Title
Optimal Hedging Under Time-Scaled Fractional Ornstein-Uhlenbeck Volatility

Author
Yan, Guangchu

Publication Date
2023

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Optimal Hedging Under Time-Scaled Fractional Ornstein-Uhlenbeck Volatility

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Guangchu Yan

Dissertation Committee:
Professor Knut Sølna, Chair
Professor Long Chen
Professor Jeff Ludwig

2023

DEDICATION

To my mother and father

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to begin by expressing my sincere gratitude to my advisor, Knut Sølna, for his exceptional guidance throughout my degree. I am truly grateful for his support and for selecting such an engaging project that has kept me highly motivated to explore further. In particular, I have learned a great deal from the four papers co-authored by him and Dr. Garnier. Without their solid, hard-core technical lemmas, it would have taken me ten times longer to achieve this.

I also want to extend my deepest appreciation to Professor Long Chen for his unwavering guidance and support since my sophomore year. Throughout my academic journey, he has been a constant source of inspiration and encouragement, and I cannot thank him enough for his mentorship. I would also like to express my appreciation to Professor Jeff Ludwig, who has been a valuable resource for me when it comes to my career questions about the financial industry. I am truly grateful for his willingness to share his expertise with me.

Following the passing of my father in 2015, I received an outpouring of support and kindness from my parents' friends, especially Hua Yan, Huanran Xia, and their family. They welcomed my mother as a member of their own family, and without their generosity, my research career may have ended prematurely. I am forever grateful for their help and support during a difficult time. I would also like to express my gratitude to my own dear friends, Yimin Zhong, Ziang Long, and Fanghui Xue. They have been exemplary research scholars and math PhD students, and their dedication to their work has been a great inspiration to me.

I also want to acknowledge the important role my family and my godfather's family has played in my life, especially my grandparents. I want to thank my mother, who has been a strong woman and my pillar of support through thick and thin. She persevered through the darkest years after my father's passing and patiently waited for me to grow up. From her, I learned how strong a woman can be, and I am incredibly proud of her. Even though my father helped me determine my path towards a PhD eight years ago, my mother deserves most of the credit for supporting me along the way.

Lastly, I would like to take a moment to reflect on my father, even though he is no longer with us. Over the years, I have come to realize that I may not have missed him every day, but I missed him so much during all the moments of ups and downs in my life. Looking back, I realize I should have talked to him more when he was alive, but I was too young to fully comprehend the fragility of life and the importance of cherishing every moment. The pain of his loss has kept me up for countless nights, but I hope that I can learn from this painful regret, to love and cherish the people in my life while they are still with me.

VITA

Guangchu Yan

EDUCATION

Doctor of Philosophy in Mathematics **2023**
University of California, Irvine *Irvine, CA*

Master of Science in Mathematics **2019**
University of California, Irvine *Irvine, CA*

Bachelor of Science in Mathematics **2016**
University of California, Irvine *Irvine, CA*

TEACHING EXPERIENCE

Teaching Assistant **2017–2023**
University of California, Irvine *Irvine, CA*

ABSTRACT OF THE DISSERTATION

Optimal Hedging Under Time-Scaled Fractional Ornstein-Uhlenbeck Volatility

By

Guangchu Yan

Doctor of Philosophy in Mathematics

University of California, Irvine, 2023

Professor Knut Sølna, Chair

In recent years, there has been growing interest in modeling volatility as a stochastic process driven by a non-Markovian process, due to empirical evidence showing that the autocorrelation function of volatility decays as a power function. This paper investigates the use of a time-scaled fractional Ornstein-Uhlenbeck process to model volatility and applies this model to derive an optimal delta hedging strategy. By incorporating non-Markovian processes into our model, we aim to provide insights into the behavior of volatility in financial markets and explore potential benefits for option pricing and risk management strategies.

Chapter 1

Introduction

In this chapter, we review previous work on option pricing. The Black-Scholes model, introduced in the 1970s, is a major breakthrough in the field, but its major weakness is the assumption of constant volatility. The local volatility model, which assumes the volatility is a deterministic function of time and underlying asset price, and the stochastic volatility model, which assumes the volatility is a stochastic process, are two major approaches to revising the Black-Scholes model. The multiscale stochastic volatility model, introduced in [8] and [9], assumes the volatility is driven by two diffusion processes, one fast-varying and one slow-varying. The authors used perturbation methods to find an asymptotic pricing formula to revise the Black-Scholes model. However, such perturbation pricing results are based on the Markov property of diffusion processes, which may need certain relaxation according to some evidence in the market. This brings the idea of using fractional Brownian motion or fractional Ornstein-Uhlenbeck process to drive the volatility, which will be introduced in section 1.3.

1.1 Black-Scholes Model

The Black-Scholes model is a seminal contribution to the theory and practice of option pricing. This model posits a semi-martingale framework for the stock price X_t, which is driven by a standard Brownian motion W_t with constant volatility σ:

$$dX_t = rX_t dt + \sigma X_t dW_t \tag{1.1}$$

Under a pricing measure $\mathbb{P}^*$ obtained via the Girsanov theorem, the option price at time t with payoff function $h(x)$ can be expressed as a discounted conditional expectation, given by

$$P(t, X_t) = e^{-r(T-t)}\mathbb{E}^*[h(X_T)|\mathcal{F}_t] \tag{1.2}$$

The Feynman-Kac formula is used to solve the partial differential equation subject to the boundary condition that $P(T, x) = h(x)$:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + rx\frac{\partial P}{\partial x} - rP = 0 \tag{1.3}$$

We note that the use of the Feymann-Kac formula requires the Markov property of such X_t as a diffusion process. This PDE can be solved by Fourier Transform and the solution for $h(x) = max(x - K, 0)$, where K is the strike price of the option, is the Black-Scholes formula for call option:

$$C(X_t, t) = X_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.4}$$

where $N(d)$ is the CDF of the standard normal distribution, r is the risk-free interest rate, $T - t$ is the time to maturity, and d_1 and d_2 are defined as:

$$d_1 = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t} \tag{1.5}$$

1.2 Multiscale Stochastic Volatility Model

The multiscale stochastic volatility model, as introduced in [8] and [9], uses asymptotic analysis with two time scales ϵ and δ, which are referred to as the fast scale and slow scale, respectively. The model defines the underlying stock price X_t and employs two diffusion processes Y_t and Z_t to drive the volatility.

$$dX_t = \mu(Y_t, Z_t)dt + f(Y_t, Z_t)X_t dW_t^{(0)} \tag{1.6}$$

$$dY_t = \frac{1}{\epsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\beta(Y_t)dW_t^{(1)} \tag{1.7}$$

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}g(Z_t)dW_t^{(2)} \tag{1.8}$$

where the volatility $f(y, z)$ is a positive function, smooth in z and such that $f^2(\cdot, z)$ is integrable with respect to the invariant distribution of Y. When ϵ is small, Y_t represents a fast-fluctuating volatility process. Such ϵ corresponds to the short mean-reversion time scale of process Y_t. When δ is small, Z_t represents a slow-fluctuating volatility process. Such δ corresponds to the long time scale $1/\delta$ of process Z_t. Under the risk-neutral pricing measure $\mathbb{P}^*$, the price evolution X_t is determined using the multi-scale Girsanov theorem:

$$dX_t = rX_t dt + f(Y_t, Z_t)X_t dW_t^{(0)} \tag{1.9}$$

$$dY_t = \left(\frac{1}{\epsilon}\alpha(Y_t) - \frac{1}{\sqrt{\epsilon}}\beta(Y_t)\Lambda_1(Y_t, Z_t)\right)dt + \frac{1}{\sqrt{\epsilon}}\beta(Y_t)dW_t^{(1)*} \tag{1.10}$$

$$dZ_t = \left(\delta c(Z_t) - \sqrt{\delta}g(Z_t)\Lambda_2(Y_t, Z_t)\right)dt + \sqrt{\delta}g(Z_t)dW_t^{(2)*} \tag{1.11}$$

where the $\mathbb{P}^*$-standard Brownian motions $\left(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*}\right)$ are correlated as follows:

$$d < W^{(0)*}, W^{(1)*} >_t = \rho_1 dt \tag{1.12}$$

$$d < W^{(0)*}, W^{(2)*} >_t = \rho_2 dt \tag{1.13}$$

$$d < W^{(1)*}, W^{(2)*} >_t = \rho_{12} dt \tag{1.14}$$

where $|\rho_1| < 1$, $|\rho_2| < 1$, $|\rho_{12}| < 1$, and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, in order to ensure positive definiteness of the covariance matrix of the three Brownian motions. By Ito's Lemma, we deduce that:

$$X_t = X_0 exp\left\{ \int_0^t \left(r - \frac{1}{2}f^2(Y_s, Z_s)\right) ds + \int_0^t f(Y_s, Z_s) dW_s^{(0)*} \right\} \tag{1.15}$$

By first-order perturbation expansion, the main result of [8] and [9] is the following: the option price $P(t, X_t, Y_t, Z_t)$ depends on variables $(r, f, \alpha, \beta, c, g, \Lambda_1, \Lambda_2)$ can be approximated by

$$\tilde{P}^{\epsilon,\delta} = P_{BS} + (T-t)\left[V_0^\delta(z)\frac{\partial}{\partial\sigma} + V_1^\delta(z)D_1\left(\frac{\partial}{\partial\sigma}\right) + V_2^\epsilon(z)D_2 + V_3^\epsilon(z)D_1D_2\right]P_{BS} \tag{1.16}$$

where T is the maturity time for European option, P_{BS} is the Black-Scholes model price (1.4), $D_k = x^k \frac{\partial^k}{\partial x^k}$ and $V_0^\delta(z)$,$V_1^\delta(z)$, $V_2^\epsilon(z)$ are groups of market parameters.

1.3 Fractional Brownian Motion and Fractional Ornstein-Uhlenbeck Process

Recent empirical evidence suggests that the auto-correlation function of volatility does not decay exponentially as expected in Markov processes, but rather follows a power decay. To address this, we introduce two non-Markov processes in this section: fractional Brownian motion and fractional Ornstein-Uhlenbeck process.

1.3.1 Fractional Brownian Motion

Definition 1.1. *A fractional Brownian motion (fBM) is a zero mean Gaussian process* B_t^H *for* $H \in (0,1)$*, called the Hurst exponent, and* σ_H *a constant with covariance property:*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{\sigma_H^2}{2}(|t^{2H}| + |s|^{2H} - |t-s|^{2H})$$

where

$$\sigma_H^2 = \frac{1}{\Gamma(2H+1)\sin(\pi H)}$$

It has an integral representation:

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} dW_s$$

where W_t *is a standard Brownian motion.*

Fractional Brownian motion has the following properties

- $B_0^H = 0$ and $\mathbb{E}[B_t^H] = 0$ for all $t \geq 0$
- B^H has homogeneous increments, i.e. $B_{t+s}^H - B_s^H$ has the same law of B_t^H for $s, t \geq 0$
- B^H is a Gaussian process and $\mathbb{E}[(B_t^H)^2] = t^{2H}$, $t \geq 0$ for all H.
- B^H has continuous trajectories.

Note that the fractional Brownian motion is not a martingale, not a Markov process, nor a stationary process. Note also that when $H = 1/2$, it is standard Brownian motion. By using

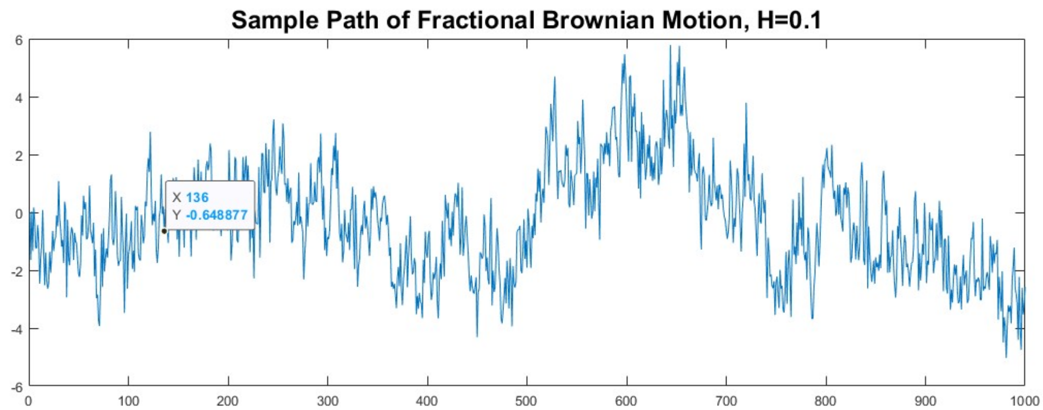

Figure 1.1: This picture shows a sample path of short-range dependent fractional Brownian motion.

the fractional Brownian motion, the roughness or the smoothness is described mathematically as short-range and long-range dependence:

Definition 1.2 (Long-range dependence)**.** *A stationary process X_t exhibits long-range dependence if the autocovariance functions $\rho(s) := cov(X_t, X_{t+s})$ satisfy:*

$$\lim_{s \to \infty} \frac{\rho(s)}{cs^{-\alpha}} = 1$$

for some constant c and $\alpha \in (0, 1)$.

Definition 1.3 (Short-range dependence)**.** *A stationary process X_t exhibits short-range dependence if the autocovariance functions $\rho(s) := cov(X_t, X_{t+s})$ satisfy:*

$$\lim_{s \to \infty} \frac{\rho(s)}{cs^{-\alpha}} = 0$$

for any constant c and $\alpha \in (0, 1)$.

For fractional Brownian motion B_t^H, the smoothness or roughness of its paths can be characterized by the Hurst exponent H:

- When $H > \frac{1}{2}$, the fractional Brownian motion exhibits long-range dependence, and its

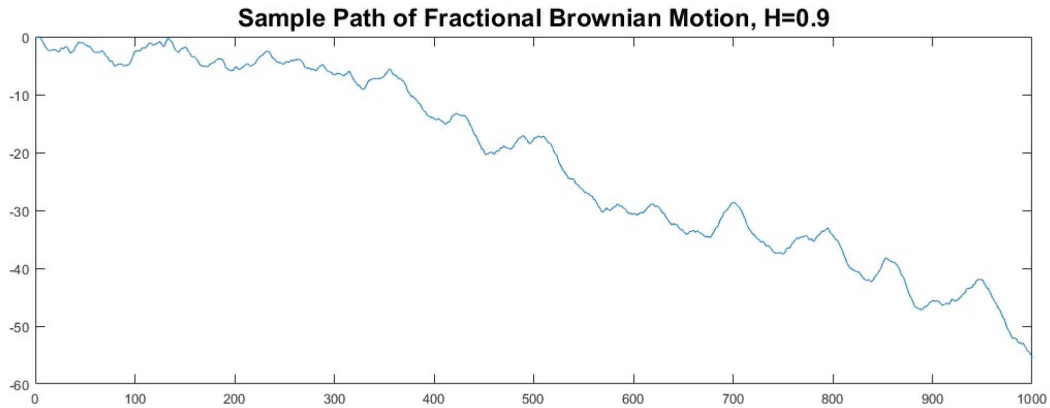

Figure 1.2: This picture shows a sample path of long-range dependent fractional Brownian motion.

paths look smoother than those of standard Brownian motion. We also call such long-range dependent process long-memory.

- When $H < \frac{1}{2}$, the fractional Brownian motion exhibits short-range dependence, and its paths look rougher than those of standard Brownian motion. We also call such short-range dependent process rough.

Note that standard Brownian motion corresponds to $H = \frac{1}{2}$. Moreover, a function $f : [0, T] \to \mathbb{R}$ is said to be Hölder continuous with exponent $0 < \gamma \leq 1$ if there exists a constant K such that:

$$|f(t) - f(s)| \leq K|t - s|^{\gamma}$$

We here note that for $H \in (0, 1)$ the fractional Brownian motion B_t^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.

The roughness of market volatility has been a topic of discussion for many years, and empirical studies have shown that the implied volatility's correlation function decays as a power function in offset. This has led to the modeling of volatility based on either long-range or short-range dependence. Several studies have found evidence of long-memory behavior in market volatility. For instance, in [1], the authors found that the dependencies of the implied volatility can be best described as a long-memory stochastic process, which is also accurate to the generalized long-run risk models. In [4], the authors modeled the price of the stock using a geometric Brownian motion driven by a fractional Ornstein Uhlenbeck process with $H > 1/2$, which corresponds to the long-range dependent case. Moreover, in [5], they computed the option price based on market data and found a match, particularly when the market is unstable. Other studies such as [3], [6], and [21] have also provided empirical evidence and discussed long-range dependence in market volatility.

However, there is also evidence to suggest that market volatility can be rough. For instance, [17] showed numerically that stochastic volatility often exhibits rough behavior, with a Hurst coefficient very close to 0 at any reasonable time scale. Similarly, [10] found an asymptotic expansion result for short-dated at-the-money volatility that contradicts non-rough volatility models, and argues that non-rough volatility can lead to arbitrage opportunities. Further discussion of rough volatility can be found in [11].

1.3.2 Fractional Ornstein-Uhlenbeck Process

We first introduce the standard Ornstein-Uhlenbeck process. The standard Ornstein-Uhlenbeck process can be defined as the following integral form:

$$Z_t = \int_{-\infty}^{t} e^{-a(t-s)} dW_t \tag{1.17}$$

for W_t a standard Brownian motion. Naturally, we define the fractional type of Ornstein-Uhlenbeck Process as the following integral form:

$$Z_t^H = \int_{-\infty}^{t} e^{-a(t-s)} dW_t^H \tag{1.18}$$

for W_t^H a fractional Brownian motion. It is a zero-mean, stationary Gaussian process with variance

$$\sigma_{ou}^2 = \mathbb{E}[(Z_t^H)^2] = \frac{1}{2}a^{-2H}\Gamma(2H+1)\sigma_H^2 \tag{1.19}$$

and covariance

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{ou}^2 \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(asx)\frac{x^{1-2H}}{1+x^2}dx \tag{1.20}$$

Here, stationarity refers to strong stationarity. We observe that the process has a zero mean and the autocovariance is independent of time, and the second moment is finite for all times, which means that this is a stationary process in the weak sense. As the process is Gaussian, it is also strictly stationary. Instead of using the non-stationary and non-martingale integral form in equation (1.18), we use the following Volterra-type integral form for the fractional Ornstein-Uhlenbeck process:

$$Z_t^H = \int_{-\infty}^t \mathcal{K}(t-s)dW_s \tag{1.21}$$

where

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+1/2)}\left[t^{H-1/2} - a\int_0^t (t-s)^{H-1/2}e^{-as}ds\right] \tag{1.22}$$

For $H \in (0, 1/2)$ the fOU process possesses short-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{ou}^2\left(1 - \frac{1}{\Gamma(2H+1)}(as)^{2H} + o((as)^{2H})\right), \qquad as \ll 1 \tag{1.23}$$

For $H \in (1/2, 1)$ the fOU process possesses long-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{ou}^2\left(\frac{1}{\Gamma(2H-1)}(as)^{2H-2} + o((as)^{2H-2})\right), \qquad as \gg 1 \tag{1.24}$$

And the kernel $\mathcal{K}$ has the following important properties.

