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New structures on embedded contact homology and applications to low-dimensional topology

Author
Wang, Luya

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by

Luya Wang

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Professor Michael Hutchings, Chair
Professor Ian Agol
Professor David Nadler

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Abstract

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Embedded contact homology (ECH) is a powerful three-manifold invariant that encodes information about the underlying contact manifold. It has been shown that embedded contact homology is isomorphic to certain versions of Seiberg-Witten Floer homology and Heegaard Floer homology. This thesis highlights the relations of ECH to other three-dimensional Floer theories. In Chapter 2, we use ideas from periodic Floer homology, a cousin of ECH, and techniques from pseudo-Anosov maps to answer a knot detection question in knot Floer homology. This is joint work with Ethan Farber and Braeden Reinoso. In Chapter 3, we show a connected sum formula for ECH by studying pseudo-holomorphic curves in the symplectization of the contact connected sum, without going through the tremendous isomorphisms to Seiberg-Witten Floer homology or Heegaard Floer homology. Our chain level description of the connected sum complex is useful for studying studying similar formulas for other contact homologies and ECH spectral invariants in the future. In Chapter 3, we give a relative version of ECH for contact three-manifolds with convex sutured boundaries. This generalizes both sutured ECH and Lipshitz's cylindrical reformulation of Heegaard Floer homology, and provides a potential framework for bordered ECH. This is joint work with Julian Chaidez, Oliver Edtmair, Yuan Yao and Ziwen Zhao.

Contents

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Chapter 1

Introduction

This thesis contains a collection of three papers that the author completed during her PhD:

1. *Fixed-point-free pseudo-Anosovs, knot Floer homology and the cinquefoil* (joint work with Ethan Farber and Braeden Reinoso)

2. *A connected sum formula for embedded contact homology*

3. *Legendrian embedded contact homology* (joint work with Julian Chaidez, Oliver Edtmair, Yuan Yao and Ziwen Zhao)

Each paper occupies a chapter. Each paper is self-contained, but we give some backgrounds and motivations here.

Chapter 2 resolves a question on whether knot Floer homology detects the cinquefoil knot $T(2,5)$. Knot Floer homology (HFK) defined by Ozsváth-Szabó and Rasmussen [98, 105] is a powerful knot invariant coming from the package of Heegaard Floer homology. Previously, HFK was known to detect the unknot [97], the trefoil knot and the figure-eight knot [40]. However, establishing such a result for the genus-two torus knot $T(2,5)$ has been very difficult, for example, because there are infinitely many genus-two hyperbolic fibered knot with the same Alexander polynomial as that of $T(2,5)$. Our approach is inspired by an isomorphism between a certain grading of HFK and the degree $d = 1$ part of the periodic Floer homology, which is a cousin of ECH [41]. The result we need from [41] can also be recovered from [3]. The essential strategy of ours is to analyze dynamical, in particular pseudo-Anosov, properties of a hyperbolic genus-two fibered knot. We show that any genus-two fibered knot in S^3 satisfying a mild condition must contain a fixed point in the interior of its fibering surface.

Chapter 3 and 4 discuss foundational aspects of embedded contact homology (ECH). Chapter 3 gives a connected sum formula for embedded contact homology by studying how pseudo-holomorphic curves behave in the symplectization of a contact connected sum. Even though the similar formulas are known in Seiberg-Witten Floer and Heegaard Floer, our proof does not go through the isomorphisms between any of ECH, Seiberg-Witten Floer and Heegaard Floer, except for a direct limit argument when passing to homology [109, 110, 111, 112, 113, 19, 24, 26, 22]. See also [76, 77, 78, 75, 79]. Our techniques can also be used to study gluing formulas along higher-genus surfaces that admit certain pseudo-holomorphic foliations in their symplectization, e.g. mixed tori [16], which will be the subjects for future works. The chain level description of the connected sum ECH complex also gives information regarding ECH spectral invariants, which will be studied in future projects. Chapter 4 constructs a relative version of embedded contact homology for convex sutured contact manifolds that we call *Legendrian ECH*. We develop intersection theory for pseudo-holomorphic curves with boundaries on Lagrangians that are symplectizations of the boundary Legendrians chosen a priori. This allows us to define a relative ECH index and obtain a relative adjunction formula. We further show analogous properties of Legendrian ECH to the ordinary ECH. In particular, we have that $\partial^2 = 0$. The Legendrian ECH generalizes both sutured ECH and Lipshitz's reformulation of Heegaard Floer homology and provides a potential framework for bordered ECH.

Chapter 2

Fixed-point-free pseudo-Anosovs, knot Floer homology and the cinquefoil

Ethan Farber, Braeden Reinoso and Luya Wang

2.1 Introduction

Recent developments in Heegaard Floer homology have highlighted intimate connections between link homology theories and the dynamics of surface diffeomorphisms (see e.g. [3], [95], [96], [41]). The aim of this paper is to use tools which may be familiar to some dynamicists, in order to study a particular open question in Heegaard Floer homology: whether knot Floer homology detects the torus knot $T(2,5)$. To that end, we answer this question in the affirmative:

Theorem A. If $\widehat{HFK}(K;\mathbb{Q}) \cong \widehat{HFK}(T(2,5);\mathbb{Q})$ as bi-graded vector spaces, then $K = T(2,5)$. In particular, $T(2,5)$ is the only genus-two L-space knot in S^3.

This is the first knot Floer detection result for a knot of genus two. Prior detection results rely on a classification of fibered knots with genus at most one [97, 40]. But, there are infinitely many genus two, hyperbolic, fibered knots with the same Alexander polynomial as that of $T(2,5)$ [92], indicating that close attention must be paid to the structure of the fibration. In this vein, our proof completes a strategy outlined by Baldwin–Hu–Sivek in [3], which uses connections between knot Floer homology and symplectic Floer homology, to reduce the question to a problem about fixed points of pseudo-Anosov maps. Our first key result is a solution to this problem.

Definition 2.1.1. Let $K \subset Y$ be a hyperbolic, fibered knot in a 3-manifold Y. The knot K specifies an open book decomposition (S, h) for Y, where $h : S \to S$ is freely isotopic to a pseudo-Anosov ψ on S. We say that K is *fixed point free* if ψ has no fixed points in the interior of S.

Theorem B. Let K be a hyperbolic, genus-two, fibered knot in S^3. If the fractional Dehn twist coefficient $c(K) \neq 0$, then K is not fixed point free.

Pseudo-Anosov maps and fractional Dehn twist coefficients are defined in Section 2.2. Theorem A follows immediately from Theorem B and the work of Baldwin–Hu–Sivek ([3], Theorem 3.5). We would like to emphasize that the proof of Theorem B is completely geometric in nature, and after the introduction we will only make passing references to Floer homology theories throughout the paper. An unfamiliar reader need not have expertise in any link homology theory to understand the proof of Theorem B.

The proof of Theorem B is broken down into two smaller theorems (Theorems B1 and B2), based on cases for the singularity type of ψ. An outline for the proof is given in subsection 2.1. We will now discuss various applications of our techniques and results.

Applications to train tracks and the dilatation spectrum

One of the central tools in the proof of Theorem B is the theory of train tracks for pseudo-Anosov braids, including a theory of "tight splitting" developed in Section 2.5. We believe the techniques we use are broadly applicable elsewhere within surface dynamics. For example:

Theorem C (cf. Theorem 2.4.2)**.** Let ψ be a pseudo-Anosov on the genus-two surface with one boundary component, with singularity type $(4; \emptyset; 3^2)$. Then, ψ is conjugate to a pseudo-Anosov carried by the train track depicted on the bottom left in Figure 2.6. A similar statement holds for the closed genus-two surface.

This result suggests a strategy for systematically studying the set of dilatations of pseudo-Anosovs in genus two. Indeed, Theorem C reduces the study of dilatations in the stratum $(4; \emptyset; 3^2)$ to the study of a special collection of maps on a single graph. One would hope for the development of a *kneading theory* generalizing the classical theory of Milnor-Thurston for interval maps (cf. [**MT**]). Applying such a theory to a small list of tracks in each of the other strata in genus two would provide a much clearer understanding of the full dilatation spectrum. This line of study was suggested to us by Chenxi Wu. Another step in this direction is the following result:

Theorem D. Let ψ be a pseudo-Anosov on the punctured disk with at least one k-pronged singularity away from the boundary with $k \geq 2$. Then ψ is carried by a standardly embedded train track τ with no joints.

See Definition 2.5.1 for the definition of a standardly embedded train track. A *loop switch* of a standardly embedded train track τ is a switch at a loop surrounding a 1-prong singularity (cf. Definition 2.5.16), and a *joint* is a loop switch that is incident to more than one expanding edge (cf. Definition 2.5.24). The track in Theorem C is the lift of a joint-less track on the punctured disk, so Theorem C may be seen as a specific case of Theorem D. See subsection 2.5 for the proofs of Theorems C and D.

Applications to the Floer homology of branched covers

For a knot $K \subset S^3$, let $\Sigma_n(K)$ denote the n-fold cyclic cover of S^3 branched along K. There has been much interest recently in the Floer homology of $\Sigma_n(K)$ in terms of K. For example, Boileau–Boyer–Gordon have studied extensively in [10] and [11] the set of all integers $n \geq 2$ such that $\Sigma_n(K)$ is an L-space (see also e.g. [68] and [103]). One question that has persisted in this area is the following:

Question 2.1.2 (Boileau–Boyer–Gordon, Moore)**.** Can $\Sigma_n(K)$ be an L-space for K a hyperbolic L-space knot?

Combining Theorem A with ([10], Corollary 1.4) yields the following complete answer to this question for $n > 2$:

Corollary 2.1.3. If K is an L-space knot and $\Sigma_n(K)$ is an L-space for some $n > 2$, then K is either $T(2,3)$ or $T(2,5)$. In particular, K is not hyperbolic.

Applications to instanton Floer homology and Dehn surgery

For a 3-manifold Y, let $R(Y) = \mathrm{Hom}(\pi_1(Y), SU(2))$ denote the $SU(2)$-representation variety. We say that a 3-manifold Y is $SU(2)$-*abelian* if $R(Y)$ contains no irreducibles. The name is motivated by the fact that Y is $SU(2)$-abelian if and only if every $\rho \in R(Y)$ has abelian image.

Following work initiated by Kronheimer–Mrowka in their proof of the Property P conjecture [72], Baldwin–Li–Sivek–Ye [6], Baldwin–Sivek [4], and Kronheimer–Mrowka [**KM2**] proved that r-surgery $S^3_r(K)$ on a nontrivial knot $K \subset S^3$ is not $SU(2)$-abelian for all slopes $r \in [0,3] \cup [4,5)$ with prime power numerator, and for some additional slopes $r \in [3,4)$.

The key theory which facilitates most of these results is the instanton Floer homology of the surgered manifold $S^3_r(K)$ (and related techniques arising from this theory, as in [**KM2**]). Combining Theorem B with ([6], Proposition 2.4) allows us to prove an analogue of Theorem A for instanton Floer homology:

Corollary 2.1.4. The cinquefoil $T(2,5)$ is the only genus-two instanton L-space knot, i.e. the only genus-two knot K for which dim $I^{\#}(S^3_r(K)) = |H_1(S^3_r(K))|$ for some $r > 0$.

Now, as described in ([6], Section 1.3), Corollary 2.1.4 completes the set of slopes r for which $S^3_r(K)$ is not $SU(2)$-abelian, to all rational numbers $r \in [0,5)$ with prime power numerator:

Corollary 2.1.5. Let $K \subset S^3$ be a nontrivial knot, and $r \in [0,5)$ a rational number with prime power numerator. Then, $S^3_r(K)$ is not $SU(2)$-abelian.

Remark 2.1.6. Note that $S^3_r(K)$ may in general be $SU(2)$-abelian for $r \geq 5$, as $S^3_5(T(2,3))$ is the lens space $L(5,1)$, which has abelian fundamental group. However, Baldwin–Li–Sivek–Ye in [6] have extended the slopes for which $S^3_r(K)$ is not $SU(2)$-abelian to some additional $r \in (5,7)$. It is an open question whether $S^3_r(K)$ is $SU(2)$-abelian for *all* rational numbers $r \in [0,5)$, though it is known to be true for $r \in [0,2]$ by work of Kronheimer–Mrowka in [**KM2**].

Applications to Khovanov homology

In [3], Baldwin–Hu–Sivek proved that Khovanov homology (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) detects the cinquefoil $T(2,5)$. Combining Theorem A with previous work of Baldwin–Dowlin–Levine–Lidman–Sazdanovic ([5], Corollary 2), we can improve Baldwin–Hu–Sivek's result from $\mathbb{Z}/2\mathbb{Z}$-coefficients to $\mathbb{Q}$-coefficients:

Corollary 2.1.7. If $Kh(K;\mathbb{Q}) \cong Kh(T(2,5);\mathbb{Q})$ as bi-graded $\mathbb{Q}$-vector spaces, then $K = T(2,5)$.

We also obtain detection results in annular Khovanov homology. One may think of $T(2,5)$ as the lift of the braid axis for the 5-braid $B = \sigma_1\sigma_2\sigma_3\sigma_4$ in S^3 seen as the double-branched cover over $\widehat{B}$ under the Birman–Hilden correspondence (see subsection 2.2 for background). From this perspective, we can adapt techniques of Binns–Martin in ([8], Theorems 10.2, 10.4, 10.7) to prove that annular Khovanov homology detects the aforementioned braid closure:

Corollary 2.1.8. Let $L \subset A \times I$ be an annular link with $AKh(L;\mathbb{Q}) \cong AKh(\widehat{B};\mathbb{Q})$. Then, L is isotopic to $\widehat{B}$ in $A \times I$.

The proof of this corollary is almost identical to those of the analogous results proved by Binns–Martin. So, we omit the proof in the present work, and instead refer the reader to [8].

Outline of the proof of Theorem B

Let K be a hyperbolic, genus-two, fibered knot in S^3 with associated open book decomposition (S, h) and pseudo-Anosov representative ψ_h. If $c(K) \neq 0$ and K is fixed point free, it follows from work of Baldwin–Hu–Sivek (see Theorem 2.2.5) that:

- ψ_h has singularity type either:
  Case 1: $(6; \emptyset; \emptyset)$, or
  Case 2: $(4; \emptyset; 3^2)$
- h is the lift of a 5-braid β with unknotted braid closure $\widehat{\beta}$
- The pseudo-Anosov representative ψ_β of the braid β has singularity type either:
  Case 1: $(3; 1^5; \emptyset)$
  Case 2: $(2; 1^5; 3)$

For our conventions of the singularity types, see Section 2.2. Cases 1 and 2 are mutually exclusive, and we will use notation $K, h, \beta, \psi_h, \psi_\beta$ as above for the rest of this outline, in either case.

Case 1 is dealt with in Section 2.3. In this case, Masur–Smillie proved in [89] that the foliations preserved by ψ_h are orientable, so that the dilatation of ψ_h is a root of the Alexander polynomial $\Delta_K(t)$. This is a special fact about the stratum $(6; \emptyset; \emptyset)$ in genus two. Using the Lefschetz fixed point theorem and basic facts about Alexander polynomials of fibered knots in S^3, we completely determine the Alexander polynomial of K and conclude that the dilatation of ψ_h coincides with the minimal dilatation λ_2 for genus-two pseudo-Anosovs. Work of Lanneau–Thifeault in [81] further implies that ψ_h is the almost unique genus-two pseudo-Anosov realizing λ_2 as its dilatation. It follows that β is (up to inverse and composing with the hyperelliptic involution) conjugate to the dilatation-minimizing 5-braid α from [44] within the mapping class group of the punctured sphere. We then show that no braid which is conjugate to α within the spherical mapping class group has unknotted closure.

Case 2 is harder: we perform our analysis using a splitting argument and a careful combinatorial analysis of train track maps. In this case, we focus on the braid β and its pseudo-Anosov representative ψ_β. We show (Theorem 2.4.2) that any pseudo-Anosov ψ_β on the five-punctured disk in the stratum $(2; 1^5; 3)$ is carried by a single canonical train track τ (cf. also Theorems C and D). To prove this result, we develop a theory of "tight splitting" in Section 2.5, which allows us to study the splitting of *all* train track maps on any given track within the stratum.

In subsection 2.4, we find a collection of braids β_n inducing special train track maps $f_n : \tau \to \tau$ on the distinguished track τ from the previous paragraph. We show, in subsection 2.4, that f_n are the *only* maps on τ which could lift to train track maps for fixed-point-free pseudo-Anosovs ψ_h in the cover. So, if β is any braid which lifts to a map h with fixed-point-free pseudo-Anosov representative ψ_h, then β is conjugate (in the mapping class group of the punctured sphere) to one of β_n. A similar argument as in Case 1 shows that no braid conjugate to β_n has unknotted closure.

2.2 Background

Mapping classes and fractional Dehn twists

Let $S = S^r_{g,n}$ be a compact surface of genus g with n marked points and r boundary components. The *mapping class group* of S is the group $\mathrm{Mod}(S)$ of isotopy classes of homeomorphisms $h : S \to S$ which fix ∂S point-wise, and permute the marked points of S, where the isotopies fix all boundary components and marked points. The *symmetric mapping class group* of S is the analogous group $\mathrm{SMod}(S)$ obtained by additionally requiring that the homeomorphisms commute with the hyperelliptic involution $\iota : S \to S$, i.e. $h \circ \iota = \iota \circ h$.

Definition 2.2.1. A *pseudo-Anosov* is a homeomorphism $\psi : S \to S$ preserving a pair of transverse singular measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ such that

$$\psi \cdot (\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u), \text{ and } \psi \cdot (\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$$

for some fixed real number $\lambda > 1$, called the *dilatation* of ψ.

We further require that each singularity p of $\mathcal{F}^u$ or $\mathcal{F}^s$ is a "k-pronged saddle," as in Figure 2.1, where $k \geq 3$ for p in the interior of S, or $k \geq 1$ for p a marked point or puncture. Along ∂S, the singular points must all have a neighborhood of the form shown on the bottom left of Figure 2.1.

By a *k-prong boundary singularity* or a *k-prong singularity on the boundary*, we mean that ψ has k singular points on a particular boundary component of S. The *singularity type* of ψ is the tuple $(b_1, ..., b_r; m_1, ..., m_n; k_1, ..., k_s)$ where the i^{th} boundary component has b_i-prongs, the i^{th} puncture or marked point has m_i-prongs, and the i^{th} interior singularity has k_i prongs. We will use $\emptyset$ if ψ has no boundary

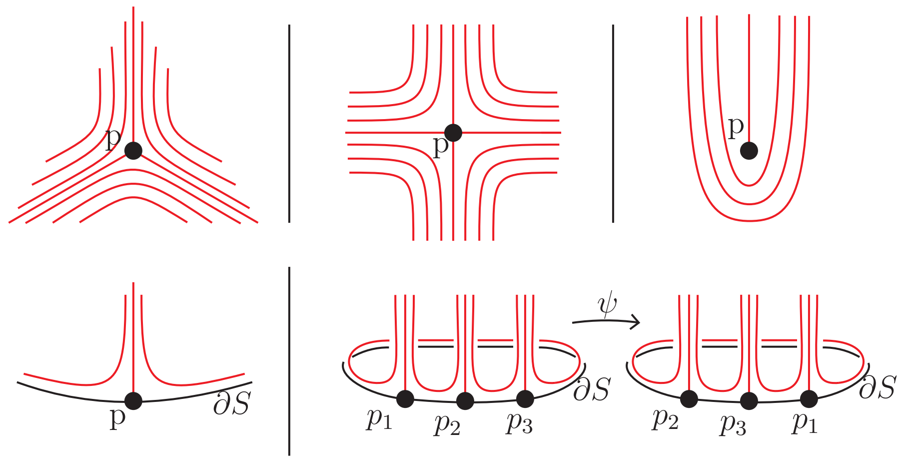


Figure 2.1: Top (left to right): a 3-pronged saddle, a 4-pronged saddle, and a 1-pronged saddle at a marked point. Bottom (left to right): the neighborhood of a boundary singularity, and a 3-pronged boundary which permutes the first prong to the second.

(or if ψ has no marked points or punctures, or interior singularities), and we will use an exponent to denote a repeated number of prongs. For example, the tuple $(3; 1^5; \emptyset)$ indicates that ψ has a 3-pronged boundary, five 1-pronged marked points or punctures, and no interior singularities. A *stratum* on S is the collection of all pseudo-Anosovs on S with a given singularity type.

Theorem 2.2.2 (Nielsen–Thurston classification)**.** Any element $h \in \mathrm{Mod}(S)$ is freely-isotopic rel. punctures (i.e. isotopic through homeomorphisms which fix the punctures, but may rotate a boundary component) to a representative ψ with at least one of the following properties:

(1) $\psi^n = \mathrm{id}$ for some n,

(2) ψ preserves the isotopy class of a multicurve C on S, or

(3) ψ is pseudo-Anosov. This case is disjoint from the previous two.

We say that such a ψ satisfying one of the properties (1)—(3) is *geometric*, and in case (3), we refer to ψ as the *pseudo-Anosov representative* of h. The representative ψ is unique for any such h, although when S has non-empty boundary, ψ will never be isotopic rel. boundary to an element of $\mathrm{Mod}(S)$, as ψ may rotate ∂S.

The *fractional Dehn twist coefficient* $c(h) = n + m/k$ is a rational number which measures the rotation of ψ along ∂S. Here, n is an integer measuring the number of full-rotations of h along ∂S; k is the number of prongs of ψ on ∂S; and ψ cyclically permutes the endpoint of the first prong to that of the $(m+1)^{th}$ prong. For example, in the bottom right of Figure 2.1, the "fractional part" of $c(h)$ is $1/3$, where $h \in \mathrm{Mod}(S)$ has pseudo-Anosov representative ψ. In particular $c(h) \in \mathbb{Z}$ if and only if the rotation number of ψ along the boundary is zero, and when ψ has a single boundary prong this is always the case.

The fractional Dehn twist coefficient can be extended analogously to braids, thought of as elements of $\mathrm{Mod}(S^1_{0,n})$ for some $n > 1$. In this case, we denote it by $c(\beta)$, where β is a braid. When K is a fibered knot with open book decomposition (S, h), we define the fractional Dehn twist coefficient $c(K)$ to be that of h, i.e. $c(K) := c(h)$. It is crucial to note that, in general, $c(K)$ is *not* the same as $c(\beta)$ for β a braid representative of K. The following theorem details a few well-known properties which we will make use of in this paper:

Theorem 2.2.3 ([49],[104],[69])**.** Let $h : S \to S$ be a mapping class with ∂S connected, and let $\beta : S^1_{0,n} \to S^1_{0,n}$ be a braid.

- $c(h)$ and $c(\beta)$ are preserved under conjugation.
- $c(D^m_{\partial S} \circ h^k) = m + kc(h)$ for any $k, m \in \mathbb{Z}$, where $D_{\partial S}$ is a Dehn twist along ∂S.
- $c(\Delta^{2m}\beta^k) = m + kc(\beta)$ for any $k, m \in \mathbb{Z}$, where $\Delta^2 = (\sigma_1 ... \sigma_{n-1})^n$.
- If β is σ_1-positive (i.e. β can be written with only positive powers of σ_1) then $c(\beta) \geq 0$.
- If β is a positive pseudo-Anosov braid, then $c(\beta) > 0$.

See [115] for more details regarding the Nielsen–Thurston classification, and [49] or [70] for more details regarding fractional Dehn twist coefficients.

Knots, braids, and the Birman–Hilden correspondence

We may use branched coverings to understand relationships between mapping class groups of different surfaces. The Birman–Hilden correspondence is a key tool for this study. For our purposes, the correspondence will help us study the mapping class groups of S^1_2 and S_2 seen as two-fold branched covers over the disk $S^1_{0,5}$ and sphere $S_{0,6}$, respectively, via the hyperelliptic involution ι. Specifically, there is a diagram:

$$\begin{array}{ccccc} \mathrm{SMod}(S_2^1) & \xrightarrow{\text{cap-off}} & \mathrm{SMod}(S_{2,1}) & \xrightarrow{\text{forget}} & \mathrm{Mod}(S_2) \\ \Theta_2^1 \downarrow & & & & \downarrow \Theta_2 \\ \mathrm{Mod}(S_{0,5}^1) & & \xrightarrow{\text{cap-off}} & & \mathrm{Mod}(S_{0,6}) \end{array}$$

Here, the map Θ_2^1 is an isomorphism, and the map Θ_2 is surjective with $\ker(\Theta_2) = \langle \iota \rangle$. The cap-off maps are both given by setting a full-twist about ∂S to 0 (geometrically, one may think about capping ∂S_2^1 with a *marked* disk), and the "forget" map forgets about the marked point on $S_{2,1}$. The cap-off map forgets about the "integer part" of the fractional Dehn twist coefficient but preserves the "fractional part" in each case (i.e. the rotation number of ψ along ∂S is the rotation number of the capped-off map $\widehat{\psi}$ at the marked point in the capping disk). See [38] for more details on the maps involved in this diagram.

Note that the braid $\Delta^2 = (\sigma_1\sigma_2\sigma_3\sigma_4)^5 \in \mathrm{Mod}(S_{0,5}^1)$ is isotopic to a full-twist about $\partial S_{0,5}^1$. It follows that, given a spherical mapping class $f \in \mathrm{Mod}(S_{0,6})$, there are $\mathbb{Z}$-many lifts of f to braids

$$...\Delta^{-4}\beta,\ \Delta^{-2}\beta,\ \beta,\ \Delta^2\beta,\ \Delta^4\beta... \in \mathrm{Mod}(S_{0,5}^1)$$

which are all related by powers of Δ^2, and are distinguished by their fractional Dehn twist coefficient $c(\beta)$, as in Theorem 2.2.3. For any such f, only finitely many such β may have braid closure $\widehat{\beta}$ an unknot. This may be seen, for example, from the following theorem of Ito–Kawamuro, which will be a key tool in this paper:

Theorem 2.2.4 (Ito–Kawamuro)**.** If the braid closure $\widehat{\beta}$ is an unknot, then $|c(\beta)| < 1$.

Moreover, one may check that Δ^2 lifts to an element of $\mathrm{SMod}(S_2^1)$ which squares to a full twist about ∂S_2^1. This implies a very useful fact: the lift of Δ^2 is freely isotopic to the hyperelliptic involution ι. One may see this in a number of different ways— for example, by noting that the full twist about ∂S_2^1 is freely isotopic to the identity, and that the lift of Δ^2 acts on $H_1(S_2^1)$ by $-\mathrm{id}$.

A genus-two, hyperbolic, fibered knot $K \subset Y$ yields an open book decomposition (S, h) for Y, where $S = S_2^1$, and $h \in \mathrm{Mod}(S_2^1)$ is freely isotopic to a pseudo-Anosov $\psi_h : S \to S$. If ψ_h is fixed point free in the interior of S, then h is symmetric (i.e. represents an element of $\mathrm{SMod}(S_2^1)$) by [3], so we may think of h as the lift of a 5-braid $\beta \in \mathrm{Mod}(S_{0,5}^1)$. From the perspective of 3-manifolds, this means that Y is the double cover of S^3 branched along the braid closure $\widehat{\beta}$. This implies, for example, that if $Y = S^3$ then $\widehat{\beta}$ is the unknot. From the perspective of knots, the original fibered knot $K = \partial S_2^1 \subset Y$ is the lift of the braid axis $\partial S_{0,5}^1 \subset S^3$ in the cover.

Baldwin–Hu–Sivek in [3] additionally computed the singularity type of ψ_h (and originally observed several of the facts mentioned in the previous paragraph). These observations are recorded as the following result, which is the starting point for many of the ideas in this paper:

Theorem 2.2.5 (Baldwin–Hu–Sivek)**.** Let $K \subset S^3$ be a hyperbolic, genus-two, fibered knot with associated open book decomposition (S, h) satisfying $c(h) \neq 0$. If the pseudo-Anosov representative ψ_h is fixed point free in the interior of S, then:

- ψ_h has singularity type either:
  Case 1: $(6; \emptyset; \emptyset)$, or
  Case 2: $(4; \emptyset; 3^2)$
- h is the lift of a 5-braid β under the Birman–Hilden correspondence, and
- as a knot in S^3, the braid closure $\widehat{\beta}$ is the unknot.

Theorem 2.2.5 yields strong constraints on the braid β. For example, because h has pseudo-Anosov mapping class, we know that β does, too, and we may determine the singularity type of its pseudo-Anosov representative ψ_β from that of ψ_h, by appealing to Lemma 3.7 of [3]. If ψ_h has singularity type $(6; \emptyset; \emptyset)$ then ψ_β has singularity type $(3; 1^5; \emptyset)$. And, if ψ_h has singularity type $(4; \emptyset; 3^2)$ then ψ_β has singularity type $(2; 1^5; 3)$.

Fibered surfaces and train tracks

For the remainder of the paper, we will denote by S' the surface S with its marked points deleted, and by $\widehat{S}$ the closed surface obtained by capping-off the boundary components of S with disks and marking a point in the interior of each disk. We will also assume that the surface S' has negative Euler characteristic.

In [7], Bestvina and Handel prove that one may associate to any geometric ψ a *fibered surface* $F \subseteq S'$. This fibered surface is decomposed into *strips* and *junctions*, where the strips are foliated by intervals, i.e. *leaves*. See Figure 2.2. Together, the leaves and junctions of F are called *decomposition elements*, and $\psi(F) \subseteq F$, sending decomposition elements into decomposition elements and, in particular, junctions into junctions. Collapsing each decomposition element to a point produces a graph G with a graph map $g : G \to G$. The vertices of G correspond to the junctions of F, and the edges of G correspond to the strips of F.

Roughly speaking, a graph map g is *efficient* if the image of no edge backtracks under any power of g. After adjusting F so that g is efficient, Bestvina and Handel

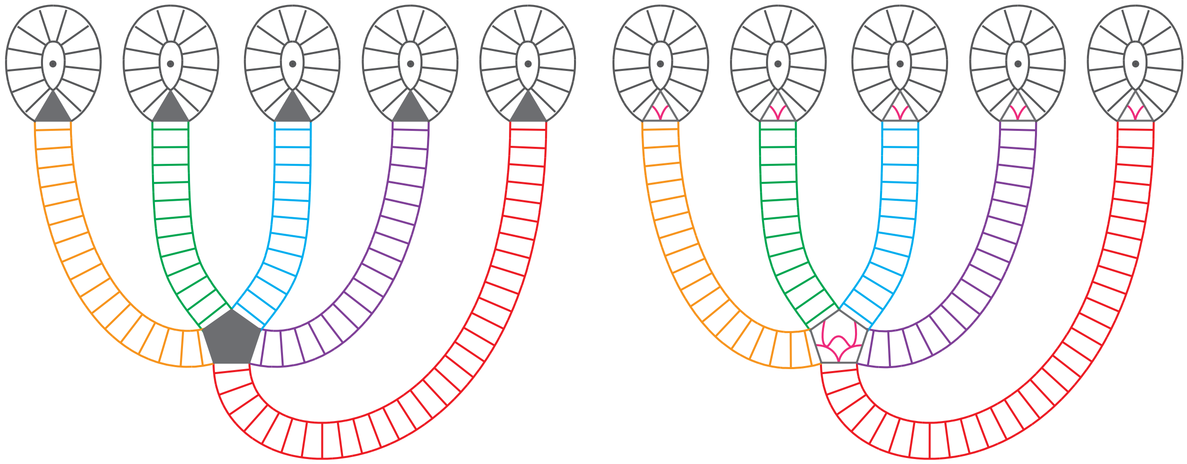


Figure 2.2: Left: the fibered surface for some geometric ψ on $S^1_{0,5}$. The shaded regions are the junctions, and the striped bands connecting them are the strips. Right: following Bestvina-Handel, one inserts additional, "infinitesimal" edges into the junctions. These will also be inserted into the graph G that one obtains by collapsing all of the decomposition elements. Their inclusion will produce the smooth analog τ of G, which is a train track. See Figure 2.3 below.

construct a "smoothed" version of G as follows. Within each junction $J \subseteq F$, one inserts additional edges that smoothly connect the strips of F and encode how images of strips under ψ pass through J.

In this way, we obtain a new graph τ smoothly embedded in the punctured surface S', called a *train track*. At each vertex s of τ, called a *switch*, there is a well-defined tangent line. Two arcs a, b of τ are *tangent* at s if $a(0) = b(0) = s$ and $a'(0) = b'(0)$. A *cusp* is the data of a pair (a, b) of adjacent arcs tangent at s. See Figure 2.3 for an example.

The following proposition appears as Proposition 3.3.5 in [7].

Proposition 2.2.6. Suppose ψ is pseudo-Anosov. Then in the capped surface, each component of $\widehat{S} \setminus \tau$ is either:

1. a disk with $k \geq 3$ cusps on its boundary, or

2. a disk with a single marked point in its interior and $k \geq 1$ cusps on its boundary.

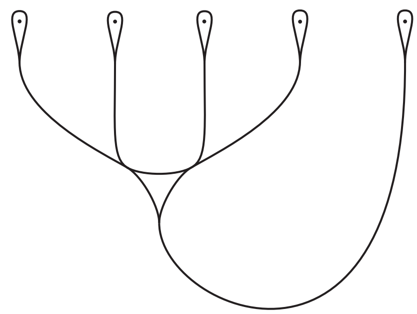


Figure 2.3: A train track τ on the five-punctured disk D_5. The components of the complement of τ consist of: five once-punctured *monogons*, i.e. disks with a single boundary cusp; a *trigon*, i.e. a disk with three boundary cusps; and an exterior once-punctured bigon. Pseudo-Anosovs carried by this track lie in the stratum $(2; 1^5; 3)$.

Remark 2.2.7. From this perspective, the cusps of a component $C \subset \widehat{S} \setminus \tau$ correspond precisely to the prongs of a singularity $p \in C$ of the invariant foliations of ψ.

Definition 2.2.8. An *edge path* in τ is a map $e : I \to \tau$ such that $e(0)$ and $e(1)$ are switches. A *train path* is an edge path that is also a smooth immersion. The *length* of a train path e is defined to be the number of edges traversed by $e(I)$, counting with multiplicity. Let $e(I) = e_1 \cdots e_k$ denote a train path whose directed image traverses first e_1, then e_2, etc. See Figure 2.4 for examples.

Definition 2.2.9. A *train track map* is a map $f : \tau \to \tau$ such that for any train path $g : I \to \tau$ the composition $f \circ g : I \to \tau$ is a train path.

Remark 2.2.10. Note that if $f : \tau \to \tau$ is a train track map, then $f(e)$ is a train path for each edge e of τ. Indeed, from this it follows that $f^k(e)$ is a train path for each $k \geq 1$, and hence f^k is a train track map for all $k \geq 1$.

The map $\psi : F \to F$, or equivalently the graph map $g : G \to G$ corresponding to ψ and F, defines a map $f : \tau \to \tau$. The fact that g is efficient implies that f is a train track map. In this case, we say that the train track τ *carries* the map ψ, and the map ψ *induces* the train track map f. The *data* of a geometric map will then be a triple (τ, ψ, f) in a commutative diagram:

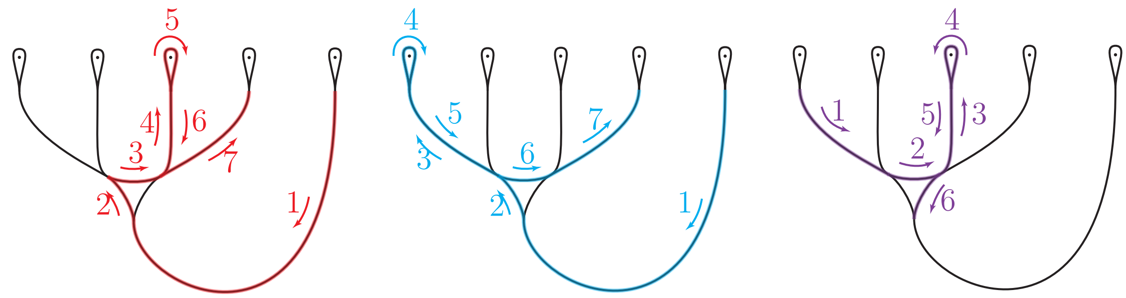


Figure 2.4: Three edge-paths on the train track τ from Figure 2.3. **Left:** an edge path of length 7 which is not a train path, since it makes several sharp turns. **Middle:** a train path of length 7 which can be "pushed off" of τ into a small neighborhood so that it does not intersect itself. **Right:** a train path of length 6 which cannot be "pushed off" of τ so that it becomes injective.

$$\begin{array}{ccc} \tau & \xrightarrow{\psi} & \psi(\tau) \\ & {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle \text{collapse}} \\ & & \tau \end{array}$$

Here, one should imagine τ being mapped forward by ψ into F, meeting the leaves of F transversely. The map f is then defined by collapsing each leaf of F to a point, while inside each junction the arcs of $\psi(\tau)$ are collapsed onto the appropriate edges of τ. See Figure 2.5.

The edges of G (other than those loops peripheral to marked points/punctures of S) are in bijection with a subset of the edges of τ, which we call the *real* edges. All other edges of τ are *infinitesimal*. In particular, all edges of τ contained in a junction of F are infinitesimal. Enumerate the edges of τ so that $e_1, \ldots, e_k$ are the real edges and $e_{k+1}, \ldots, e_n$ are the infinitesimal edges. For each pair (i, j) with $1 \leq i, j, \leq n$ define the integer

$$m_{i,j} = \text{the number of times the train path } f(e_j) \text{ traverses } e_i.$$

The *extended transition matrix* of f is the matrix $\widetilde{M}$ whose (i, j)-entry is the integer $m_{i,j}$. The *transition matrix* of f is the submatrix $M \subset \widetilde{M}$ recording the transitions between real edges of τ: in other words,

$$M = (m_{i,j}) \text{ where } 1 \leq i, j \leq k.$$

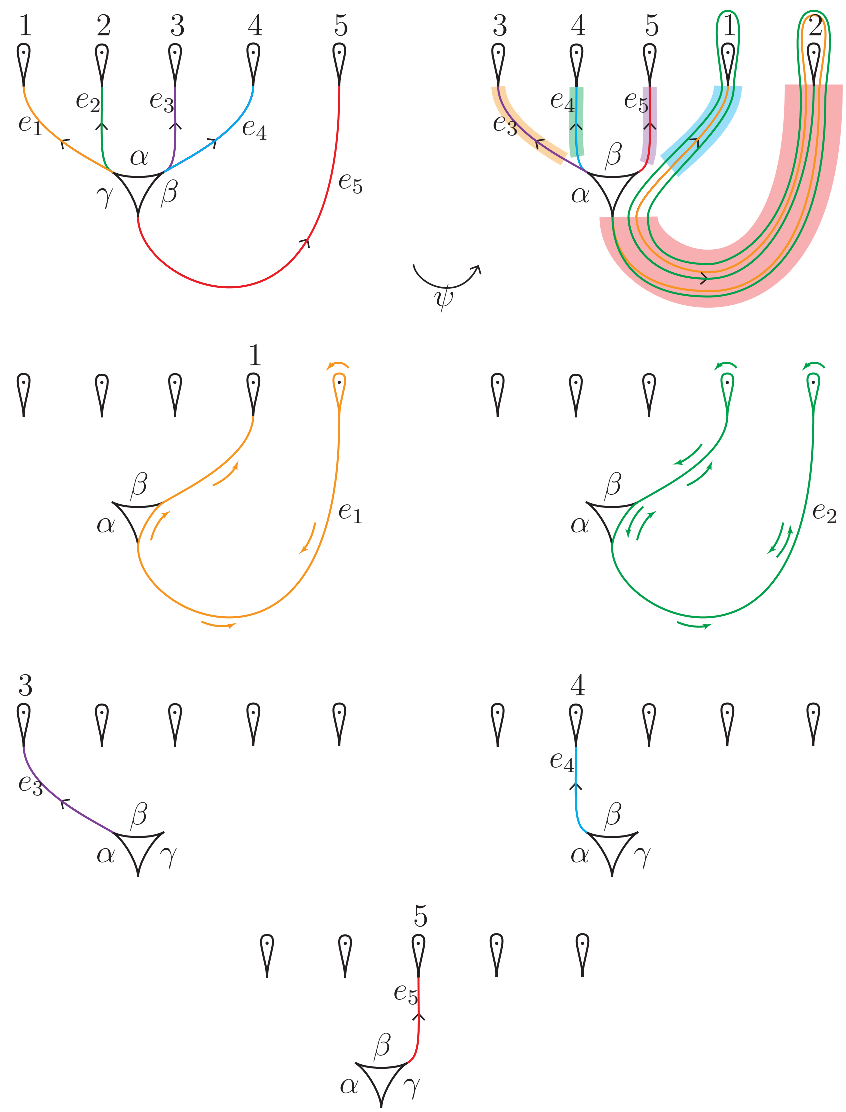


Figure 2.5: **Top row:** the train track τ from Figure 2.3 and the action of a pseudo-Anosov ψ that it carries. The real edges of τ are labeled $e_1, \ldots, e_5$. The shaded regions on the right denote the neighborhoods that deformation retract onto these edges. **Bottom three rows:** The action of $f = (\text{collapse} \circ \psi)$ on each edge of τ, depicted separately.

