Title
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Journal
Combinatorial Theory, 4(2)

ISSN
2766-1334

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Publication Date
2024

DOI
10.5070/C64264250

COMBINATORIAL THEORY **4 (2)** (2024), #14 combinatorial-theory.org

TOPOLOGICAL RECURSION FOR FULLY SIMPLE MAPS FROM CILIATED MAPS

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Submitted: Oct 19, 2021; Accepted: Jun 8, 2024; Published: Sep 30, 2024

Abstract. We solve a conjecture from the first and third authors that claims that fully simple maps, which are maps with non self-intersecting disjoint boundaries, satisfy topological recursion for the exchanged spectral curve (y, x), making use of the topological recursion for ciliated maps (building on a result from Belliard, Eynard, and the second and third authors).

Keywords. Maps, fully simple maps, enumeration, topological recursion

Mathematics Subject Classifications. 05A15, 05A19, 46L54

1. Introduction: maps, fully simple maps, and topological recursion

1.1. Maps and fully simple maps

A *map* M of genus g is the proper embedding of a finite connected graph into an oriented, topological, compact surface of genus g, so that the complement of the graph is a disjoint union of topological discs (called *faces*). We say that a face f *surrounds* a vertex v (or an edge e) when v (or e) belongs to the topological closure of f. Define an *oriented edge* to be an edge along with a choice of one of its two orientations. We say that an oriented edge is *adjacent* to a face if the face lies on its left and *incident* to a vertex if it points to this vertex. Maps may

*S.C. was supported by the Max-Planck-Gesellschaft, and currently by the CAF of Paris. S.C. wants to thank Liselotte Charbonnier and Lucie Neirac for material support.

†E.G.-F. was supported by the public grant "Jacques Hadamard" as part of the Investissement d'avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH and then received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. ERC-2016-STG 716083 "CombiTop"). She is also grateful to the Institut des Hautes Études Scientifiques (IHES) for its hospitality.

be endowed with the extra structure of an ordered tuple of distinct oriented edges, such that no two are adjacent to the same face. We refer to these oriented edges as *roots*, to their adjacent faces as *boundary faces*, and to all remaining faces as *internal faces*. The number of oriented edges adjacent to a face (resp. incident to a vertex) is called the *degree* of the face (resp. of the vertex). We denote by ∂M the disjoint union of boundary faces and by $\partial_1 M, \ldots, \partial_n M$ the boundary faces ordered as their roots are. We say a map of genus g with n boundary faces has *topology* (g, n). We say that the map is closed if it does not have boundary faces, *i.e.* $n = 0$. Throughout the article, we will keep $g \geqslant 0$ and $n \geqslant 1$ unless stated otherwise.

Two maps are equivalent if there exists an orientation-preserving homeomorphism between their underlying surfaces such that the vertices, oriented edges and faces of the first map are carried bijectively to the vertices, oriented edges and faces of the second, preserving the graph structure and the ordered tuple of roots. This also gives the notion of automorphism of a map, which is a permutation of the oriented edges arising from an orientation-preserving homeomorphism from the underlying surface to itself that preserves the graph structure and the tuple of roots[1].

$\mathbf{M}_{g,n}$ will denote the set of maps of genus g with n boundary faces and $\mathbf{M}_{g,n}(v)$ its subset of maps having v vertices. The definition of a map allows different boundary faces to be adjacent along vertices and edges, as well as for a boundary face to be adjacent to itself along vertices and edges. Informally, we call a map fully simple if such behaviour does not arise — a precise definition follows.

Definition 1.1. An oriented edge in a map is a *boundary edge* if it is adjacent to a boundary face. A map is *fully simple* if at each vertex v, at most one boundary edge is incident to v.

In previous work, the term *simple* has been used to refer to maps in which for each $i \in \{1, \ldots, n\}$, at each vertex, at most one boundary edge adjacent to $\partial_i M$ is incident [BGF20]. We stress that, unlike in fully simple maps, boundary edges adjacent to distinct boundary faces can be incident to the same vertex. Throughout the article, we use the term *ordinary* to refer to the class of all maps, so as to emphasise the distinction from the class of fully simple maps. We will be interested primarily in the following enumerations of (equivalence classes of) ordinary and fully simple maps.

Figure 1.1: Examples of connected maps of genus 2 with 3 boundary faces: on the left, an ordinary map, and on the right, a fully simple map. The root of each boundary face is indicated with an arrow.

[1]Note that for maps of topology (g, n) with $n > 0$, we have $\mathrm{Aut}(M) = \{\mathrm{Id}\}$. Automorphisms therefore will not play a role in this article, although for conceptual reasons we prefer to keep $\#\mathrm{Aut}(M)$ in the formula for weights.

Definition 1.2. Let $r \geqslant 2$ be an integer fixed throughout the article. For positive integers $k_1, k_2, \ldots, k_n$, let

$$\mathrm{Map}_{g;(k_1,\ldots,k_n)} := \sum_{\substack{M \in \mathbf{M}_{g,n} \\ \deg(\partial_i M) = k_i}} \mathscr{W}(M)$$

be the weighted enumeration of maps M with n boundary faces, such that the degree of the i^{th} boundary face is k_i for $i \in \{1, 2, \ldots, n\}$, and the internal faces have degree $\leqslant (r+1)$. The weight of a map is given by

$$\mathscr{W}(M) := \frac{\alpha^{2-2g-\#\mathcal{V}(M)}}{\#\mathrm{Aut}(M)} \, t_3^{f_3(M)} t_4^{f_4(M)} \cdots t_{r+1}^{f_{r+1}(M)}. \tag{1.1}$$

Here, $f_k(M)$ is the number of internal faces of degree k, $\mathcal{V}(M)$ is the set of vertices, and $\mathrm{Aut}(M)$ the group of automorphisms. The analogous weighted enumeration restricted to the set of fully simple maps is denoted

$$\mathrm{FSMap}_{g;(k_1,\ldots,k_n)} := \sum_{\substack{M \in \mathbf{M}_{g,n} \\ \text{fully simple} \\ \deg(\partial_i M) = k_i}} \mathscr{W}(M).$$

$\mathrm{Map}_{g;(k_1,\ldots,k_n)}$ and $\mathrm{FSMap}_{g;(k_1,\ldots,k_n)}$ are well-defined elements of $\alpha\mathbb{Z}\,[t_3, \ldots, t_{r+1}]\,[[\alpha^{-1}]]$. For brevity, our notation makes implicit the dependence on the parameters $\alpha, t_3, \ldots, t_{r+1}$. From these formal power series, one can extract the number of maps with prescribed topology, boundary face degrees and internal face degrees. We adopt the convention that $\mathrm{M}_{0,1}(1)$ contains only the map consisting of a single vertex and no edges; it is the map of genus 0 with 1 boundary of length 0, that is $\mathrm{Map}_{0;(0)} = \alpha$. Apart from this degenerate case, we always consider that boundaries have length $\geqslant 1$. For closed maps, there is no point in distinguishing ordinary and fully simple, and we denote $\mathrm{Map}_{g,\varnothing}$ the corresponding generating series.

Definition 1.3 (*Generating series of ordinary and fully simple maps*)**.** Summing over all possible lengths, we define the generating series of maps of topology (g, n) as follows:

$$W_{g,n}(x_1, \ldots, x_n) := \sum_{k_1,\ldots,k_n \geqslant 0} \frac{\mathrm{Map}_{g;(k_1,\ldots,k_n)}}{x_1^{1+k_1} \cdots x_n^{1+k_n}}.$$

We have that $W_{g,n}(x_1, \ldots, x_n) \in \alpha\mathbb{Z}[x_1^{-1}, \ldots, x_n^{-1}, t_3, \ldots, t_{r+1}][[\alpha^{-1}]]$. We also introduce the generating series for fully simple maps of topology (g, n):

$$X_{g,n}(w_1, \ldots, w_n) := \sum_{k_1,\ldots,k_n \geqslant 0} \mathrm{FSMap}_{g;(k_1,\ldots,k_n)} w_1^{k_1-1} \cdots w_n^{k_n-1},$$

and we have $X_{g,n}(w_1, \ldots, w_n) \in w_1^{-1} \cdots w_n^{-1}\mathbb{Z}[w_1, \ldots, w_n, t_3, \ldots, t_{r+1}][[\alpha^{-1}]]$.

Functional relations determining the fully simple and the ordinary generating series for topology $(0, 1)$ (discs) and $(0, 2)$ (cylinders) have been established in [BGF20], and the choice of x_i^{-1}

for ordinary maps but w_i for fully simple maps makes the formulae more elegant. In particular, a straightforward adaptation of [BGF20, Propositions 3.3. and 4.4] to include the parameter α yields:

$$X_{0,1}(\alpha^{-1}W_{0,1}(x)) = \alpha x, \qquad W_{0,1}(\alpha^{-1}X_{0,1}(w)) = \alpha w \tag{1.2}$$

and

$$\left(W_{0,2}(x_1,x_2) + \frac{1}{(x_1-x_2)^2}\right)\mathrm{d}x_1\mathrm{d}x_2 = \left(X_{0,2}(w_1,w_2) + \frac{\alpha^2}{(w_1-w_2)^2}\right)\mathrm{d}w_1\mathrm{d}w_2 \tag{1.3}$$

provided that $\alpha x_i = X_{0,1}(w_i)$ or equivalently $\alpha w_i = W_{0,1}(x_i)$.

1.2. Topological recursion for maps and fully simple maps

Topological recursion [EO07] is a procedure which appears in various contexts, and in particular for the enumeration of maps. It is a procedure which takes a *spectral curve* as input, and produces infinitely many *multi-differentials* as an output. We give now a brief definition of topological recursion. For a general introduction to the subject, see [Eyn14].

Definition 1.4 (*Spectral curve*)**.** A spectral curve $\mathcal{S}$ is a tuple $(\Sigma, x, y, \omega_{0,2})$, where:

- Σ is a Riemann surface;
- x and y are meromorphic functions from Σ to $\mathbb{P}^1$ (the Riemann sphere). We require that x is branched.
- $\omega_{0,2}(z_1, z_2)$ is a symmetric bi-differential on Σ, which has only poles of order 2 on the diagonal $z_1 = z_2$.

In the body of the article, the datum $\omega_{0,1}(z) = y(z)\mathrm{d}x(z)$ is sometimes added to the spectral curve, albeit being redundant.

The *ramification points* of $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$ are the points $a \in \Sigma$ such that $\mathrm{d}x(a) = 0$. We require in this paper that the ramification points be *simple*, that is the order of vanishing of $\mathrm{d}x$ at the ramification points is 1.

Near a ramification point $a \in \Sigma$, one defines the *local involution* σ_a:

$$\sigma_a \colon \Sigma \to \Sigma, \quad \sigma_a \neq \mathrm{Id}, \quad \sigma_a(a) = a, \quad x(\sigma_a(z)) = x(z).$$

We are now ready to present the formula of topological recursion.

Definition 1.5 (*Topological recursion*)**.** Let $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$ and $a_1, \ldots, a_k$ be its set of (simple) ramification points. We define the *kernel of recursion* near a_i by:

$$K_{a_i}(z_1, z) = \frac{1}{2}\,\frac{\int_{w=\sigma_{a_i}(z)}^{z} \omega_{0,2}(z_1, w)}{(y(z) - y(\sigma_{a_i}(z)))\mathrm{d}x(z)}.$$

From the spectral curve $\mathcal{S}$, one builds the multi-differentials $(\omega_{g,n})_{2g-2+n>0}$ according to the formula:

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{i=1}^{k} \operatorname*{Res}_{z=a_i} K_{a_i}(z_1,z)\left(\omega_{g-1,n+1}(z,\sigma_{a_i}(z),I) + \sum_{\substack{h+h'=g \\ J\sqcup J'=I}}' \omega_{h,1+\#J}(z,J)\omega_{h',1+\#J'}(\sigma_{a_i}(z),J')\right), \tag{1.4}$$

where $I = \{z_2,\dots,z_n\}$, and $\sum'$ means that the terms involving $\omega_{0,1}$ should be excluded from the sum.

Let us unfold what is implied when stating, in the context of enumeration of maps, that a model of maps satisfies topological recursion. This statement implicitely means that there exists a spectral curve $\mathcal{S}$ so that, for any $g,\ n \in \mathbb{Z}_{\geqslant 0}$ with $2g-2+n>0$, the multi-differential $\omega_{g,n}$ produced by topological recursion from $\mathcal{S}$, allows to recover the generating series of the given model of maps of genus g with n boundaries.

