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On The Decoupling Approximation in Damped Linear Systems[1]

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Abstract: The principal coordinates of a non-classically damped linear system are coupled by non-zero off-diagonal elements of the modal damping matrix. In the analysis of non-classically damped systems, a common approximation is to ignore the off-diagonal elements of the modal damping matrix. This procedure is termed the decoupling approximation. It is widely believed that if the modal damping matrix is diagonally dominant, then errors due to the decoupling approximation must be small. In addition, it is intuitively accepted that the more diagonal the modal damping matrix, the smaller will be the errors due to the decoupling approximation. Two numerical indices are proposed in this paper to measure quantitatively the degree of being diagonal in modal damping. It is demonstrated that, over a finite range, errors due to the decoupling approximation can continuously increase while the modal damping matrix becomes more and more diagonal with its off-diagonal elements decreasing in magnitude continuously. An explanation for this unexpected behavior is offered. Within a practical range of engineering applications, diagonal dominance of the modal damping matrix may not be sufficient for neglecting modal coupling in a damped system.

Key words: Linear damped systems, modal analysis, coordinate coupling, decoupling approximation.

1. INTRODUCTION

It is well known that an undamped linear vibratory system possesses classical normal modes, and that in each mode different parts of the system vibrate in a synchronous manner. The normal modes constitute a modal matrix, which defines a linear coordinate transformation that decouples the undamped system. This process of decoupling the equation of motion of an undamped vibratory system is a time-honored procedure termed modal analysis. Upon decoupling, an undamped linear system can be treated as a series of independent single-degree-of-freedom systems.

In the presence of damping, a linear system cannot be decoupled by modal analysis unless it possesses a full set of classical normal modes, in which case damping in the linear system is said to be classical. Rayleigh showed in 1894 that a system is classically damped if its damping matrix is a linear combination of its inertia and stiffness matrices. A damped system possessing such a special property is said to be proportionally damped. Subsequently, Caughey and O'Kelly (1965) established necessary and sufficient conditions under which a linear vibratory system is classically damped. When a vibratory system is classically damped, it can be decoupled by the same modal transformation that decouples the corresponding undamped system obtained when damping is ignored. Classical and in particular proportional damping is routinely assumed in design and finite-element computations.

There is no reason why damping in a linear system should be classical. Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. This assumption is violated for systems consisting of two or more parts with different levels of damping. Examples of such systems include soil-structure interacting systems (Clough and Mojtahedi, 1976), base-isolated structures (Tsai and Kelly, 1988, 1989), and systems in which coupled vibrations of structures and fluids occur. Increasing use of special energy-dissipating devices in control constitutes another important example. In fact, experimental modal testing suggests that no physical system is strictly classically damped (Sestieri and Ibrahim, 1994).

A damped system that does not possess classical normal modes is, naturally, said to be non-classically damped. A linear transformation defined by the modal matrix does not decouple a non-classically damped system. Upon modal analysis, a non-classically damped system remains coupled by the off-diagonal elements of its modal damping matrix. Ma and Caughey (1995) proved that no time-invariant linear transformations in the configuration space will decouple every non-classically damped system. Even partial decoupling, i.e. simultaneous transformation of the coefficient matrices of the equation of motion to upper triangular forms, is not ensured with time-invariant linear transformations in the configuration space (Caughey and Ma, 1993; Lee and Ma, 1997).

Classical normal modes are all real. Thus modal analysis in the classical sense involves a real transformation. Foss (1958), Velestos and Ventura (1986), and Vigneron (1986) extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state-space approach has never appealed to practising engineers. One reason usually given is that the state-space approach is computationally more involved because the dimension of the state space is twice the number of degrees-of-freedom. Another reason is that complex modal analysis still cannot decouple all non-classically damped systems. A condition of non-defective eigenvectors must be satisfied in order for complex modal analysis to achieve complete decoupling in the state space. More importantly, there is little physical insight associated with different elements of complex modal analysis.

As previously explained, a linear system can always be decoupled inertially and elastically by classical modal analysis, and any coupling occurs ultimately through damping. In the analysis of non-classically damped systems, a common approximation is to ignore the off-diagonal elements of the modal damping matrix. This procedure is termed the decoupling approximation (Meirovitch, 1967; Benaroya, 1998; Ginsberg, 2001), which amounts to neglecting coupling of the principal coordinates. Thomson et al. (1974) analyzed whether the approximation can be improved by using diagonal matrices other than the modal damping matrix with omitted off-diagonal elements. A similar approach was pursued by Felszeghy (1993). Recently, Angeles and Ostrovskaya (2002) proposed to decompose the damping matrix to extract a proportional component which approximates the original damping matrix optimally in a least-squares sense. However, it was shown by Shahruz and Ma (1988) and Shahruz (1990) that among arbitrary diagonal matrices, the one that minimizes the error bound of the decoupling approximation is the modal damping matrix with omitted off-diagonal elements. Thus, it is generally not worthwhile to go through anything more complicated than the decoupling approximation.