Definition 2.2.11. Let M be a square matrix whose entries are non-negative integers. We say that M is *Perron-Frobenius* if the entries of M^N are strictly positive, for some power N. In this case, the Perron-Frobenius theorem states that the eigenvalue of M of largest absolute value is in fact real, simple, and has an eigenvector all of whose entries are positive. We call this eigenvalue the *Perron-Frobenius eigenvalue*, and we say that such an eigenvector is *positive*.

The next theorem follows from work of Bestvina-Handel in [7].

Theorem 2.2.12. Let (τ, ψ, f) be the data of a geometric map, where τ satisfies the conclusion of Proposition 2.2.6. Let M be the transition matrix of f. Then

$$\psi \text{ is pseudo-Anosov} \iff M \text{ is Perron-Frobenius.}$$

The Perron-Frobenius eigenvalue λ of M is called the *dilatation* of ψ. There is a unique right λ-eigenvector w of M, up to scale, and its entries w_i for $i = 1 \ldots, k$ define *transverse weights* on the real edges e_i of M.

Remark 2.2.13. If a train track map F is induced by some pseudo-Anosov ψ, then such a ψ is unique up to conjugacy in $\mathrm{Mod}(\widehat{S})$. Indeed, Bestvina–Handel in [7] provide an algorithm to determine the measured foliations preserved by ψ. This will be a crucial idea in Section 2.4.

Lifted train track maps and fixed points

Let (τ, ψ, f) be the data of a pseudo-Anosov on $S = S_2^1$. One of the key ideas in this paper is to use the transition matrix M of f to study fixed points of ψ combinatorially. Our main tool to carry out this approach is the following theorem, which follows from work of Los in [88] and independently Cotton-Clay in [29]:

Theorem 2.2.14 (Los, Cotton-Clay)**.** If ψ is fixed point free in the interior of S, then $\mathrm{tr}(M) = 0$.

Our goal will be to use Theorem 2.2.14 to restrict the possible train tracks τ and maps $f : \tau \to \tau$ for a fixed-point-free ψ. In the case of genus-two, hyperbolic, fibered knots in S^3, the possible types of train tracks τ are already highly restricted by Theorem 2.2.5 and Remark 2.2.7.

Suppose we start with a genus-two, hyperbolic, fibered knot K with open book decomposition (S, h), where h is the lift of a 5-braid β with pseudo-Anosov data (ψ_β, τ, f) on the disk $S_{0,5}^1$. We may lift τ to a train track $\widetilde{\tau}$ on S which carries the pseudo-Anosov representative ψ_h of h. As a graph, $\widetilde{\tau}$ is constructed by gluing

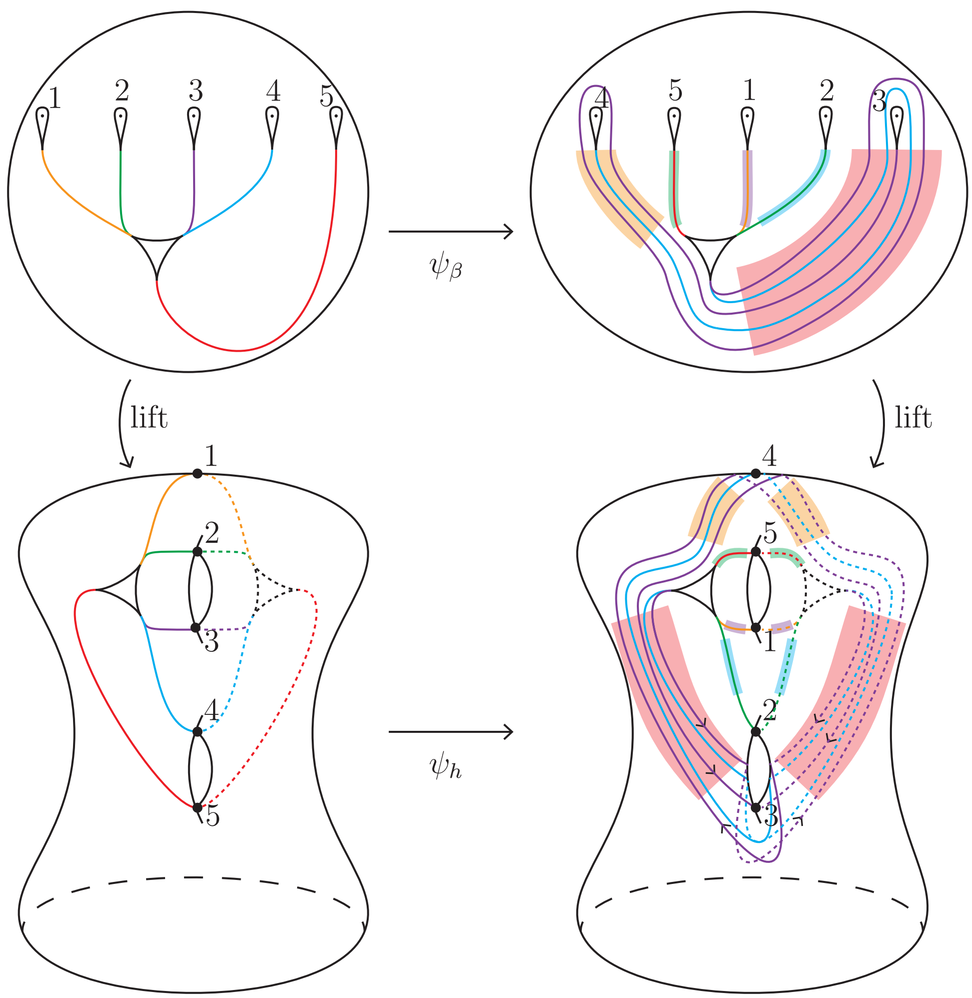


Figure 2.6: Lifting (ψ_β, τ, f) to $(\psi_h, \widetilde{\tau}, \widetilde{f})$. The dotted lines indicate the "back" of the surface. In this example, τ is the Peacock track from Figure 2.7 and β is conjugate to β_0^{-1}, from Proposition 2.4.4. Note here the 1-gons on the disk lift to 2-gons upstairs, which are smoothed out to regular points.

two copies of τ along the punctures and lifting the infinitesimal k-gons around these punctures to infinitesimal $2k$-gons upstairs, as in Figure 2.6.

For this choice of train track $\widetilde{\tau}$, we may determine the train track map $\widetilde{f} : \widetilde{\tau} \to \widetilde{\tau}$ induced by ψ_h (so that h has pseudo-Anosov data $(\psi_h, \widetilde{\tau}, \widetilde{f})$) from $f : \tau \to \tau$, up to a binary choice, as follows. For each edge $e \in \tau$, denote by e^1 or e^2 its two lifts in $\widetilde{\tau}$, and write $f(e) = e_1 ... e_n$ where each $e_i \in \tau$ is an edge. Note that if two edges $e, e' \in \tau$ form a train path ee' then exactly one of the paths $e^i e'^1$ or $e^i e'^2$ is a train path for each $i = 1, 2$. And, if $e^1 e'^i$ is a train path, then so is $e^2 e'^j$, where $i \neq j$. Set $\widetilde{f_{i_1}}(e^1) = e_1^{i_1} ... e_n^{i_n}$, where each $i_j \in \{1, 2\}$ and each $e_j^{i_j} e_{j+1}^{i_{j+1}}$ is a train path. Choosing an image for e^1 also immediately determines an image for e^2, so the maps $\widetilde{f_1}$ and $\widetilde{f_2}$ are both defined on all of $\widetilde{\tau}$. See Figure 2.6.

Note that if $\widetilde{f_i}$ is induced by ψ_i for $i = 1, 2$ then we have $\psi_1 = \iota \circ \psi_2$. In particular, at most one of $\widetilde{f_1}$ or $\widetilde{f_2}$ is induced by ψ_h, but this choice may be easily settled by examining β as a braid, rather than a mapping class on the punctured sphere. So, we will denote simply by $\widetilde{f}$ the well-defined choice of $\widetilde{f_i}$ induced by ψ_h.

2.3 The case with singularity type $(6; \emptyset; \emptyset)$

The main goal of this section is to prove the following:

Theorem B1. Let K be a genus-two, fibered, hyperbolic knot in S^3 with associated open book decomposition (S_2^1, h). If $c(h) \neq 0$ and the pseudo-Anosov representative ψ of h has singularity type $(6; \emptyset; \emptyset)$, then K is not fixed point free.

This resolves Case 1 from the outline in subsection 2.1. Together with Theorem B2 in Section 2.4, this will complete the proof of Theorem B. Before turning to the proof of Theorem B1, it will be helpful to recall the Lefschetz fixed point theorem, which will be a key ingredient in our proof:

Theorem 2.3.1 (Lefschetz fixed point theorem)**.** Let S be a compact surface and $f : S \to S$ a homeomorphism. Let $f_* : H_1(S; \mathbb{Z}) \to H_1(S; \mathbb{Z})$ denote the induced map on first homology. Then

$$2 - \mathrm{tr}(f_*) = \sum_{p = f(p)} \mathrm{Ind}(f, p).$$

We will apply the Lefschetz fixed point theorem to read off information about the action of a pseudo-Anosov on homology, from its dynamical properties. The relevant result in this vein is an index calculation due to Lanneau–Thiffeault in [81]:

Proposition 2.3.2 (Lanneau–Thiffeault)**.** Let $\psi : S \to S$ be pseudo-Anosov with orientable invariant foliations, and let p be a fixed k-prong singularity of ψ. Denote by $\psi_* : H_1(S) \to H_1(S)$ the action on homology, and denote by $\rho(\psi_*)$ the leading eigenvalue of this action, i.e. the eigenvalue with greatest absolute value.

1. If $\rho(\psi_*) < 0$ then $\mathrm{Ind}(\psi, p) = 1$; that is, every fixed point of ψ has index 1.
2. If $\rho(\psi_*) > 0$ then either:
   a) ψ fixes each prong and $\mathrm{Ind}(\psi, p) = 1 - k < 0$, or
   b) ψ cyclically permutes the prongs and $\mathrm{Ind}(\psi, p) = 1$.

We can now use Lanneau–Thiffeault's calculation to restrict the dilatation of the pseudo-Anosov representative of a potential fixed-point-free knot K in Theorem B1:

Proposition 2.3.3. Let $K \subset Y$ be a genus-two fixed-point-free knot with $c(K) \notin \mathbb{Z}$, and suppose that Y is an integer homology sphere. If the pseudo-Anosov representative ψ of K has singularity type $(6; \emptyset; \emptyset)$ then ψ achieves the minimal dilatation λ_2 among pseudo-Anosovs in genus two.

Proof. Let (S_2^1, h) be the open book decomposition of Y associated to K, and suppose ψ has no fixed points in the interior of S. Because ψ has singularity type $(6; \emptyset; \emptyset)$ by assumption, we may cap-off ψ to a pseudo-Anosov $\widehat{\psi}$ on S_2 and extend the foliations preserved by ψ over the capping disk. For this stratum on S_2, Masur-Smillie ([89]) prove that the foliations preserved by $\widehat{\psi}$, and therefore by ψ, are necessarily orientable. We will use this fact to apply the Lefschetz fixed point theorem and determine completely the Alexander polynomial of K.

Because K is fibered, the Alexander polynomial Δ_K is equal to the characteristic polynomial $\chi(\psi_*)$ of the action of ψ on homology: $\Delta_K = \chi(\psi_*)$. Because K is a genus-two fibered knot in an integer homology sphere, Δ_K is a monic, degree-four, palindromic polynomial satisfying $\Delta_K(1) = \pm 1$. Moreover, because the fractional Dehn twist coefficient $c(h) \notin \mathbb{Z}$ by assumption, we know $\widehat{\psi}$ rotates the separatrices of the 6-prong singularity p, so that $\mathrm{Ind}(\widehat{\psi}, p) = 1$ regardless of the sign of $\rho(\psi_*)$. And, because p is the unique fixed point of $\widehat{\psi}$ by assumption, it follows from the Lefschetz fixed point theorem that $\mathrm{tr}(\psi_*) = \mathrm{tr}(\widehat{\psi}_*) = 1$.

From the discussion above, we conclude that the coefficients of t^4 and t^0 in $\Delta_K(t)$ are 1, while the coefficients of t^3 and t are $-\mathrm{tr}(\psi_*) = -1$. Now, using the fact that $\Delta_K(1) = \pm 1$, we see:

$$\Delta_K(t) = t^4 - t^3 \pm t^2 - t + 1.$$

Because the foliations preserved by ψ are orientable, the dilatation $\lambda(\psi)$ is a root of Δ_K. The polynomial $t^4 - t^3 + t^2 - t + 1$, however, has no real roots. We deduce:

$$\Delta_K(t) = t^4 - t^3 - t^2 - t + 1.$$

Finally, note that this polynomial has a single root λ_2 greater than 1, which is the minimal dilatation achieved by any pseudo-Anosov on the genus-two surface, see e.g. [81]. □

We will need the following lemma to finish the proof of Proposition B1:

Lemma 2.3.4. Let $h, h' \in \mathrm{SMod}(S_2^1)$ be the lifts of braids β, β'. Suppose that the capped-off maps $\widehat{h}$ and $\widehat{h'}$ on $\widehat{S}$ are conjugate in $\mathrm{Mod}(S_2)$. Then, β is conjugate to $\Delta^{2k}\beta'$ for some $k \in \mathbb{Z}$.

Proof. Because $\widehat{h}$ and $\widehat{h'}$ are conjugate in $\mathrm{Mod}(S_2)$, and the hyperelliptic involution ι on S_2 is in the center of $\mathrm{Mod}(S_2)$, the conjugating mapping class in $\mathrm{Mod}(S_2)$ descends to the spherical mapping class group $\mathrm{Mod}(S_{0,6})$. It follows that β and β' are conjugate after capping-off to $\mathrm{Mod}(S_{0,6})$. In particular, β is conjugate to $\Delta^{2k}\beta'$ for some $k \in \mathbb{Z}$. □

Though we will not need the following corollary for our purposes, it follows quickly from Lemma 2.3.4 and we believe it to be helpful in many other contexts, as well.

Corollary 2.3.5. Let $h, h' \in \mathrm{SMod}(S_g^r)$ be the lifts of braids β, β', for $g, r \in \{1, 2\}$. Then, h and h' are conjugate in $\mathrm{Mod}(S_g^r)$ if and only if β and β' are conjugate as braids.

Proof. For simplicity, suppose $g = 2$ and $r = 1$, though the same proof works for the other cases, with minor adjustments. If β and β' are conjugate, it is clear that h and h' are conjugate, too: we may simply lift the conjugating map to S_2^1. On the other hand, suppose h and h' are conjugate in $\mathrm{Mod}(S_2^1)$. It follows that the capped-off maps $\widehat{h}$ and $\widehat{h'}$ are conjugate in $\mathrm{Mod}(S_2)$. Lemma 2.3.4 now implies that β is conjugate to $\Delta^{2k}\beta'$ for some $k \in \mathbb{Z}$. Because h and h' are conjugate in $\mathrm{Mod}(S_2^1)$, we know that $c(h) = c(h')$ (see Theorem 2.2.3). It follows that $c(\beta) = 2c(h) = 2c(h') = c(\beta')$, whereas $c(\Delta^{2k}\beta') = c(\beta') + k$, so we must have that β and β' are conjugate as braids. □

We need one last result before turning to the proof of Theorem B1.

Proposition 2.3.6. Let (S_2^1, h) be an open book decomposition with $c(h) \notin \mathbb{Z}$, such that h is symmetric and freely isotopic to a pseudo-Anosov ψ with singularity type $(6; \emptyset; \emptyset)$ and dilatation $\lambda(\psi) = \lambda_2$. Then, (S_2^1, h) is not an open book decomposition for S^3.

Proof. Because ψ has singularity type $(6; \emptyset; \emptyset)$, we may cap-off ψ to a pseudo-Anosov on S_2 with singularity type $(\emptyset; \emptyset; 6)$. Lanneau and Thiffeault [81] show that the pseudo-Anosov on S_2 with foliation type $(\emptyset; \emptyset; 6)$ and dilatation λ_2 is unique, up to conjugacy in $\mathrm{Mod}(S_2)$, inverse, and composition with the hyperelliptic involution ι on S_2. Note that the pseudo-Anosov representative ψ_α of the 5-braid $\alpha = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2$ studied by Ham–Song in [44] achieves dilatation $\lambda(\psi_\alpha) = \lambda_2$. In particular, the pseudo-Anosov representative ψ_A of the lift A of α to S_2^1 achieves minimal dilatation $\lambda(\psi_A) = \lambda_2$ with the proper singularity type. It follows that $\widehat{\psi}$ is conjugate in $\mathrm{Mod}(S_2)$ to one of $\widehat{\psi}_A^{\pm 1}$ or $\widehat{\psi}_A^{\pm 1} \circ \iota$. This further implies that $\widehat{h}$ is conjugate in $\mathrm{Mod}(S_2)$ to one of $\widehat{A}^{\pm 1}$ or $\widehat{A}^{\pm 1} \circ \iota$.

Because ι is freely isotopic to the lift of the 5-braid Δ^2 (as described in subsection 2.2), $A^{\pm 1} \circ \iota$ is freely isotopic to the lift of $\Delta^2 \alpha^{\pm 1}$. Since h is symmetric by assumption, it is the lift of a braid β. A is symmetric by construction, so Lemma 2.3.4 implies that β is conjugate as a braid to $\Delta^{2k}\alpha^{\pm 1}$ for some $k \in \mathbb{Z}$. In particular, if (S_2^1, h) is an open book decomposition for S^3, we can see that $\Delta^{2k}\alpha^{\pm 1}$ has unknotted closure for some $k \in \mathbb{Z}$. Moreover, note that the closure of $\Delta^{2k}\alpha$ is unknotted if and only if the closure of $\Delta^{-2k}\alpha^{-1}$ is also unknotted, because the unknot is amphicheiral.

By Theorem 2.2.4, if $\Delta^{2k}\alpha^{\pm 1}$ has unknotted closure, we must have $|c(\Delta^{2k}\alpha^{\pm 1})| < 1$. We may deduce that $0 < c(\alpha) < 1$ because α is a positive pseudo-Anosov braid, and $\Delta^{-2}\alpha$ is a negative pseudo-Anosov braid (see Theorem 2.2.3). In particular, $k < c(\Delta^{2k}\alpha) < k + 1$. So, it suffices to simply check that α and $\Delta^2\alpha^{-1}$ do not have unknotted closure. One may see this in a number of ways— for example by appealing to the self-linking number: the maximal self-linking number of the unknot is -1, but the self-linking numbers $sl(\alpha) = 1$ and $sl(\Delta^2\alpha^{-1}) = 9$ are both positive. □

Proof of Theorem B1. Let K, h, and ψ be as in the statement of the theorem. Recall that since K is fixed point free by assumption, h is symmetric (see Theorem 2.2.5). Since (S_2^1, h) is an open book decomposition for S^3, $|c(h)| \leq 1/2$ (see e.g. [70]). So if $c(h) \neq 0$ then $c(h) \notin \mathbb{Z}$. Proposition 2.3.3 then implies that the dilatation of ψ is $\lambda(\psi) = \lambda_2$, but this contradicts Proposition 2.3.6. □

Remark 2.3.7. Note that our argument does not show that *no* hyperbolic, genus-two, fibered knot in S^3 has the Alexander polynomial $t^4 - t^3 - t^2 - t + 1$ from the proof of Proposition 2.3.3. Rather, any such knot cannot have singularity type $(6; \emptyset; \emptyset)$. Indeed, the knots $11n_{38}$ and $11n_{102}$ on KnotInfo [87] are genus-two fibered hyperbolic knots with the given Alexander polynomial. But, their singularity types are $(1; \emptyset; 3^5)$ and $(2; \emptyset; 3^4)$, respectively, so the foliations preserved by their pseudo-Anosov representatives are not orientable. One may check that their dilatations are 1.916... and 2.751..., respectively, both of which are larger than $\lambda_2 = 1.722...$, and neither of which are roots of the given Alexander polynomial.

In the stratum $(6; \emptyset; \emptyset)$, we may additionally lift the assumption that $c(h) \neq 0$:

Proposition 2.3.8. Let K be a hyperbolic, genus-two, fibered knot in S^3, with associated open book decomposition (S_2^1, h). If $c(h) = 0$ and the pseudo-Anosov representative ψ of h has singularity-type $(6; \emptyset; \emptyset)$, then K is not fixed point free.

Proof. Suppose that ψ has no fixed points in the interior of S_2^1. As in the proof of Theorem B1, we may cap-off ψ to a pseudo-Anosov $\widehat{\psi}$ on S_2 and extend the foliations preserved by ψ. Again, we have that these foliations are orientable. Note that if $\rho(\psi_*) < 0$, then an argument identical to that of Theorem B1 will apply.

So, assuming that $\rho(\psi_*) > 0$, the Lefschetz fixed point theorem then yields $\text{tr}(\widehat{\psi}_*) = 2 - (-5) = 7$, because the unique fixed point p given by the boundary 6-prong singularity is unrotated (since $c(h) \in \mathbb{Z}$). Consider the Markov matrix M for a train track representative of $\widehat{\psi}$. It follows from a theorem of Rykken [106] that any eigenvalue of $\widehat{\psi}_*$ is also an eigenvalue of M (counting multiplicity) except for possibly eigenvalues of 0 or roots of unity. Note here that a train track representative of $\widehat{\psi}$ has 8 real edges, so that M is an 8×8 matrix, while $\widehat{\psi}_*$ is 4×4. In particular, M has at most four more eigenvalues than $\widehat{\psi}_*$, and each has absolute value at most one. Hence:

$$\text{tr}(M) \geq \text{tr}(\widehat{\psi}_*) - 4 = 7 - 4 = 3.$$

On the other hand, a well-chosen train track carrying $\widehat{\psi}$ also carries ψ. In particular, by Theorem 2.2.14, we can see that $\text{tr}(M) = 0$ because ψ is assumed to be fixed point free in the interior of S, which is a contradiction. □

2.4 The case with singularity type $(4; \emptyset; 3^2)$

The goal of this section is to prove the following theorem, which will resolve Case 2 from the outline in subsection 2.1:

Theorem B2. Let K be a genus-two, fibered, hyperbolic knot in S^3 with associated open book decomposition (S, h). If the pseudo-Anosov representative ψ of h has singularity type $(4; \emptyset; 3^2)$, then K is not fixed point free.

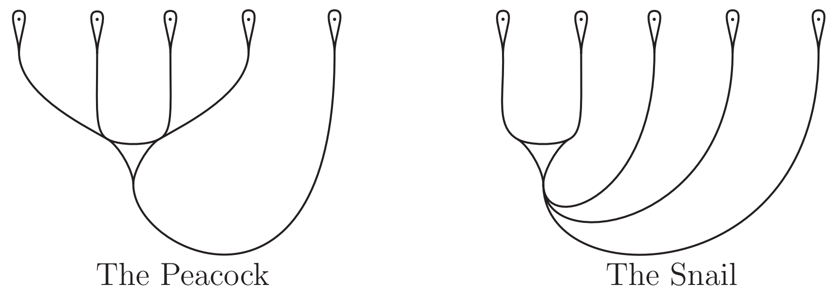


Figure 2.7: The two train track classes in the stratum $(2;1^5;3)$ with no joints.

Remark 2.4.1. Note that we do not need to require $c(h) \neq 0$ here. Between this remark and Proposition 2.3.8, one may wonder why the assumption $c(h) \neq 0$ is necessary in the statement of Theorem B: it is purely to avoid additional singularity cases. In particular, if $c(h) = 0$ then the boundary singularity may be 1-pronged or 2-pronged in Theorem 2.2.5.

The first step in proving Theorem B2 is to observe the following consequence of Theorem D, which we prove at the end of Section 2.5:

Theorem 2.4.2. Let ψ be a pseudo-Anosov on $S^1_{0,5}$ with singularity type $(2;1^5;3)$. Then ψ is conjugate to a pseudo-Anosov carried by the Peacock train track shown in Figure 2.7.

Now, to prove Theorem B2, it suffices by Theorem 2.4.2 to look at pseudo-Anosovs carried by the lift of the Peacock track. See Figure 2.6 for an image of the lifted track. We will perform a careful analysis of train track maps on this track, together with topological arguments to study a family of braids β_n inducing a special collection of train track maps. We present the relevant family of braids β_n and their corresponding train track maps in subsection 2.4. Then, in subsection 2.4, we study train track maps on the Peacock.

A family of braids lifting to fixed-point-free maps

For the remainder of this section, τ will be the Peacock train track depicted on the left in Figure 2.8, with edges and vertices labeled as in the figure (with edges oriented towards the punctures); β will be an arbitrary 5-braid with pseudo-Anosov data (ψ_β, τ, f); and h will be the lift of β to $S = S^1_2$, with pseudo-Anosov data

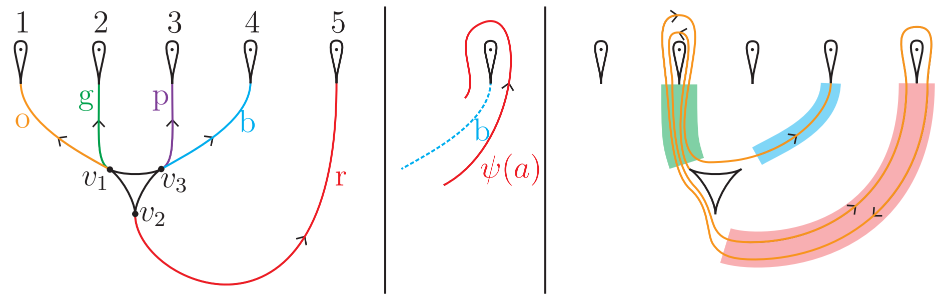


Figure 2.8: **Left:** The Peacock train track with the labels and orientations we will use in this section. **Center:** $f(a)$ passes b on the right. **Right:** $f(o) = g^+r^-g^-b^\circ$.

$(\psi_h, \widetilde{\tau}, \widetilde{f})$, where $\widetilde{\tau}$ and $\widetilde{f}$ are constructed as in subsection 2.2. An image of $\widetilde{\tau}$ in this case is shown in Figure 2.6.

Let a, b be any real edges in τ. Note that because each peripheral 1-gon in τ is adjacent to a unique real edge, the image $f(a)$ is naturally a word of the form $w_1w_2...w_n$, where $w_i \in \{o\bar{o}, g\bar{g}, p\bar{p}, b\bar{b}, r\bar{r}\}$ and $w_n \in \{o, g, p, b, r\}$.

Definition 2.4.3. For real edges a, b of τ, we say that $f(a)$ passes b *on the right* if, before collapsing down to τ, b is to the left of $\psi_\beta(a)$ as in the middle of Figure 2.8. In this case, we write the letter b^+ in place of $b\bar{b}$ in the word $f(a)$. We define passing *on the left* analogously, and denote it by b^-. If b is the last letter in $f(a)$, we write b°. When we allow for multiple possible options, we will write e.g. $b^{\pm\circ}$, $b^{+\circ}$, etc. See the right of Figure 2.8 for an example.

Here is the family of braids which we will study:

Proposition 2.4.4. Set $\beta_n = \sigma_1^{n+2}\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4^2$ for $n \geq 0$. Then, β_n^{-1} is pseudo-Anosov, and conjugate to a braid carried by τ, which induces the train track map $f_n : \tau \to \tau$ defined by:

$$f_n(o) = p^\circ \qquad f_n(g) = b^\circ \qquad f_n(r) = g^\circ$$

$$f_n(p) = \begin{cases} (r^-o^-)^{(\frac{n}{2}+1)}r^\circ & n \text{ even} \\ (r^-o^-)^{\frac{n+1}{2}}r^-o^\circ & n \text{ odd} \end{cases} \qquad f_n(b) = \begin{cases} (r^-o^-)^{\frac{n}{2}}r^-o^\circ & n \text{ even} \\ (r^-o^-)^{\frac{n+1}{2}}r^\circ & n \text{ odd} \end{cases}$$

Proof. Figure 2.9 verifies that $H\beta_0^{-1}H^{-1}$ is carried by τ and induces the train track map $f_0 : \tau \to \tau$, where the orientation-preserving homeomorphism H is given by swinging the real edge r around the train track to the other side. For $n \geq 1$, note

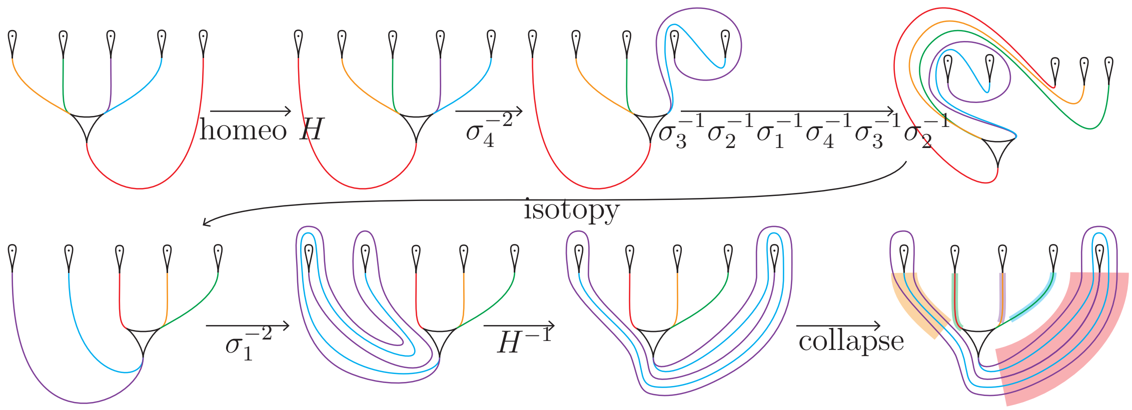


Figure 2.9: The train track map induced by $H\beta_n^{-1}H^{-1}$, where H is an orientation-preserving homeomorphism which swings r around the rest of the track.

that $\beta_n = \sigma_1\beta_{n-1}$, and the additional σ_1 simply adds more twists between the left-most edges before composing with H^{-1} in the last step. This additional twisting adds words of the form $(r^- o^-)$ to $f_n(p)$ and $f_n(b)$, and swaps which edges p and b end on, as in the map in the proposition statement.

It now remains to verify that β_n^{-1} is pseudo-Anosov. This can be seen by checking that the transition matrix M_n associated to the train track map f_n is Perron–Frobenius. When determining the matrix M_n from the map f_n given above, it may be helpful to recall that, for each real edge a, b of τ, each instance of $b^{\pm}$ in $f(a)$ records the word $b\bar{b}$, and each instance of b° records just b. Regardless of the parity of n, the transition matrix is:

$$M_n = \begin{pmatrix} 0 & 0 & n+2 & n+1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & n+3 & n+2 & 0 \end{pmatrix}$$

It is straightforward to check that M_n^7 is strictly positive for all $n \geq 0$, so M_n is Perron-Frobenius. □

It follows from Proposition 2.4.4 that any braid inducing the train track map $f_n : \tau \to \tau$ must be conjugate to β_n^{-1} within the spherical mapping class group (see Remark 2.2.13). The following proposition then implies that no braid inducing the map f_n has unknotted closure:

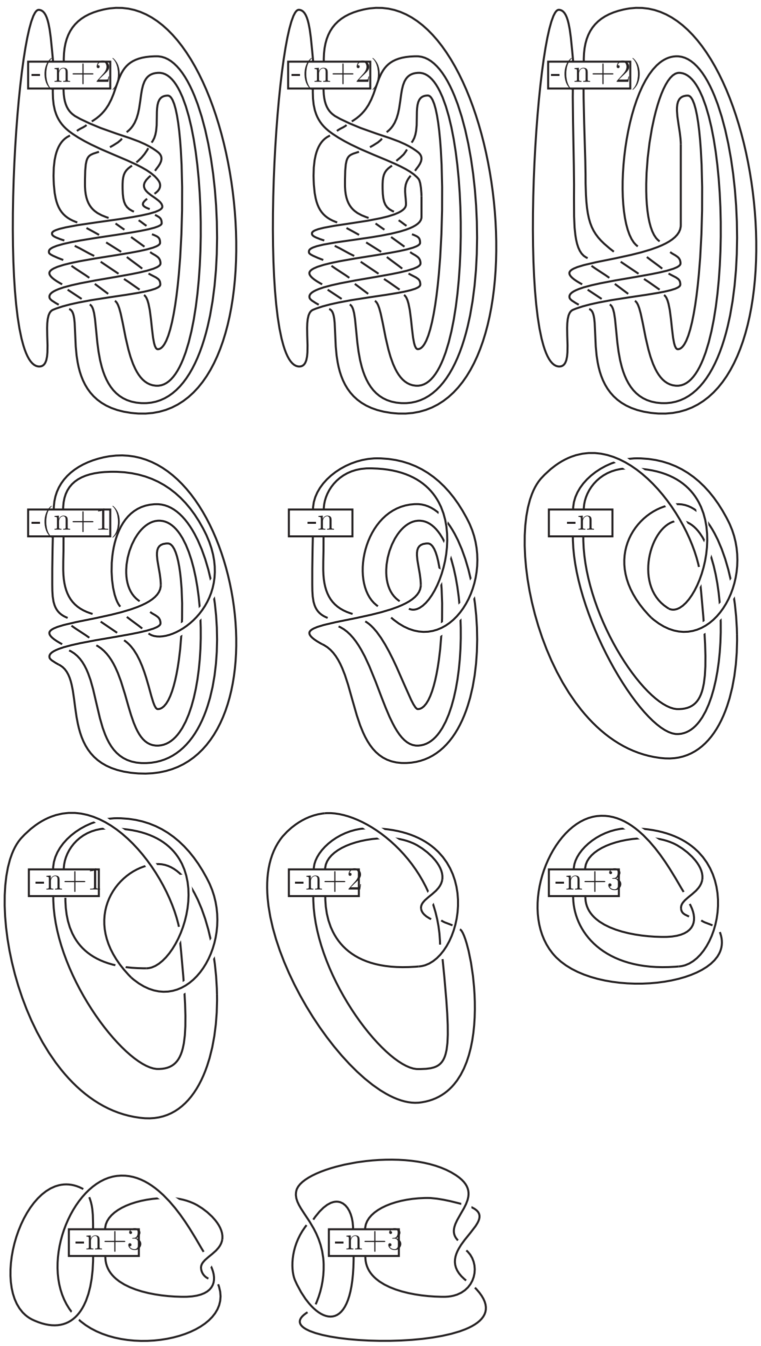


Figure 2.10: Isotopies of $\widehat{\Delta^2\beta_n^{-1}}$ to $P(3, 3-n, -2)$

Proposition 2.4.5. The braid closure $\widehat{\Delta^{2k}\beta_n^{\pm 1}}$ is not an unknot for any $k, n \in \mathbb{Z}$.

Proof. Because β_n is a positive pseudo-Anosov braid, we must have $c(\beta_n) > 0$. On the other hand, one can check that β_n is braid-isotopic to $\beta_n' = (\sigma_1\sigma_2\sigma_3\sigma_4)^2\sigma_4^{n+2}$. One can easily check that $c(\beta_n') \leq 1$ in a variety of ways— for example by verifying that $\Delta^{-2}\beta_n'$ is σ_1-negative (see Theorem 2.2.3). And, because β_n and β_n' are braid-isotopic, it follows that $c(\beta_n) \leq 1$, too. So, we have:

$$k < c(\Delta^{2k}\beta_n) \leq k+1$$

Now, by Theorem 2.2.4 and the fact that the unknot is amphicheiral, it suffices to check that neither $\widehat{\beta_n}$ nor $\widehat{\Delta^2\beta_n^{-1}}$ is an unknot for any $n \in \mathbb{Z}$. The closure $\widehat{\beta_n}$ is easily seen to be the torus knot $T(2, n+6)$. Figure 2.10 verifies that the closure $\widehat{\Delta^2\beta_n^{-1}}$ is the 3-stranded pretzel knot $P(3, 3-n, -2)$. These knots are all known to not be unknotted. □

Train track maps on the Peacock

In this subsection, we retain the notation from the previous subsection. Our remaining goal is to prove the following proposition, which, together with Propositions 2.4.4 and 2.4.5, will imply Theorem B2 and complete the proof of Theorem B:

Proposition 2.4.6. Let ψ_β be a pseudo-Anosov carried by τ, which lifts to a map ψ_h in the cover. If ψ_h is fixed point free, then β is conjugate in the spherical mapping class group to β_n or β_n^{-1} for some $n \in \mathbb{Z}$. In particular, β is conjugate as a braid to $\Delta^{2k}\beta_n^{\pm 1}$ for some $n, k \in \mathbb{Z}$.

We begin with some helpful lemmas to simplify the case analysis.

Lemma 2.4.7 (Trace Lemma)**.** If ψ_h is fixed point free, then for any real edge $a \in \tau$, we have that $a^{\pm\circ} \notin f(a)$.

