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Common zeroes of families of smooth vector fields on surfaces

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Abstract

Let Y and X denote C^k vector fields on a possibly noncompact surface with empty boundary, $1 \leq k < \infty$. Say that Y tracks X if the dynamical system it generates locally permutes integral curves of X . Let K be a locally maximal compact set of zeroes of X .

Theorem. Assume the Poincaré-Hopf index of X at K is nonzero, and the k -jet of X at each point of K is nontrivial. If \mathfrak{g} is a supersolvable Lie algebra of C^k vector fields that track X , then the elements of \mathfrak{g} have a common zero in K .

Applications are made to attractors and transformation groups.

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1 Introduction

M denotes a metrizable real analytic surface with empty boundary. The vector space of vector fields on M is $\mathcal{V}(M)$, topologized by uniform convergence on compact sets. The subspace of C^r vector fields is $\mathcal{V}^r(M)$. Here $r \in \mathbb{N}_+$ (the set of positive integers), or $r = \infty$ (infinitely differentiable), or $r = \omega$ (analytic).

X denotes a vector field on M , with zero set $Z(X)$. A compact set $K \subset Z(X)$ is a *block* of zeroes for X , or an *X -block*, if it has a precompact open neighborhood $U \subset M$ such that $Z(X) \cap \overline{U} = K$. We say U is *isolating* for X and for (X, K) .

The *index* of the X -block K is the integer $i_K(X) := i(X, U)$ defined as the Poincaré-Hopf index [20, 13] of any sufficiently close approximation to X having only finitely many zeroes in U . This number is independent of U , and is stable under perturbations of X : If $Y \in \mathcal{VM}$ is sufficiently close to X then U is isolating for Y , and $i(Y, U) = i(X, U)$ (Proposition 2.2).¹

K is *essential* if $i_K(X) \neq 0$, which every isolating neighborhood of K meets $Z(X)$. This powerful condition implies that if Y is sufficiently close to X then $Z(Y) \cap U \neq \emptyset$.

We say Y *tracks* X provided $X, Y \in \mathcal{V}^1(M)$, and the corresponding local flows $\Phi^Y = \{\Phi_t^Y\}_{t \in \mathbb{R}}$ and $\Phi^X = \{\Phi_t^X\}_{t \in \mathbb{R}}$ have the following property: For each $t \in \mathbb{R}$ the C^1 diffeomorphism $\Phi_t^Y: \mathcal{D}_t \approx \mathcal{R}_t$ maps orbits of $X|_{\mathcal{D}_t}$ to orbits of $X|_{\mathcal{R}_t}$. Equivalently: There exists $f: M \setminus Z(X) \rightarrow \mathbb{R}$ such that $X_p \neq 0 \implies [Y, X] = f(p)X_p$ (see HIRSCH, [12, Prop. 2.4] or [10, Prop. 2.3]).²

If X spans an ideal in a Lie algebra $\mathfrak{g} \subset \mathcal{V}^k(M)$, every element of \mathfrak{g} tracks X . When X is C^∞ , the set of $Y \in \mathcal{V}^\infty(M)$ that track X is an infinite dimensional Lie algebra.

Statement of the main results

Throughout the rest of this article we assume:

- $X \in \mathcal{V}^k(M)$, $k \in \mathbb{N}_+$.

X has *order* $\text{ord}_p(X) := j \in \{1, \dots, k\}$ at $p \in Z(X)$ if j is the smallest number in $\{1, \dots, k\}$ such that the j -jet of X at p is nontrivial. In other words: Some (and hence every) C^{k+1} chart $M \supset W' \approx W \subset \mathbb{R}^2$ on M , centered at p , represents $X|_W$ by a C^k map $F: W \rightarrow \mathbb{R}^2$ whose partial derivatives at the origin satisfy

$$F^{(i)}(0) = 0 \text{ for } i = 0, \dots, j-1, \quad F^{(j)}(0) \neq 0. \quad (1)$$

If no integer j has this property, X is *k -flat* at p .

When X is analytic and nontrivial on a neighborhood of a continuum $L \subset Z(X)$, the order of X is constant on L .

Theorem 1.1. *Assume $X, Y \in \mathcal{V}^k(M)$ have the following properties:*

- $K \subset Z(X)$ is an essential X -block,
- X is not k -flat at any point of K ,
- Y tracks X .