Consider ordinary maps as an example. From the weights $t_3,\dots,t_{r+1}$ of the faces, we define the potential of the model by $V(u) = \frac{u^2}{2} - \sum_{k=3}^{r+1} \frac{t_k}{k} u^k$. Let then $\mathcal{S}$ be the following spectral curve:

$$\mathcal{S}: \qquad \begin{cases} \Sigma = \mathbb{P}^1\,, \\ \mathsf{x}(\theta) = a + c(\theta+\theta^{-1})\,, \\ \mathsf{y}(\theta) = \left[V'(a+c(\theta+\theta^{-1}))\right]_{\leqslant 0}\,, \\ \omega_{0,2}(\theta_1,\theta_2) = \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{(\theta_1-\theta_2)^2}\,. \end{cases} \tag{1.5}$$

$\mathbb{P}^1$ is the Riemann sphere, the notation $[\cdots]_{\leqslant 0}$ stands for the polynomial part in the variable θ^{-1}, while c (up to a sign) and a are uniquely determined by:

$$\mathsf{y}(\theta) \underset{\theta\to\infty}{\sim} \frac{\alpha^{-1}}{x(\theta)} \sim \frac{1}{\alpha c\theta}\,, \qquad c = \mathcal{O}(\alpha^{-\frac{1}{2}})\,, \qquad a = \mathcal{O}(\alpha^{-1})\,,$$

(recall that α^{-1} is associated to the weight per vertex).

Theorem 1.6 ([Eyn16])**.** *Ordinary maps satisfy topological recursion for the spectral curve $\mathcal{S}$.*

Indeed, the generating series $W_{g,n}$ can be retrieved from the differentials $\omega_{g,n}$ by the identification:

$$\omega_{g,n}(\theta_1,\dots,\theta_n) = \left(W_{g,n}(\mathsf{x}(\theta_1),\dots,\mathsf{x}(\theta_n)) + \frac{\delta_{g,0}\,\delta_{n,2}}{(\mathsf{x}(\theta_1)-\mathsf{x}(\theta_2))^2}\right)\mathrm{d}\mathsf{x}(\theta_1)\dots\mathrm{d}\mathsf{x}(\theta_n).$$

As for fully simple maps, the first and last authors came up with the following conjecture:

Conjecture 1.7 ([BGF20])**.** Fully simple maps satisfy topological recursion. Moreover, the fully simple spectral curve is obtained from the ordinary spectral curve by exchanging the role of x and y.

This conjecture is a manifestation of the *symplectic transformation*[2] "exchange of x and y" on spectral curves $(\Sigma,\mathsf{x},\mathsf{y},\omega_{0,2}) \mapsto (\Sigma,\mathsf{y},\mathsf{x},\omega_{0,2})$.

[2] Such transformation is called symplectic because it preserves the symplectic form $\mathrm{d}\mathsf{x}\wedge\mathrm{d}\mathsf{y}$, up to sign.

1.3. Main result and outline

Our main result is Theorem 4.10 establishing topological recursion for the fully simple generating series $X_{g,n}$. This theorem, together with Theorem 4.11, proves Conjecture 1.7 of [BGF20], which was motivated by the transformations for discs and cylinders that can be justified by combinatorial methods and supported by numerical data in genus 1. The present article builds on the enumerative study of combinatorial objects called *multi-ciliated maps* carried out in [BCEGF21] and reviewed in Section 2. We show in Section 3 that multi-ciliated maps are dual to fully simple maps, and use in Section 4 analytic techniques to establish the conjecture. This approach also gives a different proof of (1.2)-(1.3) (see Theorem 4.11), *i.e.* that the fully simple spectral curve (encoding $X_{0,1}$ and $X_{0,2}$) and the ordinary spectral curve (encoding $W_{0,1}$ and $W_{0,2}$) are related by the symplectic transformation "exchange of x and y". We discuss a few consequences (some of them anticipated in [BGF20] conditionally to the former conjecture) in Section 5.

Added on revision. While finalising this work, the authors learned that B. Bychkov, P. Dunin-Barkowski, M. Kazarian and S. Shadrin had also found a proof for the fully simple conjecture [BDBKS23]. The two proofs are in essence very different. Our approach is rooted in the algebraic combinatorics treatment of a more general class of maps, while their method relies on algebraic manipulations in the Fock space formalism, where the universal relation between ordinary and fully simple generating series, formulated in terms of monotone Hurwitz numbers, can be efficiently implemented. The work [BDBKS23] can also establish topological recursion for a larger class of combinatorial problems involving a large class of universal relations, while the work [BCEGF21] suggests a more combinatorial framework to understand the ordinary-fully simple relations in a different generalised setting.

The program motivating the study of fully simple maps mentioned in Section 5.3 has been concretised and completed since we released the present work, although at present not by combinatorial means. The aforementioned universal relations can also be considered between any two collections of generating series (not necessarily coming from maps or having a combinatorial interpretation). This observation together with the flexibility of the Fock space techniques allowed us together with Shadrin and Leid [BCGF^{+}21] to find the higher-order (as well as higher-genus) generalisation of the R-transform machinery relating higher-order moments and higher-order free cumulants in free probability, solving a problem posed by Collins, Mingo, Speicher and Śniady [CMŚS07]. It also led to the formulation in [BCGF^{+}21, Conjecture 3.14] of a generalisation of the conjecture addressed in the present work, stating that any pair of collections of generating series satisfying topological recursion on a certain spectral curve and the spectral curve obtained by exchanging x and y also satisfies the ordinary-fully simple relations. This generalised conjecture was also resolved in [ABDB^{+}22] by Fock space techniques. Yet, an algebraic combinatorics approach to these developments is still to be found (for the free probability applications, there has been some progress in [Lio22]).

2. Multi-ciliated maps

In [BCEGF21], we introduced various types of generalised Kontsevich graphs. Among them, the ciliated and multi-ciliated maps are the types that we shall relate to fully simple maps. We remind the reader about their definitions; we also define their weights and their generating series; last, we present preliminary results coming from [BCEGF21].

Ciliated and multi-ciliated maps are maps with two types of vertices (black and white), which satisfy specific constraints.

Definition 2.1 (*Constraints on the vertices*)**.** *Black* and *white* vertices have the following properties:

- a black vertex $v_\bullet$ must have a degree $\deg(v_\bullet) \in \{3, \ldots, r+1\}$;
- a white vertex may have arbitrary positive degree;
- *star constraint*: for any given white vertex $v_\circ$, the faces surrounding $v_\circ$ are pairwise distinct;
- *uniqueness constraint*: each face surrounds at most one white vertex.

Note that a univalent white vertex automatically satisfies the star constraint.

Definition 2.2 (*Ciliated maps*)**.** $\mathrm{C}_{g,n}$ is the set of maps M such that

- M is a connected map of genus g;
- M has exactly n white vertices, labelled from 1 to n. They are univalent and the face surrounding the i^{th} i^{th} white vertex is called the i^{th} *marked face*. The faces that do not surround a white vertex are called *unmarked faces*.

Every such map is called *ciliated map of type* (g, n). Note that in this case, $\mathrm{Aut}(M) = \{\mathrm{Id}\}$.

The term *ciliated* comes from the fact that each of the n white vertices has degree 1, *i.e.* there is only one oriented edge incident to it. This edge attached to a white vertex can be viewed as a cilium in the corresponding marked face.

Definition 2.3 (*Multi-ciliated maps*)**.** Let $\underline{k} = (k_1, \ldots, k_n)$ be an n-tuple of positive integers. $\mathrm{S}_{g,\underline{k}}$ is the set of maps M such that

- M is a connected map of genus g;
- M has n white vertices, labelled from 1 to n. The i^{th} white vertex has degree k_i and is equipped with a choice of incident edge (the i^{th} *root*). The faces surrounding white vertices are called *marked faces*, while the others are called *unmarked faces*. The set of unmarked faces is denoted $\mathcal{F}^\circ(M)$.

Every such map is called *multi-ciliated map of type* (g, n). Again, $\mathrm{Aut}(M) = \{\mathrm{Id}\}$.

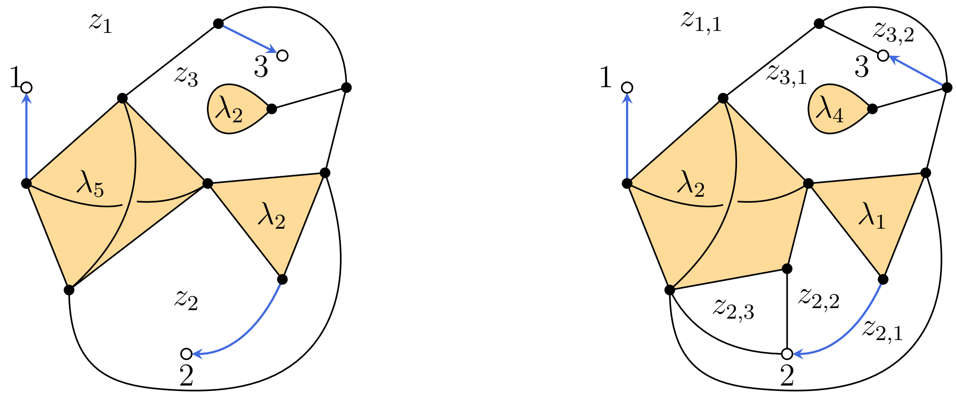


Figure 2.1: On the left, a ciliated map in $\mathbf{C}_{1,3}$; on the right, a multi-ciliated map in $\mathbf{S}_{1,(1,3,2)}$. Both maps have 3 (shaded) unmarked faces; the roots are the blue arrows. The decorations of the faces are explained in Definition 2.6.

Comparing Definitions 2.2 and 2.3, we have $\mathbf{C}_{g,n} = \mathbf{S}_{g;(1,\ldots,1)}$.

Definition 2.4 (*Degree of a map*)**.** We define the *degree* of a map M as:

$$\deg M := \#\mathcal{E}(M) - \#\mathcal{V}(M) = 2g(M) - 2 + \#\mathcal{F}(M),$$

where $g(M)$ is the genus of M and $\mathcal{V}(M)$, $\mathcal{E}(M)$ and $\mathcal{F}(M)$ respectively the sets of vertices, edges and faces of the underlying graph. We denote by

$$\mathbf{C}^{\delta}_{g,n} \subseteq \mathbf{C}_{g,n} \qquad \text{and} \qquad \mathbf{S}^{\delta}_{g,\underline{k}} \subseteq \mathbf{S}_{g,\underline{k}}$$

the corresponding subsets of maps of fixed degree δ.

It easily follows from the Euler relation (see *e.g.* [BCEGF21, Section 2.1]) that for a given topology (g,n) and degree $\delta = (2g-2+\#\mathcal{F})$, the sets $\mathbf{C}^{\delta}_{g,n}$ and $\mathbf{S}^{\delta}_{g,\underline{k}}$ are finite. We now turn to the local weights of a (multi)-ciliated map.

Definition 2.5 (*Local weights*)**.** The *potential* of the model is a polynomial of degree $r+1$, depending on the parameters $t_3, \ldots, t_{r+1}$:

$$V(u) := \frac{u^2}{2} - \sum_{j=3}^{r+1} \frac{t_j}{j}\, u^j.$$

We then introduce the *propagator*:

$$\mathscr{P}(u_1,u_2) := \frac{u_1 - u_2}{V'(u_1) - V'(u_2)},$$

and for $d \in \{3, \ldots, r+1\}$:

$$\mathscr{V}_d(u_1,\ldots,u_d) := \operatorname*{Res}_{u=\infty} \frac{V'(u)\mathrm{d}u}{\prod_{j=1}^{d}(u-u_j)} = \sum_{i=1}^{d} \frac{-V'(u_i)}{\prod_{j\neq i}(u_i-u_j)}. \tag{2.1}$$

It is manifest from the first formula that this weight is a symmetric polynomial in $u_1, \ldots, u_d$. In particular it is well-defined when some of the u_i coincide.

We fix a finite set of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$, considered as parameters. For a given n-tuple $\underline{k}$, we denote $Z_i = [z_{i,1}, \ldots, z_{i,k_i}]$ a k_i-tuple of complex variables, and $\underline{Z} = (Z_1, \ldots, Z_n)$ — square brackets are used for better parsing. It is now possible to associate a weight to a (multi)-ciliated map, by summing over decorations of unmarked faces and multiplying the local weights:

Definition 2.6. The *weight* of a (multi-)ciliated map M is given by:

$$\mathscr{W}_{\text{cil}}(M) = \sum_{U \,:\, \mathcal{F}^\circ \to \Lambda} \prod_{\substack{e \in \mathcal{E}(M) \\ e = (f_1, f_2)}} \mathscr{P}(u_{f_1}, u_{f_2}) \prod_{\substack{v \in \mathcal{V}(M) \\ \text{black}}} \mathscr{V}_{\deg(v)}(\{u_f\}_{f \mapsto v}),$$

Here, u_f is the decoration of a face: for an unmarked face it is $u_f = U(f) \in \Lambda$; for marked faces, it is $z_{i,1}$ for the face adjacent to the i^{th} root, and starting from this one, $z_{i,2}, \ldots, z_{i,k}$ for the faces encountered when travelling anticlockwise around the i^{th} vertex. The notation $e = (f_1, f_2)$ means that e is the edge surrounded by the faces f_1 and f_2, and $f \mapsto v$ means that f surrounds v.

Note that white vertices have weight 1 in this formula, and that if $\Lambda = \varnothing$, the (multi-)ciliated maps M with $\mathcal{F}^\circ(M) \neq \varnothing$ have vanishing weight: $\mathscr{W}_{\text{cil}}(M) = 0$.