Proof. First, if $a^\circ \in f(a)$, then the marked point at the end of a is fixed by ψ_β, and the lift of this marked point is fixed by ψ_h. Next, suppose that $a^\pm \in f(a)$, and recall that this means that $f(a)$ contains a word of the form $a\bar{a}$. In the lift, it follows that $\widetilde{f}(a^1)$ contains a word of the form $a^i\bar{a}^j$ for some $i, j \in \{1, 2\}$ (see the construction in subsection 2.2). Because the edges a^i and a^j are not adjacent to infinitesimal loops in the cover (the infinitesimal loops in τ lift to regular points in the cover), we can see that $i \neq j$. So, $\widetilde{f}(a^1)$ contains either a^1 or $\bar{a}^1$ as a letter. In either case, the transition matrix of $\widetilde{f}$ has non-zero trace, so ψ_h is not fixed point free by Theorem 2.2.14. □

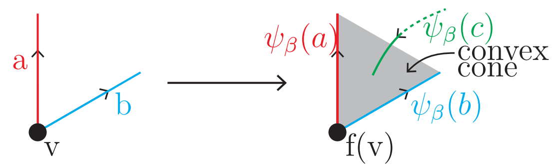


Figure 2.11: c will be absorbed into $f(v)$.

The Trace Lemma also holds in general, with the same proof, for any jointless train track with only 1-pronged punctures. Because we will use the Trace Lemma with great frequency in this section, when we invoke this lemma we will often use the shorthand "by trace."

Lemma 2.4.8. If ψ_h is fixed point free, then $f(v_i) \neq v_i$ for $i = 1, 2, 3$.

Proof. If $f(v_i) = v_i$ for some i, then $Df(r) = r^{\pm\circ}$, which is forbidden by the Trace Lemma. □

The above lemma implies that $f(v_1) \in \{v_2, v_3\}$, and this choice also determines the images $f(v_i)$ for $i = 2, 3$. Note that there is a natural horizontal symmetry of τ induced by reversing the orientation of the disk. Composing with this symmetry takes a braid to its reverse inverse, and a braid lifts to a fixed-point-free map if and only if its reverse inverse does. Hence, it suffices to choose either one of the images $f(v_1)$ as above. Therefore, without loss of generality, $f(v_1) = v_3$.

Lemma 2.4.9. Let a, b be any two real edges of τ with a and b adjacent at initial vertex v. Then, for any real edge c of τ, $\psi_\beta(c)$ does not intersect the convex cone determined by the initial segments of $\psi_\beta(a)$ and $\psi_\beta(b)$. See Figure 2.11 for reference.

Proof. By injectivity of ψ_β, we know that $\psi_\beta(c)$ may not cross $\psi_\beta(b)$ or $\psi_\beta(a)$. If $\psi_\beta(c)$ enters the convex cone X on the initial segments of $\psi_\beta(a)$ and $\psi_\beta(b)$, then $\psi_\beta(c)$ must either have its endpoint inside X or must leave X. The first case is not possible because c ends at a vertex of an infinitesimal monogon by assumption. Therefore $\psi_\beta(c)$ must enter and leave X. Let A and B denote the strips of the fibered surface F that collapse onto a and b, respectively. The arc $\psi_\beta(c)$ lies transverse to the fibers of F. Assume without loss of generality that $\psi_\beta(c)$ enters X along A. Since it must exit X, $\psi_\beta(c)$ must subsequently traverse either A or B. Neither case is possible, however, since a and b form a cusp: the arc $\psi_\beta(c)$ is forced to be non-smooth. □

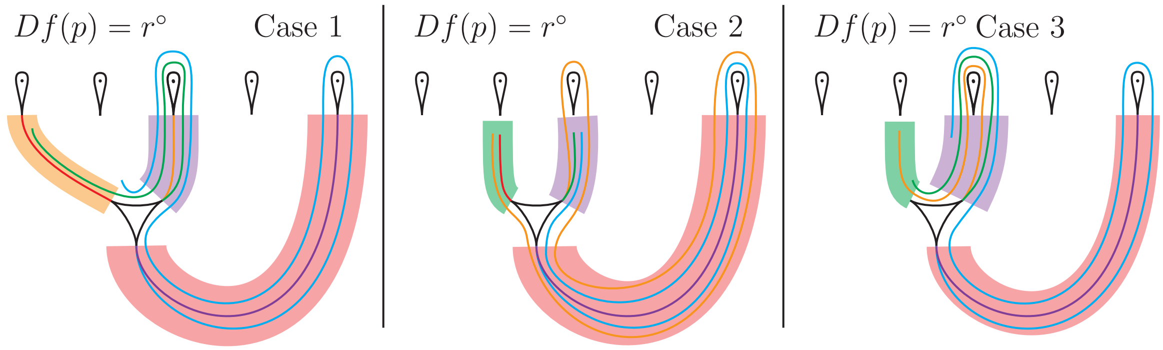


Figure 2.12: Cases for the proof of Lemma 2.4.11.

When a situation as in Lemma 2.4.9 arises, we say that the edge c is *absorbed into* $f(v)$. In practice, an edge being absorbed into a vertex is much easier to spot visually than by formal definition: the typical picture is the one depicted in Figure 2.11.

For the following arguments, recall that we assume without loss of generality that $f(v_1) = v_3$. The figures provided in each proof below are single scenarios appearing in each main case, not exhaustive images of every possibility. We *strongly* encourage the reader to draw by hand the train track maps which are written in words, as they read through each argument.

Lemma 2.4.10. If ψ_h is fixed point free, then $Df(r) \notin \{o^+, g^+\}$.

Proof. If $Df(r) = o^+$, then we must have $f(r) = o^+ r^{\pm\circ}...$, which is forbidden by the Trace Lemma. The same argument applies to g^+. □

Lemma 2.4.11. If ψ_h is fixed point free, then $Df(p) = r^-$.

Proof. First suppose that $Df(p) = r^+$. Then, the second letter in $f(p)$ is either p or b. The former is not allowed by trace. In the latter case, note that then $f(b) = r^+ b^{\pm\circ}...$ since p is to the left of b, which is again ruled out by trace.

Now, suppose that $f(p) = r^\circ$. Note that then $Df(b) = r^+$, and this further implies by trace that $f(b) = r^+ p^{\pm\circ}...$ It then follows that $Df(o) = p^{\pm\circ}$, so we will check these three possible cases individually. See Figure 2.12.

Case 1: $f(o) = p^\circ$. In this case, $f(b) = r^+ p^+ ...$ and $Df(g) = p^+$. By trace, it follows that $f(g) = p^+ o^{\pm\circ}...$, and this in turn forces $Df(r) = o^{\pm\circ}$. By Lemma 2.4.10, we know $Df(r) \neq o^+$, so $Df(r) = o^{-\circ}$. If $Df(r) = o^\circ$, then the real edges o, p, and

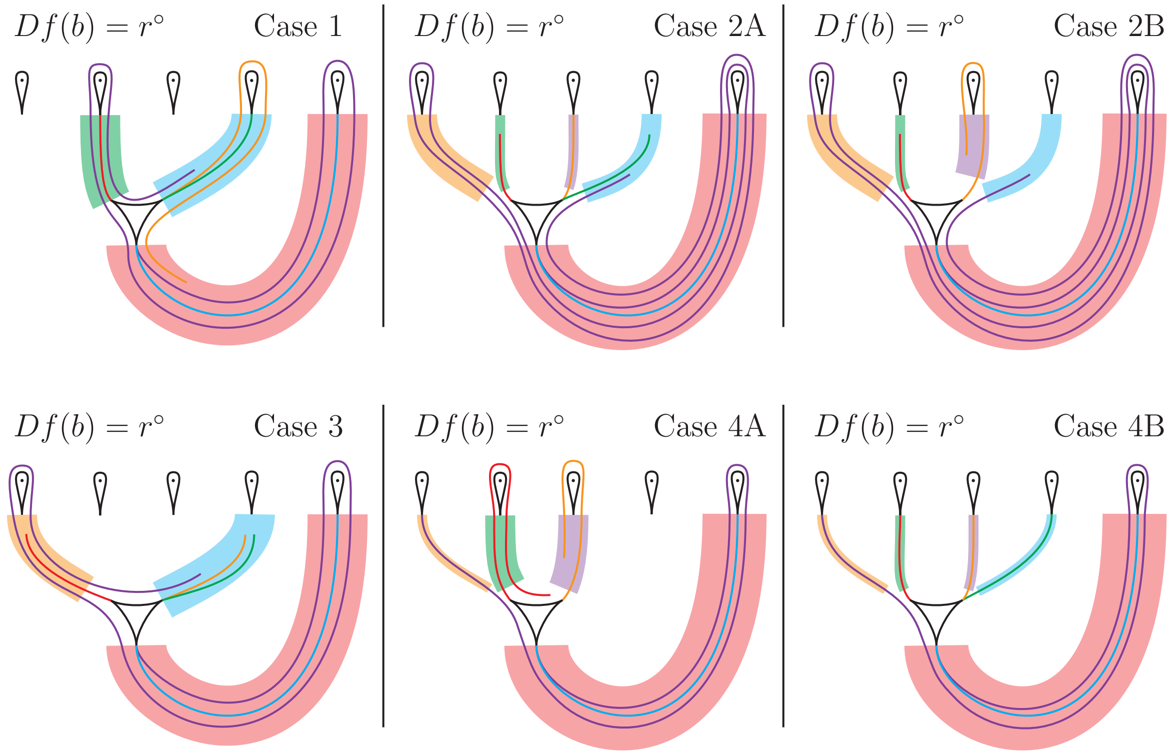


Figure 2.13: Cases for the proof of Lemma 2.4.12.

r are permuted, implying that the transition matrix of f is not Perron-Frobenius. Finally, if $Df(r) = o^-$, we can see that $f(r) = o^-p^-r^{\pm\circ}...$, which is ruled out by trace.

Case 2: $Df(o) = p^-$. In this case, we must have $f(o) = p^-r^-g^{\pm\circ}...$ by trace (or else $f(o)$ is absorbed into either $f(v_3)$ or $f(v_1)$, if it goes "inside" b or g, respectively). This further implies that $f(g) = p^-r^-g^{\pm\circ}...$, which is ruled out by trace.

Case 3: $Df(o) = p^+$. Here, we have $f(o) = p^+g^{\pm\circ}...$, and therefore $f(b) = r^+p^+g^{\pm\circ}...$ In particular, this forces $f(g) = p^+g^{\pm\circ}...$, which is ruled out by trace. □

Lemma 2.4.12. If ψ_h is fixed point free, then $Df(b) = r^-$.

Proof. By Lemma 2.4.11, we may assume $Df(p) = r^-$. Note that $Df(b) = r^+$ is not possible, because then one of b or p will be absorbed into $f(v_3)$.

So, suppose $f(b) = r^\circ$. We branch along cases for the second letter in $f(p)$. Note that if $f(p) = r^-g^{+\circ}...$, then $Df(r) = g^+$, which contradicts Lemma 2.4.10. So, there are four cases left to consider, shown in Figure 2.13.

Case 1: $f(p) = r^-g^-...$ In this case, we must have $f(p) = r^-g^-b^{\pm\circ}...$, because otherwise p is absorbed into $f(v_3)$ (if it follows r next) or contributes trace (if it follows p next). Now note that $Df(o) = b^{\pm\circ}$. If $Df(o) = b^{+\circ}$ then $f(g) = b^+g^{\pm\circ}...$, which is ruled out by trace. So, $Df(o) = b^-$ and we must have $f(o) = b^-r^-g^{\pm\circ}...$ by trace. Note that $Df(g) \neq b^+$ by Lemma 2.4.9, so $Df(g) = b^{-\circ}$. If $Df(g) = b^-$, then we have $f(g) = b^-r^-g^{\pm\circ}...$, which is ruled out by trace. So, $Df(g) = b^\circ$. Finally, consider cases for $Df(r)$. We know $Df(r) = g^{\pm\circ}$, and by Lemma 2.4.10, we must have $Df(r) = g^{-\circ}$. If $Df(r) = g^\circ$, then the transition matrix is not Perron-Frobenius. If $Df(r) = g^-$, then $f(r) = g^-b^-r^{\pm\circ}...$, which is ruled out by trace.

Case 2: $f(p) = r^-o^+...$ In this case, $f(p) = r^-o^+r^+b^{\pm\circ}...$ by trace and, by Lemma 2.4.10, it follows that $Df(r) = g^{-\circ}$. Now, look at cases for $Df(o)$.

If $Df(o) = b^{+\circ}$ then $f(g) = b^+g^{\pm\circ}...$, which is ruled out by trace. And, if $Df(o) = b^-$ then $f(o) = b^-r^-o^{\pm\circ}...$, also ruled out by trace. If $Df(o) = p^-$ then similarly $f(o) = p^-r^-o^{\pm\circ}...$, which is again ruled out by trace. There are two remaining subcases to consider:

Subcase 2A: $Df(o) = p^\circ$. Here, consider cases for $Df(g)$: either $Df(g) = p^+$ or $Df(g) = b^{\pm\circ}$. If $Df(g) = p^+$ then $f(g) = p^+g^{\pm\circ}...$ which is ruled out by trace. We cannot have $Df(g) = b^+$ because then g is absorbed into $f(v_1)$. Finally, we cannot have $Df(g) = b^{-\circ}$ because then either p is absorbed into $f(v_1)$ (if $Df(g) = b^\circ$ or $Df(g) = b^-$ with p outside g) or g is absorbed into $f(v_3)$ (if $Df(g) = b^-$ and g is outside p).

Subcase 2B: $Df(o) = p^+$. Here, consider cases for $Df(r)$: either $Df(r) = g^\circ$ or $Df(r) = g^-$, by Lemma 2.4.10. If $Df(r) = g^\circ$ then o will be absorbed into $f(v_3)$. If $Df(r) = g^-$, then either o is inside r, in which case $f(r) = g^-p^-r^{\pm\circ}...$ (which is ruled out by trace); or, r is inside o, in which case o will be absorbed into $f(v_3)$.

Case 3: $f(p) = r^-o^-...$ In this case, we must have $f(p) = r^-o^-b^{\pm\circ}...$ (otherwise p will be absorbed into $f(v_3)$ or contribute trace), which forces $Df(o) = b^{\pm\circ}$ and $Df(g) = b^{\pm\circ}$, as well. If $Df(o) = b^+$, then either $f(o) = b^+o^{\pm\circ}...$ (which is ruled out by trace), or $f(o) = b^+g^{\pm\circ}...$ in which case $f(g) = b^+g^{\pm\circ}...$, too (which is again ruled out by trace). And, if $Df(o) = b^-$, then $f(o) = b^-r^-o^{\pm\circ}...$, which is ruled out by trace.

So, we must have $f(o) = b^\circ$. Note here that we must have $Df(r) = o^{-\circ}$, by Lemma 2.4.10, and $Df(g) = b^+$. Now, look at r: if $Df(r) = o^-$, then $f(r) = o^-b^-r^{\pm\circ}...$ which is ruled out by trace. Finally, if $Df(r) = o^\circ$, then either g is inside p, in which case g is absorbed into $f(v_3)$, or p is inside g, in which case p will be absorbed into $f(v_1)$.

Case 4: $f(p) = r^-o^\circ$ In this case, note that $Df(r) = g^{-\circ}$ by Lemma 2.4.10, so consider subcases for $Df(r)$.

Subcase 4A: $Df(r) = g^-$. Here, consider cases for $Df(o)$. If $Df(o) = b^{+\circ}$ then $f(g) = b^+g^{\pm\circ}$..., which is ruled out by trace. If $Df(o) = b^-$ then $f(o) = b^-r^-o^-$..., again ruled out by trace. If $Df(o) = p^{-\circ}$ then $f(r) = g^-p^-r^{\pm\circ}$..., ruled out by trace. And finally, if $Df(o) = p^+$ then either o is inside r in which case $f(r) = g^-p^-r^{\pm\circ}$... (which is ruled out trace), or o is outside r in which case it will be absorbed into $f(v_3)$.

Subcase 4B: $Df(r) = g^\circ$. Look first at $Df(o)$. If $Df(o) = p^+$ or $Df(o) = b^+$ then o will be absorbed into $f(v_3)$. If $Df(o) = p^-$ or $Df(o) = b^-$ then $f(o) = p^-r^-o^{\pm\circ}$... or $f(o) = b^-r^-o^{\pm\circ}$..., both of which are ruled out trace. So, we must have either $Df(o) = p^\circ$ or $Df(o) = b^\circ$ and then it follows that $Df(g) = b^\circ$ or $Df(g) = p^\circ$, respectively, as well, after some simple analysis on $Df(g)$. But, note that $Df(o) = b^\circ$ and $Df(g) = p^\circ$ is not possible, since ψ_β is orientation-preserving. And, $Df(o) = p^\circ$ and $Df(g) = b^\circ$ is not possible, because then the transition matrix is not Perron-Frobenius. □

Lemma 2.4.13. If ψ_h is fixed point free, then $Df(r) = g^{-\circ}$.

Proof. By Lemma 2.4.10, we know that $Df(r) \notin \{g^+, r^+\}$, so we just need to show that $Df(r) \neq o^{-\circ}$. Suppose otherwise, i.e. $Df(r) = o^{-\circ}$. By Lemmas 2.4.11 and 2.4.12, we know that $Df(p) = Df(b) = r^-$, and then because $Df(r) = o^{-\circ}$ by assumption, it follows that $f(p) = r^-o^-$... and $f(b) = r^-o^-$... From here, we must have $f(b) = r^-o^-p^{\pm\circ}$... by trace, but then $f(p) = r^-o^-p^{\pm\circ}$, which is ruled out by trace. □

We are finally ready to prove Proposition 2.4.6.

Proof of Proposition 2.4.6. Note by the discussion after Lemma 2.4.8, it suffices to assume $f(v_1) = v_3$. By Lemmas 2.4.11 and 2.4.12, we know $Df(p) = Df(b) = r^-$, and by Lemma 2.4.13 we know $Df(r) = g^{-\circ}$. From here, we branch along cases for $Df(o)$. The cases $Df(o) = b^{\pm\circ}$ can be ruled out quickly as follows.

If $Df(o) = b^+$, then $f(o) = b^+g^{\pm\circ}$... by trace. But, then $f(g) = b^+g^{\pm\circ}$..., which is ruled out by trace. If $Df(o) = b^\circ$, then $f(g) = b^+g^{\pm\circ}$... because $Df(r) = g^{-\circ}$, and this is ruled out by trace. Finally, if $Df(o) = b^-$, then $f(o) = b^-r^-g^{\pm\circ}$... by trace. Because $Df(r) = g^{-\circ}$, it follows that $f(p) = r^-g^-$... and $f(b) = r^-g^-$... From here, we must have $f(b) = r^-g^-p^{\pm\circ}$... by trace. But, then $f(p) = r^-g^-p^{\pm\circ}$, too, which is ruled out by trace.

And, note that if $Df(o) = p^-$, then $f(o) = p^-r^-g^{\pm\circ}$... by trace. Here, we must have $f(p) = r^-g^-p^{\pm\circ}$... because $Df(r) = g^{-\circ}$, which is ruled out by trace. So, we have two cases left to consider, shown in Figure 2.14.

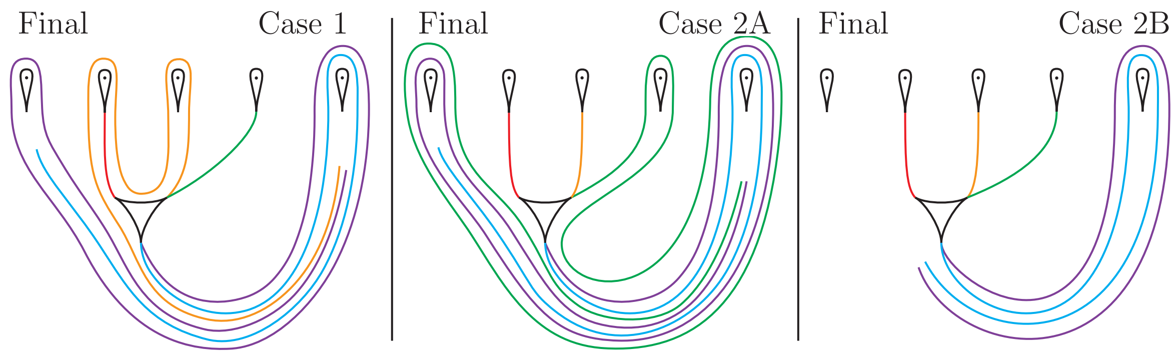


Figure 2.14: Cases for the proof of Proposition 2.4.6. In this figure, we have chosen to omit the shaded collapsing regions for readability.

Case 1: $Df(o) = p^+$. In this case, we have $f(o) = p^+g^{\pm\circ}$... by trace. If $f(o) = p^+g^{-\circ}$... then $f(r)$ is either absorbed into $f(v_1)$ or passes over r. So $f(o) = p^+g^+r^{\pm\circ}$....

Next, consider $Df(g)$. If $Df(g) = b^+$ then g is absorbed into $f(v_1)$, and the case $Df(g) = p^+$ is ruled out quickly by trace. So, we have $Df(g) = b^{-\circ}$. In either case, note that for both of $f(p)$ and $f(b)$, the second letter is in the set $\{o^{\pm\circ}, g^-\}$ because $Df(r) = g^{-\circ}$. If $f(p) = r^-g^-$... then $f(p) = r^-g^-p^{\pm\circ}$..., which is ruled out by trace. So, $f(p) = r^-o^{\pm\circ}$...

Similarly, if $f(b) = r^-g^-$... then either b is outside o, in which case b will be absorbed into $f(v_1)$, or o is outside b, in which case o will be absorbed into $f(v_3)$. So, we must have $f(b) = r^-o^{\pm\circ}$..., too.

Now, if $f(p) = r^-o^{+\circ}$... then $f(b) = r^-o^+r^+b^{\pm\circ}$... which is ruled out by trace. So, we must have $f(p) = r^-o^-r^{\pm\circ}$... And, if $f(p) = r^-o^-r^{-\circ}$... then $f(o) = b^+g^+r^-o^{\pm\circ}$..., which is ruled out by trace. So, we have instead $f(p) = r^-o^-r^+g^-p^{\pm\circ}$... because $Df(r) = g^{-\circ}$, which is again ruled out by trace.

Case 2: $Df(o) = p^\circ$. Here, we branch along subcases for $Df(g)$. Note that if $Df(g) = b^+$ then g is absorbed into $f(v_1)$, and if $Df(g) = p^+$ then $f(g) = p^+g^{\pm\circ}$..., which is ruled out by trace. So we have two remaining subcases to consider:

Subcase 2A: $Df(g) = b^-$. Because $Df(r) = g^{-\circ}$, note that if $f(p) = r^-g^-$... then p is absorbed into $f(v_1)$. A similar argument applies to $f(b)$, so we must have both $f(p) = r^-o^{\pm\circ}$... and $f(b) = r^-o^{\pm\circ}$... Now, if $f(b) = r^-o^+$... then either b is absorbed into $f(v_3)$, or $f(b) = r^-o^+r^+b^{\pm\circ}$, which is ruled out by trace. So, either $f(b) = r^-o^-r^{\pm\circ}$... or $f(b) = r^-o^\circ$.

In the first case, note that if $f(b) = r^-o^-r^+$..., then either b is absorbed into $f(v_3)$, absorbed into $f(v_1)$, or $f(b) = r^-o^-r^+o^+r^+b^{\pm\circ}$..., which is ruled out by trace.

So, we must have either $f(b) = r^-o^-r^-...$ or $f(b) = r^-o^-r^\circ$.

We can then see that $f(b) = (r^-o^-)^k r^-o^\circ$ or $f(b) = (r^-o^-)^{k+1}r^\circ$ for some $k \geq 0$. The argument that follows will not depend on k (with large k, all remaining strands will just turn more times along r and o), so for simplicity suppose either $f(b) = r^-o^\circ$ or $f(b) = r^-o^-r^\circ$.

First, suppose $f(b) = r^-o^\circ$. Then, we must have $f(p) = r^-o^-r^{\pm\circ}...$ and $f(g) = b^-r^-o^-...$ Note here that if $f(p) = r^-o^-r^{+\circ}...$ then g will be absorbed into $f(v_3)$. But, if $f(p) = r^-o^-r^-...$, then p will be absorbed into $f(v_3)$. This same argument will work for arbitrary k after several twists around r and o.

Next, suppose $f(b) = r^-o^-r^\circ$. The argument is very similar in this case. We must have $f(p) = r^-o^-r^-o^{\pm\circ}...$ and $f(g) = b^-r^-o^-r^-o^{\pm\circ}...$ If $f(p) = r^-o^-r^-o^{-\circ}...$ then g will be absorbed into $f(v_3)$. And, if $f(p) = r^-o^-r^-o^+$ then either g will be absorbed into $f(v_3)$ or p will be absorbed into $f(v_1)$. As before, the same argument will work for arbitrary k after several additional twists around r and o.

Subcase 2B: $Df(g) = b^\circ$. We cannot have $Df(r) = g^-$ since then r will be absorbed into $f(v_1)$. So, we must have $Df(r) = g^\circ$. From here, note that $f(b) = r^-o^{-\circ}...$ because otherwise b will be absorbed into either $f(v_1)$ or $f(v_3)$. In either case, it follows that $f(p) = r^-o^-r^{-\circ}...$ because otherwise p will be absorbed into either $f(v_1)$ or $f(v_3)$. Iterating the same argument, it is now easy to see that $f(b) = (r^-o^-)^{\frac{n}{2}}r^-o^\circ$ or $f(b) = (r^-o^-)^{\frac{n+1}{2}}r^\circ$, and $f(p) = (r^-o^-)^{(\frac{n}{2}+1)}r^\circ$ or $f(p) = (r^-o^-)^{\frac{n+1}{2}}r^-o^\circ$ for some $n \geq 0$.

One may observe that these final train track maps match identically with the ones given in Proposition 2.4.4. Proposition 2.4.4 and the subsequent discussion then implies that the braid β is conjugate in the spherical mapping class group to β_n^{-1} for some n. □

2.5 The tight splitting

This section is devoted to developing a tool which will be integral to the proof of Theorem 2.4.2: a specialized form of "splitting," which will allow us to restrict our attention to pseudo-Anosovs carried by a single train track.

Standardly embedded tracks

We first describe a particular class of train tracks on the punctured disk, called *standardly embedded* tracks, which will aid in the description of our splitting procedure. Standardly embedded train tracks have previously appeared in the work of

Ko–Los–Song ([71]), Cho–Ham ([15]), and Ham–Song ([44]), who used them to study pseudo-Anosovs on $S^1_{0,n}$ for small n.

Definition 2.5.1. An *infinitesimal polygon* of a train track τ is a connected component of $S^1_{0,n} \setminus \tau$ whose boundary consists of finitely many infinitesimal edges of τ. A train track τ on $S^1_{0,n}$ is *standardly embedded* if the following conditions hold:

1. Every component of $S^1_{0,n} \setminus \tau$ is an infinitesimal polygon, except for the one containing $\partial S^1_{0,n}$.
2. If two edges of τ are tangent at a switch, then either both are real or both are infinitesimal.
3. Cusps only occur at vertices of infinitesimal polygons.

Figure 2.3 is an example of a standardly embedded track, and Figure 2.5 shows a pseudo-Anosov carried by this track, as well as the induced train track map. Every train track may be adjusted to a standarly embedded one, and this adjustment does not affect which pseudo-Anosovs the track carries. So, we have:

Proposition 2.5.2. Every pseudo-Anosov on $S^1_{0,n}$ is carried by a standardly embedded train track.

We adapt the following definition from Ham–Song's notion of an elementary folding map [44].

Definition 2.5.3. Let $\tau, \tau_1 \hookrightarrow S^1_{0,n}$ be standardly embedded train tracks. A *Markov map* is a graph map $p : \tau_1 \to \tau$ that maps vertices to vertices, and is locally injective away from the preimages of vertices. An *elementary folding map* is a smooth Markov map such that for exactly one real edge α, the image $p(\alpha)$ has word length 2, while the images of all other edges have word length 1. We require that the distinguished edge α belong to a cusp (α, β) of τ_1, and that $p(\alpha)$ be of the form

$$p(\alpha) = p(\beta) \cdot a,$$

where a is a real edge joined to $p(\beta)$ by an infinitesimal edge.

For the purposes of this paper, an elementary folding map $p : \tau_1 \to \tau$ will be the identity map away from the distinguished real edge α. See Figure 2.15.

Remark 2.5.4. An elementary folding map in our terminology is the composition of two elementary moves in Ham-Song's terminology [44].

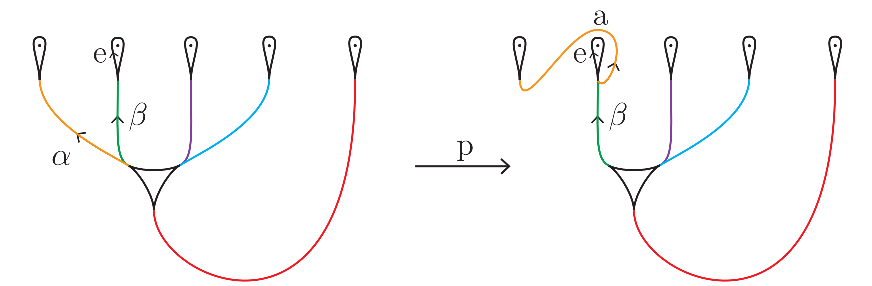


Figure 2.15: An example of an elementary folding map. The map p is the identity except at the edge α, which is mapped as a directed path to $\beta \cdot e \cdot a$.

The tight splitting

We are ready to introduce a specialized variant of a classical operation on train tracks, known as *splitting* (cf. Section 2.1 of [**HP**]). Our variant, which we call *tight splitting*, takes as input the data (τ, ψ, f) of a pseudo-Anosov ψ with an invariant standardly embedded train track τ and outputs another train track τ_1 that is also invariant under ψ.

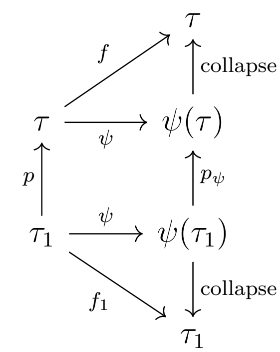
More precisely, suppose that (τ, ψ, f) is the data of a pseudo-Anosov ψ on $S^1_{0,n}$ carried by the standardly embedded train track τ:

$$\begin{array}{ccc} & & \tau \\ & \overset{f}{\nearrow} & \uparrow \text{collapse} \\ \tau & \xrightarrow[\psi]{} & \psi(\tau) \end{array}$$

Suppose further that $\tau_1 \hookrightarrow S^1_{0,n}$ is another standardly embedded train track such that there exists an elementary folding map $p : \tau_1 \to \tau$. Then there is a well-defined elementary folding map $p_\psi : \psi(\tau_1) \to \psi(\tau)$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \tau \\ & \overset{f}{\nearrow} & \uparrow \text{collapse} \\ \tau & \xrightarrow[\psi]{} & \psi(\tau) \\ p\uparrow & & \uparrow p_\psi \\ \tau_1 & \xrightarrow{\psi} & \psi(\tau_1) \end{array}$$

If τ_1 were invariant under ψ, we would then be able to complete the above commutative diagram as follows:



Unfortunately, τ_1 need not be invariant under ψ, as Example 2.5.5 shows. We introduce tight splitting to deal with this problem by taking into account the train track map $f : \tau \to \tau$. In particular, we will show that if (τ, ψ, f) is the data of a pseudo-Anosov acting on a standardly embedded train track τ, then there is always a split $p : \tau_1 \to \tau$ such that τ_1 is still invariant under ψ. See Proposition 2.5.10.

Example 2.5.5. Consider the pseudo-Anosov $\psi : S^1_{0,5} \to S^1_{0,5}$ represented in Figure 2.16. By computing a right eigenvector for the Perron-Frobenius eigenvalue of the transition matrix M, we see that edge e_2 has smaller transverse measure than edge e_3. Following Harer-Penner, we can perform a split by peeling e_2 away from e_3 (truthfully, this is two splits since we need to also peel e_2 away from the non-expanding edge connecting it to e_3). The resulting train track is shown on the left of Figure 2.17. Unhappily, τ_1 is not invariant under ψ.

After some thought, one might notice that a different split does produce an invariant train track for ψ. Indeed, we can see from Figure 2.16 that e_4 has smaller transverse measure than e_5, and performing the corresponding pair of splits (first over the black edge connecting e_4 to e_5, then over e_5) produces an invariant train track τ_2, as desired. See Figure 2.18. This second split differs from the first in that it is compatible with the action of ψ on τ: all paths in the image $\psi(\tau)$ that collapse onto e_4 also collapse onto e_5. It is this property that allows us to easily isotope strands in $\psi(\tau)$ to lie transverse to the leaves of the fibered neighborhood of τ_2. See Definition 2.5.8 and Proposition 2.5.10 for more.

Let $\tau \hookrightarrow S^1_{0,n}$ be standardly embedded, and let $v \in \tau$ be a switch. The *link* of v is the collection $\mathrm{Lk}(v)$ of edges of τ incident to v. The elements of $\mathrm{Lk}(v)$ inherit a natural counterclockwise cyclic order $e_1, \ldots, e_k$. A subset $C \subseteq \mathrm{Lk}(v)$ is *connected* if whenever $e_i, e_j \in C$ and $i < j$, then either

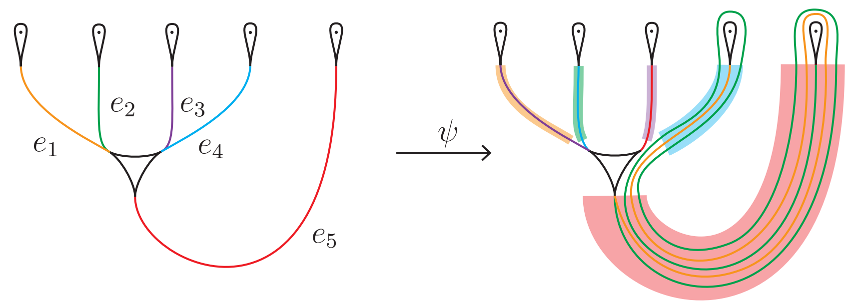


Figure 2.16: The action of a particular pseudo-Anosov $\psi : S^1_{0,5} \to S^1_{0,5}$ on an invariant train track. Each of the loop edges contains a puncture.

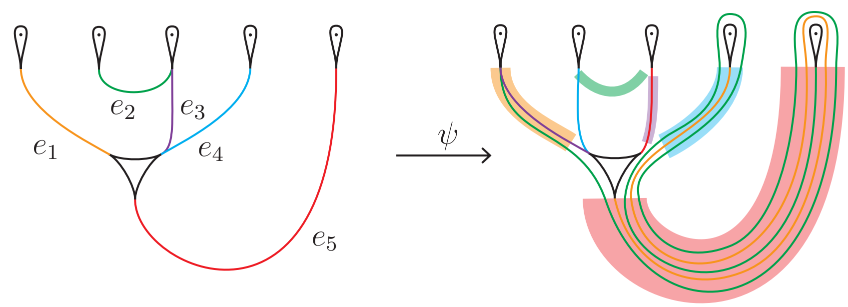


Figure 2.17: The action of ψ after naïvely splitting e_2 over e_3. On the left is the new track τ_1, and on the right is the image $\psi(\tau_1)$, up to isotopy. The only difference between the righthand images of this figure and Figure 2.16 is that the image of of e_2 has been peeled back along that of e_3 and now starts at the leftmost loop. As we can see, τ_1 is not invariant under ψ.

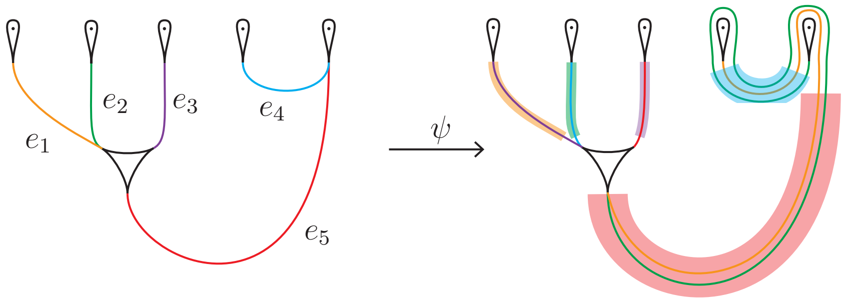


Figure 2.18: The action of ψ after more carefully splitting τ. On the left is the train track τ_2, and on the right is $\psi(\tau_2)$. We obtain a new train track that is still invariant under ψ. This is an example of a "tight split."

1. $e_{i+1}, \ldots, e_{j-1} \in C$, or
2. $e_{j+1}, \ldots, e_k, e_1, \ldots, e_{i-1} \in C$.

The collections

$$R(v) = \{\text{real edges in Lk}(v)\}, \quad I(v) = \{\text{infinitesimal edges in Lk}(v)\}$$

are connected. We index the elements of $\mathrm{Lk}(v)$ so that the real edges are $e_1, \ldots, e_m$ under the cyclic order. In other words, from the perspective of v facing its real edges, e_1 is the real edge furthest to the right and e_m is the edge furthest to the left.

Definition 2.5.6. The *right extremal edge* of v is $r(v) = e_1$, and the *left extremal edge* is $l(v) = e_m$. If $R(v) = \{e\}$ is a singleton, then we set $e = l(v) = r(v)$.

If v is a switch at an infinitesimal loop of τ, we treat each end of the loop as a distinct element of $\mathrm{Lk}(v)$. Hence $I(v)$ always consists of two elements, i_l and i_r. These are defined so that, under the cyclic order, we have

$$l(v) < i_l < i_r < r(v).$$

Definition 2.5.7. We denote by v_l the switch of τ at the other end of i_l from v. Similarly, we denote by v_r the switch of τ at the other end of i_r from v. In the case that v is at a loop of τ, we set $v_l = v_r = v$.

From now on, we set the convention that, for a given switch v of τ, all edges in $R(v)$ are oriented into v as paths.

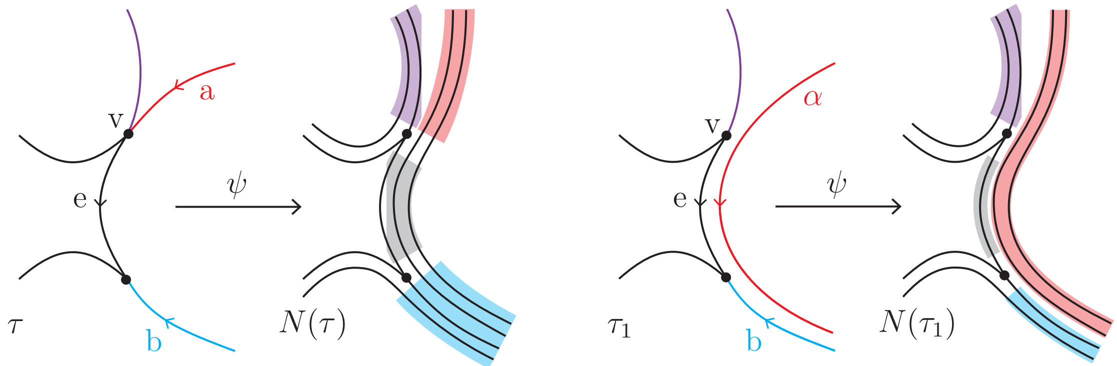


Figure 2.19: **Left:** part of a train track τ and the image of a pseudo-Anosov ψ carried by τ. Here ψ induces a train track map $f : \tau \to \tau$ for which v splits tightly to the right. **Right:** The train track τ_1 after r-splitting v, and the action of ψ on τ_1. Note in particular that ψ has not changed, only τ and its fibered neighborhood $N(\tau)$. In each subfigure, the highlighted regions are collapsed by a deformation retraction onto the corresponding edges.

