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THE GENERALIZED PARTIAL-WAVE DISPERSION RELATION
AND THE N/D METHOD*

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ABSTRACT

We show that a meaningful dispersion relation and associated N/D integral equations can be formulated for any partial-wave amplitude, even though the ordinary partial-wave dispersion relation with a finite number of subtractions is violated by models like that of Veneziano. It is also shown that the arbitrary cutoff of the N/D method, which is often introduced to represent the high-energy behavior of the input "potential," can be eliminated in some models (including that of Veneziano); in terms of Regge parameters.

1. INTRODUCTION

The N/D method¹⁻³ linearizes the nonlinear integral equation derived from unitarity and the partial-wave dispersion relation with a finite number of subtractions. Given contributions from subtractions and unphysical cuts (the "potential") as well as the inelastic factor and CDD pole parameters (if necessary), the N/D integral equations allow calculation of the elastic phase shift. Nevertheless, N/D methods have difficulties which, if not resolved, destroy their usefulness. In this article we resolve some of the difficulties and partially resolve the remaining ones.

The first question about the N/D method is its validity. The usual N/D method is based on the partial-wave dispersion relation with a finite number of subtractions (throughout this article we refer to the latter as "the ordinary partial-wave dispersion relation"), which is satisfied if the Mandelstam representation⁴ with a finite number of subtractions can be written down. It has become realized recently that both the Mandelstam representation and the ordinary partial-wave dispersion relation with a finite number of subtractions are violated in models with indefinitely rising Regge trajectories, so the validity of the N/D method becomes questionable. We will show in Sec. 2 and Sec. 3 of this article that the unitarity bound⁵ in the physical region guarantees that a generalized partial-wave dispersion relation can always be formulated, and the N/D integral equations can then be derived from the generalized partial-wave dispersion relation.

A second question about the N/D method is how to determine the CDD pole parameters. It is known⁶⁻⁹ that the origin of CDD poles¹⁰ in a one-channel N/D calculation can be interpreted as the effect of inelastic channels. We thus can expect to avoid the appearance of CDD poles by considering a multichannel formulation, if the potentials for each channel are given. Thus the question of CDD pole parameters is replaced by the question of obtaining potentials from reasonable physical models for each channel. In this article we will not consider CDD poles, but restrict our attention to a method for obtaining the input potentials.

A physical crucial question for the N/D method, evident from the previous paragraph, is how the input potential is to be obtained. The behavior of the input potential in the low-energy region can be estimated in a variety of ways (we will discuss these briefly in Sec. 2). Our major concern is how to obtain the high-energy behavior of the potential, because the solution of N/D equations depends critically thereon even if the low-energy behavior is given. It is this mathematical property of the N/D integral equation together with the difficulty (discussed in some detail in Sec. 4) of obtaining the high-energy behavior of the input potential that in previous N/D methods¹¹ has led to arbitrary cutoff parameters or to arbitrary assumptions about the properties of distant singularities (the high-energy behavior of the input potential is closely related to distant singularities in crossed channels). We shall resolve this difficulty for a certain class of models by expressing the high-

energy behavior of the input potential in terms of Regge parameters. The class of model, though not completely general, is large enough to contain the Veneziano model.¹²⁻¹⁵ These matters are discussed in Secs. 4 through 6.

A "restricted" partial-wave dispersion relation is defined in Sec. 4, and in Sec. 5 constraints on Regge parameters are inferred from assumptions used to formulate this "restricted" relation. It turns out that some of our assumptions force Regge residues to have nonsense-wrong-signature zeros and also force the intercept of the leading Regge trajectory to be less than one. In Sec. 6 a definition is given of "partial-wave duality" for models which satisfy the restricted partial-wave dispersion relation, and it is shown¹⁶ that the high-energy tail of the input potential can be expressed in terms of Regge parameters if partial-wave duality is imposed. Thus for models obeying partial-wave duality the N/D integral equations connect low-energy parameters (from the solution of the N/D equations and the input low-energy behavior of the potential) and high-energy parameters (from the input high-energy tail of the potentials). We note that this connection between low- and high-energy parameters in the N/D method is based explicitly on the nonlinear content of unitarity, whereas the role of unitarity in the finite energy sum rule¹⁷ is not explicit.

We only consider spinless, equal mass particles, the generalization to include spins and unequal mass particles being straightforward, though complicated.