- $\mathcal{K}$ is non-negative, and $\mathcal{K} \in L^2(0,\infty)$. But $\mathcal{K} \in L^1(0,\infty)$, only if $H \in (0, 1/2)$

- For small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+1/2)} \left(t^{H-1/2} + \mathcal{O}(t^{H+1/2})\right) \tag{1.25}$$

- For large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H-1/2)} \left(t^{H-3/2} + \mathcal{O}(t^{H-5/2})\right) \tag{1.26}$$

Chapter 2

Option Pricing by Fractional Stochastic Volatility

This section presents three corrected price formulas. For a more detailed summary, see [15]. Specifically, [12] proposes a pricing formula based on slow-varying volatility for $H \in (0, 1)$. The second paper, [13], assumes a fast-varying volatility with $H \in (0, 1/2)$, while [14] assumes $H \in (1/2, 1)$ as a complement.

For a generalized model that encompasses both slow and fast-varying volatility cases, the pricing is described by the following stochastic differential equation

$$dX_t = \sigma_t X_t dW_t^* \tag{2.1}$$

$$\sigma_t = F(Z_t) \tag{2.2}$$

where W_t^* is a process driven by two independent and standard Brownian motions.

$$W_t^* = \rho W_t + \sqrt{1 - \rho^2} B_t \tag{2.3}$$

F is assumed to be 1-1, smooth, positive valued with $F(0) = 0$ and $F'(0) = 1$. In option pricing on slow-varying volatility, we define

$$Z_t = Z_t^\delta = \int_{-\infty}^t \mathcal{K}^\delta(t-s)dW_s, \qquad \mathcal{K}^\delta(t) = \delta^{1/2}\mathcal{K}(\delta t) \tag{2.4}$$

We assume F is a smooth, one-to-one, positive-valued function with positive valued derivative such that $F(0) = 0$ and $F'(0) = 1$. In the fast-varying case, we define:

$$Z_t = Z_t^\epsilon = \int_{-\infty}^t \mathcal{K}^\epsilon(t-s)dW_s, \qquad \mathcal{K}^\epsilon(t) = \frac{1}{\sqrt{\epsilon}}\mathcal{K}(t/\epsilon) \tag{2.5}$$

2.1 Fast Varying Volatility Pricing $H < 1/2$

This is the case when we accepts the volatility is a function of fast-varying rough fOU process defined by (2.5). We note in advance that the following correction term has ϵ with fixed power $\frac{1}{2}$. The following proposition is the main result of paper [13]:

Proposition 2.1. *We have*

$$\mathbb{E}[h(X_T)|\mathcal{F}_t] = M_t = Q_t(X_t) + o(\sqrt{\epsilon}) \tag{2.6}$$

where

$$Q_t^\epsilon(x) = Q_t^{(0)}(x) + \epsilon^{1/2}Q_t^{(1)}(x) \tag{2.7}$$

$Q_t^{(0)}(x)$ *is deterministic and given by the Black–Scholes formula with constant volatility* $\bar{\sigma}$

$$\mathcal{L}_{BS}(\bar{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x) \tag{2.8}$$

with

$$\bar{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_{ou}z)^2 p(z)dz \tag{2.9}$$

p(z) is the standard normal distribution pdf. $Q_t^{(1)}(x)$ *is deterministic*

$$Q_t^{(1)}(x) = (T-t)\bar{D}\left(x\partial_x(x^2\partial_x^2)Q_t^{(0)}(x)\right) \tag{2.10}$$

with coefficient

$$\bar{D} = \sigma_{ou}\int_0^\infty \left[\iint_{\mathbb{R}^2} F(\sigma_{ou})(FF')(\sigma_{ou}z')p_{\mathcal{C}_Z(s)}(z,z')dzdz'\right]\mathcal{K}(s)ds \tag{2.11}$$

$p_{\mathcal{C}_Z(s)}(z,z')$ *is the pdf of the bi-variate normal distribution with mean zero and covariance matrix* $\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}$ *and* $C_Z(s)$ *is defined as the following:*

$$C_Z(s) = \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(sx)\frac{x^{1-2H}}{1+x^2}dx \tag{2.12}$$

2.2 Fast Varying Volatility Pricing $H > 1/2$

This is the case when we accepts (2.5) and $H > 1/2$. The following proposition is the main result of paper [14]:

Proposition 2.2. *We have*

$$\mathbb{E}[h(X_T)|\mathcal{F}_t] = M_t = Q_t(X_t) + o(\epsilon^{1-H}) \tag{2.13}$$

where

$$Q_t^\epsilon(x) = Q_t^{(0)}(x) + (x^2\partial_x^2)Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x) \tag{2.14}$$

$Q_t^{(0)}(x)$ *is deterministic and given by the Black–Scholes formula with constant volatility* $\bar{\sigma}$

$$\mathcal{L}_{BS}(\bar{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x) \tag{2.15}$$

with

$$\bar{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_{ou} z)^2 p(z) dz, \qquad \tilde{\sigma} = \langle F \rangle = \int_{\mathbb{R}} F(\sigma_{ou} z) p(z) dz \tag{2.16}$$

p(z) is the standard normal distribution pdf. The random component ϕ_t^ϵ is given by

$$\phi_t^\epsilon = \mathbb{E}\left[\frac{1}{2}\int_t^T (\sigma_s^\epsilon)^2 - \bar{\sigma}^2 ds \middle| \mathcal{F}_t\right] \tag{2.17}$$

And $Q_t^{(1)}(x)$ is deterministic

$$Q_t^{(1)}(x) = \left(x\partial_x(x^2\partial_x^2 Q_t^{(0)}(x))\right) D_t \tag{2.18}$$

where

$$D_t = \bar{D}(T-t)^{H+1/2}, \qquad \bar{D} = \frac{\langle FF' \rangle}{\Gamma(H+3/2)} = \frac{1}{\Gamma(H+3/2)}\sigma_{ou}\int_{\mathbb{R}} FF'(\sigma_{ou} z) p(z) dz \tag{2.19}$$

We also note that $\theta_t = \frac{\langle FF' \rangle}{\Gamma(H+1/2)}(T-t)^{H-1/2}$, and such variable is introduced in [14]

2.3 Slow Varying Volatility Pricing

In [12], there are two pricing propositions based on slightly different volatility definitions. Proposition 3.1 has a leading time scale of order 1, while Proposition 6.1 includes a leading order correction of δ^H, resulting in improved pricing. The stochastic volatility of Proposition 6.1 is also more consistent with fast-varying volatility. We now introduce the following proposition:

Proposition 2.3. *When δ is small, let $\sigma_0 = F(Z_0^\delta)$, $p_0 = F'(Z_0^\delta)$, then*

$$\mathbb{E}[h(X_T)|\mathcal{F}_t] = M_t = Q_t(X_t) + \mathcal{O}(\delta^{2H}) \tag{2.20}$$

where

$$Q_t(x) = Q_t^{(0)}(x) + \sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x) + \delta^H \rho p_0 Q_t^{(1)}(x) \tag{2.21}$$

$Q_t^{(0)}(x)$ *is deterministic and given by the Black–Scholes formula with constant volatility* σ_0

$$\mathcal{L}_{BS}(\sigma_0) Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x) \tag{2.22}$$

ϕ_t *is a random term depends on time* t*,*

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T Z_s^\delta - Z_0^\delta ds \middle| \mathcal{F}_t\right] \tag{2.23}$$

And $Q_t^{(1)}(x)$ *is deterministic*

$$Q_t^{(1)}(x) = \sigma_0^2 x \partial_x (x^2 \partial_x^2) Q_t^{(0)}(x) D_{t,T}, \qquad D_{t,T} = \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} \tag{2.24}$$

We also note that $\theta_{t,T} = (T-t)^{H+1/2}/\Gamma(H+3/2)$*, and such variable will be introduced in the proof in Appendix.*

The proof of the above proposition is omitted by the authors. However, a thorough proof is provided in Appendix B. It is important to note that in the case of slow-varying volatility, $D_{t,T}$ defines $Q_t^{(1)}(x)$, which is dependent on time and the Hurst exponent.

There are four major differences among these three pricing formulas,

- In the case of fast-varying volatility, we can obtain perturbation results by expanding with respect to the root mean square of the volatility process, which is averaged with respect to the invariant distribution. On the other hand, for the slow-varying volatility case, we can expand with respect to the initial volatility at time 0, denoted by σ_0.
- In both proposition 1 and proposition 3, the corrected pricing depends on the random

terms ϕ_t^δ or ϕ_t^ϵ, while the whole asymptotic pricing is deterministic in proposition 2. For market data calibration purpose, the existence of such random term requires updating as time changes. For fast-varying rough volatility, as the coefficient of $Q_t^{(1)}$ does not depend on t, such timely-vary update is not necessary.

- In proposition 2, the coefficient $\bar{D}$ does not depend on Hurst exponent while in the remaining cases, their analogous coefficients do depend on it. Such power difference is thoroughly discussed in section 6 [13].

- The leading order correction term in proposition 2 has a fixed $\sqrt{\epsilon}$ order, while in the other two cases, the leading order is strictly less than $1/2$ and it depends on Hurst exponent. We argue that the convergence property of kernel function leads such compromise in proposition 2 and further discussion can be found in [13] section 6.

Chapter 3

Optimal Delta Hedging

3.1 Introduction to Delta Hedging

Consider two portfolios: portfolio $P(t, X_t)$ which is the one call option and portfolio $V_t = \delta_t X_t + b_t$, where $\delta_t X_t$ denotes the amount of the underlyings and b_t is the amount deposit in the bank. The replication of these two portfolio gives

$$P(t, X_t) = \delta_t X_t + b_t \tag{3.1}$$

Then we define the total hedging cost from 0 to t:

$$E_t = P_0 + \int_0^t (dP_t - \delta_t dX_t) = P(t, X_t) - \int_0^t \delta_t dX_t \tag{3.2}$$

The above term we choose such that δ_t is called Delta in finance Greeks. Besides, such continuous δ_t we choose to hedge is called a dynamic (DA) hedging strategy. Note that, in the stochastic volatility modeling, the cost is not zero, since suppose the function $Q^{(0)}(t, X_t, \sigma)$ is the solution to the corresponding Black-Scholes PDE, and Z_s is the volatility process, then

$$dE_t = dP_t - a_t dX_t = \left(\partial_t + \frac{1}{2}\sigma^2(Z_s)(x^2\partial_x^2)\right) Q_t^{(0)}(X_t)dt + \partial_x Q_t^{(0)}(X_t)dX_t - \delta_t dX_t \tag{3.3}$$

$$= \frac{1}{2}\left(\sigma^2(Z_s) - \sigma^2\right)\left(x^2\partial_x^2\right) Q_s^{(0)}(X_s)dt \tag{3.4}$$

This is non-zero when $\sigma(Z_s) \neq \sigma$. We remark that this cost is related to Vega: option price derivative with respect to volatility σ: $\partial_\sigma Q_t^{(0)}(x)$. The identity $(T-t)\sigma(x^2\partial_x^2)Q_t^{(0)}(x) = \partial_\sigma Q_t^{(0)}(x)$ gives

$$dE_t = \frac{1}{2}\left(\sigma^2(Z_s) - \sigma^2\right)\frac{\partial_\sigma Q_t^{(0)}(x)}{\sigma(T-t)}dt \tag{3.5}$$

In this paper, we aim to find the optimal delta hedging strategy that minimizes the variance of E_t.

The history of minimum variance hedging dates back to 1991, when [7] used a stochastic model with two diffusion stochastic differential equations to find the optimal strategy using the orthogonal projection method. However, this setting lacked stochastic volatility. In a follow-up paper [23], a similar approach was used to find a general contingent claim, and the hedging error was measured discretely in [18], where a non-Markovian model was employed, and the weak convergence approach was used to find the asymptotic distribution of the cost function. Hedging error control can also be viewed as an optimization problem related to the Hamilton-Jacobian-Bell equation, with the related PDE obtained in [19] for some special cases. Multiple assets delta hedging strategies are discussed in [22], known as the Delta-Sigma hedging, while the recent work in [2] analyzes the same hedging error for asset prices driven by n-dimensional Brownian motions and includes simulation results.

In this paper, we first compare the following two delta hedging strategies that were introduced in [16]:

- Hull-White Scheme (HW):

$$\delta_t^{HW} = \partial_x Q(t, x; \sigma^*), \qquad P_t^{HW} = Q(t, x; \sigma^*) \tag{3.6}$$

  $Q(t, x; \sigma)$ is the perturbation result for some fixed historical volatility σ^* that varies in different situations. In fast-varying volatility case, we choose $\sigma^* = \bar{\sigma}$, the effective volatility and in slow-varying volatility case, we choose $\sigma^* = \sigma_0$, the initial volatility. HW scheme uses corrected pricing to model $P^{HW}(t, X_t)$ and take the partial derivative to each component of the corrected pricing.

- Black-Scholes Scheme (BS):

$$\delta_t^{BS} = \partial_x Q^{(0)}(t, x; \sigma)|_{\sigma=\sigma(t,x)}, \qquad P_t^{BS} = Q(t, x; \sigma) \tag{3.7}$$

  with the implied volatility $\sigma(t, x)$ solving

$$Q(t, x; \sigma) = Q^{(0)}(t, x; \sigma(t, x)) \tag{3.8}$$

  The Black-Scholes strategy uses the corrected price as HW scheme. Instead, the delta uses the implied volatility computed by using the corrected price. We denote that the main result in [16] is that when $H < 1/2$, for fast varying volatility model, the optimal strategy among all delta hedging strategies is the BS scheme.

Based on our computation of the BS scheme, we have found that in order to achieve the minimum variance of the hedging cost, it is sometimes necessary to add a correction term to the BS scheme delta, depending on the assumptions made about the volatility. This corrected version is referred to as the Corrected Black-Scholes scheme (C):

- For fast-varying volatility where H ranging on $(1/2, 1)$, the Corrected BS scheme is defined as the following:

$$\delta^C(t,x) = \delta^{BS}(t,x) + \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2}\frac{H-1/2}{H+1/2}(x\partial_x^2)Q_t^{(0)}(X_t)\theta_t \tag{3.9}$$

for $Q_t^{(0)}(x)$, $\bar{\sigma}$, $\tilde{\sigma}$ and θ_t defined in Proposition 2.2.

- For slow-varying volatility where H ranging on $(0,1)$, the Corrected BS scheme is defined as the following:

$$\delta^C(t,x) = \delta^{BS}(t,x) + \delta^H \rho p_0 \frac{H+1/2}{H+3/2}(x\partial_x^2)Q_t^{(0)}(X_t)\theta_{t,T} \tag{3.10}$$

for $Q_t^{(0)}(x)$, $\theta_{t,T}$ and p_0 defined in Proposition 2.3.

The main results in this chapter are the followings:

- For the slow-varying volatility model, we find that

$$Var[E_t^{BS}|\mathcal{F}_0] \leq Var[E_t^{HW}|\mathcal{F}_0] \tag{3.11}$$

and the degree to which the BS scheme outperforms the HW scheme depends on the Hurst exponent H and the correlation ρ. The optimal strategy among all is a corrected BS scheme (5), which we demonstrate through numerical illustration in Section 7.

- For the fast-varying long-memory volatility model, we also find that

$$Var[E_t^{BS}|\mathcal{F}_0] \leq Var[E_t^{HW}|\mathcal{F}_0] \tag{3.12}$$

and the degree to which the BS scheme outperforms the HW scheme depends on not only the Hurst exponent H and the correlation ρ, but also the ratio between $\bar{\sigma}$ and $\tilde{\sigma}$. The optimal strategy among all is also a corrected Black-Scholes scheme, which we again confirm through numerical illustration in Section 7.

Our main finding in this paper is different from that of [16], where the optimal scheme is the Black-Scholes scheme without any correction, but it coincides when $H = 1/2$ in equation (6). This is because the power of the time-to-maturity adjustment term $Q_t^{(1)}(x)$ in the fast-varying rough pricing formula does not depend on the Hurst exponent. For a more detailed comparison of the pricing formulas, see [15].

3.2 Fast-varying Volatility Hedging $H < 1/2$

In this section, we summarize the main results of [16] where the authors addressed the problem of optimal hedging in the presence of fast-varying rough fractional Ornstein-Uhlenbeck volatility. They showed that the BS scheme (3.7)-(3.8) has the minimal variance conditioned on the filtration $\mathcal{F}_0$ among all possible DA schemes . They also provided asymptotic computations of HW and BS schemes. The major results of their paper are listed below for completion of this subject. We omit their proofs.

