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# THE POLYHEDRAL TREE COMPLEX

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**Abstract.** The tree complex is a simplicial complex defined in recent work of Belk, Lanier, Margalit, and Winarski with applications to mapping class groups and complex dynamics. This article introduces a connection between this setting and the convex polytopes known as associahedra and cyclohedra. Specifically, we describe a characterization of these polytopes using planar embeddings of trees and show that the tree complex is the barycentric subdivision of a polyhedral cell complex for which the cells are products of associahedra and cyclohedra.

**Keywords.** Associahedra, cyclohedra, planar trees, mapping class groups

**Mathematics Subject Classifications.** 05C05, 05C10, 20F65, 52B11

## Introduction

Convex polytopes which arise from combinatorial structures form a rich area of study with an increasing number of applications. Associahedra, cyclohedra, and permutahedra are some of the classic examples [BT94, Sim03, Zie95], while more recent cases include the use of generalized associahedra in the study of cluster algebras [FR07] and the appearance of amplituhedra in work on scattering amplitudes [AHBHY18]. In each of these examples, the convex polytope can be combinatorially defined by describing the partially ordered set of its faces, i.e. the face poset. The main goal of this article is to introduce a polytope arising from a partial order on a certain type of trees and examine an associated cell complex.

A *marked  $n$ -tree* (or simply an  *$n$ -tree*) is a tree with marked vertices  $v_1, \dots, v_n$ , along with some number of unmarked and unlabeled vertices, such that all vertices of valence 1 or 2 are marked. While there is no stated restriction on the unmarked vertices, one can show that an  $n$ -tree has at most  $2n - 2$  vertices in total. A *planar  $n$ -tree* is the free isotopy class of a planar embedding  $\Gamma \rightarrow \mathbb{C}$ , where  $\Gamma$  is an  $n$ -tree. In other words, a planar  $n$ -tree consists of an  $n$ -tree together with the cyclic counter-clockwise ordering of edges incident to a vertex.

There is a natural partial order on the set of planar  $n$ -trees, defined by declaring that  $\Gamma_1 \leq \Gamma_2$  if there is a collection of subtrees in  $\Gamma_1$ , each of which contains at most one marked vertex,

such that contracting each subtree to a point yields  $\Gamma_2$ . Note that the direction of this partial order is perhaps the reverse of what you might expect; unlike the poset structure for Outer space [CV86], for example, contracting a subtree corresponds to moving up in the partial order. Within this partially ordered set, each planar  $n$ -tree has a corresponding *lower set* which consists of all elements below that tree in the partial order; *upper sets* are defined similarly. Our first main theorem identifies the lower sets as familiar combinatorial objects.

**Theorem A** (Theorem 3.9). *The lower set of a planar  $n$ -tree is the face poset of a convex polytope which can be expressed as a product of associahedra and cyclohedra.*

Given a fixed set of points  $z_1, \dots, z_n$  in the complex plane, analogously define a *planted  $n$ -tree* to be the *relative* isotopy class of a planar embedding  $\Gamma \rightarrow \mathbb{C}$  where the marked vertex  $v_i$  is sent to  $z_i$  and the images of the marked points are fixed under isotopy. One can similarly define a partial order by contraction on the set of planted  $n$  trees and refer to it as the *planted tree poset*.

The inspiration for studying  $n$ -trees comes in part from complex dynamics. The *Hubbard tree* for a postcritically finite polynomial [DH85, DH84] can be viewed as representing a planted  $n$ -tree where the  $n$  marked points correspond to the postcritical set for the polynomial. In a recent article on complex dynamics by Belk, Lanier, Margalit, and Winarski [BLMW22], the authors define the (*simplicial*) *tree complex* as the geometric realization of the planted tree poset and use this complex to study Hubbard trees.

The pure mapping class group for the  $n$ -punctured plane acts naturally on the set of planted  $n$ -trees (and thus the simplicial tree complex), and there is a one-to-one correspondence between the orbits of this action and the set of planar  $n$ -trees. Moreover, this action is equivariant with respect to the partial order, so each lower set in the planted tree poset may be interpreted using Theorem A. As a consequence, the simplicial tree complex can be viewed as the result of subdividing a simpler polyhedral cell complex.

**Theorem B** (Theorem 4.6). *The planted tree poset is the face poset of a cell complex which we call the polyhedral tree complex. Each cell can be expressed as a product of associahedra and cyclohedra; in particular, the top dimensional cells are all products of cyclohedra. Furthermore, the simplicial tree complex is the barycentric subdivision of the polyhedral tree complex.*

Finally, define a planted  $n$ -tree to be *reduced* if it has no edges between unmarked vertices. This leads to an equivalence relation on the set of planted  $n$ -trees by declaring that two trees are equivalent if contracting all the edges between unmarked vertices in each one yields the same reduced tree. Each equivalence class for this relation is then canonically labeled by a reduced tree, and the original partial order on planted  $n$ -trees induces a partial order on the set of equivalence classes. The final theorem examines the structure of this poset and demonstrates a connection with the noncrossing hypertree poset introduced in [McC].

**Theorem C** (Theorems 5.10 and 5.13). *Let  $\Gamma$  be a reduced  $n$ -tree with equivalence class  $[\Gamma]$ . Then the lower set of  $[\Gamma]$  is isomorphic to the face poset for a product of simplices and the upper set of  $[\Gamma]$  is isomorphic to a product of noncrossing hypertree posets.*

The combinatorial transition from planted trees to reduced trees has a topological interpretation in which the polyhedral tree complex is analogously transformed into a polysimplicial

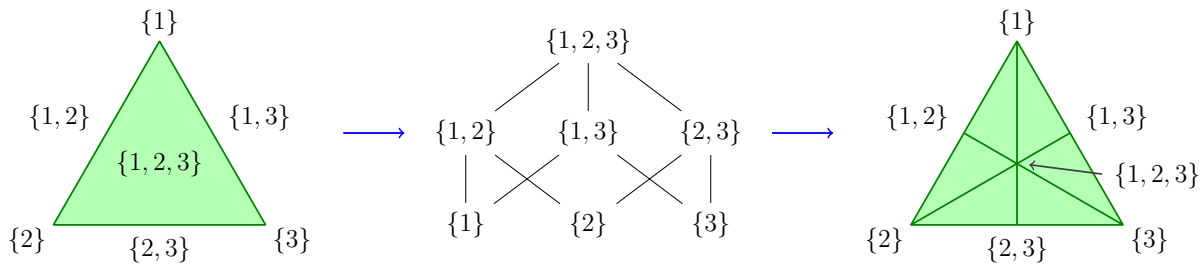


Figure 1.1: The face poset of a 2-simplex is isomorphic to  $\text{Bool}_3^*$ , and its order complex is a barycentrically subdivided 2-simplex.

complex. This resulting complex is closely connected to both the cactus complex [Nek14] and the dual braid complex [Bra01, BM10]; the details will be given in a future article.

This article begins with some preliminaries on posets in Section 1, followed in Section 2 by the introduction of planar  $n$ -trees and their contractions. Section 3 defines associahedra and cyclohedra and uses them to prove Theorem A. Section 4 concerns planted  $n$ -trees and the proof of Theorem B. Finally, reduced  $n$ -trees and are discussed and Theorem C is proven in Section 5.

## 1. Preliminaries

We begin with a few useful facts and examples regarding partially ordered sets and their associated cell complexes. Given a cell complex  $X$ , the *face poset*  $P(X)$  is the set of all nonempty faces of  $X$ , partially ordered by inclusion. Given a poset  $P$ , the *order complex* (or *geometric realization*)  $\Delta(P)$  is the simplicial complex with vertices corresponding to elements of  $P$  and a  $k$ -simplex on vertices  $x_1, \dots, x_{k+1}$  for each chain  $x_1 \leq \dots \leq x_{k+1}$  in  $P$ . These two operations are not quite inverses of one another, but they are closely related: if  $X$  is a polytopal cell complex (its cells are convex polytopes which intersect in smaller-dimensional convex polytopes), then the simplicial complex  $\Delta(P(X))$  is the barycentric subdivision of  $X$ . For an in-depth exploration of these tools, see [Wac07].

The *Boolean lattice*, denoted  $\text{Bool}_n$ , is the set of all subsets of  $\{1, 2, \dots, n\}$ , partially ordered by inclusion. As a useful shorthand, let  $\text{Bool}_n^*$  denote the subposet of all nonempty subsets.

**Example 1.1.** The  $n$ -dimensional simplex may be realized as the set of all points  $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$  with positive entries such that  $x_1 + \dots + x_{n+1} = 1$ . Then each face of dimension  $k$  may be described by specifying a nonempty subset of  $k$  coordinates which are nonzero, and it follows that the face poset of the  $n$ -simplex is isomorphic to  $\text{Bool}_n^*$ . Furthermore, the order complex of  $\text{Bool}_n^*$  is a barycentrically subdivided  $n$ -simplex. Each subset of size  $k$  labels the barycenter of a  $k$ -dimensional face, and each of the  $n!$  maximal chains in  $\text{Bool}_n^*$  labels an  $n$ -simplex in the new simplicial cell structure. See Figure 1.1 for an example in dimension 2.

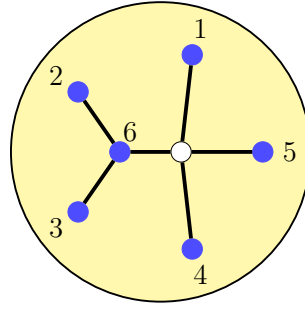


Figure 2.1: A marked 6-tree.

**Definition 1.2.** If  $P$  is a poset with  $a, b \in P$  and  $a \leq b$ , then the *interval* between  $a$  and  $b$ , denoted  $[a, b]$ , is the subposet of all  $x \in P$  such that  $a \leq x \leq b$ . The *upper set* of  $a$ , denoted  $\uparrow(a)$ , is the set of all  $x \in P$  such that  $a \leq x$ , and the *lower set* of  $b$ , denoted  $\downarrow(b)$ , is the set of all  $x \in P$  such that  $x \leq b$ .