¹Equivalently: $i(X, U)$ is the intersection number of $X|_U$ with the zero section of the tangent bundle (C. BONATTI [3]). If X is generated by a smooth local flow ϕ then $i(X, U)$ equals the fixed-point index $I(\phi_t|_U)$ of A. DOLD [5] for sufficiently small $t > 0$.

²Article [10] is a preliminary version of [12].

Then $Z(Y) \cap K \neq \emptyset$.

When X and Y are commuting analytic vector fields, this is a special case of a remarkable theorem of C. BONATTI [3]— the inspiration for the present paper.

Theorem 1.1 is proved by demonstrating the strong form of the contrapositive stated below.

A *line field* Λ on a set $N \subset M$ is a (continuous) section $p \mapsto \Lambda_p$ of the fibre bundle over N whose fibre over $p \in N$ is the circle of unoriented lines through the origin in the tangent space $T_p(M)$. If $Y_p \in \Lambda_p$ for all $p \in N$ then Λ *controls* Y in N .

Theorem 1.2. *Assume:*

- (a) $X \in \mathcal{V}^k(M)$ is not k -flat at any point of the X -block K ,
- (b) $Y \in \mathcal{V}^k(M)$ tracks X ,
- (c) $Z(Y) \cap K = \emptyset$,
- (d) $U \subset M$ is an isolating neighborhood for (X, K) .

Then:

- (i) $i_K(X) = 0$.
- (ii) K has only finitely many components, and each component is a C^k -embedded circle.
- (iii) X is controlled by a unique line field in U .
- (iv) X has index zero at each component of K .
- (v) X can be C^k -approximated by vector fields that have no zeroes in U and agree with X outside U .

The is in Section 3.

Applications

Let $\mathfrak{g} \subset \mathcal{V}^k(M)$ denote a Lie algebra— a linear subspace closed under Lie brackets. The zero set of \mathfrak{g} is defined as $Z(\mathfrak{g}) := \bigcap_{Y \in \mathfrak{g}} Z(Y)$. If every $Y \in \mathfrak{g}$ tracks X , then \mathfrak{g} *tracks* X . We call \mathfrak{g} *supersolvable* if it is faithfully represented by upper triangular real matrices.

Theorem 1.3. *Assume:*

- (a) K is an essential X -block,
- (b) X is not k -flat at any point of K ,
- (c) $\mathfrak{g} \subset \mathcal{V}^k(M)$ is a supersolvable Lie algebra tracking X .

Then $Z(\mathfrak{g}) \cap K \neq \emptyset$.

Related theorems and counterexamples are discussed in HIRSCH [12].

Theorem 1.4. *Suppose the local flow of $X \in \mathcal{V}^\omega(M)$ has a compact attractor $P \subset M$ with Euler characteristic $\chi(P) \neq 0$. Then there exists $k \in \mathbb{N}_+$ with the following property: If $\mathfrak{h} \subset \mathcal{V}^k(M)$ is a supersolvable Lie algebra that tracks X , then $Z(\mathfrak{h}) \cap Z(X) \cap P \neq \emptyset$.*

Proof. If $P = M$ then M is a closed surface and $\chi(M) \neq 0$. Therefore $Z(X)$ is an essential X -block by Poincaré's Theorem [20], and the conclusion follows from Theorem 1.2.

Suppose $P \neq M$. The basin of attraction of P contains a smooth compact surface N with boundary such that Y is inwardly transverse to ∂N . The interior $N_0 := N \setminus \partial N$ is a precompact open set which is positively invariant under Φ^Y (F. WILSON [25, Th. 2.2]). Therefore

$$t > s \geq 0 \implies \Phi_t^Y(N_0) \subset \Phi_s^Y(N_0), \quad \bigcap_{t \geq 0} \Phi_t^Y(N_0) = P.$$

The inclusion maps $P \hookrightarrow N_0 \hookrightarrow N$ induce isomorphisms of Čech cohomology groups, hence $\chi(N_0) \neq 0$. The conclusion follows from Theorem 1.3 applied to $X := Y|_{N_0}$ and the Lie algebra $\mathfrak{g} \subset \mathcal{V}^k(N_0)$ comprising the restrictions of the vector fields in \mathfrak{h} to N_0 . ■

Example 1.5. Assume $P \subset \mathbb{R}^2$ is a compact global attractor for $X \in \mathcal{V}^\omega(\mathbb{R}^2)$ and $\mathfrak{h} \subset \mathcal{V}^k(\mathbb{R}^2)$ is a supersolvable Lie algebra tracking Y . Then:

- $Z(\mathfrak{h}) \cap Z(X) \cap P \neq \emptyset$.