3.2.1 HW Scheme (HW)

In this case, for (3.2), we take

$$P_t^{HW} = Q_t(X_t) = Q_t^{(0)}(x) + \epsilon^{1/2}\rho Q_t^{(1)}(x) \tag{3.13}$$

$$\delta^{HW}(t, X_t) = \partial_x Q(t,x)\big|_{x=X_t} = \partial_x \left(Q_t^{(0)}(x) + \epsilon^{1/2}\rho Q_t^{(1)}(x)\right)\big|_{x=X_t} \tag{3.14}$$

All terms above, we refer to Proposition 2.1. Then we take the financing cost of the portfolio:

$$E_t^{HW} = P_t^{HW} - \int_0^t \delta^{HW}(s, X_s) dX_s \tag{3.15}$$

The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.1. *The mean extra hedging cost beyond the corrected price is zero:*

$$\lim_{\epsilon \longrightarrow 0} \mathbb{E}\left[\left(\epsilon^{-1/2}\mathbb{E}[E_t^{HW} - E_0^{HW}|\mathcal{F}_0]\right)^2\right]^{1/2} = 0 \tag{3.16}$$

with $E_0^{HW} = P(0, X_0)$. *The variance of the cost fluctuations satisfies*

$$\lim_{\epsilon \longrightarrow 0} \mathbb{E}\left[\left|\epsilon^{-1} Var[E_t^{HW} - E_0^{HW}|\mathcal{F}_0] - \mathcal{V}_t^{(3)}(X_0)\right|\right] = 0 \tag{3.17}$$

where

$$\mathcal{V}_t^{(3)}(x_0) = \bar{\Gamma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left((x^2\partial_x^2) Q_s^{(0)}\left(x_0 e^{\bar{\sigma}\sqrt{s}z - \bar{\sigma}^2 s/2}\right)\right) \tag{3.18}$$

Here $p(z)$ *is the pdf of the standard normal distribution,* $\bar{\Gamma}$ *is the parameter*

$$\bar{\Gamma}^2 = 2\sigma_Z^2 \int_0^\infty \int_s^\infty \left[\iint_{\mathbb{R}^2} FF'(\sigma_Z z) FF'(\sigma_Z z') p_{\mathcal{C}_{\mathcal{K}}(s,s')}(z, z') dz dz'\right] \mathcal{K}(s)\mathcal{K}(s') ds' ds \tag{3.19}$$

and $p_{\mathcal{C}}$ *is the pdf of the bivariate normal distribution with covariance matrix defined in Proposition 2.1 and*

$$\mathcal{C}_{\mathcal{K}}(s, s') = \int_0^\infty \mathcal{K}(s + v)\mathcal{K}(s' + v) dv \tag{3.20}$$

3.2.2 BS Scheme (BS)

In this case, the rough fast-varying volatility BS scheme is defined as the following.

$$\delta^{BS}(t,x) = \partial_x Q^{(0)}(t,x;\sigma)|_{\sigma=\sigma(t,x)} \tag{3.21}$$

with implied volatility $\sigma(t,x)$ solving

$$Q(t,x) = Q^{(0)}(t,x;\sigma(t,x)) \tag{3.22}$$

The implied volatility $\sigma(t,x)$ is such that

$$Q^{(0)}(t,x;\sigma(t,x)) = Q(t,x) = Q_t^{(0)}(x) + \epsilon^{1/2}\rho Q_t^{(1)}(x) \tag{3.23}$$

Then we take the financing cost of the portfolio:

$$E_t^{BS} = P_t^{BS} - \int_0^t \delta^{BS}(s,X_s)dX_s \tag{3.24}$$

The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.2. *The mean extra hedging cost beyond the corrected price is zero:*

$$\lim_{\epsilon\longrightarrow 0} \mathbb{E}\left[\left(\epsilon^{-1/2}\mathbb{E}[E_t^{BS} - E_0^{BS}|\mathcal{F}_0]\right)^2\right]^{1/2} = 0 \tag{3.25}$$

with $E_0^{BS} = P(0,X_0)$. *The variance of the cost fluctuations satisfies*

$$\lim_{\epsilon\longrightarrow 0} \mathbb{E}\left[\left|\epsilon^{-1}Var[E_t^{BS} - E_0^{BS}|\mathcal{F}_0] - \tilde{\mathcal{V}}_t^{(1)}(X_0) - 2\tilde{\mathcal{V}}_t^{(2)}(X_0) - \tilde{\mathcal{V}}_t^{(3)}(X_0)\right|\right] = 0 \tag{3.26}$$

where

$$\tilde{\mathcal{V}}_t^{(1)}(x_0) = \rho^2 \bar{D}^2 \bar{\sigma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left(\tilde{\mathcal{H}}_s(x_0 e^{\bar{\sigma}\sqrt{s}z - \bar{\sigma}^2 s/2})\right)^2, \tag{3.27}$$

$$\tilde{\mathcal{V}}_t^{(2)}(x_0) = \rho^2 \bar{D}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \tilde{\mathcal{H}}_s(x_0 e^{\bar{\sigma}\sqrt{s}z - \bar{\sigma}^2 s/2}) \left((x^2\partial_x^2) Q_s^{(0)}(x_0 e^{\bar{\sigma}\sqrt{s}z - \bar{\sigma}^2 s/2})\right), \tag{3.28}$$

$$\tilde{\mathcal{V}}_t^{(3)}(x_0) = \bar{\Gamma}^2 \int_{\mathbb{R}} dz p(z) \int_0^t ds \left((x^2\partial_x^2) Q_s^{(0)} \left(x_0 e^{\bar{\sigma}\sqrt{s}z - \bar{\sigma}^2 s/2}\right)\right)^2 \tag{3.29}$$

where $\bar{\Gamma}$ *is defined by* (3.19) *and* $\tilde{\mathcal{H}}_s$ *is defined by:*

$$\tilde{\mathcal{H}}(x) = \frac{1}{\bar{D}} \left((x\partial_x) - \left(\frac{x\partial_x \partial_\sigma Q^{(0)}(s,x;\bar{\sigma})}{\partial_\sigma Q^{(0)}(s,x;\bar{\sigma})}\right)\right) Q^{(1)}(s,x;\bar{\sigma}) \tag{3.30}$$

3.2.3 Optimal Delta Hedging on Rough Fast-varying Volatility Mode

The following proposition proves that the the BS scheme is the optimal hedging strategy among all possible DA hedging scheme.

Proposition 3.3. *For any smooth and bounded* $a_t = \mathcal{A}(t, X_t)$*, as the delta hedging strategy indicating the number of underlyings to hedge, the following cost function:*

$$E_t^* = P(t, X_t) - \int_0^t a_s dX_s \tag{3.31}$$

has minimum variance with leading order $\epsilon^{1/2}$*:*

$$E_0^* = P(0, X_0), \qquad Var[E_t^{BS}|\mathcal{F}_0] \leq Var[E_t^*|\mathcal{F}_0] \tag{3.32}$$

for any $t \in [0, T]$

It is worth noting that in this section, the pricing formula (3.13) and the asymptotic variance terms (3.27)-(3.29) are independent of the Hurst exponent H, despite the assumption of short-range dependent volatility. However, in the cases of fast-fluctuating long-memory and slow-varying volatility, the Hurst exponent plays a significant role. The next two sections will highlight the importance of the Hurst exponent in these cases.

3.3 Fast-varying Volatility Hedging $H > 1/2$

In this section, we analyze the delta hedging problem for European options under a long-memory fast-varying volatility model. We derive the asymptotic variances for the HW, BS, and corrected BS delta-hedging schemes. Our main result is to prove that for all delta-hedging schemes under fast-varying long-memory volatility, all cost processes are martingales and the corrected BS scheme achieves the minimum variance.

3.3.1 HW Scheme (HW)

In this case, for (3.2), we take,

$$P_t^{HW} = Q_t(X_t) = Q_t^{(0)}(x) + (x^2\partial_x^2)Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x) \tag{3.33}$$

$$\delta^{HW}(t, X_t) = \partial_x Q(t,x)\big|_{x=X_t} = \partial_x \left(Q_t^{(0)}(x) + (x^2\partial_x^2)Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x) \right)\big|_{x=X_t} \tag{3.34}$$

then we define that

$$E_t^{HW} = P_t^{HW} - \int_0^t \delta^{HW}(s, X_s)dX_s \tag{3.35}$$

In other words, we use the historical volatility $\bar{\sigma}$ and the corrected formula. The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.4. *The cost of the HW hedging strategy satisfies:*

$$\lim_{\epsilon \to 0} \epsilon^{H-1} \mathbb{E}\left[\left(\mathbb{E}[E_t^{HW} - E_0^{HW} | \mathcal{F}_0] \right)^2 \right]^{1/2} = 0 \tag{3.36}$$

where $E_0^{HW} = P(0, X_0)$. The asymptotic variance of the cost fluctuations satisfies:

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\left| Var[\epsilon^{H-1}(E_t^{HW} - E_0^{HW}) | \mathcal{F}_0] - \mathcal{V}_t^{(3)} \right| \right] = 0 \tag{3.37}$$

where

$$\mathcal{V}_t^{(3)} = \frac{\langle FF' \rangle^2}{\Gamma(H+1/2)^2} \int_{\mathbb{R}} \int_0^t \left((x^2 \partial_x^2) Q_s^{(0)} (x_0 e^{\sigma_0 \sqrt{s} z - \sigma_0 s/2}) \right)^2 (T-t)^{2H-1} ds p(z) dz \tag{3.38}$$

where $p(z)$ is the pdf of standard normal distribution. We note that such $\mathcal{V}_t^{(3)}$ is an analogy to the variance approximation of the HW scheme in [16]. The major difference is that the power of time-to-maturity depends on the Hurst exponent here. For a numerical illustration, we refer the Figure C.1

Proof. By (43) in [14], we know that, up to leading order:

$$d[Q_t^{(0)}(x) + (x^2 \partial_x^2) Q_t^{(0)}(x) \phi_t^\epsilon + \epsilon^{1-H} \tilde{\sigma} \rho Q_t^{(1)}(x)] \tag{3.39}$$

$$= dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} + \epsilon^{1-H} \tilde{\sigma} \rho dN_t^{(2)} \tag{3.40}$$

where $dR_t^{(j)}$ are higher order terms and the martingale terms are defined as the following:

$$dN_t^{(0)} = (x\partial_x) Q_t^{(0)}(X_t) \sigma_t^\epsilon dW_t^* \tag{3.41}$$

$$dN_t^{(1)} = (x^2 \partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\epsilon + (x\partial_x(x^2 \partial_x^2)) Q_t^{(0)}(X_t) \sigma_t^\epsilon \phi_t^\epsilon dW_t^* \tag{3.42}$$

$$dN_t^{(2)} = (x\partial_x) Q_t^{(1)}(X_t) \sigma_t^\epsilon dW_t^* \tag{3.43}$$

By Lemma A.6, we get:

$$dV_t^{HW} - a_s^{HW} dX_s = (x^2\partial_x^2)Q_t^{(0)}(X_t)d\psi_t^\epsilon \tag{3.44}$$

By defining:

$$\hat{N}_t^\epsilon = \int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\epsilon \tag{3.45}$$

we get,

$$E_t^{HW} = E_0^{HW} + \hat{N}_t^\epsilon \tag{3.46}$$

Thus, by Lemma B.2 and Lemma B.5 in [14], and Lemma A.11 in [16] we have the following leading order computation

$$\begin{aligned}
\epsilon^{-2(1-H)}Var[\hat{N}_t|\mathcal{F}_0] &= \epsilon^{-2(1-H)}Var\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\epsilon|\mathcal{F}_0\right] && (3.47)\\
&= \epsilon^{-2(1-H)}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_t^\epsilon)^2 \, ds|\mathcal{F}_0\right] && (3.48)\\
&= \epsilon^{-2(1-H)}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 \left(\epsilon^{1-H}\theta_t + \tilde{\theta}_t^\epsilon\right)^2 ds|\mathcal{F}_0\right] && (3.49)\\
&= \mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_t)^2 \, ds\right] && (3.50)
\end{aligned}$$

where θ_t is defined in Proposition 2.2. Then by Lemma A.3, we get up to leading order,

$$\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_t)^2 \, ds\right] = \frac{\langle FF'\rangle^2}{\Gamma(H+1/2)^2}\int_{\mathbb{R}}\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(x_0 e^{\sigma_0\sqrt{s}z-\sigma_0 s/2})\right)^2 (T-t)^{2H-1}dsp(z)dz \tag{3.51}$$

□

3.3.2 BS Scheme (BS)

The long-memory fast-varying volatility BS scheme is defined as the following,

$$\delta^{BS}(t,x) = \partial_x Q^{(0)}(t,x;\sigma)|_{\sigma=\sigma(t,x)} \tag{3.52}$$

with implied volatility $\sigma(t,x)$ solving

$$Q(t,x) = Q^{(0)}(t,x;\sigma(t,x)) \tag{3.53}$$

The implied volatility $\sigma(t,x)$ is such that

$$Q^{(0)}(t,x;\sigma(t,x)) = Q(t,x) = Q_t^{(0)}(x) + (x^2\partial_x^2)Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x) \tag{3.54}$$

Then we have the following computation of its variance

$$E_t^{BS} = P_t^{BS} - \int_0^t \delta^{BS}(s,X_s)dX_s \tag{3.55}$$

The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.5. *The cost of the BS hedging strategy satisfies:*

$$\lim_{\epsilon\longrightarrow 0} \epsilon^{H-1}\mathbb{E}\left[\left(\mathbb{E}[E_t^{BS} - E_0^{BS}|\mathcal{F}_0]\right)^2\right]^{1/2} = 0 \tag{3.56}$$

where $E_0^{HW} = P(0,X_0)$*. The asymptotic variance of the cost fluctuations satisfies:*

$$\lim_{\epsilon\longrightarrow 0} \mathbb{E}\left[\left|Var[\epsilon^{H-1}(E_t^{BS} - E_0^{BS})|\mathcal{F}_0] - \left(1-\bar{\rho}^2\frac{2H}{(H+1/2)^2}\right)\mathcal{V}_t^{(3)}\right|\right] = 0 \tag{3.57}$$

where $\bar{\rho} = \rho\frac{\tilde{\sigma}}{\bar{\sigma}}$. It further implies that with leading order:

$$Var[E_t^{BS}|\mathcal{F}_0] \leq Var[E_t^{HW}|\mathcal{F}_0] \tag{3.58}$$

Proof. Consider the asymptotic expansion of $Q^{(0)}(t,x;\sigma(t,x))$:

$$Q^{(0)}(t,x;\sigma(t,x)) = Q^{(0)}(t,x;\sigma_0) + \left(\partial_\sigma Q^{(0)}(t,x;\sigma)\right)|_{\sigma=\sigma_0}(\sigma(t,x)-\sigma_0) + \mathcal{O}(\delta^{2H}) \tag{3.59}$$

We know $\sigma(t,x)-\sigma_0$ from:

$$\sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x,\sigma_0) = \left(\partial_\sigma Q^{(0)}(t,x;\sigma)\right)|_{\sigma=\sigma_0}(\sigma(t,x)-\sigma_0) + \mathcal{O}(\delta^{2H}) \tag{3.60}$$

which implies that

$$\sigma(t,x)-\sigma_0 = \frac{(x^2\partial_x^2)\,Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x)}{(\partial_\sigma Q^{(0)}(t,x;\sigma))\,|_{\sigma=\sigma_0}} + \mathcal{O}(\delta^{2H}) \tag{3.61}$$

Then with leading order

$$\delta^{BS}(t,x) = \partial_x\left(Q^{(0)}(t,x;\sigma_0) + \partial_\sigma Q^{(0)}(t,x;\sigma_0)(\sigma-\sigma_0)\right)\Big|_{\sigma=\sigma(t,x)} \tag{3.62}$$

$$= \partial_x Q^{(0)}(t,x;\sigma_0) + \frac{\partial_{x\sigma}^2 Q^{(0)}(t,x;\sigma_0)}{\partial_\sigma Q^{(0)}(t,x;\sigma_0)}\left(\sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x,\sigma_0)\right) \tag{3.63}$$

By (43) in [14], we get

$$dE_t^{BS} = dP^{BS}(t, X_t) - \delta^{BS}(s, X_s)dX_s \tag{3.64}$$

$$= dN_t^{(0)} + dN_t^{(1)} + \epsilon^{1-H}\tilde{\sigma}\rho dN_t^{(2)} - \delta^{BS}(s, X_s)dX_s \tag{3.65}$$

$$= dN_t^{(1)} + \epsilon^{1-H}\tilde{\sigma}\rho dN_t^{(2)} - \frac{\partial^2_{x\sigma}Q^{(0)}(t,x;\sigma_0)}{\partial_\sigma Q^{(0)}(t,x;\sigma_0)}\left(\left(x^2\partial_x^2\right)Q_t^{(0)}(X_t)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(X_t)\right)X_t\sigma_t^\epsilon dW_t^* \tag{3.66}$$

where

$$dN_t^{(1)} = (x^2\partial_x^2)Q_t^{(0)}(X_t)d\psi_t^\epsilon + (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\epsilon\phi_t^\epsilon dW_t^* \tag{3.67}$$

$$dN_t^{(2)} = (x\partial_x)Q_t^{(1)}(X_t)\sigma_t^\epsilon dW_t^* \tag{3.68}$$

By Lemma A.6,

$$dE_t^{BS} = dP^{BS}(t, X_t) - \delta^{BS}(s, X_s)dX_s \tag{3.69}$$

$$= (x^2\partial_x^2)Q_t^{(0)}(X_t)d\psi_t^\epsilon - \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}(x^2\partial_x^2)Q_t^{(0)}(X_t)\theta_t\sigma_t^\epsilon dW_t^* \tag{3.70}$$

Then by Lemma A.8, we know that

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\left(\frac{2H}{(H+1/2)^2}\right)\right] \tag{3.71}$$

And it further implies that,

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] \leq Var\left[E_t^{HW}|\mathcal{F}_0\right] \tag{3.72}$$

□

3.3.3 Corrected Black-Scholes (C)

Now we consider a corrected BS scheme that is the candidate to be the optimal hedging strategy. We choose the portfolio to be the corrected price:

$$Q(t,x) = Q_t^{(0)}(x) + (x^2\partial_x^2)Q_t^{(0)}(x)\phi_t^\epsilon + \epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x) \tag{3.73}$$

And the delta of this scheme is BS delta plus a corrected term:

$$\delta^C(t,x) = \delta^{BS}(t,x) + \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2}\frac{H-1/2}{H+1/2}(x\partial_x^2)Q_t^{(0)}(X_t)\theta_t \tag{3.74}$$

Then we define:

$$E_t^C = P_t^C - \int_0^t \delta^C(s,X_s)dX_s \tag{3.75}$$

and the following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.6. *The cost of the Corrected BS hedging strategy satisfies:*

$$\lim_{\epsilon\longrightarrow 0}\epsilon^{H-1}\mathbb{E}\left[\left(\mathbb{E}[E_t^C - E_0^C|\mathcal{F}_0]\right)^2\right]^{1/2} = 0 \tag{3.76}$$

where $E_0^C = P(0,X_0)$*. The asymptotic variance of the cost fluctuations satisfies:*

$$\lim_{\epsilon\longrightarrow 0}\mathbb{E}\left[\left|Var[\epsilon^{H-1}(E_t^C - E_0^C)|\mathcal{F}_0] - (1-\bar{\rho}^2)\mathcal{V}_t^{(3)}\right|\right] = 0 \tag{3.77}$$

where $\bar{\rho} = \rho\frac{\tilde{\sigma}}{\bar{\sigma}}$*. It further implies that with leading order:*

$$Var[E_t^C|\mathcal{F}_0] \le Var[E_t^{HW}|\mathcal{F}_0] \tag{3.78}$$

Proof. For similar computation as BS scheme, we get:

$$\begin{aligned} dE_t^C =& dP(t, X_t) - \delta^C(s, X_s)dX_s & (3.79)\\ =& dE_t^{BS} - (\delta^C(s, X_s) - \delta^{BS}(s, X_s))dX_s & (3.80)\\ =& dE_t^{BS} - \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2}\frac{H-1/2}{H+1/2}(x^2\partial_x^2)Q_t^{(0)}(X_t)\theta_t\sigma_t^\epsilon dW_t^* & (3.81) \end{aligned}$$

which implies that with leading order

$$E_t^C = E_0^C + \sigma_0 p_0 \int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta - \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2}(x^2\partial_x^2)Q_t^{(0)}(X_t)\theta_t\sigma_t^\epsilon dW_t^* \quad (3.82)$$

Let $\bar{\rho} = \rho\frac{\tilde{\sigma}}{\bar{\sigma}}$, with a similar computation in Lemma A.8, we get,

$$Var\left[E_t^C|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - 2\bar{\rho}^2 + \bar{\rho}^2\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \bar{\rho}^2\right] \quad (3.83)$$