2. THE GENERALIZED PARTIAL-WAVE DISPERSION RELATION

It is shown in this section that a generalized partial-wave dispersion relation can be defined for any partial-wave amplitude even though the ordinary partial-wave dispersion relation with a finite number of subtractions is violated in models with indefinitely rising trajectories, such as the Veneziano model.¹²⁻¹⁵

A partial-wave amplitude $A_\ell(s)$ is defined as

$$A_\ell(s) = \frac{S_\ell(s) - 1}{2i\rho(s)} = \frac{1}{32\pi} \int_{-1}^1 dz_s P_\ell(z_s) A(s, t, u)$$

where

$$\rho(s) = [(s - 4\mu^2)/s]^{\frac{1}{2}}$$

and S_ℓ is the partial wave S-matrix element, $A(s, t, u)$ is the scattering amplitude and z_s is the cosine of the s-channel scattering angle defined as

$$z_s = 1 + \frac{2t}{s - 4\mu^2} = -1 - \frac{2u}{s - 4\mu^2}$$

The unitarity bound requires that $|S_\ell| \leq 1$, so⁵

$$\lim_{s \rightarrow +\infty} \frac{A_\ell(s)}{s^\epsilon} = 0 \quad \text{for some positive } \epsilon. \quad (2.1)$$

For simplicity we only consider the case

$$\lim_{s \rightarrow +\infty} s^\epsilon A_\ell(s) = 0 \quad \text{for some positive } \epsilon \quad (2.2)$$

in this and the next section; the generalization to partial-wave amplitudes which satisfy Eq. (2.1) but not (2.2), is straightforward.

The constraint (2.2) allows us to define a function $V_\ell(s)$ as

$$V_\ell(s) = A_\ell(s) - \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} - \sum_i^N \frac{g_i}{s - s_i}$$

N being a finite integer, (2.3)

where the s_i 's ($0 < s_i < 4\mu^2$) are the positions of the bound-state poles contained in the partial-wave amplitude $A_\ell(s)$, and the g_i 's are the pole residues:

$$g_i = \lim_{s \rightarrow s_i} (s - s_i) A_\ell(s).$$

The potential function $V_\ell(s)$ is free from poles and also does not contain the s-channel normal thresholds, since $V_\ell(s)$ is real in the region $s \geq 4\mu^2$. Transposing the terms of (2.3),

$$A_\ell(s) = V_\ell(s) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} + \sum_i^N \frac{g_i}{s - s_i}, \quad (2.4)$$

we call Eq. (2.4) the generalized partial-wave dispersion relation. We note that Eq. (2.4) may contain one subtraction if the asymptotic behavior of Eq. (2.1) is used.

The behavior of the potential $V_\ell(s)$ in the low-energy s-channel physical region may be estimated from the assumption

of the dominance of several nearby singularities of the cross channels. The validity of such methods can be checked by inserting the experimental phase shift for $A_\ell(s)$ in both sides of Eq. (2.4). The above procedure cannot be distinguished from that used in the analysis¹⁸ of ordinary partial-wave dispersion relation. Therefore these previous analyses can be carried over directly to the generalized partial-wave dispersion relation.

We will see in the next section that the N/D integral equations can also be derived from the generalized partial-wave dispersion relation.

3. DERIVATION OF THE N/D INTEGRAL EQUATIONS

We now discuss the possibility of decomposing a partial-wave amplitude into the N/D form. For simplicity as discussed in Sec. 2, we only consider the case

$$\lim_{s \rightarrow +\infty} s^\epsilon A_\ell(s) = 0 \quad \text{for some positive } \epsilon, \quad (3.1)$$

the generalization being straightforward. The generalized partial-wave dispersion relation is

$$A_\ell(s) = \sum_i^N \frac{g_i}{s - s_i} + V_\ell(s) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} \quad (3.2)$$

where

$$4\mu^2 > s_i > 0 \quad (i = 1, \dots, N).$$

The amplitude $A_\ell(s)$ can be parametrized as

$$A_\ell(s) = |A_\ell(s)| \cdot e^{-i\theta_\ell(s)}$$

where $\theta_\ell(s)$ is real. We are aiming at the construction of the N/D integral equation from the R-function method.¹ If we want to employ the Frye-Warnock¹⁹ method, we need to consider the phase of $S_\ell(s)$ instead of that of $A_\ell(s)$. In the remaining discussion of this article we consider only the R-function method.

The zeros of $A_\ell(s)$ require attention. Certain of the zeros of $A_\ell(s)$ may correspond to CDD poles,¹⁰ and certain may not. On the other hand, CDD poles can be interpreted either as poles of the D function or zeros of the N function