The shape of the weights in Definitions 2.5 and 2.6 might seem mysterious. However, they find their inspiration in a formal hermitian matrix model, that generalises naturally the matrix model used by Kontsevich in his proof of Witten's conjecture [Kon92]. It is a formal hermitian matrix model with external field, whose potential is given by V; and whose external field is given by $\operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ (see [BCEGF21, Section 4.1]). The weights exposed above come from the application of Wick's theorem to the matrix integral.

Definition 2.7 (*Generating series of (multi-)ciliated maps*)**.**

$$\begin{aligned} C_{g,n}(Z) &= \sum_{M \in \mathbf{C}_{g,n}} \alpha^{-\deg M}\, \mathscr{W}_{\text{cil}}(M) \\ &= \sum_{\delta \geqslant 2g-2+n} \alpha^{-\delta} \sum_{M \in \mathbf{C}^{\delta}_{g,n}} \mathscr{W}_{\text{cil}}(M), \\ S_{g;\underline{k}}(\underline{Z}) &= \sum_{M \in \mathbf{S}_{g;\underline{k}}} \alpha^{-\deg M}\, \mathscr{W}_{\text{cil}}(M) \\ &= \sum_{\delta \geqslant 2g-2+n} \alpha^{-\delta} \sum_{M \in \mathbf{S}^{\delta}_{g;\underline{k}}} \mathscr{W}_{\text{cil}}(M). \end{aligned}$$

Those generating series are well-defined formal series in α^{-1}, except $C_{0,1}$ which in addition contains the term α. The dependence on α^{-1}, λs and ts has been omitted from the notation. We now recall a key recursion on the degree of white vertices for multi-ciliated generating series.

Lemma 2.8. *[BCEGF21, Lemma 2.25] If $k_1 \geqslant 2$, set $\underline{k}' = (k_1 - 1, k_2, \ldots, k_n)$. We have:*

$$S_{g;\underline{k}}(Z_1, \ldots, Z_n) = \frac{1}{\alpha} \frac{S_{g;\underline{k}'}([z_{1,1}, z_{1,3}, \ldots, z_{1,k_1}], Z_2, \ldots, Z_n) - S_{g;\underline{k}'}([z_{1,2}, \ldots, z_{1,k_1}], Z_2, \ldots, Z_n)}{V'(z_{1,1}) - V'(z_{1,2})} + \delta_{g,0}\delta_{n,1}\delta_{k_1,2}\mathscr{P}(z_{1,1}, z_{1,2}).$$

Applying this formula recursively, we can express the generating series of multi-ciliated maps in terms of the generating series of ciliated maps:

Lemma 2.9. *[BCEGF21, Theorem 2.23] Recall that $Z_i = [z_{i,1}, \ldots, z_{i,k_i}]$ for $i \in \{1, \ldots, n\}$. We have:*

$$S_{g;\underline{k}}(Z_1, \ldots, Z_n) = \frac{1}{\alpha^{k_1 + \ldots + k_n - n}} \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{C_{g,n}(z_{1,j_1}, \ldots, z_{n,j_n}) + \delta_{g,0}\delta_{n,1}\alpha z_{1,j_1}}{\prod_{i=1}^{n} \prod_{\substack{l_i=1 \\ l_i \neq j_i}}^{k_i} \big(V'(z_{i,j_i}) - V'(z_{i,l_i})\big)}. \tag{2.2}$$

3. Relating fully simple maps and multi-ciliated maps

The aim of this section is to show that multi-ciliated maps are dual to fully simple maps, based on a simple duality argument. It is convenient to do so via the permutation model for maps presented *e.g.* in [LZ04]. Here is a schematic summary of what we will present in this section at the level of combinatorial objects (if this duality is clear for some readers, this section can be skipped) and what we will present in the next section at the level of generating series.

(A) Ordinary multi-ciliated maps	$\longrightarrow$	(B) Ordinary maps with fixed boundary lengths
(C) Multi-ciliated maps	$\longrightarrow$	(D) Fully simple maps with fixed boundary lengths

The arrows represent duality at the level of combinatorial objects and specialisation to 0 of parameters associated to faces (see Section 4 for details) at the level of generating series.

Let us also use this diagram to briefly comment on the general context of these results. The left-hand side corresponds to the realm of generalised Kontsevich graphs, while the right-hand side corresponds to the usual 1-hermitian matrix model. More concretely, the graphs that Kontsevich introduced to prove Witten's conjecture correspond to (C) for the case $r = 2$, and generating series of ordinary maps within (A) are given by the usual correlators of the 1-hermitian matrix model. It was shown in [BGF20] that some other concrete correlators of the 1-hermitian matrix model recover generating series of fully simple maps within (D). This work provides an interpolation between Kontsevich matrix models from the left-hand side and usual 1-hermitian matrix models of the right-hand side by showing that the matrix models used as inspiration in [BCEGF21] allow to reach every corner (although in that work they were introduced to study (C), *i.e.* a generalisation of Kontsevich' work to $r > 2$, and not (A) or the right-hand side).

3.1. Permutation model for fully simple maps

An unrooted map can be encoded into a triple $(\sigma_0, \sigma_1, \sigma_2)$ of permutations acting on the set $\vec{\mathcal{E}}$ of oriented edges, in which

- σ_0 rotates each oriented edge anticlockwise around the vertex it is incident to;
- σ_1 is the fixed-point-free involution swapping the two oriented edges with same underlying edge;
- σ_2 rotates each oriented edge anticlockwise around the face it is adjacent to (*i.e.* located to its left).

The vertices, edges and faces of the map correspond respectively to the cycles of σ_0, σ_1 and σ_2.

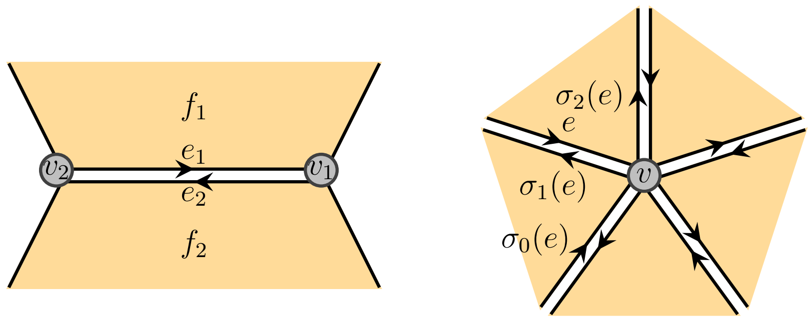


Figure 3.1: The left panel depicts the local structure of an edge in a map. The oriented edges e_1 and e_2 are indicated by the arrows. With our conventions, e_i is *adjacent* to face f_i and *incident* to vertex v_i for $i = 1, 2$. The right panel depicts the local structure of a vertex in a map, including the action of the permutations $\sigma_0, \sigma_1, \sigma_2$ on an oriented edge e.

It follows that $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \mathrm{Id}$. This can easily be adapted to describe rooted maps.

Lemma 3.1. *A rooted map can be encoded by a triple $(\sigma_0, \sigma_1, \sigma_2)$ of permutations in $\mathfrak{S}(\vec{\mathcal{E}})$ and a tuple $\mathcal{R} \in \vec{\mathcal{E}}^n$ such that*

- *σ_1 is a fixed-point-free involution;*
- *$\sigma_0 \circ \sigma_1 \circ \sigma_2 = \mathrm{Id}$;*
- *no two elements of $\mathcal{R}$ belong to the same cycle of σ_2;*
- *the group generated by $\sigma_0, \sigma_1, \sigma_2$ acts transitively on $\vec{\mathcal{E}}$ (connectedness).*

The data $(\sigma_0, \sigma_1, \sigma_2; \mathcal{R})$ and $(\sigma_0', \sigma_1', \sigma_2'; \mathcal{R}')$ define equivalent maps if and only if there exists a bijection $\phi : \vec{\mathcal{E}} \to \vec{\mathcal{E}}$ that sends $\mathcal{R}$ to $\mathcal{R}'$ and satisfies $\sigma_i' = \phi \circ \sigma_i \circ \phi^{-1}$ for $i \in \{0, 1, 2\}$.

3.1.1 Characterisation of simplicity in the permutation model

This permutation model allows the following characterisation of simple maps. Suppose that a map is given by the data $(\sigma_0, \sigma_1, \sigma_2; \mathcal{R})$ with $\mathcal{R} = (e_1, \ldots, e_n)$. For $i \in \{1, \ldots, n\}$, define the set $\mathcal{B}_i \subseteq \vec{\mathcal{E}}$ to be the σ_2-orbits of e_i. We observe that $\mathcal{B}_i$ naturally corresponds to the set of boundary edges around the i^{th} marked face, and this face is simple if and only if the elements of $\mathcal{B}_i$ belong to pairwise distinct σ_0-orbits.

Let us describe the characterisation of simple maps in a slightly different way, using a notation that will later be useful. For $i \in \{1, \ldots, n\}$, denote by $\sigma_0^{\partial_i} \in \mathfrak{S}(\mathcal{B}_i)$ the permutation obtained by expressing $\sigma_0 \in \mathfrak{S}(\vec{\mathcal{E}})$ as a union of disjoint cycles and deleting those elements that do not lie in $\mathcal{B}_i$. If $e \in \mathcal{B}_i$ is an oriented edge incident to the vertex v, then $\sigma_0^{\partial_i}(e)$ is the next oriented edge in $\mathcal{B}_i$ incident to v that is encountered when turning anticlockwise around v. Then the i^{th} boundary face is simple if and only if the permutation $\sigma_0^{\partial_i}$ is the identity permutation.

3.1.2 Characterisation of full simplicity in the permutation model

For fully simple maps, define the set $\mathcal{B} \subseteq \vec{\mathcal{E}}$ to be the union of the σ_2-orbits of the elements of $\mathcal{R}$. We observe that $\mathcal{B}$ naturally corresponds to the set of boundary edges. Then, the map is fully simple if and only if the elements of $\mathcal{B}$ belong to pairwise distinct σ_0-orbits. Equivalently, denote by $\sigma_0^{\partial} \in \mathfrak{S}(\mathcal{B})$ the permutation obtained by expressing $\sigma_0 \in \mathfrak{S}(\vec{\mathcal{E}})$ as a union of disjoint cycles and deleting those elements that do not lie in $\mathcal{B}$. If $e \in \mathcal{B}$ is an oriented edge incident to the vertex v, then $\sigma_0^{\partial}(e)$ is the next oriented edge in $\mathcal{B}$ incident to v that is encountered when turning anticlockwise around v. A map is then fully simple if and only if the permutation σ_0^{∂} is the identity permutation.

3.2. Characterisation of multi-ciliated maps in the permutation model

Next, we show that the star constraint concept on white vertices of multi-ciliated maps is the dual of simplicity, and that adding furthemore the uniqueness constraint, we get the dual concept of full simplicity. We recall that $\vec{\mathcal{E}}$ is the set of oriented edges.

Hereon we will consider multi-ciliated maps for which we relax the conditions imposed on white vertices: we call them *ordinary multi-ciliated maps* if their white vertices do not necessarily satisfy the star and uniqueness constraints, so as to distinguish them from multi-ciliated maps, which carry those conditions by definition.

Lemma 3.2. *An ordinary multi-ciliated map with n white vertices is encoded into a triple $(\sigma_0', \sigma_1', \sigma_2')$ of permutations in $\mathfrak{S}(\vec{\mathcal{E}})$ and a tuple $\mathcal{R}' \in (\vec{\mathcal{E}})^n$ such that*

- *σ_1' is a fixed-point-free involution;*
- *$\sigma_0' \circ \sigma_1' \circ \sigma_2' = \mathrm{Id}$; and*
- *no two elements of $\mathcal{R}'$ lie in the same cycle of σ_0';*
- *the group generated by $\sigma_0', \sigma_1', \sigma_2'$ acts transitively on $\vec{\mathcal{E}}$.*

The data $(\sigma_0', \sigma_1', \sigma_2'; \mathcal{R}')$ *and* $(\sigma_0'', \sigma_1'', \sigma_2''; \mathcal{R}'')$ *define equivalent maps if and only if there exists a bijection* $\phi : \vec{\mathcal{E}} \to \vec{\mathcal{E}}$ *that sends* $\mathcal{R}'$ *to* $\mathcal{R}''$ *and satisfies* $\sigma_i'' = \phi \circ \sigma_i' \circ \phi^{-1}$ *for* $i \in \{0,1,2\}$.

Proof. We take $(\sigma_0', \sigma_1', \sigma_2')$ the triple of permutations encoding the underlying map. For each $i \in \{1, \dots, n\}$, we take e_i to be the i^{th} root. We then set $\mathcal{R}' = (e_1, \dots, e_n)$. By construction all the conditions announced in the lemma are satisfied. □

In the rest of the subsection, ordinary multi-ciliated maps M' are replaced by a corresponding permutation model $(\sigma_0', \sigma_1', \sigma_2'; \mathcal{R}')$ with $\mathcal{R}' = (e_1, \dots, e_n)$.