Intuitively speaking, the errors due to the decoupling approximation should be small if the off-diagonal elements of the modal damping matrix are small in magnitude. This condition is routinely used to justify the decoupling approximation (Prater and Singh, 1986; Nair and Singh, 1986; Sharuz and Ma, 1988; Sharuz, 1990; Tong et al., 1994; Gawronski, 1998). The purpose of this paper is to show numerically that small off-diagonal elements of the modal damping matrix are not sufficient to neglect modal coupling by the decoupling approximation. In fact, within a practical range of engineering applications, coupling effects can increase continuously as the off-diagonal elements of the modal damping matrix continuously decrease in magnitude. The organization of the paper is as follows. Section 2 provides the theoretical background of the decoupling approximation and introduces a quantitative measure of the effect of modal coupling in discrete vibratory systems. Diagonal Dominance and diagonality of matrices are discussed in section 3. Two numerical indices are proposed to measure quantitatively the diagonality of real square matrices. Contrary to widely accepted beliefs, it will be shown in section 3 that small off-diagonal elements are not sufficient to ensure small errors due to the decoupling approximation. An example is provided to show that the errors due to the decoupling approximation may continuously increase as the off-diagonal elements of the modal damping matrix continuously decrease in magnitude. An explanation for this observation is offered in section 4. In section 5, a summary of findings is provided.

2. THE DECOUPLING APPROXIMATION

The equation of motion of an n-degree-of-freedom linear system with viscous damping under external excitation can be written in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$. The generalized coordinate $\mathbf{x}$ and excitation $\mathbf{f}(t)$ are real n-dimensional column vectors. The mass matrix $\mathbf{M}$, the damping matrix $\mathbf{C}$, and the stiffness matrix $\mathbf{K}$ are real symmetric matrices of order n. For passive systems, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite. These assumptions are not arbitrary, but in fact have solid footing in the theory of Lagrangian dynamics. Associated with the undamped system is a generalized eigenvalue problem (Meirovitch, 1967)

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}. \tag{2}$$

Owing to the definiteness of the coefficient matrices, the n eigenvalues are real and positive, and the corresponding eigenvectors are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$ such that

$\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j = 0$ and $\mathbf{u}_i^T\mathbf{K}\mathbf{u}_j = 0$ for $i \neq j$. Moreover, the eigenvectors can always be chosen real. Define the modal matrix associated with system (1) by

$$\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]. \tag{3}$$

Upon normalization, the orthogonality of the modes can be expressed in a matrix form:

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}, \tag{4}$$

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathbf{\Omega}. \tag{5}$$

The diagonal matrix $\mathbf{\Omega}$ is referred to as the spectral matrix and it contains the natural frequencies squared such that $\omega_i^2 = \lambda_i$. Define a modal transformation by

$$\mathbf{x} = \mathbf{U}\mathbf{q}. \tag{6}$$

In terms of the principal coordinate $\mathbf{q}$, the equation of motion of a damped system takes the canonical form

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{g}(t) \tag{7}$$

with initial conditions $\mathbf{q}(0) = \mathbf{U}^T\mathbf{M}\mathbf{x}_0$, $\dot{\mathbf{q}}(0) = \mathbf{U}^T\mathbf{M}\dot{\mathbf{x}}_0$ and excitation $\mathbf{g}(t) = \mathbf{U}^T\mathbf{f}(t)$. The symmetric matrix

$$\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U} \tag{8}$$

is referred to as the modal damping matrix. Owing to the orthogonality of the modes, any system can be decoupled inertially and elastically through modal transformation. The modal damping matrix is diagonal if, and only if, system (1) is classically damped (Caughey and O'Kelly, 1965). The equation of motion of a non-classically damped system in principal coordinate is coupled by the off-diagonal elements of $\mathbf{D}$.

Modal damping can always be written in the form

$$\mathbf{D} = \mathbf{D}_d + \mathbf{D}_o, \tag{9}$$

where $\mathbf{D}_d = \text{diag}[d_{11}, d_{22}, \ldots, d_{nn}]$ is a diagonal matrix composed of the diagonal elements of $\mathbf{D}$, and $\mathbf{D}_o$ is a matrix with zero diagonal elements and whose off-diagonal elements coincide with those in $\mathbf{D}$. The decoupling approximation amounts to simply neglecting $\mathbf{D}_o$ and thus replacing $\mathbf{D}$ by $\mathbf{D}_d$. The system response by decoupling approximation satisfies the decoupled equation

$$\ddot{\mathbf{q}}_a(t) + \mathbf{D}_d\dot{\mathbf{q}}_a(t) + \mathbf{\Omega}\mathbf{q}_a(t) = \mathbf{g}(t) \tag{10}$$

with initial conditions $\mathbf{q}_a(0) = \mathbf{q}(0)$, $\dot{\mathbf{q}}_a(0) = \dot{\mathbf{q}}(0)$. The error due to the decoupling approximation is the difference of the exact and approximate solutions:

$$\mathbf{e}(t) = \mathbf{q}(t) - \mathbf{q}_a(t). \tag{11}$$

Subtract equation (10) from (7) to obtain

$$\ddot{\mathbf{e}}(t) + \mathbf{D}_d\dot{\mathbf{e}}(t) + \mathbf{\Omega}\mathbf{e}(t) = -\mathbf{D}_o\dot{\mathbf{q}}(t) \tag{12}$$

with initial conditions $\mathbf{e}(0) = \mathbf{0}$, $\dot{\mathbf{e}}(0) = \mathbf{0}$. For the remainder of the paper, an excitation of the form

$$\mathbf{g}(t) = \widehat{\mathbf{g}}g(t), \tag{13}$$

where $\widehat{\mathbf{g}}$ is a constant vector of amplitudes, is assumed. An excitation of this form is appropriate for many applications, for example in earthquake engineering. Often, one may further assume that $g(t)$ is a harmonic function. Numerical data suggests that errors due to the decoupling approximation are comparable or smaller for non-harmonic excitations (Warburton and Soni, 1977; Ajavakom, 2005).