□

3.3.4 Optimal Delta Hedging on Fast-varying Long-memory Volatility Model

In the following proposition, we show that under variance with filtration $\mathcal{F}_0$ this measure, the corrected BS scheme is the optimal one among all:

Proposition 3.7. *For any smooth and bounded $a_t = \mathcal{A}(t, X_t)$, as the delta hedging strategy indicating the number of underlyings to hedge, the following cost function:*

$$E_t^* = P(t, X_t) - \int_0^t a_s dX_s \quad (3.84)$$

has minimum variance with leading order ϵ^{1-H}:

$$E_0^* = P(0, X_0), \qquad Var[E_t^C|\mathcal{F}_0] \le Var[E_t^*|\mathcal{F}_0] \tag{3.85}$$

for any $t \in [0, T]$

Proof. Now for the optimal part: We start from

$$E_t^* = Q(t, X_t) - \int_0^t \delta^{HW}(s, X_s)dX_s + \int_0^t (\delta^{HW}(s, X_s) - a_s)dX_s = E_t^{HW} + \int_0^t (\delta^{HW}(s, X_s) - a_s)dX_s \tag{3.86}$$

For leading order approximation, we consider

$$a_s \in \mathcal{A}(t, x) = \partial_x Q_t^{(0)}(x) + \epsilon^{1-H}\mathcal{A}_1(t, x) \tag{3.87}$$

Then by (**??**),

$$E_t^* = E_0^{HW} + \hat{N}_t + \epsilon^{1-H}\int_0^t \hat{\mathcal{A}}(s, x)\sigma_s^\epsilon dW_s^* \tag{3.88}$$

Thus, if we define $\epsilon^{1-H}\int_0^t \hat{\mathcal{A}}(s, x)\sigma_s^\epsilon dW_s^* = N_t$ and

$$Var[E_t^*|\mathcal{F}_0] = Var[\hat{N}_t + N_t|\mathcal{F}_0] \tag{3.89}$$

Then by Lemma B.2, Lemma B.5, Lemma B.6 in [14], with leading order computation

$$\mathbb{E}[\hat{N}_t N_t|\mathcal{F}_0] = \mathbb{E}\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\epsilon^{1-H}\hat{\mathcal{A}}(s, x)\sigma_s^\epsilon \rho\epsilon^{1-H}\theta_s ds|\mathcal{F}_0\right] \tag{3.90}$$

$$= \epsilon^{2-2H}\rho\tilde{\sigma}\mathbb{E}\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_s\hat{\mathcal{A}}(s, x)ds|\mathcal{F}_0\right] \tag{3.91}$$

$$= \epsilon^{2-2H}\rho\tilde{\sigma}\mathbb{E}\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_s\hat{\mathcal{A}}(s, x)ds|\mathcal{F}_0\right] \tag{3.92}$$

$$\mathbb{E}[\hat{N}_t^2|\mathcal{F}_0] = \mathbb{E}\left[\left(\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\epsilon\right)^2 |\mathcal{F}_0\right] \tag{3.93}$$

$$= \epsilon^{2-2H}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 \theta_s^2 ds|\mathcal{F}_0\right] \tag{3.94}$$

$$\mathbb{E}[N_t^2|\mathcal{F}_0] = \mathbb{E}\left[\left(\epsilon^{1-H}\int_0^t \hat{\mathcal{A}}(s,x)\sigma_s^\epsilon dW_s^*\right)^2 |\mathcal{F}_0\right] \tag{3.95}$$

$$= \epsilon^{2-2H}\mathbb{E}\left[\int_0^t \left(\hat{\mathcal{A}}(s,x)\right)^2 (\sigma_s^\epsilon)^2\, ds|\mathcal{F}_0\right] \tag{3.96}$$

$$= \epsilon^{2-2H}\bar{\sigma}^2\mathbb{E}\left[\int_0^t \left(\hat{\mathcal{A}}(s,x)\right)^2 ds|\mathcal{F}_0\right] \tag{3.97}$$

Then,

$$\check{\rho}_t = Corr(N_t, \hat{N}_t|\mathcal{F}_0) = \frac{\mathbb{E}[\hat{N}_t N_t|\mathcal{F}_0]}{\sqrt{\mathbb{E}[\hat{N}_t^2|\mathcal{F}_0]\mathbb{E}[N_t^2|\mathcal{F}_0]}} \tag{3.98}$$

And, for

$$\check{\rho}_t \le |\rho|\frac{\tilde{\sigma}}{\bar{\sigma}} = \bar{\rho}, \qquad \check{\alpha}_t = \sqrt{\frac{Var[N_t^2|\mathcal{F}_0]}{Var[\hat{N}_t^2|\mathcal{F}_0]}} \tag{3.99}$$

we can achieve the following inequalities

$$Var[E_t^*|\mathcal{F}_0] = Var[\hat{N}_t^2|\mathcal{F}_0](1+2\check{\rho}_t\check{\alpha}_t+\check{\alpha}_t^2) \ge Var[\hat{N}_t^2|\mathcal{F}_0](1-2\bar{\rho}\check{\alpha}_t+\check{\alpha}_t^2) \ge Var[\hat{N}_t^2|\mathcal{F}_0](1-\bar{\rho}^2) \tag{3.100}$$

and such minimum is achieve by Corrected BS scheme according to (3.71) □

3.4 Slow-varying Volatility Delta Hedging

This section analyzes the delta hedging problem for European options under a slow-varying volatility model for all $H \in (0,1)$. Asymptotic variances for the HW, BS, and corrected BS delta-hedging schemes are derived, and the main result is that the corrected BS scheme achieves the minimum variance for all delta-hedging schemes under slow-varying volatility.

3.4.1 HW Scheme (HW)

In this case, for (3.2) we take,

$$P_t^{HW} = Q_t(X_t) = Q_t^{(0)}(X_t) + \sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(X_t) + \delta^H \rho p_0 Q_t^{(1)}(X_t) \tag{3.101}$$

$$\delta^{HW}(t, X_t) = \partial_x Q(t,x)\big|_{x=X_t} = \partial_x \left(Q_t^{(0)}(x) + \sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x) + \delta^H \rho p_0 Q_t^{(1)}(x) \right) \big|_{x=X_t} \tag{3.102}$$

then we define that

$$E_t^{HW} = P_t^{HW} - \int_0^t \delta^{HW}(s, X_s) dX_s \tag{3.103}$$

The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$

Proposition 3.8. *The cost of the HW hedging strategy satisfies:*

$$\lim_{\delta \to 0} \delta^{-H} \mathbb{E}\left[\left(\mathbb{E}[E_t^{HW} - E_0^{HW} | \mathcal{F}_0] \right)^2 \right]^{1/2} = 0 \tag{3.104}$$

where $E_0^{HW} = P(0, X_0)$*. The asymptotic variance of the cost fluctuations satisfies:*

$$\lim_{\delta \to 0} \mathbb{E}\left[\left| Var[\delta^{-H}(E_t^{HW} - E_0^{HW}) | \mathcal{F}_0] - \mathcal{V}_t^{(3)} \right| \right] = 0 \tag{3.105}$$

where

$$\mathcal{V}_t^{(3)} = \frac{\sigma_0^2 p_0^2}{\Gamma(H+3/2)^2} \int_{\mathbb{R}} \int_0^t \left((x^2\partial_x^2) Q_s^{(0)}(x_0 e^{\sigma_0 \sqrt{s} z - \sigma_0 s/2}) \right)^2 (T-t)^{2H+1} ds p(z) dz \tag{3.106}$$

where $p(z)$ *is the pdf of standard normal distribution. We note that such* $\mathcal{V}_t^{(3)}$ *is an analogy to the variance approximation of the HW scheme in [16]. The major difference is that the power of time-to-maturity depends on the Hurst exponent here. For a numerical illustration, we refer the Figure C.2.*

Proof. By the (3.101), we know that:

$$dP_t^{HW} = dQ_t(X_t) = dQ_t^{(0)}(X_t) + d\sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(X_t) + d\delta^H \rho p_0 Q_t^{(1)}(X_t) \tag{3.107}$$

By the proof in Appendix B,

$$d[Q_t^{(0)}(X_t) + \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t)\phi_t^\delta + \delta^H \rho p_0 Q_t^{(1)}(X_t)] \tag{3.108}$$

$$= dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dR_t^{(4)} + dR_t^{(5)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} + \delta^H \rho p_0 dN_t^{(2)} \tag{3.109}$$

where $dR_t^{(j)}$ are higher order terms and the martingale terms are defined as the following:

$$dN_t^{(0)} = (x\partial_x) Q_t^{(0)}(X_t)\sigma_t^\delta dW_t^* \tag{3.110}$$

$$\sigma_0 p_0 dN_t^{(1)} = \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta + \sigma_0 p_0 (x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(X_t)\sigma_t^\delta \phi_t^\delta dW_t^* \tag{3.111}$$

$$dN_t^{(2)} = (x\partial_x) Q_t^{(1)}(X_t)\sigma_t^\delta dW_t^* \tag{3.112}$$

We then notice that by Lemma A.6, we have:

$$dE_t^{HW} = dP_t^{HW} - \delta_s^{HW} dX_s = \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta \tag{3.113}$$

We further denote that

$$\hat{N}_t = \sigma_0 p_0 \int_0^t (x^2\partial_x^2) Q_s^{(0)}(X_s) d\psi_s^\delta \tag{3.114}$$

which implies

$$E_t^{HW} = E_0^{HW} + \hat{N}_t \tag{3.115}$$

As we know ψ_s^δ is a martingale in Lemma A.1, we can conclude that the first asymptotic result is true. Now, it suffices to find $\delta^{-2H} Var[\hat{N}_t|\mathcal{F}_0]$. By Lemma A.1 and Lemma A.3, we have the following leading order computation:

$$\delta^{-2H}Var[\hat{N}_t|\mathcal{F}_0] = \delta^{-2H}Var\left[\int_0^t \sigma_0 p_0(x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta|\mathcal{F}_0\right] \tag{3.116}$$

$$= \mathbb{E}\left[\int_0^t \left(\sigma_0 p_0(x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2(\theta_{s,T})^2 ds|\mathcal{F}_0\right] \tag{3.117}$$

$$= \sigma_0^2 p_0^2 \int_{\mathbb{R}}\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(x_0 e^{\sigma_0\sqrt{s}z-\sigma_0^2 s/2})\right)^2 p(z)(\theta_{s,T})^2 dsdz \tag{3.118}$$

$$= \frac{\sigma_0^2 p_0^2}{\Gamma(H+3/2)^2}\int_{\mathbb{R}}\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(x_0 e^{\sigma_0\sqrt{s}z-\sigma_0^2 s/2})\right)^2 (T-s)^{2H+1}dsp(z)dz \tag{3.119}$$

□

3.4.2 BS Scheme (BS)

We define the BS scheme delta on slow-varying volatility as the following:

$$\delta^{BS}(t,x) = \partial_x Q^{(0)}(t,x;\sigma)|_{\sigma=\sigma(t,x)} \tag{3.120}$$

with implied volatility $\sigma(t,x)$ solving

$$P_t^{BS} = Q(t,x) = Q^{(0)}(t,x;\sigma(t,x)) \tag{3.121}$$

The implied volatility $\sigma(t,x)$ is such that

$$Q^{(0)}(t,x;\sigma(t,x)) = Q(t,x) = Q_t^{(0)}(x,\sigma_0) + \sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2)Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x,\sigma_0) \tag{3.122}$$

and similarly we define:

$$E_t^{BS} = P_t^{BS} - \int_0^t \delta^{BS}(s,X_s)dX_s \tag{3.123}$$

The following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$:

Proposition 3.9. *The cost of the BS hedging strategy satisfies:*

$$\lim_{\delta \longrightarrow 0} \delta^{-H} \mathbb{E}\left[\left(\mathbb{E}[E_t^{BS} - E_0^{BS} | \mathcal{F}_0]\right)^2\right]^{1/2} = 0 \tag{3.124}$$

where $E_0^{BS} = P(0, X_0)$*. The asymptotic variance of the cost fluctuations satisfies:*

$$\lim_{\delta \longrightarrow 0} \mathbb{E}\left[\left| Var[\delta^{-H}(E_t^{BS} - E_0^{BS}) | \mathcal{F}_0] - \left(1 - \rho^2 \frac{2H+2}{(H+3/2)^2}\right) \mathcal{V}_t^{(3)} \right|\right] = 0 \tag{3.125}$$

which further implies that with leading order:

$$Var[E_t^{BS} | \mathcal{F}_0] \leq Var[E_t^{HW} | \mathcal{F}_0] \tag{3.126}$$

Proof. Consider the asymptotic expansion of $Q^{(0)}(t, x; \sigma(t, x))$:

$$Q^{(0)}(t, x; \sigma(t, x)) = Q^{(0)}(t, x; \sigma_0) + \left(\partial_\sigma Q^{(0)}(t, x; \sigma)\right)|_{\sigma=\sigma_0} (\sigma(t, x) - \sigma_0) + \mathcal{O}(\delta^{2H}) \tag{3.127}$$

We know $\sigma(t, x) - \sigma_0$ from:

$$\sigma_0 p_0 \phi_t^\delta (x^2 \partial_x^2 Q_t^{(0)}(x, \sigma_0)) + \delta^H \rho p_0 Q_t^{(1)}(x, \sigma_0) = \left(\partial_\sigma Q^{(0)}(t, x; \sigma)\right)|_{\sigma=\sigma_0} (\sigma(t, x) - \sigma_0) + \mathcal{O}(\delta^{2H}) \tag{3.128}$$

Then it implies that

$$\sigma(t, x) - \sigma_0 = \frac{\sigma_0 p_0 \phi_t^\delta (x^2 \partial_x^2) Q_t^{(0)}(x, \sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x, \sigma_0)}{(\partial_\sigma Q^{(0)}(t, x; \sigma))|_{\sigma=\sigma_0}} + \mathcal{O}(\delta^{2H}) \tag{3.129}$$

Then with leading order computation

$$\delta^{BS}(t,x) = \partial_x \left(Q^{(0)}(t,x;\sigma_0) + \partial_\sigma Q^{(0)}(t,x;\sigma_0)(\sigma-\sigma_0)\right)\Big|_{\sigma=\sigma(t,x)} \tag{3.130}$$

$$= \partial_x Q^{(0)}(t,x;\sigma_0) + \frac{\partial^2_{x\sigma} Q^{(0)}(t,x;\sigma_0)}{\partial_\sigma Q^{(0)}(t,x;\sigma_0)} \left(\sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x,\sigma_0)\right) \tag{3.131}$$

which implies with leading order

$$\delta^{BS}(t,x) dX_s \tag{3.132}$$

$$= (x\partial_x) Q^{(0)}(t,x;\sigma_0)\sigma_t^\delta dW_t^* + \frac{\partial^2_{x\sigma} Q^{(0)}}{\partial_\sigma Q^{(0)}} \left(\sigma_0 p_0 x \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 x Q_t^{(1)}(x,\sigma_0)\right) \sigma_t^\delta dW_t^* \tag{3.133}$$

where we used a shortened notation $\frac{\partial^2_{x\sigma} Q^{(0)}}{\partial_\sigma Q^{(0)}}$. By Lemma A.6, we have

$$dE_t^{BS} = dP^{BS}(t,X_t) - \delta^{BS}(s,X_s) dX_s \tag{3.134}$$

$$= dE^{HW}(t,X_t) + \delta^H \rho p_0 dN_t^{(2)} - \frac{\partial^2_{x\sigma} Q^{(0)}}{\partial_\sigma Q^{(0)}} \left(\delta^H \rho p_0 x Q_t^{(1)}(x,\sigma_0)\right)\Big|_{x=X_t} \sigma_t^\delta dW_t^* \tag{3.135}$$

$$= \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta + \delta^H \rho p_0 dN_t^{(2)} - \frac{\partial^2_{x\sigma} Q^{(0)}}{\partial_\sigma Q^{(0)}} \left(\delta^H \rho p_0 x Q_t^{(1)}(x,\sigma_0)\right)\Big|_{x=X_t} \sigma_t^\delta dW_t^* \tag{3.136}$$

Then by Lemma A.7 the leading order computation gives,

$$dE_t^{BS} = dP^{BS}(t,X_t) - \delta^{BS}(s,X_s) dX_s \tag{3.137}$$

$$= \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta + \delta^H \rho p_0 dN_t^{(2)} - \frac{\partial^2_{x\sigma} Q^{(0)}}{\partial_\sigma Q^{(0)}} \left(\delta^H \rho p_0 x Q_t^{(1)}(x,\sigma_0)\right)\Big|_{x=X_t} \sigma_t^\delta dW_t^* \tag{3.138}$$

$$= \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta - \delta^H \rho p_0 \sigma_0^2 \frac{(x^2\partial_x^2) Q_t^{(0)}(X_t)}{\sigma_0^2 (T-t)} D_{t,T} \sigma_t^\delta dW_t^* \tag{3.139}$$

$$= \sigma_0 p_0 (x^2\partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta - \frac{\delta^H \rho p_0}{H+3/2} (x^2\partial_x^2) Q_t^{(0)}(X_t) \theta_{t,T} \sigma_t^\delta dW_t^* \tag{3.140}$$

which implies:

$$E_t^{BS} = \sigma_0 p_0 \int_0^t (x^2\partial_x^2) Q_s^{(0)}(X_s) d\psi_s^\delta - \frac{\delta^H \rho p_0}{H+3/2} \int_0^t (x^2\partial_x^2) Q_s^{(0)}(X_s) \theta_{s,T} \sigma_s^\delta dW_s^* \tag{3.141}$$

Above shows that E_t^{BS} is a martingale and hereby the first asymptotic result holds. By Lemma A.7, we can conclude, up to a leading order:

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \rho^2\left(\frac{2H+2}{(H+3/2)^2}\right)\right] \tag{3.142}$$

which further denotes that

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] \leq Var\left[E_t^{HW}|\mathcal{F}_0\right] \tag{3.143}$$

$\square$

3.4.3 Corrected Black-Scholes (C)

Now we consider a corrected BS scheme that is the candidate to be the optimal hedging strategy under the assumption that the volatility is a function of a slow-varying fractional Ornstein-Uhlenbeck process. We choose the portfolio to be the corrected price:

$$P_t^C = Q(t,x) = Q_t^{(0)}(x,\sigma_0) + \sigma_0 p_0 \phi_t^\delta (x^2\partial_x^2) Q_t^{(0)}(x,\sigma_0) + \delta^H \rho p_0 Q_t^{(1)}(x,\sigma_0) \tag{3.144}$$

And the delta of this scheme is BS delta plus a corrected term:

$$\delta^C(t,x) = \delta^{BS}(t,x) + \delta^H \rho p_0 \frac{H+1/2}{H+3/2}(x\partial_x^2) Q_t^{(0)}(X_t)\theta_{t,T} \tag{3.145}$$

Then we define:

$$E_t^C = P_t^C - \int_0^t \delta^C(s, X_s) dX_s \tag{3.146}$$

and the following proposition shows the asymptotic results of its mean and variance conditioning on $\mathcal{F}_0$.