3.2.1 Star constraint

The permutation model allows the following characterisation of the star-constraint. For $i \in \{1, \dots, n\}$, define the set $\mathcal{B}_i' \subseteq \vec{\mathcal{E}}$ to be the σ_0'-orbit of the e_i. We observe that it corresponds naturally to the set of oriented edges incident to the i^{th} white vertex. Denote by $(\sigma_2')^{\partial_i} \in \mathfrak{S}(\mathcal{B}_i')$ the permutation obtained by expressing $\sigma_2' \in \mathfrak{S}(\vec{\mathcal{E}})$ as a union of disjoint cycles and deleting those elements that do not lie in $\mathcal{B}_i'$. If $e \in \mathcal{B}_i'$ is an oriented edge adjacent to a marked face f, then $(\sigma_2')^{\partial_i}(e)$ is the next oriented edge in $\mathcal{B}_i'$ met when turning anticlockwise around f. The i^{th} white vertex satisfies the star constraint if and only if the permutation $(\sigma_2')^{\partial_i}$ is the identity permutation.

Lemma 3.3 (Star = dual of simplicity)**.** *The* i^{th} *white vertex of an ordinary multi-ciliated map satisfies the star constraint if and only if the* i^{th} *boundary face of the dual map is simple.*

Proof. Let $M' = (\sigma_2', \sigma_1', \sigma_0'; \mathcal{R}')$ be an ordinary multi-ciliated map of type (g, n). Define the map $M = (\sigma_0, \sigma_1, \sigma_2; \mathcal{R}) = (\sigma_2', \sigma_1', \sigma_0'; \mathcal{R}')$ as the dual of M'. The white vertices of M' correspond to the boundary faces of M; the black ones correspond to the internal faces of M. Then:

- M is of genus g since M' is of genus g;
- M has n labelled boundary faces since M' has n labelled white vertices;
- the boundary faces of M are rooted: the root of the i^{th} boundary face is the dual oriented edge to the i^{th} root edge of M'.

The i^{th} white vertex of M' satisfies the star constraint if and only if the permutation $(\sigma_2')^{\partial_i}$ is the identity permutation, *i.e.* the permutation $\sigma_0^{\partial_i}$ is the identity permutation. This is the property defining the simplicity of the i^{th} boundary face. □

3.2.2 Uniqueness constraint

In the permutation model, having the star and the uniqueness constraints simultaneously is characterised as follows. Define $\mathcal{B}' \subseteq \vec{\mathcal{E}}$ to be the union of the σ_0'-orbits of e_i, for $i \in \{1, \dots, n\}$. Denote by $(\sigma_2')^{\partial} \in \mathfrak{S}(\mathcal{B}')$ the permutation obtained by expressing $\sigma_2' \in \mathfrak{S}(\vec{\mathcal{E}})$ as a union of disjoint cycles and deleting those elements that do not lie in $\mathcal{B}'$. If $e \in \mathcal{B}'$ is an oriented edge adjacent to a marked face f, then $(\sigma_2')^{\partial}(e)$ is the next oriented edge in $\mathcal{B}'$ that is met when turning anticlockwise around f. Then the vertices of an ordinary multi-ciliated map satisfy the star and uniqueness constraints if and only if the permutation $(\sigma_2')^{\partial}$ is the identity permutation.

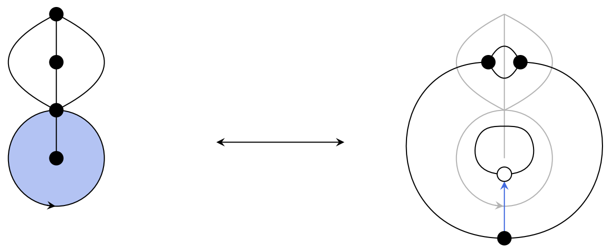


Figure 3.2: On the left, a non simple map: a vertex has two boundary incident edges. On the right, the dual ordinary multiciliated map: two corners of the white vertex belong to the same face, so it does not satisfy the star constraint.

Lemma 3.4 (Uniqueness and star = dual of full simplicity)**.** *The white vertices of an ordinary multi-ciliated map satisfy the uniqueness and star constraints if and only if the dual map is fully simple.*

Proof. As in the previous proof, the property that $\sigma_0^\partial = (\sigma_2')^\partial$ is the identity permutation matches the characterisation of full-simplicity in the permutation model. □

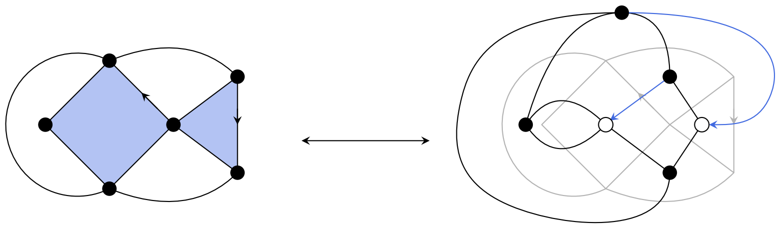


Figure 3.3: On the left, a (simple but) non fully simple map: two boundaries share a common vertex. On the right, the dual ordinary multi-ciliated map. The two white vertices are adjacent to a same face, hence they do not satisfy the uniqueness constraint.

4. Topological recursion for fully simple maps

We use multi-ciliated maps in order to prove that fully simple maps satisfy topological recursion. For this purpose we will only need to specialise the set of undecorated face weights to $\Lambda = \{0\}$, *i.e.* take $N = 1$ and $\lambda_1 = 0$. We first recall from [BCEGF21] the topological recursion formula for ciliated maps, specialised to $\Lambda = \{0\}$; we then show how to use this result to prove that fully simple maps satisfy topological recursion; eventually, we discuss in greater detail the disc and the cylinder case.

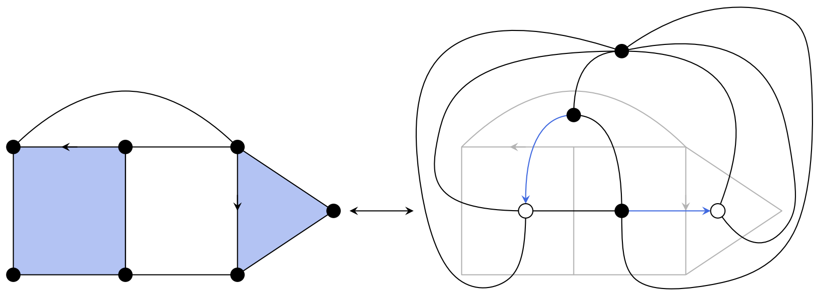


Figure 3.4: On the left, a fully simple map; on the right, the dual multi-ciliated map satisfying the star and uniqueness constraints.

4.1. Topological recursion for ciliated maps

If f is a formal power series in α^{-1}, we denote by $[f]_d$ the coefficient of α^{-d}. We define the following spectral curve, which is a specialisation of the spectral curve for ciliated maps obtained in [BCEGF21, Section 3.1].

Lemma 4.1. *($N = 1$, $\lambda_1 = 0$ specialisation of [BCEGF21, Theorem 3.3]). For any $t_3, \ldots, t_{r+1} \in \mathbb{C}$ with non-zero t_{r+1}, there exists a unique polynomial Q of degree r with coefficients in $\mathbb{C}[[\alpha^{-1}]]$, as well as $a \in \alpha^{-1}\mathbb{C}[[\alpha^{-1}]]$ satisfying:*

$$Q(\zeta) \underset{\zeta\to\infty}{=} V'\left(\zeta + \frac{1}{\alpha}\frac{1}{Q'(a)(\zeta - a)}\right) + \mathcal{O}(\zeta^{-1}), \qquad Q(a) = 0.$$

Let us explain how to work with the lemma in practice. Recall that the coefficients $t_3, \ldots, t_{r+1}$ in V and the parameter λ_1 are given. The first condition determines the top coefficient of the polynomial Q to be $-t_{r+1}$ and give algebraic equations for the other coefficients of Q depending on the auxiliary parameter a and the t_js. The second one gives an algebraic equation of degree $2r - 1$ for a depending on the coefficients of Q. The condition $a = \mathcal{O}(\alpha^{-1})$ selects a unique solution to this system of algebraic equations, which can be computed perturbatively in α^{-1}. Note that at leading order, we have $Q(\zeta) = V'(\zeta) + \mathcal{O}(\alpha^{-1})$.

Example 4.2. For $r = 2$ (triangulations) we find from the first condition

$$Q(\zeta) = q_0 + \zeta - t_3\zeta^2, \qquad q_0 = -\frac{2t_3}{\alpha}\frac{1}{1 - 2t_3 a}.$$

The auxiliary parameter a then satisfies the cubic equation $-\frac{2t_3}{\alpha}\frac{1}{1-2t_3 a} + a - t_3 a^2 = 0$, and its solution which is $\mathcal{O}(\alpha^{-1})$ is

$$a = 2t_3\alpha^{-1} + 12t_3^3\alpha^{-2} + 128t_3^5\alpha^{-3} + 1680t_3^7\alpha^{-4} + 24576t_3^9\alpha^{-5} + 384384t_3^{11}\alpha^{-6} + \mathcal{O}(\alpha^{-7}).$$

This is turn gives

$$-q_0 = 2t_3\alpha^{-1} + 8t_3^3\alpha^{-2} + 80t_3^5\alpha^{-3} + 1024t_3^7\alpha^{-4} + 14784t_3^9\alpha^{-5} + 229376t_3^{11}\alpha^{-6} + \mathcal{O}(\alpha^{-7}).$$

Example 4.3. For $r = 3$ (triangles and quadrangles) we find from the first condition

$$Q(\zeta) = q_0 + q_1\zeta - t_3\zeta^2 - t_4\zeta^3,$$

with parameters

$$q_0 = -\frac{2t_3 + 3t_4 a}{\alpha(q_1 - 2t_3 a - 3t_4 a^2)}, \qquad q_1 = 1 - \frac{3t_4}{\alpha}\frac{1}{q_1 - 2t_3 a - 3t_4 a^2}.$$

The auxiliary parameter a then satisfies the quintic equation $q_0 + q_1 a - t_3 a^2 - t_4 a^3 = 0$. We have $q_1 = 1 + \mathcal{O}(\alpha^{-1})$. Eliminating q_0, it turns into a system of two algebraic equations for q_1 and a. The solution is not explicit, but we can compute it perturbatively

$$\begin{aligned} a &= 2t_3\alpha^{-1} + 6t_3(3t_4 + 2t_3^2)\alpha^{-2} + 4t_3(5t_4 + 8t_3^2)(9t_4 + 4t_3^2)\alpha^{-3} + \mathcal{O}(\alpha^{-4}), \\ -q_0 &= 2t_3\alpha^{-1} + 4t_3(3t_4 + 2t_3^2)\alpha^{-2} + 4t_3(3t_4 + 5t_3^2)(9t_4 + 4t_3^2)\alpha^{-3} + \mathcal{O}(\alpha^{-4}), \\ q_1 &= 1 - 3t_4\alpha^{-1} - 3t_4(3t_4 + 4t_3^2)\alpha^{-2} - 6t_4(9t_4^2 + 42t_4t_3^2 + 20t_3^4)\alpha^{-3} + \mathcal{O}(\alpha^{-4}). \end{aligned}$$

If $t_3 = 0$ (quadrangulations), one can easily prove by induction on the order in α^{-1} that $a = 0$ and $q_0 = 0$. Then, q_1 satisfies a quadratic equation and we find

$$q_1 = \frac{1 + \sqrt{1 - 12t_4\alpha^{-1}}}{2}.$$

Definition 4.4. We introduce $\zeta \in \mathbb{C}(z)[[\alpha^{-1}]]$, the unique formal power series whose coefficients are rational functions of z determined by:

$$Q(\zeta) = V'(z), \qquad [\zeta]_0 = z. \tag{4.1}$$

The *spectral curve* for the weighted enumeration of ciliated maps with unmarked face parameters $\Lambda = \{0\}$ is $\mathcal{S} = (\mathbb{P}^1, x, y, \Gamma_{0,2})$, where the meromorphic maps $x, y : \mathbb{P}^1 \to \mathbb{P}^1$ and the bidifferential $\Gamma_{0,2}$ are defined by:

$$\begin{cases} x(\zeta) = Q(\zeta), \\ y(\zeta) = \alpha\,\zeta + \frac{1}{Q'(a)(\zeta - a)}, \\ \Gamma_{0,1}(\zeta) = y(\zeta)\mathrm{d}x(\zeta), \\ \Gamma_{0,2}(\zeta_1, \zeta_2) = \frac{\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2}{(\zeta_1 - \zeta_2)^2}. \end{cases}$$

In Lemma 4.1, ζ is a placeholder for the variable of the polynomial Q. In Definition 4.4, ζ plays the role of a uniformising coordinate on the underlying Riemann sphere $\mathbb{P}^1$. It differs from the variables z used in the generating series of multi-ciliated maps, but there is an invertible functional relation between them which is specified by the first part of Definition 4.4. $\Gamma_{0,2}$ is the unique fundamental bidifferential of the second kind on $\mathbb{P}^1$ (called *standard bidifferential* for short). To be precise, $\mathcal{S}$ is a family of spectral curves parametrised by the formal parameter α^{-1} and the complex parameters $t_3, \ldots, t_r$.