Upon Fourier transformation of equation (7), the particular solution in principal coordinates can be written as

$$\mathbf{Q}(i\omega) = (\mathbf{\Omega} - \omega^2\mathbf{I} + i\omega\mathbf{D})^{-1}\widehat{\mathbf{g}}G(i\omega). \tag{14}$$

As a standard notation, Fourier transforms are denoted by capital letters, $\mathbf{I}$ represents the $n \times n$ identity matrix and $i = \sqrt{-1}$. Introduce the frequency response matrix

$$\mathbf{H}(i\omega) = (\mathbf{\Omega} - \omega^2\mathbf{I} + i\omega\mathbf{D})^{-1}. \tag{15}$$

Thus the system response in the frequency domain becomes

$$\mathbf{Q}(i\omega) = \mathbf{H}(i\omega)\widehat{\mathbf{g}}G(i\omega). \tag{16}$$

Similarly, apply Fourier transformation to equation (10) to obtain

$$\mathbf{Q}_a(i\omega) = \mathbf{H}_a(i\omega)\widehat{\mathbf{g}}G(i\omega), \tag{17}$$

where the diagonal frequency response matrix $\mathbf{H}_a(i\omega)$ is given by

$$\mathbf{H}_a(i\omega) = (\mathbf{\Omega} - \omega^2\mathbf{I} + i\omega\mathbf{D}_d)^{-1}. \tag{18}$$

Note that the inverse of $\mathbf{\Omega} - \omega^2\mathbf{I} + i\omega\mathbf{D}$ or $\mathbf{\Omega} - \omega^2\mathbf{I} + i\omega\mathbf{D}_d$ exists for any frequency ω if $\mathbf{C}$ is positive definite. Upon Fourier transformation, equation (12) becomes

$$\mathbf{E}(i\omega) = -i\omega\mathbf{H}_a(i\omega)\mathbf{D}_o\mathbf{Q}(i\omega). \tag{19}$$

Combine equations (16) and (19) to yield

$$\mathbf{E}(i\omega) = -i\omega\mathbf{H}_a(i\omega)\mathbf{D}_o\mathbf{H}(i\omega)\widehat{\mathbf{g}}G(i\omega). \tag{20}$$

From equations (16) and (20), a numerical index can be defined to measure quantitatively the effect of modal coupling in system (7):

$$\chi(i\omega) = \frac{\|\mathbf{E}(i\omega)\|_2}{\|\mathbf{Q}(i\omega)\|_2} = \frac{\left\|i\omega\mathbf{H}_a(i\omega)\mathbf{D}_o\mathbf{H}(i\omega)\widehat{\mathbf{g}}\right\|_2}{\left\|\mathbf{H}(i\omega)\widehat{\mathbf{g}}\right\|_2}. \tag{21}$$

At any frequency ω, $\chi(i\omega)$ may be interpreted as the relative steady-state error of decoupling approximation under a harmonic excitation with frequency ω and spatial distribution $\widehat{\mathbf{g}}$. The choice of Euclidean norm in equation (21) is for convenience only. Any other vector norm may be used to yield similar results in subsequent sections.

3. DIAGONAL DOMINANCE OF THE DAMPING MATRIX

It is generally accepted that errors due to the decoupling approximation must be small if the off-diagonal elements of the modal damping matrix **D** are small (Prater and Singh, 1986; Nair and Singh, 1986; Shahruz and Ma, 1988; Shahruz, 1990; Tong et al., 1994). In addition, the errors should decrease as **D** becomes more and more diagonal. But the meanings of these terms are not clear. How can one quantify the property of being diagonal? When is a matrix more diagonal than another? These issues will first be clarified in this section.

3.1. Diagonality of Modal Damping

The damping matrix **D** is said to be diagonally dominant (Horn and Johnson, 1985) if

$$|d_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|d_{ij}\right| \tag{22}$$

for all $i = 1, ..., n$. It is said to be strongly diagonally dominant if

$$|d_{ii}| \gg \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|d_{ij}\right| \tag{23}$$

for all $i = 1, ..., n$. These definitions of diagonal dominance have solid footing in linear algebra and many important properties of diagonally dominant matrices have been established. For example if **D** is diagonal dominant, then the real parts of its eigenvalues have the same sign as the diagonal entries. Berman and Plemmons (1994) generalized the concept of diagonal dominance. The matrix **D** is diagonally dominant in a generalized sense if there exist scalars $\alpha_i \neq 0$ such that

$$|d_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_i}{\alpha_j} \left|d_{ij}\right|, \quad |d_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_j}{\alpha_i} \left|d_{ij}\right| \tag{24}$$

for all $i = 1, ..., n$. Clearly, a diagonally dominant matrix is diagonally dominant in the generalized sense. Recall the definitions of $\mathbf{D}_d$ and $\mathbf{D}_o$ in equation (9). Let $|\mathbf{D}_d| = \mathrm{diag}[|d_{11}|, |d_{22}|, \ldots, |d_{nn}|]$ and similarly let $|\mathbf{D}_o|$ be a matrix whose elements are the absolute values of those in $\mathbf{D}_o$. It can be shown (Graham, 1987) that if the spectral radius (largest absolute value of any eigenvalue) of $\left|\mathbf{D}_d^{-1}\right| |\mathbf{D}_o|$ satisfies

$$\sigma \left(\left|\mathbf{D}_d^{-1}\right| |\mathbf{D}_o|\right) < 1 \tag{25}$$

then $\mathbf{D}$ is diagonally dominant in the generalized sense. The concept of generalized diagonal dominance is not only important from a mathematical point of view, but it also finds broad applications in multivariable control theory.