Proposition 3.10.

$$\lim_{\delta \longrightarrow 0} \delta^{-H} \mathbb{E}\left[\left(\mathbb{E}[E_t^C - E_0^C | \mathcal{F}_0] \right)^2 \right]^{1/2} = 0 \tag{3.147}$$

where $E_0^C = P(0, X_0)$. *The asymptotic variance of the cost fluctuations satisfies:*

$$\lim_{\delta \longrightarrow 0} \mathbb{E}\left[\left| Var[\delta^{-H}(E_t^C - E_0^C) | \mathcal{F}_0] - (1-\rho^2)\mathcal{V}_t^{(3)} \right| \right] = 0 \tag{3.148}$$

which further implies that with leading order:

$$Var[E_t^C | \mathcal{F}_0] \leq Var[E_t^{HW} | \mathcal{F}_0] \tag{3.149}$$

Proof. For similar computation as BS scheme, we get:

$$\begin{aligned}
dE_t^C =& dP(t, X_t) - \delta^C(s, X_s) dX_s & (3.150)\\
=& dE_t^{BS} - (\delta^C(s, X_s) - \delta^{BS}(s, X_s)) dX_s & (3.151)\\
=& dE_t^{BS} - \delta^H \rho p_0 \frac{H+1/2}{H+3/2} (x \partial_x^2) Q_t^{(0)}(X_t) \theta_{t,T} dX_s & (3.152)\\
=& dE_t^{BS} - \delta^H \rho p_0 \frac{H+1/2}{H+3/2} (x^2 \partial_x^2) Q_t^{(0)}(X_t) \theta_{t,T} \sigma_t^\delta dW_t^* & (3.153)
\end{aligned}$$

By BS scheme result, we have

$$dE_t^{BS} = \sigma_0 p_0 (x^2 \partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\delta - \frac{\delta^H \rho p_0}{H+3/2} (x^2 \partial_x^2) Q_t^{(0)}(X_t) \theta_{t,T} \sigma_t^\delta dW_t^* \tag{3.154}$$

which implies that with leading order

$$E_t^C = E_0^C + \sigma_0 p_0 \int_0^t (x^2\partial_x^2) Q_s^{(0)}(X_s) d\psi_s^\delta - \delta^H \rho p_0 \int_0^t (x^2\partial_x^2) Q_s^{(0)}(X_s) \theta_{s,T} \sigma_s^\delta dW_s^* \tag{3.155}$$

We observe that E_t^C is a martingale and with a similar computation in Lemma A.7 we get

$$Var\left[E_t^C|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - 2\rho^2 + \rho^2\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \rho^2\right] \tag{3.156}$$

□

3.4.4 Optimal Delta Hedging on Slow-varying Volatility Model

Now for the optimal part, in the following proposition, we show that under the measure: variance with filtration $\mathcal{F}_0$, the corrected BS scheme is the optimal one among all possible DA schemes:

Proposition 3.11. *For any smooth and bounded $a_t = \mathcal{A}(t, X_t)$, as the delta hedging strategy indicating the number of underlyings to hedge, the following cost function:*

$$E_t^* = P(t, X_t) - \int_0^t a_s dX_s \tag{3.157}$$

has minimum variance with leading order δ^H:

$$E_0^* = P(0, X_0), \qquad Var[E_t^C|\mathcal{F}_0] \leq Var[E_t^*|\mathcal{F}_0] \tag{3.158}$$

for any $t \in [0, T]$

Proof. For

$$E_t^* = Q(t,X_t) - \int_0^t \delta^{HW}(s,X_s)dX_s + \int_0^t (\delta^{HW}(s,X_s) - a_s)dX_s = E_t^{HW} + \int_0^t (\delta^{HW}(s,X_s) - a_s)dX_s \tag{3.159}$$

we first consider

$$a_s \in \mathcal{A}(t,x) = \partial_x Q_t^{(0)}(x) + \delta^H \mathcal{A}_1(t,x) \tag{3.160}$$

Then for some $\hat{\mathcal{A}}(s,x)$

$$E_t^* = E_0^{HW} + \hat{N}_t + \delta^H \int_0^t \hat{\mathcal{A}}(s,x)\sigma_s^\delta dW_s^* \tag{3.161}$$

Thus, if we define $\delta^H \int_0^t \hat{\mathcal{A}}(s,x)\sigma_s^\delta dW_s^* = N_t$

$$Var[E_t^*|\mathcal{F}_0] = Var[\hat{N}_t + N_t|\mathcal{F}_0] \tag{3.162}$$

Then, with leading order computation:

$$\mathbb{E}[\hat{N}_t N_t|\mathcal{F}_0] = \mathbb{E}\left[\int_0^t \sigma_0 p_0 (x^2\partial_x^2)Q_s^{(0)}(X_s)\delta^H \theta_{s,T}\delta^H \hat{\mathcal{A}}(s,x)\sigma_s^\delta \rho ds|\mathcal{F}_0\right] \tag{3.163}$$

$$= \delta^{2H}\sigma_0 p_0 \rho \mathbb{E}\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_{s,T}\hat{\mathcal{A}}(s,x)\sigma_s^\delta ds|\mathcal{F}_0\right] \tag{3.164}$$

$$\mathbb{E}[\hat{N}_t^2|\mathcal{F}_0] = \mathbb{E}\left[\left(\int_0^t \sigma_0 p_0 (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta\right)^2 |\mathcal{F}_0\right] \tag{3.165}$$

$$= \delta^{2H}\sigma_0^2 p_0^2 \mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 \theta_{s,T}^2 ds|\mathcal{F}_0\right] \tag{3.166}$$

$$\mathbb{E}[N_t^2|\mathcal{F}_0] = \mathbb{E}\left[\left(\delta^H \int_0^t \hat{\mathcal{A}}(s,x)\sigma_s^\delta dW_s^*\right)^2 |\mathcal{F}_0\right] \tag{3.167}$$

$$= \delta^{2H}\mathbb{E}\left[\int_0^t \left(\hat{\mathcal{A}}(s,x)\sigma_s^\delta\right)^2 ds|\mathcal{F}_0\right] \tag{3.168}$$

Then,

$$\check{\rho}_t = Corr(N_t, \hat{N}_t|\mathcal{F}_0) = \frac{\mathbb{E}[\hat{N}_t N_t|\mathcal{F}_0]}{\sqrt{\mathbb{E}[\hat{N}_t^2|\mathcal{F}_0]\mathbb{E}[N_t^2|\mathcal{F}_0]}} \tag{3.169}$$

And we can see that $|\check{\rho}_t| \leq |\rho|$. If we let

$$\check{\alpha}_t = \sqrt{\frac{Var[N_t^2|\mathcal{F}_0]}{Var[\hat{N}_t^2|\mathcal{F}_0]}} \tag{3.170}$$

$$Var[E_t^*|\mathcal{F}_0] = Var[\hat{N}_t^2|\mathcal{F}_0](1+2\rho\check{\alpha}_t+\check{\alpha}_t^2) \geq Var[\hat{N}_t^2|\mathcal{F}_0](1-2|\rho|\check{\alpha}_t+\check{\alpha}_t^2) \geq Var[\hat{N}_t^2|\mathcal{F}_0](1-|\rho|^2) \tag{3.171}$$

And in Proposition 6, we proved such minimum is $Var[E_t^C|\mathcal{F}_0]$ $\square$

Chapter 4

Numerical Illustration

In this section, we assess the effectiveness of various hedging strategies through simulations of underlying price paths. We analyze the performance of these strategies separately for slow-varying volatility and fast-varying long-memory volatility. Unlike the hedging simulation and analysis in [16], both the slow-varying and fast-varying models feature a middle term with a random component ϕ_t. Since ϕ_t has an expected value of zero, we evaluate the performance of each scheme by taking the expected value of this term.

4.1 Fast-varying Long-memory Volatility Simulation

In this section, we evaluate different hedging strategies under the assumption of fast-varying long-memory volatility. We use the asymptotic formula (2.14) and assume a middle term of zero. Specifically, we analyze the performance of the HW, BS, and Corrected BS schemes. It is important to note that for all three strategies, we define $\mathcal{D} = \tilde{\sigma}\langle FF' \rangle / \bar{\sigma}^2$:

- HW Scheme: By Black-Scholes Greeks computation:

$$
\begin{aligned}
\delta^{HW}(t,x) = \partial_x Q(t,x) &= \left(\partial_x Q_t^{(0)}(x) + \epsilon^{1-H}\tilde{\sigma}\rho\partial_x Q_t^{(1)}(x)\right) \\
&= \partial_x Q_t^{(0)}(x) + \epsilon^{1-H}\rho\mathcal{D}\frac{(T-t)^H}{\Gamma(H+3/2)}\frac{(d_2^2-1)(x^2\partial_x^2)Q_t^{(0)}(x)}{x\sqrt{T-t}} \\
&= \partial_x Q_t^{(0)}(x) + \frac{\epsilon^{1-H}\rho\mathcal{D}K}{\sqrt{2\pi}\Gamma(H+3/2)}\frac{e^{-d_2^2}}{x\sqrt{T-t}}\left[(d_2^2-1)(T-t)^{H-1/2}\right]
\end{aligned}
$$

- BS Scheme:

$$
\begin{aligned}
\delta^{BS}(t,x) &= \partial_x Q^{(0)}(t,x) + \frac{\partial_{x\sigma}^2 Q^{(0)}(t,x)}{\partial_\sigma Q^{(0)}(t,x)}\left(\epsilon^{1-H}\tilde{\sigma}\rho Q_t^{(1)}(x)\right) \\
&= \partial_x Q_t^{(0)}(x) + \epsilon^{1-H}\rho\mathcal{D}\frac{(T-t)^H}{\Gamma(H+3/2)}\frac{d_2^2(x^2\partial_x^2)Q_t^{(0)}(x)}{x\sqrt{T-t}} \\
&= \partial_x Q_t^{(0)}(x) + \frac{\epsilon^{1-H}\rho\mathcal{D}K}{\sqrt{2\pi}\Gamma(H+3/2)}\frac{e^{-d_2^2}}{x\sqrt{T-t}}\left[d_2^2(T-t)^{H-1/2}\right]
\end{aligned}
$$

- Corrected BS Scheme:

$$
\begin{aligned}
\delta^{C}(t,x) &= \delta^{BS}(t,x) + \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2}\frac{H-1/2}{H+1/2}(x\partial_x^2)Q_t^{(0)}(X_t)\theta_t \\
&= \partial_x Q_t^{(0)}(x) + \epsilon^{1-H}\rho\mathcal{D}\frac{(T-t)^H}{\Gamma(H+3/2)}\frac{(d_2^2+H-1/2)(x^2\partial_x^2)Q_t^{(0)}(x)}{x\sqrt{T-t}} \\
&= \partial_x Q_t^{(0)}(x) + \frac{\epsilon^{1-H}\rho\mathcal{D}K}{\sqrt{2\pi}\Gamma(H+3/2)}\frac{e^{-d_2^2}}{x\sqrt{T-t}}\left[(d_2^2+H+\frac{1}{2})(T-t)^{H-1/2}\right]
\end{aligned}
$$

As in Section 6.1, we tune the parameter $\mathcal{D}_2$ while keeping other variables fixed. We can then summarize the three hedging deltas as follows:

$$
\delta^A(t,x) = \partial_x Q_t^{(0)}(x) + \mathcal{D}_1\frac{e^{-d_2^2}}{x\sqrt{T-t}}g^A(H,d_2,T-t), \qquad A = HW, BS, C \tag{4.1}
$$

where

$$
\mathcal{D}_1 = \frac{\epsilon^{1-H}\rho\mathcal{D}K}{\sqrt{2\pi}\Gamma(H+3/2)}, \qquad d_2 = \frac{\log(x/K)-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{4.2}
$$

and

$$g^{HW} = (d_2^2 - 1)(T-t)^{H-1/2}, \qquad g^{BS} = d_2^2 (T-t)^{H-1/2}, \qquad g^{C} = (d_2^2 + H - \frac{1}{2})(T-t)^{H-1/2} \tag{4.3}$$

Note that in the case of long-memory volatility, the Hurst exponent H is in the range $(1/2, 1)$. So, it is important to note that $g^{HW} < g^{BS} < g^{C}$. When $H = 1/2$, the corrected BS scheme is identical to the BS scheme, which is consistent with the result in [16] for $H = 1/2$.

4.2 Slow-varying Volatility Simulation

In this section, we consider hedging strategies under the assumption of slow-varying volatility. We assume that the middle term is zero. We evaluate the performance of the HW, BS, and Corrected BS schemes accordingly.

- HW Scheme:

$$\begin{aligned}
\delta^{HW}(t,x) &= \partial_x Q(t,x) = \left(\partial_x Q_t^{(0)}(x) + \delta^H \rho p_0 \partial_x Q_t^{(1)}(x)\right) \\
&= \partial_x Q_t^{(0)}(x) + \delta^H \rho p_0 \frac{(T-t)^{H+1}}{\Gamma(H+5/2)} \frac{(d_2^2 - 1)(x^2 \partial_x^2) Q_t^{(0)}(x)}{x\sqrt{T-t}} \\
&= \partial_x Q_t^{(0)}(x) + \frac{\rho \delta^H p_0 K}{\sqrt{2\pi}\sigma_0 \Gamma(H+5/2)} \frac{e^{-d_2^2}}{x\sqrt{T-t}} \left[(d_2^2 - 1)(T-t)^{H+1/2}\right]
\end{aligned}$$

- BS Scheme:

$$\begin{aligned}
\delta^{BS}(t,x) &= \partial_x Q^{(0)}(t,x) + \frac{\partial_{x\sigma}^2 Q^{(0)}(t,x)}{\partial_\sigma Q^{(0)}(t,x)} \left(\delta^H \rho p_0 Q_t^{(1)}(x)\right) \\
&= \partial_x Q_t^{(0)}(x) + \delta^H \rho p_0 \frac{(T-t)^{H+1}}{\Gamma(H+5/2)} \frac{d_2^2 (x^2 \partial_x^2) Q_t^{(0)}(x)}{x\sqrt{T-t}} \\
&= \partial_x Q_t^{(0)}(x) + \frac{\rho \delta^H p_0 K}{\sqrt{2\pi}\sigma_0 \Gamma(H+5/2)} \frac{e^{-d_2^2}}{x\sqrt{T-t}} \left[d_2^2 (T-t)^{H+1/2}\right]
\end{aligned}$$

- Corrected BS Scheme:

$$\begin{aligned}
\delta^C(t,x) &= \delta^{BS}(t,x) + \delta^H \rho p_0 \frac{H+1/2}{H+3/2}(x\partial_x^2)Q_t^{(0)}(X_t)\theta_{t,T} \\
&= \partial_x Q_t^{(0)}(x) + \delta^H \rho p_0 \frac{(T-t)^{H+1}}{\Gamma(H+5/2)} \left(\frac{(d_2^2 + H + \frac{1}{2})(x^2\partial_x^2)Q_t^{(0)}(x)}{x\sqrt{T-t}} \right) \\
&= \partial_x Q_t^{(0)}(x) + \frac{\rho\delta^H p_0 K}{\sqrt{2\pi}\sigma_0\Gamma(H+5/2)} \frac{e^{-d_2^2}}{x\sqrt{T-t}} \left[(d_2^2 + H + \frac{1}{2})(T-t)^{H+1/2} \right]
\end{aligned}$$

In our numerical simulation and optimization, we only consider the optimization on the variable $\mathcal{D}_2$ defined below. Thus, we can summarize three hedging deltas as the following:

$$\delta^B(t,x) = \partial_x Q_t^{(0)}(x) + \mathcal{D}_2 \frac{e^{-d_2^2}}{x\sqrt{T-t}} g^B(H, d_2, T-t), \qquad B = HW, BS, C \tag{4.4}$$

where

$$\mathcal{D}_2 = \frac{\rho\delta^H p_0 K}{\sqrt{2\pi}\sigma_0\Gamma(H+5/2)}, \qquad d_2 = \frac{\log(x/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{4.5}$$

and

$$g^{HW} = (d_2^2 - 1)(T-t)^{H+1/2}, \qquad g^{BS} = d_2^2(T-t)^{H+1/2}, \qquad g^C = (d_2^2 + H + \frac{1}{2})(T-t)^{H+1/2} \tag{4.6}$$

Note that for any moneyness in our simulation case, $g^{HW} < g^{BS} < g^C$.

4.3 Numerical Method Interpretation

In this section, our goal is to compare the cost of hedging with the volatility fluctuation by computing the expression:

$$E_T = h(X_T) - \int_0^T \delta(t, X_t) dX_t \tag{4.7}$$

We use the relative measure:

$$C^C(T, X_T) = \frac{Std[E_T]}{Q^{(0)}(X_0, \sigma)} \tag{4.8}$$

where $\sigma = \sigma_0$ for the slow-varying case, and $\sigma = \bar{\sigma}$ for the fast-varying case. We simulate one path of the fOU process using the kernel introduced in [15]. Based on the resulting volatility path, we use the ExpfOU process from Section 7 of [16]:

$$F(z) = \bar{\sigma} \exp\left(\frac{wz}{\sigma_Z} - w^2\right) \tag{4.9}$$

Here, $w > 0$ is the fluctuation parameter that measures the typical amplitude of the relative fluctuations of the volatility. We then use a step-by-step simulation to obtain independent price trajectories X_t and determine the optimal constant $\mathcal{D}_1$ or $\mathcal{D}_2$ respectively, that minimizes the cost over moneyness.

All Figures 4.1-4.4, we use parameters $\sigma = 0.5$, $\rho = -0.5$ and scale $w = 0.4$.

Figure 4.1 illustrates the fast-varying standard Ornstein-Uhlenbeck process with $\epsilon = 0.05$, where the volatility process is neither short-range dependent nor long-range dependent. As a result, the roughness has no effect on differentiating the two curves, (BS) and (C). This observation is in line with our theoretical result that when $H = 1/2$, the two schemes are essentially the same, and the matching curve lies below the (HW) scheme curve. Specifically, the variance of E_t^{BS} and E_t^C conditioned on $\mathcal{F}_0$ are given by:

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \bar{\rho}^2\left(\frac{2H}{(H+1/2)^2}\right)\right] \tag{4.10}$$

$$Var\left[E_t^{C}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \bar{\rho}^2\right] \tag{4.11}$$

It is worth noting that this is a similar result to Figure 9.1 in [16].

In Figure 4.2, it can be seen that, as the Hurst exponent is close to 1, there is a trend that the long-memory property pulls the (BS) scheme out ant from the function $\frac{2H}{(H+1/2)^2}$ is a decreasing function ranging from $\frac{8}{9}$ to 1 for $H \in (1/2, 1)$. Figure 3 shows their difference when H is close to 1.