**Example 1.3.** Let  $A$  and  $B$  be elements of  $\text{Bool}_n$  such that  $A \leq B$ ,  $|A| = k$ , and  $|B| = \ell$ . Then the interval  $[A, B]$  consists of all subsets of  $B$  which contain  $A$ , and so  $[A, B]$  is isomorphic to  $\text{Bool}_{\ell-k}$ . Similarly, the lower set  $\downarrow(A)$  is isomorphic to  $\text{Bool}_k$  and the upper set  $\uparrow(A)$  is isomorphic to  $\text{Bool}_{n-k}$ .

## 2. Planar trees

This section introduces planar  $n$ -trees and an associated partial order.

**Definition 2.1.** A *tree* is a finite, contractible, 1-dimensional simplicial complex. Connected subcomplexes of a tree are *subtrees* and a collection of disjoint subtrees is a *subforest*. The number of edges incident to a vertex  $v$  is referred to as its *valence*  $\text{val}(v)$ ; a vertex is a *leaf* if it has valence 1 and an *interior vertex* otherwise. Given any two vertices  $v_1$  and  $v_2$  in a tree, there is a unique path subcomplex from  $v_1$  to  $v_2$ , which we refer to as a *geodesic* and denote by  $\gamma(v_1, v_2)$ .

**Definition 2.2.** Given an integer  $n \geq 3$ , a *marked  $n$ -tree* (or simply an  *$n$ -tree*) consists of a tree  $\Gamma$  with a collection of marked vertices  $v_1, \dots, v_n$  which includes each vertex of valence 1 or 2 in  $\Gamma$ ; see Figure 2.1 for an example with  $n = 6$ . Since every leaf of an  $n$ -tree is labeled, the identity map is the only isomorphism between  $n$ -trees which preserves the labels on marked vertices. Using the Euler characteristic, one can show that an  $n$ -tree has between  $n$  and  $2n - 2$  vertices.

**Definition 2.3.** Let  $\Gamma_1$  and  $\Gamma_2$  be  $n$ -trees with marked vertices  $v_1, \dots, v_n$ . A *contraction*  $f: \Gamma_1 \rightarrow \Gamma_2$  is a surjective cellular map with the property that, for all  $i$ ,  $f^{-1}(v_i)$  is a subtree of  $\Gamma_1$  which contains  $v_i$ . Regarding contractions up to isotopy within each edge, each such map can be determined in a purely combinatorial manner. In other words, a contraction is obtained from an  $n$ -tree by specifying some number of subtrees, each of which contains at most one marked vertex, and retracting each subtree to a point.

In more general cases (e.g. if unmarked vertices of valence 2 were allowed), there may be several different contractions from one given tree to another. With  $n$ -trees, however, the existence of contractions is far more restrictive.

**Lemma 2.4.** *Let  $\Gamma_1$  and  $\Gamma_2$  be  $n$ -trees. If there is a contraction  $\Gamma_1 \rightarrow \Gamma_2$ , then it is unique. Consequently, contracting two different subforests of the same  $n$ -tree must result in different trees.*

*Proof.* Suppose that  $f: \Gamma_1 \rightarrow \Gamma_2$  is a contraction. By definition,  $f$  sends each marked vertex in  $\Gamma_1$  to the corresponding marked vertex in  $\Gamma_2$ . We will show that the image of each unmarked vertex is also completely determined.

If  $u$  is an unmarked vertex, then we know that the valence of  $u$  is at least three, and so the complement of  $u$  in  $\Gamma_1$  consists of at least three connected components. Fix  $v_i, v_j$ , and  $v_k$  to be marked vertices in three distinct connected components of  $\Gamma_1 - \{u\}$ . By construction, we know that  $u$  is the unique vertex which lies in the common intersection of the geodesics  $\gamma(v_i, v_j)$ ,  $\gamma(v_j, v_k)$ , and  $\gamma(v_i, v_k)$ ; if there were another, then  $\Gamma_1$  would have a cycle.

Now, since  $f$  is a contraction, we know that  $f$  sends  $v_i, v_j$ , and  $v_k$  to the corresponding vertices in  $\Gamma_2$  and that the images  $f(\gamma(v_i, v_j))$ ,  $f(\gamma(v_j, v_k))$ , and  $f(\gamma(v_i, v_k))$  yield three paths between these three vertices. Since  $\Gamma_2$  is also a tree, the common intersection of these paths contains a single point, which must be  $f(u)$ . Thus, the image of each vertex under  $f$  is completely determined by  $\Gamma_1$  and  $\Gamma_2$ , so the contraction is unique.  $\square$

As a consequence of Lemma 2.4, contractions of a given  $n$ -tree correspond exactly to subforests for which each subtree contains at most one marked point.

**Definition 2.5.** A subtree of an  $n$ -tree  $\Gamma$  is *unmarked* if it contains no marked vertices and *singly-marked* if it contains exactly one. That is, a subforest of  $\Gamma$  corresponds to a contraction of  $\Gamma$  if and only if its components are all unmarked or singly-marked subtrees. If  $\Gamma_1 \rightarrow \Gamma_2$  is a contraction of  $n$ -trees, define  $F(\Gamma_1, \Gamma_2)$  to be the unique subforest of  $\Gamma_1$  consisting of unmarked and/or singly-marked subtrees which can be contracted to obtain  $\Gamma_2$ .

Before moving on, we record a useful corollary of Lemma 2.4 and Definition 2.5.

**Corollary 2.6.** *Let  $\Gamma, \Gamma_1$ , and  $\Gamma_2$  be  $n$ -trees.*

1. *If there are contractions  $\Gamma_1 \rightarrow \Gamma$  and  $\Gamma_2 \rightarrow \Gamma$ , then there is a contraction  $\Gamma_1 \rightarrow \Gamma_2$  if and only if  $F(\Gamma_2, \Gamma)$  is a subforest of  $F(\Gamma_1, \Gamma)$ .*
2. *If there are contractions  $\Gamma \rightarrow \Gamma_1$  and  $\Gamma \rightarrow \Gamma_2$ , then there is a contraction  $\Gamma_1 \rightarrow \Gamma_2$  if and only if  $F(\Gamma, \Gamma_1)$  is a subforest of  $F(\Gamma, \Gamma_2)$ .*

We are interested in two types of planar embeddings for marked  $n$ -trees. First, we consider the case of planar embeddings up to free isotopy. In Section 4, we instead examine embeddings up to relative isotopy fixing the marked points.

**Definition 2.7.** Let  $\Gamma$  be an  $n$ -tree. A *planar  $n$ -tree* is a free isotopy class of planar embeddings  $\phi: \Gamma \rightarrow \mathbb{C}$ . That is, a planar  $n$ -tree is just an  $n$ -tree together with a cyclic counter-clockwise ordering of the edges incident to each vertex. When the context is clear, we omit explicit references to embeddings or markings, and instead refer to a planar  $n$ -tree simply as  $\Gamma$ .

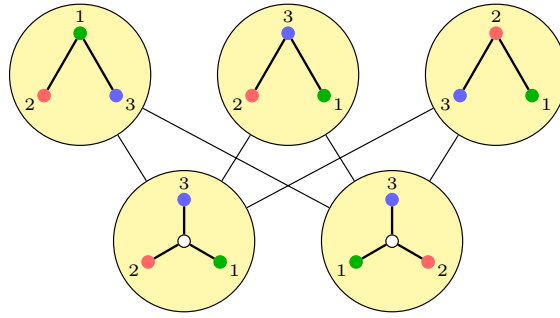


Figure 2.2: The planar tree poset  $\text{PNR}_3$ .

Planar  $n$ -trees are distinct from those studied in [BHV01], for example, since we include the ordering of edges at each vertex.

**Definition 2.8.** Let  $\phi_1: \Gamma_1 \rightarrow \mathbb{C}$  and  $\phi_2: \Gamma_2 \rightarrow \mathbb{C}$  be planar  $n$ -trees. If  $f: \Gamma_1 \rightarrow \Gamma_2$  is a contraction such that  $\phi_1$  and  $\phi_2 \circ f$  are isotopic, then we say that  $f$  is a *planar contraction*. If this is the case, we abuse notation to omit the embedding and write  $\Gamma_1 \leq \Gamma_2$ . This determines a partial order on the set of all  $n$ -trees which we call the *planar tree poset* and denote by  $\text{PNR}_n$ . See Figure 2.2 for an example when  $n = 3$ .

**Theorem 2.9.** If  $\Gamma_1 \leq \Gamma_2$  in  $\text{PNR}_n$  and the subforest  $F(\Gamma_1, \Gamma_2)$  has  $k$  edges, then the interval  $[\Gamma_1, \Gamma_2]$  is isomorphic to the Boolean lattice  $\text{Bool}_k$ .

*Proof.* Let  $\Gamma_1, \Gamma_2 \in \text{PNR}_n$  with  $\Gamma_1 \leq \Gamma_2$ . By the second part of Corollary 2.6, the planar  $n$ -trees lying between  $\Gamma_1$  and  $\Gamma_2$  in  $\text{PNR}_n$  correspond exactly to the subforests of  $F(\Gamma_1, \Gamma_2)$ , each of which is chosen by selecting a subset of the  $k$  edges. This gives a bijection from the interval  $[\Gamma_1, \Gamma_2]$  to the Boolean lattice  $\text{Bool}_k$ , and this is an isomorphism since  $\Gamma' \leq \Gamma''$  in  $[\Gamma_1, \Gamma_2]$  if and only if  $F(\Gamma_1, \Gamma')$  is a subforest of  $F(\Gamma_1, \Gamma'')$  by Corollary 2.6. See Figure 2.3 for an example.  $\square$

### 3. Associahedra and cyclohedra

The lower sets in the planar tree poset are closely related to two important types of convex polytopes: associahedra and cyclohedra. For an excellent overview of both, see [Dev03, Section 1].

