(\frac{1}{N} - 1)} \Phi \left(-z^* + \sigma \frac{\sqrt{T}}{N} \right) - e^{-rT} K \Phi \left(-z^* \right) \\
\epsilon =& \frac{e^{-rT}}{2} \Phi(d)^{\frac{1}{2}} \Bigg[\left(\exp \left(2\Delta r + \sigma^2 \frac{\Delta^2}{T} \right) \Phi(d - 2\sigma\Delta) \right) \\
& \left((N^2 - N) \left(\exp \left(-\frac{\sigma^2 \Delta}{T} \right) - 1 \right) + N \left(\exp \left(\sigma^2 \Delta - \frac{\sigma^2 \Delta^2}{T} \right) - 1 \right) \right) \Bigg]^{\frac{1}{2}}
\end{aligned}
$$

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