Although the concept of diagonal dominance has been commonly accepted, numerical indices for quantifying the degree of being diagonal have not been reported in the open literature. Based upon equation (22), an index of diagonality may be readily defined as

$$\rho(\mathbf{D}) = \frac{\sum_{i=1}^{n} |d_{ii}|}{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left|d_{ij}\right|}. \tag{26}$$

Clearly, $0 \leq \rho(\mathbf{D}) \leq \infty$ for any matrix $\mathbf{D}$. If $\mathbf{D}$ is diagonally dominant, then $\rho(\mathbf{D}) \geq 1$. A large value of $\rho(\mathbf{D})$ indicates a more diagonal matrix and, for a diagonal matrix, $\rho(\mathbf{D}) = \infty$. Another index of diagonality may be based upon the spectral radius of $\left|\mathbf{D}_d^{-1}\right| |\mathbf{D}_o|$ in equation (25) and defined as

$$\rho_1(\mathbf{D}) = \sigma \left(\left|\mathbf{D}_d^{-1}\right| |\mathbf{D}_o|\right). \tag{27}$$

If $\mathbf{D}$ is diagonally dominant in the generalized sense, $0 \leq \rho_1(\mathbf{D}) \leq 1$. When $\mathbf{D}$ is diagonal, $\rho_1(\mathbf{D}) = 0$. If $|\mathbf{D}|$ is positive (or irreducible), then $\rho_1(\mathbf{D})$ is monotonic increasing as the off-diagonal elements of $\mathbf{D}$ increase in magnitude (Graham, 1987). Thus a small value of $\rho_1(\mathbf{D})$ indicates a more diagonal matrix and the two indices $\rho(\mathbf{D})$ and $\rho_1(\mathbf{D})$ have opposite trends. An advantage of using $\rho_1(\mathbf{D})$ is that it lies within a finite range. On the other hand, $\rho(\mathbf{D})$ can be computed more readily.

It is certainly possible to define other indices of diagonality. However, it will become evident that the choice of an index of diagonality of $\mathbf{D}$ is of minor significance in the characterization of modal coupling. It must be kept in mind that neither $\rho(\mathbf{D})$ nor $\rho_1(\mathbf{D})$ are intended for measuring the errors due to the decoupling approximation, only how diagonal the modal damping matrix $\mathbf{D}$ is.

3.2. Inadequacy of Diagonal Dominance of Damping in Decoupling

As stated above, it is widely believed that if the off-diagonal elements of the modal damping matrix $\mathbf{D}$ are small, they can be neglected and errors due to the decoupling approximation will be small. In addition, the errors should decrease as $\mathbf{D}$ becomes more and more diagonal. Numerical examples can however be constructed to yield contradictory results: diagonal dominance can continuously increase while errors due to the decoupling approximation also continuously increase.

Example. The following example is provided by Ajavakom (2005). Consider two four-degree-of-freedom systems of the form (7). System 1 is governed by

$$\ddot{\mathbf{q}} + \mathbf{D}_1\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t), \tag{28}$$

where the spectral matrix, the modal damping matrix, and the excitation are given by

$$\mathbf{\Omega}_1 = \operatorname{diag}\left[\ 3.95^2,\quad 3.98^2,\quad 4.00^2,\quad 4.10^2\ \right], \tag{29}$$

$$\mathbf{D}_1 = \begin{bmatrix} 1.61 & -0.1865 & -0.1742 & 0.3838 \\ -0.1865 & 1.7 & 0.3792 & -0.1773 \\ -0.1742 & 0.3792 & 1.8 & -0.1742 \\ 0.3838 & -0.1773 & -0.1742 & 1.75 \end{bmatrix}, \tag{30}$$

$$\mathbf{g}(t) = \widehat{\mathbf{g}}\exp(i\omega t) = [\ 1\quad 1\quad 1\quad 1\ ]^T \exp(i4.16t). \tag{31}$$

The equation of motion of System 2 has the form

$$\ddot{\mathbf{q}} + \mathbf{D}_2\dot{\mathbf{q}} + \mathbf{\Omega}_1\mathbf{q} = \mathbf{g}(t), \tag{32}$$

which differs from equation (28) only in the off-diagonal elements of the modal damping matrix

$$\mathbf{D}_2 = \begin{bmatrix} 1.61 & 0.0009 & 0.04 & 0.041 \\ 0.0009 & 1.7 & 0.0227 & 0.0376 \\ 0.04 & 0.0227 & 1.8 & 0.04 \\ 0.041 & 0.0376 & 0.04 & 1.75 \end{bmatrix}. \tag{33}$$

It can be observed that both $\mathbf{D}_1$ and $\mathbf{D}_2$ satisfy condition (22) and therefore are diagonally dominant. In fact, $\mathbf{D}_2$ is strongly diagonally dominant since it satisfies condition (23). Utilizing the index of diagonality proposed in equation (26), it is found that

$$\rho(\mathbf{D}_1) = 2.3 \ll 18.8 = \rho(\mathbf{D}_2). \tag{34}$$

Thus $\mathbf{D}_2$ is more diagonal than $\mathbf{D}_1$. This is perhaps obvious by inspection since the off-diagonal elements of $\mathbf{D}_2$ are at least an-order-of-magnitude smaller. The second index of diagonality defined in equation (27) yields a consistent result:

$$\rho_1(\mathbf{D}_1) = 0.43 \gg 0.055 = \rho_1(\mathbf{D}_2). \tag{35}$$

Intuitively, one would expect System 2 to yield a smaller error due to the decoupling approximation than System 1. However, calculation of the steady-state errors due to the decoupling approximation yields an opposite result:

$$\chi_1(i\omega) = 2.8\% < 5.3\% = \chi_2(i\omega). \tag{36}$$

In other words, System 2 has a larger steady-state error than System 1. Hence, errors due to the decoupling approximation can be larger for systems whose modal damping matrix has a higher degree of diagonality.