In Figure 4.3, we can see that for slow-varying volatility rough case, the corrected Black-Scholes scheme is indeed the optimal one. We can also observe the same optimality in Figure 4.4. We hereby lists the relationship of the conditional variance of three schemes:

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\left(\frac{2H+2}{(H+3/2)^2}\right)\right] \tag{4.12}$$

$$Var\left[E_t^{C}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\right] \tag{4.13}$$

We further note that function $\frac{2H+2}{(H+3/2)^2}$ is also a decreasing function on $(0, 1)$ ranging from 0.64 to 8/9. We argue that we cannot really see such trends comparing Figure 4.3 and Figure 4.4 because we assumes D_1 is independent to moneyness and this assumption would slightly change the picture.

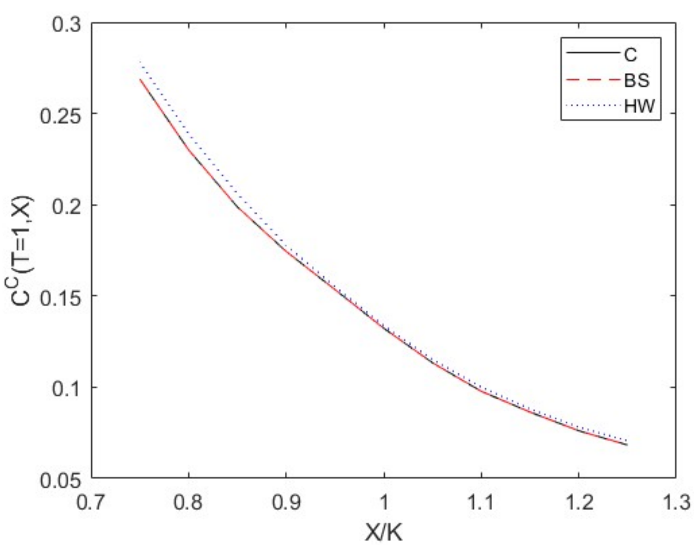

Figure 4.1: This picture shows the relative error standard deviation when $H = 0.5$, ϵ=0.05. In this case, the BS scheme is the same as the corrected BS scheme, and this picture coincides with Figure 9.1 in [16]

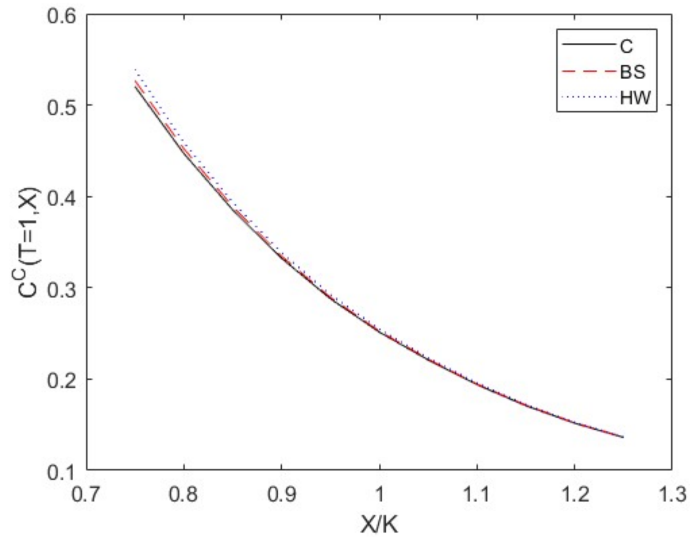

Figure 4.2: This picture shows the relative error standard deviation when $H = 0.9$, ϵ=0.05. In this case, the corrected BS scheme is the optimal one and slightly better than the BS scheme

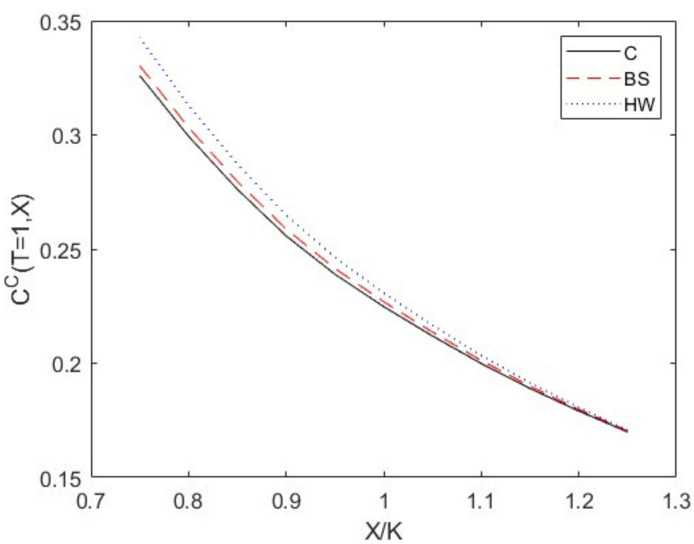

Figure 4.3: This picture shows the relative error standard deviation when $H = 0.1$, δ=0.05. We observe that the corrected BS scheme is the optimal one of all.

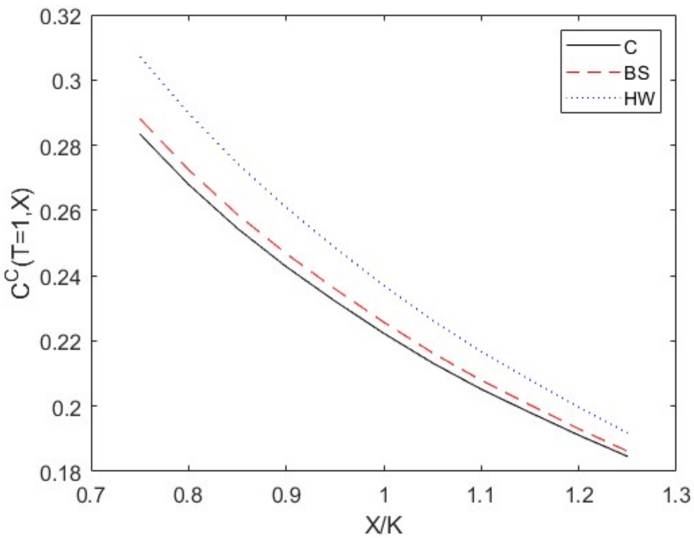

Figure 4.4: This picture shows the relative error standard deviation when $H = 0.5$, δ=0.05. In this case, the corrected BS scheme is the optimal one.

4.4 The Corrected Black-Scholes Scheme and Leverage Effect

In this section, we discuss the connection of our work to other empirical studies. In previous sections, we demonstrated the optimality of the corrected Black-Scholes scheme through simulated volatility and price paths. While this only serves as evidence of the correctness of our theory, it emphasizes the significance of our findings. The corrected Black-Scholes delta is the Black-Scholes delta with an added correction term, which is expressed as follows:

$$\delta^C(t,x) = \delta^{BS}(t,x) + \rho C^H_{t,T}(x\partial_x^2)Q_t^{(0)}(x) \tag{4.14}$$

which can be also written as

$$\delta^C(t,x) = \delta^{BS}(t,x) + \rho \tilde{C}^H_{t,T}\frac{\partial_\sigma Q_t^{(0)}(x)}{x\sqrt{T-t}} \tag{4.15}$$

The value of $C^H_{t,T}$ or adjusted $\tilde{C}^H_{t,T}$ is a positive constant that depends on the Hurst exponent, time to maturity, and other constants that vary in different cases. From (4.14) we can see that the constant ρ determines the corrected Black-Scholes scheme, it shows that the existence of the leverage effect leads to a correction to the BS scheme.

The leverage effect is typically negative due to the relationship between stock prices and implied volatility. As stock prices rise, investors may perceive the stock to be less risky, leading to lower implied volatility, while falling stock prices may lead to higher implied volatility as investors perceive the stock to be riskier. This correlation between stock prices and implied volatility results in a negative correlation between the two variables, leading to the negative leverage effect. In [20], the authors estimated call options with varying moneyness and time to maturity and found that empirically, $\delta^{MV} - \delta^{BS}$ is negative. Assuming $\delta^C - \delta^{BS}$ is also negative, and with the constant $\tilde{C}^H_{t,T}$ being a positive constant, we obtain $\rho < 0$. Our corrected BS scheme bridges the gap between these empirical findings.

4.5 Conclusion and Future Research Directions

This thesis concludes the delta hedging problem for the timely-scaled fractional Ornstein-Uhlenbeck process volatility in three cases: slow-fluctuating for $H \in (0, 1)$, fast-varying for $H < 1/2$, and fast-varying for $H > 1/2$. Our main contribution is demonstrating that in all cases, the corrected Black-Scholes (BS) scheme achieves minimal variance and is the optimal delta-hedging strategy. Notably, the corrected BS scheme coincides with the BS scheme in the special case of fast-varying rough volatility, which was previously solved in [16]. We validate our results using simulated volatility and underlying price paths. We provide empirical evidence of the existence of a correction term to the BS delta, even though we did not conduct experiments with real market data.

There are several avenues for future research. Firstly, we could explore a more comprehensive pricing formula by assuming the volatility is driven by two correlated paths, one fast and one slow. We can set up a multi-scale volatility model similar to (1.6)-(1.8) as the following:

$$dX_t = \sigma_t^{\epsilon,\delta} X_t dW_t^{(0)} \tag{4.16}$$

$$\sigma_t^{\epsilon,\delta} = F(Y_t^{\epsilon}, Z_t^{\delta}) \tag{4.17}$$

F is assumed to be 1-1, smooth, positive valued with $F(0,0) = 0$ and $F_y(0,0) = 1$ and $F_z(0,0) = 1$. And

$$Y_t^{\epsilon} = \int_{-\infty}^{t} \mathcal{K}^{\epsilon}(t-s) dW_s^{(1)}, \qquad \mathcal{K}^{\epsilon}(t) = \frac{1}{\sqrt{\epsilon}} \mathcal{K}(t/\epsilon) \tag{4.18}$$

$$Z_t^{\delta} = \int_{-\infty}^{t} \mathcal{K}^{\delta}(t-s) dW_s^{(2)}, \qquad \mathcal{K}^{\delta}(t) = \delta^{1/2} \mathcal{K}(\delta t) \tag{4.19}$$

where

$$d < W^{(0)}, W^{(1)} >_s = \rho_1 ds \tag{4.20}$$

$$d < W^{(0)}, W^{(2)} >_s = \rho_2 ds \tag{4.21}$$

$$d < W^{(1)}, W^{(2)} >_s = \rho_{12} ds \tag{4.22}$$

Once the multi-scale volatility pricing is determined, the optimality problem of delta hedging should be very interesting and once solved, it would make a final conclusion to this topic.

We can further investigate the empirical relevance of the corrected term in the Black-Scholes delta and develop a method for market calibration. It is important to note that our formulas, regardless of whether the volatility is fast-varying or slow-varying, rely on accurate calibration of historical volatility, either σ_0 or $\bar{\sigma}$. In practice, however, obtaining historical volatility is difficult, while implied volatility is more readily available. Moreover, while we asymptotically expand the implied volatility in the Black-Scholes scheme computation, the Hull-White scheme is not an effective strategy to implement in practice. Therefore, it is of interest to determine how to calibrate the corrected BS scheme based on the Black-Scholes scheme. In [20], the authors found that the difference between the minimum variance delta and BS delta is approximately a quadratic function of δ_{BS}:

$$\delta_{MV} = \delta_{BS} + \frac{\partial_\sigma Q_t^{(0)}(x)\big|_{\sigma=\sigma(t,x)}}{x\sqrt{T-t}}(a + b\delta_{BS} + c\delta_{BS}^2) \tag{4.23}$$

where $\partial_\sigma Q_t^{(0)}(x)\big|_{\sigma=\sigma(t,x)}$ is Vega based on implied volatility. To explore this further, we can investigate how to interpret $\partial_\sigma Q_t^{(0)}(x)$ using Black-Scholes delta and Vega. Further research can be conducted to test the performance of our approach using empirical data.

Appendix A

Lemma A.1. *For slow-varying volatility pricing, we have the following leading order approximation, for $H \in (0,1)$*

$$\theta^{\delta}_{t,T} = \delta^H \frac{1}{\Gamma(H+3/2)}(T-t)^{H+1/2} + \mathcal{O}(\delta^{2H}), \qquad D^{\delta}_{t,T} = \delta^H \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} + \mathcal{O}(\delta^{2H}) \tag{A.1}$$

Proof. We here compute $d < \psi^{\delta}, W >_t = \theta^{\delta}_{t,T} dt = \delta^H \theta_{t,T} dt$. We recall the definitions;

$$\psi^{\delta}_t = \mathbb{E}\left[\int_0^T Z^{\delta}_s - Z^{\delta}_0 ds \middle| \mathcal{F}_t\right], \qquad Z^{\delta}_t = \int_{-\infty}^t \mathcal{K}^{\delta}(t-s) dW_s$$

Consider

$$\psi^{\delta}_t = \mathbb{E}\left[\int_0^T Z^{\delta}_s - Z^{\delta}_0 ds \middle| \mathcal{F}_t\right] = \int_0^t Z^{\delta}_s ds + \int_t^T \mathbb{E}\left[Z^{\delta}_s \middle| \mathcal{F}_t\right] ds - T Z^{\delta}_0 \tag{A.2}$$

For $0 \le t \le s$

$$\begin{aligned}
\mathbb{E}\left[Z^{\delta}_s \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\int_{-\infty}^s \mathcal{K}^{\delta}(s-u) dW_u \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_{-\infty}^t \mathcal{K}^{\delta}(s-u) dW_u \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^s \mathcal{K}^{\delta}(s-u) dW_u \middle| \mathcal{F}_t\right] \\
&= \int_{-\infty}^t \mathcal{K}^{\delta}(s-u) dW_u
\end{aligned}$$

Then

$$
\begin{aligned}
\psi_t^\delta &= \int_0^t Z_s^\delta ds + \int_t^T \mathbb{E}\left[Z_s^\delta \middle| \mathcal{F}_t\right] ds - TZ_0^\delta \\
&= \int_0^t \int_{-\infty}^s \mathcal{K}^\delta(s-u) dW_u ds + \int_t^T \int_{-\infty}^t \mathcal{K}^\delta(s-u) dW_u ds - TZ_0^\delta \\
&= \int_0^t \int_{-\infty}^s \mathcal{K}^\delta(s-u) dW_u ds + \int_t^T \int_{-\infty}^s \mathcal{K}^\delta(s-u) dW_u ds + \int_t^T \int_s^t \mathcal{K}^\delta(s-u) dW_u ds - TZ_0^\delta \\
&= \int_0^T \int_{-\infty}^s \mathcal{K}^\delta(s-u) dW_u ds + \int_0^t \int_u^T \mathcal{K}^\delta(s-u) ds dW_u - TZ_0^\delta
\end{aligned}
$$

which implies that

$$
d < \psi^\delta, W >_t = \theta_{t,T}^\delta dt = \left(\int_t^T \mathcal{K}^\delta(s-t) ds\right) dt = \left(\int_0^{T-t} \mathcal{K}^\delta(s) ds\right) dt \tag{A.3}
$$

$$
d < \psi^\delta >_t = \left(\int_0^{T-t} \mathcal{K}^\delta(s) ds\right)^2 dt \tag{A.4}
$$

Since for small times $at \ll 1$

$$
\mathcal{K}(t) = \frac{1}{\Gamma(H+1/2)} \left(t^{H-1/2} + \mathcal{O}\left(t^{H+1/2}\right)\right) \tag{A.5}
$$

it implies that

$$
\begin{aligned}
\theta_{t,T}^\delta &= \int_0^{T-t} \mathcal{K}^\delta(s) ds \\
&= \delta^{1/2} \int_0^{T-t} \mathcal{K}(\delta s) ds \\
&= \delta^{1/2} \frac{1}{\Gamma(H+1/2)} \int_0^{T-t} \delta^{H-1/2} s^{H-1/2} + \mathcal{O}\left(\delta^{H+1/2} s^{H+1/2}\right) ds \\
&= \delta^H \frac{1}{\Gamma(H+3/2)} (T-t)^{H+1/2} + \mathcal{O}(\delta^{2H}) \\
&= \delta^H \theta_{t,T} + \mathcal{O}(\delta^{2H})
\end{aligned}
$$

if we define that $\theta_{t,T} = \frac{1}{\Gamma(H+3/2)}(T-t)^{H+1/2}$. Then $D_{t,T}^\delta = \delta^H \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} + \mathcal{O}(\delta^{2H})$ follows from the fact that $\partial D^{\delta}_{t,T}/\partial t = -\theta^{\delta}_{t,T}$. And we pick the leading order $D_{t,T} = \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)}$ $\square$

Lemma A.2. *For $j = 1, ..., 5$*

$$\lim_{\delta \longrightarrow 0} \delta^{-H} \mathbb{E}[|R^{(j)}_T - R^{(j)}_t|^2]^{1/2} = 0 \tag{A.6}$$

where

$$dR^{(1)}_t = \left(\sigma_0 g^{\delta}_t + \frac{1}{2}\left(\sigma^{\delta}_t - \sigma_0\right)^2\right)(x^2\partial^2_x Q^{(0)}_t)(X_t)dt \tag{A.7}$$

$$dR^{(2)}_t = \sigma_0 p_0 \left[\sigma_0\left(\sigma^{\delta}_t - \sigma_0\right) + \frac{1}{2}(\sigma^{\delta}_t - \sigma_0)^2\right](x^2\partial^2_x(x^2\partial^2_x))Q^{(0)}_t(X_t)\phi^{\delta}_t dt \tag{A.8}$$

$$dR^{(3)}_t = \sigma_0 p_0 \rho (x\partial_x(x^2\partial^2_x))Q^{(0)}_t(X_t)\left(\sigma^{\delta}_t - \sigma_0\right)\theta^{\delta}_{t,T} dt \tag{A.9}$$

$$dR^{(4)}_t = \rho\sigma_0^2 p_0 (x\partial_x(x^2\partial^2_x))Q^{(0)}_t(X_t)\mathcal{O}(\delta^{2H})dt \tag{A.10}$$

$$dR^{(5)}_T = \delta^H \rho p_0 \left[\sigma_0\left((\sigma^{\delta}_t - \sigma_0\right) + \frac{1}{2}\left(\sigma^{\delta}_t - \sigma_0\right)^2\right](x^2\partial^2_x Q^{(1)}_t)(X_t)dt \tag{A.11}$$

Proof. Consider

$$g^{\delta}_s = \sigma^{\delta}_s - \sigma_0 - F'(Z^{\delta}_0)(Z^{\delta}_s - Z^{\delta}_0) \implies |g^{\delta}_s| \leq \frac{1}{2}\|F''\|_{\infty}(Z^{\delta}_t - Z^{\delta}_0)^2 = \mathcal{O}(\delta^{2H}) \tag{A.12}$$

and

$$|\sigma^{\delta}_s - \sigma_0| = |\sigma^{\delta}_s - \sigma_0| \leq \|F'\|_{\infty}(Z^{\delta}_s - Z^{\delta}_0) + \mathcal{O}(\delta^{2H}) \tag{A.13}$$