Proof. $\chi(P) \neq 0$ because \mathbb{R}^2 is contractible and P is a global attractor, so Theorem 1.4 yields the conclusion. ■

Corollary 1.6. *Let G be a connected Lie group whose Lie algebra is supersolvable, with an effective C^∞ action on a compact surface M . If $\chi(M) \neq 0$ and the action is analytic on a normal 1-dimensional Lie subgroup, then G fixes a point of M .*

The special case in which M is compact and G acts analytically is due to HIRSCH & WEINSTEIN [7].

Proof. The action of G on M induces an isomorphism θ from the Lie algebra \mathfrak{a} of G onto a subalgebra $\mathfrak{g} \subset \mathcal{V}^\infty(M)$. Let $Y \in \mathfrak{a}$ span the Lie algebra of H and set $\theta(Y) = X \in \mathfrak{g}$. Then $X \in \mathcal{V}^\omega(M)$ and \mathfrak{g} tracks X , whence the conclusion from Theorem 1.4. ■

Related results on Lie group actions and Lie algebras of vector fields can be found in the articles [1, 2, 4, 8, 9, 11, 15, 17, 18, 19, 22, 23, 24].

2 Index calculations

Proposition 2.1. *Assume $Y, X \in \mathcal{V}^k(M)$ and Y tracks X .*

(i) $Z(X)$ is invariant under Φ^Y .

(ii) If $p, q \in Z(X)$ are in the same orbit of Φ^Y , then $\text{ord}_p(X) = \text{ord}_q(X)$. ■

Proof. Follows from the definition of tracking. ■

These properties of the index function are crucial:

Proposition 2.2 (STABILITY). *Let $U \subset M$ be isolating for X .*

(a) If $i(X, U) \neq 0$ then $Z(X) \cap U \neq \emptyset$.

(b) If Y is sufficiently close to X then $i(Y, U) = i(X, U)$.

(c) Let $\{X^t\}_{t \in [0,1]}$ be a deformation of X . If each X^t is nonsingular on the frontier of U , then $i(X^t, U) = i(X, U)$.

Proof. This is Theorem 3.9 of [12]. ■

Proposition 2.3. *Assume $Y, Y' \in \mathcal{V}(M)$ and $U \subset M$ is isolating for both Y and Y' . Assume $N := \overline{U}$ is a compact C^1 surface such that Y_p and Y'_p are linearly dependent at all $p \in \partial N$. Then $i(Y, U) = i(Y', U)$.*

Proof. This consequence of Proposition 2.2 is a special case of [10, Prop. 3.12] or [12, Prop. 3.11]. ■

Proposition 2.4. *Let $K \subset M$ be a block of zeroes for $X \in \mathcal{V}^k(M)$, and $U \subset M$ an isolating neighborhood for (X, K) . If $X|U$ is controlled by a line field Λ on U , then $i_K(X) = 0$.*

Proof. By shrinking U slightly, we assume $N := \overline{U}$ is a compact C^1 surface.

Suppose Λ is an orientable line field. Then Λ controls a nonsingular vector field Y on N , and $i(Y, U) = 0$ because $Z(Y) = \emptyset$. Evidently U is isolating for both X and Y , and X_p, Y_p are linearly dependent at all $p \in N$. Proposition 2.3 implies $i(X, U) = i(Y, U) = 0$, hence $i_K(X) = 0$.

Now suppose Λ is nonorientable. There is a double covering $\pi: \tilde{V} \rightarrow V$ of an open neighborhood $V \subset M$ of N , isolating for (X, K) , such that Λ lifts to an orientable line field on \tilde{V} . The orientable case shows that the vector field \tilde{X} on \tilde{V} that projects to $X|V$ under π has index zero in \tilde{V} .

Fix $X_1 \in \mathcal{V}(V)$ with $Z(X_1)$ finite, and such that the sum of the Poincaré-Hopf indices of the zeroes of X_1 equals $i(X, V)$. Define $\tilde{X}_1 \in \mathcal{V}(\tilde{V})$ to be the vector field projecting to X_1 under π . Each zero p of X_1 is the image under π of exactly two zeroes q_1, q_2 of \tilde{X}_1 , and the Poincaré-Hopf indices of \tilde{X} at q_1 and q_2 both equal the Poincaré-Hopf index of X at p . Therefore

$$0 = i(\tilde{X}_1, \tilde{V}) = 2i(X_1, V) = 2i(X, V) = 2i_K(X),$$

completing the proof. ■

Other calculations of indices can be found in [5, 6, 14, 16, 21].