This example can be extended. Consider a series of systems

$$\ddot{\mathbf{q}} + \mathbf{D}_\alpha \dot{\mathbf{q}} + \mathbf{\Omega}_1 \mathbf{q} = \mathbf{g}(t) \tag{37}$$

indexed by a parameter α in such a way that $\mathbf{D}_\alpha$ is linearly interpolated between $\mathbf{D}_1$ and $\mathbf{D}_2$:

$$\mathbf{D}_\alpha = (1-\alpha)\mathbf{D}_1 + \alpha\mathbf{D}_2, \qquad 0 \le \alpha \le 1. \tag{38}$$

As α increases from 0 to 1, the diagonal entries of $\mathbf{D}_\alpha$ remain constant while the index of diagonality $\rho(\mathbf{D}_\alpha)$ increases continuously from $\rho(\mathbf{D}_1) = 2.3$ to $\rho(\mathbf{D}_2) = 18.8$. The second index of diagonality $\rho_1(\mathbf{D}_\alpha)$ exhibits a consistent behavior; it decreases continuously from $\rho_1(\mathbf{D}_1) = 0.43$ to $\rho_1(\mathbf{D}_2) = 0.055$ as α increases from 0 to 1. The relative error due to the decoupling approximation $\chi(i\omega)$ can be computed for each system in the series using equation (21). In Figure 1, the steady-state error is plotted against the index of diagonality $\rho(\mathbf{D}_\alpha)$. It can be observed that as the modal damping matrix $\mathbf{D}_\alpha$ becomes more diagonal, the error due to the decoupling approximation increases. In fact, the steady-state error due to the decoupling approximation continuously increases from 2.8% to 5.3% as the index of diagonality ρ continuously increases from 2.3 to 18.8.

If the choice of an index of diagonality is of minor importance, one should be able to obtain results consistent with Figure 1 using the second index of diagonality $\rho_1(\mathbf{D}_\alpha)$. As a measure of diagonality, $\rho_1(\mathbf{D}_\alpha)$ and $\rho(\mathbf{D}_\alpha)$ have opposite trends. For this reason, the steady-state error due to the decoupling approximation $\chi(i\omega)$ is plotted against the reciprocal of $\rho_1(\mathbf{D}_\alpha)$ in Figure 2. As expected, the error curves in Figures 1 and 2 are very similar. Both demonstrate that as diagonality of the modal damping matrix continuously increases, errors due to the decoupling approximation can continuously increase as well. In short, diagonal dominance of the damping matrix may not be sufficient to ensure small errors due to the decoupling approximation.

It should be mentioned that although only a limited set of data is presented, extensive numerical simulations have been performed by the authors to support any quantitative observations herein.

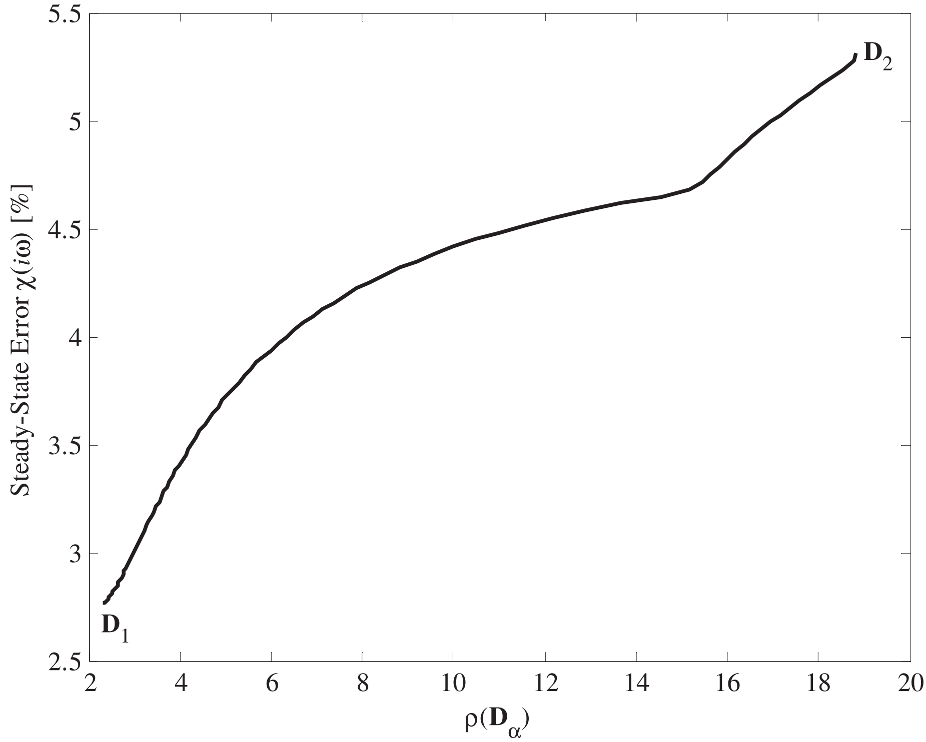

Figure 1. Steady-state error due to the decoupling approximation vs. diagonality ρ of the damping matrix.