(the reverse of this statement is not true). If we have a D function with poles, we can always introduce a new set of D and N functions, by multiplying the original D and N functions by an appropriate polynomial, such that the new D function is free from poles but has different asymptotic behavior, while the new N function contains additional zeros. In this section we concentrate on the D function which is free from all poles, and whose zeros are in one-to-one correspondence with the bound state poles of $A_\ell(s)$. The CDD poles (if any) are represented by zeros of the N function, the asymptotic behavior of the D function having a close connection thereto. We defer further discussion of the CDD ambiguity to the end of the section. We first want to show that if a partial-wave amplitude $A_\ell(s)$ is given, we can construct a D function, under certain assumptions, such that the D function is power-bounded, has the phase of $A_\ell(s)$ in the region $s \geq 4\mu^2$, is real in the region $s < 4\mu^2$, and its zeros are in one-to-one correspondence with the bound-state poles of $A_\ell(s)$. Assuming that

$$\lim_{s \rightarrow +\infty} |\theta_\ell(s)| < +\infty,$$

we define a function $D_\ell(s)$ as

$$D_\ell(s) = D_\ell(0) \cdot \left\{ \prod_{i=1}^N \frac{s_i - s}{s_i} \right\} \exp \left\{ \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\theta_\ell(s')}{s'(s' - s)} \right\}, \quad (3.3)$$

where the s_i 's are the positions of poles of $A_\ell(s)$. This function $D_\ell(s)$ may have some poles at $s = \bar{s}_i$ ($i=1, \dots$) when

$$\lim_{\epsilon \rightarrow 0} \{\theta_\ell(\bar{s}_i + \epsilon) - \theta_\ell(\bar{s}_i - \epsilon)\} = -\pi.$$

We assume that $\theta_\ell(s)$ in the asymptotic energy region is determined by Regge phase. The Regge phase derived from Regge poles is continuous. Thus we may assume that the number of possible poles contained in Eq. (3.3) is finite. If such poles appear in Eq. (3.3), we multiply $D_\ell(s)$ by an adequate polynomial to remove these poles.

We consider this new function as the function $D_\ell(s)$. The function $D_\ell(s)$ has the phase $\theta_\ell(s_\pm)$ in the region $s_\pm \geq 4\mu^2$, is real in the region $s < 4\mu^2$, and its zeros are in one-to-one correspondence with poles of $A_\ell(s)$. The function $D_\ell(s)$ is power-bounded, so it satisfies a dispersion relation with a finite number of subtractions. If $\theta_\ell(s)$ is unbounded as $s \rightarrow +\infty$, the existence of a function $D_\ell(s)$, power-bounded, with the phase $\theta_\ell(s_\pm)$ in the region $s_\pm \geq 4\mu^2$ and real in the region $s < 4\mu^2$, is an open question. We note that this problem concerning the existence of $D_\ell(s)$ is not specially associated with the generalized partial-wave dispersion relation but already occurs for the N/D method based on the ordinary partial-wave dispersion relation. We also note that an unbounded $\theta_\ell(s)$ implies that the ratio $\text{Re } A_\ell(s)/\text{Im } A_\ell(s)$ oscillates as $s \rightarrow +\infty$. Here we assume that either $\theta_\ell(s)$ is bounded or that, if unbounded, a function $D_\ell(s)$, with the properties enumerated above, nevertheless exists. A function $N_\ell(s)$ is next defined as

$$N_\ell(s) = A_\ell(s) \cdot D_\ell(s), \quad (3.4)$$

so we have [note that $\text{Im } A_\ell(s) = \text{Im } V_\ell(s)$ for $s < s_L$]

$$\text{Im } N_\ell(s) = \begin{cases} 0 & \text{for } s \geq 4\mu^2 \\ D_\ell(s) \text{Im } V_\ell(s) & \text{for } s < s_L, \end{cases} \quad (3.5)$$

where s_L is the position of the nearest left-hand singularity of $A_\ell(s)$.

We consider below for simplicity the case that $D_\ell(s)$ satisfies a dispersion relation with one subtraction, but no poles. The N function may contain zeros. This case corresponds to the absence of CDD poles, since it can be shown¹⁶ that the solution is completely determined by giving the potential, the inelastic function (the R -function), and (if required) the subtraction constant in the generalized partial-wave dispersion relation. The generalization to include CDD poles is straightforward and is discussed briefly at the end of this section.

The unitarity relation can be written as

$$\text{Im} \left\{ \frac{1}{A_\ell(s)} \right\} = -\rho(s) \cdot R_\ell(s), \quad (3.6)$$

where

$$\rho(s) = \frac{1}{2} [(s - 4\mu^2)/s]^{\frac{1}{2}}$$

and

$$R_\ell(s) = \frac{\sigma_\ell^{\text{total}}(s)}{\sigma_\ell^{\text{elastic}}(s)} = \frac{\text{Im } A_\ell(s)}{\rho(s) \cdot |A_\ell(s)|^2}.$$

From Eqs. (3.4), (3.5), and (3.6), we have the relation

$$\text{Im } D_\ell(s) = \begin{cases} -\rho(s) \cdot R_\ell(s) \cdot N_\ell(s) & \text{for } s \geq 4\mu^2 \\ 0 & \text{for } s < 4\mu^2. \end{cases} \quad (3.7)$$

The dispersion relation for $D_\ell(s)$ (only the case with one subtraction is considered here) can be written with the help of Eq. (3.7) as

$$D_\ell(s) = D_\ell(0) - \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\rho(s') R_\ell(s') N_\ell(s')}{s'(s' - s)}. \quad (3.8)$$