Definition 2.5.8. Let $\tau \hookrightarrow S^1_{0,n}$ be a standardly embedded train track. Let v be a switch of τ. Fix a train track map $f : \tau \to \tau$. We say that v *splits tightly to the left* or *l-splits* if for every real edge $x \subseteq \tau$ the following two conditions hold:

1. Whenever $l(v)$ appears in the train path $f(x)$, it is followed by $\overline{r(v_l)}$, and
2. whenever $\overline{l(v)}$ appears in the train path $f(x)$, it is preceded by $r(v_l)$.

Similarly, we say that v *splits tightly to the right* or *r-splits* if for every real edge $x \subseteq \tau$ the following two conditions hold:

1. Whenever $r(v)$ appears in the train path $f(x)$, it is followed by $\overline{l(v_r)}$, and
2. whenever $\overline{r(v)}$ appears in the train path $f(x)$, it is preceded by $l(v_r)$.

In either case, we say that v *splits tightly*. See Figures 2.19 and 2.20.

If v splits tightly, we define a new train track that maps to τ by an elementary folding map. In this way, we view splitting as an inverse operation to folding. In what follows we will restrict our attention to the case that v tightly splits to the left: all definitions are analogous if v splits tightly to the right. To obtain these analogous statements and proofs, one need only replace all l's with r's and vice versa.

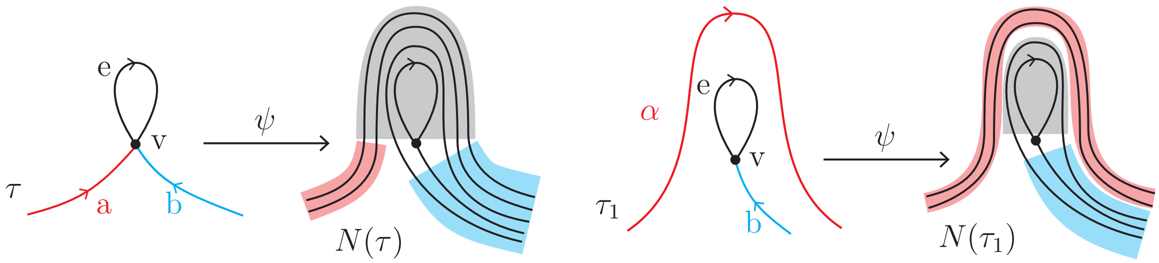


Figure 2.20: **Left:** another train track τ and map $f : \tau \to \tau$ for which v splits tightly to the right. **Right:** the train track τ_1 after r-splitting v.

Suppose v l-splits. Define τ_v^l to be the standardly embedded train track obtained by deleting $l(v)$ and replacing it with a real edge α such that

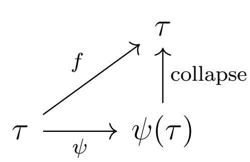
1. As a directed edge, $\alpha(0) = l(v)(0)$ and $\alpha(1) = r(v_l)(1)$.
2. The edge α forms a bigon (i.e. a two-cusped disk) with the train path $l(v) \cdot \overline{r(v_l)}$, and there is an isotopy rel the punctures of $S^1_{0,n}$ so that α lies transverse to the leaves of the fibered neighborhood of τ.

The standardly embedded track τ_v^l comes equipped with a natural elementary folding map $p : \tau_v^l \to \tau$, defined by

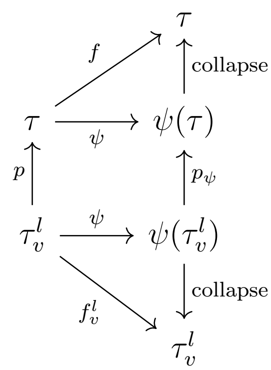
$$p(x) = \begin{cases} x & \text{if } x \neq \alpha \\ l(v) \cdot \overline{r(v_l)} & \text{if } x = \alpha. \end{cases}$$

Definition 2.5.9. If v splits tightly to the left, then the map $p : \tau_v^l \to \tau$ is called a *tight left split* or an *l-split* of τ. We analogously define the *tight right split* or *r-split* $p : \tau_v^r \to \tau$.

Proposition 2.5.10. Suppose (τ, ψ, f) is the data of a pseudo-Anosov carried by the standardly embedded train track τ:



If v l-splits, then τ_v^l carries ψ. Hence the above diagram may be completed to the commutative diagram



where f_v^l is a train track map.

Proof. Let $F \subseteq S^1_{0,n}$ be a fibered surface for ψ from which the Bestvina-Handel algorithm produces τ. Let L, I, and R denote the strips of F collapsing to the (unoriented) edges $l(v), i_l$, and $r(v_l)$ of τ, respectively. Deleting L and replacing it with a strip A collapsing to α produces a new fibered surface F' from which the algorithm produces τ_v^l. The fact that F' is a fibered surface for ψ follows from the fact that v l-splits: any strip of $\psi(F)$ passing through L in fact passes through all three of L, I, and R in order, and hence after an isotopy we may arrange for the strip to pass through A instead. Furthermore, since α is isotopic to $l(v) \cdot i_l \cdot \overline{r(l_v)}$ and $\psi(L)$, $\psi(I)$, and $\psi(R)$ may be isotoped into F', it follows that $\psi(A)$ may be isotoped into F' as well. □

Proposition 2.5.11. Suppose that v l-splits and let M and M_v be the transition matrices of $f : \tau \to \tau$ and $f_v^l : \tau_v^l \to \tau_v^l$, respectively. Then

$$M_v = P^{-1} M P,$$

where P is the transition matrix of the elementary folding map $p : \tau_v^l \to \tau$: that is, if $l(v)$ is the jth edge and $r(v_l)$ is the ith edge, then we have

$$P = I_n + D_{i,j},$$

where τ has n real edges, I_n is the identity, and $D_{i,j}$ is the square matrix with a 1 in the (i, j)-entry and 0's elsewhere.

Proof. We will argue that we have the following commutative diagram:

$$\begin{array}{ccc} \tau_v^l & \xrightarrow{f_v^l} & \tau_v^l \\ \big\downarrow p & & \big\downarrow p \\ \tau & \xrightarrow{f} & \tau \end{array}$$

From this the claim will follow, since each of the arrows is a Markov map, and so upon passing to transition matrices we obtain the relation

$$PM_v = MP.$$

Suppose x is an edge of τ_v^l. By the definition of p we have

$$(f \circ p)(x) = \begin{cases} f(x) & \text{if } x \neq \alpha \\ f(l(v)) \cdot f\left(\overline{r(v_l)}\right) & \text{if } x = \alpha. \end{cases}$$

On the other hand, we must understand the map $f_v^l : \tau_v^l \to \tau_v^l$ in order to analyze the composition $p \circ f_v^l$. For any edge $y \in \tau$, define $f'(y)$ to be the word obtained from the train path $f(y)$ by replacing each instance of $l(v) \cdot \overline{r(v_l)}$ with α and each instance of $r(v_l) \cdot \overline{l(v)}$ with $\overline{\alpha}$. In other words, $f'(x)$ is the unique word such that

$$p(f'(x)) = f(x).$$

If $x \neq \alpha$ is an edge of τ_v^l, then $f_v^l(x) = f'(x)$. If $x = \alpha$, then $f_v^l(x) = f_v^l(\alpha) = f'(l(v)) \cdot f'\left(\overline{r(v_l)}\right)$. In either case, we obtain the formula

$$(p \circ f_v^l)(x) = \begin{cases} f(x) & \text{if } x \neq \alpha. \\ f(l(v)) \cdot f\left(\overline{r(v_l)}\right) & \text{if } x = \alpha. \end{cases}$$

This agrees with the formula for $f \circ p$, so the proof is complete. □

Recall that by the Perron-Frobenius theorem, the dilatation of ψ is a simple eigenvalue of the transition matrix M, and there exists a positive right λ-eigenvector μ of M. For a fixed choice of μ we will denote by $\mu(x)$ the entry of μ corresponding to the real edge x.

Corollary 2.5.12. Let (τ, ψ, f) be the data of a pseudo-Anosov carried by a standardly embedded train track. Let M be the transition matrix for $f : \tau \to \tau$, and let λ be the dilatation of f. Fix a positive right λ-eigenvector μ of M. If v l-splits then $\mu_v = P^{-1}\mu$ is a positive right λ-eigenvector of M_v. Consequently,

$$\mu(l(v)) < \mu(r(v_l)).$$

Proof. Since $M_v = P^{-1}MP$, it immediately follows that $\mu_v = P^{-1}\mu$ is a right λ-eigenvector of M_v. At least one entry of μ_v is positive, since $\mu_v(\alpha) = \mu(l(v)) > 0$. Therefore μ_v is positive, since the Perron-Frobenius theorem states that λ is a simple eigenvalue of M_v and has a positive eigenvector.

To see that $\mu(l(v)) < \mu(r(v_l))$, observe that

$$0 < \mu_v(r(v_l)) = \mu(r(v_l)) - \mu(l(v)).$$

□

Example 2.5.13. Here is an extended example of a sequence of tight splittings. The maps appearing in this example are closely related to the maps studied in Section 2.4. Let (τ, ψ, f) be the data of the pseudo-Anosov represented in Figure 2.21. The transition matrix for $f : \tau \to \tau$ is

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of M_1 is $\chi(t) = (t+1)(t^4 - t^3 - t^2 - t + 1)$. The dilatation of ψ is the root λ of this polynomial with largest absolute value, as in subsection 2.2. A positive right λ-eigenvector for M_1 is

$$\mu_1 = \begin{pmatrix} \mu_1(e_1) \\ \mu_1(e_2) \\ \mu_1(e_3) \\ \mu_1(e_4) \\ \mu_1(e_5) \end{pmatrix} = \begin{pmatrix} 2 + 5\lambda - \lambda^2 - \lambda^3 \\ -2 - 2\lambda + \lambda^2 + \lambda^3 \\ 1 + \lambda + 4\lambda^2 - 2\lambda^3 \\ -1 - \lambda - \lambda^2 + 2\lambda^3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.537... \\ 2.628... \\ 4.370... \\ 4.526... \\ 3 \end{pmatrix}$$

One can see that the vertex at loop 5 splits tightly to the left. Performing this l-split produces the track τ_2, which also carries ψ. See Figure 2.21. The transition matrix of the l-split $p_1 : \tau_2 \to \tau_1$ is

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I_5 + D_{4,5}$$

and the transition matrix for $f_2 : \tau_2 \to \tau_2$ is

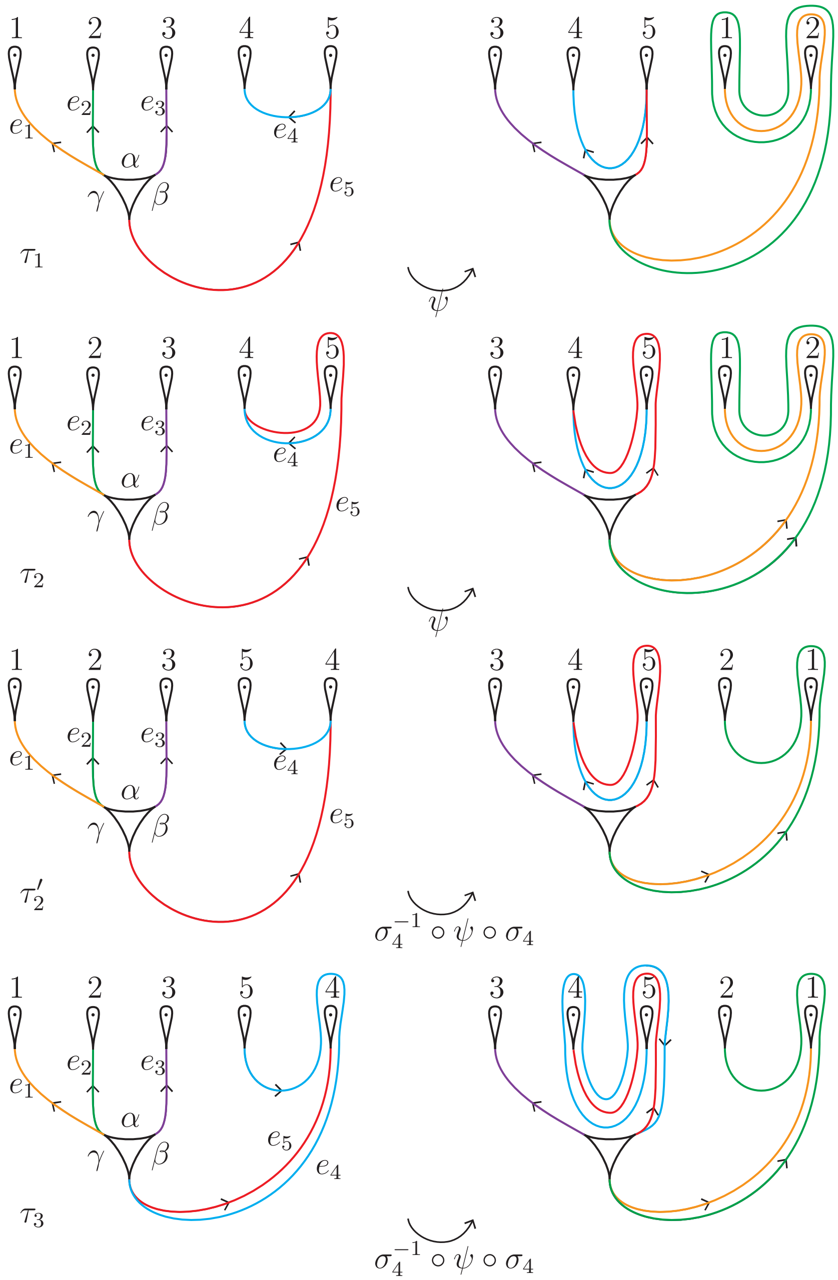


Figure 2.21: The track τ_1, τ_2 carries ψ. The track $\tau_2' = \sigma_4^{-1}(\tau_2)$ carries $\sigma_4^{-1} \circ \psi \circ \sigma_4$. The track τ_3 carries $\sigma_4^{-1} \circ \psi \circ \sigma_4$.

$$M_2 = P_1^{-1} M_1 P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

which has right λ-eigenvector

$$\mu_2 = P_1^{-1}\mu_1 = \begin{pmatrix} \mu_2(e_1) \\ \mu_2(e_2) \\ \mu_2(e_3) \\ \mu_2(e_4) \\ \mu_2(e_5) \end{pmatrix} = \begin{pmatrix} \mu_1(e_1) \\ \mu_1(e_2) \\ \mu_1(e_3) \\ \mu_1(e_4) - \mu_1(e_5) \\ \mu_1(e_5) \end{pmatrix} = \begin{pmatrix} 2.537... \\ 2.628... \\ 4.370... \\ 1.526... \\ 3 \end{pmatrix}$$

We may conjugate by σ_4^{-1} to obtain the track τ_2', which is slightly easier to read. See Figure 2.21. This move is a standardizing braid move in the language of [71]. It is not a tight splitting and is purely cosmetic. It does not alter the transition matrix or any other relevant dynamical information.

We can now see that the switch at loop 4 splits tightly to the right. Performing this r-split produces the track τ_3, which also carries $\sigma_4^{-1} \circ \psi \circ \sigma_4$. See Figure 2.21. The transition matrix of the r-split $p_2 : \tau_3 \to \tau_2$ is

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = I_5 + D_{5,4}$$

and the transition matrix for $f_3 : \tau_3 \to \tau_3$ is

$$M_3 = P_2^{-1} M_2 P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has right λ-eigenvector

$$\mu_3 = P_2^{-1}\mu_2 = \begin{pmatrix} \mu_2(e_1) \\ \mu_2(e_2) \\ \mu_2(e_3) \\ \mu_2(e_4) \\ \mu_2(e_5) - \mu_2(e_4) \end{pmatrix} = \begin{pmatrix} \mu_1(e_1) \\ \mu_1(e_2) \\ \mu_1(e_3) \\ \mu_1(e_4) - \mu_1(e_5) \\ 2\mu_1(e_4) - \mu_1(e_4) \end{pmatrix} = \begin{pmatrix} 2.537... \\ 2.638... \\ 4.370... \\ 1.526... \\ 1.473... \end{pmatrix}$$

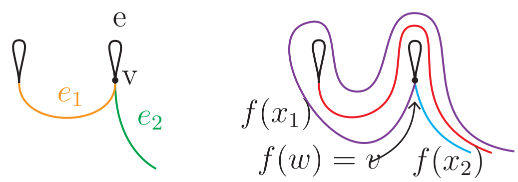


Figure 2.22: An example of a rigid switch. On the left is the switch, on the right the image of the map near the switch.

Switch rigidity

In this section we investigate when a tight splitting is possible at a given switch, identifying the essential obstruction. We call this obstruction *switch rigidity* and show that it is uncommon. Indeed, the orbit of every switch contains a switch that is tightly splittable (cf. Proposition 2.5.20).

Let v be a switch of the train track τ. Recall that $\mathrm{Lk}(v)$ is the set of edges of τ incident to v. A Markov map $f : \tau \to \tau$ induces a map $Df : \mathrm{Lk}(v) \to \mathrm{Lk}(f(v))$ as follows. Orient all edges in $\mathrm{Lk}(v)$ and $\mathrm{Lk}(f(v))$ away from v and $f(v)$, respectively. Then

$$Df(a) = b \text{ if } f(a) \text{ begins with } b.$$

As a consequence of the Bestvina-Handel algorithm, all elements of $R(v)$ belong to the same *gate*: that is, there exists an integer $k \geq 1$ such that $(Df)^k = D(f^k)$ is constant on $R(v)$.

Definition 2.5.14. Let $\tau \hookrightarrow S^1_{0,n}$ be standardly embedded, and let $f : \tau \to \tau$ be a train track map. Let v be a switch of τ such that $R(v)$ is not a singleton, and set $R(v) = \{e_1, \dots, e_k\}$. Let w be the switch of τ such that $f(w) = v$. We say that v is *rigid* if there exist $x_1, \dots, x_k \in R(w)$ such that

$$Df(x_i) = e_i \text{ for all } i.$$

Lemma 2.5.15. Let (τ, ψ, f) be the data of the pseudo-Anosov ψ on $S^1_{0,n}$ carried by the standardly embedded τ. Let w be a switch of τ. Write $\alpha = r(w)$, $\beta = l(w)$, and $v = f(w)$. For any $c \in R(v)$ between $Df(\alpha)$ and $Df(\beta)$, there exists $\gamma \in R(w)$ such that $Df(\gamma) = c$. In other words, the set $Df(\mathrm{Lk}(w)) \subseteq \mathrm{Lk}(v)$ is connected.

Proof. Suppose $c \in R(v)$ is between $Df(\alpha)$ and $Df(\beta)$. Since ψ is pseudo-Anosov, f is surjective. Hence there exists a real edge γ such that $f(\gamma)$ collapses onto c. But since ψ is a homeomorphism, $\psi(\gamma)$ cannot intersect $\psi(\alpha \cup \beta)$, so γ must be incident to w. In other words, $c = Df(\gamma)$. □

Definition 2.5.16. We say a switch v of τ is a *loop switch* if it is incident to an infinitesimal loop.

The next lemma says that switch rigidity is the only barrier to the existence of a tight splitting at a loop switch. Note that if v is a loop switch, then $v_l = v_r = v$.

Lemma 2.5.17. Let (τ, ψ, f) be the data of a pseudo-Anosov ψ on $S^1_{0,n}$ carried by the standardly embedded τ. Let v be a loop switch, and suppose that $R(v)$ is not a singleton. Then exactly one of the following three possibilities is true.

1. The switch v splits tightly to the left.

2. The switch v splits tightly to the right.

3. The switch v is rigid.

Proof. Let w be the loop switch of τ such that $f(w) = v$. If either (1) or (2) holds then v cannot be rigid: for example, if v l-splits then there does not exist $x \in R(w)$ such that $Df(x) = l(v)$. On the other hand, if v is not rigid then Lemma 2.5.15 implies that at least one of $l(v), r(v)$ is not in the image $Df(\mathrm{Lk}(w))$.

Assume without loss of generality that $l(v) \notin Df(\mathrm{Lk}(w))$. Then any appearance of $l(v)$ in an image train path is in fact an appearance of $l(v) \cdot \overline{x}$, up to orientation. Here x is some edge in $R(v)$ that might vary. If x is always $r(v)$ then v l-splits. Otherwise, we claim that v r-splits.

Indeed, suppose that there exists a real edge $y \subseteq \tau$ such that $f(y)$ contains $l(v) \cdot \overline{x}$, up to orientation, for some real edge $x \neq r(v)$. Lemma 2.5.15 implies that $Df(\mathrm{Lk}(w))$ is a subset of the real edges between $l(v)$ and x. In particular, $r(v) \notin Df(\mathrm{Lk}(w))$. Let z be a real edge such that $f(z)$ contains $r(v)$, up to orientation. Since ψ is a homeomorphism and $f(z)$ is a train path, the appearance of $r(v)$ in $f(z)$ must be followed by $\overline{l(v)}$, due to the existence of $\psi(y)$. In other words, v r-splits.

Thus we have established that (1) or (2) holds if and only if (3) does not hold. It remains to show that (1) and (2) are mutually exclusive. Corollary 2.5.12 says that if v l-splits then $\mu(l(v)) < \mu(r(v))$. It follows that if (1) holds then (2) cannot. The proof is complete. □

The same argument gives the following proposition for a switch not at a loop.

Proposition 2.5.18. Let (τ, ψ, f) be the data of a pseudo-Anosov ψ on $S^1_{0,n}$ carried by the standardly embedded τ. Let v be a switch of τ, and suppose that $R(v)$ is not a singleton. Suppose additionally that $R(v_l)$ and $R(v_r)$ are singletons. Then at least one of the following three possibilities is true.

1. The switch v splits tightly to the left.
2. The switch v splits tightly to the right.
3. The switch v is rigid.

Moreover, case (3) is disjoint from cases (1) and (2).

Lemma 2.5.17 says that if we cannot split at a particular switch v, then it is rigid. The natural next step is to consider the preimage switch v_1 causing v to be rigid. If v_1 is also rigid, we look at its preimage v_2. It might happen that we never find a splittable switch. In this case, the periodic orbit of v consists of a cycle of rigid switches.

Definition 2.5.19. A *rigid cycle* of length k is a collection of rigid switches $v_1, \dots, v_k \in \tau$ such that $f(v_j) = v_{j-1}$ for all j, where the indices are taken modulo k.

Proposition 2.5.20. Rigid cycles do not exist.

Proof. Let $v \in \tau$ be a switch. Since τ is standardly embedded, every element of $R(v)$ belongs to the same gate of v, hence there exists $k \geq 1$ such that $(Df)^k$ is constant on $R(v)$. In fact, for all $n \geq k$ we have that $(Df)^n$ is constant on $R(v)$. On the other hand, if v belonged to a rigid cycle of length n then $(Df)^n : R(v) \to R(v)$ would be the identity map, a contradiction. □

Corollary 2.5.21. Let $v \in \tau$ be a switch such that $R(v)$ is not a singleton. Then some iterated preimage switch w of v is not rigid.

It is well-known that if (τ, ψ, f) is the data of a pseudo-Anosov, then f permutes the infinitesimal k-gons for each k (cf. [7]). We obtain the following corollary, which will be of central importance in the following section. The *real valence* of a switch v is the cardinality of $R(v)$.

Corollary 2.5.22. Let n_k denote the maximal real valence of a switch at an infinitesimal k-gon of τ, where $k \geq 1$. If $n_k > 1$ then there exists a switch of valence n_k at such a k-gon which is not rigid.

Proof. The infinitesimal k-gons are permuted by f. If every such maximal valence switch is rigid, then they must form a rigid cycle, since real valence cannot decrease when passing to the preimage of a rigid switch. This is impossible, since rigid cycles do not exist. □

The proofs of Theorems 2.4.2, C, and D

In this subsection, we will use the theory of *tight splitting* developed above to prove Theorems D, and see 2.4.2 as a consequence. Though Theorem D itself is more general than necessary to prove Theorem 2.4.2, we believe it has wider-reaching applications to surface dynamics.

Definition 2.5.23. Let $\tau \hookrightarrow S^1_{0,n}$ be a standardly embedded train track. We say a real edge e of τ is a *stem* if at least one end of e is incident to an infinitesimal k-gon, where $k \geq 2$.

Definition 2.5.24. Let $\tau \hookrightarrow S^1_{0,n}$ be a standardly embedded train track. We say a loop switch $v \in \tau$ is a *joint* if $|R(v)| \geq 2$.

Theorem D. Let ψ be a pseudo-Anosov on $S^1_{0,n}$ with at least one k-pronged singularity away from the boundary with $k \geq 2$. Then ψ is carried by a train track τ with no joints.

The central argument in the proof of Theorem D hinges on finding a maximal-valence vertex v near a puncture, and then using Corollary 2.5.22 to tightly split at v. Before diving into the proof, we observe one crucial lemma. Although well-known to experts, the authors could not find a complete proof of Lemma 2.5.25 in the literature. For the sake of completeness, we have included a proof which arose from a helpful conversation with Karl Winsor.

Lemma 2.5.25. For any fixed n and $B > 0$, there is a finite number of Perron-Frobenius matrices of size n and spectral radius at most B. In particular, there is a finite number of Perron-Frobenius matrices of a given size with a particular Perron-Frobenius eigenvalue.

Proof. Fix $n \geq 2$, and let M be an $n \times n$ Perron-Frobenius matrix. Write $M_{i,j}$ for the (i, j)th entry of M, and $C_j(M)$ for the jth column of M. An exercise in matrix algebra shows that for each integer $k \geq 1$,

$$C_j(M^k) = \sum_{i=1}^{n} (M^{k-1})_{i,j} \cdot C_i(M).$$

It is well-known (cf. [124]) that M^{n^2-2n+2} has all entries positive. Hence the smallest column sum of M^{n^2-2n+3} is at least the sum $\|M\|_1$ of all the entries of M. It is not hard to see that the smallest column sum of a Perron-Frobenius matrix is a lower bound on its spectral radius $\rho(M)$. We now have

$$\rho(M)^{n^2-2n+3} = \rho\left(M^{n^2-2n+3}\right) \geq \|M\|_1 .$$

In particular, $\rho(M) \geq \|M\|_1^{\frac{1}{n^2-2n+3}}$. Since there are only finitely many integer-valued matrices M with $\|M\|_1$ below a given bound, the result follows. □

Proof of Theorem D. Let $\tau_0 \hookrightarrow S^1_{0,n}$ be a standardly embedded train track carrying ψ. We will algorithmically perform a finite sequence of tight splittings on τ_0 to produce the desired track τ with no joints.

Let J denote the number of cusps at the loop switches of τ, i.e. $J = \sum_v (|R(v)| - 1)$, where v ranges over the loop switches of τ. If $J = 0$ then there is nothing to prove, so assume $J \geq 1$. By Corollary 2.5.22 there exists a loop switch of τ_0 of maximal valence that can be tightly split. Therefore, we introduce the following simple algorithm.

1. Initialize $\tau = \tau_0$ and $\mathcal{M} = \{M_0\}$, where M_0 is the transition matrix associated to the data (τ_0, ψ, f_0).
2. Find a loop switch of τ of maximal valence that is not rigid, and split it, obtaining the data (τ_1, ψ, f_1) with transition matrix M_1. Set $\tau = \tau_1$.
3. If J has decreased by one, return the data (τ_1, ψ, f_1).
4. If J has not decreased, add M_1 to $\mathcal{M}$ and repeat Steps 2 and 3 with (τ_1, ψ, f_1).

We claim that this algorithm terminates in finitely many steps, and returns a train track τ with one fewer joint than τ_0. First, note that splitting at a loop switch v_0 either preserves J or decreases it by one. Indeed, let b be the real edge that is split over, i.e. the edge whose transverse weight is reduced (cf. Corollary 2.5.12). Let v_b denote the switch at the other end of b. The tight splitting transfers a cusp from the splitting switch v_0 to v_b. Thus, in the formula

$$J = \sum_{v \text{ a loop switch}} (|R(v)| - 1),$$

the contribution from v_0 decreases by one, whereas the contribution from v_b either (1) increases by one, if v_b is itself a loop switch; or (2) does not change, if v_b is not a loop switch. In particular, if b is a stem, then splitting over b at the loop switch v_0 will always reduce J by one.

It remains to show that, by repeatedly applying the above algorithm, we will eventually split over a stem. Indeed, by Lemma 2.5.25 there are only finitely many possible transition matrices that can appear, hence we will eventually produce a matrix $M_j = M_i \in \mathcal{M}$. Since this matrix is Perron-Frobenius, the dilatation λ of ψ is an eigenvalue with strictly positive eigenvectors μ_i and μ_j. Moreover, λ is simple, so in fact μ_j is a scalar multiple of μ_i. According to Corollary 2.5.12, each tight splitting reduces one of the entries of this eigenvector, so recurring to a matrix in $\mathcal{M}$ implies that every entry of μ has been reduced, i.e. that every real edge of τ_0 has been split over. In particular, the stems of τ_0 have been split over. The preceding paragraph now implies that the algorithm must terminate in finite time.

Repeating this algorithm sufficiently many times will eventually reduce J to 0, proving the theorem. □

proof of Theorem 2.4.2. Note that in the stratum $(2; 1^5; 3)$, there are only two classes of standardly-embedded train tracks without joints: those shown in Figure 2.7. By Theorem D, any pseudo-Anosov in this stratum is conjugate to one carried by either the Peacock or the Snail. We will argue that any pseudo-Anosov ψ carried by the Snail tightly splits to one carried by the Peacock.

First, observe that ψ must split at the unique valence-3 switch of the infinitesimal triangle in the Snail, by Corollary 2.5.22. Either a left or right split at this vertex yields a pseudo-Anosov ψ' conjugate to ψ, and carried by a track τ' with a unique two-valent vertex v at a puncture. This vertex v is again splittable by Corollary 2.5.22. At v, note that ψ' splits either to another map carried by τ', with strictly smaller edge weight on the edge running between two punctures, or to a map carried by the Peacock. In particular, after sufficiently many splits, ψ' splits to a pseudo-Anosov carried by the Peacock. □

proof of Theorem C. Note that if $\psi : S \to S$ has the given singularity type, we may cap-off ψ to a pseudo-Anosov $\widehat{\psi}$ on the closed genus-two surface $\widehat{S}$ and extend the foliations preserved by ψ along the capping disk. In this case, the 4-prong singularity p in the capping disk is the unique 4-prong singularity of $\widehat{\psi}$. In particular, $\widehat{\psi}$ commutes with the hyperelliptic involution ι on $\widehat{S}$ and p is fixed by ι, as in e.g. Lemma 3.7 of [3]. And, because p is fixed by ι, we see that ψ commutes with the hyperelliptic involution on S, as well. We may then quotient ψ to a pseudo-Anosov 5-braid β, and from here the techniques of Section 2.5 apply. Theorem 2.4.2 implies that β is carried by the "Peacock" train track depicted in Figure 2.7, and we can then lift this track to S as described in subsection 2.2. □

Chapter 3

A connected sum formula for embedded contact homology

Luya Wang

3.1 Introduction

Embedded contact homology

Embedded contact homology (ECH) is a three-manifold invariant introduced by Hutchings in [51, 53]. We assume for simplicity that our three-manifolds are connected. Given a contact three-manifold (Y, ξ), fix a contact form λ such that $\ker(\lambda) = \xi$ and all Reeb orbits associated to λ are nondegenerate. Given $\Gamma \in H_1(Y)$, the ECH chain complex is a free $\mathbb{F}$-module generated by certain finite sets of Reeb orbits in the homology class Γ, where $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. The ECH differential counts pseudo-holomorphic curves with asymptotic ends at Reeb orbits and ECH index one in the symplectization of the contact three-manifold, with respect to a certain choice of $\mathbb{R}$-invariant almost complex structure J. It is shown in [59, 61] that $\partial^2 = 0$ for the ECH differential. Let $ECC_*(Y, \lambda, \Gamma, J)$ denote the ECH chain complex and let $ECH_*(Y, \lambda, \Gamma, J)$ denote the homology of the ECH chain complex. We often suppress the notation Γ and understand that ECH decomposes as

$$ECH_*(Y, \lambda, J) = \oplus_{\Gamma \in H_1(Y)} ECH_*(Y, \lambda, \Gamma, J).$$

ECH is a priori dependent on the choices of contact form λ and almost complex structure J. In [109, 110, 111, 112, 113], Taubes showed an isomorphism between

ECH and a certain version of Seiberg-Witten Floer cohomology:

$$ECH_*(Y, \lambda, \Gamma, J) \cong \widehat{HM}^{-*}(Y, \mathfrak{s}(\xi) + \mathrm{PD}(\Gamma)) \tag{3.1.1}$$

as relatively graded $\mathbb{F}$-modules, where $\mathfrak{s}(\xi)$ is the spin-c structure determined by the 2-plane field ξ. For the definition of Seiberg-Witten Floer homology, see [73]. The isomorphism (3.1.1) establishes the well-definedness of ECH. In particular, ECH is independent of the choices of almost complex structures and contact forms and is sometimes denoted as $ECH_*(Y, \xi, \Gamma)$.

ECH is also equipped with a degree -2 chain map called the *U map*. The chain level U map is defined by counting ECH index 2 curves passing through a generic base point, which depends on the choice of the base point. The induced U map on homology does not depend on such a choice and endows ECH with an $\mathbb{F}[U]$-module structure.

The main theorem

Given two contact three-manifolds (Y_1, ξ_1) and (Y_2, ξ_2), one can form their contact connected sum $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ by the Weinstein one-handle attachment [119]. Up to contactomorphism, this is a well-defined operation.

Define *the derived tensor product* of two chain complexes C_1 and C_2 over $\mathbb{F}[U]$ to be

$$C_1 \widetilde{\otimes}_{\mathbb{F}[U]} C_2 := H_*(C_1 \otimes_{\mathbb{F}} C_2 \xrightarrow{U_1 \otimes id + id \otimes U_2} C_1 \otimes_{\mathbb{F}} C_2[-1]),$$

where the right hand side denotes the homology of the *mapping cone* of the chain map $U_1 \otimes id + id \otimes U_2$ from $C_1 \otimes_{\mathbb{F}} C_2$ to $C_1 \otimes_{\mathbb{F}} C_2[-1]$, also sometimes denoted as $Cone(U_1 \otimes id + id \otimes U_2)$ in our paper.

We now state the main theorem:

Theorem 1. *Let (Y_1, λ_1) and (Y_2, λ_2) be two closed connected contact three-manifolds with given nondegenerate contact forms λ_1 and λ_2. Then,*

$$ECH(Y_1 \# Y_2, \xi_1 \# \xi_2, \Gamma_1 + \Gamma_2) \cong (ECH(Y_1, \xi_1, \Gamma_1) \widetilde{\otimes}_{\mathbb{F}[U]} ECH(Y_2, \xi_2, \Gamma_2) \tag{3.1.2}$$

as $\mathbb{F}$-modules.

Remark 2. Similar theorems to our main theorem are known for Seiberg-Witten Floer homology [9, 79, 84] and Heegaard Floer homology [100, 99]. See [109, 110, 111, 112, 113, 19, 24, 26, 22] for the isomorphisms between ECH and Seiberg-Witten Floer and between ECH and Heegaard Floer. See also [76, 77, 78, 75, 79]. Our motivations to have a connected sum formula proven in ECH include potential generalizations to other contact homologies and obtaining more information on quantitative invariants in ECH. More details will be discussed in Section 3.1.

Theorem 1 can also be used to compute ECH of subcritical surgery. Note that the subcritical surgery that is attaching a 1-handle to the three-manifold Y itself is simply a self connected sum. Topologically, this is the same as performing a connected sum with $S^1 \times S^2$. The ECH of $S^1 \times S^2$ has been computed in Section 12.2.1 in [58]. One can also understand the U maps on $ECH(S^1 \times S^2)$ from Seiberg-Witten theory [73, Section 36]. Given a contact structure ξ_0 on $S^1 \times S^2$, let $\Gamma_0 \in H_1(S^1 \times S^2)$ be the homology class such that $c_1(\xi) + 2PD(\Gamma_0) = 0$. Then,

$$ECH(S^1 \times S^2, \xi_0, \Gamma) = \begin{cases} (\mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]) \otimes H_*(S^1; \mathbb{Z}), \Gamma = \Gamma_0 \\ 0, \text{ otherwise} \end{cases}$$

as $\mathbb{F}[U]$-modules. Now by using (3.1.2) and Künneth formula, we have:

Corollary 3. Given a closed contact three-manifold (Y, ξ),

$$ECH(Y\#(S^1 \times S^2), \xi\#\xi_0, \Gamma + \Gamma_0) \cong ECH(Y, \xi, \Gamma) \otimes_{\mathbb{F}} H_*(S^1; \mathbb{Z})$$

as $\mathbb{F}$-modules.

Ideas of the proof of the main theorem

To understand ECH chain complex of the contact connected sum $(Y_1\#Y_2, \xi_1\#\xi_2)$, one needs to understand not only the contact structure but also the contact form on the connected sum. Let λ_i be a nondegenerate contact form such that $\ker \lambda_i = \xi_i$ for $i = 1, 2$. Given two Darboux charts in Y_1 and Y_2, a particular model of the contact connected sum $(Y_1\#Y_2, \lambda_1\#\lambda_2)$ is carefully described in [39] and depends on various choices. In particular, the connected sum sphere S_+ which is contained in the ascending manifold of the Weinstein one-handle contains a hyperbolic Reeb orbit which is the equator of S_+. This hyperbolic is denoted as the *special hyperbolic orbit h* throughout our paper. In addition, one may adjust the size of the Weinstein one-handle in order to control the radius of S_+. Let $(Y_1\#_R Y_2, \lambda_1\#_R\lambda_2)$ denote the resulting contact connected sum with a connected sum sphere of radius R. More details about the contact connected sum operation will be given in Section 3.3.

The proof of Theorem 1 goes through a chain level statement on the filtered ECH chain complex. Recall that there is a filtration on ECH chain complex by the symplectic action functional integrating the contact form over Reeb orbits. The filtered ECH chain complex $ECC^L(Y, \lambda, \Gamma, J)$ is generated by orbit sets up to symplectic action L. To ease notations, we suppress Γ and J from input of ECH chain complex when it is clear.