In addition that the deterministic function $Q^{(0)}_t(x)$ satisfies:

$$\left|\partial^k_x Q^{(0)}_t(x)\right| \leq C\left(1 + \frac{1}{(T-t)^{(k-1)/2}}\right)$$

and the proof is similar to the proof of [12] Proposition 3.1. $\square$

Lemma A.3. *Let $f(t,x)$ be smooth bounded and with bounded derivatives. Let X_t and $\tilde{X}_t$*

be defined as the followings:

$$dX_t = \sigma_t^\delta X_t dW_t^*, \qquad d\tilde{X}_t = \sigma_0 \tilde{X}_t dW_t^*$$

Then

$$\lim_{\delta \longrightarrow 0} \mathbb{E}\left[\left|\mathbb{E}[f(t,X_t) - f(t,\tilde{X}_t)|\mathcal{F}_0]\right|^2\right]^{1/2} = 0 \tag{A.14}$$

Proof. Suppose $f(t,x) = h(x)$ and $T = t$ and we know that $Q_0^{(0)}(X_0) = \mathbb{E}[h(\tilde{X}_t)|\mathcal{F}_0]$, we know that:

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}[h(X_t)|\mathcal{F}_0] - \mathbb{E}[h(\tilde{X}_t)|\mathcal{F}_0]\right|^2\right]^{1/2} \\
\leq &\mathbb{E}\left[\left|\mathbb{E}[h(X_t)|\mathcal{F}_0] - Q_0^{(0)}(X_0) - \sigma_0 p_0 \phi_0^\delta (x^2\partial_x^2) Q_0^{(0)}(X_0) - \delta^H \rho p_0 Q_0^{(1)}(X_0)\right|^2\right]^{1/2} \\
&+ \mathbb{E}\left[\left|\sigma_0 p_0 \phi_0^\delta (x^2\partial_x^2) Q_0^{(0)}(X_0) + \delta^H \rho p_0 Q_0^{(1)}(X_0)\right|^2\right]^{1/2}
\end{aligned}$$

Take limit for both sides, we get the desired result. □

Lemma A.4. *The solution to*

$$\mathcal{L}_{BS}(\sigma_0) Q_t^{(1)}(x) = -\sigma_0^2 (x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(x)\theta_{t,T}, \qquad Q_T^{(1)}(x) = 0 \tag{A.15}$$

is

$$Q_t^{(1)}(X_t) = -\sigma_0^2 \left(x\partial_x(x^2\partial_x^2)\right) Q_t^{(0)}(X_t) D_{t,T}, \qquad D_{t,T} = \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} \tag{A.16}$$

Proof. Suppose we define

$$Q_t^{(1)}(X_t) = -\sigma_0^2 \left(x\partial_x(x^2\partial_x^2)\right) Q_t^{(0)}(X_t) D_{t,T}$$

for some deterministic term $D_{t,T}$.

$$\begin{aligned}
\mathcal{L}_{BS}(\sigma_0)Q_t^{(1)}(x) &= \mathcal{L}_{BS}(\sigma_0)\left[\sigma_0^2\left(x\partial_x(x^2\partial_x^2)\right)Q_t^{(0)}(X_t)D_{t,T}\right] \\
&= \sigma_0^2\left(x\partial_x(x^2\partial_x^2)\right)\mathcal{L}_{BS}(\sigma_0)Q_t^{(0)}(X_t)\times D_{t,T} + \sigma_0^2\left(x\partial_x(x^2\partial_x^2)\right)Q_t^{(0)}(X_t)\times\frac{\partial}{\partial t}D_{t,T} \\
&= \sigma_0^2\left(x\partial_x(x^2\partial_x^2)\right)Q_t^{(0)}(X_t)\times\frac{\partial D_{t,T}}{\partial t}
\end{aligned}$$

Then we can conclude that

$$\theta_{t,T} = (T-t)^{H+1/2}\frac{H+3/2}{\Gamma(H+5/2)} \implies D_{t,T} = \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} \tag{A.17}$$

□

Lemma A.5. *For $t \in (0,T)$, define:*

$$\eta_t^\delta = \delta^H\int_0^t \sigma_s^\delta - \sigma_0 ds \tag{A.18}$$

then we have

$$\limsup_{\delta\longrightarrow 0}\delta^{-H}\sup_{t\in[0,T]}\mathbb{E}\left[\left(\eta_t^\delta\right)^2\right]^{1/2} = 0 \tag{A.19}$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\left(\eta_t^\delta\right)^2\right] &= \delta^{2H}\mathbb{E}\left[\left(\int_0^t \sigma_s^\delta - \sigma_0 ds\right)^2\right] \\
&\leq \delta^{2H}\mathbb{E}\left[\int_0^t \left(\sigma_s^\delta - \sigma_0\right)^2 ds\right] \\
&\leq \delta^{2H}C\|F'\|_\infty\mathbb{E}\left[\int_0^t \left(Z_s^\delta - Z_0\right)^2 ds\right] \\
&\leq \delta^{2H}C\|F'\|_\infty\mathbb{E}\left[\int_0^t \sigma_H^2(\delta s)^{2H} + \mathcal{O}(\delta^{2H})ds\right] \\
&\leq \tilde{C}\|F'\|_\infty\delta^{2H} + \mathcal{O}(\delta^{2H})
\end{aligned}$$

□

Lemma A.6. *By Greeks formulas, we have the following two identities:*

$$\frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} x(x^2\partial_x^2) Q_t^{(0)}(x) = (x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(x) \tag{A.20}$$

$$(x\partial_x)(x\partial_x)(x^2\partial_x^2) Q_t^{(0)}(x) = \frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} x Q_t^{(1)}(x) - \frac{(x^2\partial_x^2) Q_t^{(0)}(x)}{\sigma^2(T-t)} \tag{A.21}$$

Proof. By option Greeks: For

$$d_1 = \frac{\log(x/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = \frac{\log(x/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{A.22}$$

We have

$$\begin{aligned}
(x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(x) &= 2(x^2\partial_x^2) Q_t^{(0)}(x) + (x^3\partial_x^3) Q_t^{(0)}(x) \\
&= 2(x^2\partial_x^2) Q_t^{(0)}(x) + x^3 \left[-\frac{\partial_x^2 Q_t^{(0)}(x)}{x} \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right) \right] \\
&= 2(x^2\partial_x^2) Q_t^{(0)}(x) - (x^2\partial_x^2) Q_t^{(0)}(x) \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right) \\
&= \left[1 - \frac{d_1}{\sigma\sqrt{T-t}} \right] (x^2\partial_x^2) Q_t^{(0)}(x) \\
&= x \frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} (x^2\partial_x^2 Q_t^{(0)}(x))
\end{aligned}$$

In fact,

$$x \frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} (x^2\partial_x^2 Q_t^{(0)}(x)) = x \frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\sigma(T-t)} \tag{A.23}$$

For the second identity:

$$\frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} x(x\partial_x)(x^2\partial_x^2) Q_t^{(0)}(x) = \frac{\partial^2_{x\sigma} Q^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)} x^2 \frac{\partial^2_{x\sigma} Q_t^{(0)}(x)}{\sigma(T-t)} = \frac{d_2^2 \partial_\sigma Q_t^{(0)}(x)}{\sigma^3(T-t)^2} \tag{A.24}$$

$$
\begin{aligned}
(x\partial_x)(x\partial_x)(x^2\partial_x^2)Q_t^{(0)}(x) &= (x\partial_x)\left(x\frac{\partial^2_{x\sigma}Q_t^{(0)}(x)}{\sigma(T-t)}\right)\\
&= \frac{x}{\sigma(T-t)}\partial^2_{x\sigma}Q_t^{(0)}(x) + \frac{x^2}{\sigma(T-t)}\partial^3_{x^2\sigma}Q_t^{(0)}(x)\\
&= \frac{d_2^2\partial_\sigma Q_t^{(0)}(x)}{\sigma^3(T-t)^2} - \frac{\partial_\sigma Q_t^{(0)}(x)}{\sigma^3(T-t)^2}\\
&= \frac{d_2^2\partial_\sigma Q_t^{(0)}(x)}{\sigma^3(T-t)^2} - \frac{(x^2\partial_x^2)Q_t^{(0)}(x)}{\sigma^2(T-t)}\\
&= \frac{(d_2^2-1)(x^2\partial_x^2)Q_t^{(0)}(x)}{\sigma^2(T-t)}
\end{aligned}
$$

Then we conclude:

$$
(x\partial_x)(x\partial_x)(x^2\partial_x^2)Q_t^{(0)}(x) = \frac{\partial^2_{x\sigma}Q_t^{(0)}(x)}{\partial_\sigma Q_t^{(0)}(x)}xQ_t^{(1)}(x) - \frac{(x^2\partial_x^2)Q_t^{(0)}(x)}{\sigma^2(T-t)} \tag{A.25}
$$

□

Lemma A.7. *For any $t \in [0,T]$,*

$$
E_t^{BS} = \sigma_0 p_0 \int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta - \frac{\delta^H \rho p_0}{H+3/2}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_{s,T}\sigma_s^\delta dW_s^* \tag{A.26}
$$

Then with leading order we know that

$$
Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\left(\frac{2H+2}{(H+3/2)^2}\right)\right] \tag{A.27}
$$

Proof. Consider computing $Var\left[E_t^{BS}|\mathcal{F}_0\right]$ and recall that

$$
Var\left[E_t^{HW}|\mathcal{F}_0\right] = Var\left[\sigma_0 p_0 \int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta|\mathcal{F}_0\right] \tag{A.28}
$$

By Lemma A.1:

$$Var\left[\sigma_0 p_0 \int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta|\mathcal{F}_0\right] = \delta^{2H}\sigma_0^2 p_0^2 \mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_{t,T})^2\, ds|\mathcal{F}_0\right] \tag{A.29}$$

On the other hand, we can compute the following leading order term by Lemma A.5:

$$\begin{aligned} Var\left[\frac{\delta^H\rho p_0}{H+3/2}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_{s,T}\sigma_s^\delta dW_s^*|\mathcal{F}_0\right] &= \frac{\delta^{2H}\rho^2 p_0^2}{H+3/2}\mathbb{E}\left[\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_{s,T}\sigma_s^\delta dW_s^*|\mathcal{F}_0\right] \\ &= \frac{\delta^{2H}\rho^2\sigma_0^2 p_0^2}{H+3/2}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_{t,T})^2\, ds|\mathcal{F}_0\right] \end{aligned}$$

Same, by Lemma A.5 we can compute the following covariance

$$\begin{aligned} &\mathbb{E}\left[\left(\sigma_0 p_0\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)d\psi_s^\delta\right)\left(\frac{\delta^H\rho p_0}{H+3/2}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_{s,T}\sigma_s^\delta dW_s^*\right)|\mathcal{F}_0\right] \\ =&\frac{\delta^H\rho\sigma_0^2 p_0^2}{H+3/2}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\right)^2 (\theta_{t,T})^2\, ds|\mathcal{F}_0\right] \end{aligned}$$

Thus,

$$\begin{aligned} Var\left[E_t^{BS}|\mathcal{F}_0\right] &= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-2\rho^2\frac{1}{H+3/2}+\rho^2\frac{1}{(H+3/2)^2}\right] \\ &= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\left(\frac{2(H+3/2)}{(H+3/2)^2}-\frac{1}{(H+3/2)^2}\right)\right] \\ &= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\left(\frac{2H+2}{(H+3/2)^2}\right)\right] \end{aligned}$$

□

Lemma A.8. *For any $t \in [0,T]$,*

$$E_t^{BS} = \int_0^t (x^2\partial_x^2 Q_s^{(0)})(X_s)d\psi_s^\epsilon - \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\sigma_s^\epsilon dW_s^* \tag{A.30}$$

Then with leading order we know that

$$Var\left[E_t^{BS}|\mathcal{F}_0\right] = Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1-\rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\left(\frac{2H}{(H+1/2)^2}\right)\right] \tag{A.31}$$

Proof. Then the hedging cost is:

$$E_t^{BS} = \int_0^t (x^2\partial_x^2 Q_s^{(0)})(X_s)d\psi_s^\epsilon - \frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\sigma_s^\epsilon dW_s^* \tag{A.32}$$

Then we compute $Var[E_t^{BS}|\mathcal{F}_0]$, then with leading order computation by Lemma B.2 in [14]

$$\begin{aligned}
Var\left[\int_0^t (x^2\partial_x^2 Q_s^{(0)})(X_s)d\psi_s^\epsilon|\mathcal{F}_0\right] &= \mathbb{E}\left[\int_0^t \left((x^2\partial_x^2 Q_s^{(0)})(X_s)\right)^2(\theta_t^\epsilon)^2\,ds\right] \\
&= \mathbb{E}\left[\int_0^t \left((x^2\partial_x^2 Q_s^{(0)})(X_s)\right)^2\left(\epsilon^{1-H}\theta_t+\tilde{\theta}_t^\epsilon\right)^2 ds\right] \\
&= \epsilon^{2-2H}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2 Q_s^{(0)})(X_s)\theta_t\right)^2 ds\right]
\end{aligned}$$

By Lemma B.4, we have the following leading order computation:

$$\begin{aligned}
Var\left[\frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\sigma_s^\epsilon dW_s^*|\mathcal{F}_0\right] &= \mathbb{E}\left[\left(\frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\sigma_s^\epsilon dW_s^*\right)^2\right] \\
&= \frac{\epsilon^{2-2H}\tilde{\sigma}^2\rho^2}{\bar{\sigma}^2(H+1/2)^2}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\right)^2 ds\right]
\end{aligned}$$

And by Lemma B.4 in [14], we have the following leading order computation:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t (x^2\partial_x^2 Q_s^{(0)})(X_s)d\psi_s^\epsilon\right)\left(\frac{\epsilon^{1-H}\tilde{\sigma}\rho}{\bar{\sigma}^2(H+1/2)}\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t\sigma_s^\epsilon dW_s^*\right)|\mathcal{F}_0\right] \\
=&\frac{\epsilon^{1-H}\tilde{\sigma}^2\rho}{\bar{\sigma}^2(H+1/2)}\mathbb{E}\left[\left(\int_0^t (x^2\partial_x^2 Q_s^{(0)})(X_s)d\psi_s^\epsilon\right)\left(\int_0^t (x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_t dW_s^*\right)|\mathcal{F}_0\right] \\
=&\frac{\epsilon^{1-H}\tilde{\sigma}^2\rho^2}{\bar{\sigma}^2(H+1/2)}\mathbb{E}\left[\int_0^t \left((x^2\partial_x^2)Q_s^{(0)}(X_s)\theta_s\right)^2 ds\right]
\end{aligned}$$

Then,

$$\begin{aligned}
Var\left[E_t^{BS}|\mathcal{F}_0\right] &= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - 2\rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\frac{\Gamma(H+1/2)}{\Gamma(H+3/2)} + \rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\frac{\Gamma(H+1/2)^2}{\Gamma(H+3/2)^2}\right] \\
&= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - 2\rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\frac{1}{(H+1/2)} + \rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\frac{1}{(H+1/2)^2}\right] \\
&= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\left(\frac{2(H+1/2)}{(H+1/2)^2} - \frac{1}{(H+1/2)^2}\right)\right] \\
&= Var\left[E_t^{HW}|\mathcal{F}_0\right]\left[1 - \rho^2\frac{\tilde{\sigma}^2}{\bar{\sigma}^2}\left(\frac{2H}{(H+1/2)^2}\right)\right]
\end{aligned}$$

$\square$

Appendix B

In this Appendix B, we prove the Proposition 2.3, where the author omitted the proof in [12]. We need some asymptotic expansion results for the hedging analysis.

Proposition B.1. *When δ is small, let $\sigma_0 = F(Z_0^\delta)$, $p_0 = F'(Z_0^\delta)$, then*

$$\mathbb{E}[h(X_T)|\mathcal{F}_t] = M_t = Q_t(X_t) + \mathcal{O}(\delta^{2H}) \tag{B.1}$$

where

$$Q_t(x) = Q_t^{(0)}(x) + \sigma_0 p_0 \phi_t^\delta (x^2 \partial_x^2 Q_t^{(0)}(x)) + \delta^H \rho p_0 Q_t^{(1)}(x) \tag{B.2}$$

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T Z_s^\delta - Z_0^\delta ds \big| \mathcal{F}_t\right] \tag{B.3}$$

and

$$Q_t^{(1)}(x) = \sigma_0^2 x \partial_x (x^2 \partial_x^2 Q_t^{(0)}(x)) D_{t,T}, \qquad D_{t,T} = \frac{(T-t)^{H+3/2}}{\Gamma(H+5/2)} \tag{B.4}$$

Proof. Consider

$$dQ_t^{(0)}(X_t) = \partial_t Q_t^{(0)}(X_t)dt + \frac{1}{2}(\sigma_t^\delta)^2 (x^2 \partial_x^2 Q_t^{(0)})(X_t)dt + (x\partial_x Q_t^{(0)})(X_t)\sigma_t^\delta dW_t^* \tag{B.5}$$

$$= \frac{1}{2}\left((\sigma_t^\delta)^2 - \sigma_0^2\right)(x^2 \partial_x^2 Q_t^{(0)})(X_t)dt + (x\partial_x Q_t^{(0)})(X_t)\sigma_t^\delta dW_t^* \tag{B.6}$$

And as we set up $\sigma_t^\delta = F(Z_t^\delta)$, $\sigma_0 = F(Z_0^\delta)$, $p_t^\delta = F'(Z_t^\delta)$, $p_0 = F'(Z_0^\delta)$

$$\sigma_t^\delta = F(Z_t^\delta) = F(Z_0^\delta) + F'(Z_0^\delta)(Z_t^\delta - Z_0^\delta) + \sigma_t^\delta - \sigma_0 - F'(Z_0^\delta)(Z_t^\delta - Z_0^\delta) \tag{B.7}$$

$$= \sigma_0 + p_0(Z_t^\delta - Z_0^\delta) + \sigma_t^\delta - \sigma_0 - p_0(Z_t^\delta - Z_0^\delta) \tag{B.8}$$

Define $g_t^\delta = \sigma_t^\delta - \sigma_0 - p_0(Z_t^\delta - Z_0^\delta)$ we have

$$\sigma_t^\delta = \sigma_0 + p_0(Z_t^\delta - Z_0^\delta) + g_t^\delta \tag{B.9}$$

it implies that

$$(\sigma_t^\delta)^2 = (\sigma_0 + p_0(Z_t^\delta - Z_0^\delta) + g_t^\delta)^2 \tag{B.10}$$

$$= (\sigma_0)^2 + 2\sigma_0 \left(p_0(Z_t^\delta - Z_0^\delta) + g_t^\delta\right) + (\sigma_t^\delta - \sigma_0)^2 \tag{B.11}$$