3 Proof of Theorem 1.2

Lemma 3.1. *Let $L \subset K$ be a component. Then X has the same order at each point of L .*

Proof. The sets $L_j = \{p \in L: \text{ord}_p(X) = j\}$, $j \in \{1, \dots, k\}$ are relatively open in L and mutually disjoint. As L is connected and covered by the L_j , it coincides with one of them. ■

Let $p \in K$ be arbitrary. Choose a C^{k+1} flowbox h_p for Y centered at p :

$$h_p: W'_p \approx W_p = J_p \times J'_p \subset \mathbb{R}^2, \quad h(p) = (0, 0). \quad (2)$$

This means W'_p is open in M , $J_p, J'_p \subset \mathbb{R}$ are open intervals around 0, and the C^{k+1} diffeomorphism h_p transforms $Y|_{W'_p}$ to the constant vector field $\frac{\partial}{\partial x}|_{W_p}$, where x, y are the usual planar coordinates. Notice that $h_p(K \cap W'_p) = J_p \times \{0\}$ because K is Y -invariant.

The transform of $X|_{W'_p}$ by h_p is a C^k vector field

$$\hat{X}(p) \in \mathcal{V}^k(W_p), \quad (x, y) \mapsto F_p(x, y),$$

where $F_p: W_p \rightarrow \mathbb{R}^2$ is C^k .

Set $\text{ord}_p(X) = l \in \{1, \dots, k\}$. The partials of F_p satisfy

$$F_p^{(i)}(0) = 0 \text{ for } i = 0, \dots, l-1, \quad F_p^{(l)}(0) \neq 0.$$

In a sufficiently small open disk

$$D := D_p \subset \mathbb{R}^2, \quad (0, 0) \in D_p, \quad (3)$$

the l 'th order Taylor expansion of F_p about $(0, 0)$ takes the form

$$F_p(x, y) = y^l g(x, y), \quad g(x, y) \neq (0, 0) \quad (4)$$

with $g: D_p \rightarrow \mathbb{R}^2$ continuous. Therefore

$$Z(\hat{X}(p)) = F_p^{-1}(0, 0) = J_p \times \{0\},$$

whence $K \cap W'_p$ is an open arc, relatively closed in W'_p .

It follows that K has an open cover by open arcs. Thus K is a compact 1-manifold having only finitely many components, each of which is a topological circle. The restriction of Φ^Y to any component $L \subset K$ is a smooth flow with no fixed points. Therefore L is a periodic orbit of Φ^Y , and is thus a smooth submanifold. This proves Theorem 1.2(ii).

Lemma 3.2. *\hat{X} is controlled by a unique line field $\Lambda(p)$ on D_p .*

Proof. Consider the unit vector field \hat{F} on $D' := D \setminus J \times \{0\}$, as

$$\hat{F}(x, y) := \text{sign}(y) \frac{F(x, y)}{\|F(x, y)\|}$$

where $\|\cdot\|$ denotes the Euclidean norm. Equation (4) implies

$$\lim_{y \rightarrow 0} \hat{F}(x, y) = \frac{g(x, 0)}{\|g(x, 0)\|} \text{ uniformly in } D'.$$

Therefore \hat{F} extends to a unique continuous map $\tilde{F}: D \rightarrow \mathbf{S}^1$ (the unit circle). The desired line field sends $(x, y) \in D$ to the line through $(0, 0)$ spanned by $\tilde{F}_{(x, y)}$. ■

Next we prove 1.2(iii). For each $p \in K$ let $V_p := h_p^{-1}(D_p)$, with notation as in Equations (2), (3). Define $V := \bigcap_{p \in K} V_p$. Let $\Lambda(p)$ be the line field defined in Lemma 3.2. The pullback of $\Lambda(p)$ by h_p is the unique line field $\Gamma(p)$ on V_p controlling $X|_{V_p}$. The unique line field Γ on V that restricts to $\Gamma(p)$ for each $p \in K$ has the required properties.

Parts (i) and (iv) of 1.2 are consequences of (iii) and Proposition 2.4.

Part (v) follows from Propositions 3.13 and 3.14 of [12] when U is connected, and this implies the general case because the compact set K is covered by finitely many components of U .

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