Let

$$C_o := ECC(Y_1, \lambda_1) \otimes_{\mathbb{F}} ECC(Y_2, \lambda_2)$$

denote all orbit sets in $Y_1 \sqcup Y_2$ counted as ECH generators. Now note that up to action L, we may ignore orbits that cross the connected sum region by shrinking the connected sum sphere to be of a small enough radius $R(L)$, since these orbits would have actions greater or equal to L by a compactness argument as in Lemma 41. Therefore, there is an obvious identification on the vector space level between the filtered ECH complex $ECC^L(Y_1 \#_{R(L)} Y_2, \lambda_1 \#_{R(L)} \lambda_2)$ and the filtered mapping cone complex

$$Cone^L(U_1 \otimes id + id \otimes U_2) := C_o^L \oplus C_h^L,$$

where $C_h \cong C_o$ by appending the special hyperbolic orbit h and the superscript L denotes the filtration. In fact, one may find a chain homotopy equivalence that is triangular with respect to this obvious identification of the vector spaces:

Proposition 4. Given two closed connected contact three-manifolds (Y_1, λ_1) and (Y_2, λ_2) with nondegenerate contact forms λ_i, there exists a strictly decreasing function $R : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{L \to \infty} R(L) = 0,$$

such that there is a chain homotopy equivalence

$$f : ECC^L(Y_1 \#_{R(L)} Y_2, \lambda_1 \#_{R(L)} \lambda_2) \longrightarrow Cone^L(U_1 \otimes id + id \otimes U_2).$$

More details about the identification on the vector space level will be provided in Section 3.4. The chain homotopy equivalence in Proposition 4 will be constructed in Section 3.6.

Further directions

From the perspective of Symplectic Field Theory (SFT) constructed by Eliashberg, Givental and Hofer [35], one could also try to prove a connected sum formula in linearized contact homology. This is studied by Bourgeois and van-Koert [13]. We hope that our connected sum formula for embedded contact homology could give ideas to such formulas for other contact homologies that involve higher genus curves and more asymptotic ends.

It would also be interesting to study the ECH formulas and cobordism maps under additional contact surgeries. One natural candidate that is suitable for the techniques of the present paper is to study how ECH behaves when cut along mixed tori studied by [16].

Theorem 1 computes ECH of a connected sum as an $\mathbb{F}$-module. However, the model of the derived tensor product on the right hand side of (3.1.2) has a natural U-action, $U_1 \otimes id$, which is homotopic to $id \otimes U_2$. It is natural to ask whether Theorem 1 holds over $\mathbb{F}[U]$-modules. This amounts to studying ECH U maps on the chain level on the connected sum, which is also important for studying more refined invariants in ECH, such as the ECH spectral invariants. This will be discussed in future works, but we present here a conjecture.

Suppose λ is nondegenerate. Let $0 \neq \sigma \in ECH(Y, \lambda, \Gamma)$. Define $c_\sigma(Y, \lambda)$ to be the infimum over $L \in \mathbb{R}$ such that σ is in the image of the inclusion-induced map

$$ECH^L(Y, \lambda, \Gamma) \longrightarrow ECH(Y, \lambda, \Gamma).$$

Recall that there is a canonical element $c(\xi) := [\emptyset] \in ECH(Y, \lambda, 0)$ called the *ECH contact invariant*.

Definition 5. If (Y, λ) is a closed connected contact three-manifold with the contact invariant $c(\xi) \neq 0$ and if k is a nonnegative integer, then define the *k-th ECH spectral invariant* to be

$$c_k(Y, \lambda) := \inf\{c_\sigma(Y, \lambda) | \sigma \in ECH(Y, \lambda, 0), U^k \sigma = [\emptyset]\}.$$

Conjecture 6.

$$\lim_{R \to 0} c_k((Y_1 \#_R Y_2, \lambda_1 \#_R \lambda_2)) = \max\{c_i(Y_1, \lambda_1) + c_j(Y_2, \lambda_2) | i + j = k\}.$$

Organization. Section 3.2 reviews basic definitions of embedded contact homology. Section 3.3 discusses the Reeb dynamics of the contact connected sum and the asymptotic behaviors of pseudo-holomorphic curves in its symplectization. Section 3.4 discusses how to ignore potential Reeb orbits that cross the connected sum region and identifies a filtered ECH chain complex of a connected sum with a filtered mapping cone complex associated to the ECH U maps on the level of vector spaces. Section 3.5 relates some of the new ECH differentials in the connected sum to the ECH differentials in the original contact three-manifolds. Section 3.6 relates the remaining differentials to the ECH U maps in the original contact three-manifolds and constructs a chain homotopy equivalence that proves Proposition 4. Section 3.7 uses a direct limit argument similar to that in [94] to prove the main result Theorem 1.

3.2 An overview of ECH

In this section we give a very quick overview of ECH. See [51, 55, 53] for more details. Let Y be a closed connected three-manifold with a contact form λ. Let R be the *Reeb vector field* determined by $d\lambda(R,\cdot) = 0$ and $\lambda(R) = 1$. Over each closed Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$, where T is the period of the orbit, the linearized Reeb flow determines a symplectic linear map

$$d\psi_T : T_{\gamma(0)}Y \longrightarrow T_{\gamma(T)}Y.$$

If $d\psi_T$ does not have 1 as an eigenvalue, we say that γ is *nondegenerate*. A contact form λ is said to be *nondegenerate* if all Reeb orbits are nondegenerate. Note that this is a generic condition and from now on we fix λ to be nondegenerate.

Let $\mathcal{P}$ be the set of embedded Reeb orbits of the Reeb vector field associated to λ. The ECH chain complex $ECC(Y,\lambda,J)$ is generated as a $\mathbb{F}$ vector space by *orbit sets* $\gamma = \{(\gamma_i, m_i)\}$, such that:

- $\gamma_i \in \mathcal{P}$ are distinct Reeb orbits;
- $m_i \in \mathbb{Z}^+$ is the covering multiplicity of the orbit γ_i;
- if γ_i is hyperbolic, i.e. the linearized return map has real eigenvalues, then $m_i = 1$.

We call orbit sets satisfying the above criteria *admissible*.

Let $(\mathbb{R}_s \times Y, d(e^s\lambda))$ be the *symplectization* of (Y,λ). Let J be a λ-*adapted* almost complex structure. This means that J is $\mathbb{R}$-invariant, $J(\partial_s) = R$ where R is the Reeb vector field associated to λ, and J sends the contact structure $\xi := ker(\lambda)$ to itself, positively rotating ξ with respect to the orientation on ξ given by $d\lambda$. We consider J-holomorphic curves $u : (\Sigma, j) \to (\mathbb{R}\times Y, J)$, where (Σ, j) is a punctured Riemann surface, up to biholomorphisms. A *positive asymptotic end* of u at a Reeb orbit γ is a puncture with a neighborhood that can be given coordinates of a *positive half-cylinder* $(s,t) \in [0,\infty) \times (\mathbb{R}/T\mathbb{Z})$ such that $j(\partial s) = t$, $\lim_{s\to\infty} \pi_{\mathbb{R}}(u(s,t)) = \infty$ and $\lim_{s\to\infty} \pi_Y(u(s,\cdot)) = \gamma$. A *negative asymptotic end* is defined analogously, where the neighborhood of a negative puncture is identified with a *negative half-cylinder* $(-\infty, 0] \times (\mathbb{R}/T\mathbb{Z})$.

The ECH differential counts *J-holomorphic currents* of *ECH index* one. A *J-holomorphic current* is a finite set of pairs $\mathcal{C} = \{(C_k, d_k)\}$, where C_k are distinct, connected, somewhere injective J-holomorphic curves in $(\mathbb{R}\times Y, d(e^s\lambda))$ and $d_k \in \mathbb{Z}^+$. We say that $\mathcal{C}$ is *positively asymptotic to* an orbit set $\alpha = \{(\alpha_i, m_i)\}$, if C_k has positive ends at covers of α_i with multiplicity $cov(C_k)$ and $\sum_k cov(C_k) = m_i$. *Negative* *asymptotic* as currents is defined analogously. We now review ingredients of the ECH index and related properties.

The Conley-Zehnder index

In this subsection we discuss a classical quantity that measures the "winding" of the linearized return map of a given Reeb orbit, with respect to a given trivialization.

Definition 7. Let τ be a trivialization of the contact structure ξ over the Reeb orbit γ, parameterized as $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$. The linearization of the Reeb flow $d\psi_t : T_{\gamma(0)}Y \to T_{\gamma(t)}Y$ induces a family of 2×2 symplectic matrices $\phi_t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ with respect to τ. The *Conley-Zehnder index over* γ denoted as $CZ_\tau(\gamma) \in \mathbb{Z}$ is the Conley-Zehnder index of the family of symplectic matrices $\phi_{t\in[0,T]}$.

Since we assumed the contact form we started with is *nondegenerate*, the linearized return map $\phi_T(\gamma)$ does not have 1 as an eigenvalue[1]. If $\phi_T(\gamma)$ has eigenvalues in the unit circle, we call γ *elliptic*. Otherwise, γ is *hyperbolic*. If γ is hyperbolic, we may choose $v \in \mathbb{R}^2$ to be an eigenvector of ϕ_T and the family $\{\phi_t(v)\}_{t\in[0,T]}$ rotates by $k\pi$, where $k \in \mathbb{Z}$. In this case,

$$CZ_\tau(\gamma) = k.$$

If γ is elliptic, we may adjust the trivialization τ so that ϕ_t is a rotation by angle $2\pi\theta_t$, where θ_t is continuous in $t \in [0, T]$ and $\theta_0 = 0$. In this case,

$$CZ_\tau(\gamma) = 2\lfloor\theta\rfloor + 1.$$

The relative first Chern class

The relative first Chern class is a generalization of the usual first Chern class of a complex line bundle ξ over a curve with boundaries.

Definition 8. Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be orbit sets with $[\alpha] = [\beta] \in H_1(Y)$, where $[\alpha] = \sum_i m_i[\alpha_i]$ and $[\beta] = \sum_j n_j[\beta_j]$. Then $H_2(Y, \alpha, \beta)$ denotes the set of relative homology classes of 2-chains Z in Y such that

$$\partial Z = \sum_i m_i\alpha_i - \sum_j n_j\beta_j.$$

[1]Indeed, we may not have any roots of unity as eigenvalues of the linearized return map since we may consider covers of Reeb orbits.

Definition 9. Fix a trivialization τ of the contact structure ξ over the Reeb orbits. Let $Z \in H_2(Y, \alpha, \beta)$ where α and β are orbit sets, and $f : S \to Y$ be a smooth representative of Z. The *relative first Chern class*

$$c_\tau(Z) := c_1(\xi|_Z, \tau) \in \mathbb{Z}$$

counts algebraically the zeros of a generic section ψ of the bundle $f^*\xi \to S$ transverse to the zero section, which is non-vanishing and has zero winding number with respect to τ on the boundary.

The relative first Chern class $c_\tau(Z)$ is well-defined and changes under the relative second homology class by

$$c_\tau(Z) - c_\tau(Z') = \langle c_1(\xi), Z - Z' \rangle \tag{3.2.1}$$

where $c_1(\xi) \in H^2(Y; \mathbb{Z})$ is the ordinary first Chern class. Let $\tau' := (\{\tau_i^{+\prime}\}, \{\tau_j^{-\prime}\})$ be another trivialization over the positive orbit set $\alpha := \{(a_i, m_i)\}$ and negative orbit set $\beta := (\{b_j, n_j\})$, then

$$c_\tau(Z) - c_{\tau'}(Z) = \sum_i m_i(\tau_i^+ - \tau_i^{+\prime}) - \sum_j n_j(\tau_j^- - \tau_j^{-\prime}). \tag{3.2.2}$$

The relative intersection pairing

The relative intersection pairing associates an intersection number to a pair of relative second homology classes. It is closely related to the behavior of the *asymptotic linking number* which will be defined in Section 3.3.

Definition 10. Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be orbit sets with $[\alpha] = [\beta] \in H_1(Y)$ and let $Z \in H_2(Y, \alpha, \beta)$. An *admissible representative* of Z is a smooth map $f : S \to [-1, 1] \times Y$, where S is a compact oriented surface with boundary, such that:

- $f|_{\partial S}$ consists of positively oriented covers of $\{1\} \times \alpha_i$ with total multiplicity m_i and negatively oriented covers of $\{-1\} \times \beta_j$ with total multiplicity n_j;
- the composition of f with the projection $[-1, 1] \times Y \to Y$ represents the class Z;
- $f|_{\dot{S}}$ is an embedding and f is transverse to $\{-1, 1\} \times Y$.

Now fix a trivialization τ of the contact structures ξ over all Reeb orbits.

Definition 11. Let $Z \in H_2(Y, \alpha, \beta)$ and $Z' \in H_2(Y, \alpha', \beta')$. The *relative intersection pairing* $Q_\tau(Z, Z') : H_2(Y, \alpha, \beta) \times H_2(Y, \alpha', \beta')$ is defined by

$$Q_\tau(Z, Z') := \#(\dot{S} \cap \dot{S}') - l_\tau(S, S'),$$

where S and S' are admissible representatives of Z and Z', and the interiors $\dot{S}$ and $\dot{S}'$ are transverse and do not intersect near the boundary. The term $l_\tau(S, S')$ is the *asymptotic linking number*, which will be defined in Section 3.3.

An alternative definition for $Q_\tau(Z, Z')$ is to find representatives S of Z and S' of Z', called *τ-representatives*, such that the projection of their intersections under the restricted projection maps $(1 - \epsilon, 1] \times Y \to Y$ and $[-1, -1 + \epsilon) \times Y \to Y$ are embeddings, and that near each orbit γ_i at their ends, S and S' are unions of rays emanating from points of γ_i, and the rays do not intersect and do not rotate with respect to the trivialization $\tau|_{\gamma_i}$. One can check that these two definitions are the same (Lemma 8.5 in [51]). In fact, both definitions will be useful to us.

Both definitions of $Q_\tau(Z, Z')$ are well-defined by Lemma 2.5 and 8.5 in [51]. In particular, Q_τ is independent of the choice of admissible representatives of the relative second homology classes. Moreover, under change of second homology class,

$$Q_\tau(Z_1, Z') - Q_\tau(Z_2, Z') = (Z_1 - Z_2) \cdot [\alpha'] \tag{3.2.3}$$

and under change of trivilizations,

$$Q_\tau(Z, Z') - Q_{\tau'}(Z, Z') = \sum_i m_i {m_i}'(\tau_i^+ - {\tau_i^+}') - \sum_j n_j {n_j}'(\tau_j^- - {\tau_j^-}'). \tag{3.2.4}$$

We call $Q_\tau(Z) := Q_\tau(Z, Z)$ the *relative self-intersection pairing.*

Notation 12. In the ECH literature, we often abuse notation and let C denote the holomorphic curve $u : (C, j) \to (X, J)$. In addition, we often write $c_\tau(C) := c_\tau([C])$ and $Q_\tau(C) := Q_\tau([C])$.

The Fredholm index and the ECH index

Let $\mathcal{M}(\gamma_1^+, \cdots, \gamma_k^+; \gamma_1^-, \cdots, \gamma_l^-; J)$ denote the moduli space of J-holomorphic curves with positive ends at $\gamma_1^+, \cdots, \gamma_k^+$ and negative ends at $\gamma_1^-, \cdots, \gamma_l^-$.

Definition 13. The *Fredholm index* of a curve $C \in \mathcal{M}({\gamma_1}^+, \cdots, {\gamma_k}^+; {\gamma_1}^-, \cdots, {\gamma_l}^-; J)$ is defined by

$$\operatorname{ind}(C) := -\chi(C) + 2c_\tau(C) + \sum_{i=1}^{k} CZ_\tau(\gamma_i^+) - \sum_{j=1}^{l} CZ_\tau(\gamma_j^-).$$

If J is generic and C is simple, then the above moduli space is a manifold near C of dimension $ind(C)$ by [32]. Now, let $\mathcal{M}(\alpha, \beta)$ denote the set of J-holomorphic currents that approach $\alpha = \{(\alpha_i, m_i)\}$ as a current for $s \to +\infty$ and $\beta = \{(\beta_j, n_j)\}$ as a current for $s \to -\infty$.

Definition 14. Let $Z \in H_2(Y, \alpha, \beta)$, then the *ECH index* is

$$I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(\alpha, \beta),$$

where

$$CZ_\tau^I(\alpha, \beta) := \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} CZ_\tau(\beta_j^k).$$

Let $\mathcal{C} \in \mathcal{M}(\alpha, \beta)$. Then we define $I(\mathcal{C}) := I(\alpha, \beta, [\mathcal{C}])$.

ECH index has many useful properties. The following inequality relates it to the Fredholm index (See [53]).

Theorem 15. *Let α and β be orbit sets and $C \in \mathcal{M}(\alpha, \beta)$ be a somewhere injective curve, then*

$$ind(C) \leq I(C)$$

with equality if and only if C is embedded.

The inequality in Theorem 15 has in particular the following applications.

Theorem 16 (Low-index currents)**.** *[53, Proposition 3.7] Suppose J is generic. Let α and β be orbit sets and $\mathcal{C} = \{(C_i, d_i)\} \in \mathcal{M}(\alpha, \beta)$, be a not necessarily somewhere injective J-holomorphic current. Then:*

- *$I(\mathcal{C}) \geq 0$, with equality if and only if each C_i is a trivial cylinder with multiplicity;*
- *If $I(\mathcal{C}) = 1$, then $\mathcal{C} = C_0 \sqcup C_1$, where $I(C_0) = 0$ and C_1 is an embedded curve with $I(C_1) = ind(C_1) = 1$;*
- *If $I(C) = 2$, and α and β are admissible orbit sets, then $\mathcal{C} = C_0 \sqcup C_2$, where $I(C_0) = 0$ and C_2 is an embedded curve with $I(C_2) = ind(C_2) = 2$.*

In particular, whenever we are counting low-index currents, we may assume that the underlying curves, other than potentially branched covers of trivial cylinders, are embedded. Another useful result relating the different quantities introduced in the previous subsections is the relative adjunction formula, which often helps with ECH index computations.

Proposition 17 (Relative adjunction formula)**.** [55, Proposition 4.9] Let $C \in \mathcal{M}(\alpha;\beta;J)$ be somewhere injective. Then C has finitely many singularities and

$$c_\tau(C) = \chi(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C), \tag{3.2.5}$$

where $\delta(C)$ is an algebraic count of singularities and $w_\tau(C)$ is the asymptotic writhe to be discussed in Section 3.3.

The ECH differential and the U map

The ECH differential $\langle \partial\alpha, \beta\rangle$ is defined to be the mod 2 count of the ECH index 1 pseudo-holomorphic currents u in $\mathbb{R}\times M$ from α to β. It is shown in [59, 61] that $\partial^2 = 0$, and in [109, 110, 111, 112, 113] that the homology $ECH(Y,\lambda,J)$ of this chain complex does not depend on the choice of J or λ.

In addition, ECH is equipped with a degree -2 chain map given a base point $z \in Y$ not on a Reeb orbit. Given admissible orbit sets α and β, we define

$$U_z : ECC_*(Y,\lambda,J) \longrightarrow ECC_{*-2}(Y,\lambda,J),$$

where $\langle U\alpha,\beta\rangle$ is the mod 2 count of pseudo-holomorphic currents of ECH index $I = 2$ from α to β passing through $(0,z) \in \mathbb{R}\times Y$. Furthermore, one can show that U_z is a chain map and that up to chain homotopy, U_z does not depend on the choice of the base point z. More precisely, let $z' \in Y$ be a different choice of the base point, then it is shown in [66] that there is a chain homotopy between U_z and $U_{z'}$. We will describe this further in Section 3.6.

The filtered ECH

We can associate to a Reeb orbit γ a *symplectic action*

$$\mathcal{A}(\gamma) := \int_\gamma \lambda,$$

and to each ECH generator $\alpha = \{(\alpha_i, m_i)\}$ the action

$$\mathcal{A}(\alpha) := \sum_i m_i \mathcal{A}(\alpha_i).$$

By Stokes' theorem and the definition of a symplectization-adapted almost complex structure J, we know that ECH differential decreases the symplectic action. Therefore, one may define the *filtered* version $ECC^L(Y,\lambda,J)$ to be the subcomplex generated by orbit sets α such that $\mathcal{A}(\alpha) < L$. For $L < L'$, under the inclusion-induced map

$$i_J^{L,L'} : ECH_*^L(Y,\lambda,J) \longrightarrow ECH_*^{L'}(Y,\lambda,J),$$

we may recover

$$ECH_*(Y, \lambda, J) = \lim_{L \to \infty} ECH_*^L(Y, \lambda, J). \tag{3.2.6}$$

In the following theorem, we review definitions from [64] and summarize their results regarding the filtered ECH that we need.

Definition 18. Given an action L, a contact form λ is *L-nondegenerate* if all orbits of action less than L are nondegenerate, and there are no orbit sets of action exactly L.

Definition 19. An *admissible deformation* is a smooth family $\rho = \{(\lambda_t, L, J_t, r_t) \mid t \in [0,1]\}$ such that for each $t \in [0,1]$:

- λ_t is an L_t-nondegenerate contact form on Y;
- J_t is a symplectization-admissible almost complex structure for λ_t;
- r_t is a positive real number.

Definition 20. Given an action L, an almost complex structure is *ECH^L-generic* if the ECH differential is well-defined on admissible orbit sets with actions less than L and satisfies $\partial^2 = 0$.

Theorem 21. *[64, Theorem 1.3] Let Y be a closed connected oriented three-manifold, then:*

1. *If λ is an L-nondegenerate contact form on Y, then $ECH_*^L(Y, \lambda, J)$ does not depend on the choice of ECH^L-generic J;*
2. *If $L < L'$ and λ is L'-nondegenerate, then $i_J^{L,L'}$ induces a well-defined map*
$$i^{L,L'} : ECH_*^L(Y, \lambda) \longrightarrow ECH_*^{L'}(Y, \lambda);$$
3. *If λ is a nondegenerate contact form on Y, then $ECH_*(Y, \lambda, J)$ does not depend on the choice of ECH-generic J, so it may be denoted by $ECH_*(Y, \lambda)$.*

3.3 The connected sum for contact three-manifolds

Let (Y_1, λ_1) and (Y_2, λ_2) be two closed connected contact three-manifolds with specified contact forms λ_i. In this paper, we consider their connected sum via the 4-dimensional Weinstein 1-handle attachment as in [119]. We follow [119] and [39] for the explicit descriptions of the Reeb dynamics on the Weinstein 1-handle model and the pseudo-holomorphic curves in the symplectization. See also [102, 13]. We give a version here for completeness and to put into our setting.

Weinstein 1-handle model and the Reeb flow

Consider

$$\mathbb{C}^2 = \mathbb{R}^4 = \{(x, y, z, w)\}$$

and the standard symplectic form

$$\omega := dx \wedge dy + dz \wedge dw$$

on it. Consider the Liouville vector field

$$X = \frac{1}{2}x\partial x + \frac{1}{2}y\partial y + 2z\partial z - w\partial w.$$

Now consider the function

$$f(x, y, z, w) = \frac{1}{4}x^2 + \frac{1}{4}y^2 + z^2 - \frac{1}{2}w^2.$$

The hypersurface $\{f = 1\}$ is of contact-type. This is because

$$df(X) = \frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 + w^2 > 0,$$

so X is transverse to $\{f = 1\}$. This is the contact manifold that contains the *ascending sphere*

$$S_+ := \{w = 0\} \cap \{f = 1\}$$

as in [119]. We call S_+ the *connected sum sphere*. Similarly, we have the contact-type hypersurface $\{f = -1\}$ and the *descending sphere*

$$S_- = \{x = y = z = 0\} \cap \{f = -1\}.$$

We will see in Lemma 24 that our resulting connected sum $Y_1 \# Y_2$ contains the ascending sphere S_+. Therefore, in order to understand the Reeb dynamics of $Y_1 \# Y_2$, we focus on the contact-type hypersurface $\{f = 1\}$.

We may compute that the contact form

$$\alpha = i_X\omega|_{f=1} = \frac{1}{2}xdy - \frac{1}{2}ydx + 2zdw + wdz|_{f=1}$$

and that $d\alpha = \omega$. We also compute the Reeb vector field

$$R = \kappa(\frac{1}{2}x\partial y - \frac{1}{2}y\partial x + 2z\partial w + w\partial z)$$

where $\kappa := 1/(1 + 3z^2 + \frac{3}{2}w^2)$. Therefore, one can see that the equator

$$h := \{x^2 + y^2 = 4, z = w = 0\}$$

of S_+ is the only orbit in the ascending manifold of the 1-handle. To compute the Reeb flow at the orbit h, we need to solve the linear ODE

$$\dot{\eta(t)} = A\eta(t),$$

where

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

and $\eta(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix}$. We have the solution $\eta(t) = e^{At}\eta(0)$, where

$$\Phi(t) := e^{At} = \begin{pmatrix} -cos(t/2) & sin(t/2) & 0 & 0 \\ sin(t/2) & cos(t/2) & 0 & 0 \\ 0 & 0 & cosh(\sqrt{2}t) & sinh(\sqrt{2}t) \\ 0 & 0 & \sqrt{2}sinh(\sqrt{2}t) & \sqrt{2}cosh(\sqrt{2}t) \end{pmatrix} \tag{3.3.1}$$

Note that the contact structure restricted to h is

$$\xi|_h = \langle \partial z, \partial w \rangle.$$

Now consider the trivialization τ_0 over h given by the second factor of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Under this trivialization,

$$CZ_{\tau_0}(h) = 0.$$

This is because the flow does not rotate the contact plane $\langle \partial z, \partial w \rangle$ by looking at the lower 2×2 matrix of $\Phi(t)$.

Once we have the model for the connected sum “tube”, the following proposition by Weinstein tells us how to glue two isotropic setups together via the tube that is the Weinstein 1-handle in our case. See [119, 17] for more details.

Definition 22. An *isotropic setup* is a quintuple $(V, \omega, x, Y, \Lambda)$ where (V, ω) is a symplectic manifold with Liouville vector field x and $\omega = d\lambda$; $Y \subset V$ is a codimension-1 hypersurface transverse to x; and $\Lambda \subset Y$ is a closed isotropic submanifold for the contact structure $ker(\lambda|_Y)$.

Proposition 23. [119, Proposition 4.2] Let $(V_i, \omega_i, x_i, Y_i, \Lambda_i)$, $i \in \{1, 2\}$ be two isotropic setups. Given a diffeomorphism $f : \Lambda_1 \to \Lambda_2$ covered by an isomorphism Φ of their symplectic subnormal bundles $(T\Lambda_i)^\omega / T\Lambda_i \subset \xi_i$, there exists an isomorphism of isotropic setups

$$F : (W_1, \omega_1, x_1, Y_1, \Lambda_1) \longrightarrow (W_2, \omega_2, x_2, Y_2, \Lambda_2)$$

between neighborhoods W_i of Λ_i in V_i inducing f and Φ.

Then, one may glue the descending sphere in the standard 1-handle as above to the product $[0, 1] \times (Y_1 \sqcup Y_2)$ along neighborhoods of the isotropic spheres in $\{1\} \times (Y_1 \sqcup Y_2)$. This 1-handle attachment yields $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ where the contact structure $\xi_1 \# \xi_2$ is the same as ξ_1 and ξ_2 when restricted to the complement of the neighborhoods of the isotropic spheres (Theorem 5.1 in [119]). In fact, the proof of Theorem 5.1 in [119] shows the following.

Lemma 24. Let $(Y_1 \# Y_2, \lambda_1 \# \lambda_2)$ be obtained by attaching the Weinstein one-handle defined above along the isotropic sphere in the contact manifold $(Y_1 \sqcup Y_2, \lambda_1 \sqcup \lambda_2)$. Then, the contact form $(Y_1 \# Y_2, \lambda_1 \# \lambda_2)$ differs from that on $(Y_1 \sqcup Y_2, \lambda_1 \sqcup \lambda_2)$ only on the neighborhood of the isotropic sphere where the surgery takes place. In addition, $\lambda_1 \# \lambda_2$ can be perturbed to be non-degenerate if $\lambda_1 \sqcup \lambda_2$ is non-degenerate.

Proof. Consider the following isotropic setup

$$(V, \omega, x, Y, \Lambda) = ([0, 1] \times (Y_1 \sqcup Y_2), d(e^t(\lambda_1 \sqcup \lambda_2)), \partial t, \{1\} \times (Y_1 \sqcup Y_2), S^0 = \{p\} \sqcup \{q\}).$$

where $p \in Y_1$ and $q \in Y_2$ are the two points away from orbits on which we are performing the connected sum operation. Consider the second isotropic setup given by the standard Weinstein handle described in Section 3.3

$$(V', \omega', x', Y', \Lambda') = (\mathbb{R}^4, \omega, X, \{f = -1\}, S_-)$$

Now applying Proposition 23 gives an isomorphism of isotropic setups and results in the contact manifold $(Y_1 \# Y_2, \lambda_1 \# \lambda_2)$. The strict contactomorphism away from the surgery region is given by flowing along ∂t. This is because $\mathcal{L}_{\partial t}\omega = \omega$ since ∂t is a Liouville vector field, so $i_{\partial t}\omega$ is preserved under flowing along ∂t. Now, by construction $\lambda_1 \# \lambda_2$ is nondegenerate on the complement of the surgery region.

The linearized flow in (3.3.1) shows that h is also nondegenerate. For other orbits potentially created during the connected sum operation, they must intersect the boundary spheres $S^2_p \sqcup S^2_q$ of a neighborhood of $\Lambda = \{p\} \sqcup \{q\}$. Therefore, we may find a C^∞-small perturbation supported in an arbitrarily small neighborhood of the spheres $S^2_p \sqcup S^2_q$ as in [39] so that $\lambda_1 \# \lambda_2$ is nondegenerate while maintaining the results in Theorem 25 in the next section. □

Almost complex structures

In this subsection, we discuss the pseudo-holomorphic behaviors in the symplectization of the connected sum region $\mathbb{R} \times S^2 \subset \mathbb{R} \times (Y_1 \# Y_2)$. This has been studied in [102, 39]. Fix a generic λ_i-adapted almost complex structure J_i on $\mathbb{R} \times (Y_i, \lambda_i)$ and let

$$\lambda' := \lambda_1 \# \lambda_2$$

be the connected sum contact form described in Lemma 24. Fish-Siefring found a λ'-adapted almost complex structure such that one may see a pair of pseudo-holomorphic planes asymptotic to h in $\mathbb{R} \times S^2$. We continue to follow the paper [39], now for information regarding the pseudo-holomorphic curves.

Theorem 25. *[39, Theorem 5.1] Let $Y := Y_1 \sqcup Y_2$ be a three-manifold equipped with a nondegenerate contact form λ. Let Y' be the connected sum manifold equipped with a nondegenerate contact form λ' given Lemma 24. Then, given any λ'-adapted J, there exists a λ'-adapted J' agreeing with J outside the surgery region such that:*

1. *There exists a pair of embedded, disjoint J'-holomorphic planes P_N and P_S both asymptotic to h.*

2. *The planes P_N and P_S approach h in* opposite directions*, that is, their leading asymptotic coefficients are of opposite signs (see Section 3.3).*

3. *The planes P_N and P_S project to the northern and southern hemispheres of the connected sum sphere S_+ under $\pi : \mathbb{R} \times Y' \to Y'$. Together with $\mathbb{R} \times h$, their $\mathbb{R}$-translations foliate $\mathbb{R} \times S_+$.*

Proof. Consider the 1-form

$$\alpha = i_X \omega|_{f=1} = \frac{1}{2} x dy - \frac{1}{2} y dx + 2z dw + w dz|_{f=1}$$

on $\mathbb{R} \times S_+$ considered in Section 3.3. The coordinates in [39] are related to ours by the following coordinate change Φ:

$$\begin{aligned} x(\rho,\phi,\theta) &= \sin\theta\cos\phi, \\ y(\rho,\phi,\theta) &= \sin\theta\sin\phi, \\ z(\rho,\phi,\theta) &= 4\cos\theta, \\ w(\rho,\phi,\theta) &= \frac{3}{8}\rho. \end{aligned}$$

Now, at the sphere $w = \rho = 0$,

$$\Phi^*\alpha = \frac{1}{2}\sin^2\theta d\phi + 3\cos\theta d\phi, \tag{3.3.2}$$

which agrees with the contact form given in [39] up to a constant. Therefore, the proof of Lemma 5.7 in [39] goes through. □

Now we compute the ECH index of the planes in Theorem 25.

Lemma 26. $I(P_S) = I(P_N) = ind(P_S) = ind(P_N) = 1.$

Proof. Fix trivialization τ_0 as in Section 3.3, then

$$c_{\tau_0}(P_S) = 1$$

by considering the vector field

$$zx\partial y - zy\partial x + (1-z^2)\partial w$$

which has one positive zero at $z = -1$. The zero is positive by the Poincaré-Hopf theorem. This vector field is in the contact structure $\ker\alpha$ restricted to P_S and tangent bundle of $P_S|_h$ while being normal to the boundary, and under the Reeb flow this vector field does not rotate with respect to the contact structure restricted to h. Therefore,

$$ind(P_S) = -\chi(C) + 2c_{\tau_0}(C) + CZ_{\tau_0}(h) = -1 + 2 + 0 = 1.$$

Now, to compute the ECH index, we first need to compute $Q_{\tau_0}(P_S)$. Since our trivialization τ_0 is the same trivialization associated to the constant section given by ∂z, one can see that

$$Q_{\tau_0}(P_S) = 0.$$

Therefore,

$$I(P_S) = c_{\tau_0}(P_S) + Q_{\tau_0}(P_S) + CZ^I_{\tau_0}(P_S) = 1 + 0 + 0 = 1.$$

Note that we did not need to compute the Fredholm index above, since if $I = 1$, then $ind = 1$ by Theorem 16. □

To see that all the curves are transversely cut out with the almost complex structure obtained in Theorem 25, we first observe that in the open set W defined by $\mathbb{R} \times (Y_1 \# Y_2)$ minus the symplectization of the connected sum sphere $\mathbb{R} \times S^2$, we can perturb our complex structure to be generic. Therefore, all the curves except for the two $ind = 1$ planes bounding the hyperbolic orbit h are transversely cut out, because they all intersect W. Then, we observe that for P_N and P_S which are embedded, we have automatic transversality (e.g. Proposition A.1 in [123]). This is because

$$ind(P_S) > c_N(P_S) = \frac{ind(P_S) - 2 + 2g + \#\Gamma_0}{2} = 0,$$

where $\#\Gamma_0$ denotes the number of asymptotic ends with even Conley-Zehnder indices. Note that the parity of the Conley-Zehnder index is independent of the choice of trivialization. The calculation is the same for P_N.

Remark 27. Another useful way to obtain the contact connected sum is the following. First, drill two convex sutured balls $(D^2 \times I, \Gamma)$, where the suture is $\Gamma = \partial D^2 \times \{1/2\}$, from the two contact three-manifolds Y_1 and Y_2. Turn one of the concave sutured manifolds, for example, $Y_1 \backslash (D^2 \times I, \Gamma)$, into a convex sutured manifold $Y_1(1)$ as described in [27]. This concave-to-convex process introduces a positive hyperbolic orbit. Finally, glue $Y_1(1)$ and $Y_2 \backslash (D^2 \times I, \Gamma)$ together to obtain a contact connected sum $Y_1 \# Y_2$. Although this construction also results in exactly one extra positive hyperbolic orbit up to large action and yields isotopic contact structures as the Weinstein construction above, it is not a priori clear how the almost complex structure on its symplectization corresponds to that coming from the Weinstein 1-handle attachment.

Asymptotic neighborhood of the special hyperbolic orbit h

The asymptotic neighborhood of a Reeb orbit encodes a lot of information about the potential pseudo-holomorphic curves that could asymptote to it. In order to understand the asymptotic neighborhood of h, we first review the asymptotic operator associated to a Reeb orbit. For more details, see for example [107] and [59].

Let C be a somewhere injective J-holomorphic curve with an asymptotic end at an embedded Reeb orbit γ. By rescaling the s and t coordinates on $\mathbb{R} \times Y$ near the Reeb orbit γ, we may assume that γ has period 1. The almost complex structure J on $\xi|_\gamma$ defines a family of 2×2 matrices J_t such that $J_t^2 = -1$ where $t \in \mathbb{R}/\mathbb{Z}$. The linearized Reeb flow $\Psi(t)$ along γ gives a symplectic connection

$$\nabla_t^R = \partial t + S_t$$

on the $\xi|_\gamma$, where S_t is a symmetric matrix for each $t \in \mathbb{R}/\mathbb{Z}$.

Definition 28. The asymptotic operator[2] is defined as

$$L_\gamma := J_t \nabla_t^R : C^\infty(S^1, \gamma^*\xi) \to C^\infty(S^1, \gamma^*\xi).$$

More specifically, let τ be a complex linear, symplectic trivialization of $\xi|_\gamma$. For each $t \in S^1$, define

$$S_t := -J_0 \frac{d\Psi(t)}{dt} \Psi^{-1}(t).$$

Then we may write

$$L_\gamma = J_0 \frac{d}{dt} + S_t.$$

Note L_γ is self-adjoint.

Let $\eta(t)$ be an eigenfunction of L_γ with eigenvalue λ. Then η solves the ODE

$$\frac{d\eta(t)}{dt} = J_0(S_t - \lambda)\eta(t),$$

and hence η is nonvanishing, if it's nonzero. Therefore, we may define $\mathrm{wind}_\tau(\eta)$ to be the winding number of the loop $\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ around zero. The above discussion generalizes to multiple covers γ^d of a simple orbit γ. Since we will focus on the hyperbolic orbit h which does not allow multiple covers as ECH generators, we refer the readers to [107, 59] for further discussions.

We recall the following useful properties of eigenvalues and winding numbers associated to an asymptotic operator.

Lemma 29. [51, Lemma 6.4] and [45, Chapter 3]

1. If η, η' are eigenvalues of the asymptotic operator L corresponding to eigenvalues $\lambda \leq \lambda'$, then $\mathrm{wind}(\eta) \geq \mathrm{wind}(\eta')$.
2. For each winding number w, the space of eigenfunctions with winding number w is 2-dimensional.
3. If γ is nondegenerate, then $\mathrm{wind}_\tau(\phi) \leq \lfloor CZ_\tau(\gamma^d)/2 \rfloor$ for $\lambda > 0$ i.e. λ is an eigenvalue associated to an asymptotic operator at a positive end, and $\mathrm{wind}_\tau(\phi) \geq \lceil CZ_\tau(\gamma^d)/2 \rceil$ for $\lambda < 0$ i.e. λ is an eigenvalue associated to an asymptotic operator at a negative end.

[2]Our asymptotic operator is defined with an opposite sign from others in the literature, e.g. [46].

Now we review the asymptotic expansion of an *asymptotically cylindrical pseudo-holomorphic curve* at a given nondegenenerate orbit. From now on, we focus on the case when a curve is positively asymptotic to a Reeb orbit, i.e. $s >> 0$. The case for the negative asymptotic end is completely analogous. The following definition is from [107].