**Definition 3.1.** Let  $n \geq 3$ , define  $z_k = e^{i\pi k/n}$  for every integer  $k$ , and let  $P_n$  denote the regular  $n$ -gon obtained by taking the convex hull in  $\mathbb{C}$  of the points  $z_1, \dots, z_n$ . A straight line segment between non-adjacent vertices of  $P_n$  is called a *diagonal*, and a *partial triangulation* of  $P_n$  is a (possibly empty) set of pairwise non-intersecting diagonals. The set  $\text{TRI}_n$  of all partial triangulations of  $P_n$  is a partially ordered set under reverse containment; given  $\tau_1$  and  $\tau_2$  in  $\text{TRI}_n$ , we say that  $\tau_1 \leq \tau_2$  if  $\tau_1$  contains  $\tau_2$ . Note that the minimal elements of  $\text{TRI}_n$  under this partial order are the actual triangulations of  $P_n$ . Finally, a partial triangulation of  $P_{2n}$  is *centrally symmetric* if it is invariant under rotation by  $\pi$ . The set of all centrally symmetric elements, denoted  $\overline{\text{TRI}}_n$ , is a subset of  $\text{TRI}_{2n}$ .

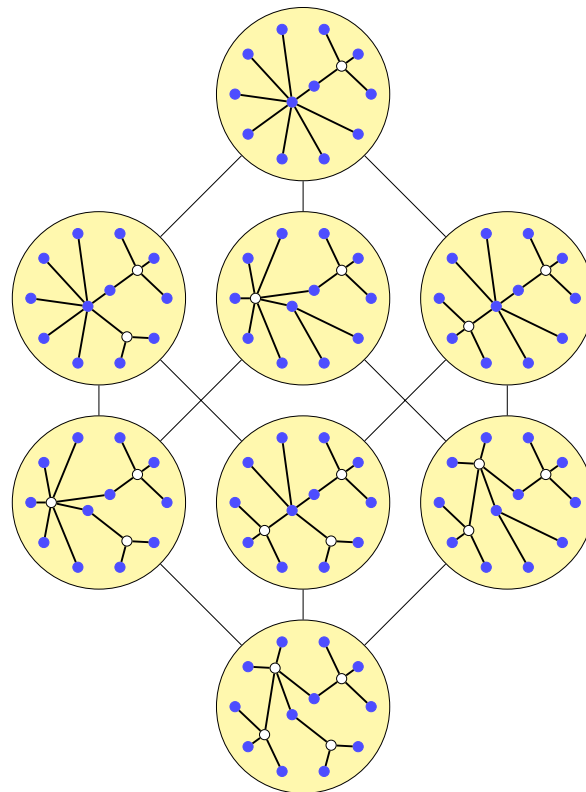


Figure 2.3: An interval in  $\text{PNR}_{12}$ . In this figure and several others which follow, we omit the vertex labels and instead distinguish marked vertices by their location in the plane.

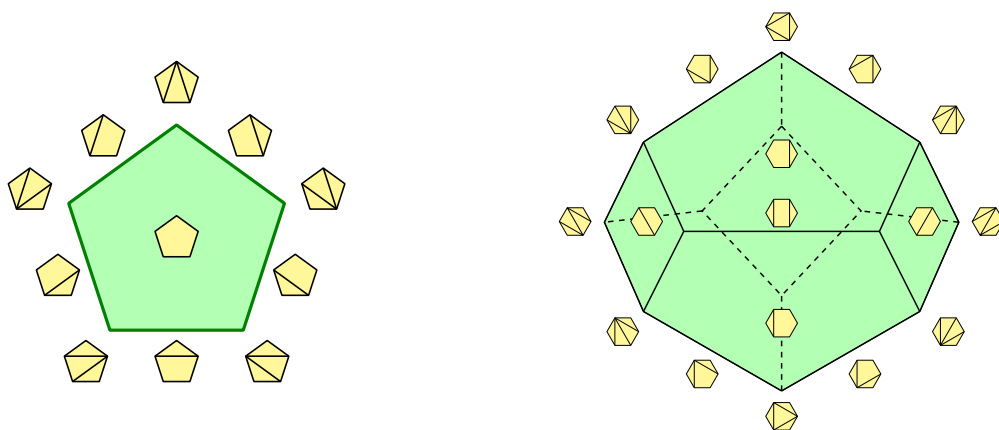


Figure 3.1: The associahedra  $K_4$  and  $K_5$ , with some of the front-facing cells labeled.



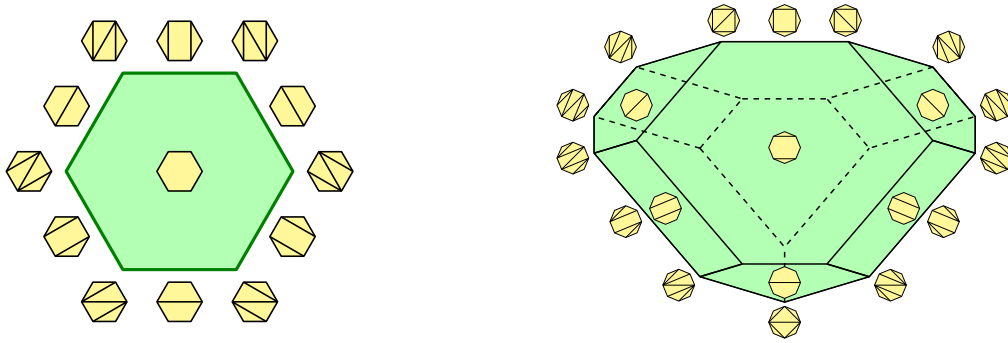


Figure 3.2: The cyclohedra  $W_3$  and  $W_4$ , with some of the front-facing cells labeled.

**Definition 3.2.** The *associahedron*  $K_n$  is an  $(n - 2)$ -dimensional convex polytope with face poset isomorphic to  $\overline{\text{TRI}}_{n+1}$ . For example,  $K_3$  is a line segment,  $K_4$  is a pentagon, and  $K_5$  is a polyhedron with 14 vertices and 9 faces: six pentagons and three squares. See Figure 3.1. In general, faces of the associahedron may be expressed as products of lower-dimensional associahedra.

The associahedron was initially given a combinatorial description by Tamari in 1951 before being rediscovered in a topological context by Stasheff in the 1960s [Sta12]. Associahedra and their generalizations play important roles in several areas of mathematics, including the studies of cluster algebras [FR07],  $A_\infty$ -algebras and  $A_\infty$ -categories (such as the Fukaya category of a symplectic manifold [Aur14]), and moduli spaces of disks [Dev99]. In physics, they also appear in the studies of open string theory and scattering amplitudes [AHBHY18].

**Definition 3.3.** The *cyclohedron*  $W_n$  is an  $(n - 1)$ -dimensional convex polytope with face poset isomorphic to  $\overline{\text{TRI}}_n$ . For example,  $W_2$  is a line segment,  $W_3$  is a hexagon, and  $W_4$  is a polyhedron with 20 vertices and 12 faces: four hexagons, four pentagons, and four squares. See Figure 3.2. More broadly, faces of the cyclohedron are products of lower-dimensional associahedra and (at most one) cyclohedra.

Among the many generalizations of the associahedron, cyclohedra are perhaps the closest relative. They first appeared in the context of knot invariants, where they were given a description by Bott and Taubes [BT94]. Realizations of the cyclohedron as a convex polytope later followed in the work of Markl [Mar99] and Simion [Sim03].

In the planar tree poset, the face posets of associahedra and cyclohedra appear as the lower sets of particular types of trees.

**Definition 3.4.** If  $\Gamma$  is a planar  $n$ -tree with a unique interior vertex, then  $\Gamma$  is a *star*. Since each leaf must be marked, there are two types of stars;  $\Gamma$  is a *marked star* if the unique interior vertex is marked and an *unmarked star* otherwise. Let  $*_n^\circ$  denote the unmarked star with  $n$  marked points, labeled  $1, \dots, n$  in counter-clockwise order around the unique unmarked interior vertex. Let  $*_n^\bullet$  denote the marked star with  $n$  marked points: a single interior vertex labeled  $n$ , and  $n - 1$  leaves labeled  $1, \dots, n - 1$  in counter-clockwise order around the unique marked interior vertex. See Figure 3.3.

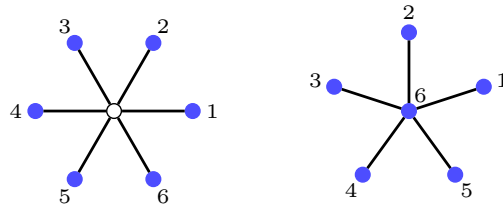


Figure 3.3: The unmarked star  $*_6^\circ$  and the marked star  $*_6^\bullet$ .

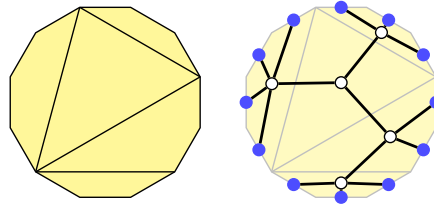


Figure 3.4: A partial triangulation in  $\text{TRI}_{12}$  and the associated planar tree in  $\downarrow(*_{12}^\circ)$ .

*Remark 3.5.* The trees which lie below  $*_{n+1}^\circ$  or  $*_{n+1}^\bullet$  in the partial order admit a canonical planar embedding. Let  $\Gamma$  be a planar  $n$ -tree in the lower set  $\downarrow(*_{n+1}^\circ)$ . Then each marked point of  $\Gamma$  is a leaf, so one can see that there is a natural representation of  $\Gamma$  in  $\mathbb{C}$  where each marked point labeled  $i$  is sent to  $(z_i + z_{i+1})/2$  (i.e. the midpoint of a side of  $P_{n+1}$ ) and the entire tree is contained within the unit disk. Similarly, if  $\Gamma$  is a planar  $n$ -tree in  $\downarrow(*_{n+1}^\bullet)$ , then there is a canonical planar embedding which sends the  $n$  (marked) leaves to the  $n$  edges of  $P_{n+1}$  as above and which sends the unique marked interior vertex to the origin. In both cases above, the embedding is unique up to isotopy fixing the marked points.

Equivalent versions of the following two lemmas have appeared previously in the literature, although we prove them here for the sake of completeness.