A function $C_\ell(s)$ may be defined as

$$C_\ell(s) = N_\ell(s) - V_\ell(s) \cdot D_\ell(s).$$

From Eqs. (3.5) and (3.7), we see that

$$\text{Im } C_\ell(s) = \begin{cases} \rho(s) \cdot R_\ell(s) \cdot V_\ell(s) \cdot N_\ell(s) & \text{for } s \geq 4\mu^2 \\ 0 & \text{for } s < 4\mu^2. \end{cases}$$

As mentioned in Sec. 2 and at the beginning of this section, we only consider, for simplicity, the case that the asymptotic condition of Eq. (3.1) is satisfied. Therefore

$$\lim_{s \rightarrow \infty} C_\ell(s) = 0,$$

since

$$D_\ell(s) \xrightarrow{s \rightarrow \infty} \text{const.}$$

$$N_\ell(s), \text{ and } V_\ell(s) \xrightarrow{s \rightarrow \infty} 0.$$

At the other asymptotic direction on the s -plane, the function $A_\ell(s)$ is not necessarily power-bounded, but the function $C_\ell(s)$ always vanishes at such limits. To see this explicitly, we write

$$s = |s| \cdot e^{i\theta},$$

and suppose at some limit $|s| \rightarrow +\infty$ with $\theta = \Psi \neq 2m\pi$ ($m = 0, 1, 2, \dots$), we have the relation

$$\lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi \\ (m=0, 1, 2, \dots)}} |s|^n / A_\ell(|s|e^{-i\theta}) = 0 \quad \text{for any positive integer } n.$$

This expression just implies that we are assuming $A_\ell(s)$ is not power-bounded at the direction $\theta = \Psi \neq 2m\pi$, $|s| \rightarrow +\infty$. From the definition of $V_\ell(s)$

$$V_\ell(s) = A_\ell(s) - \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} - \sum_i^N \frac{g_i}{s - s_i}$$

and the constant asymptotic behavior of $D_\ell(s)$ (we are considering this case as mentioned before), we have the relation

$$\begin{aligned} & \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi}} C_\ell(|s|e^{i\theta}) \\ &= \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi}} \{N_\ell(|s|e^{i\theta}) - V_\ell(|s|e^{i\theta}) \cdot D_\ell(|s|e^{i\theta})\} \\ &= \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi}} \{D_\ell(|s|e^{i\theta}) \cdot [A_\ell(|s|e^{i\theta}) - V_\ell(|s|e^{i\theta})]\} \\ &= \left\{ \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi}} D_\ell(|s|e^{i\theta}) \right\} \cdot \left\{ \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi \neq 2m\pi}} \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} + \sum_i^N \frac{g_i}{s - s_i} \right\} = 0. \end{aligned}$$

The above discussion implies that $C_\ell(s)$ satisfies a dispersion relation with no subtraction, i.e.

$$\begin{aligned} C_\ell(s) &= N_\ell(s) - V_\ell(s) \cdot D_\ell(s) \\ &= \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\rho(s') R_\ell(s') V_\ell(s') N_\ell(s')}{s' - s} \end{aligned}$$

Substituting Eq. (3.8) into this relation, we have

$$N_\ell(s) = D_\ell(0) \cdot V_\ell(s) + \int_{4\mu^2}^{\infty} ds' K_\ell(s; s') N_\ell(s') \quad (3.9)$$

where

$$K_\ell(s; s') = \frac{1}{\pi} \rho(s') \cdot R_\ell(s') \cdot \frac{s' V_\ell(s') - s V_\ell(s)}{s' - s}$$

As mentioned before, this $N_\ell(s)$ may contain some zeros, but these zeros have nothing to do with CDD poles, since the ratio $N_\ell(s)/D_\ell(s)$ is determined completely¹⁶ by giving $V_\ell(s)$ and $R_\ell(s)$. Equations (3.8) and (3.9) are the N/D integral equations and their form is the same as the N/D integral equation^{2,3} associated with an ordinary partial-wave dispersion relation.

We have demonstrated the derivation of the N/D integral equation for the simplest case, i.e., the case that $D_\ell(s)$ satisfies a dispersion relation with one subtraction and no poles. For the case that $D_\ell(s)$ does not have poles but $n+1$ subtractions are required to write down a dispersion relation

for $D_\ell(s)$, we can go through a similar argument as in the case with one subtraction and derive the N/D integral equations. In this case the dispersion relation for $C_\ell(s)$ requires n subtractions, and we have $2n+1$ arbitrary parameters (i.e., $2n+1$ subtraction constants) in the N/D integral equations. Only $2n$ of $2n+1$ parameters are nontrivial, since one parameter will set the scale. This case corresponds to the existence of n CDD poles.