For $g \geqslant 0$ and $n \geqslant 1$, we define the n-differential

$$\begin{aligned}\Gamma_{g,n}(\zeta_1,\dots,\zeta_n) &= \left(C_{g,n}(z_1,\dots,z_n) + \frac{\delta_{g,0}\delta_{n,2}}{\big(x(\zeta_1)-x(\zeta_2)\big)^2}\right)\mathrm{d}x(\zeta_1)\cdots\mathrm{d}x(\zeta_n) \\ &\quad + \delta_{g,0}\delta_{n,1}\left(\alpha\, z_1 + \sum_{j=1}^{N}\frac{1}{V'(z_1)-V'(\lambda_j)}\right)\mathrm{d}x(\zeta_1), \qquad (4.2)\end{aligned}$$

where ζ_i and z_i are related as in (4.1). It was proved in [BCEGF21] that ciliated maps satisfy topological recursion for the spectral curve of Definition 4.4:

Theorem 4.5. *[BCEGF21, Theorems 3.7 and 3.18 for $N = 1$] Let $t_3,\dots,t_{r+1} \in \mathbb{C}$ with t_{r+1} non-zero, chosen such that the polynomial $Q'(\zeta)$ has simple roots. Then, the differentials $\Gamma_{g,n}$ can be analytically continued to meromorphic n-forms on the spectral curve, still denoted $\Gamma_{g,n}$. Besides, $\Gamma_{0,2}$ is the standard bidifferential on the spectral curve, and $\Gamma_{g,n}$ for $2g-2+n > 0$ satisfy the topological recursion for the spectral curve of Definition 4.4. The generating series $C_{g,n}$ are retrieved by expansion when $z_i \to \infty$.*

The roots of the polynomial $Q'(\zeta)$ are the *ramification points* of the spectral curve. Given ζ_0, we define $\{\zeta_0^{(0)},\zeta_0^{(1)},\dots,\zeta_0^{(r-1)}\}$ as the set of roots of $Q(\zeta)-Q(\zeta_0)$, where $\zeta_0^{(0)} = \zeta_0$. For generic parameters, the branched cover x has $r-1$ simple ramification points $\rho_1,\dots,\rho_{r-1}$, *i.e.*

$$Q'(\rho_k) = 0, \qquad Q''(\rho_k) \neq 0,$$

and hence the theorem applies. For ζ near ρ_k, we can always choose the labellings of points in $x^{-1}(x(\zeta))$ so that $\rho_k = \rho_k^{(k)}$ (in the previous notation applied to $\zeta_0 = \rho_k$ and to $\zeta_0 = \zeta$)[3]; since the ramification points are simple, we have $\rho_k^{(l)} \neq \rho_k$ for $l \neq 0, k$. To each ramification point we associate the recursion kernel:

$$K_{\rho_k}(\zeta_1,\zeta) = \frac{1}{2}\,\frac{\int_{\zeta^{(k)}}^{\zeta}\Gamma_{0,2}(\zeta_1,\cdot)}{\Gamma_{0,1}(\zeta)-\Gamma_{0,1}(\zeta^{(k)})}\,, \qquad (4.3)$$

which is defined locally for ζ near ρ_k and globally for $\zeta_1 \in \mathbb{P}^1$. The topological recursion formula allows the computation of $\Gamma_{g,n}$ by induction on $2g-2+n > 0$:

$$\Gamma_{g,n}(\zeta_1,\dots,\zeta_n) = \sum_{k=1}^{r-1}\operatorname*{Res}_{\zeta=\rho_k} K_{\rho_k}(\zeta_1,\zeta)\left(\Gamma_{g-1,n+1}(\zeta,\zeta^{(k)},I) + \sideset{}{'}\sum_{\substack{h+h'=g \\ J\sqcup J'=I}}\Gamma_{h,1+\#J}(\zeta,J)\,\Gamma_{h',1+\#J'}(\zeta^{(k)},J')\right), \qquad (4.4)$$

where $I = \{\zeta_2,\dots,\zeta_n\}$ and $\sum'$ means that terms involving $\Gamma_{0,1}$ should be excluded from the sum.

[3] That is, $x^{-1}(x(\zeta)) \coloneqq \{\zeta^{(0)} = \zeta, \zeta^{(1)},\dots,\zeta^{(r-1)}\}$ and $\zeta^{(k)},\zeta^{(0)} = \zeta \to \rho_k^{(k)} = \rho_k^{(0)} = \rho_k$ when $\zeta \to \rho_k$, for ζ near the simple ramification point ρ_k (and $\zeta^{(k)} \neq \zeta^{(0)} = \zeta$ as long as $\zeta \neq \rho_k$).

4.2. Relating fully simple and ciliated generating series

The following key observation relates the enumeration of fully simple maps to the one of multi-ciliated maps. If $k \geqslant 0$, let $[0^k]$ be the k-tuple whose elements are all zero.

Lemma 4.6. *We have for any $g \geqslant 0$, $n \geqslant 1$ and $k_1, \ldots, k_n \geqslant 0$:*

$$\mathrm{FSMap}_{g;(k_1,\ldots,k_n)} = S_{g;(k_1,\ldots,k_n)}\big([0^{k_1}], \ldots, [0^{k_n}]\big)\big|_{\Lambda=\{0\}} + \delta_{g,0}\delta_{n,1}\delta_{k_1,0}\alpha. \tag{4.5}$$

Remark 4.7. The additional term for the disc case comes from the degenerate fully simple map in $\mathbf{M}_{0,1}(1)$, which has no equivalent among multi-ciliated maps.

Proof. From Lemma 3.4, multi-ciliated maps are dual to fully simple maps — we rigorously characterised this correspondence in Section 3.2. The perimeter of the i^{th} boundary face of a fully simple map $M \in \mathbf{M}_{g,n}$ corresponds to the degree of the i^{th} white vertex of $M' \in \mathbf{S}_{g;(k_1,\ldots,k_n)}$, *i.e.* $\deg(\partial_i M) = k_i$. We shall now compare the weights in their enumeration. Recall the Definition 2.5 for the potential V and $\mathscr{V}_d$, especially its expression via a residue. We observe that

$$\mathscr{V}_d(u_1, \ldots, u_d)\big|_{u_k=0} = t_d, \qquad d \in \{3, \ldots, r+1\},$$
$$\mathscr{P}(u_1, u_2)\big|_{u_k=0} = 1\,.$$

This can be used to evaluate the weight $\mathscr{W}_{\text{cil}}(M')$ of a multi-ciliated map M' at $u_k = 0$ — recall that the us are either equal to λs or to zs which are in the present situation all set to zero. Thus, the local weight t_d for a black vertex of degree d in the multi-ciliated map M' can be interpreted as a local weight for an internal face of degree d in the dual fully-simple map M. Unlike $\mathscr{W}_{\text{cil}}(M')$ in Definition 2.6, the weight $\mathscr{W}(M)$ introduced in (1.1) contains a factor

$$\alpha^{2-2g(M)-\#\mathcal{V}(M)} = \alpha^{2-2g(M')-\#\mathcal{F}(M')} = \alpha^{-\deg M'}.$$

We deduce that this specialisation retrieves the weights for the standard notion of maps, in the form

$$\alpha^{-\deg M'}\,\mathscr{W}_{\text{cil}}(M')\Big|_{u_k=0} = \mathscr{W}(M).$$

□

Lemma 4.8. *We introduce $\mathfrak{z}(w) \in \mathbb{C}[w][[\alpha^{-1}]]$ uniquely determined by*

$$\begin{cases} V'(\mathfrak{z}(w)) = \frac{w}{\alpha}\,, \\ \mathfrak{z}(w) = \frac{w}{\alpha} + \mathcal{O}(\alpha^{-2})\,. \end{cases}$$

Then, for any $g \geqslant 0$ and $n \geqslant 1$:

$$X_{g,n}(w_1, \ldots, w_n) = C_{g,n}(\mathfrak{z}(w_1), \ldots, \mathfrak{z}(w_n))|_{\Lambda=\{0\}} + \delta_{g,0}\delta_{n,1}\,\alpha\left(\frac{1}{w_1} + \mathfrak{z}(w_1)\right). \tag{4.6}$$

Proof. From the definition of $X_{g,n}$ and Equation (4.5), we have

$$\begin{aligned} X_{g,n}(w_1, \ldots, w_n) &= \sum_{k_1,\ldots,k_n \geqslant 1} w_1^{k_1-1} \cdots w_n^{k_n-1}\, \mathrm{FSMap}_{g;(k_1,\ldots,k_n)} + \delta_{g,0}\delta_{n,1}\frac{\alpha}{w_1} \\ &= \sum_{k_1,\ldots,k_n \geqslant 1} w_1^{k_1-1} \cdots w_n^{k_n-1}\; S_{g;(k_1,\ldots,k_n)}\big([0^{k_1}], \ldots, [0^{k_n}]\big)\big|_{\Lambda=\{0\}} + \delta_{g,0}\delta_{n,1}\frac{\alpha}{w_1}. \end{aligned} \tag{4.7}$$

The expression for $S_{g;(k_1,\ldots,k_n)}(Z_1,\ldots,Z_n)$ in Lemma 2.9 relating multi-ciliated and ciliated generating series is not directly adapted to set simultaneously all variables $z_{i,j}$ to zero, since $V'(0)=0$, so we would have singularities in the denominators. We first rewrite it using the change of formal variable $\mathfrak{z}(w)\in\mathbb{C}[w][[\alpha^{-1}]]$ announced in the lemma, which is easily seen to exist and to be unique. Then, similarly to (2.1) we can write

$$\begin{aligned}&S_{g;(k_1,\ldots,k_n)}(Z_1,\ldots,Z_n)|_{\Lambda=\{0\}}\\&=\operatorname*{Res}_{w_1=0}\cdots\operatorname*{Res}_{w_n=0}\frac{C_{g,n}(\mathfrak{z}(w_1),\ldots,\mathfrak{z}(w_n))\mathrm{d}w_1\cdots\mathrm{d}w_n+\delta_{g,0}\delta_{n,1}\alpha\,\mathfrak{z}(w_1)\mathrm{d}w_1}{\prod_{i=1}^{n}\prod_{j=1}^{k_i}(w_i-\alpha V'(z_{i,j}))}\Big|_{\Lambda=\{0\}}.\end{aligned}$$

Now we can set $z_{i,j}$ to zero and this yields

$$\begin{aligned}&S_{g;(k_1,\ldots,k_n)}([0]^{k_1},\ldots,[0]^{k_n})|_{\Lambda=\{0\}}\\&=\operatorname*{Res}_{w_1=0}\cdots\operatorname*{Res}_{w_n=0}\frac{C_{g,n}(\mathfrak{z}(w_1),\ldots,\mathfrak{z}(w_n))\mathrm{d}w_1\cdots\mathrm{d}w_n+\delta_{g,0}\delta_{n,1}\alpha\mathfrak{z}(w_1)\mathrm{d}w_1}{\prod_{i=1}^{n}w_i^{k_i}}\\&=\frac{1}{(k_1-1)!}\frac{\partial^{k_1-1}}{\partial w_1^{k_1-1}}\cdots\frac{1}{(k_n-1)!}\frac{\partial^{k_n-1}}{\partial w_n^{k_n-1}}\Big(C_{g,n}(\mathfrak{z}(w_1),\ldots,\mathfrak{z}(w_n))+\delta_{g,0}\delta_{n,1}\alpha\,\mathfrak{z}(w_1)\Big)\Big|_{\substack{w_i=0\\\Lambda=\{0\}}}.\end{aligned}\tag{4.8}$$

Inserting it in the formula (4.7) and observing that $\mathfrak{z}(0)=0$, we recognise $X_{g,n}$ as a Taylor expansion of $C_{g,n}$ around 0:

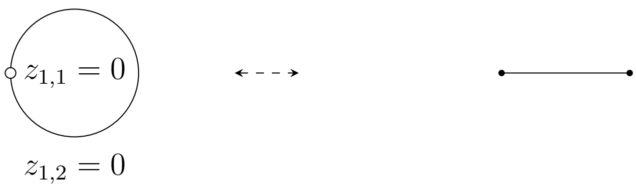
$$\begin{aligned}&X_{g,n}(w_1,\ldots,w_n)\\&=\sum_{k_1,\ldots,k_n\geqslant 1}\left[\prod_{i=1}^{n}\frac{w_i^{k_i-1}}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial w_i^{k_i-1}}\right]\Big(C_{g,n}(\mathfrak{z}(w_1),\ldots,\mathfrak{z}(w_n))+\delta_{g,0}\delta_{n,1}\alpha\,\mathfrak{z}(w_1)\Big)\Big|_{\substack{w_i=0\\\Lambda=\{0\}}}\\&=\Big(C_{g,n}(\mathfrak{z}(w_1),\ldots,\mathfrak{z}(w_n))+\delta_{g,0}\delta_{n,1}\,\alpha\mathfrak{z}(w_1)\Big)\Big|_{\Lambda=\{0\}}.\end{aligned}$$

The equalities hold as formal series in α^{-1} (or formal Laurent series for $(g,n)\neq(0,1)$) and $(w_i)_{i=1}^{n}$. □

Remark 4.9. The additional term of Lemma 2.8 for the case $(g,n,\underline{k})=(0,1,(2))$ comes from the term

$$\mathscr{P}(z_{1,1},z_{1,2})|_{z_{1,1}=z_{1,2}=0}=\mathscr{P}(0,0)=1,$$

which corresponds to the special multi-ciliated map, dual to the degenerate fully simple map in $\mathbf{M}_{0,1}(2)$:



Multi-ciliated map Corresponding fully simple map

This special case yields the additional term of Lemma 2.9 and hence of the previous equations.