Definition 30. Let γ be a Reeb orbit. Let u be a J-holomorphic curve positively asymptotic to γ. Let $U : [R,\infty) \times S^1 \to \gamma^*\xi$ be a smooth map satisfying $U(s,t) \in \xi|_{\gamma(t)}$ for all $(s,t) \in [R,\infty) \times S^1$, where R is a large real number. Then, U is an *asymptotic representative* of u if there exists a proper embedding $\psi : [R,\infty) \times S^1 \to \mathbb{R} \times S^1$ asymptotic to the identity, so that

$$u(\psi(s,t)) = (s, \exp_{\gamma(t)} U(s,t)) = \widetilde{\exp}_{(s,\gamma(t))}(0, U(s,t))$$

for all $(s,t) \in [R,\infty) \times S^1$, where exp and $\widetilde{\exp}$ are the exponential maps associated to a Riemannian metric on Y and its $\mathbb{R}$-invariant lift to $\mathbb{R} \times Y$.

The asymptotic representative encodes the isotopy class of a braid ζ that is the intersection of the pseudo-holomorphic curve asymptotic to γ and a tubular neighborhood μ of γ at $s >> 0$. Hofer-Wysocki-Zehnder [46] showed that the map U can be written in the form

$$U(s,t) = e^{-\lambda s}(\eta(t) + r(s,t)) \tag{3.3.3}$$

for an eigenvalue $\lambda > 0$ and eigenfunction $\eta(t)$ of L_γ and an error term $r(s,t)$ converging exponentially to zero as $s \to \infty$. The thesis of Siefring analyzed the difference of two pseudo-holomorphic half-cylinders asymptotic to the same Reeb orbit and as a result generalized the asymptotic form in Equation (3.3.3) to "higher orders" [107].

Theorem 31. *[107, Theorem 2.2] Let u, v be J-holomorphic curves with a positive asymptotic end at γ and the maps $U, V : [R,\infty) \times S^1 \to C^\infty(\gamma^*\xi)$ be their asymptotic representatives. Assume $U - V$ does not vanish identically. Then there exists a positive eigenvalue λ of the asymptotic operator L_γ with an eigenfunction $\eta(t)$ such that*

$$U(s,t) - V(s,t) = e^{-\lambda s}(\eta(t) + r(s,t)), \tag{3.3.4}$$

where the map $r(s,t)$ satisfies the decay estimate

$$|\nabla_s^i \nabla_t^j r(s,t)| \leq M_{ij} e^{-ds} \tag{3.3.5}$$

for every $(i,j) \in \mathbb{N}^2$, where M_{ij} and d consist positive constants.

A consequence of the above theorem of Siefring is that, for any positive integer N, we may expand a positive end of a pseudoholomorphic curve by

$$u(s,t) = \sum_{i}^{N} e^{-\lambda_i s} a_i \eta_i(t) + o_\infty(\lambda_N), \tag{3.3.6}$$

where $a_i \in \mathbb{R}$ keeps track of the scalar of the eigenfunction $\eta_i(t)$ and o_∞ is a function $f : [R, \infty) \times S^1 \to \mathbb{R}^2$ satisfying the decay estimate (3.3.5), c.f. Theorem 2.3 in [107]. We call a_i the *asymptotic coefficients* of the asymptotic expansion of u at γ. An immediate corollary is the following.

Corollary 32. Let u, v be distinct pseudo-holomorphic curves with a common asymptotic end at γ. Then the asymptotic coefficients of the asymptotic expansions of u and v at γ differ at some term.

To fix notations, from now on, let (λ_i, η_i) be the corresponding eigenvalues and eigenvectors of the asymptotic operator L_h associated to the hyperbolic orbit h. Fix some integer $N >> 0$. Let the asymptotic expansion of P_N be

$$\sum_{i}^{N} e^{-\lambda_i s} a_i \eta_i(t) + o_\infty(\lambda_N)$$

and the asymptotic expansion of P_S be

$$\sum_{i}^{N} e^{-\lambda_i s} b_i \eta_i(t) + o_\infty(\lambda_N).$$

Notice that we use the same index i, and some of a_i and b_i could be zero. In addition, a_0 and b_0 are of opposite signs by Theorem 25(2).

By Lemma 29(3), we see that $\text{wind}_{\tau_0}(h) \leq 0$ when the special hyperbolic orbit h is at a positive end. In Lemma 37, we show that this bound is achieved. However, before that, we need to introduce a few more definitions in preparation of the proof.

Specifically, let ζ be a braid in a tubular neighborhood μ of γ determined by a pseudo-holomorphic curve u asymptotic to γ as above. Identify

$$\mu \cong S^1 \times \mathbb{R}^2 \cong A \times (0,1) \subset \mathbb{R}^3$$

with respect to a trivialization τ, where A denotes an annulus. Let γ be an embedded Reeb orbit such that C has positive ends of multiplicities $q_1, \ldots, q_n$ at γ with total multiplicity m. Each end of a pseudo-holomorphic curve determines a braid component ζ_i of the braid ζ with q_i strands, where $\zeta = im(u) \cap \{s\} \times \mu$ for $s >> 0$ as before.

Definition 33. The *writhe* $w_\tau(\zeta) \in \mathbb{Z}$ is defined to be one half the signed count of crossings under the projection $A \times (0,1) \to A$.

Definition 34. Let ζ_1 and ζ_2 be two disjoint braids in the neighborhood of γ. Then the *linking number* $l_\tau(\zeta_1, \zeta_2) \in \mathbb{Z}$ is defined to be one half the signed count of crossings between ζ_1 and ζ_2 under the projection $A \times (0,1) \to A$.

In light of the braid picture, the winding number of an eigenfunction η_ζ associated to the braid ζ around the underlying simple orbit γ is exactly

$$\text{wind}_\tau(\eta_\zeta) = l_\tau(\zeta, \gamma).$$

Definition 35. Let S and S' be admissible representatives of $H_2(Y, \alpha, \beta)$ and $H_2(Y, \alpha', \beta')$. Then,

$$w_\tau(S) := \sum_i w_\tau(\zeta_i^+) - \sum_j w_\tau(\zeta_j^-),$$

$$\text{wind}_\tau(S) := \sum_i \text{wind}_\tau(\zeta_i^+) - \sum_j \text{wind}_\tau(\zeta_j^-),$$

$$l_\tau(S, S') := \sum_i l_\tau(\zeta_i^+, \zeta_i^{+\prime}) - \sum_j l_\tau(\zeta_j^-, \zeta_j^{-\prime}),$$

where ζ_i^+ and $\zeta_i^{+\prime}$ are braids defined by the intersections of S and S' with $(\{1-\epsilon\} \times Y)$, and ζ_j^- and $\zeta_j^{-\prime}$ similarly with $\{-1+\epsilon\} \times Y$.

We have the following "linking bound" lemma that first appeared in [51].

Lemma 36. [53, Lemma 5.5(b)] Let

$$\rho_i := \left\lfloor \frac{CZ_\tau(\gamma^{q_i})}{2} \right\rfloor.$$

If $i \neq j$, then ζ_i and ζ_j are disjoint and

$$l_\tau(\zeta_i, \zeta_j) \leq \max(\rho_i q_j, \rho_j q_i).$$

In particular, any two braids ζ_i and ζ_j associated to the asymptotic neighborhood at h have bounded linking number $l_{\tau_0}(\zeta_i, \zeta_j) \leq 0$ since $CZ_{\tau_0}(h) = 0$.

Lemma 37. $\text{wind}_{\tau_0}(P_S) = \text{wind}_{\tau_0}(P_N) = 0$.

Proof. We prove the lemma for P_S. The proof for P_N is exactly the same. Applying the relative adjunction formula in Proposition 17 to $C := P_S \sqcup P'_S$, where P'_S is an s-translation of P_S, we get

$$c_{\tau_0}(C) = \chi(C) + Q_{\tau_0}(C) + w_{\tau_0}(C) - 2\delta(C).$$

By the construction of almost complex structure in Theorem 25 that admits the foliation of $\mathbb{R} \times S^2$ where P_S is a leaf, we have that $\delta(C) = 0$. We also know that $Q_{\tau_0}(C) = 0$ since Q_τ is quadratic and $Q_{\tau_0}(P_S) = 0$. In addition, $w_{\tau_0}(P_S) = 0$. This is because the multiplicity of h is one and therefore $w_{\tau_0}(\zeta_h) = 0$ where ζ_h is the braid at the intersection of P_S and $\{1-\epsilon\} \cap Y$. Now we have that

$$2c_{\tau_0}(P_S) = 2 + 2\,\mathrm{wind}_{\tau_0}(P_S),$$

since

$$w_{\tau_0}(P_S \sqcup P'_S) = 4w_{\tau_0}(P_S) + 2\,\mathrm{wind}_{\tau_0}(P_S) = 2\,\mathrm{wind}_{\tau_0}(P_S),$$

where the first equality is justified in the proof of Proposition 8.4 in [51]. Therefore,

$$\mathrm{wind}_{\tau_0}(P_S) = 0.$$

□

The above lemma is needed for our specific case when the almost complex structure on $\mathbb{R} \times S^2$ is not generic as constructed in Theorem 25. When we have generic almost complex structure, the following proposition says that the extremal bound of the winding number is always achieved.

Proposition 38. [61, Proposition 3.2] If the symplectization-adapted almost complex structure J on $\mathbb{R} \times Y$ is generic, then for any Fredholm index 1, connected, non-multiply-covered J-holomorphic curve C having a positive end at γ, the winding number of the leading eigenfunction of the asymptotic expansion of C achieves the equality in Lemma 29(3).

3.4 Correspondence of Reeb orbits and filtered ECH complex

In this section, we discuss how to correspond the Reeb orbits in the closed manifolds $Y_1 \# Y_2$ to the ones in Y_1 and Y_2. We show that up to a sufficiently large action L, we may ignore orbits crossing the connected sum sphere. This allows an identification on the vector space level of a filtered ECH complex of the connected sum and a filtered mapping cone complex.

The mapping cone

Given a chain map between two chain complexes, one can form its *mapping cone*. In the following, we review this construction in our setting. We define

$$C_o := ECC(Y_1 \sqcup Y_2, \lambda_1 \sqcup \lambda_2) = ECC(Y_1, \lambda_1) \otimes_{\mathbb{F}} ECC(Y_2, \lambda_2).$$

Definition 39. Given the chain map $\varphi : C_o \to C_o$ defined by

$$\varphi := U_1 \otimes id + id \otimes U_2,$$

one can form the *mapping cone* $Cone(\varphi)$ by the following. As a vector space,

$$Cone(\varphi) := C_o \oplus C_o[-1],$$

where $C_o[-1]^* := C_o^{*-1}$. The differential of $Cone(\varphi)$ is given by

$$\partial_{cone} := \begin{pmatrix} \partial_1 \otimes id + id \otimes \partial_2 & 0 \\ \varphi & \partial_1 \otimes id + id \otimes \partial_2 \end{pmatrix}.$$

Let $C_h := C_o \otimes h$ denote the collection of all orbit sets in $Y_1 \sqcup Y_2$ concatenating with the special hyperbolic orbit h. By Theorem 25 and Lemma 26, we know that h bounds $I = 1$ planes P_N and P_S. Therefore, one can think of h as an element that increases the relative grading by one. We can then identify $C_o[-1]$ with C_h and obtain that

$$Cone(\varphi) = C_0 \oplus C_h$$

as vector spaces. Furthermore, we define a filtered mapping cone complex.

Definition 40. Given a real number L, we define

$$Cone^L(\varphi) := C_o^L \oplus C_h^L = C_o^L \oplus C_o^{L-\mathcal{A}(h)}.$$

One may check that ∂_{cone} preserves $Cone^L(\varphi)$, since both the differential and the U map decrease the symplectic action by Stokes' theorem and the definition of a λ-adapted almost complex structure. Therefore, $Cone^L(\varphi)$ is a subcomplex of $Cone(\varphi)$. In addition, we observe that

$$\lim_{L \to \infty} Cone^L(\varphi) = Cone(\varphi)$$

given by the obvious inclusion maps.

Filtered ECH of a connected sum

We use filtered ECH to ignore potential orbits of large symplectic actions formed during the connected sum procedure, which cross the connected sum S^2. The following lemma is folklore.

Lemma 41. Let $L > 0$. Let p be a point which is not on a Reeb orbit of action $< L$. Then there exists a radius $R(L)$ such that any Reeb trajectory which starts on the ball B of radius $R(L)$, leaves B and returns to B, has symplectic action greater or equal L.

Proof. Suppose not. We take a sequence of balls B_n around p of radius converging to 0, and Reeb trajectories γ_n of action less than L which start on B_n, leave B_n, and return to B_n. We can then pass to a subsequence so that the Reeb trajectories γ_n converge to a Reeb orbit γ of action less than L which passes through p. Contradiction. □

The above lemma tells us that up to an action L, we may ignore orbits that pass through the connected sum region other than the special hyperbolic orbit h. More precisely, using Lemma 41, we may find a strictly decreasing smooth function $R : \mathbb{R} \to \mathbb{R}$ such that $\lim_{L\to\infty} R(L) = 0$ and

$$ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)}) \cong C_o^L \oplus C_h^L = Cone^L \tag{3.4.1}$$

as vector spaces, where $\alpha_{R(L)} := \lambda_1\#_{R(L)}\lambda_2$ denotes the contact form on $Y_1\#_{R(L)}Y_2$ where the function f in Section 3.3 is scaled by $R(L)$. Note that any orbit intersecting a connected sum sphere of radius r would also intersect a connected sum sphere of radius $r' > r$, so we may assume $R(L)$ in Lemma 41 is non-increasing. Furthermore, we can modify $R(L)$ to be smooth and strictly decreasing. Therefore, by (3.4.1), we obtain an isomorphism on the level of vector spaces of the following theorem.

Proposition 42 (Proposition 4)**.** Given two closed connected contact three-manifolds (Y_1, λ_1) and (Y_2, λ_2) with nondegenerate contact forms λ_i. Let R be the decreasing function of L defined above. Then there is a chain homotopy equivalence

$$f : ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)}) \to Cone^L(U_1 \otimes id + id \otimes U_2).$$

The main goal of Section 3.5 and Section 3.6 is to prove Proposition 42 by studying the connected sum differentials. Then, in Section 3.7, we will pass the filtered ECH chain complexes $ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)})$ to a direct limit, where we consider smaller and smaller connected sum spheres as we let the symplectic action L go to infinity, to obtain the main theorem.

3.5 The differentials for ECH of a connected sum

There are four types of curves to consider for the differential of $ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)})$:

- $I = 1$ curves in $\mathbb{R} \times (Y_1\#_{R(L)}Y_2)$ with positive ends in C_o^L and negative ends in C_o^L, denoted as ∂_{oo};
- $I = 1$ curves in $\mathbb{R} \times (Y_1\#_{R(L)}Y_2)$ with positive ends in C_h^L and negative ends in C_o^L, denoted as ∂_{oh};
- $I = 1$ curves in $\mathbb{R} \times (Y_1\#_{R(L)}Y_2)$ with positive ends in C_o^L and negative ends in C_h^L, denoted as ∂_{ho};
- $I = 1$ curves in $\mathbb{R} \times (Y_1\#_{R(L)}Y_2)$ with positive ends in C_h^L and negative ends in C_h^L, denoted as ∂_{hh}.

We organize this as

$$\partial_\# = \begin{pmatrix} \partial_{oo} & \partial_{oh} \\ \partial_{ho} & \partial_{hh} \end{pmatrix}.$$

In this section, we describe how the differentials $\partial_{oo}, \partial_{oh}$ and ∂_{hh} behave in terms of the original differentials in $ECC^L(Y_1, \lambda_1)$ and $ECC^L(Y_2, \lambda_2)$. In particular, we show that the connected sum differentials ∂_{oo} and ∂_{hh} are identified with the original differentials on Y_1 and Y_2, considering the additional trivial cylinder over h. The differential ∂_{oh} is reminiscent of the linearized contact homology case [13], where the special hyperbolic orbit can be assumed to have small action and hence cannot admit a curve positively asymptotic to it while having negative asymptotics. In our case where a curve may have multiple positive asymptotic ends, we need to study the asymptotic behavior near the neighborhood of the hyperbolic orbit h more closely. The differential ∂_{ho} involves the original U maps in $ECC^L(Y_1, \lambda_1)$ and $ECC^L(Y_2, \lambda_2)$ and will be discussed in Section 3.6.

No-crossing lemma

This lemma appears in the draft of the connected sum formula for linearized contact homology [13] and we give essentially the same proof here using the relative intersection number Q_τ and the asymptotic linking number l_τ.

Lemma 43 (No-crossing)**.** *Let C be a connected, embedded J-holomorphic curve in the symplectization $\mathbb{R} \times (Y_1\#_{R(L)}Y_2, \alpha_{R(L)})$ with ends that do not intersect the connected sum S^2. Then the image of C cannot have ends in both Y_1 and Y_2 away from the connected sum region.*

Proof. Suppose we have a curve C with positive ends at α and negative ends at β, such that $\alpha \cup \beta$ lie in both Y_1 and Y_2 away from the Darboux balls used for the connected sum operation. Then C must intersect $\mathbb{R} \times S_+$. In particular, C must intersect one of the holomorphic planes that are the translations of P_N the P_S given in Theorem 25 or the trivial cylinder over h. We may assume without loss of generality that C intersects P_N or C intersects the trivial cylinder over h. Therefore, we have $\#(\dot{C} \cap \dot{P_N}) \geq 1$ or $\#(\dot{C} \cap \dot{(\mathbb{R} \times h)}) \geq 1$.

Suppose $\#(\dot{C} \cap \dot{P_N}) \geq 1$. Recall that

$$Q_\tau(C, P_N) = \#(\dot{S} \cap \dot{S}') - l_\tau(S, S'),$$

where S and S' are any admissible representatives of $[C]$ and $[P_N]$, and the interiors $\dot{S}$ and $\dot{S}'$ are transverse and do not intersect near the boundary. We have that

$$l_\tau(S, S') = 0$$

since by assumptions the ends of C do not overlap with the end h of P_N.

Now it suffices to show that

$$Q_\tau(C, P_N) = 0.$$

By a usual transversality argument, one can find another representative of $[C] \in H_2(Y, \alpha, \beta)$, which is still denoted as S, such that the intersections of S and S' miss a neighborhood of the north pole on P_N. Therefore, we may define a homotopy $S_{t\in[0,1]}$ moving all possible interior intersections of S and S' to be on P_S. Therefore,

$$\#(\dot{S} \cap \dot{S}') = \#(\dot{S}_1 \cap \dot{S}') = 0,$$

so $Q_\tau(C, P_N) = 0$. The arguments for $Q_\tau(C, \mathbb{R} \times h) = 0$ is entirely analogous. □

∂_{oo}: differential from orbit sets without h to orbit sets without h

The discussions in Section 3.4 shows that, by using filtered ECH up to filtration L, we may focus only on orbits of actions less than L in $(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$ and all such orbits are away from the connected sum S^2 except for the special hyperbolic orbit h. We then use the no-crossing lemma to show that the part of the connected sum differential not approaching the hyperbolic orbit h at all is the same as the sum of the original differential ∂_i on $ECC^L(Y_i, \lambda_1)$:

Lemma 44. We have that the ECH differential

$$\partial_{oo} = \partial_1 \otimes id + id \otimes \partial_2$$

when restricted to the subcomplex $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$.

Proof. By Lemma 41, orbits that intersect the connected sum S^2 have energy $\geq L$, except for the special hyperbolic orbit h. To study ∂_{oo}, we may focus on the connected, embedded, $I = 1$ curves given by Theorem 16. Then by the Lemma 43, the curves counted by ∂_{oo} are exactly those that do not pass through $\mathbb{R} \times S^2$. These are exactly the $I = 1$ curves either completely in $\mathbb{R} \times Y_1$ or completely in $\mathbb{R} \times Y_2$, i.e. those counted by $\partial_1 \otimes id + id \otimes \partial_2$. □

∂_{oh}: differential from orbit sets with h to orbit sets without h

The heuristics come from "local energy" of curves near the asymptotic end at the special hyperbolic orbit h. In this subsection, we eliminate all curves that have a positive asymptotic end at h and have no negative asymptotic end at h, other than the two planes P_S and P_N and their translations bounding h as described in Section 3.3.

Definition 45. A J-holomorphic curve C *approaches* h *from the south (resp. north)* if its leading asymptotic coefficient has the same sign as that of P_S (resp. P_N).

Proposition 46. Let J be generic in the open set $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2$. Any embedded, Fredholm index 1, connected curve with positive ends in C_h and negative ends in C_o is either P_S or P_N, or a translation of them in the ∂s direction.

Proof. Suppose C is an embedded, Fredholm index 1, connected J-holomorphic curve with a positive asymptotic end at h. In particular, C is not a trivial cylinder by Theorem 16. Suppose C is not a translation of P_S or P_N. In particular,

$$C \cap (\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2) \neq \emptyset,$$

so J is generic for C. By Lemma 29(3) and Proposition 38, we know that the leading eigenfunction ϕ of C at h has

$$\text{wind}_{\tau_0}(\phi) = \lfloor CZ_{\tau_0}(h)/2 \rfloor = 0.$$

Notice that this is a calculation that holds for any C satisfying the assumptions.

Without loss of generality, assume that C approaches h from the south. In particular, the leading asymptotic coefficient of C has the same sign as that of

P_S. The proof for C approaching from the north is exactly the same. Now by the definition of Q_τ and intersection positivity, we have that

$$Q_{\tau_0}(C, P_S) + l_{\tau_0}(C, P_S) = \#(\dot{C} \cap \dot{P}_S) \geq 0, \tag{3.5.1}$$

since both C and P_S are admissible representatives of their respective relative second homology classes.

Note that the leading eigenvalues of C and P_S both achieve the minimal possible eigenvalue of L_h by Lemma 37 for curves completely inside the $\mathbb{R} \times S^2$ region, i.e. P_S and P_N, and by Lemma 29(3) and Proposition 38 for curves that intersect the $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2$ region hence they are generic so they also achieve minimal winding numbers. Let the asymptotic expansions of C and P_S at h be:

$$C = \sum_i^N e^{-\lambda_i s} c_i \eta_i + o_\infty(\lambda_N) \tag{3.5.2}$$

$$P_S = \sum_i^N e^{-\lambda_i s} b_i \eta_i + o_\infty(\lambda_N) \tag{3.5.3}$$

for some integer $N >> 0$, where (λ_i, η_i) are the corresponding eigenvalues and eigenfunctions of the asymptotic operator L_h as discussed in Section 3.3. From the above discussions on minimal winding numbers, we have that $c_0 \neq 0$ and $b_0 \neq 0$. In addition, by the assumption that C approaches h from the south, we know that c_0 and b_0 have the same sign. Now we have two possibilities.

If $c_0 = b_0$, then we may find the first $i > 0$ such that $c_i \neq b_i$ by Corollary 32 and the assumption that $C \not\equiv P_S$. Now by Lemma 29(3), we know that the winding number that is 0 is achieved by the corresponding eigenfunctions of both the maximal negative and minimal positive eigenvalues. Since $\lambda_0 < \lambda_i$, by Lemma 29(1), we have that $\text{wind}(\eta_i) \leq \text{wind}(\eta_0)$. Therefore, now by Lemma 29(2), $\text{wind}(\eta_i) < \text{wind}(\eta_0) = 0$. Therefore, we have that

$$l_{\tau_0}(C, P_S) \leq \text{wind}(\eta_i) \leq -1$$

as in the proof of Lemma 6.9 in [51].

If $c_0 \neq b_0$, then we can shift P_S in the ∂s direction to some $s' \neq s$ corresponding to the curve $P_{S'}$ so that $c_0 e^{-\lambda_0 s} = b_0 e^{-\lambda_0 s'}$, since c_0 and b_0 have the same sign. Then some higher order $i > 1$ term will contribute to at least an additional -1 to $\text{wind}(\eta_0)$ by using the same arguments as in the above paragraph. Therefore, in this case we also have

$$l_{\tau_0}(C, P_{S'}) \leq -1.$$

Recall that $Q_{\tau_0}(C, P_S)$ is well-defined under relative second homology class. Now by the inequality in (3.5.1), it suffices to show that $Q_{\tau_0}(C, P_S) = 0$. Combined with the fact that the connected sum S^2 is null-homologous, it suffices to show $Q_{\tau_0}(C, P_N) = 0$ by (3.2.3).

By assumption, C and P_N have their leading asymptotic eigenfunctions of opposite signs. Therefore, $l_{\tau_0}(C, P_N) = 0$. We then have that

$$Q_{\tau_0}(C, P_N) = \#(\dot{C} \cap \dot{P_N}) - l_{\tau_0}(C, P_N) = \#(\dot{C} \cap \dot{P_N}).$$

In particular, $\#(\dot{C} \cap \dot{P_N})$ is independent of the admissible representative Σ of $[C]$, since Q_{τ_0} is. Now, by a usual transversality argument, we may find an admissible representative Σ of $[C]$ such that Σ does not pass through a neighborhood of the north pole of the connected sum S^2. Therefore, we may define a homotopy $\Sigma_{t\in[0,1]}$ moving all possible interior intersections of Σ and P_N to be on P_S. Therefore,

$$\#(\dot{C} \cap \dot{P_N}) = \#(\dot{\Sigma}_1 \cap \dot{P_N}) = 0,$$

so $Q_{\tau_0}(C, P_N) = 0$. This gives a contradiction to (3.5.1). □

Corollary 47. We have that the ECH differential $\partial_{oh} = 0$ when restricted to the subcomplex $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$.

Proof. By Lemma 43, we may restrict our attention to orbits that lie in $C_o^L \oplus C_h^L$. By Proposition 46, the only ECH index 1 curves that have a positive asymptotic end at h and no negative asymptotic end at h are P_S and P_N up to $\mathbb{R}$-translation. This immediately means that $\partial_{oh} = 0$ on $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$, since we are working over $\mathbb{F}$. □

∂_{hh}: differential from orbit sets with h to orbit sets with h

In this subsection, we show that when restricted to $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$, any ECH index 1 current that have both a positive asymptotic end and a negative asymptotic end at h must separate out a trivial cylinder over h.

Proposition 48. Let J be generic in the open set $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2$. Any embedded, Fredholm index 1 curve with positive ends in C_h and negative ends in C_h must contain a trivial cylinder over h.

Proof. The proof is exactly the same as that of Proposition 46. Consider an embedded, Fredholm index 1, connected holomorphic curve C with a positive end at h and a negative end at h. In particular, Proposition 38 still holds. Now consider again the equation

$$Q_{\tau_0}(C, P_S) + l_{\tau_0}(C, P_S) = \#(\dot{C} \cap \dot{P}_S).$$

As in the proof of Proposition 46, the left hand side is strictly negative while by positivity of intersections, the right hand side is greater or equal to zero. Therefore, the index 1 curve C was disconnected. In order for C to have a positive end at h, we know the only possibility is for C to contain a trivial cylinder over h or one of the P_S and P_N planes by Proposition 46. However, the latter case can be ruled out since both P_S and P_N are of Fredholm index 1 and the Fredholm index is additive. □

Corollary 49. We have that the ECH differential

$$\partial_{hh} = h\partial_{oo}\frac{1}{h}$$

when restricted to the subcomplex $ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)})$.

Proof. By Lemma 43, we may restrict our attention to orbits that lie in $C_o^L \oplus C_h^L$. By Proposition 48, we know that any current C counted by ∂_{hh} must contain a trivial cylinder over h, which is of ECH index 0. The remaining components of C contribute exactly to currents counted by ∂_{oo}. □

3.6 The chain homotopy equivalence

In this section, we discuss the last piece of the differential in the connected sum, i.e. ∂_{ho}, which is the differential from orbit sets without h to orbit sets with h. This uses a similar chain homotopy defined in the proof to show that the ECH U map is independent of the change of base points (Section 2.5 in [66]). Roughly speaking, the differential ∂_{ho} in the connected sum corresponds to U maps in the original contact three-manifolds Y_1 and Y_2, up to a "chain homotopy" K discussed in Section 3.6. This map K will in turn help us construct the chain homotopy equivalence between the filtered ECH chain complex of the connected sum and the filtered mapping cone in Section 3.6.

A "chain homotopy" induced by the change of base point

First, we need a lemma analogous to Lemma 44 for the U maps. Let $U_{\#,z_i}$ denote the U map in $\mathbb{R}\times(Y_1\#_{R(L)}Y_2)$ with base point z_i contained in Y_i away from the connected sum region. A similar proof to that of Lemma 44 gives the following lemma, since

$I = 2$ curves with asymptotic ends on ECH generators are also embedded by Theorem 16 and satisfy the no-crossing lemma.

Lemma 50. The moduli space of curves counted by $U_{\#,z_i}$ with asymptotic ends in C_o is exactly the same as the moduli space of curves counted by $I = 2$ curves in $\mathbb{R} \times Y_i$ passing through the point z_i, for $i \in \{1, 2\}$. In particular,

$$U_1 \otimes id + id \otimes U_2 = (U_{\#,z_1})_{oo} + (U_{\#,z_2})_{oo}$$

when restricted to the filtered subcomplex $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$.

There are very strong restrictions on which curves could have a positive end at h, as shown in the previous section. When we require such curves to pass through a point p in P_N, this naturally gives more restrictions to the curves. In fact, in the following, we obtain that the only curve with a positive end at h that passes through p is P_N.

Lemma 51. Let J be generic in the open set $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2$. Let C be an $I = 1$ curve that has positive ends in C_h, negative ends in $C_o \oplus C_h$, and passes through a given point $p \in P_N$. Then $C \equiv P_N$.

Proof. This is a combination of Proposition 46 and Proposition 48. □

Remark 52. We can also pick p on P_S. Note that p is not generic in the sense that it lies on a Fredholm index 1 curve.

Definition 53. Let $\mathcal{M}_1(\alpha; \beta; p)$ denote the moduli space of $I = 1$ curves in $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2)$ with positive asymptotic ends at the orbit set α and negative asymptotic ends at orbit set β, passing through $\{0\} \times A$, where $A := [z_1, p) \cup (p, z_2]$ is required to not pass through P_S or any Reeb orbit, which is possible since there are only countably many Reeb orbits. See Figure 3.1.

Lemma 54. Let J be generic in the open set $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2) \setminus \mathbb{R} \times S^2$. Then $\mathcal{M}_1(\alpha; \beta; p)$ is a compact 0-dimensional manifold.

Proof. Consider the moduli space $\widehat{\mathcal{M}}_1(\alpha; \beta; p)$ consisting of $I = 1$ curves in $\mathbb{R} \times (Y_1 \#_{R(L)} Y_2)$ with positive asymptotic ends at the orbit set α and negative asymptotic ends at the orbit set β, passing through $\{0\} \times (A \cup \{p\})$. Observe that the moduli space $\mathcal{M}_1(\alpha; \beta; p)$ is the same as $\widehat{\mathcal{M}}_1(\alpha; \beta; p)$, except we are subtracting the holomorphic plane P_N by Lemma 51. Since $\widehat{\mathcal{M}}_1(\alpha; \beta; p)$ is a compact 0-dimensional manifold (Section 2.5 of [66]), we have that $\mathcal{M}_1(\alpha; \beta; p)$ is also a compact 0-dimensional manifold. □

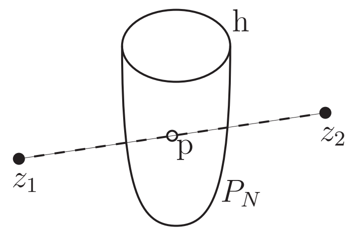


Figure 3.1: The base point p on the pseudo-holomorphic plane P_N

In fact, by relating to the change of base point homotopy in the setting of [66], we obtain more information. First, we define the following related map.

Definition 55. Let p be a point on P_N. Given an orbit set α, the linear map K is defined as

$$K\alpha := \sum_{\beta} \sum_{C \in \mathcal{M}_1(\alpha;\beta;p)} \beta,$$

where β ranges over all orbit sets.

Lemma 56. We have that $K_{oh} = 0$ when restricted to the subcomplex $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$.

Proof. Recall again that we may restrict our attention to $C_o^L \oplus C_h^L$ by the no-crossing Lemma 43. Then we apply Proposition 46 and the definition of the map K. In particular, P_S also does not intersect the arc A, so there is no curve counted by K_{oh}. □

Lemma 57. When restricted to the subcomplex $ECC^L(Y_1 \#_{R(L)} Y_2, \alpha_{R(L)})$, we have the equation

$$U_1 \otimes id + id \otimes U_2 + \frac{1}{h}\partial_{ho} = \partial_{oo} K_{oo} + K_{oo}\partial_{oo}. \tag{3.6.1}$$

Proof. The chain homotopy $\widehat{K}$ induced by the change of base points in [66] gives us that

$$(U_{\#,z_1})_{oo} + (U_{\#,z_2})_{oo} = \partial_{oo}\widehat{K}_{oo} + \widehat{K}_{oo}\partial_{oo} + \partial_{oh}\widehat{K}_{ho} + \widehat{K}_{oh}\partial_{ho}, \tag{3.6.2}$$

where $\widehat{K}$ is the degree -1 map defined by counting $I = 1$ curves passing through the path $\{0\} \times (A \cup \{p\})$ as above. Restricting (3.6.2) to appropriate ends up with action $< L$ and applying Lemma 50 gives us that

$$U_1 \otimes id + id \otimes U_2 = \partial_{oo}\widehat{K}_{oo} + \widehat{K}_{oo}\partial_{oo} + \partial_{oh}\widehat{K}_{ho} + \widehat{K}_{oh}\partial_{ho}. \tag{3.6.3}$$

Recall by Lemma 41, we have that $ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)}) \cong C_o^L \oplus C_h^L = Cone^L$ as vector spaces as in (3.4.1). By the definition of K and Lemma 43, we immediately have that $K_{oo} = \widehat{K}_{oo}$ on the filtered complex $Cone^L$. By Lemma 51, we have that

$$K_{oh} = \widehat{K}_{oh} - \frac{1}{h}$$

on $Cone^L$. Moreover, by Lemma 56, we have that $K_{oh} = 0$ hence

$$\widehat{K}_{oh} = \frac{1}{h}$$

on $Cone^L$. Recall $\partial_{oh} = 0$ by Corollary 47 on $Cone^L$. Therefore, we obtain Equation (3.6.1) on $Cone^L = ECC^L(Y_1\#_{R(L)}Y_2, \alpha_{R(L)})$. □

Remark 58. There is another proof of Lemma 57 by directly analyzing the possible buildings in the breaking when one considers SFT compactness as following. Consider

$$\mathcal{M}_2^L := \{I = 2 \text{ curves passing through } [z_1, p) \cup (p, z_2] \text{ with ends on orbit sets of actions} < L\},$$

which is compact because the corresponding moduli space with curves passing through p is compact (see Section 2.5 in[66]). We consider the boundary of $\mathcal{M}_2^L$. One may show that gluing the two $I = 1$ levels gives back a curve passing through the same side of the connected sum S^2. Then, the boundary of $\mathcal{M}_2^L$ constitutes buildings ∂K, $K\partial$, $I = 2$ curves passing through basepoint z_1 or z_2, and $I = 2$ buildings that pass through p. Now the $I = 2$ buildings that pass through p are exactly the buildings that have two $I = 1$ levels, and the bottom level constitutes of trivial cylinders and P_N, by Lemma 51. See Figure 3.2. Now since $\mathcal{M}_2^L$ is a compact 1-dimensional manifold, we have that

$$\partial \mathcal{M}_2^L = (\partial K)_{oo} + (K\partial)_{oo} + (U_{\#,z_1})_{oo} + (U_{\#,z_2})_{oo} + \frac{1}{h}\partial_{ho} = 0,$$

which is equivalent to (3.6.1).

Proof of the main theorem

Now we construct a chain homotopy equivalence $F : (C^L_{cone}, \partial_{cone}) \to (C^L_{\#}, \partial_{\#})$ in order to conclude the proof of Proposition 42.

Lemma 59. The map $F = \begin{pmatrix} id & 0 \\ hK_{oo} & h \end{pmatrix}$ is a chain homotopy equivalence.

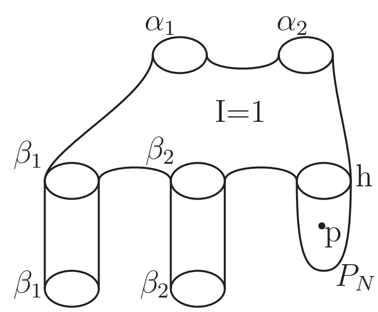


Figure 3.2: An example of an $I = 2$ building that passes through p.

Proof. Since the diagonal constitutes of the identity maps on vector spaces and F is lower triangular, it suffices to show that F is a chain map, i.e. $F\partial_{cone} = \partial_{\#}F$.

This amounts to checking the following equation:

$$\begin{pmatrix} id & 0 \\ hK_{oo} & h \end{pmatrix} \begin{pmatrix} \partial_{oo} & 0 \\ U_1 \otimes id + id \otimes U_2 & \partial_{oo} \end{pmatrix} = \begin{pmatrix} \partial_{oo} & \partial_{oh} \\ \partial_{ho} & \partial_{hh} \end{pmatrix} \begin{pmatrix} id & 0 \\ hK_{oo} & h \end{pmatrix}$$

Or, equivalently:

1. $id \circ \partial_{oo} + 0 = \partial_{oo} \circ id + \partial_{oh} hK_{oo}$. This is true by Corollary 47.

2. $0 + 0 = 0 + \partial_{oh} h$. This is true by Corollary 47.

3. $hK_{oo}\partial_{oo} + h(U_1 \otimes id + id \otimes U_2) = \partial_{ho} \circ id + \partial_{hh} hK_{oo}$. This is true by Corollary 47, Corollary 49 and Lemma 57.

4. $0 + h\partial_{oo} = 0 + \partial_{hh} h$. This is true by Corollary 49.

□

Proof of Proposition 42. The inverse of F in Lemma 59 gives the chain homotopy equivalence f. □

3.7 The direct limit

We have shown the filtered version Proposition 42 of the main theorem. Now to obtain Theorem 1, we use a direct limit argument similar to that in [94] which involves Seiberg-Witten theory. First, we construct an isomorphism on the cobordism that interpolates between connected sums with necks of different radii up to a certain threshold. The following lemma is useful for constructing the cobordism.

Lemma 60. Let α and α' be two contact forms of the same contact structure ξ on a contact manifold Y, then the convex combination $\lambda := (1-s)\alpha + s\alpha'$, where $s \in [0,1]$, is a contact form.

Proof. We have that

$$d\lambda = -ds \wedge \alpha + (1-s)d\alpha + ds \wedge \alpha' + sd\alpha'$$

and

$$\lambda \wedge d\lambda = (1-s)^2\alpha \wedge d\alpha + \alpha \wedge ds \wedge \alpha' + (1-s)s\alpha \wedge d\alpha' + s(1-s)\alpha' \wedge d\alpha + s^2\alpha' \wedge d\alpha'.$$

Now since $\ker\alpha = \ker\alpha'$, we have that $\alpha = \varphi\alpha'$ for some positive function $\varphi : Y \to \mathbb{R}_{>0}$. Therefore,

$$\lambda \wedge d\lambda = (1-s)(1 + \frac{s}{\varphi})\alpha \wedge d\alpha + s(s + (1-s)\varphi)\alpha' \wedge d\alpha' > 0$$

since α and α' are contact forms. □

Proposition 61. Fix L such that no orbit sets in Y_1 and Y_2 have action L. Then for

$$r' < r \leq R(L),$$

there is a cobordism map

$$g^L_{r,r'} : ECH^L_*(Y_1\#_r Y_2, \alpha_r) \to ECH^L_*(Y_1\#_{r'} Y_2, \alpha_{r'}), \tag{3.7.1}$$

which is an isomorphism.