4. THE RESTRICTED PARTIAL-WAVE DISPERSION RELATION

The N/D integral equations are derived in the previous section, but such equations are not useful unless we find reasonable ways to approximate the input potential. It is a mathematical property of the N/D integral equations that both the low-energy and high-energy behavior of the input potential is important in determining the structure of the output solution at low energy. We should not confuse this statement with the principle of nearby singularity dominance in the analysis of ordinary or generalized partial-wave dispersion relations. In such analysis the required inputs are the low-energy behavior of the potential $V_\ell(s)$ and the behavior of $\text{Im } A_\ell(s)$ in the region $s \geq 4\mu^2$, and the output is the behavior of $\text{Re } A_\ell(s)$ in the low-energy region. On the other hand, the outputs of the N/D integral equations are both $\text{Im } A_\ell(s)$ and $\text{Re } A_\ell(s)$ in the low-energy region, and we are saying that the required inputs are the behavior of the potential $V_\ell(s)$ in the low-, intermediate-, and high-energy regions, as well as the inelastic function $R_\ell(s)$ (and CDD pole parameters if necessary). If the behavior of the potential $V_\ell(s)$ in the intermediate- and high-energy region can be expressed in terms of high-energy phenomenological parameters (e.g., Regge parameters) without referring to the knowledge about the low-energy phase shift, the N/D method will provide some connection between the low- and high-energy phenomenological parameters, which cannot be obtained from the above mentioned analysis of the generalized

partial-wave dispersion relation. One may think that the high-energy behavior of the potential $V_\ell(s)$ can be expressed in terms of Regge parameters from the relation

$$V_\ell(s) \approx \text{Re } A_\ell(s) - \frac{1}{\pi} P \int_C^\infty ds' \frac{\text{Im } A_\ell(s')}{s' - s}$$

for $s \gg C \gg 4\mu^2$ (4.1)

by inserting the partial-wave projected Regge asymptotic behavior for $\text{Re } A_\ell(s)$ and $\text{Im } A_\ell(s')$. But such high-energy behavior of $V_\ell(s)$ cannot be extrapolated down to the intermediate-energy region, since at $s = C$ $V_\ell(s)$ of Eq. (4.1) has a spurious singularity which should not exist in the potential $V_\ell(s)$ from its definition Eq. (2.3). The intermediate-energy behavior of $V_\ell(s)$, as can be seen from the more general expression, is

$$V_\ell(s) = \text{Re } A_\ell(s) - \frac{1}{\pi} P \int_{4\mu^2}^\infty ds' \frac{\text{Im } A_\ell(s')}{s' - s}, \quad (4.2)$$

which is sensitive to the input behavior of $\text{Im } A_\ell(s')$ in the low-energy region. Therefore the N/D method gives no useful information beyond the generalized partial-wave dispersion relation, if Eqs. (4.1) and (4.2) are used to estimate the behavior of $V_\ell(s)$ in the high- and intermediate-energy region. In the remaining part of this article, we shall mainly be concerned with the possibility of expressing $V_\ell(s)$ in the high- and intermediate-energy region in terms of high-energy phenomenological parameters alone.

The low-energy behavior of the potential $V_\ell(s)$ is dominated by nearby singularities in crossed channels. On the other hand, the intermediate- and high-energy behavior of the potential is controlled by both distant and nearby singularities of crossed channels, and only if the distant singularities are very weak, is the nearby contribution dominant. In previous N/D calculations,¹¹ the difficulty of estimating the contribution of distant singularities is handled by neglecting them completely (e.g., in models that approximate the potential by several nearby poles), by introducing some arbitrary regulating functions to approximate the discontinuity across the left-hand cut of the partial-wave amplitude $A_\ell(s)$, or by inserting some arbitrary cutoff parameters into the calculation. All such approaches, of course, lack a profound physical basis. Starting in this section we will investigate the possibilities of expressing the intermediate and high-energy behavior of $V_\ell(s)$ in terms of high-energy Regge parameters, and as will be shown in Sec. 6, we succeed for a certain class of model, which contains the Veneziano model^{12-15,21} with all trajectory intercepts less than one. In this section we begin by formulating "the restricted partial-wave dispersion relation," which applies to a certain class of model (such as that of Veneziano).

A partial-wave amplitude $A_\ell(s)$ is said to satisfy "the restricted partial-wave dispersion relation" if the following four properties are satisfied:

- (1) A scattering amplitude $A(s,t,u)$ is a linear combination of three functions $A_{st}(s,t)$, $A_{tu}(t,u)$, and $A_{us}(u,s)$,

each having special properties. Let (x,y,z) be an arbitrary cyclic permutation of (s,t,u) . The function $A_{xy}(x,y)$ contains the x and y channel normal thresholds and poles, but does not contain those for the z channel. Although the function $A_{xy}(x,y)$ is allowed to contain spurious singularities which are not contained in the scattering amplitude $A(s,t,u)$ [which will be cancelled in the linear combination that constructs $A(s,t,u)$], we assume that such spurious singularities are absent from A_{xy} in the region $x \leq 0$ with y fixed and $y \leq 0$ with x fixed.