For $g \geqslant 0$ and $n \geqslant 1$, we define the specialisation of the n-differential (4.2) to $\Lambda = \{0\}$:

$$\chi_{g,n}(\zeta_1,\ldots,\zeta_n) \coloneqq \Gamma_{g,n}(\zeta_1,\ldots,\zeta_n)\big|_{\Lambda=\{0\}}\,.$$

As a consequence of Lemma 4.8, this is also:

$$\begin{aligned}\chi_{g,n}(\zeta_1,\ldots,\zeta_n) &= \left(X_{g,n}(w_1,\ldots,w_n) + \frac{\delta_{g,0}\delta_{n,2}}{\big(x(\zeta_1)-x(\zeta_2)\big)^2}\right)\mathrm{d}x(\zeta_1)\cdots\mathrm{d}x(\zeta_n)\\ &= \left(\alpha^{-n}\,X_{g,n}(w_1,\ldots,w_n) + \frac{\delta_{g,0}\delta_{n,2}}{\big(w(\zeta_1)-w(\zeta_2)\big)^2}\right)\mathrm{d}w_1\cdots\mathrm{d}w_n,\end{aligned} \tag{4.9}$$

since we have $x(\zeta_i) = Q(\zeta_i) = V'(z_i) = \frac{w_i}{\alpha}$. The specialisation of Theorem 4.5 now implies our main result, *i.e.* that the fully simple generating series satisfy the topological recursion.

Theorem 4.10. *For any $t_3,\ldots,t_{r+1} \in \mathbb{C}$ with $t_{r+1} \neq 0$ such that the ramification points are simple (this holds for generic parameters), the n-differentials $\chi_{g,n}$ can be analytically continued to meromorphic n-forms on the spectral curve of Definition 4.4. The analytic continuations, still denoted $\chi_{g,n}$, satisfy the topological recursion on this spectral curve. The generating series $X_{g,n}$ are retrieved by expansion when $w_i \to 0$.*

4.3. Spectral curve for fully simple maps

The purpose of this section is to derive the spectral curve governing fully simple maps, based on the previous sections.

Theorem 4.11. *The fully simple spectral curve is obtained from the ordinary spectral curve* (1.5) *by exchanging the role of* x *and* y*, and Conjecture 1.7 holds.*

Remark 4.12. The topological recursion in Theorem 4.5 and Theorem 4.10 is stated for (generic) parameters, so that the corresponding spectral curve has only simple ramification points. This assumption can be waived using the continuity properties of topological recursion with respect to parameters that have been anticipated in [BE13] and thoroughly studied in [BBC$^+$23].

Proof of Theorem 4.11. We start from the spectral curve given in Definition 4.4. Besides, the variables $w \in \alpha\mathbb{C}(\zeta)[[\alpha^{-1}]]$, resp. $\mathfrak{z}(w) = z \in \mathbb{C}(\zeta)[[\alpha^{-1}]]$) that should be used to extract the fully simple (resp. ciliated) generating series are determined by:

$$\frac{w}{\alpha} = V'(z) = Q(\zeta), \qquad \frac{w}{\alpha} = z + \mathcal{O}(\alpha^{-2}) = \zeta + \mathcal{O}(\alpha^{-2}),$$

where Q is the polynomial from Lemma 4.1. Let $\hat{\chi}_{g,n}$ be the multi-differentials computed by the topological recursion on the rescaled spectral curve:

$$\hat{\mathcal{S}}\;:\quad \begin{cases} \hat{x}(\zeta) = x(\zeta),\\ \hat{y}(\zeta) = \frac{y(\zeta)}{\alpha} = \zeta + \frac{1}{\alpha}\frac{1}{Q'(a)(\zeta-a)},\\ \hat{\chi}_{0,1}(\zeta) = \hat{y}(\zeta)\mathrm{d}\hat{x}(\zeta),\\ \hat{\chi}_{0,2}(\zeta_1,\zeta_2) = \chi_{0,2}(\zeta_1,\zeta_2).\end{cases}$$

As the only effect of this rescaling on the topological recursion is to multiply the recursion kernel by α (compare with (4.3) and (4.4)), and the topology (g,n) is reached after $2g-2+n$ steps of the recursion, we have:

$$\hat{\chi}_{g,n}(\zeta_1,\ldots,\zeta_n) = \alpha^{2g-2+n}\chi_{g,n}(\zeta_1,\ldots,\zeta_n)\,.$$

Taking into account the α^{-n} present in (4.9) and coming back to Definition 1.3 for $X_{g,n}$, we see that $\hat{\chi}_{g,n}$ are generating series of fully simple maps with modified weights:

$$\hat{\mathscr{W}}(M) = \frac{\alpha^{-\#\mathcal{V}(M)}}{\#\,\mathrm{Aut}(M)} t_3^{f_3(M)}\cdots t_{r+1}^{f_{r+1}(M)}.$$

This choice for the weight is the one made for the enumeration of ordinary maps *e.g.* in [Eyn16, Chapter 3] with a formal parameter that keeps track of the number of vertices which is the inverse of our α, i.e. $t=\alpha^{-1}$. More precisely, for any $g \geqslant 0$ and $n \geqslant 1$ we have:

$$\hat{\chi}_{g,n}(\zeta_1,\ldots,\zeta_n) = \sum_{\substack{M\in\mathbf{M}_{g,n}\\ \text{fully simple}}} \hat{\mathscr{W}}(M)\, w_1^{\deg(\partial_1 M)-1}\cdots w_n^{\deg(\partial_n M)-1}\mathrm{d}w_1\cdots\mathrm{d}w_n, \tag{4.10}$$

where $w_i = \alpha Q(\zeta_i)$ as specified in Lemma 4.8, and the equation should be understood as the equality of the all-order series expansion of the left-hand side when $w_i \to 0$ with the formal series on the right-hand side.

Now, we introduce a different uniformising coordinate on the Riemann sphere, which we call θ and is related to ζ by:

$$\zeta(\theta) = a + c\theta^{-1}, \qquad c \coloneqq \big(\alpha\, Q'(a)\big)^{-\frac{1}{2}}. \tag{4.11}$$

It can be checked that $c \in \mathbb{C}[[\alpha^{-\frac{1}{2}}]]$; in particular

$$c = \mathcal{O}(\alpha^{-\frac{1}{2}}). \tag{4.12}$$

We then find

$$\hat{\mathcal{S}}\ : \qquad \begin{cases} \hat{x}(\zeta(\theta)) = Q\big(a+c\theta^{-1}\big)\,, \\ \hat{y}(\zeta(\theta)) = a + c(\theta+\theta^{-1})\,, \\ \hat{\chi}_{0,1}(\zeta(\theta)) = \hat{y}(\zeta(\theta))\,\mathrm{d}\hat{x}(\zeta(\theta))\,, \\ \hat{\chi}_{0,2}(\zeta(\theta_1),\zeta(\theta_2)) = \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{(\theta_1-\theta_2)^2}\,. \end{cases} \tag{4.13}$$

The characterisation of the polynomial Q from Lemma 4.1 can be rewritten as

$$Q\big(a+c\theta^{-1}\big) \underset{\theta\to 0}{=} V'\big(a+c(\theta+\theta^{-1})\big) + \mathcal{O}(\theta).$$

In other words:

$$Q\big(a+c\theta^{-1}\big) = \Big[V'\big(a+c(\theta+\theta^{-1})\big)\Big]_{\leqslant 0}\,,$$

where $[\cdots]_{\leqslant 0}$ is the polynomial part in the variable θ^{-1}. Besides, we have the constraints

$$Q(a) = 0 \qquad \text{and} \qquad \lim_{\theta\to\infty} \theta\, Q(a + c\theta^{-1}) = cQ'(a) = (\alpha c)^{-1}. \tag{4.14}$$

The first one is a reminder from Lemma 4.1 while the second one follows from the definition of c in (4.11). For comparison, the spectral curve for ordinary maps is[4]:

$$\mathcal{S}\ :\qquad \begin{cases} \mathsf{x}(\theta) = a + c(\theta + \theta^{-1})\,, \\ \mathsf{y}(\theta) = \big[V'(a + c(\theta+\theta^{-1}))\big]_{\leqslant 0}\,, \\ \omega_{0,1}(\theta) = \mathsf{y}(\theta)\mathrm{d}\mathsf{x}(\theta)\,, \\ \omega_{0,2}(\theta_1,\theta_2) = \frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{(\theta_1-\theta_2)^2}\,, \end{cases} \tag{4.15}$$

where c (up to a sign) and a are uniquely determined by the conditions

$$\begin{aligned} &\mathsf{y}(\theta) \underset{\theta\to\infty}{\sim} \frac{\alpha^{-1}}{x(\theta)} \sim \frac{1}{\alpha c\theta}\,, \\ &c = \mathcal{O}(\alpha^{-\frac{1}{2}})\,, \\ &a = \mathcal{O}(\alpha^{-1})\,, \end{aligned} \tag{4.16}$$

and α^{-1} is the weight per vertex. We recognise

$$\mathsf{y}(\theta) = \hat{x}(\zeta(\theta)), \qquad \mathsf{x}(\theta) = \hat{y}(\zeta(\theta)), \qquad \omega_{0,2}(\theta_1,\theta_2) = \hat{\chi}_{0,2}(\zeta(\theta_1),\zeta(\theta_2)),$$

with parameters (a, c) determined in an identical way: the first condition of (4.16) is equivalent to (4.14), the second condition is equivalent to (4.12) (the sign ambiguity amounts to the choice of squareroot), and the third condition matches the last condition in Lemma 4.1.

It is well-known that topological recursion on the spectral curve (4.15) computes the generating series of ordinary maps. More precisely, for $g \geqslant 0$ and $n \geqslant 1$, let us define:

$$\omega_{g,n}(\theta_1,\ldots,\theta_n) \coloneqq \bigg(W_{g,n}(\mathsf{x}(\theta_1),\ldots,\mathsf{x}(\theta_n)) + \frac{\delta_{g,0}\delta_{n,2}}{\big(\mathsf{x}(\theta_1)-\mathsf{x}(\theta_2)\big)^2}\bigg)\mathrm{d}\mathsf{x}(\theta_1)\cdots\mathrm{d}\mathsf{x}(\theta_n). \tag{4.17}$$

It is established in [Eyn04, Eyn16] that for any $t_3,\ldots,t_{r+1} \in \mathbb{C}$ with $t_{r+1} \neq 0$ and such that Q' has only simple zeros, the multi-differentials $\omega_{g,n}$ can be analytically continued to meromorphic n-forms on the spectral curve (4.15). If we still denote $\omega_{g,n}$ the analytic continuations, $\omega_{0,2}$ is the standard bidifferential and $\omega_{g,n}$ for $2g-2+n > 0$ is computed by the topological recursion on this spectral curve. This explains the formulation of the claim, and concludes the proof, given Theorem 4.10. □

[4]See *e.g.* [Eyn16, Section 3.1.3]. Note that the y in [Eyn16] is ours minus $\frac{V'(\mathsf{x})}{2}$, but adding a rational function of x to y does not alter the result of topological recursion, as is manifest in the formula (4.3) of the recursion kernel. We use this y to obtain the right combinatorial interpretation for $(0,1)$ without any shift: $\omega_{0,1}(\theta) = \mathsf{y}(\theta)\mathrm{d}\mathsf{x}(\theta) = W_{0,1}(\theta)\mathrm{d}\mathsf{x}(\theta)$, as in [BGF20]. In [Eyn16], the triple (α^{-1}, a, c) was denoted (t, α, γ).

5. Applications

In this section, we briefly explain some motivation for this work in the form of some consequences of Theorems 4.10-4.11 which pave the way for future investigations.

5.1. Symplectic invariance

5.1.1 Context

Let $\mathcal{S} = (\mathcal{C}, x, y, \omega_{0,2})$ be a spectral curve and P the set of zeroes of $\mathrm{d}x$, the topological recursion constructs[5] multi-differentials $\omega_{g,n}$ indexed by $g \geqslant 0$ and $n \geqslant 1$, but also the following numbers — called *free energies* — indexed by $g \geqslant 2$:

$$\mathfrak{F}_g[\mathcal{S}] = \frac{1}{2-2g} \sum_{\rho \in P} \operatorname*{Res}_{z=\rho} \left(\int_{o_\rho}^{z} y\mathrm{d}x \right) \omega_{g,1}(z). \tag{5.1}$$

Here, $o_\rho \in \mathcal{C}$ is an arbitrary point in a small contractible neighborhood of ρ and we integrate from o_ρ to z in such a neighborhood.