4. QUANTITATIVE ASSESSMENT OF MODAL COUPLING

It has been observed that, contrary to widely accepted beliefs, errors due to the decoupling approximation can even increase as the modal damping matrix **D** becomes more and more diagonal. In order to explain this unexpected behavior in modal coupling, new theoretical developments will be pursued. Based upon the early work of Park et al. (1994), a theory of modal coupling will be developed using complex algebra.

In the frequency domain, the error due to the decoupling approximation is given by equation (19). Intuitively, a greater degree of coupling in the system should always be associated with a larger magnitude of the error $\mathbf{E}(i\omega)$ in the decoupling approximation. The kth component of $\mathbf{E}(i\omega)$ is a measure of the error due to the decoupling approximation in the kth mode:

$$E_k(i\omega) = \frac{-i\omega}{\omega_k^2 - \omega^2 + i\omega d_{kk}} \sum_{\substack{l=1 \\ l \neq k}}^{n} d_{kl} Q_l(i\omega). \tag{39}$$

In the above expression, the term

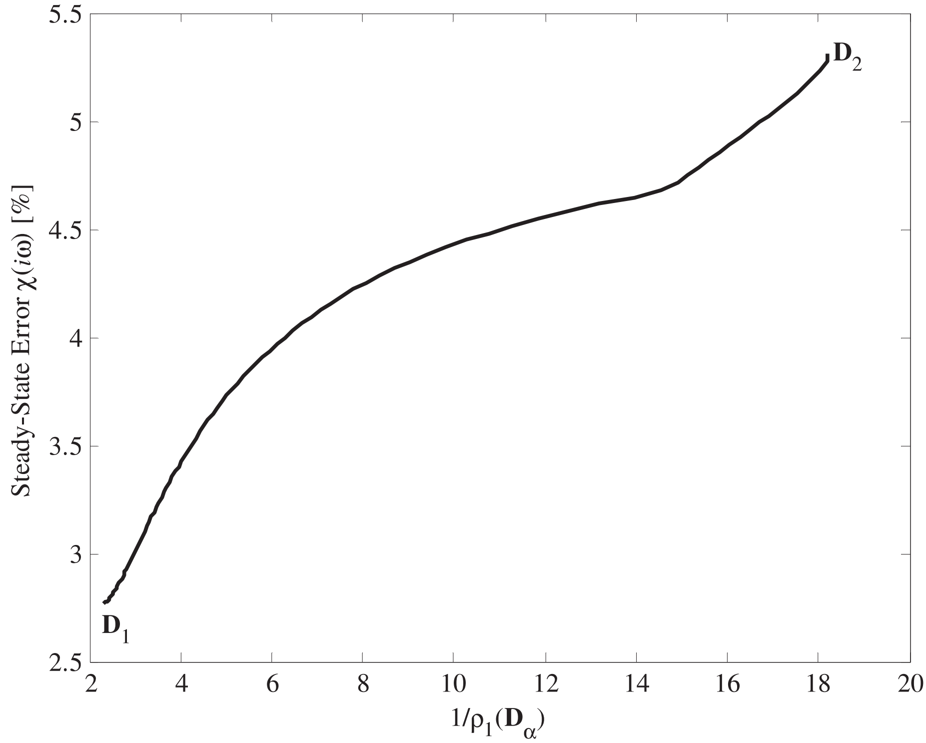

Figure 2. Steady-state error due to the decoupling approximation vs. diagonality $1/\rho_1$ of the damping matrix.

$$\varepsilon_{kl}(i\omega) = \frac{-i\omega}{\omega_k^2 - \omega^2 + i\omega d_{kk}} d_{kl} Q_l(i\omega), \quad k \neq l. \tag{40}$$

is a measure of the error due to the decoupling approximation in the kth mode that is caused by the lth mode. For this reason, $\varepsilon_{kl}(i\omega)$ may be referred to as a coefficient of coupling error in the kth mode caused by the lth mode, or coupling coefficient for short. The overall error due to the decoupling approximation in the kth mode is then given by the complex sum of $n-1$ associated coupling coefficients

$$E_k(i\omega) = \sum_{\substack{l=1 \\ l\neq k}}^{n} \varepsilon_{kl}(i\omega). \tag{41}$$

The magnitude of the error $E_k(i\omega)$ depends on the magnitude and angular orientation of the coupling coefficients $\varepsilon_{kl}(i\omega)$ in the complex plane. The magnitude of the coupling coefficients can readily be calculated as

$$|\varepsilon_{kl}(i\omega)| = \frac{\omega}{\sqrt{(\omega_k^2 - \omega^2)^2 + \omega^2 d_{kk}^2}} \left| d_{kl} Q_l(i\omega) \right|, \quad k \neq l. \tag{42}$$

Table 1. Systems used as end-states in the example.