(2) The function A_{xy} has Regge asymptotic behavior in the limit x (or y) $\rightarrow \pm\infty$ with y (or x) fixed, but it damps out faster than any power at the limit x (or y) $\rightarrow \pm\infty$ with z fixed.

We need to consider certain implications of assumption (1) before going on to state assumptions (3) and (4). The function $A_{st}(s,t)$ can be written in terms of s and z_s , where z_s is the cosine of the scattering angle of the s channel:

$$z_s = 1 + \frac{2t}{s - 4\mu^2} = -1 - \frac{2u}{s - 4\mu^2}.$$

In the s -channel physical region, i.e., $s \geq 4\mu^2$, we look at the singularities of $A_{st}(s,z_s)$ in the complex z_s -plane with s fixed. From assumption (1) all the singularities of A_{st} with s fixed are in the region $t > 0$, so the singularities of A_{st} in the z_s plane are in the region $z_s > 1$. We thus can draw a Lehman ellipse²² with foci at $s_z = \pm 1$ for A_{st} , implying that A_{st} has a partial-wave expansion in the s -channel physical region. The functions A_{tu} and A_{us} also have their own partial-wave expansions in the s -channel physical

region, by a similar argument, and of course the corresponding result is true in the t - and u -channel physical regions as well. We denote the partial-wave amplitudes of A_{st} , A_{tu} , and A_{us} in the s -channel physical region as $A_\ell^{st}(s)$, $A_\ell^{tu}(s)$, and $A_\ell^{us}(s)$ respectively. Assumptions (3) and (4) will now be formulated in the s channel, but they, of course, are also true in the t and u channels.

(3) The partial-wave amplitudes $A_\ell^{st}(s)$ and A_ℓ^{us} are power-bounded in the s plane, although such is not required for the amplitude $A_\ell^{tu}(s)$.

(4) The following asymptotic condition holds:

$$\lim_{s \rightarrow +\infty} \frac{A_\ell^{xy}(s)}{A_\ell(s)} < +\infty.$$

We note that models which satisfy the ordinary partial-wave dispersion relation are not necessarily special cases of the restricted partial-wave dispersion relation, since assumption (1) is not required for the ordinary partial-wave dispersion relation [assumption (2) is rather independent of the partial-wave dispersion relations]. On the other hand, assumption (3) is more general than the ordinary partial-wave dispersion relation in some sense, since $A_\ell(s)$ is not required to be power-bounded in the complex s plane.

The partial-wave amplitude $A_\ell(s)$ satisfies the unitarity bound, as discussed in Sec. 2,

$$\lim_{s \rightarrow +\infty} \frac{A_\ell(s)}{s^\epsilon}.$$

From assumptions (3) and (4) the amplitudes $A_\ell^{st}(s)$ and $A_\ell^{us}(s)$ satisfy dispersion relations with at most one subtraction. We only consider the simplest case in this section, i.e.,

$$\lim_{s \rightarrow +\infty} A_\ell(s) \cdot s^\epsilon = 0 \quad \text{for some positive } \epsilon,$$

but the generalization is obvious. The dispersion relations for $A_\ell^{st}(s)$ and $A_\ell^{us}(s)$ are then

$$A_\ell^{st}(s) = \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_\ell^{st}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell^{st}(s')}{s' - s} \quad (4.2)$$

$$A_\ell^{us}(s) = \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_\ell^{us}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell^{us}(s')}{s' - s}$$

The amplitude $A_\ell(s)$ can be written from assumption (1) as

$$A_\ell(s) = A_\ell^{st}(s) + A_\ell^{tu}(s) + A_\ell^{us}(s).$$

Since the function $A_\ell^{tu}(s)$ does not contain any singularities in the region $s \geq 4\mu^2$ and also does not contain any s-channel bound state poles [from assumption (1)], the potential $V_\ell(s)$ of the generalized partial-wave dispersion relation is

$$V_\ell(s) = A_\ell^{tu}(s) + \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_\ell^{st}(s') + \text{Im } A_\ell^{us}(s')}{s' - s} \quad (4.3)$$

It has been shown in the Veneziano model¹²⁻¹⁵ that the functions $A_\ell^{st}(s)$ and $A_\ell^{us}(s)$ satisfy²³ the dispersion relations of Eq.(4.2), but the function $A_\ell^{tu}(s)$ is not power-bounded at the limit $s \rightarrow -\infty$.²⁴ Therefore the Veneziano model satisfies the restricted partial-wave dispersion relation, but violates the ordinary partial-wave dispersion relation. We note that if the Pomeranchuk intercept of Wong's version of the Veneziano model²¹ is less than one, it satisfies the restricted partial-wave dispersion relation, but Wong's version of the Veneziano model with Pomeranchuk intercept one violates our condition. We also note that the Virasoro model²⁵ violates the restricted partial-wave dispersion relation. We will see these points clearly in the next section.