Proof. Consider the admissible deformation

$$\rho := \{(\lambda_t := (1-t)\alpha_{r'} + t\alpha_r, L, J_t, \mu) | t \in [0,1]\}, \tag{3.7.2}$$

where μ is sufficiently large. Note that λ_t is a contact form since a convex combination of contact forms of the same contact structure is still a contact form by Lemma 60.

Now, observe by Lemma 41 that for $r' < r \leq R(L)$, we have that any orbit intersecting the connected sum S^2 has to have action $\geq L$, except for the special hyperbolic orbit whose action stays $<< L$. Since the symplectic cobordism $([0,1] \times (Y_1\#Y_2), d(e^t\lambda_t))$ does not change orbits away from the connected sum region by the proof of Lemma 24, no orbit set of action equal to L appears for any λ_t.

By Lemma 3.7 and Section 3.5 in [64], since λ_t has no orbit sets of action L and we may perturb J to be ECH^L-generic, we have that for each $\Gamma \in H_1(Y)$, a well-defined isomorphism

$$ECH_*^L(Y, \lambda_t, \Gamma) \cong \widehat{HM}_L^{-*}(Y, \lambda_t, \mathfrak{s}_{\xi,\Gamma}), \tag{3.7.3}$$

where $\mathfrak{s}_{\xi,\Gamma}$ is the spin-c structure $\mathfrak{s}(\xi)+\mathrm{PD}(\Gamma)$. For the definition of $\widehat{HM}_L^{-*}(Y, \lambda_t, \mathfrak{s}_{\xi,\Gamma})$, see [64], whose exact definition we do not need in this proof.

By Lemma 3.4 in [64], the admissible deformation ρ gives an isomorphism

$$\Phi : \widehat{HM}_L^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}; \lambda_0, J_0, \mu) \xrightarrow{\simeq} \widehat{HM}_L^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}; \lambda_1, J_1, \mu). \tag{3.7.4}$$

Composing the above isomorphisms (3.7.3) and (3.7.4) gives the desired map $g^L_{r,r'}$. It is shown in Section 3.3 in [64] that $g^L_{r,r'}$ is well-defined. □

Given L, when $r > R(L)$, we no longer have the nice control of $ECC^L(Y_1\#_r Y_2)$, since there might be orbits of large action (despite being $< L$) that cross the connected sum S^2. However, we still have the following cobordism map induced by an exact symplectic cobordism given in [64].

Theorem 62 (Corollary 5.3(a) in [64]). *Let (X, λ) be an exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-), where $\lambda_\pm$ is L-nondegenerate. Let $J_\pm$ be a symplectization-adapted almost complex structure for $\lambda_\pm$. Suppose ρ is sufficiently large. Fix appropriate 2-forms $\mu_\pm$ and perturbations needed to define the chain complex $\widehat{CM}^*(Y_\pm; \lambda_\pm, J_\pm, \rho)$. Then there is a well-defined map*

$$\widehat{HM}^*_L(X, \lambda) : \widehat{HM}^*_L(Y_+; \lambda_+, J_+, \rho) \longrightarrow \widehat{HM}^*_L(Y_-; \lambda_-, J_-, \rho) \tag{3.7.5}$$

depending only on $X, \lambda, L, \rho, J_\pm, \mu_\pm$ and the perturbations, such that

$$\begin{array}{ccc} \widehat{HM}^*_L(Y_+; \lambda_+, J_+, \rho) & \xrightarrow{i^{L,L'}} & \widehat{HM}^*_{L'}(Y_+; \lambda_+, J_+, \rho) \\ \Big\downarrow{\scriptstyle \widehat{HM}^*_L(X,\lambda)} & & \Big\downarrow{\scriptstyle \widehat{HM}^*_{L'}(X,\lambda)} \\ \widehat{HM}^*_L(Y_-; \lambda_-, J_-, \rho) & \xrightarrow{i^{L,L'}} & \widehat{HM}^*_{L'}(Y_-; \lambda_-, J_-, \rho) \end{array} \tag{3.7.6}$$

commutes for $L < L'$ and $\lambda_\pm$ which are L-nondegenerate, where $i^{L,L'}$ are induced by inclusions of chain complexes.

Therefore, we may consider the product exact symplectic cobordism

$$(X := [0,1]_t \times (Y_1\#Y_2), \lambda_t),$$

where λ_t is defined in (3.7.2), from $Y_1\#_{r'}Y_2$ to $Y_1\#_r Y_2$ where $r' < r$. After passing through the isomorphism to Seiberg-Witten as in (3.7.3), Theorem 62 gives the following cobordism map:

$$ECH_*^L(X,\lambda) : ECH_*^L(Y_1\#_r Y_2, \alpha_r) \longrightarrow ECH_*^L(Y_1\#_{r'}Y_2, \alpha_{r'}). \tag{3.7.7}$$

Going back to the case when $r \leq R(L)$, the cobordism map induced by admissible deformation in Proposition 61, together with the cobordism map induced by inclusion, gives the following commutative diagram (as in proof of Lemma 3.7 in [64]) for $L_1 < L_2$ and $R(L_1) > R(L_2)$:

$$\begin{array}{ccc} ECH_*^{L_1}(Y_1\#_{R(L_1)}Y_2, \alpha_{R(L_1)}) & \xrightarrow{i^{L_1,L_2}} & ECH_*^{L_2}(Y_1\#_{R(L_1)}Y_2, \alpha_{R(L_1)}) \\ {\scriptstyle g^{L_1}_{R(L_1),R(L_2)}}\downarrow & & \downarrow{\scriptstyle g^{L_2}_{R(L_1),R(L_2)}} \\ ECH_*^{L_1}(Y_1\#_{R(L_2)}Y_2, \alpha_{R(L_2)}) & \xrightarrow{i^{L_1,L_2}} & ECH_*^{L_2}(Y_1\#_{R(L_2)}Y_2, \alpha_{R(L_2)}), \end{array} \tag{3.7.8}$$

where i^{L_1,L_2} are inclusion induced cobordism maps defined in Theorem 21. Furthermore, by Lemma 5.6 in [64], the map $g^{L_i}_{R(L_1),R(L_2)}$ identifies with the exact cobordism map $ECH_*^L(X,\lambda)$ in (3.7.7) when we consider $L = L_i$, $r = R(L_1)$ and $r' = R(L_2)$.

Now we check that (3.7.8) gives a directed system, similar to that in [94]. We need to check that we have a well-defined composition for the cobordism maps

$$\Phi^{L_1,L_2}(R(L_1),R(L_2)) : ECH_*^{L_1}(Y_1\#_{R(L_1)}Y_2, \alpha_{R(L_1)}) \to ECH_*^{L_2}(Y_1\#_{R(L_2)}Y_2, \alpha_{R(L_2)}) \tag{3.7.9}$$

defined by either path in (3.7.8). For $L_1 < L_2 < L_3$, we define $r := R(L_1)$, $r' := R(L_2)$ and $r'' := R(L_3)$. Then the composition of (3.7.9) is given by the following four-fold commutative diagram.

$$\begin{array}{ccccc} ECH_*^{L_1}(Y_1\#_r Y_2, \alpha_r) & \xrightarrow{i^{L_1,L_2}} & ECH_*^{L_2}(Y_1\#_r Y_2, \alpha_r) & \xrightarrow{i^{L_2,L_3}} & ECH_*^{L_3}(Y_1\#_r Y_2, \alpha_r) \\ {\scriptstyle g^{L_1}_{r,r'}}\downarrow & & \downarrow{\scriptstyle g^{L_2}_{r,r'}} & & \downarrow{\scriptstyle g^{L_3}_{r,r'}} \\ ECH_*^{L_1}(Y_1\#_{r'} Y_2, \alpha_{r'}) & \xrightarrow{i^{L_1,L_2}} & ECH_*^{L_2}(Y_1\#_{r'} Y_2, \alpha_{r'}) & \xrightarrow{i^{L_2,L_3}} & ECH_*^{L_3}(Y_1\#_{r'} Y_2, \alpha_{r'}) \\ {\scriptstyle g^{L_1}_{r',r''}}\downarrow & & \downarrow{\scriptstyle g^{L_2}_{r',r''}} & & \downarrow{\scriptstyle g^{L_3}_{r',r''}} \\ ECH_*^{L_1}(Y_1\#_{r''} Y_2, \alpha_{r''}) & \xrightarrow{i^{L_1,L_2}} & ECH_*^{L_2}(Y_1\#_{r''} Y_2, \alpha_{r''}) & \xrightarrow{i^{L_2,L_3}} & ECH_*^{L_3}(Y_1\#_{r''} Y_2, \alpha_{r''}) \end{array}$$

Now, we may pass the filtered ECH complexes to the direct limit with respect to the above maps. This requires some algebraic manipulations as in proof of Theorem 7.1 in [94], with which we will conclude the proof of our main theorem.

Proof of Theorem 1. In this proof we suppress the notation of the connected sum radius in $Y_1\#Y_2$ when it is encoded in the contact form, in order to simplify notations. Let $L(r)$ denote the value of L such that $R(L) = r$, where $R(L)$ is the strictly decreasing function defined in Section 3.4. Therefore,

$$ECH^L(Y_1\#Y_2, \alpha_{R(L)}) = ECH^{L(r)}(Y_1\#Y_2, \alpha_r).$$

Now,

$$\begin{aligned} ECH(Y_1\#Y_2, \xi_1\#\xi_2) &= \lim_{r\to 0} ECH(Y_1\#Y_2, \alpha_r) \\ &= \lim_{r\to 0}\lim_{L\to\infty} ECH^L(Y_1\#Y_2, \alpha_r) \\ &= \lim_{r\to 0} ECH^{L(r)}(Y_1\#Y_2, \alpha_r) \\ &= \lim_{L\to\infty} ECH^L(Y_1\#Y_2, \alpha_{R(L)}) \\ &= H_*(Cone(U_1 \otimes id + id \otimes U_2)). \end{aligned} \tag{3.7.10}$$

The first equation is because ECH is independent of the choice of contact forms [109]. The second equation is given by (3.2.6) and Theorem 21(1). The third equation comes from consideration of the following map:

$$\Psi : \lim_{r\to 0} ECH^{L(r)}(Y_1\#Y_2, \alpha_r) \longrightarrow \lim_{r\to 0}\lim_{L\to\infty} ECH^L(Y_1\#Y_2, \alpha_r) \tag{3.7.11}$$

by sending the equivalence class of an element $d_r \in ECH^{L(r)}(Y_1\#Y_2, \alpha_r)$ under $\lim_{r\to 0}$ to the equivalence class of d_r under $\lim_{r\to 0}\lim_{L\to\infty}$. We need to show that Ψ is well-defined and a bijection. To establish well-definedness, consider d_r and $d_{r'}$ in $\lim_{r\to 0} ECH^{L(r)}(Y_1\#Y_2, \alpha_r)$ where $r > r'$ and $d_r \sim d_{r'}$. This means that there is a common element $d_{r''} \in ECH^{L(r'')}(Y_1\#Y_2, \alpha_{r''})$ such that both d_r and $d_{r'}$ are mapped to under the direct limit. Then $\Psi(d_r) \sim \Psi(d_{r'})$ by composing the maps

$$ECH^{L(r)}(Y_1\#Y_2, \alpha_r) \longrightarrow ECH^{L(r'')}(Y_1\#Y_2, \alpha_r) \longrightarrow ECH^{L(r'')}(Y_1\#Y_2, \alpha_{r''}) \tag{3.7.12}$$

and

$$ECH^{L(r')}(Y_1\#Y_2, \alpha_{r'}) \longrightarrow ECH^{L(r'')}(Y_1\#Y_2, \alpha_{r'}) \longrightarrow ECH^{L(r'')}(Y_1\#Y_2, \alpha_{r''}) \tag{3.7.13}$$

given by the right followed by down composition of the commutative diagram (3.7.8). Now we show that Ψ is injective. Suppose $\Psi(d) = 0$. That means there exists L_0 such that a representative $\widetilde{\Psi(d)} \in ECH^{L_0}(Y_1\#Y_2, \alpha_{r_0})$, where $[\widetilde{\Psi(d)}] = \Psi(d)$, is zero for some r_0. See Figure 3.3. If $r_0 \leq R(L_0)$, then we are done, since for $r \leq R(L)$,

$$ECH^L(Y_1\#Y_2, \alpha_r) = ECH^L(Y_1\#Y_2, \alpha_{R(L))})$$

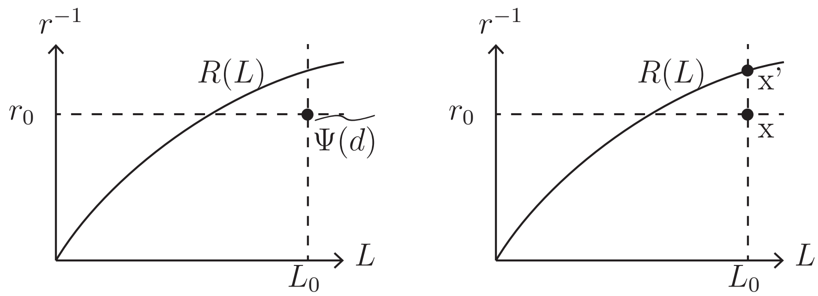


Figure 3.3: Schematic illustration of the proof involving direct limits in Theorem 1. In the region above $R(L)$, i.e. when $r \leq R(L)$, we have isomorphisms of $ECH^L_*(Y_1\#_r Y_2, \alpha_r)$ given a fixed value of L by Proposition 61.

by Proposition 61, so (3.7.11) is a bijection. Suppose $r_0 > R(L_0)$. Then $\widetilde{\Psi(d)}$ is mapped to zero in $ECH^{L_0}(Y_1\#Y_2, \alpha_{R(L_0)})$ under the map in (3.7.7). Therefore, $d \sim 0$. To show that Ψ is surjective, let

$$y \in \lim_{L\to\infty} ECH^L(Y_1\#Y_2, \alpha_{r_0})$$

for some r_0. Similar to the injective case, if $r_0 \leq R(L_0)$, then we are done. Suppose $r_0 > R(L_0)$. Now let $x \in ECH^{L_0}(Y_1\#Y_2, \alpha_{r_0})$ be such that $[x] = y$. Let

$$x' \in ECH^{L_0}(Y_1\#Y_2, \alpha_{R(L)})$$

be the image of x when taking the limit as $r \to 0$ defined by the exact cobordism map as in (3.7.7). Then $[y] = \Psi([x'])$.

The fourth equation is by the definition of $L(r)$. The fifth equation is by Proposition 42 and the fact that taking direct limit commutes with taking homology.

Finally, note that the mapping cone in (3.7.10) is of the map

$$U_1 \otimes id + id \otimes U_2 : ECC(Y_1, \lambda_1) \otimes_{\mathbb{F}} ECC(Y_2, \lambda_2) \longrightarrow ECC(Y_1, \lambda_1) \otimes_{\mathbb{F}} ECC(Y_2, \lambda_2)[-1].$$

Over $\mathbb{F}$, the homology of $Cone(U_1 \otimes id + id \otimes U_2)$ is isomorphic to the homology of the mapping cone of the induced map $(U_1 \otimes id + id \otimes U_2)_*$ on homology, by observing that Künneth formula is natural over $\mathbb{F}$. This concludes the proof of the main theorem. □

Chapter 4

Legendrian embedded contact homology

Julian Chaidez, Oliver Edtmair, Luya Wang, Yuan Yao, Ziwen Zhao

4.1 Introduction

The embedded contact homology of a closed contact 3-manifold (Y, ξ) is a Floer theoretic invariant originally introduced by Hutchings [52]. These homology groups, denoted by

$$ECH(Y, \xi)$$

are, roughly speaking, computed as the homology of a chain complex freely generated by certain finite sets of closed simple Reeb orbits (with multiplicity). The differential counts certain embedded, possibly disconnected J-holomorphic curves in the symplectization of the contact manifold that can have arbitrary genus.

Since its inception, embedded contact homology and its associated invariants have been applied to many of the fundamental problems in 3-dimensional Reeb dynamics, to dramatic effect. These applications include a simple formal proof of the 3-dimensional Weinstein conjecture by [114], and the Arnold chord conjecture by Hutchings-Taubes [63, 65], and the existence of broken book decompositions [18]. The spectral invariants derived from the ECH groups (and related PFH groups), called the ECH capacities, have been used to prove the smooth closing lemma for Reeb flows [67] and area preserving surface maps [31, 33], and many results on symplectic embeddings [90, 54, 30].

Embedded contact homology can morally be regarded as a flavor of symplectic field theory (SFT), as formulated by Eliashberg-Givental-Hofer [36]. SFT is, broadly speaking, a framework for constructing invariants of contact manifolds in any dimension, acquired from chain complexes generated by Reeb orbits. Variants of SFT include cylindrical contact homology (cf. [**hn2014**, 93]), linearized contact homology (cf. [12]) and the contact homology algebra (cf. [101]).

Many flavors of SFT can be extended to invariants of a pair (Y, Λ) of a closed contact manifold Y and a closed Legendrian sub-manifold $\Lambda \subset Y$. The goal of this paper is to initiate the study of a corresponding Legendrian version of embedded contact homology.

Standard ECH

We begin this introduction with a review of standard ECH, largely drawing on [52]. We discuss many of the key ideas that allow one to define ECH and to show that the differential in ECH, which counts certain J-holomorphic curves, defines a chain complex. Besides serving as a review, this will also provide a road map of the results required to construct Legendrian ECH.

Holomorphic Currents

Let (Y, λ) be a closed contact 3-manifold with a non-degenerate contact form λ. We start by considering the symplectization of Y, i.e. the cylindrical manifold

$$\mathbb{R}_s \times Y$$

There is a natural class of translation invariant complex structures J on $\mathbb{R} \times Y$, which sends ∂_s to the Reeb vector-field R of λ and positively preserves the contact structure $\ker \lambda$. We call such almost complex structure *λ-adapted*.

Recall that a *J-holomorphic curve C* from a punctured Riemann surface (without boundary) is an equivalence class of map

$$u : (\Sigma, j) \to (\mathbb{R} \times Y, J) \qquad \text{satisfying} \qquad du \circ j = J \circ du$$

modulo holomorphic reparametrization of the domain (Σ, j). We say that C is proper if the map u is proper and finite energy if the integral of $u^* d\lambda$ is finite. If u is proper and finite energy, then u converges (in an appropriate sense) to the cylinder over a collection of Reeb orbits

$$\Gamma_+ = (\gamma_1^+, \dots, \gamma_k^+) \qquad \text{and} \qquad \Gamma_- = (\gamma_1^-, \dots, \gamma_k^-)$$

near $+\infty \times Y$ and $-\infty \times Y$, respectively. The *Fredholm index* $\text{ind}(u)$ of u is the Fredholm index of a certain linearized Cauchy-Riemann operator associated to u. It is given by the formula

$$\text{ind}(u) = -\chi(\Sigma)+2c_\tau(u)+\text{CZ}^{ind}_\tau(\Gamma_+)-\text{CZ}^{ind}_\tau(\Gamma_-) \qquad \text{where} \qquad \text{CZ}^{ind}_\tau(\Gamma_\pm) := \sum_i CZ_\tau(\gamma_i^\pm)$$

Here $c_\tau(u)$ is the relative first Chern number of $u^*\xi$ with respect to a trivialization τ of ξ along Γ_+ and Γ_-, and $CZ_\tau(\gamma)$ is the Conley-Zehnder index of the linearized Reeb flow around γ in the trivialization τ. Standard transversality result states that there is a generic class of *regular* J such that, if u is somewhere injective, the moduli space of proper, finite energy, somewhere injective J-holomorphic maps v near u asymptotic to $\Gamma_\pm$ at $\pm\infty$ is a manifold of dimension $\text{ind}(u)$.

A *J-holomorphic current* $\mathcal{C} = \{(C_i, m_i)\}$ in Y is a finite collection of connected, proper, finite energy, somewhere injective J-holomorphic curves C_i and positive integer multiplicities m_i.

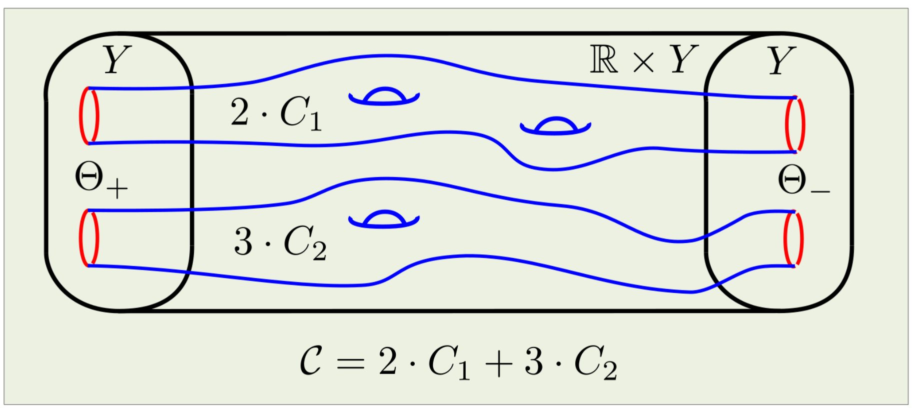


Figure 4.1: An depiction of a J-holomorphic current.

An *orbit set* $\Theta = \{(\gamma_i, n_i)\}$ is, similarly, a collection of simple closed Reeb orbits γ_i at $\pm\infty$. In analogy to curves, every J-holomorphic current is asymptotic orbit sets at $\pm\infty$. We note here we must count with multiplicities, e.g. if the J-holomorphic current $(C, 1)$ is positively asymptotic to the simple Reeb orbit γ, then $(C, 2)$ is positively asymptotic to $(\gamma, 2)$. We denote the moduli space of J-holomorphic currents asymptotic to α at $+\infty$ and β at $-\infty$ by

$$\mathcal{M}(\alpha, \beta)$$

Note that this moduli space admits an $\mathbb{R}$-action given by $\mathbb{R}$-translation in $\mathbb{R} \times Y$. We denote the quotient by $\mathcal{M}(\alpha, \beta)/\mathbb{R}$.

ECH Index

The key ingredient of ECH that differentiates it from other versions of symplectic field theory is the *ECH index*. It may be viewed as a map

$$I : \mathcal{M}(\alpha, \beta) \to \mathbb{Z}$$

On a J-holomorphic current $\mathcal{C}$ asymptotic to α at $+\infty$ and β at $-\infty$, the ECH index is given by

$$I(\mathcal{C}) = c_\tau(\mathcal{C}) + Q_\tau(\mathcal{C}) + \mathrm{CZ}_\tau^{ECH}(\alpha) - \mathrm{CZ}_\tau^{ECH}(\beta). \tag{4.1.1}$$

Here Q_τ is the relative self-intersection number [50, §2.4] that counts intersections between $\mathcal{C}$ and a push-off of $\mathcal{C}$ determined by τ, and CZ_τ^{ECH} is a Conley-Zehnder index term given by

$$\mathrm{CZ}_\tau^{ECH}(\Theta) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mathrm{CZ}(\gamma_i^j) \qquad \text{for} \qquad \Theta = \{(\gamma_i, m_i)\} \tag{4.1.2}$$

The fundamental property that the ECH index satisfies is the following index inequality.

Theorem 1 (Index Inequality)**.** [50, 55] Let C is a somewhere injective J-holomorphic curve in $\mathbb{R} \times Y$, for a compatible J and let $\delta(C)$ denote the count of singularities (with multiplicity) of C. Then

$$\mathrm{ind}(C) \leq I(C) - 2\delta(C)$$

This inequality places stringent constraints on curves and currents of low ECH index. Let us discuss the main ingredients of the proof of this inequality, as their Legendrian analogues will be the main topic of this paper.

The first ingredient of Theorem 1 is the writhe bound. Recall that the *writhe* $w(\zeta)$ of a braid ζ in $S^1 \times D^2$ is an isotopy invariant that can be computed as a signed count of the self-intersections of the image $\pi(\zeta)$ under the projection $\pi : S^1 \times D^2 \to S^1 \times [-1, 1]$. If C is a somewhere injective J-holomorphic curve asymptotic to (covers of) a simple orbit γ at some subset of its punctures, then $\mathcal{C}$ determines a braid in a tubular neighborhood of γ (due to J-holomorphic curve asymptotics established by Siefring [108]).

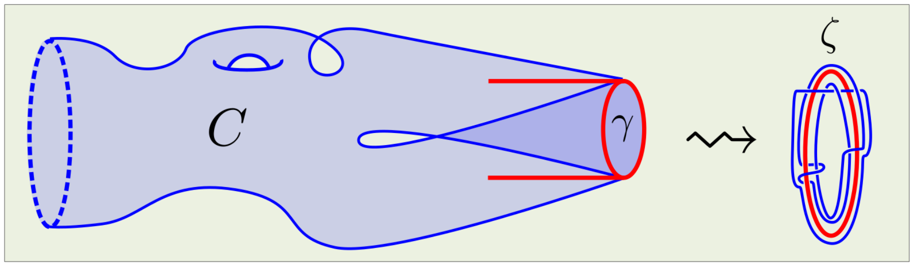


Figure 4.2: The braid near a simple orbit γ determined by a curve (or current) C.

Given a choice of trivialization τ of ξ, any braid near γ is identified with a braid in $S^1 \times D^2$. The writhe of this braid is denoted by $w_\tau(\mathcal{C};\gamma)$, and we define the writhe of $\mathcal{C}$ as

$$w_\tau(\mathcal{C}) = \sum_{(\gamma,m)\in\alpha} w_\tau(\mathcal{C};\gamma) - \sum_{(\gamma,m)\in\beta} w_\tau(\mathcal{C};\gamma)$$

The writhe bound estimates the writhe of $\mathcal{C}$ in terms of the difference between the Conley-Zehnder terms in the ECH index and Fredhom index.

Theorem 2 (Writhe Bound)**.** [55] Let C be a somewhere injective J-holomorphic curve in $\mathbb{R} \times Y$. Then

$$w_\tau(C) \leq CZ_\tau^{ECH}(C) - CZ_\tau^{ind}(C).$$

The second ingredient to Theorem 1 is the adjunction inequality. This is a version of the classical adjunction inequality in complex geometry, tailored to the setting of ECH.

Theorem 3 (Adjunction)**.** Let C be a somewhere injective J-holomorphic curve in $\mathbb{R} \times Y$. Then

$$c_\tau(C) = \chi(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C)$$

The index inequality is a short calculation using these two bounds.

ECH Complex

We are now ready to explain the construction of the ECH chain complex, using the properties of the ECH index discussed above.

Remark 63 (Orientations)**.** In this paper, we work without orientation for simplicity. In particular, we use $\mathbb{Z}/2$-coefficient to define all ECH groups.

An *ECH generator* is an orbit set Θ such that any hyperbolic orbit γ_i in Θ has multiplicity 1. The ECH chain complex is simply the free vector-space over $\mathbb{Z}/2$ generated by ECH generators.

$$ECC(Y,\lambda) := \mathbb{Z}/2\big\langle \text{ECH generators } \Theta \text{ of } (Y,\lambda)\big\rangle$$

The differential on the ECH complex is defined by counting currents. To formalize this, we require the following classification of low ECH index currents deducible from the index bound.

Theorem 4 (Low-Index Currents)**.** Let J be a regular, compatible almost complex structure on $\mathbb{R} \times Y$ and let $\mathcal{C}$ be a J-holomorphic current of ECH index $I(\mathcal{C}) \leq 2$. Then

- $I(\mathcal{C}) \geq 0$ with equality only if $\mathcal{C}$ is a union of trivial cylinders.
- If $I(\mathcal{C}) = 1$ then $\mathcal{C} = C \sqcup \mathcal{T}$ where C is Fredholm index 1 and embedded, and $\mathcal{T}$ is a union of trivial cylinders with multiplicity.
- If $I(\mathcal{C}) = 2$ and is asymptotic to ECH generators at $\pm\infty$, then $\mathcal{C} = C \sqcup \mathcal{T}$ where C is Fredholm index 2 and embedded, and $\mathcal{T}$ is a union of trivial cylinders with multiplicity.

This classification can, in turn, be used to deduce compactness properties of low ECH index moduli spaces.

Theorem 5. [52, §5.3-5.4] Let J be a regular, compatible almost complex structure on $\mathbb{R} \times Y$ and let $\mathcal{M}_k(\Theta, \Xi)$ be the space of ECH index k J-holomorphic currents from Θ to Ξ. Then

- the space $\mathcal{M}_1(\Theta, \Xi)/\mathbb{R}$ is 0-dimensional and compact
- the space $\mathcal{M}_2(\Theta, \Xi)/\mathbb{R}$ is a 1-manifold with a compact truncation[1] $\mathcal{M}_2'(\Theta, \Xi)/\mathbb{R}$ with a map

$$\Pi : \partial\mathcal{M}_2'(\Theta,\Xi)/\mathbb{R} \to \bigsqcup_{\Theta'} \mathcal{M}_1(\Theta,\Theta')/\mathbb{R} \times \mathcal{M}_1(\Theta',\Xi)/\mathbb{R}$$

[1] It is a technical point that we cannot guarantee a priori that the moduli space of index 2 currents is compact. The truncation may be viewed as a replacement for the compactification.

- The inverse image $\Pi^{-1}(\mathcal{C}, \mathcal{C}')$ of a pair of currents in $\mathcal{M}_1(\Theta, \Theta')/\mathbb{R} \times \mathcal{M}_1(\Theta', \Xi)/\mathbb{R}$ has an odd number of points if and only if the orbit set Θ' is an ECH generator.

Most of Theorem 5 follows from Theorem 4, a type of Gromov compactness due to Taubes and a bound on the topological complexity of low ECH index curves [52]. However, the last point require a delicate obstruction bundle gluing argument that is well beyond the scope of this introduction. However, we will remark that this analysis requires certain *partition conditions* obeyed by low ECH index currents, which restrict the braids that can appear at their ends. For a more detailed explanation of partition conditions, see [52].

By applying Theorem 5, one can simply define the ECH differential as the count of ECH index 1 curves, modulo reparametrization, for a regular choice of J.

$$\partial : ECC(Y, \lambda) \to ECC(Y, \lambda) \quad \text{be given by} \quad \partial\Theta = \sum_{\Xi} \#(\mathcal{M}_1(\Theta, \Xi)/\mathbb{R}) \cdot \Xi \mod 2$$

It is a simple consequence of Theorem 5 that ∂ is well-defined and that $\partial \circ \partial = 0$.

Legendrian ECH And Main Results

We now move to the main topic of this paper, providing an overview of the construction of Legendrian ECH. We shall see that all of the constructions in ordinary ECH have generalizations to the Legendrian setting.

Holomorphic Currents With Boundary

Let Y be a contact 3-manifold with convex sutured boundary ∂Y and a non-degenerate, adapted contact form λ. We refer the reader to [28] for a detailed treatment of sutured contact manifolds. The boundary of Y divides as

$$\partial_- Y \qquad \partial_\sigma Y \quad \text{and} \quad \partial_+ Y$$

where the Reeb vector-field is inward normal, tangent and outward normal respectively. Also fix a closed (possibly disconnected) Legendrian $\Lambda \subset \partial Y$ decomposing as

$$\Lambda_+ \subset \partial_+ Y \qquad \text{and} \qquad \Lambda_- \subset \partial_- Y$$

We assume that that these are exact Lagrangians in the Liouville domains $(\partial_\pm Y, \lambda|_{\partial_\pm Y})$, so that

$$\lambda|_{\partial_\pm Y} \text{ vanish along } \Lambda_\pm$$

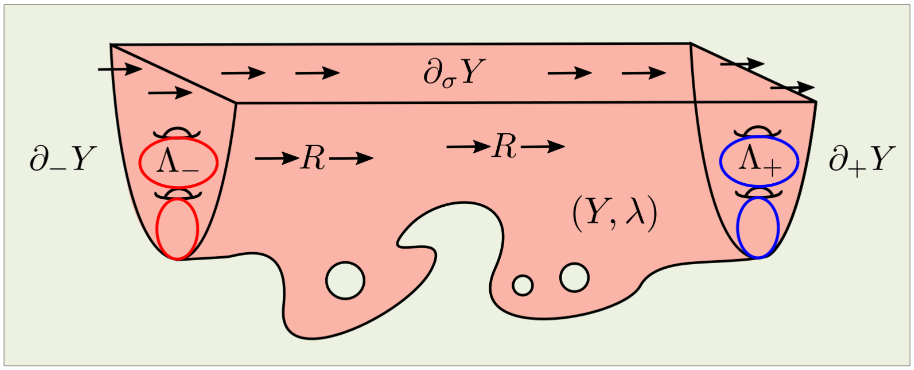


Figure 4.3: A contact manifold Y with convex boundary and Legendrians $\Lambda_\pm \subset \partial_\pm Y$.

We have included Figure 4.3 below to better illustrate this setup for the reader. There is a natural class of *tailored* almost complex structures on $\mathbb{R} \times Y$, which we write as $\mathcal{J}_T(\mathbb{R} \times Y)$, that are compatible and also satisfy a set of assumptions near ∂Y that gurantee that J-holomorphic curves do not cross the boundary. As defined in [28] tailoered almost complex structures exist in great abundance (see Section 4.3 for a description).

We consider finite energy, proper J-holomorphic maps from a punctured Riemann surface (Σ, j) with boundary to the symplectization of Y, with boundary on the symplectization of Λ.

$$u : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \qquad \text{with} \qquad du \circ j = J \circ du \tag{4.1.3}$$

A J-holomorphic current $\mathcal{C}$ in (Y, Λ) is defined exactly as in the closed case, using J-holomorphic maps with boundary (4.1.3). These curves are now asymptotic at $\pm\infty$ to *orbit-chord sets*

$$\Xi = \{(\gamma_i, m_i)\}$$

Here γ_i is either a simple closed Reeb orbit or a Reeb chord of Λ, and m_i is a multiplicity. We continue to denote set of J-holomorphic currents asymptotic to orbit-chord sets Θ at $+\infty$ and Ξ at $-\infty$ by

$$\mathcal{M}(\Theta, \Xi)$$

We will discuss the foundations of J-holomorphic currents with boundary in our setting more fully in Section 4.3.

Legendrian ECH Index

We are now ready to introduce the Legendrian ECH index, generalizing the ECH index from the closed case.

Definition 6 (Definition 189)**.** The *(Legendrian) ECH index* of a J-holomorphic current $\mathcal{C}$ in the pair (Y, Λ) from the orbit-chord set Θ to the orbit-chord set Ξ is

$$I(\mathcal{C}) = \frac{1}{2} \cdot \mu_\tau(\mathcal{C}) + Q_\tau(\mathcal{C}) + CZ_\tau^{ECH}(\Theta) - CZ_\tau^{ECH}(\Xi)$$

where the terms in the index I are as follows.

- τ is a trivialization of the bundle pair $(\xi, T\Lambda)$ over Θ and Ξ (see Definition 92)
- μ_τ is the (relative) Maslov number (see Definition 98)
- Q_τ is the *relative self intersection number* with respect to τ (see Definition 114)
- CZ_τ^{ECH} is a Conley-Zehnder term associated to the orbit-chord set (see Definition 175).

All of the terms in Definition 6 directly correspond to (and generalize) the terms in (4.1.1). The Legendrian ECH index satisfies a direct generalization of the ECH index inequality.

Theorem 7 (Legendrian Index Inequality)**.** Let C be a somewhere injective, J-holomorphic curve with boundary in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ for tailored J. Then

$$\mathrm{ind}(C) \leq I(C) - 2\delta(C) - \epsilon(C)$$

Here $\delta(C)$ and $\epsilon(C)$ are the counts of the singularities of C in its interior and its boundary, respectively.

This result is a consequence of Legendrian generalizations of adjunction and the writhe bound.

Theorem 8 (Legendrian Adjunction, §4.2)**.** Let C be a proper, finite energy, somewhere injective curve in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$. Then

$$\frac{1}{2} \cdot \mu_\tau(C) = \bar{\chi}(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C) - \epsilon(C)$$

Note that $\bar{\chi}$ denotes a corrected Euler characteristic that only counts boundary punctures with a factor of $\frac{1}{2}$.

To study the writhe, we further adapt Siefring's [108] asymptotic formula for J-holomorphic curves asymptotic to chords.

Theorem 9. Let $u_i : [0, \infty) \times [0, 1] \to \mathbb{R} \times Y$ be a collection of J-holomorphic strips asymptotic to a Reeb chord $\gamma(t)$. Here γ connects between Legendrians L_1 and L_2, and u_i maps the boundary of the strip to $\mathbb{R} \times L_i$. Then there exists a neighborhood U of γ, a smooth embedding $\Phi : \mathbb{R} \times U \to \mathbb{R} \times Y$, proper reparametrizations $\phi_i : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ exponentially asymptotic to the identity, and positive integers N_i so that for large values of s,

$$\Phi \circ u_i \circ \psi_i(s, t) = (s, t, \sum_{k=1}^{N_i} e^{\lambda_{i,j} s} e_{i,j}(t))$$

where $\lambda_{i,j}$ are negative eigenvalues of the asymptotic operator associated to γ, and the $e_{i,f}(t)$ are the corresponding eigenfunctions.

This theorem is discussed in detail in the Appendix. We will delay the precise statement of the writhe bound to Section 4.6.

Legendrian ECH Complex

Finally, we present the construction of the Legendrian ECH chain complex of (Y, Λ), mirroring the construction in the closed setting. These claims will be revisited and proven in Section 4.7.

Definition 64. An *ECH generator* of (Y, Λ) is an orbit-chord set $\Theta = \{(\gamma_i, m_i)\} \cup \{(c_i, n_i)\}$ where

- Every hyperbolic orbit γ_i has multiplicity 1.
- Every chord c_i is multiplicity 1.
- There is at most one Reeb chord incident to L in Θ for each connected component L of Λ

As with the closed case, we define the differential by counting ECH index 1 curves. We again need a classification of low ECH index curves and an accompanying compactness statement.

Theorem 10 (Low-Index Currents With Boundary)**.** Let J be a regular, tailored almost complex structure on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ and let $\mathcal{C}$ be a J-holomorphic current of ECH index $I(\mathcal{C}) \leq 2$. Then

- $I(\mathcal{C}) \geq 0$ with equality only if $\mathcal{C}$ is a union of trivial cylinders and strips with multiplicity.
- If $I(\mathcal{C}) = 1$ then $\mathcal{C} = C \sqcup \mathcal{T}$ where C is Fredholm index 1 and embedded, and $\mathcal{T}$ is a union of trivial cylinders and strips with multiplicity.
- If $I(\mathcal{C}) = 2$ and is asymptotic to ECH generators at $\pm\infty$, then $\mathcal{C} = C \sqcup \mathcal{T}$ where C is Fredholm index 2 and embedded, and $\mathcal{T}$ is a union of trivial cylinders and strips with multiplicity.