**Lemma 3.6** ([Dev99]). *The face poset of the associahedron  $K_n$  is isomorphic to  $\downarrow(*_{n+1}^\circ)$ .*

*Proof.* Let  $\tau$  be a partial triangulation in  $\text{TRI}_{n+1}$ ; we will construct a corresponding “dual” planar tree in  $\downarrow(*_{n+1}^\circ)$ . For each  $k \in \{1, \dots, n+1\}$ , place a marked vertex labeled  $k$  at  $(z_k + z_{k+1})/2$ . For each polygonal region of  $P_{n+1}$  formed by  $\tau$ , place an unmarked vertex at its barycenter. Add an edge from each unmarked vertex to the marked vertices labeling the corners of the corresponding polygonal region and to the unmarked vertices labeling adjacent polygonal regions. The resulting connected planar graph must be a tree since cutting  $P_{n+1}$  along any diagonal in  $\tau$  splits the polygon into two pieces, and this corresponds to deletion of any edge disconnecting the graph. Moreover, this process is reversible since each planar tree in  $\downarrow(*_{n+1}^\circ)$  can be drawn on  $P_{n+1}$  by Remark 3.5, from which we may recover the corresponding partial triangulation. Thus, we have defined a bijection  $\phi: \text{TRI}_{n+1} \rightarrow \text{PNR}_{n+1}$ . See Figure 3.4 for an example. Removing a diagonal in  $\tau$  corresponds exactly to contracting an edge in  $\phi(\tau)$ , so  $\phi$  is a poset isomorphism.  $\square$

**Lemma 3.7** ([Dev03]). *The face poset of the cyclohedron  $W_n$  is isomorphic to  $\downarrow(*_{n+1}^\bullet)$ .*

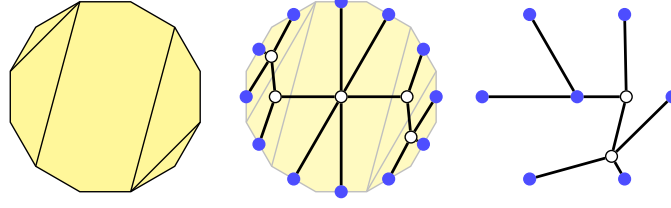


Figure 3.5: A centrally symmetric partial triangulation in  $\text{TRI}_{12}$  and its associated planar trees in  $\downarrow(*_{12}^\circ)$  and  $\downarrow(*_7^\bullet)$ .

*Proof.* Let  $\overline{\downarrow(*_{2n}^\circ)}$  denote the subposet of trees in  $\downarrow(*_{2n}^\circ)$  which are invariant under rotation by  $\pi$  and note that this subposet is isomorphic to  $\overline{\text{TRI}_n}$  by Lemma 3.6. Each  $\Gamma \in \overline{\downarrow(*_{2n}^\circ)}$  can be canonically represented in the polygon  $P_{2n}$ , where the center point is either an unmarked vertex of  $\Gamma$  or the midpoint of an edge in  $\Gamma$ . Either way, delete the center point, take the quotient of the tree by a  $\pi$  rotation, and replace the missing point with a marked vertex to obtain a new graph  $\Gamma'$ . The marked vertices labeled  $k$  and  $n+k$  in  $\Gamma$  are identified to a single marked vertex labeled  $k$  in  $\Gamma'$ . Label the new marked interior vertex by  $n+1$  and note that  $\Gamma'$  is a planar tree with  $n+1$  marked points. Moreover, we know that  $\Gamma$  contained a subtree which could be contracted to obtain  $*_{2n}^\circ$ , so by contracting the corresponding subtree of  $\Gamma'$  (which now contains the marked point  $n+1$ ), we obtain  $*_{n+1}^\bullet$ , and thus  $\Gamma' \in \downarrow(*_{n+1}^\bullet)$ . See Figure 3.5 for an example.

Define  $\psi: \overline{\downarrow(*_{2n}^\circ)} \rightarrow \downarrow(*_{n+1}^\bullet)$  by  $\psi(\Gamma) = \Gamma'$ . The quotient by  $\pi$  respects edge-contraction (so  $\psi$  is an order embedding) and can be undone to obtain a centrally symmetric planar tree (so  $\psi$  is a bijection). Therefore,  $\psi$  is an isomorphism.  $\square$

These lemmas provide the necessary tools to examine the lower set  $\downarrow(\Gamma)$  for an arbitrary tree  $\Gamma \in \text{PNR}_n$ .

**Definition 3.8.** Let  $\Gamma$  be a planar  $n$ -tree and let  $\Gamma_1$  be a subtree of  $\Gamma$ . Then the (closed) *neighborhood* of  $\Gamma_1$  in  $\Gamma$ , denoted  $\text{nbhd}(\Gamma_1)$ , is the smallest subtree containing  $\Gamma_1$  and all of its incident edges. In particular, if  $v$  is a vertex of  $\Gamma$  with valence  $k$ , then  $\text{nbhd}(v)$  is a star. With the appropriate labeling,  $\text{nbhd}(v)$  is isomorphic to either  $*_k^\circ$  (if  $v$  is unmarked) or  $*_k^\bullet$  (if  $v$  is marked).

**Theorem 3.9.** Let  $\Gamma$  be a planar  $n$ -tree with unmarked interior vertices  $u_1, \dots, u_k$  and marked interior vertices  $v_1, \dots, v_\ell$ . Then

$$\downarrow(\Gamma) \cong \prod_{1 \leq i \leq k} \downarrow(*_{\text{val}(u_i)}^\circ) \times \prod_{1 \leq j \leq \ell} \downarrow(*_{\text{val}(v_j)}^\bullet)$$

and therefore  $\downarrow(\Gamma)$  is the face poset of a convex polytope which can be expressed as a product of associahedra and cyclohedra.

*Proof.* Let  $\Gamma$  be a planar  $n$ -tree, let  $u$  be an unmarked interior vertex of  $\Gamma$ , and fix an identification between  $\text{nbhd}(u)$  and  $*_{\text{val}(u)}^\circ$  as described in Definition 3.8. For each  $\Gamma' \in \downarrow(\Gamma)$ , we know that some subtree  $\Gamma'_u$  of  $\Gamma'$  contracts to  $u$  and so  $\text{nbhd}(\Gamma'_u)$  contracts to  $\text{nbhd}(u)$ . Abusing notation slightly, we denote this by writing  $\text{nbhd}(\Gamma'_u) \in \downarrow(*_{\text{val}(u)}^\circ)$ . Similarly, we can see that each interior marked vertex  $v$  of  $\Gamma$  has a corresponding subtree  $\Gamma'_v$  of  $\Gamma'$  such that  $\text{nbhd}(\Gamma'_v) \in \downarrow(*_{\text{val}(v)}^\bullet)$ .

Now, let  $u_1, \dots, u_k$  and  $v_1, \dots, v_\ell$  be the unmarked and marked interior vertices of  $\Gamma$ , respectively. Identify each  $\text{nbhd}(u_i)$  with  $*_{\text{val}(u_i)}^\circ$  and each  $\text{nbhd}(v_i)$  with  $*_{\text{val}(v_i)}^\bullet$  as before and define the function

$$\psi: \downarrow(\Gamma) \rightarrow \prod_{1 \leq i \leq k} \downarrow(*_{\text{val}(u_i)}^\circ) \times \prod_{1 \leq j \leq \ell} \downarrow(*_{\text{val}(v_j)}^\bullet)$$

by declaring  $\psi(\Gamma') = (\text{nbhd}(\Gamma'_{u_1}), \dots, \text{nbhd}(\Gamma'_{u_k}), \text{nbhd}(\Gamma'_{v_1}), \dots, \text{nbhd}(\Gamma'_{v_\ell}))$ . We will show that  $\psi$  is a poset isomorphism.

Given any  $u_i$  and an element  $\Lambda_i \in \downarrow(*_{\text{val}(u_i)}^\circ)$ , we can use the identification fixed previously to replace  $\text{nbhd}(u_i)$  with  $\Lambda_i$ , and the resulting planar  $n$ -tree will be an element of  $\downarrow(\Gamma)$  since, by construction, it can be contracted to  $\Gamma$ . Similarly, we can replace  $\text{nbhd}(v_j)$  with any element of  $\downarrow(*_{\text{val}(v_j)}^\bullet)$  to obtain an element of  $\downarrow(\Gamma)$ . Moreover, these replacements do not interfere with one another, so any  $(k + \ell)$ -tuple in the codomain of  $\psi$  determines a unique  $n$ -tree  $\Gamma'$  in  $\downarrow(\Gamma)$ . Finally, note that  $\psi(\Gamma')$  returns our original  $(k + \ell)$ -tuple by construction and therefore  $\psi$  is a bijection.

Next, let  $\Gamma'$  and  $\Gamma''$  be elements of  $\downarrow(\Gamma)$ . We know by the first part of Corollary 2.6 that  $\Gamma' \leq \Gamma''$  if and only if  $F(\Gamma'', \Gamma)$  is a subforest of  $F(\Gamma', \Gamma)$ . This is equivalent to saying that  $\Gamma''_{u_i} \leq \Gamma'_{u_i}$  and  $\Gamma''_{v_j} \leq \Gamma'_{v_j}$  for each  $i$  and  $j$ , which is then equivalent to saying that  $\text{nbhd}(\Gamma''_{u_i}) \leq \text{nbhd}(\Gamma'_{u_i})$  in  $*_{\text{val}(u_i)}^\circ$  and  $\text{nbhd}(\Gamma''_{v_j}) \leq \text{nbhd}(\Gamma'_{v_j})$  in  $*_{\text{val}(v_j)}^\bullet$  for each  $i$  and  $j$ . We have thus shown that  $\Gamma' \leq \Gamma''$  if and only if  $\psi(\Gamma') \leq \psi(\Gamma'')$ , so  $\psi$  is an order embedding and therefore an isomorphism.  $\square$

**Example 3.10.** If  $\Gamma$  is the planar 6-tree depicted in Figure 2.1, then the lower set  $\downarrow(\Gamma)$  is isomorphic to the direct product  $\downarrow(*_4^\circ) \times \downarrow(*_3^\bullet)$ , which is isomorphic to the face poset for the product of  $K_3 \times W_2$ , i.e. a hexagonal prism.