5. THE CONSTRAINTS ON REGGE PARAMETERS FROM THE
RESTRICTED PARTIAL-WAVE DISPERSION RELATION

Assumption (1) of Sec. 4 is a strong assumption. When combined with assumption (2) of Sec. 4, there is implied a set of constraints on Regge parameters. We consider only Regge poles, although the analysis can be generalized to Regge cuts.

The Regge asymptotic behavior of the scattering amplitude may be parameterized as

$$A(s,t,u) \xrightarrow[\substack{s \rightarrow +\infty \\ x \text{ fixed}}]{} \sum_i \frac{-\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \cdot \left\{ \tau_i \left(\sigma_x \frac{s}{\bar{s}_i} \right)^{\alpha_i(x)} + \left(-\sigma_x \frac{s}{\bar{s}_i} \right)^{\alpha_i(x)} \right\} \quad (5.1)$$

where $\sigma_t = +1$, $\sigma_u = -1$, the subscript i denotes the different Regge poles, and $\alpha_i(x)$ and $\beta_i(x)$ are the Regge trajectory and residue functions respectively, both assumed real in the region $t \leq 4\mu^2$. The symbol $\tau_i = \pm 1$ is the signature of the i th Regge trajectory and \bar{s}_i is a real constant. Assumption (1) of Sec. 4 says that we can decompose $A(s,t,u)$ into a linear combination of A_{st} , A_{tu} , and A_{us} as

$$A(s,t,u) = A_{st}(s,t) + A_{tu}(t,u) + A_{us}(u,s).$$

Assumption (2) means that

$$A_{sx} \xrightarrow[\substack{s \rightarrow +\infty \\ x \text{ fixed}}]{} \sum_i \frac{-\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \cdot \left(-\sigma_x \frac{s}{\bar{s}_i} \right)^{\alpha_i(x)} \quad (5.2)$$

$$A_{tu} \xrightarrow[\substack{s \rightarrow +\infty \\ x \text{ fixed}}]{} \sum_i \frac{-\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \cdot \left(\sigma_x \frac{s}{\bar{s}_i} \right)^{\alpha_i(x)}, \quad (5.3)$$

where $\sigma_t = +1$ and $\sigma_u = -1$. The requirement [from assumption (1)] about the absence of the spurious singularities of A_{sx} and A_{tu} in the region $x \leq 0$ with s or y fixed (if $x = t$, $y = u$, if $x = u$, $y = t$) forces us to choose $\alpha_i(x)$ and $\beta_i(x)$ in Eqs. (5.2) and (5.3) such that

$$\alpha_i(0) < 1 \quad \text{if } \beta_i(0) \neq 0 \quad (5.4)$$

and

$$\lim_{\alpha_i(x) \rightarrow 0} \left| \frac{\beta_i(x)}{\sin \pi \alpha_i(x)} \right| < +\infty \quad (x \leq 0) \quad (5.5)$$

$$\lim_{\alpha_i(x) \rightarrow -N} \left| \frac{\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \right| < +\infty$$

for $N = 1, 2, 3, \dots$, $x \leq 0$.

For the leading trajectory $i = P$ it is obvious from the optical theorem and the experimental total cross section that

$$\beta_P(0) \neq 0$$

so the constraint of Eq. (5.4) requires that the intercept of the leading trajectory should be less than one. If the class of models of Sec. 4 contains Pomeranchuk poles, this implies that the Pomeranchuk intercept (of this class of models) must

be less than one. The constraints of Eq. (5.5) imply not only the existence of ghost killing factors at the right signature points

$$\tau_i \alpha_i(x) = 0, -2, -4, \dots,$$

but also the existence of zeros in the residue $\beta_i(x)/\Gamma[\alpha_i(x) + 1]$ at the nonsense wrong signature points, i.e., the points

$$\tau_i \alpha_i(x) = -1, -3, -5, \dots$$

The argument of this section makes clear why Wong's version²¹ with unit Pomeranchuk intercept and Virasoro's model²⁵ do not satisfy the restricted partial-wave dispersion relation. The function A_{sx} of Wong's version with unit Pomeranchuk intercept has a spurious singularity (a pole) at $x = 0$, which partially violates assumption (1). Virasoro's model does not satisfy assumptions (1) through (4).