System Parameters	System 1	System 2
Spectral matrix	$\mathbf{\Omega}_1$ in equation (29)	$\mathbf{\Omega}_1$ in equation (29)
Modal damping matrix	$\mathbf{D}_1$ in equation (30)	$\mathbf{D}_2$ in equation (33)
Harmonic excitation	$\mathbf{g}(t)$ in equation (31)	$\mathbf{g}(t)$ in equation (31)
Driving frequency	$\omega = 4.16$	$\omega = 4.16$
Relative error due to the decoupling approximation	$\chi_1(i\omega) = 2.8\%$ in equation (35)	$\chi_2(i\omega) = 5.3\%$ in equation (36)

Clearly, each coupling coefficient is large in magnitude if the off-diagonal elements d_{kl} of the modal damping matrix $\mathbf{D}$ are large compared to its diagonal elements. However, large coupling coefficients need not generate a large error due to the decoupling approximation. Depending on the angular orientation of $\varepsilon_{kl}(i\omega)$, the coupling coefficients may cancel out to produce a small overall error $E_k(i\omega)$. On the other hand, relatively small coupling coefficients can align in the complex plane to produce an unexpectedly large overall error.

Any quantitative assessment of modal coupling based solely on diagonal dominance of $\mathbf{D}$ would be inaccurate since such assessment does not take into account the alignment of coupling coefficients in the complex plane. This is the reason why small off-diagonal elements in $\mathbf{D}$ are not sufficient to neglect modal coupling.

4.1. Explanation of Observations in Previous Example

In section 3.2 an example was constructed to demonstrate that, over a finite range, it is possible for errors due to the decoupling approximation to continuously increase while $\mathbf{D}$ becomes more and more diagonal with its off-diagonal elements decreasing in magnitude continuously. To be specific, a series of four-degree-of-freedom systems are defined in Example 1 between two end-states. The two end-states, denoted by Systems 1 and 2, have the specifications listed in Table 1.

Systems 1 and 2 differ only in the modal damping matrix. It can be checked that while $\mathbf{D}_1$ and $\mathbf{D}_2$ have the same diagonal, the off-diagonal elements of $\mathbf{D}_1$ are significantly larger than those of $\mathbf{D}_2$ in magnitude. However, errors due to the decoupling approximation in System 2 are appreciably greater. This surprising result can be fully explained by an examination of the error vectors in the complex plane. The coupling coefficients $\varepsilon_{kl}(i\omega)$ are first evaluated for both Systems 1 and 2. With these coupling coefficients, the overall errors $E_k(i\omega)$ in each mode due to the decoupling approximation are plotted in Figure 3 for System 1. Similarly, the overall errors $E_k(i\omega)$ in each mode due to the decoupling approximation are plotted in Figure 4 for System 2. Although the coupling coefficients in System 1 are often larger in magnitude than those of System 2, the angular orientations of the coupling coefficients are such that they align to produce a diminished overall error vector $E_k(i\omega)$ in each of the four modes. This is the reason why errors due to the decoupling approximation in System 1 are smaller. In other words, modal coupling in System 1 is smaller.

When a series of systems is defined with Systems 1 and 2 as the end-states by equation (38), each intermediate damping matrix $\mathbf{D}_\alpha$ is more diagonally dominant than $\mathbf{D}_1$. However, the error vector $E_k(i\omega)$ for each mode of the intermediate system with $0 < \alpha < 1$ has larger

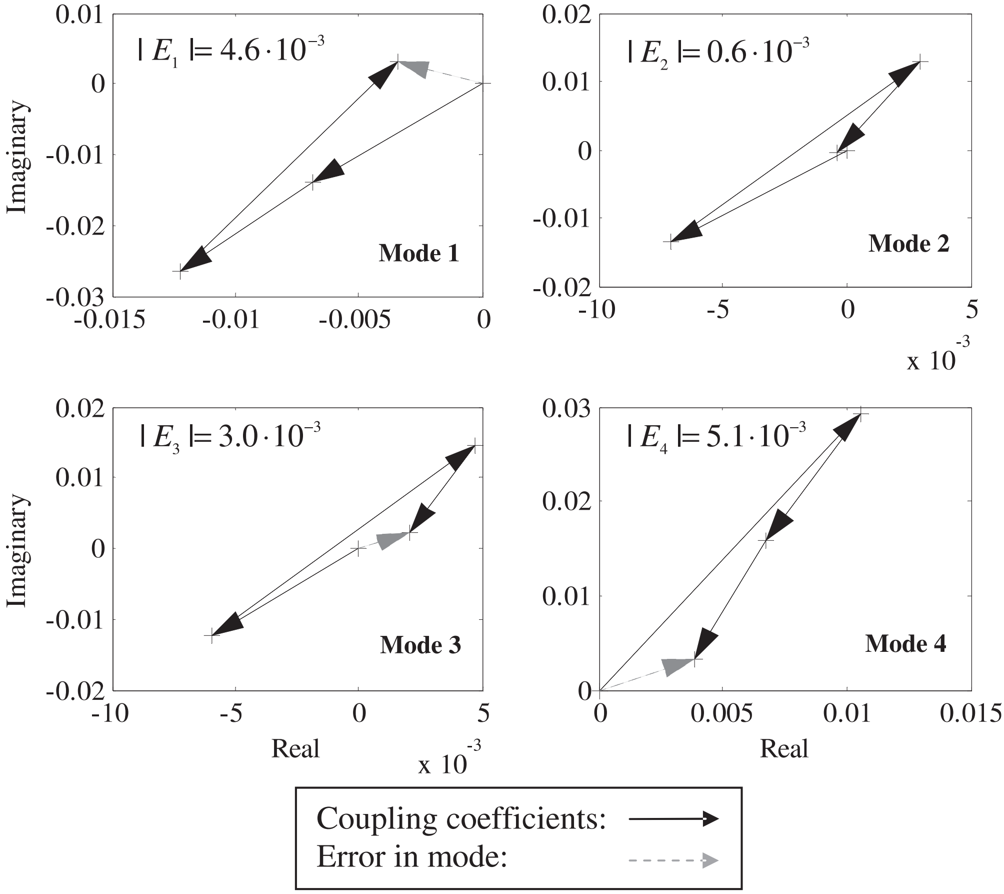

Figure 3. Errors in System 1 due to the decoupling approximation.

magnitude than that for System 1, as illustrated in Figure 1. It is now clear that the degree of diagonal dominance or the index of diagonality of **D** is not an accurate quantitative indicator of modal coupling. Complex coupling coefficients should be used to assess modal coupling quantitatively.