*Remark 3.11.* As Theorem 3.9 suggests,  $\text{Pnr}_n$  is the face poset of a regular cell complex in which each cell is a convex polytope. As depicted in Figure 2.2,  $\text{Pnr}_3$  is the face poset of a 1-dimensional complex with 3 edges and 2 vertices (i.e. a theta graph). Meanwhile,  $\text{Pnr}_4$  is the face poset of a 2-dimensional complex consisting of 20 2-cells (eight hexagons and twelve squares), 30 edges, and 12 vertices.

### 4. Tree complexes

In this section, we consider planar tree embeddings up to relative isotopy fixing the marked vertices pointwise. This infinite set of trees admits a similar partial order by contraction and forms the face poset for a contractible cell complex.

**Definition 4.1.** Fix a set of points  $P = \{z_1, \dots, z_n\}$  in  $\mathbb{C}$ , where  $n \geq 3$ . A *planted  $n$ -tree* is a relative isotopy class of embeddings  $\phi: \Gamma \rightarrow \mathbb{C}$  where  $\Gamma$  is an  $n$ -tree such that each marked point  $v_i$  in  $\Gamma$  is sent to  $z_i$  in  $P$ . Following Definition 2.8, we say that if  $\phi_1: \Gamma_1 \rightarrow \mathbb{C}$  and  $\phi_2: \Gamma_2 \rightarrow \mathbb{C}$  are planted  $n$ -trees and  $f: \Gamma_1 \rightarrow \Gamma_2$  is a contraction such that  $\phi_1$  and  $\phi_2 \circ f$  are isotopic relative to the marked points, then  $f$  is a *planted contraction*. If such a contraction exists, we write  $\Gamma_1 \leq \Gamma_2$  and observe that this determines a partial order on the set of all planted  $n$ -trees. We refer to this partially ordered set as the *planted tree poset* and denote it  $\text{PTD}_n$ .

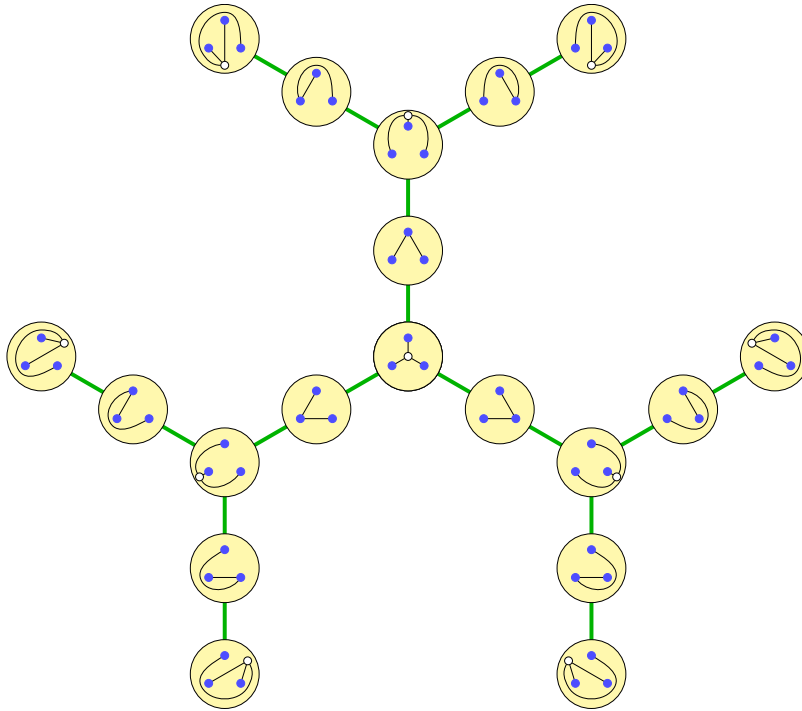


Figure 4.1: A piece of the simplicial tree complex  $\mathcal{S}_3$ .

It is worth noting that the combinatorial structure of the planted tree poset does not depend on our choice of  $P$ . More explicitly, if  $\text{PTD}_n(P)$  and  $\text{PTD}_n(P')$  are the posets determined by two sets  $P$  and  $P'$  of  $n$  points in  $\mathbb{C}$ , then any homeomorphism  $\mathbb{C} \rightarrow \mathbb{C}$  which takes  $P$  to  $P'$  induces an isomorphism  $\text{PTD}_n(P) \rightarrow \text{PTD}_n(P')$ .

**Definition 4.2.** The *simplicial tree complex*  $\mathcal{S}_n$  is the order complex of the planted tree poset  $\text{PTD}_n$ . For example,  $\mathcal{S}_3$  is isomorphic to the infinite bipartite tree  $T_{2,3}$ , depicted in Figure 4.1.

This complex appeared in recent work of Belk, Lanier, Margalit, and Winarski on complex dynamics [BLMW22]. In this setting, each postcritically finite complex polynomial (i.e. one in which the critical points have finite forward orbits) has an associated *Hubbard tree*. The simplicial tree complex then acts as a useful tool for studying transformations of polynomials via their Hubbard trees. Using a result of Penner [Pen96], the authors in [BLMW22] describe an embedding of the simplicial tree complex as a spine for the Teichmüller space of the  $(n+1)$ -punctured sphere. As a consequence, they conclude that the simplicial tree complex is contractible. The focus of this article is to examine the combinatorial structure of the simplicial tree complex and introduce a simpler polyhedral cell structure.

Any self-homeomorphism of the plane which fixes  $P$  pointwise sends each planted  $n$ -tree to another planted  $n$ -tree, and the only homeomorphisms which send a tree to itself are those which are isotopic to the identity. Moreover, if  $\phi_1: \Gamma_1 \rightarrow \mathbb{C}$  and  $\phi_2: \Gamma_2 \rightarrow \mathbb{C}$  are planted  $n$ -trees, then  $\phi_1$  and  $\phi_2$  are freely isotopic if and only if there is a homeomorphism  $g: \mathbb{C} \rightarrow \mathbb{C}$  such that  $\phi_1$  and  $g \circ \phi_2$  are isotopic relative to the marked points. In other words, if we let  $\mathbb{C}_P$  denote the  $n$ -punctured plane  $\mathbb{C} - P$ , then the pure mapping class group  $\text{PMod}(\mathbb{C}_P)$  acts freely on  $\text{PTD}_n$ , and

the orbits under this action correspond exactly to planar  $n$ -trees. This action is equivariant with respect to the partial order on the planted tree poset, which is to say that for any  $g \in \text{PMod}(\mathbb{C}_P)$ ,  $\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$  if and only if  $g\Gamma_1 \leq g\Gamma_2$ .

**Definition 4.3.** Define the order-preserving surjective map  $p: \text{PTD}_n \rightarrow \text{PNR}_n$  by sending each planted  $n$ -tree  $\phi: \Gamma \rightarrow \mathbb{C}$  to its free isotopy class in  $\text{PNR}_n$ , and observe that the preimages under this map are the orbits of the action by  $\text{PMod}(\mathbb{C}_P)$ . In particular, the partial order on the planar tree poset matches the partial order on the orbits:  $p(\Gamma_1) \leq p(\Gamma_2)$  in  $\text{PNR}_n$  if and only if  $g\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$  for some  $g \in \text{PMod}(\mathbb{C}_P)$ .

Our first step is to strengthen an observation from the definition above.

**Lemma 4.4.** *Let  $\Gamma_1$  and  $\Gamma_2$  be planted  $n$ -trees. Then  $p(\Gamma_1) \leq p(\Gamma_2)$  in  $\text{PNR}_n$  if and only if there is a unique  $g \in \text{PMod}(\mathbb{C}_P)$  such that  $g\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$ .*

*Proof.* Let  $\Gamma_1$  and  $\Gamma_2$  be planted  $n$ -trees with planar embeddings  $\phi_1: \Gamma_1 \rightarrow \mathbb{C}$  and  $\phi_2: \Gamma_2 \rightarrow \mathbb{C}$  respectively. We can see from Definition 4.3 that  $p(\Gamma_1) \leq p(\Gamma_2)$  in  $\text{PNR}_n$  if and only if  $g\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$  for some  $g$  in the pure mapping class group  $\text{PMod}(\mathbb{C}_P)$ ; all that remains is to show that  $g$  must be unique. Suppose  $p(\Gamma_1) \leq p(\Gamma_2)$  and that  $g\Gamma_1 \leq \Gamma_2$  and  $h\Gamma_1 \leq \Gamma_2$  for some  $g, h \in \text{PMod}(\mathbb{C}_P)$ . By Lemma 2.4, both of these inequalities must be realized by the same contraction  $f: \Gamma_1 \rightarrow \Gamma_2$ . We then know that  $g \circ \phi_1$  is relatively isotopic to  $\phi_2 \circ f$  since  $g\Gamma_1 \leq \Gamma_2$  and  $h \circ \phi_1$  is relatively isotopic to  $\phi_2 \circ f$  since  $h\Gamma_1 \leq \Gamma_2$ . Therefore,  $g \circ \phi_1$  is isotopic to  $h \circ \phi_2$  relative to the marked points, and since  $\text{PMod}(\mathbb{C}_P)$  acts freely on the planted tree poset, we may conclude that  $g = h$ .  $\square$

The preceding lemma implies that  $p$  restricts to injective maps on the lower set  $\downarrow(\Gamma)$  and the upper set  $\uparrow(\Gamma)$  (recall Definition 1.2).

**Lemma 4.5.** *Let  $\Gamma$  be a planted  $n$ -tree. Then  $\downarrow(\Gamma)$  is isomorphic to  $\downarrow(p(\Gamma))$  and  $\uparrow(\Gamma)$  is isomorphic to  $\uparrow(p(\Gamma))$ .*

*Proof.* Let  $\Gamma$  be a planted  $n$ -tree. By Lemma 4.4, we know that every element of  $\downarrow(p(\Gamma))$  has a unique preimage under  $p$  which lies in  $\downarrow(\Gamma)$ . In particular, this lemma tells us that restricting  $p$  to a map  $\downarrow(\Gamma) \rightarrow \downarrow(p(\Gamma))$  yields a surjective order embedding, i.e. a poset isomorphism. By noting that  $g\Gamma_1 \leq \Gamma_2$  if and only if  $\Gamma_1 \leq g^{-1}\Gamma_2$ , we obtain the analogous result for  $\uparrow(\Gamma)$  as well.  $\square$

Now that we have established the connection between planted and planar trees, we are ready to introduce a simplified cell structure for the simplicial tree complex.