Remark 5.1. If $y\mathrm{d}x$ is meromorphic on a connected curve $\mathcal{C}$, we can also choose o_ρ independent of ρ, and (5.1) does not depend on the path of integration from o to z since $\mathrm{Res}_{z=\rho}\, \omega_{g,1} = 0$ for any $\rho \in P$, see *e.g.* [EO07].

Let $\check{\mathcal{S}} = (\mathcal{C}, y, x, \omega_{0,2})$ be the spectral curve where the role of x and y are exchanged, and $\check{P}$ be the set of zeroes of $\mathrm{d}y$. It is expected that for reasonable spectral curves, we have the equality

$$\mathfrak{F}_g[\mathcal{S}] = \mathfrak{F}_g[\check{\mathcal{S}}], \tag{5.2}$$

even though the multi-differentials constructed by the topological recursion for $\mathcal{S}$ and $\check{\mathcal{S}}$ are different. This property is called symplectic invariance (for the exchange transformation $(x, y) \mapsto (y, x)$). It is deep and still mysterious. In applications of topological recursion in Gromov–Witten theory of toric Calabi–Yau threefolds [BKMP09, EO15], it corresponds for instance to the framing invariance of the closed sector. The precise meaning of "reasonable", *i.e.* the minimal assumptions on the spectral curve under which (5.2) is expected to hold (perhaps after adding certain explicit terms on the right-hand side) are not known. Eynard and Orantin have proposed in [EO08, EO13] a rather involved derivation via the two-matrix model; yet, their result does not seem to always apply in cases of interest. Understanding better the origin of symplectic invariance, formulating it precisely and obtaining its proof under the weakest possible assumptions remains admittedly a fundamental and open problem in the theory of topological recursion.

Here, we are in position to give an interpretation of some pieces of the puzzle when $\mathcal{S} = (\mathbb{P}^1, \mathsf{x}, \mathsf{y}, \omega_{0,2})$ is the spectral curve (4.15) governing ordinary maps. In that case, $\check{\mathcal{S}} = (\mathbb{P}^1, \mathsf{y}, \mathsf{x}, \hat{\omega}_{0,2})$ is the spectral curve (4.13) which we have proved to govern fully simple maps, and numerically, (5.2) does not seem to hold as such.

[5] There are assumptions on the spectral curve for this construction to be well-defined, we refer to [BKS24] for a discussion and the weakest currently known set of assumptions.

5.1.2 Free energy computations

We shall compute both sides of (5.2) for our two spectral curves, in terms of ordinary and fully simple generating series in topology $(g,1)$. We use the letters ω (resp. $\breve{\omega}$) for the multi-differentials associated to the spectral curve $\mathcal{S}$ (resp. $\check{\mathcal{S}}$), so in fact $\breve{\omega}$ coincide with $\hat{\chi}$ of Section 4.3. We denote $P = \{-1,1\}$ the set of zeroes of $\mathsf{x}'(\theta) = c(1-\theta^{-2})$ and $\check{P}$ the zeroes of $\mathsf{y}'(\theta)$. By a continuity argument, it is sufficient to prove the result for $t_3,\ldots,t_{r+1}$ such that the zeroes of y' are simple, *i.e.* $\#\check{P} = r$. Notice that

$$\mathsf{y}(\theta) = \big[V'(\mathsf{x}(\theta))\big]_{\leqslant 0} = V'(\mathsf{x}(\theta)) + \mathcal{O}(\theta), \tag{5.3}$$

where the $\mathcal{O}(\theta)$ is in fact a polynomial in θ.

Let us fix $g \geqslant 2$. According to the basic properties of the topological recursion [EO07], $\omega_{g,1}$ (resp. $\breve{\omega}_{g,1}$) is a meromorphic 1-form with poles at P (resp. $\check{P}$) and zero residues. Then, we can introduce the rational functions

$$\Phi_{g,1}(\theta) = \int_{\infty}^{\theta} \omega_{g,1}, \qquad \check{\Phi}_{g,1}(\theta) = \int_{\infty}^{\theta} \breve{\omega}_{g,1}.$$

Another basic property is the linear loop equation (see *e.g.* [BS17]), which states that

$$\sum_{\tilde{\theta} \in \mathsf{x}^{-1}(\mathsf{x}(\theta))} \omega_{g,1}(\tilde{\theta}) \tag{5.4}$$

is holomorphic near P, for $g > 0$. But here $\mathsf{x}^{-1}(\mathsf{x}(\theta)) = \{\theta, \theta^{-1}\}$; in particular, the involution $\theta \mapsto \theta^{-1}$ giving the second point in this fiber is globally defined on $\mathbb{P}^1$. Therefore, the left-hand side of (5.4) is a holomorphic 1-form on $\mathbb{P}^1$, hence

$$\omega_{g,1}(\theta) + \omega_{g,1}(\theta^{-1}) = 0, \quad \text{for } g > 0. \tag{5.5}$$

See also [Eyn16]. Note that the linear loop equation for $\breve{\omega}_{g,1}$ states that $\sum_{\tilde{\theta} \in \mathsf{y}^{-1}(\mathsf{y}(\theta))} \breve{\omega}_{g,1}(\tilde{\theta})$ is holomorphic near $\check{P}$, which does not lead to any formula for $\breve{\omega}_{g,1}(\theta^{-1})$. We also mention the property obtained in [EO07]:

$$\sum_{\rho \in \check{P}} \operatorname*{Res}_{\theta=\rho} \mathsf{x}(\theta)\mathsf{y}(\theta)\breve{\omega}_{g,1}(\theta) = 0. \tag{5.6}$$

An analogous one is also true for $\omega_{g,1}$ but we will not need it.

We recall that topological recursion for ordinary maps (see (4.17)) and for fully simple maps (see (4.10)) yields the expansions:

$$\begin{aligned}
\omega_{g,1}(\theta) &\underset{\theta\to\infty}{=} \sum_{k=1}^{r+1} \mathrm{Map}_{g;(k)} \frac{\mathrm{dx}(\theta)}{x(\theta)^{k+1}} + \mathcal{O}\bigg(\frac{\mathrm{dx}(\theta)}{\mathsf{x}(\theta)^{r+3}}\bigg), \\
\check{\omega}_{g,1}(\theta) &\underset{\theta\to\infty}{=} \sum_{k=1}^{r+1} \mathrm{FSMap}_{g;(k)}\, \mathsf{y}(\theta)^{k-1}\mathrm{dy}(\theta) + \mathcal{O}\big(\mathsf{y}(\theta)^{r+1}\mathrm{dy}(\theta)\big) \\
&\underset{\theta\to\infty}{=} -\sum_{k=1}^{r+1} \sum_{\substack{\ell_1,\ldots,\ell_k\geqslant 0\\ \ell_1+\cdots+\ell_k+k\leqslant r+1}} \frac{(\ell_1+\cdots+\ell_k+k)\,\mathrm{FSMap}_{g;(k)}}{k\,\mathsf{x}(\theta)^{\ell_1+\cdots+\ell_k+k+1}} \bigg[\prod_{i=1}^{k} \mathrm{Map}_{0,(\ell_i)}\bigg]\mathrm{dx}(\theta) \\
&\qquad + \mathcal{O}\bigg(\frac{\mathrm{dx}(\theta)}{\mathsf{x}(\theta)^{r+3}}\bigg).
\end{aligned} \tag{5.7}$$

We could truncate the sum in the last line using $\mathsf{y}(\theta) = \mathcal{O}\big(\mathsf{x}(\theta)^{-1}\big)$ when $\theta \to \infty$. We will see that the expansion of $\check{\omega}_{g,1}(\theta)$ near $\theta = 0$ plays a role for the computation of $\mathfrak{F}_g[\check{\mathcal{S}}]$. We therefore introduce a name for its coefficients:

$$\check{\omega}_{g,1}(\theta) \underset{\theta\to 0}{=} \sum_{k=1}^{r+1} \mathrm{Rest}_{g,(k)} \frac{\mathrm{dx}(\theta)}{\mathsf{x}(\theta)^{k+1}} + \mathcal{O}\bigg(\frac{\mathrm{dx}(\theta)}{\mathsf{x}(\theta)^{r+3}}\bigg).$$

We are ready to compute $\mathfrak{F}_g[\mathcal{S}]$, starting from (5.1).

Lemma 5.2. *For $g \geqslant 2$, the generating series of closed maps of genus g satisfy:*

$$(2-2g)\mathfrak{F}_g[\mathcal{S}] = \alpha\partial_\alpha \mathrm{Map}_{g,\varnothing} = -\frac{\mathrm{Map}_{g;(2)}}{2} + \sum_{k=3}^{r+1} t_k \frac{\mathrm{Map}_{g;(k)}}{k}.$$

Proof. Integration by parts in (5.1) yields:

$$(2-2g)\mathfrak{F}_g[\mathcal{S}] = \sum_{\rho\in P} \operatorname*{Res}_{\theta=\rho} \bigg(\int_\infty^\theta \mathsf{y}\mathrm{dx}\bigg)\omega_{g,1}(\theta) = -\sum_{\rho\in P} \operatorname*{Res}_{\theta=\rho} \Phi_{g,1}(\theta)\, \mathsf{y}(\theta)\mathrm{dx}(\theta).$$

The 1-form $\omega_{0,1}(\theta) = \mathsf{y}(\theta)\mathrm{dx}(\theta)$ has a simple pole at $\theta = \infty$ and a pole of order $r+2$ at $\theta = 0$. Besides, the function $\Phi_{g,1}(\theta)$ has a simple zero at $\theta = \infty$. Moving contours we deduce that

$$(2-2g)\mathfrak{F}_g[\mathcal{S}] = \operatorname*{Res}_{\theta=0} \Phi_{g,1}(\theta)\, \mathsf{y}(\theta)\mathrm{dx}(\theta) = \operatorname*{Res}_{\theta=0} \Phi_{g,1}(\theta)\, \mathrm{d}V(\mathsf{x}(\theta)),$$

where we have used (5.3). We then perform the change of variable $\theta \mapsto \theta^{-1}$ and use that x is invariant while $\Phi_{g,1}$ is antiinvariant (by integration of (5.5)) to find

$$(2-2g)\mathfrak{F}_g[\mathcal{S}] = -\operatorname*{Res}_{\theta=\infty} \Phi_{g,1}(\theta)\, \mathrm{d}V(\mathsf{x}(\theta)) = \operatorname*{Res}_{\theta=\infty} V(\mathsf{x}(\theta))\omega_{g,1}(\theta).$$

We rather use the local coordinate x near $\theta = \infty$ and insert the expansion (5.7) and Definition 2.5 of the potential, which results in:

$$\begin{aligned}(2-2g)\mathfrak{F}_g[\mathcal{S}] &= \operatorname*{Res}_{x=\infty} \mathrm{d}x \left(\frac{x^2}{2} - \sum_{m=3}^{r+1} \frac{t_m}{m} x^m\right)\left(\sum_{k=1}^{r+1} \mathrm{Map}_{g;(k)} \frac{\mathrm{d}x}{x^{k+1}}\right) \\ &= -\frac{\mathrm{Map}_{g;(2)}}{2} + \sum_{k=3}^{r+1} t_k \frac{\mathrm{Map}_{g;(k)}}{k}.\end{aligned} \tag{5.8}$$

The weight of an ordinary map M includes a factor $\alpha^{\deg M} = \alpha^{2-2g(M)-\#\mathcal{V}(M)} = \alpha^{-\#\mathcal{E}(M)+\#\mathcal{F}(M)}$. Closed ordinary maps of genus g with a marked (non-oriented) edge are in bijection with ordinary maps of genus g with an (unrooted) boundary face of degree 2: just glue the two edges of the boundary face. Closed ordinary maps of genus g with a marked (unrooted) face are in bijection with ordinary maps of genus g with an (unrooted) boundary face of degree $k \in \{3, \ldots, r+1\}$. All together, these observations imply that

$$\alpha\partial_\alpha \mathrm{Map}_{g,\varnothing} = -\frac{\mathrm{Map}_{g;(2)}}{2} + \sum_{k=3}^{r+1} t_k \frac{\mathrm{Map}_{g;(k)}}{k} = (2-2g)\mathfrak{F}_g[\mathcal{S}]. \qquad \square$$

We now turn to the free energy for $\check{\mathcal{S}}$. It is not directly expressed in terms of FSMap.