If we define that

$$(x\partial_x Q_t^{(0)})(X_t)\sigma_t^\delta dW_t^* = dN_t^{(0)} \tag{B.12}$$

to simplify the notation. Then

$$\begin{aligned}
dQ_t^{(0)}(X_t) =& \frac{1}{2}\left[2\sigma_0 \left(p_0(Z_t^\delta - Z_0^\delta) + g_t^\delta\right) + \left(\sigma_t^\delta - \sigma_0\right)^2\right](x^2\partial_x^2 Q_t^{(0)})(X_t)dt + dN_t^{(0)} \\
=& \sigma_0 \left(p_0(Z_t^\delta - Z_0^\delta) + g_t^\delta\right)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2 (x^2\partial_x^2 Q_t^{(0)})(X_t)dt + dN_t^{(0)} \\
=& \sigma_0 p_0(Z_t^\delta - Z_0^\delta)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt + \left(\sigma_0 g_t^\delta + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2\right)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt + dN_t^{(0)} \\
=& \sigma_0 p_0(Z_t^\delta - Z_0^\delta)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt + dR_t^{(1)} + dN_t^{(0)}
\end{aligned}$$

Here we define

$$dR_t^{(1)} = \left(\sigma_0 g_t^\delta + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2\right)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt \tag{B.13}$$

to simplify the notation. Then we introduce

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T Z_s^\delta - Z_0^\delta ds \big| \mathcal{F}_t\right], \qquad \psi_t^\delta = \mathbb{E}\left[\int_0^T Z_s^\delta - Z_0^\delta ds \big| \mathcal{F}_t\right] \tag{B.14}$$

where ψ_t^δ is a martingale. Now

$$(Z_t^\delta - Z_0^\delta)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt = (x^2\partial_x^2 Q_t^{(0)})(X_t)d\psi_t^\delta - (x^2\partial_x^2 Q_t^{(0)})(X_t)d\phi_t^\delta \tag{B.15}$$

It implies that

$$\begin{aligned} dQ_t^{(0)}(X_t) =& \sigma_0 p_0 (Z_t^\delta - Z_0^\delta)(x^2\partial_x^2 Q_t^{(0)})(X_t)dt + dR_t^{(1)} + dN_t^{(0)} \\ =& \sigma_0 p_0 (x^2\partial_x^2 Q_t^{(0)})(X_t)d\psi_t^\delta - \sigma_0 p_0 (x^2\partial_x^2 Q_t^{(0)})(X_t)d\phi_t^\delta + dR_t^{(1)} + dN_t^{(0)} \end{aligned}$$

By Ito's formula

$$\begin{aligned} d[(x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta] =& (x^2\partial_x^2 Q_t^{(0)})(X_t)d\phi_t^\delta \\ &+ (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\phi_t^\delta dW_t^* \\ &+ \frac{1}{2}(x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\phi_t^\delta dt \\ &+ (x_2\partial_x^2\partial_t)Q_t^{(0)}(X_t)\phi_t^\delta dt \\ &+ (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta d < \phi^\delta, W^* >_t \\ =& (x^2\partial_x^2 Q_t^{(0)})(X_t)d\phi_t^\delta \\ &+ (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\phi_t^\delta dW_t^* \\ &+ \left[\sigma_0\left(\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}(\sigma_t^\delta - \sigma_0)^2\right](x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)\phi_t^\delta dt \\ &+ (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta \rho d < \psi^\delta, W >_t \end{aligned}$$

Then

$$
\begin{aligned}
d\left[Q_t^{(0)}(X_t)+\sigma_0 p_0(x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta\right] =&\sigma_0 p_0(x^2\partial_x^2 Q_t^{(0)})(X_t)d\psi_t^\delta \\
&+\sigma_0 p_0(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\phi_t^\delta dW_t^* \\
&+\sigma_0 p_0\left[\sigma_0\left(\sigma_t^\delta-\sigma_0\right)+\frac{1}{2}(\sigma_t^\delta-\sigma_0)^2\right](x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)\phi_t^\delta dt \\
&+\sigma_0 p_0(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\rho d<\psi^\delta,W>_t \\
&+dR_t^{(1)}+dN_t^{(0)}
\end{aligned}
$$

As we define the martingale term:

$$
\sigma_0 p_0 dN_t^{(1)}=\sigma_0 p_0(x^2\partial_x^2 Q_t^{(0)})(X_t)d\psi_t^\delta+\sigma_0 p_0(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\phi_t^\delta dW_t^* \tag{B.16}
$$

$$
\begin{aligned}
d\left[Q_t^{(0)}(X_t)+\sigma_0 p_0(x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta\right] =&\sigma_0 p_0\left[\sigma_0\left(\sigma_t^\delta-\sigma_0\right)+\frac{1}{2}(\sigma_t^\delta-\sigma_0)^2\right](x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)\phi_t^\delta dt \\
&+\sigma_0 p_0(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\rho d<\psi^\delta,W>_t \\
&+dR_t^{(1)}+dN_t^{(0)}+\sigma_0 p_0 dN_t^{(1)}
\end{aligned}
$$

To simplify the notation, we furthermore denote that

$$
dR_t^{(2)} =\sigma_0 p_0\left[\sigma_0\left(\sigma_t^\delta-\sigma_0\right)+\frac{1}{2}(\sigma_t^\delta-\sigma_0)^2\right](x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)\phi_t^\delta dt \tag{B.17}
$$

to simplify the notation. Thus,

$$
\begin{aligned}
d\left[Q_t^{(0)}(X_t)+\sigma_0 p_0(x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta\right] =&\sigma_0 p_0(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\delta\rho d<\psi^\delta,W>_t \\
&+dR_t^{(1)}+dR_t^{(2)}+dN_t^{(0)}+\sigma_0 p_0 dN_t^{(1)}
\end{aligned}
$$

We first denote this to be $d<\psi^\delta,W>_t=\theta_{t,T}^\delta dt$ where $\theta_{t,T}^\delta$ is deterministic. By Lemma A.1

we know that $\theta^{\delta}_{t,T} = \delta^H \theta_{t,T} + \mathcal{O}(\delta^{2H})$. Then we can rewrite by (5):

$$
\begin{aligned}
d\left[Q_t^{(0)}(X_t) + \sigma_0 p_0 (x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^{\delta}\right] =& \sigma_0 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\left(\sigma_0 + p_0(Z_t^{\delta} - Z_0^{\delta}) + g_t^{\delta}\right)\rho\theta^{\delta}_{t,T}dt \\
&+ dR_t^{(1)} + dR_t^{(2)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} \\
=& \sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\rho\theta^{\delta}_{t,T}dt \\
&+ \sigma_0 p_0 \rho (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\left(\sigma_t^{\delta} - \sigma_0\right)\theta^{\delta}_{t,T}dt \\
&+ dR_t^{(1)} + dR_t^{(2)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} \\
=& \delta^H \sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\rho\theta_{t,T}dt \\
&+ \rho\sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\mathcal{O}(\delta^{2H})dt \\
&+ dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} \\
=& \delta^H \sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\rho\theta_{t,T}dt \\
&+ dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dR_t^{(4)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)}
\end{aligned}
$$

where

$$
dR_t^{(3)} = \sigma_0 p_0 \rho (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\left(\sigma_t^{\delta} - \sigma_0\right)\theta^{\delta}_{t,T}dt \tag{B.18}
$$

$$
dR_t^{(4)} = \rho\sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)\mathcal{O}(\delta^{2H})dt \tag{B.19}
$$

By Ito's Lemma:

$$
dQ_t^{(1)}(X_t) = \mathcal{L}_{BS}(\sigma_0)Q_t^{(1)}(X_t)dt + \left[\sigma_0\left(\sigma_t^{\delta} - \sigma_0\right) + \frac{1}{2}\left(\sigma_t^{\delta} - \sigma_0\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t)dt + (x\partial_x Q_t^{(1)})(X_t)\sigma_t^{\delta}dW_t^* \tag{B.20}
$$

We define that:

$$
\mathcal{L}_{BS}(\sigma_0)Q_t^{(1)}(x) = -\sigma_0^2(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(x)\theta_{t,T}, \qquad Q_T^{(1)}(x) = 0 \tag{B.21}
$$

and

$$dN_t^{(2)} = (x\partial_x Q_t^{(1)})(X_t)\sigma_t^\delta dW_t^* \tag{B.22}$$

Then

$$dQ_t^{(1)}(X_t) = -\sigma_0^2(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(x)\theta_{t,T}dt + \left[\sigma_0\left(\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t)dt + dN_t^{(2)} \tag{B.23}$$

which implies that

$$\begin{aligned}
\delta^H \rho p_0 dQ_t^{(1)}(X_t) =& \delta^H \rho p_0 \mathcal{L}_{BS}(\sigma_0) Q_t^{(1)}(X_t) \\
&+ \delta^H \rho p_0 \left[\sigma_0\left((\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t)dt \\
&+ \delta^H \rho p_0 dN_t^{(2)} \\
=& -\delta^H \rho p_0 \sigma_0^2(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(x)\theta_{t,T}dt \\
&+ \delta^H \rho p_0 \left[\sigma_0\left((\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\sigma_t^\delta - \sigma_0\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t)dt \\
&+ \delta^H \rho p_0 dN_t^{(2)}
\end{aligned}$$

Then the sum is

$$
\begin{aligned}
d[Q_t^{(0)}(X_t) + \sigma_0 p_0 (x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta + \delta^H \rho p_0 Q_t^{(1)}(X_t)] \\
=&\delta^H \rho \sigma_0^2 p_0 (x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(X_t)\theta_{t,T} dt \\
&+ dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dR_t^{(4)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} \\
&- \delta^H \rho p_0 \sigma_0^2 (x\partial_x(x^2\partial_x^2)) Q_t^{(0)}(x)\theta_{t,T} dt \\
&+ \delta^H \rho p_0 \left[\sigma_0 \left((\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\tilde{g}_t^\delta\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t) dt \\
&+ \delta^H \rho p_0 dN_t^{(2)} \\
=& + dR_t^{(1)} + dR_t^{(2)} + dR_t^{(3)} + dR_t^{(4)} + dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} \\
&+ \delta^H \rho p_0 \left[\sigma_0 \left((\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\tilde{g}_t^\delta\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t) dt \\
&+ \delta^H \rho p_0 dN_t^{(2)}
\end{aligned}
$$

And we finally arrive that:

$$
d[Q_t^{(0)}(X_t) + \sigma_0 p_0 (x^2\partial_x^2 Q_t^{(0)})(X_t)\phi_t^\delta + \delta^H \rho p_0 Q_t^{(1)}(X_t)] = dN_t^{(0)} + \sigma_0 p_0 dN_t^{(1)} + \delta^H \rho p_0 dN_t^{(2)} + \sum_{i=1}^{5} dR_t^{(i)} \tag{B.24}
$$

where we define:

$$
dR_T^{(5)} = \delta^H \rho p_0 \left[\sigma_0 \left((\sigma_t^\delta - \sigma_0\right) + \frac{1}{2}\left(\tilde{g}_t^\delta\right)^2\right](x^2\partial_x^2 Q_t^{(1)})(X_t) dt \tag{B.25}
$$

Thus, we know that

$$
M_t = \mathbb{E}[h(X_T)|\mathcal{F}_t] = \mathbb{E}[N_T|\mathcal{F}_t] + \mathbb{E}[R_T|\mathcal{F}_t] = Q_t(X_t) + \mathbb{E}[R_T - R_t|\mathcal{F}_t] \tag{B.26}
$$

In appendix Lemma A.2, we prove that

$$\lim_{\delta \longrightarrow 0} \mathbb{E}\left[\left(\int_t^T d(R_s^{(1)} + R_s^{(2)} + R_s^{(3)} + R_s^{(4)} + R_s^{(5)})\right)^2\right]^{1/2} = 0$$

Then we can conclude the asymptotic formula (B.2) $\square$

Appendix C

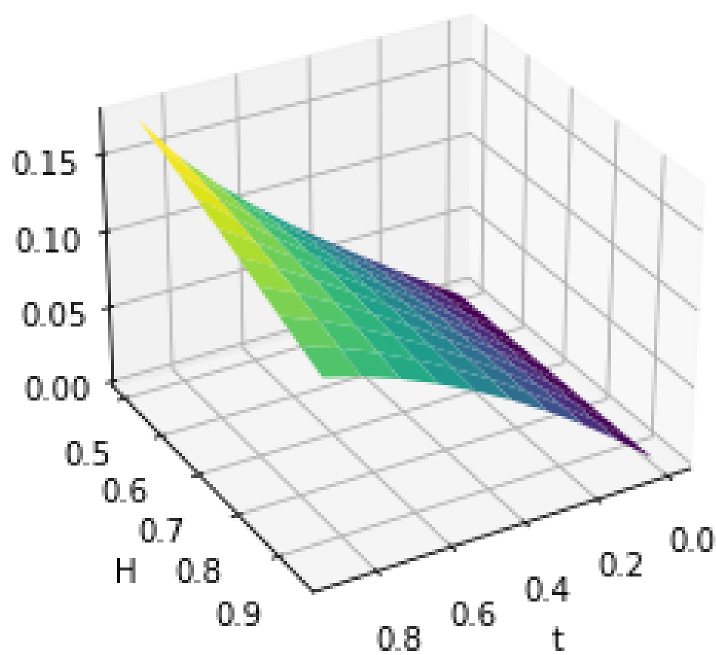

Figure C.1: This plot depicts the asymptotic approximation of the HW scheme (3.51) for the fast-varying long-memory case, as a function of time t and Hurst exponent H. The option is assumed to be at the money, and $\bar{\sigma} = 0.2$, while all other parameters are set to 1. We observe that the curves increase as time to maturity approaches, which aligns with market expectations. It is noteworthy that the roughness of the volatility paths has a diminishing effect on the approximation.

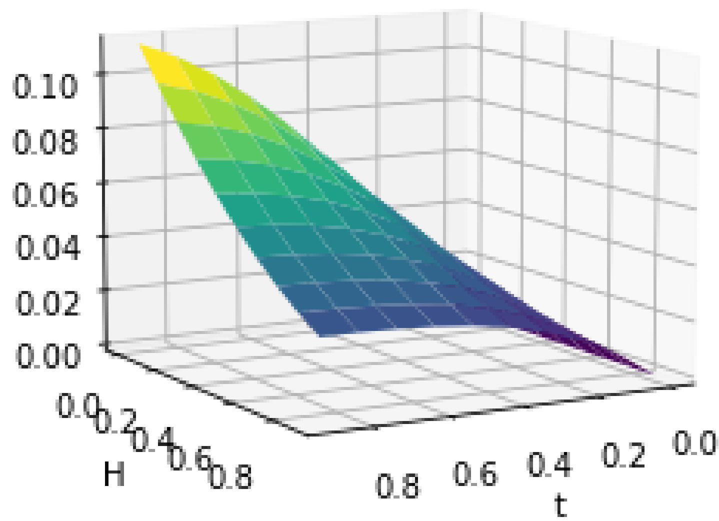

Figure C.2: This plot depicts the asymptotic approximation of the HW scheme (3.119) in the slow-varying volatility case. It shows the function as a plot of time t and Hurst exponent H, with the option being at the money and $\sigma_0 = 0.2$, while all other parameters are set to 1. As expected in the market, we observe that the curves increase as the time to maturity approaches. Additionally, we can see that the curves decrease for H values between $(0, 1)$, without splitting the case into $H < 0.5$ or $H > 0.5$.

Bibliography

[1] T. Bollerslev, D. Osterrieder, N. Sizova, and G. Tauchen. Risk and return: Long-run relations, fractional cointegration, and return predictability. *Journal of Financial Economics*, 108(2):409–424, 2013.

[2] G. Bouzianis and L. P. Hughston. Optimal hedging in incomplete markets. *Applied Mathematical Finance*, 27:265 – 287, 2020.

[3] F. J. Breidt, N. Crato, and P. J. F. de Lima. The detection and estimation of long memory in stochastic volatility. *Journal of Econometrics*, 83:325–348, 1998.

[4] A. Chronopoulou and F. Viens. Estimation and pricing under long-memory stochastic volatility. *Annals of Finance*, 8:379–403, 2012.

[5] A. Chronopoulou and F. Viens. Stochastic volatility and option pricing with long-memory in discrete and continuous time. *Quantitative Finance*, 12:635 – 649, 2012.

[6] R. Cont. Long range dependence in financial markets. *Fractals in Engineering*, pages 159–179, 2005.

[7] D. Duffie and H. Richardson. Mean-variance hedging in continuous time. *The Annals of Applied Probability*, 1(1):1–15, 1991.

[8] J.-P. Fouque, G. Papanicolaou, and R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, 2000.

[9] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*. Cambridge University Press, 2011.

[10] M. Fukasawa. Volatility has to be rough. *Quantitative Finance*, 21:1 – 8, 2020.

[11] H. Funahashi and M. Kijima. Does the hurst index matter for option prices under fractional volatility? *Annals of Finance*, 13:55–374, 2017.

[12] J. Garnier and K. Solna. Correction to black-scholes formula due to fractional stochastic volatility. *SIAM Math. Finance*, 8:560–588, 2017.

[13] J. Garnier and K. Solna. Option pricing under fast-varying and rough stochastic volatility. *Annals of Finance*, 14:489–516, 2018.

[14] J. Garnier and K. Solna. Option pricing under fast-varying long-memory stochastic volatility. *Math. Finance*, 29:39–83, 2019.

[15] J. Garnier and K. Solna. Implied volatility structure in turbulent and long-memory markets. In *Frontiers in Applied Mathematics and Statistics*, 2020.

[16] J. Garnier and K. Solna. Optimal hedging under fast-varying stochastic volatility. *SIAM Math. Finance*, 11:274–320, 2020.

[17] J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *Quantitative Finance*, 18:933–949, 2018.

[18] T. Hayashi and P. A. Mykland. Evaluating hedging errors: An asymptotic approach. *Mathematical Finance*, 15:309–343, 2005.

[19] C. Hipp and M. I. Taksar. Hedging in incomplete markets and optimal control. 2005.

[20] J. Hull and A. White. Optimal delta hedging for options. *Journal of Banking Finance*, 82:180–190, 2017.

[21] G. Oh, S. Kim, and C. Eom. Long-term memory and volatility clustering in high-frequency price changes. *Physica A-statistical Mechanics and Its Applications*, 387:1247–1254, 2008.

[22] W. Poklewski-Koziell. Stochastic volatility models: calibration, pricing and hedging. 2012.

[23] M. Schweizer. Mean variance for general claims. *The Annals of Applied Probability*, 2(1):171–179, 1992.