**Theorem 4.6** (Theorem B). *The planted tree poset  $\text{PTD}_n$  is the face poset of a regular CW-complex  $\mathcal{P}_n$  which we call the polyhedral tree complex, in which each planted  $n$ -tree  $\Gamma$  labels a cell which is isomorphic to the convex polytope labeled by the planar  $n$ -tree  $p(\Gamma)$ , and gluing relations are given by the partial order in  $\text{PTD}_n$ . In particular, the top-dimensional cells are products of cyclohedra. Consequently, the simplicial tree complex is the barycentric subdivision of the polyhedral tree complex.*

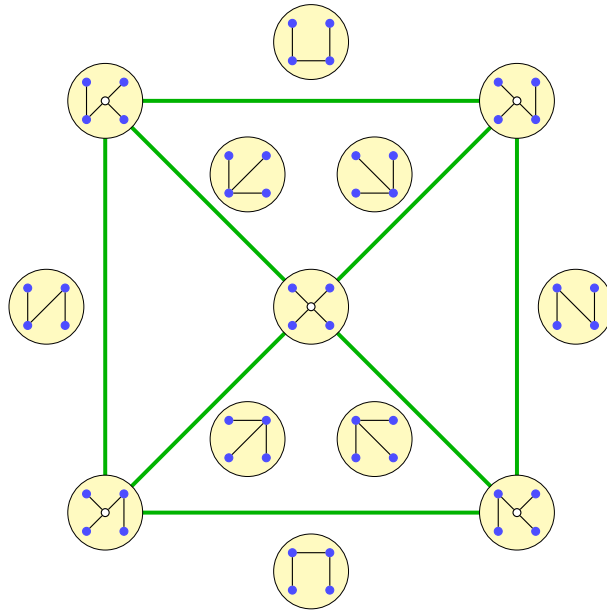


Figure 4.2: The link of a vertex in  $\mathcal{P}_4$ .

*Proof.* The first claim follows from the definitions and Lemma 4.5. The fact that top-dimensional cells can be expressed as products of cyclohedra is a consequence of Theorem 3.9. The final claim follows from the relationship between order complexes and their face posets.  $\square$

Since the simplicial tree complex is contractible [BLMW22], we may immediately conclude the following corollary.

**Corollary 4.7.** *The polyhedral tree complex is contractible and admits a free and cocompact action by the pure mapping class group for the  $n$ -punctured plane. The quotient by this action, a finite cell complex with  $P_{NR_n}$  as its face poset, is thus a classifying space for  $\text{PMod}(\mathbb{C}_P)$ .*

**Example 4.8.** To demonstrate the relative simplicity of the polyhedral cell structure, let  $\Gamma$  be a planted 6-tree such that  $p(\Gamma)$  is the planar 6-tree shown in Figure 2.1. Then  $\Gamma$  labels a hexagonal prism in the polyhedral tree complex, and the barycentric subdivision of this prism (i.e. the corresponding subcomplex in the simplicial tree complex) consists of 72 tetrahedra which meet at a common vertex.

We close the section with two illustrative examples.

**Example 4.9.** The simplicial tree complex  $\mathcal{S}_3$  is the bipartite tree  $T_{2,3}$ , whereas the polyhedral tree complex  $\mathcal{P}_3$  is the regular trivalent tree  $T_3$  - see Figure 4.1. The quotient of  $\mathcal{P}_3$  by the  $\text{PMod}(\mathbb{C}_P)$  action is the theta graph described in Remark 3.11.

**Example 4.10.** The polyhedral tree complex  $\mathcal{P}_4$  is a two-dimensional cell complex which can be thought of as the universal cover of the complex with  $P_{NR_4}$  as its face poset, described in Remark 3.11. The link of a vertex in  $\mathcal{P}_4$  is a graph with five vertices and eight edges - see Figure 4.2.

### 5. Marked and unmarked edges

The final section of this article concerns an equivalence relation on the set of planted  $n$ -trees where elements are considered up to contraction of unmarked edges.

**Definition 5.1.** Let  $\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$ . If each component in the subforest  $F(\Gamma_1, \Gamma_2)$  is singly-marked, we write  $\Gamma_1 \leq_m \Gamma_2$ . If each component in the subforest is unmarked, then we write  $\Gamma_1 \leq_u \Gamma_2$ .

**Lemma 5.2.** Suppose that  $\Gamma_1 \leq \Gamma_2$  in  $\text{PTD}_n$ . Then there is a unique  $\Gamma_3 \in \text{PTD}_n$  such that  $\Gamma_1 \leq_u \Gamma_3$  and  $\Gamma_3 \leq_m \Gamma_2$ .

*Proof.* If some subforest of  $\Gamma_1$  can be contracted to obtain  $\Gamma_2$ , then we may instead contract only the unmarked edges to obtain  $\Gamma_3 \in \text{PTD}_n$  such that  $\Gamma_1 \leq_u \Gamma_3$  and  $\Gamma_3 \leq_m \Gamma_2$ . Moreover,  $\Gamma_3$  is the unique tree with this property; if we contracted a proper subset of the unmarked edges to obtain  $\Gamma_3$ , then we would need to contract such an edge in  $\Gamma_3$  to obtain  $\Gamma_2$ , meaning that  $\Gamma_3 \leq \Gamma_2$ , but  $\Gamma_3 \not\leq_m \Gamma_2$ .  $\square$

**Lemma 5.3.** Suppose that  $\Gamma_u \leq_u \Gamma$  and  $\Gamma_m \leq_m \Gamma$  in  $\text{PTD}_n$ . Then the meet  $\Gamma_u \wedge \Gamma_m$  exists in  $\text{PTD}_n$ , and  $\Gamma_u \wedge \Gamma_m \leq_m \Gamma_u$  and  $\Gamma_u \wedge \Gamma_m \leq_u \Gamma_m$ .

*Proof.* If  $\Gamma_u$  is obtained from  $\Gamma$  by expanding some collection of unmarked vertices into subtrees and  $\Gamma_m$  is obtained from  $\Gamma$  by expanding some collection of marked vertices into subtrees, each containing exactly one marked point, then these expansions necessarily occur at disjoint sets of vertices, so they do not interfere with one another and may be performed simultaneously to obtain  $\Gamma_u \wedge \Gamma_m$ . By construction, we can further see that  $\Gamma_u \wedge \Gamma_m \leq_m \Gamma_u$  and  $\Gamma_u \wedge \Gamma_m \leq_u \Gamma_m$ .  $\square$

These lemmas allow us to examine a modified version of the planted tree poset, in which we consider  $n$ -trees up to contraction of unmarked edges.

**Definition 5.4.** A planted  $n$ -tree is *reduced* if it has no edges between unmarked vertices. For each planted  $n$ -tree  $\Gamma$ , there is a unique reduced tree in  $\uparrow(\Gamma)$ , denoted  $r(\Gamma)$ , obtained by contracting all edges between unmarked vertices. Define an equivalence relation on  $\text{PTD}_n$  by declaring  $\Gamma_1 \sim \Gamma_2$  if and only if  $r(\Gamma_1) = r(\Gamma_2)$ , let  $[\Gamma]$  denote the equivalence class of  $\Gamma$  under this relation (for which  $r(\Gamma)$  is the maximum element) and let  $\text{RED}_n$  denote the set of equivalence classes  $\text{PTD}_n / \sim$ .

**Lemma 5.5.** Declaring that  $[\Gamma_1] \leq [\Gamma_2]$  in  $\text{RED}_n$  if and only if  $\Gamma'_1 \leq \Gamma'_2$  in  $\text{PTD}_n$  for some  $\Gamma'_1 \in [\Gamma_1]$  and  $\Gamma'_2 \in [\Gamma_2]$  yields a partial order for  $\text{RED}_n$ .

*Proof.* By Lemma 5.2, we know that if  $\Gamma'_1 \in [\Gamma_1]$  and  $\Gamma'_2 \in [\Gamma_2]$  with  $\Gamma'_1 \leq \Gamma'_2$ , then there is some  $\Gamma''_1 \in [\Gamma_1]$  with  $\Gamma''_1 \leq_m \Gamma'_2$ . Thus, it suffices to consider contractions involving marked edges in  $\text{PTD}_n$ .

It follows immediately from the definition that this relation is reflexive, and anti-symmetry follows from our remark above: if  $[\Gamma_1] \leq [\Gamma_2]$ , then  $\Gamma'_1 \leq_m \Gamma'_2$  for some  $\Gamma'_1, \Gamma'_2 \in \text{PTD}_n$ , and so  $\Gamma'_1$  has at least as many edges with exactly one marked endpoint as  $\Gamma'_2$  does. Since the number



of such edges is constant in each equivalence class, we know that  $[\Gamma_2] \leq [\Gamma_1]$  only if  $\Gamma'_1$  and  $\Gamma'_2$  actually have the same number of edges with one marked endpoint, in which case  $[\Gamma_1] = [\Gamma_2]$ .

All that remains is to demonstrate transitivity. Suppose that  $[\Gamma_1] \leq [\Gamma_2]$  and  $[\Gamma_2] \leq [\Gamma_3]$  in  $\text{RED}_n$ . Without loss of generality, we may write  $\Gamma_1 \leq_m \Gamma_2$  and  $\Gamma'_2 \leq_m \Gamma_3$ , where  $\Gamma'_2 \in [\Gamma_2]$ . By Lemma 5.3, we know that  $\Gamma_1 \wedge \Gamma'_2 \leq_u \Gamma_1$  and  $\Gamma_1 \wedge \Gamma'_2 \leq_m \Gamma'_2$ . By transitivity of the partial order for  $\text{PTD}_n$ , we then have that  $\Gamma_1 \wedge \Gamma'_2 \leq_m \Gamma_3$ , so  $[\Gamma_1] \leq [\Gamma_3]$  and we are done.  $\square$

**Definition 5.6.** Equipped with the partial order from Lemma 5.5, we refer to  $\text{RED}_n$  as the *reduced tree poset*.