Lemma 5.3. *For $g \geqslant 2$, we have*

$$(2-2g)\mathfrak{F}_g[\check{\mathcal{S}}] = -\frac{\mathrm{Rest}_{g,(2)}}{2} + \sum_{k=3}^{r+1} \frac{\mathrm{Rest}_{g,(k)}}{k}. \tag{5.9}$$

Proof. Writing $\mathsf{x}\mathrm{d}\mathsf{y} = -\mathsf{y}\mathrm{d}\mathsf{x} + \mathrm{d}(\mathsf{x}\mathsf{y})$ and with the help of (5.6), we compute:

$$\begin{aligned}(2-2g)\mathfrak{F}_g[\check{\mathcal{S}}] &= \sum_{\rho \in \check{P}} \operatorname*{Res}_{\theta=\rho} \left(\int_{o_\rho}^{\theta} \mathsf{x}\mathrm{d}\mathsf{y}\right)\check{\omega}_{g,1}(\theta) = -\sum_{\rho \in \check{P}} \operatorname*{Res}_{\theta=\rho} \left(\int_{o_\rho}^{\theta} \mathsf{y}\mathrm{d}\mathsf{x}\right)\check{\omega}_{g,1}(\theta) \\ &= \sum_{\rho \in \check{P}} \operatorname*{Res}_{\theta=\rho} \check{\Phi}_{g,1}(\theta)\, \mathsf{y}(\theta)\mathrm{d}\mathsf{x}(\theta) = -\operatorname*{Res}_{\theta=0,\infty} \check{\Phi}_{g,1}(\theta)\, \mathsf{y}(\theta)\mathrm{d}\mathsf{x}(\theta).\end{aligned} \tag{5.10}$$

When $\theta \to \infty$, we have

$$\mathsf{y}(\theta)\mathrm{d}\mathsf{x}(\theta) = \mathcal{O}(\theta^{-1}\mathrm{d}\theta), \qquad \check{\Phi}_{g,1}(\theta) = \mathcal{O}(\theta^{-1}),$$

therefore $\theta = \infty$ does not contribute to the residue, and we find:

$$\begin{aligned}(2-2g)\mathfrak{F}_g[\check{\mathcal{S}}] &= -\operatorname*{Res}_{\theta=0} \check{\Phi}_{g,1}(\theta)\, \mathrm{d}V(\mathsf{x}(\theta)) \\ &= \operatorname*{Res}_{\theta=0} V(\mathsf{x}(\theta))\, \check{\omega}_{g,1}(\theta) \\ &= \operatorname*{Res}_{x=\infty} \left(\frac{x^2}{2} - \sum_{m=3}^{r+1} t_m \frac{x^m}{m}\right)\left(\sum_{k=1}^{r+1} \mathrm{Rest}_{g;(k)} \frac{\mathrm{d}x}{x^{k+1}}\right) \\ &= -\frac{\mathrm{Rest}_{g,(2)}}{2} + \sum_{k=3}^{r+1} \frac{\mathrm{Rest}_{g,(k)}}{k}.\end{aligned} \qquad \square$$

5.1.3 Comment

Lemma 5.2 relates the enumeration of closed maps of genus g to the enumeration of ordinary maps of genus g with 1 boundary face. One may try to apply a simplification procedure to this boundary face, so as to relate it further to a fully simple enumeration. However, in doing so, many topologies lower than $(g, 1)$ may appear. To understand if symplectic invariance is true (or true up to additional terms), we would need to find a combinatorial interpretation of the generating series $\mathrm{Rest}_{g,(k)}$, stored in the $\theta \to 0$ series expansion of $\check{\omega}_{g,1}$. The fully simple enumeration itself is stored in the expansion at $\theta = \infty$. The particular form it takes in (5.7) has a clear combinatorial meaning: one can attach ordinary disks at each vertex of a simple boundary face to make it ordinary. However, an ordinary face can be obtained from a simple face in different ways as well, which would involve maps of lower topologies. Therefore, the combinatorial meaning of (5.9) is at present not elucidated although we expect there should be one.

Theorem 4.11 had received a conditional proof in [BGF20], provided a milder version of symplectic invariance was true for the topological recursion for the matrix model with external field, from a combinatorial interpretation of the partition function governing the matrix model with external field. Indeed, the latter is a generating series for fully simple maps. The definition of ciliated maps from [BCEGF21] was also motivated by the matrix model with external field, seen as a particular generalisation of the Kontsevich matrix model. The latter is relevant in the study of the r-spin intersection numbers [Wit93, FSZ10] on the moduli space of curves, while the $r = 2$ case is the one introduced by Kontsevich in his proof of Witten's conjecture [Kon92]. This suggests that the concrete combinatorial tools developed to relate fully simple and ordinary maps could have an extension to the full generality of the matrix model with external field; concretely, we mean to the situation where the set of parameters Λ is not specialised to $\{0\}$, which therefore encompasses a larger family of spectral curves.

5.2. Enumeration of fully simple maps

The enumeration of fully simple maps of genus 0 was explicitly given by Krikun [Kri07] for triangulations (only $t_3 \neq 0$). His formula was later extended by Bernardi–Fusy to planar quadrangulations (only $t_4 \neq 0$) and boundary faces of even degrees with a bijective approach [BF18]. Using the predictions coming from the conjectural topological recursion those formulae were conjecturally generalised for any boundary face degrees [BGF20, Conjecture 1.9]. That closed formula is now proved for $n \leqslant 4$ from a straightforward application of two steps of the topological recursion. The enumeration of discs and cylinders for any structure of internal faces was already established in [BGF20].

In general, possibly disconnected ordinary maps are known to be related to possibly disconnected fully simple maps via monotone Hurwitz numbers. This relation was established using Weingarten calculus in [BGF20] and using bijective methods in [BCDGF19]. These formulae allow to compute the number of fully simple maps with certain constraints, if one is already able to compute the number of ordinary maps and (strictly or weakly) monotone Hurwitz numbers. The enumeration of connected maps in terms of the disconnected ones is possible making use of inclusion-exclusion formulae. The advantage of Theorem 4.11 is that it solves directly the enumeration of connected fully simple maps for any topology, recursively on $2g-2+n$, and for any structure of the internal faces. For instance, it implies the enumeration of quadrangulations of topology $(1,1)$:

Corollary 5.4. *For $m \in \mathbb{Z}_{\geqslant 0}$, let $\phi_m = c^{2m}\frac{1+(m-1)\sqrt{1-12t_4}}{1-12t_4}$, where we $c^2 = \frac{1-\sqrt{1-12t_4}}{6t_4}$. Then,*

$$\mathrm{Map}_{1;(2(m+1))}\big|_{\alpha=1} = \frac{(2m+1)!}{6\,m!^2}\,\phi_m, \quad \text{for } m \geqslant 0, \tag{5.11}$$

$$\mathrm{FSMap}_{1;(2m)}\big|_{\alpha=1} = \frac{(3m)!\,t_4^{m+1}}{4\,m!(2m-1)!}\,\phi_{3m+1}, \quad \text{for } m \geqslant 1. \tag{5.12}$$

The details of the proof and how to extract closed formulae from this corollary are detailed in [BGF20, Section 5.2.3].

Another advantage of Theorem 4.11 is the fact that the topological recursion implies that the differentials that it produces are endowed with strong structures, such as the following.

Corollary 5.5. *For $2g-2+n \geqslant 1$, the formal series*

$$X_{g,n}(w(\zeta_1),\ldots,w(\zeta_n))\,x'(\zeta_1)\ldots x'(\zeta_n)$$

is a rational function such that in each variable, the poles are located at the ramification points $\rho_1,\ldots,\rho_{r-1}$ and are of order at most $6g-4+2n$.

For a proof of this statement, see [BCCGF24, Corollary 2.15]. In particular, we can apply this result for triangulations and quadrangulations, using the computations of Section 4.1.

Example 5.6. For $r=2$ (triangulations). We have $x'(\zeta) = Q'(\zeta) = 1-2t_3\zeta$, so the spectral curve has one ramification point $\rho_1 = \frac{1}{2t_3}$. We can deduce that for $2g-2+n > 0$, $X_{g,n}(w(\zeta_1),\ldots,w(\zeta_n))\,x'(\zeta_1)\ldots x'(\zeta_n)$ is a rational function whose poles are all located at $\frac{1}{2t_3}$ in each variable.

Example 5.7. For $r=3$ and $t_3=0$ (quadrangulations). In this case,

$$x'(\zeta) = Q'(\zeta) = \frac{1+\sqrt{1-12t_4\alpha^{-1}}}{2} - 3t_4\zeta^2.$$

There are two ramification points:

$$\{\rho_1,\rho_2\} = \left\{\pm\sqrt{\frac{1+\sqrt{1-12t_4\alpha^{-1}}}{6t_4}}\right\}.$$

Those are the only possible poles in each variable for the rational function

$$X_{g,n}(w(\zeta_1),\ldots,w(\zeta_n))\,x'(\zeta_1)\ldots x'(\zeta_n)$$

(where again, $2g-2+n$ is supposed to be strictly positive).

5.3. Functional relations and connection to free probability

In free probability theory, the notion of independence from classical probability is replaced by a notion of freeness, which is particularly well-adapted to study non-commutative probability spaces. Free cumulants are crucial objects that allow to characterise freeness in a simple way. Random matrices in the large size limit constitute an important class of free random variables. In [MS06, MŚS07, CMŚS07] a notion of higher order freeness was introduced to study these questions more finely. While first order free cumulants are defined in terms of moments using non-crossing partitions, the definition of higher order free cumulants involve intricate combinatorial objects, called non-crossing partitioned permutations.

For $n = 1$, the R-transform machinery [Voi86] gives a relation between the generating series of moments and of free cumulants, by functional inversion. For $n = 2$, a functional relation between the ordinary and the free generating series was also found [CMŚS07], already in a quite complicated way. Similar functional relations are unfortunately not known for $n \geqslant 3$, which leaves us with a rather complicated theory to compute with. In [BGF20, Section 11.2], for an arbitrary (formal) unitarily-invariant measure of the space of Hermitian matrices, the classical identification of moments of products of traces of Hermitian matrices with generating series of ordinary maps [BIPZ78] was extended to an identification of free cumulants for the same measure with the generating series of planar fully simple maps. The formulae for discs and cylinders (1.2)-(1.3) recover the R-transform machinery for $n = 1, 2$. In particular, the formula for cylinders gives an intrinsic, geometric meaning to the functional relation for $n = 2$.

The present work allows to recursively compute higher order free cumulants in certain unitary invariant matrix models, namely for measures of the form $\mathcal{Z}_N^{-1} \mathrm{d}M\, e^{-N \mathrm{Tr}\, V(M)}$. We expect that this approach could *in fine* extend the R-transform machinery for any order n for those measures. For instance, for $n = 3$, we have already established the desired functional relation between ordinary and fully simple pairs of pants, that is of topology $(0, 3)$:

Corollary 5.8. *Let* $\omega_{0,2}(\theta_1, \theta_2) = \chi_{0,2}(\theta_1, \theta_2)$ *be the standard bidifferential and set* $\alpha = 1$. *Then, we have the following relation of ordinary and fully simple pairs of pants:*

$$\omega_{0,3}(\theta_1, \theta_2, \theta_3) + \chi_{0,3}(\theta_1, \theta_2, \theta_3) = \operatorname*{Res}_{\theta=\theta_1,\theta_2,\theta_3} \frac{\omega_{0,2}(\theta, \theta_1)\omega_{0,2}(\theta, \theta_2)\omega_{0,2}(\theta, \theta_3)}{\mathrm{dx}(\theta)\mathrm{dy}(\theta)} \tag{5.13}$$

$$= \mathrm{d}_1 \Big[\frac{\omega_{0,2}(\theta_1, \theta_2)\omega_{0,2}(\theta_1, \theta_3)}{\mathrm{dx}(\theta_1)\mathrm{dy}(\theta_1)} \Big] + \mathrm{d}_2 \Big[\frac{\omega_{0,2}(\theta_2, \theta_1)\omega_{0,2}(\theta_2, \theta_3)}{\mathrm{dx}(\theta_2)\mathrm{dy}(\theta_2)} \Big] + \mathrm{d}_3 \Big[\frac{\omega_{0,2}(\theta_3, \theta_1)\omega_{0,2}(\theta_3, \theta_2)}{\mathrm{dx}(\theta_3)\mathrm{dy}(\theta_3)} \Big].$$

This corollary follows from Theorem 4.11 and the details exposed in [BGF20, Section 6]. Even if free cumulants are so far only defined for $g = 0$, our work suggests that there should exist a universal theory of approximate higher order free cumulants taking into account higher genus corrections. For a compact introduction to all the necessary objects to understand this connection to free probability precisely, the reader could consult [GF18, Section 1.6] or many other more extended sources written by experts in free probability [NS06, MS17].

For general (formal) unitarily-invariant measures, the underlying combinatorial objects (in ordinary or fully simple flavor) are the stuffed maps introduced in [Bor14]. It was proved that stuffed maps satisfy a generalisation of the topological recursion, called blobbed topological recursion [BS17], where the initial data of the spectral curve is enriched by symmetric holomorphic forms in n variables $(\phi_{g,n})_{2g-2+n>0}$. In [BGF20] it was conjectured that after the same symplectic exchange transformation, and a transformation of the blobs still to be described, blobbed topological recursion will enumerate fully simple stuffed maps. This conjecture already follows for the base topologies $(0,1)$ and $(0,2)$ from the formulae (1.2)-(1.3) for discs and cylinders, since the base topologies are not altered by the blobs. It may be possible, either by studying multi-ciliated stuffed maps, or by substitution methods (at least in genus 0), to extend the results of the present article to the case of stuffed maps. The solution of this problem would allow the compute higher order free cumulants in the full generality of [CMŚS07], and progressing towards such a solution is an important motivation for the present work.

Acknowledgements

All the authors thank the anonymous referees for useful comments, and are grateful to Raphaël Belliard, Norman Do, Bertrand Eynard, Danilo Lewański and Ellena Moskowsky for numerous discussions and computations concerning fully simple maps.

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