6. PARTIAL-WAVE DUALITY

We have introduced in Sec. 4 the restricted partial-wave dispersion relation, which is satisfied by the Veneziano model including Wong's version with the Pomeranchuk intercept less than one, but even within a model obeying the restricted partial-wave dispersion relation the N/D method still cannot be made useful. In this section we will show that if we add an additional assumption (the assumption of "partial-wave duality") to assumptions (1) through (4) of Sec. 4, we can express the high-energy behavior of the input potential in terms of Regge parameters which can be obtained, in principle, from phenomenological fitting of the high-energy scattering data, and this high-energy behavior of $V_\ell(s)$ can be extrapolated down to the intermediate energy region without introducing spurious singularities. The Veneziano model with all the intercepts of Regge trajectories less than one satisfy the additional requirement.

We again only consider for simplicity the case

$$\lim_{s \rightarrow +\infty} s^\epsilon A_\ell(s) = 0 \quad \text{for some positive } \epsilon,$$

a condition that holds when all Regge trajectories have non-vanishing slopes and intercepts less than one. Statements (1) through (4) of Sec. 4 are assumed. The partial-wave amplitude $A_\ell^{sx}(s)$ ($x = t, u$) satisfies a dispersion relation as shown in Eq. (4.1)

$$\begin{aligned}
 & A_{\ell}^{SX}(s) \\
 &= \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_{\ell}^{SX}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_{\ell}^{SX}(s')}{s' - s} \\
 & \qquad \qquad \qquad (x = t, u) . \qquad (6.1)
 \end{aligned}$$

We now introduce the additional assumption

(5) that the following asymptotic relations hold,

$$\lim_{s \rightarrow +\infty} \frac{\int_{\text{L.H.C.}} ds' \frac{\text{Im } A_{\ell}^{SX}(s')}{s' - s}}{\int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_{\ell}^{SX}(s')}{s' - s}} = 0 \qquad (x = t, u) \qquad (6.2)$$

and

$$\lim_{s \rightarrow +\infty} \left| \frac{A_{\ell}^{SX}(s)}{A_{\ell}^{tu}(s)} \right| < +\infty \qquad (x = t, u) . \qquad (6.3)$$

The relation (6.3) holds if the Regge trajectories have nonvanishing slopes, since then both $A_{\ell}^{SX}(s)$ and $A_{\ell}^{tu}(s)$ have the asymptotic behavior $1/s^{1-b} \ln s$ (b is the intercept of the leading Regge trajectory), which are obtained by partial-wave projecting the Regge asymptotic behavior of A_{SX} and A_{tu} respectively. We call the asymptotic behavior of $A_{\ell}^{xy}(s)$ at the limit $s \rightarrow +\infty$ the partial-wave Regge asymptotic behavior, since it is caused by the Regge asymptotic behavior of A_{xy} . Equation (6.2) implies

$$\lim_{s \rightarrow +\infty} A_{\ell}^{SX}(s) = \lim_{s \rightarrow +\infty} \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_{\ell}^{SX}(s')}{s' - s} .$$

This means that the partial-wave Regge asymptotic behavior of $A_{\ell}^{SX}(s)$ is dual to the "pure" s-channel singularities, i.e., the right-hand singularities of $A_{\ell}^{SX}(s)$. In this sense we call assumption (5) the assumption of partial-wave duality.

The potential $V_{\ell}(s)$ of the models which satisfy assumptions (1) through (4) is defined in Eq. (4.2) as

$$V_{\ell}(s) = A_{\ell}^{tu}(s) + \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_{\ell}^{st}(s') + \text{Im } A_{\ell}^{su}(s')}{s' - s} .$$

If assumption (5) is imposed, we see that

$$\lim_{s \rightarrow +\infty} V_{\ell}(s) = \lim_{s \rightarrow +\infty} A_{\ell}^{tu}(s) .$$

Since the asymptotic behavior of $A_{\ell}^{tu}(s)$ can be expressed, by partial-wave projecting the Regge asymptotic behavior of A_{tu} , in terms of Regge parameters which can be obtained from phenomenological fitting of the high-energy scattering data, so is the high-energy behavior of $V_{\ell}(s)$. To obtain the behavior of $V_{\ell}(s)$ in the intermediate-energy region, we must assume that partial-wave duality holds down to that energy region, i.e., then we can write

$$V_{\ell}(s) \approx A_{\ell}^{tu}(s)$$

in the intermediate-energy region. This method is used in Ref. (16) to express the behavior of the $\pi\pi$ p-wave potential in terms of Pomeranchuk parameters in the intermediate and high-energy region.

The Veneziano model satisfies²³ the assumption of partial-wave duality, since the discontinuity of $A_{\ell}^{sx}(s)$ ($x = t, u$) across the left-hand cut decreases fast enough to allow Eq. (6.2) to hold, and Eq. (6.3) holds since all the Regge trajectories of that model are linear. These arguments can be generalized to Wong's model with the Pomeranchuk intercept less than one.

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FOOTNOTES AND REFERENCES

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