*Remark 5.7.* Suppose that  $\Gamma_1$  and  $\Gamma_2$  are reduced  $n$ -trees and that  $[\Gamma_1] < [\Gamma_2]$  in  $\text{RED}_n$ . There are three possibilities for the relationship between  $\Gamma_1$  and  $\Gamma_2$ .

1. (contraction) If  $\Gamma_1 < \Gamma_2$  in  $\text{PTD}_n$ , then  $\Gamma_2$  is obtained from  $\Gamma_1$  by contracting one singly-marked edge.

If  $\Gamma_2$  is not a contraction of  $\Gamma_1$ , then there exist  $\Gamma'_1$  and  $\Gamma'_2$  with  $\Gamma'_1 \leq_u \Gamma_1$  and  $\Gamma'_2 \leq_u \Gamma_2$  such that  $\Gamma'_2$  can be obtained from  $\Gamma'_1$  by contracting some number of singly-marked edges. Since each can be contracted independently, it suffices to consider the case when there is only one contracted singly-marked edge  $e$  in  $\Gamma'_1$ .

2. (slide) If the unmarked end of  $e$  is adjacent to exactly one unmarked vertex, then  $\Gamma_2$  is obtained from  $\Gamma_1$  by “sliding” some number of edges along  $e$ .
3. (split) If the unmarked end of  $e$  is adjacent to multiple unmarked vertices, then  $\Gamma_2$  is obtained from  $\Gamma_1$  by “splitting”  $e$  into some number of copies with different unmarked endpoints.

Thus, when  $[\Gamma_1] < [\Gamma_2]$  in  $\text{RED}_n$ , we know that  $\Gamma_2$  is obtained from  $\Gamma_1$  by a sequence of contractions, slides, and splits along singly-marked edges. See Figure 5.1.

*Remark 5.8.* Let  $\Gamma$  be a reduced  $n$ -tree and suppose that  $[\Gamma]$  lies in either the lower set or upper set for  $[\ast_n^\bullet]$  or  $[\ast_n^\circ]$ . By Remark 5.7, we then know that  $\Gamma$  is related to either a marked or unmarked star by a sequence of contractions, slides, and splits. In the same spirit as Remark 3.5, note that each of these three operations (and their reverses) do not disturb the canonical planar embedding of a star onto a disk. Thus,  $\Gamma$  can be represented in the plane as a tree with certain marked points on the unit circle ( $v_1, \dots, v_{n-1}$  if  $\Gamma$  is related to  $\ast_n^\bullet$  and  $v_1, \dots, v_n$  if related to  $\ast_n^\circ$ ) and the rest of the tree on the interior of the unit disk.

For the remainder of this article, we consider the structure of lower sets and upper sets in the reduced tree poset.

**Lemma 5.9.** *The lower set  $\downarrow([\ast_n^\bullet])$  is isomorphic to  $\text{Bool}_{L_{n-1}}^*$ .*

*Proof.* Let  $[\Gamma] \in \downarrow([\ast_n^\bullet])$ , where  $\Gamma$  is chosen to be the maximum element of its equivalence class. By Remark 5.8, we may draw  $\Gamma$  so that the marked points  $v_1, \dots, v_{n-1}$  are on the boundary of the unit disk and  $v_n$  lies at the origin. Then  $\Gamma$  divides the disk into  $n - 1$  regions; let  $\Omega_i$  denote the

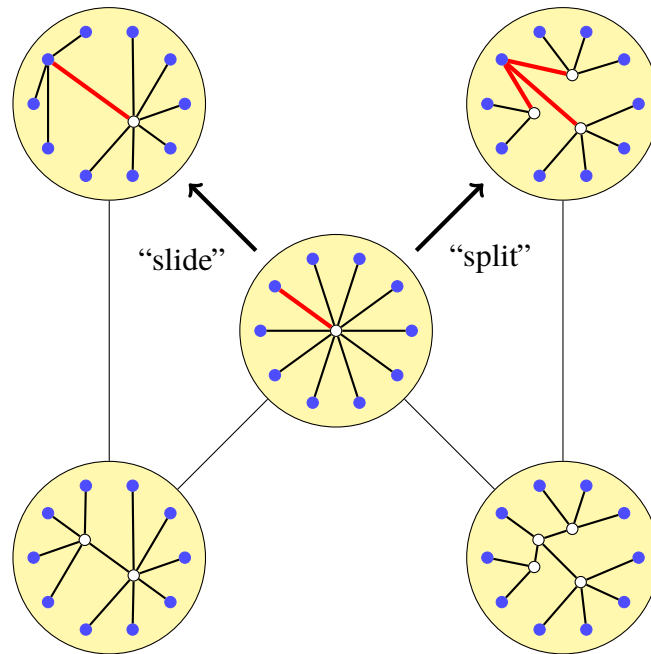


Figure 5.1: A slide and a split for  $*_{10}^{\circ}$ . The thin lines connecting trees indicate the partial order in  $\text{PTD}_{10}$ , and the relevant singly-marked edge is thicker and colored red.

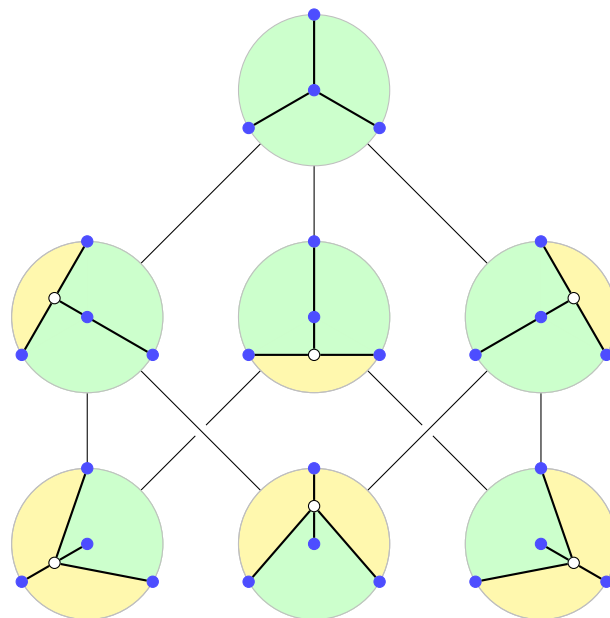


Figure 5.2: The lower set  $\downarrow([*_{4}^{\bullet}])$ , where each equivalence class is represented by its maximal element and the regions adjacent to the unique interior marked point are shaded green.

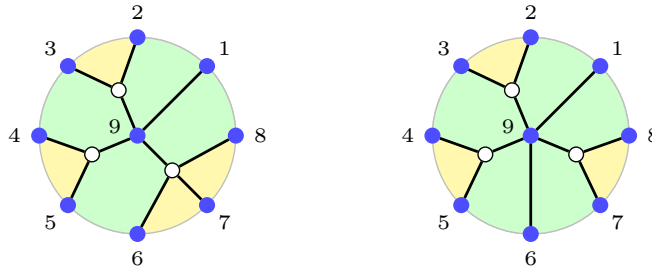


Figure 5.3: These reduced 9-trees correspond to the subsets  $\{1, 3, 5, 8\} \subseteq \{1, 3, 5, 6, 8\}$  in  $\text{Bool}_8^*$ , where the relevant regions are shaded green. Note that the second tree can be obtained from the first by performing a slide.

region which is incident to  $v_i$  and  $v_{i+1}$ . The vertex  $v_n$  is then incident to some nonempty subset of  $\{\Omega_1, \dots, \Omega_{n-1}\}$ , which we can naturally identify with an element of  $\text{Bool}_{n-1}^*$ . Note that this subset is actually well-defined on the equivalence class  $[\Gamma]$  since the contraction or expansion of unmarked edges does not change which regions are incident to  $v_n$ . Define  $f: \downarrow([\ast_n^\bullet]) \rightarrow \text{Bool}_{n-1}^*$  to be the function which sends  $[\Gamma]$  to this corresponding subset of  $\{1, \dots, n-1\}$ . If  $\Gamma_1$  and  $\Gamma_2$  are reduced  $n$ -trees such that  $[\Gamma_1], [\Gamma_2] \in \downarrow([\ast_n^\bullet])$  and  $[\Gamma_1] \leq [\Gamma_2]$ , then  $\Gamma_2$  is obtained from  $\Gamma_1$  by a sequence of contractions, slides, and/or splits by Remark 5.7, and since each of these increases the number of regions incident to  $v_n$ , it follows that  $f(\Gamma_1) \subseteq f(\Gamma_2)$  and thus  $f$  is order-preserving.

Next, we build an inverse for  $f$  by constructing a reduced  $n$ -tree for each nonempty  $A \subset \{1, \dots, n-1\}$ . Begin with  $v_1, \dots, v_{n-1}$  labeling points around the boundary of the unit disk and let  $v_n$  be at the origin. For each maximal string of consecutive cyclically-ordered elements in  $\{1, \dots, n-1\} - A$ , introduce a new unmarked vertex near the corresponding boundary vertices, then connect it to those vertices and  $v_n$ . For example, if  $i$  and  $i+k$  (reduced mod  $n-1$ ) are in  $A$ , but  $i+1, i+2, \dots, i+k-1$  are not, then we would introduce a single unmarked vertex associated to this string of elements and connect it to  $v_{i+1}, v_{i+2}, \dots, v_{i+k-1}$  and  $v_n$ . Finally, we connect any remaining boundary vertices directly to  $v_n$ . The result is a reduced planted  $n$ -tree such that  $v_n$  is adjacent to the desired regions. Moreover, this inverse is order-preserving as well - see Figure 5.3 for an illustrative example. Therefore,  $f$  is an isomorphism from  $\downarrow([\ast_n^\bullet])$  to  $\text{Bool}_{n-1}^*$  and the proof is complete.  $\square$

Using Lemmas 3.7 and 5.9, together with the fact that  $\text{Bool}_{n-1}^*$  is the face poset for an  $(n-2)$ -simplex, we can see that the combinatorial process of passing from planted  $n$ -trees to reduced  $n$ -trees induces a topological transformation of a cyclohedron into a simplex of equal dimension. This procedure has been described previously by Devadoss: the  $n$ -dimensional cyclohedron can be obtained from the Coxeter simplex for the affine Coxeter group of type  $\tilde{A}_{n-1}$  by iteratively truncating a certain sequence of faces [Dev03].