5. CONCLUSIONS

A common procedure in the analysis of non-classically damped linear systems is to neglect the off-diagonal elements in the modal damping matrix. This procedure is termed the decoupling approximation. The errors due to the decoupling approximation have been analyzed and a quantitative measure of the effect of modal coupling has been derived. Two numerical indices have been proposed to quantitatively measure diagonal dominance in real square matrices. It has been shown that diagonal dominance of damping would not be a sufficient condition for neglecting modal coupling. Over a finite range, it is even possible for errors due

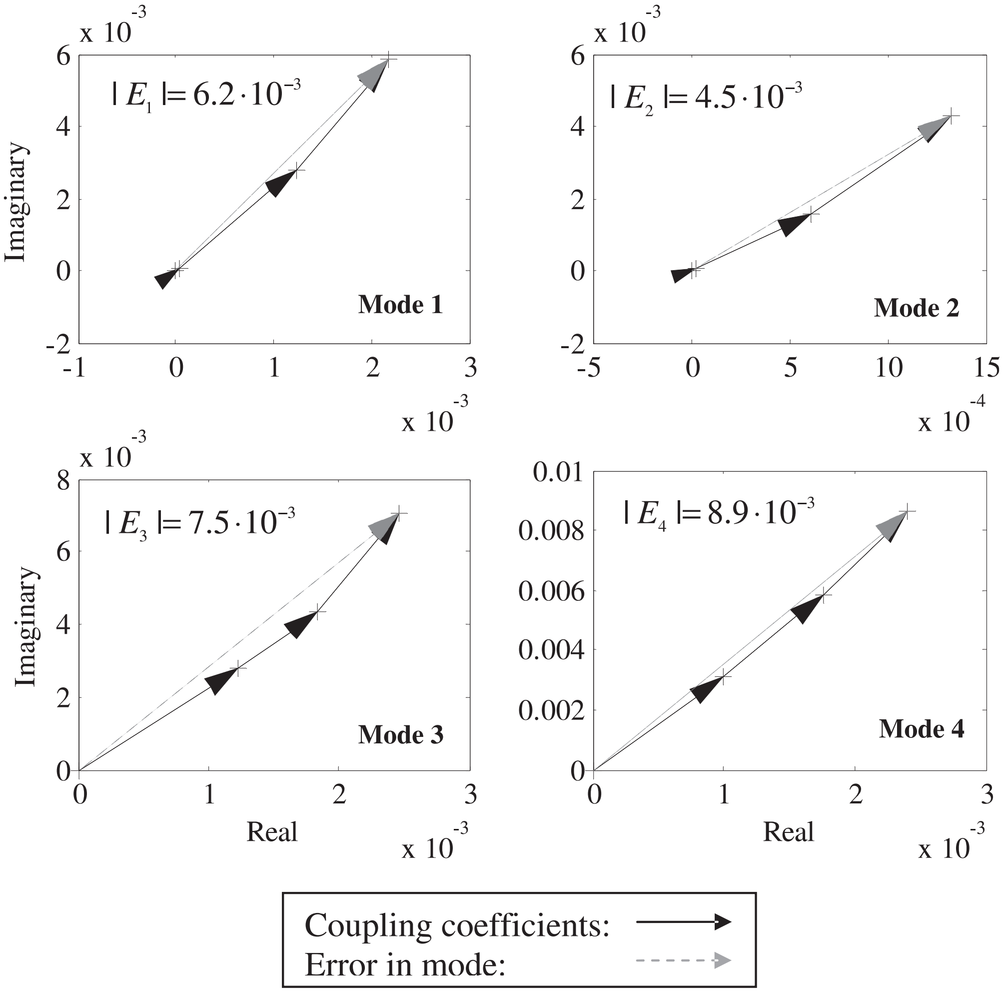

Figure 4. Errors in System 2 due to the decoupling approximation.

to the decoupling approximation to continuously increase while the modal damping matrix becomes more and more diagonal with its off-diagonal elements decreasing in magnitude continuously.

Complex algebra has been used to explain this unexpected behavior and to quantitatively assess modal coupling. Complex coupling coefficients have been defined. The error in each mode has been shown to depend on the magnitude and angular orientation of the associated coupling coefficients in the complex plane. When a modal damping matrix becomes more and more diagonal, the coupling coefficients become smaller and smaller in magnitude. However, small coupling coefficients can align in the complex plane to generate an unexpectedly large overall error due to the decoupling approximation. On the other hand, large coupling coefficients can cancel out and produce an unexpectedly small error. Although a limited set of data is presented herein, extensive calculations have been performed by the authors, and all numerical simulations have yielded qualitatively identical results on the characteristics of modal coupling.

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