More generally, Theorem 3.9 and Lemma 4.5 imply that  $\downarrow(\Gamma)$  is the face poset for a product of associahedra and cyclohedra for any planted  $n$ -tree  $\Gamma$ . The following theorem gives the corresponding topological transformation for  $\downarrow([\Gamma])$ .

**Theorem 5.10** (Theorem C, first claim). *Let  $\Gamma$  be a reduced  $n$ -tree with marked interior vertices  $v_1, \dots, v_\ell$ . Then*

$$\downarrow([\Gamma]) \cong \prod_{i=1}^{\ell} \text{Bool}_{\text{val}^*(v_i)}$$

and so the lower set  $\downarrow([\Gamma])$  is isomorphic to the face poset for a product of simplices.

*Proof.* The main idea for this proof is similar to that of Theorem 3.9: the structure of  $\downarrow([\Gamma])$  is determined by what the neighborhoods of interior vertices for  $\Gamma$  look like. The additional wrinkle in this case is that we are now dealing with equivalence classes of trees, so it is not immediately clear that such an argument is valid.

To that end, note that  $[\ast_n^\circ]$  is the unique element of  $\downarrow([\ast_n^\circ])$  since the only trees below  $\ast_n^\circ$  in the partial order are obtained by expanding the unique unmarked vertex, and hence lie in the same equivalence class. So when considering elements of  $\downarrow([\Gamma])$ , we need only concern ourselves with marked interior vertices. Note also that the neighborhood of such an interior vertex remains unchanged under contraction of unmarked edges, so there is a well-defined meaning for the neighborhoods of marked interior vertices of  $[\Gamma]$ .

From here, the proof is identical to that of Theorem 3.9. Since the neighborhoods of interior vertices are well-defined up to equivalence and the expansions of distinct interior vertices can be considered independently, we know that  $\downarrow([\Gamma])$  is isomorphic to the product of the lower sets of the neighborhoods of its interior vertices. Finally, we know that  $\downarrow([\ast_n^\circ])$  is trivial by our argument above and that  $\downarrow([\ast_n^\bullet]) \cong \text{Bool}_{n-1}^*$  by Lemma 5.9, so  $\downarrow([\Gamma])$  is isomorphic to the face poset of a product of simplices.  $\square$

The second half of Theorem C concerns the upper sets of  $\text{RED}_n$ , which can also be connected to a well-understood object: the poset of noncrossing hypertrees. We give a specialized definition here; more background may be found in [McC].

**Definition 5.11.** Let  $n \geq 3$ , define  $z_k = e^{i\pi k/n}$  for each integer  $k$ , and let  $V = \{z_1, \dots, z_n\}$ . A *hyperedge* is a subset of  $V$  with at least two elements and can be drawn in the plane as the convex hull of its vertices; a *hypergraph* is a collection of hyperedges on the vertex set  $V$ . A *noncrossing hypertree* is a hypergraph with no embedded loops such that the intersection of any pair of hyperedges is either empty or consists of a single vertex. The set of all noncrossing hypertrees on this vertex set admits a partial order: if  $\sigma$  and  $\tau$  are noncrossing hypertrees, then we say that  $\sigma \leq \tau$  if each hyperedge of  $\sigma$  is a subset of a hyperedge of  $\tau$ . Denote the partially ordered set of noncrossing hypertrees by  $\text{NCHT}_n$ .

**Lemma 5.12.** *The upper set  $\uparrow([\ast_n^\circ])$  is isomorphic to the dual of  $\text{NCHT}_n$ .*

*Proof.* Let  $[\Gamma] \in \uparrow([\ast_n^\circ])$ , where  $\Gamma$  is chosen to be the maximum element of its equivalence class. By Remark 5.8, we may draw  $\Gamma$  so that the marked points  $v_1, \dots, v_n$  are on the boundary of the unit disk and the rest of the tree lies in the interior. Since  $\Gamma$  is reduced, we know that each unmarked vertex  $u$  with valence  $k$  in  $\Gamma$  has a neighborhood  $\text{nbhd}(u)$  which is isomorphic to  $\ast_k^\circ$ , up to relabeling. The leaves of  $\text{nbhd}(u)$  then determine a hyperedge, and the set of all such hyperedges gives a hypergraph on the vertex set  $V = \{v_1, \dots, v_n\}$ . In fact, this is a noncrossing

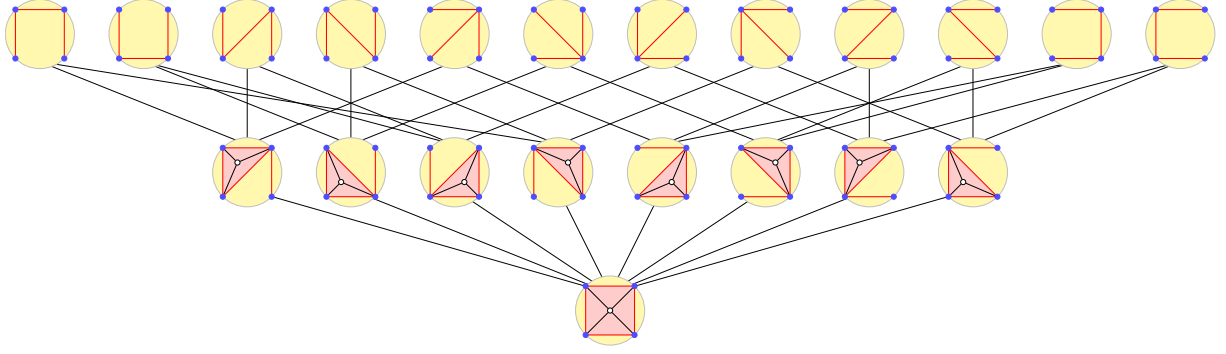


Figure 5.4: The upper set  $\uparrow([\ast_4^\circ])$ , where each equivalence class is represented by its maximal element and the corresponding noncrossing hypertree is superimposed.

(since  $\Gamma$  is embedded in the plane) hypertree (since  $\Gamma$  is a tree); let  $f: \uparrow([\ast_n^\circ]) \rightarrow \text{NCHT}_n^d$  be the function which sends  $[\Gamma]$  to the corresponding noncrossing hypertree, where  $\text{NCHT}_n^d$  denotes the dual of the noncrossing hypertree poset.

Constructing an inverse for  $f$  is straightforward. Fix a noncrossing hypertree. For each hyperedge with at least three vertices, introduce an unmarked vertex at the barycenter and connect it to each of the (marked) vertices at the corners via a straight edge. For each hyperedge with two vertices (i.e. a typical edge), simply leave it as an edge between two marked vertices. Since we began with a noncrossing hypertree, this graph is an embedded tree, and in particular it is reduced since no two unmarked vertices are adjacent. So  $f$  is a bijection.

Finally, we can see that  $f$  is an order embedding. If  $\Gamma_1$  and  $\Gamma_2$  are reduced trees and  $[\Gamma_1] \leq [\Gamma_2]$  in  $\uparrow([\ast_n^\circ])$ , then by Remark 5.7, this is equivalent to saying that  $\Gamma_2$  can be obtained from  $\Gamma_1$  via a sequence of contractions, slides, and splits. We can see by definition that each of these serves to break apart the corresponding hyperedges into smaller pieces, and vice versa. Thus,  $[\Gamma_1] \leq [\Gamma_2]$  in  $\uparrow([\ast_n^\circ])$  if and only if  $f([\Gamma_1]) \leq f([\Gamma_2])$  in  $\text{NCHT}_n^d$ , so  $f$  is a poset isomorphism.  $\square$

**Theorem 5.13** (Theorem C, second claim). *Let  $\Gamma$  be a reduced  $n$ -tree with unmarked vertices  $u_1, \dots, u_k$ . Then*

$$\uparrow([\Gamma]) \cong \prod_{i=1}^k \text{NCHT}_{\text{val}(u_i)}.$$

*Proof.* The argument is similar to that of Theorem 5.10. Note first that  $\uparrow([\ast_n^\bullet])$  is trivial since no contractions, slides, or splits can be performed without unmarked vertices. So the structure of  $\uparrow([\Gamma])$  is determined entirely by the unmarked vertices of  $\Gamma$ , which are isolated since  $\Gamma$  is assumed to be reduced.

For each unmarked vertex  $u_i$  in  $\Gamma$ , the subtree  $\text{nbhd}(u_i)$  can be identified with  $\ast_{\text{val}(u_i)}^\circ$ . As in the case of Theorem 5.10, these unmarked vertices contribute independently to the structure of  $\uparrow([\Gamma])$ . More specifically, we can see that

$$\uparrow([\Gamma]) \cong \prod_{i=1}^k \uparrow([\ast_{\text{val}(u_i)}^\circ]) \cong \prod_{i=1}^k \text{NCHT}_{\text{val}(u_i)},$$

where the second isomorphism is due to Lemma 5.12. □

*Remark 5.14.* Each cell in the polyhedral tree complex is a product of associahedra and cyclohedra by Theorem 4.6, so Theorems 5.10 and 5.13 suggest that by passing from planted  $n$ -trees to reduced  $n$ -trees, the polyhedral tree complex is transformed into a polysimplicial cell complex for which each vertex link is described by the poset of noncrossing hypertrees. This transformation can be viewed on each cell as a truncation of certain faces, but in such a way that the truncations of adjacent cells are compatible. As it turns out, this resulting complex is isomorphic to the cactus complex defined in [Nek14] and is homeomorphic to a subspace of the dual braid complex described in [BM10]. The connections between these three complexes will be made explicit in a future article.

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