Theorem 11. Let J be a regular, tailored almost complex structure on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ and let $\mathcal{M}_k(\Theta, \Xi)$ be the space of ECH index k J-holomorphic currents in (Y, Λ) from Θ to Ξ. Then

- the space $\mathcal{M}_1(\Theta, \Xi)/\mathbb{R}$ is 0-dimensional and compact
- the space $\mathcal{M}_2(\Theta, \Xi)/\mathbb{R}$ is a 1-manifold with a compact truncation[2] $\mathcal{M}_2'(\Theta, \Xi)/\mathbb{R}$ with a map

$$\Pi : \partial \mathcal{M}_2'(\Theta, \Xi)/\mathbb{R} \to \bigsqcup_{\Theta'} \mathcal{M}_1(\Theta, \Theta')/\mathbb{R} \times \mathcal{M}_1(\Theta', \Xi)/\mathbb{R}$$

- The inverse image $\Pi^{-1}(\mathcal{C}, \mathcal{C}')$ of a pair of currents in $\mathcal{M}_1(\Theta, \Theta')/\mathbb{R} \times \mathcal{M}_1(\Theta', \Xi)/\mathbb{R}$ has an odd number of points if and only if the orbit-chord set Θ' is an ECH generator.

We will give detailed proofs of these claims in Section 4.7. They are analogous to the closed case.

Remark 65. Note that, due to the lack of multiply covered Reeb chords (in contrast to orbits), there is no need to carry out a new obstruction bundle gluing strategy. Instead, the work in [60] and [62] will yield the last claim in Theorem 11 after some minor modifications. We will discuss this in 4.7.

[2]It is a technical point that we cannot guarantee a priori that the moduli space of index 2 currents is compact. The truncation may be viewed as a replacement for the compactification.

Finally, we can state the definition of Legendrian embedded contact homology.

Definition 66 (Legendrian ECH)**.** The *ECH chain complex* $ECC(Y,\Lambda,\lambda)$ of (Y,Λ,λ) is the free $\mathbb{Z}/2$-module generated by ECH generators.

$$ECC(Y,\Lambda,\lambda) = \mathbb{Z}/2\big\langle \text{ECH generators } \Theta\big\rangle$$

The differential ∂_J with respect to a regular, tailored almost complex structure J on $(\mathbb{R}\times Y, \mathbb{R}\times\Lambda)$ is given by the count of ECH index 1 J-holomorphic currents.

$$\partial_J\Theta = \sum_{\Xi} \#_2\mathcal{M}_1(\Theta,\Xi)/\mathbb{R}\cdot\Xi$$

The *Legendrian embedded contact homology* $ECH(Y,\Lambda;\lambda,J)$ is the homology $ECC(Y,\Lambda,\lambda)$.

We prove that the differential defined above gives rise to a chain complex (i.e. $\partial^2 = 0$) in Section 4.7.

Remark 67 (Invariance)**.** If Y is closed, then $ECH(Y,\lambda,J)$ is independent of λ and J up to canonical isomorphism. Thus we may write

$$ECH(Y) := ECH(Y,\lambda,J) \qquad \text{for any choice of } \lambda, J$$

This is due to a chain level correspondence with the hat flavor $\widehat{HM}(Y)$ of Mrowka-Kronheimer's monopole Floer homology. An analogous proof of invariance in the Legendrian setting is beyond the scope of this paper and is an interesting topic for future work.

Remark 68 (Component Grading)**.** Let Λ and K be Legendrians in ∂Y satisfying the hypotheses of our construction, and suppose that $\Lambda\subset K$. Then this induces an injective map

$$\iota_\Lambda^K : ECH(Y,\Lambda,\lambda,J) \to ECH(Y,K,\lambda,J)$$

If Λ is the empty set, then $ECH(Y,\emptyset,\lambda,J)$ is simply the sutured ECH $ECH(Y)$ of Colin-Ghiggini-Honda-Hutchings [28]. In particular, there is always an inclusion

$$ECH(Y) \to ECH(Y,\Lambda,\lambda,J)$$

Remark 69. Given a homology class $A\in H_1(Y,\Lambda;\mathbb{Z})$, there is a sub-group

$$ECH_A(Y,\Lambda,\lambda,J) \subset ECH(Y,\Lambda,\lambda,J)$$

generated by orbit sets Θ in the homology class A. This induces a direct sum decomposition

$$ECH(Y,\Lambda,\lambda,J) = \bigoplus_{A\in H_1(Y,\Lambda)} ECH_A(Y,\Lambda,\lambda,J)$$

Remark 70 (Reeb Chord Filtration)**.** Let $\mathfrak{z}$ be a Reeb chord connecting $\partial_- Y$ to $\partial_+ Y$ that is disjoint from Λ. Then there is an associated holomorphic sub-manifold $Z \subset \mathbb{R}\times Y$ given by

$$Z = \mathbb{R}\times\mathfrak{z}$$

Given this choice, we can define an extended ECH complex

$$\overline{ECC}(Y,\Lambda,\lambda,J) = \mathbb{F}_2[t]\otimes ECC(Y,\Lambda,\lambda,J)$$

and a differential on $\overline{ECH}(Y,\Lambda,\lambda,J)$ (as a module over $\mathbb{F}_2[t]$) determined by J and $\mathfrak{z}$.

$$\partial_{J,\mathfrak{z}}(\Theta) = \sum_{\Xi}\Big(\sum_{\mathcal{C}\in\mathcal{M}_1(\Theta,\Xi)} t^{\mathcal{C}\cdot Z}\cdot\Xi\Big)$$

Here $\mathcal{C}\cdot Z$ denotes the count (with multiplicity) of interior intersections between $\mathcal{C}$ and Z. By intersection positivity, this must always be positive. This defines an extended homology

$$\overline{ECH}(Y,\Lambda,\lambda,J) := H(\overline{ECC}(Y,\Lambda,\lambda,J))$$

We can extract further homology groups by studying the associated graded to the t-filtrartion. As we will discuss in §4.1, our construction is related to a construction of Heegaard-Floer homology.

Remark 71 (Previous Work)**.** The Legendrian ECH index has appeared in the works of Colin-Ghiggini-Honda [23, 25] in a more limited context, in the process of establishing an isomorphism between ECH and Heegaard Floer homology.

This work is a natural elaboration on [23]. In particular, we fully develop a general theory of holomorphic currents with boundary, adjunction, writhe bound and the Legendrian ECH index that goes beyond the specialized context of [23, 25]. In our version of the index inequality, our holomorphic currents, for instance, have no restrictions on their asymptotic braids and are permitted to have boundary singularities.[3]

[3]However we must impose more restrictions if we want the differential to square to zero, see Section 4.7 for details.

Legendrian ECH In The Closed Case

The Legendrian ECH setup is, at first glance, highly constrained. We now explain how to use our framework to associate ECH groups to *any* pair

$$(Y, \Lambda)$$

of a closed contact 3-manifold Y with contact form α and a closed Legendrian $\Lambda \subset Y$. Crucially, these ECH groups essentially count Reeb chords of Λ with respect to the initial contact form.

To start, choose an arbitrary metric g on Λ and choose a small $\epsilon > 0$. Let $D^*\Lambda$ denote the unit codisk bundle. The Weinstein neighborhood theorem for Legendrians states that, for small ϵ and after scaling the metric to shrink the codisk bundle, there is an embedding

$$\iota : B := [-\epsilon, \epsilon]_t \times D^*\Lambda \to Y \qquad \text{such that} \qquad \iota(0, \Lambda) = \Lambda \text{ and } \iota^*\alpha = dt + \lambda_{\text{std}}$$

The complement $M = Y \setminus \iota(\text{int}(B))$ of the interior of B in Y is a *concave* sutured contact manifold (see [28, Def. 4.2]) with a decomposition of the boundary into three pieces

$$\partial_- M = \epsilon \times D^*\Lambda \qquad \partial_+ M = -\epsilon \times D^*\Lambda \quad \text{and} \quad \partial_\circ M = [-\epsilon, \epsilon] \times S^*\Lambda$$

Note that the Reeb vector-field of α points out of $\partial_+ M$ and into $\partial_- M$. This is the reason for the sign reversal in the notation. There are two copies of Λ on ∂M given by

$$\Lambda_- := \epsilon \times \Lambda \subset \partial_- M \qquad \text{and} \qquad \Lambda_+ := \epsilon \times \Lambda \subset \partial_+ M$$

The next step is to apply the *concave-to-convex* operation described in [28, §4.2]. In particular, there is a plug U that one can attach to a neighborhood of $\partial_\circ M$ to acquire a convex sutured contact manifold

$$\check{Y} := M \cup U \qquad \text{with contact form } \check{\alpha}$$

The plug only modifies ∂M near $\partial_\circ M$, so that $\Lambda_+ \subset \partial_+ \check{Y}$ and $\Lambda_- \subset \partial_- \check{Y}$. Moreover, every Reeb chord c from Λ_- to Λ_+ arises as a sub-chord of a self Reeb chords of Λ, and the lengths differ by an error of 2ϵ. We can now make the following definition.

Definition 72. The *Legendrian embedded contact homology* $ECH^L(Y, \Lambda)$ of (Y, α) and Λ is the Legendrian ECH of $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ of action filtration below L.

$$ECH^L(Y, \Lambda; \delta) := ECH^L(\check{Y}, \check{\Lambda}; \check{\alpha}, J)$$

Here δ denotes the set of all choices made during the construction: the metric g, the parameter ϵ, the embedding ι and the tailored almost complex structure J.

The groups in Definition 72 certainly depend on the specific choices, since Reeb chords can appear and disappear depending on the size of $\iota(B)$ in Y. Addressing this issue is far beyond the scope of this paper. However, let us state an optimistic conjecture in this direction.

Definition 73. A choice of data $\delta = (g, \epsilon, \iota, J)$ is *L-admissible* if

- Every Reeb orbit of length less than or equal to L is in $\check{Y}$.
- The Reeb chords $\Lambda_+ \to \Lambda_-$ in $\check{Y}$ of length less than or equal to L are in bijection with the Reeb chords of $\Lambda \to \Lambda$ in Y of length less than or equal to L

These criteria are always achievable by shrinking ι and scaling the metric to reduce the size of B.

Conjecture 74. Let (Y, Λ) be a pair of a closed contact 3-manifold Y, a closed Legendrian link $\Lambda \subset Y$ and a contact form α that is non-degenerate for (Y, Λ). Then

- (Well-Defined) If δ and δ' are two choices of L-admissible data, then there is a natural isomorphism
$$ECH^L(Y, \Lambda; \delta) \simeq ECH^L(Y, \Lambda; \delta')$$
The resulting group $ECH^L(Y, \Lambda)$ is the filtered ECH of (Y, Λ).
- (Filtration Map) For any $K > L$, there is a map
$$\iota_L^K : ECH(Y, \Lambda) \to ECH^K(Y, \Lambda) \qquad \text{such that} \qquad \iota_L^M = \iota_K^M \circ \iota_L^K$$
- (Colimit) The colimit of the filtered ECH groups of (Y, Λ) over L
$$ECH(Y, \Lambda) := \underset{L}{\operatorname{colim}} ECH^L(Y, \Lambda)$$
depends only on (Y, Λ) up to contactomorphism of the pair.

Remark 75. It is likely that the maps in Conjecture 74 would be built from cobordism maps arising in a Seiberg-Witten based model for Legendrian ECH. In particular, Conjecture 74 is of a similar difficulty to the invariance proof discussed in Remark 67.

Remark 76. In [28] they use sutured ECH to define an (conjectured) invariant of Legendrian in closed 3-manifolds by considering sutured ECH of the same convex sutured manifold. However, we expect our invariant to be different from theirs because we allow Reeb chords in our chain complex.

Motivation And Future Directions

This paper lays the groundwork for several future projects on the structure of ECH, each of which provides ample motivation for the development of our theory. We conclude this introduction by giving an overview of these motivating projects.

Circle-Valued Gradient Flows

Let M be a closed 3-manifold equipped with a circle-valued Morse function. That is, a smooth function

$$f : M \to S^1$$

with isolated critical points p that each have non-degenerate Hessian. Assume also that f has no index 0 or index 3 critical points. In [57, 56], Hutchings and Lee defined a 3-manifold invariant

$$I_3 : \mathrm{Spin}^c(M) \to \mathbb{Z}$$

from the set of spin-c structures on M to the integers, via counts of configurations of closed orbits and flow lines of the gradient vector field of f. This invariant was motivated by and related to a number of other previously known invariants.

First, in a series of papers [118, 117], Turaev introduced a form of Reidemeister torsion, later dubbed *Turaev torsion*, which is also a map

$$\tau_M : \mathrm{Spin}^c(M) \to \mathbb{Z}.$$

Hutchings-Lee proved in [56] that $\tau_M = I_3$. On the other hand, rapid developments in low-dimensional topology and gauge theory contemporaneous to [118, 117] lead to the introduction of the *Seiberg-Witten invariant*

$$\mathrm{SW}_M : \mathrm{Spin}^c(M) \to \mathbb{Z}$$

This invariant is defined using a signed and weighted count of solutions to the 3-dimensional Seiberg-Witten equation (cf. [83]). Turaev established in [116] that $\mathrm{SW}_M = \tau_M$ (up to sign), proving through indirect means that

$$I_3 = \mathrm{SW}_M \tag{4.1.4}$$

Through the equality (4.1.4), one is lead to the following question.

Question 77. Is there a direct proof of the equality $I_3 = \mathrm{SW}_M$ that does not use Turaev torsion?

Moreover, the Seiberg-Witten invariant was categorified by Kronheimer-Mrowka's monopole Floer homology $HM_\bullet$ [74] and other variants of Seiberg-Witten-Floer theory. This suggests the following (roughly formulated) question, which is related to Question 77.

Question 78. Is there a Floer homology theory $FH_\bullet(Y, f)$ of 3-manifolds Y with a circle valued Morse function $f : Y \to S^1$ as above such that

(a) $FH_\bullet$ categorifes I_3, i.e. it is computed as the homology of a complex generated by counts of configurations of gradient flow lines and closed orbits as in I_3.

(b) $FH_\bullet$ is isomorphic to (the appropriate flavor of) monopole Floer homology.

Embedded contact homology, and its sister theory periodic Floer homology (PFH), answer both of these questions when the Morse function f has no critical points. In this case, M may be viewed as a mapping torus of a map

$$\phi : \Sigma \to \Sigma \qquad \text{where} \qquad \Sigma = f^{-1}(0)$$

and the generators of the hypothetical Floer homology groups $FH_\bullet$ must be configurations of periodic points of ϕ. The PFH groups $PFH_\bullet$ provides just such a theory and a result of Lee-Taubes [82] states that $PFH_\bullet$ and $HM_\bullet$ are isomorphic[4].

In the general case where f can have critical points, Questions 77 and 78 are still open. However, there is a potential approach based on a slight generalization of the constructions in this paper. Choose a metric g on M such that f is harmonic (this is possible if f has no index 0 or 3 critical points). Assume that each index 2 critical point p has a Morse chart where

$$f(x, y, z) = 2x^2 - y^2 - z^2.$$

In this chart, we can remove a standard neighborhood U and introduce a boundary component to M. The new boundary has corners, and on the smooth components the gradient vector-field is either orthogonal to or tangent to the boundary. This neighborhood is depicted in Figure 4.4.

We can perform this neighborhood removal around each critical point of f (using an analogous local model near the index 1 points) to acquire a new space $Y \subset M$. This space is equipped with a stable Hamiltonian structure with 2-form ω_f given by the Hodge dual of df and stabilizing 1-form $\theta_f = df$. The boundary of Y contains a

[4]Note that in the general case, the isomorphism between $HM_\bullet$ and $PFH_\bullet$ uses variants of both Floer groups with appropriate twisted Novikov coefficients.

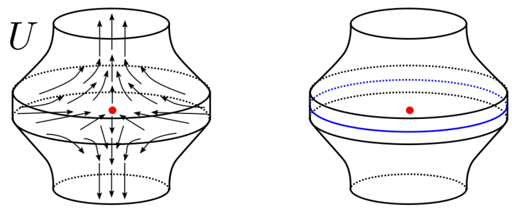


Figure 4.4: A standard neighborhood U of an index 2 critical point. The critical point is in red and the descending sphere is in blue. Note the regions where the gradient points in and out of the boundary are level surfaces of f.

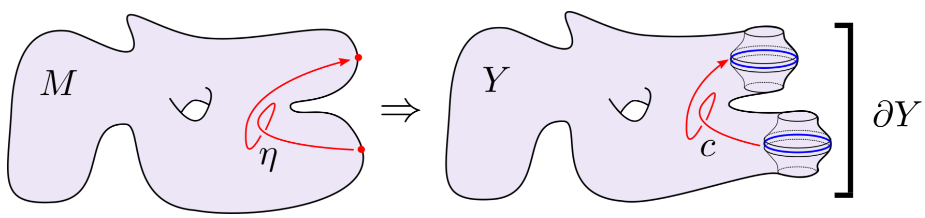


Figure 4.5: The space acquired by removing standard neighborhoods around each critical point from Y.

natural 1-dimensional sub-manifold Λ given as the union of the ascending sphere of the index 1 critical points and the descending spheres of the index 2 critical points. This sub-manifold satisfies

$$T\Lambda \subset \ker(\theta_f) \qquad \text{and} \qquad \omega_f|_{T\Lambda} = 0$$

In other words, Λ is the analogue of a Legendrian in the stable Hamiltonian manifold (Y, ω_f, θ_f). Moreover, any gradient flow line η from an index 1 critical point p to an index 2 critical point q becomes a chord c connecting the corresponding components of Λ.

The pair (Y, Λ) very much resembles a stable Hamiltonian analogue of the setup used in this paper to define Legendrian ECH. Thus, one approach to addressing Question 78 is to use the methods of this paper to develop a PFH version of our Legendrian ECH theory for this setting. There are some significant technical challenges to carrying out this program. For example, the boundary of Y is naturally *concave* sutured, rather than convex, and thus we do not have immediate access to an appropriate maximum principle. This could be addressed by adapting the *concave-to-convex* operation in [28] to our stable Hamiltonian setting and establish a maximum principle for J-holomorphic curves near the boundary. We hope to pursue this in future work.

Heegaard-Floer Homology And Embedded Contact Homology

It has been shown that embedded contact homology is isomorphic to (the appropriate flavors of) several other Floer homologies, notably Heegaard-Floer homology and monopole Floer homology. The isomorphisms are, however, highly non-trivial to construct. For instance, the construction of the isomorphism relating ECH to Heegaard-Floer theory (through open book decompositions) occupies four long papers due to Colin-Ghiggini-Honda [20, 23, 25, 21].

The work of Colin-Ghiggini-Honda utilizes a cylindrical reformulation of Heegaard-Floer homology due to Lipshitz [85] that we now describe in broad terms. The construction begins with a *pointed Heegaard diagram* that we write as follows.

$$(\Sigma, \alpha, \beta, \mathfrak{z})$$

This consists of a closed orientable surface Σ of genus g, a distinguished point $\mathfrak{z} \in \Sigma$ and two collections of g simple, non-separating, closed curves

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g \qquad \text{and} \qquad \beta = \beta_1 \cup \cdots \cup \beta_g$$

Recall that $(\Sigma, \alpha, \beta, \mathfrak{z})$ determines a 3-manifold M (uniquely, up to diffeomorphism). More precisely, we take the 3-manifold

$$Y = [0,1] \times \Sigma \qquad \text{with curves} \qquad \Lambda_\alpha = (0 \times \alpha) \quad \text{and} \quad \Lambda_\beta = (1 \times \beta)$$

and attach 2-handles to each curve in Λ_α and Λ_β. The resulting manifold has a boundary consisting of two 2-spheres, and after attaching 3-handles to these areas, we acquire M.

The space $Y = [0,1]_t \times \Sigma$ may be viewed as a stable Hamiltonian manifold with the two-form ω equal to an area form on Σ and stabilizing 1-form dt. The analogue of the Reeb vector-field is $R = \partial_t$. Moreover, Λ_α and Λ_β are Legendrians in the sense that

$$\omega|_{\Lambda_\pm} = 0 \qquad \text{and} \qquad dt|_{\Lambda_\pm} = 0$$

In this picture, Reeb chords between Λ_α and Λ_β are equivalent to intersection points in $\alpha \cap \beta$. The symplectization of Y is given the symplectic manifold

$$W = \mathbb{R}_s \times Y \qquad \text{with symplectic form} \qquad \Omega = ds \wedge dt + \omega_\Sigma$$

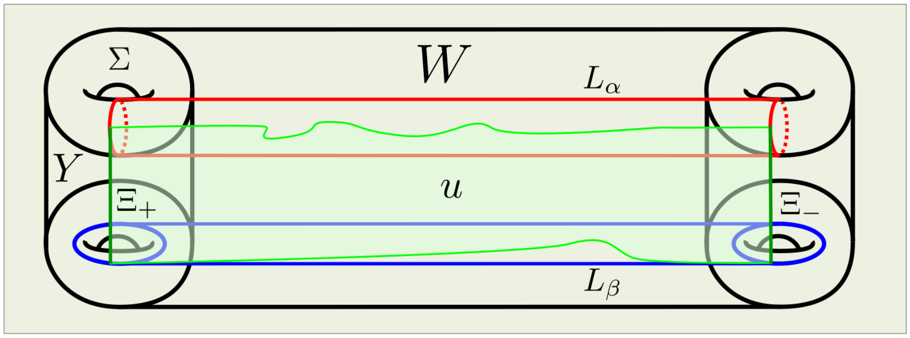


Figure 4.6: A picture of the setup of Lipshitz' cylindrical formulation of Heegaard-Floer homology.

The cylindrical sub-manifolds $L_\alpha = \mathbb{R} \times \Lambda_\alpha$ and $L_\beta = \mathbb{R} \times \Lambda_\beta$ are both Lagrangians. There is a natural class of compatible almost complex structures J on W: those that are translation invariant and that satisfy $J(\partial_s) = \partial_t$ and $J(T\Sigma) = T\Sigma$. Finally, note that the marked point $\mathfrak{z}$ determines a J-holomorphic strip

$$Z = \mathbb{R} \times [0,1] \times \mathfrak{z}$$

The formulation of Heegaard-Floer homology of Lipshitz [85] can now be described as follows. The chain complex $CF_{\text{Lip}}(\Sigma, \alpha, \beta, \mathfrak{z})$ is generated by sets of the form

$$\Xi = \{c_1, \ldots, c_g\}$$

where Ξ consists of chords c_i from α_i to $\beta_{\sigma(j)}$ for some permutation σ, or equivalently a set of intersection points $p_i \in \alpha_i \cap \beta_{\sigma(i)}$. The differential ∂ counts pseudo-holomorphic curves of the following form. Let Γ and Ξ be two generators. Let S with a Riemann surface with $2g$ boundary punctures, g of which we label positive and g of which we label negative. We consider J-holomorphic maps

$$u : (S, \partial S) \to (W, L_\alpha \cup L_\beta)$$

of Fredholm index 1 (modulo translation), where the positive punctures are asymptotic to the chords corresponding to Γ at $+\infty$ of the $\mathbb{R}$ direction in W; the negative punctures are asymptotic to the chords corresponding to Ξ at $-\infty$ of the $\mathbb{R}$ direction in W; the image of S under u is embedded; and $u(S)$ is disjoint from Z. For more technical formulations of these conditions see section 1 in [85]. For a depiction of this setup, see Figure 4.6.

Theorem 79. *[85] The homology groups of $CF_{Lip}(\Sigma, \alpha, \beta, \mathfrak{z})$ are isomorphic to the hat version of the Heegaard-Floer homology groups.*

$$H(CF_{Lip}(\Sigma, \alpha, \beta, \mathfrak{z})) \simeq \widehat{HF}(Y)$$

The setup for Lipshitz' construction is very similar to the one for Legendrian ECH. In fact, it may be viewed as a special case of our construction adapted to the stable Hamiltonian manifold

$$(Y, \omega, \theta) = ([0,1] \times \Sigma, \omega_\Sigma, dt) \qquad \text{with "Legendrians"} \qquad \Lambda_\alpha \cup \Lambda_\beta \subset \partial Y$$

This is the perspective adopted by Colin-Ghiggini-Honda, and in [23, §4] they prove the following claim:

Claim 80. [23, §4] The differential $CF_{\text{Lip}}(\Sigma, \alpha, \beta, \mathfrak{z})$ precisely counts ECH index 1 holomorphic curves C in $\mathbb{R} \times Y$ with boundary $\mathbb{R} \times (\Lambda_\alpha \cup \Lambda_\beta)$.

The theory articulated in this paper can therefore be viewed as providing a common framework for describing both Heegaard-Floer theory and embedded contact homology as special cases of a single construction, when one includes only the Reeb chords or Reeb orbits as generators.

Bordered Embedded Contact Homology And Gluing Formulas

In the paper [86], Lipshitz-Oszvath-Thurston formulate a theory of Heegaard-Floer homology for 3-manifolds with boundary, called *bordered Heegaard-Floer homology.*

Roughly speaking, the bordered Heegaard-Floer homology groups depend on a 3-manifold Y and on a diffeomorphism $\phi : F \simeq \partial Y$ of ∂Y with a surface F determined by a datum called a *matched circle* $\mathcal{Z}$. The hat version of this theory comes in two flavors.

$$\widehat{CFA}(Y, \phi) \qquad \text{and} \qquad \widehat{CFD}(Y, \phi)$$

which are, respectively, an A_∞-module and a dg-module over a certain free dg-algebra $\mathcal{A}$ associated to $\mathcal{Z}$. A fundamental application of this theory is the following gluing result.

Theorem 81. *[86] If $Y = M \cup_F N$ is a closed union of two 3-manifolds with boundary via the identifications of $\phi : F \simeq \partial M$ and $\psi : F \simeq \partial N$ with F, then*

$$\widehat{HF}(Y) \simeq H(\widehat{CFA}(M, \phi) \otimes_{\mathcal{A}} \widehat{CFD}(N, \psi))$$

Here $\otimes_{\mathcal{A}}$ denotes the tensor product as A_∞-modules over $\mathcal{A}$.

This gluing formula has since become a fundamental tool in computing Heegaard-Floer invariants and their knot counterparts. In particular, they are useful in studying submanifolds e.g. knots and surfaces [48, 42].

Although many variants of symplectic field theory can be formulated for contact manifolds with boundary (e.g. [28]), a general gluing formula in the spirit of Theorem 81 has yet to appear. As a first step, one can try to prove Theorem 81 where the sutured Heegaard-Floer groups are replaced with variants of ECH for manifolds with boundary.

To further understand what bordered ECH might look like, we observe that the two types of hat bordered Heegaard-Floer complexes are constructed by extending Lipshitz' cylindrical formulation [85] in two ways.

First, the complexes are computed using a *bordered Heegaard diagram* (Σ, α, β) for Y, where Σ has boundary and some α curves are permitted to have boundary in $\partial\Sigma$. Second, the bordered complexes incorporate Reeb chords between the Legendrians (i.e. points) $\alpha \cap \partial\Sigma$ in the matched circle $\mathcal{Z}$, which is identified with $\partial\Sigma$. The dg-algebra $\mathcal{A}$ associated to $\mathcal{Z}$ is, in fact, a variant of the Chekanov-Eliashberg dg-algebra of the Legendrians in $\mathcal{Z}$. The holomorphic curves counted in bordered Heegaard-Floer theory are allowed to limit to chords of $\alpha \cap \partial\Sigma$ at boundary punctures, and the chain complexes themselves are (roughly speaking) augmented by changing the ordinary Heegaard-Floer complexes to have coefficients in $\mathcal{A}$.

In light of these observations, we expect that a bordered theory of ECH will require an analogous extension of Legendrian ECH that incorporates Legendrians $\Lambda \subset \partial_\pm Y$ with boundary on $\partial\Lambda$ contained in $\partial(\partial_\pm Y) \simeq \sigma$, the suture, and Reeb chords between the points $\partial\Lambda \subset \sigma$. The precise formulation of this theory will be the subject of future work based on this paper.

4.2 Intersection Theory

In this section, we describe an intersection theory for surfaces in symplectic cobordisms with boundary on a Lagrangian cobordism in dimension 4. This mirrors the theory for surfaces without Lagrangian boundary described by Hutchings in [55].

Bundle Pairs

We begin by introducing the notion of a bundle pair with punctures over a punctured Riemann surface with boundary.

Definition 82. A *symplectic bundle pair with punctures* $(E, F) \to (\Sigma, \partial\Sigma)$ consists of

(a) A compact, oriented surface with boundary and corners

$$\Sigma \qquad \text{with} \qquad \partial\Sigma = \partial_\star\Sigma \cup \partial_\circ\Sigma$$

where $\partial_\star\Sigma$ and $\partial_\circ\Sigma$ are each unions of smooth strata of the boundary.

(b) A bundle pair of a symplectic vector-bundle and a Lagrangian sub-bundle

$$(E, \omega) \to \Sigma \qquad \text{and} \qquad F \to \partial_\circ\Sigma \quad \text{with} \quad F \subset E|_{\partial_\circ\Sigma}$$

We refer to $\partial_\circ\Sigma$ as the *Lagrangian boundary* and to $\partial_\star\Sigma$ as the *puncture boundary*. We note by our previous notation $\partial_\star\Sigma$ is the same as $\partial_\pm\Sigma$. We equip both $\partial_\circ\Sigma$ and $\partial_\star\Sigma$ with the induced boundary orientation.

Definition 83. A *puncture trivialization* τ of $(E, F) \to (\Sigma, \partial\Sigma)$ consists of the following. First a symplectic trivialization of E over $\partial_\star\Sigma$, which we write as

$$E|_{\partial_\star\Sigma} \simeq \mathbb{C}^n$$

so that

$$F|_{\partial_\star\Sigma\cap\partial_\circ\Sigma} \simeq \mathbb{R}^n \subset \mathbb{C}^n.$$

We fix the convention of thinking of τ as a map from $E \to \partial_\star\Sigma$ to $\mathbb{C}^n \to \partial_\star\Sigma$.

Bundle pairs with punctures admit an integer invariant analogous to the Chern class, generalizing the Maslov number of an ordinary bundle pair.

Proposition 84 (Maslov Number)**.** For any symplectic bundle pair with punctures $(E, F) \to (\Sigma, \partial\Sigma)$ and puncture trivialization τ, there is a well-defined Maslov number

$$\mu(E, F; \tau) \in \mathbb{Z}$$

Furthermore, the Maslov number is uniquely determined by the following axioms.

(a) (Isomorphism) The Maslov number is invariant under isomorphism of the pair and trivialization.

$$\mu(E, F; \tau) = \mu(E', F'; \tau') \qquad \text{if} \qquad (E, F; \tau) \simeq (E', F'; \tau')$$

(b) (Direct Sum) The Maslov number is additive with respect to direct sum.

$$\mu(E \oplus E', F \oplus F'; \tau \oplus \tau') = \mu(E, F; \tau) + \mu(E', F'; \tau')$$

(c) (Disjoint Union) The Maslov number is additive under disjoint union.

$$\mu(E \cup E', F \cup F'; \tau \cup \tau') = \mu(E, F; \tau) + \mu(E', F'; \tau')$$

(d) (Puncture Gluing) Let $(E, F) \to (\Sigma, \partial\Sigma)$ and $(E', F') \to (\Sigma', \partial\Sigma')$ be two bundle pairs with punctures. Let $R \subset \partial_\star\Sigma$ and $R' \subset \partial_\star\Sigma'$ be components of the puncture boundary. Let $-R'$ denote R' with reverse orientation. Assume we have a bundle pair isomorphism

$$\psi : (E|_R) \simeq (E'|_{-R'}) \qquad \text{covering a diffeomorphism} \qquad R \simeq -R'$$

Further, composed with trivializations τ and τ', the map ψ takes the form

$$\tau'^{-1} \circ \psi \circ \tau : (\mathbb{C}^n, \mathbb{R}^n) \to (\mathbb{C}^n, \mathbb{R}^n)$$

over R and $-R'$. Then

$$\mu(E \cup_\psi E', F \cup_\psi F', \tau \cup \tau') = \mu(E, F, \tau) + \mu(E', F', \tau')$$

Here $(E \cup_\psi E', F \cup_\psi F')$ is the bundle pair over $(\Sigma, \partial\Sigma)$ acquired by gluing of (E, F) and (E', F'), and $\tau \cup_\psi \tau'$ is the glued trivialization.

(e) (Normalization) If $\Sigma = D$ is the 2-disk with $\partial\Sigma = \partial_\circ\Sigma = S^1$, and (E, F) is the bundle pair

$$E = D \times \mathbb{C} \quad \text{and} \quad F = \{e^{i\theta}\mathbb{R} : \theta \in \partial D \simeq \mathbb{R}/\pi\mathbb{Z}\}$$

then the Maslov number is given by $\mu(E, F; \tau) = 1$.

Proof. To prove existence, let $\bar{F} \to \partial\Sigma$ denote the Lagrangian sub-bundle of $E|_{\partial\Sigma}$ given by

$$\bar{F}|_{\partial_\circ\Sigma} = F \quad \text{and} \quad \bar{F}|_{\partial_\star\Sigma} = \tau^{-1}(\mathbb{R}^n)$$

Then $\mu(E, F; \tau)$ is simply the standard Maslov number $\mu(E, \bar{F})$ of the pair $(E, \bar{F})$ (see [91, §C.3]).

$$\mu(E, F; \tau) = \mu(E, \bar{F})$$

The properties (a)-(e) except (d) follow immediately from the analogous properties of the ordinary Maslov number (see [91, Theorem C.3.5(a)-(d)]).

To see (d), consider for $(E, \bar{F}) \to (\Sigma, \partial\Sigma)$, there is already a trivialization of E over the boundary punctures. Extend this to a trivialization of E over the entire boundary. This is possible since all complex bundles over S^1 are trivial. In the interior of Σ, we cut out a disk with boundary $(D, \partial D)$, then the trivialization of E can be extended to $E \setminus D$. We further choose a family of Lagrangians over $E|_{\partial D}$, which we denote by $\widetilde{F}$. Then by the additivity property of the Maslov index we have

$$\mu(E, F; \tau) = \mu(E|_{\Sigma\setminus D}, \bar{F} \cup \widetilde{F}|_{\partial\Sigma\cup-\partial D}) + \mu(E|_D, \widetilde{F}).$$

Further more, we observe $E|_{\Sigma\setminus D} \simeq \Sigma \setminus D \times \mathbb{C}^n$, hence $\mu(E|_{\Sigma\setminus D}, \bar{F} \cup \widetilde{F}|_{\partial\Sigma\cup-\partial D})$ is the sum of the Maslov indices of the Lagrangians $\bar{F} \cup \widetilde{F} \subset \mathbb{C}^n$ along $\partial\Sigma \cup -\partial D$. An entirely analogous construction holds for $(E'\bar{F}') \to (\Sigma', \partial\Sigma')$ - we just add prime to all of our previous symbols. Then the act of gluing induces a gluing between $(\partial\Sigma \setminus D, \partial\Sigma \cup -\partial D)$ and $(\partial\Sigma' \setminus D', \partial\Sigma' \cup -\partial D')$, then this implies that

$$\mu(E|_{\Sigma\setminus D}, \bar{F}\cup\widetilde{F}|_{\partial\Sigma\cup-\partial D})+\mu(E'|_{\Sigma'\setminus D'}, \bar{F}'\cup\widetilde{F}'|_{\partial\Sigma\cup-\partial D}) = \mu(E\cup E'|_{(\Sigma\setminus D)\cup_\psi(\Sigma'\setminus D')}, F\cup_\psi F'\cup\widetilde{F}\cup\widetilde{F}')$$

because the contributions for the Lagrangians F and F' cancel due to our assumptions. The additivity formula then follows from the composition formula in the regular Maslov index case.

To prove uniqueness, we argue as follows. First, consider the case where Σ is the half-disk

$$\Sigma = \mathbb{D} \cap \mathbb{H} \quad \text{with} \quad \partial_\star\Sigma = \partial\mathbb{D} \cap \mathbb{H} \text{ and } \partial_\circ\Sigma = \mathbb{D} \cap \mathbb{R} \tag{4.2.1}$$

$$E = \mathbb{C}^n \qquad F = \mathbb{R}^n \quad \text{and} \quad \tau \text{ is the tautological trivialization}$$

Then by gluing (E, F) to itself along $\partial_\star\Sigma$ and applying axiom (d), we find that

$$\mu(E, F; \tau) = \frac{1}{2} \cdot \mu(\mathbb{C}^n, \mathbb{R}^n) = 0$$

Second, the uniqueness of the ordinary Maslov index [91, Theorem C.3.5] implies that for any bundle pair $(E, F) \to (\Sigma, \partial\Sigma)$ with $\partial_\star\Sigma = \emptyset$, we have

$$\mu(E, F) = \mu(E, F; \tau)$$

Finally, any bundle pair with punctures $(E, F) \to (\Sigma, \partial\Sigma)$ transforms into a bundle pair with no puncture boundary by gluing on trivial bundles over the half-disk along $\partial_\star\Sigma$. Thus the gluing axiom (d) determines $\mu(E, F; \tau)$. Uniqueness then follows from the uniqueness of the Maslov index for surfaces without boundary punctures. □

Remark 85. In the case where $\partial_\star\Sigma = \emptyset$, we omit the (empty) trivialization from the notation and denote the Maslov number by $\mu(E, F)$.

Example 86 (Special Cases)**.** The following integer invariants can be viewed as special cases of the Maslov number.

(a) The 1st Chern number of a bundle $E \to \Sigma$ of a surface with boundary with respect to a trivialization τ along $\partial\Sigma$ is

$$c_1(E, \tau) = \frac{1}{2} \cdot \mu(E, \emptyset; \tau).$$

In keeping with our previous conventions, we take $\partial_*\Sigma$ to be the entire boundary of Σ, and $\partial_o\Sigma = \emptyset$. In particular, if Σ is closed then $c_1(E) = \frac{1}{2} \cdot \mu(E, \emptyset)$.

(b) [91, Thm C.3.6] Let Σ be a compact surface with boundary and let $F : \partial\Sigma \to \mathrm{LGr}(2n)$ be a set of loops in the Lagrangian Grassmannian. Then

$$\mu(F) := \mu(\Sigma \times \mathbb{C}^n, F)$$

(c) The Maslov index of a loop $\Phi : S^1 \to \mathrm{Sp}(2n)$ is

$$\mu(\Phi) := \frac{1}{2} \cdot \mu(D \times \mathbb{C}^n, F) \quad \text{where} \quad F_\theta := \Phi_\theta(\mathbb{R}^n) \subset \mathbb{C}^n$$

Example 87 (Maslov Class)**.** Let (X, L) be a pair of a symplectic manifold and a Legrangain sub-manifold $L \subset X$. The *Maslov class*

$$\mu(X, L) : H_2(X, L) \to \mathbb{Z}$$

is the map defined so that the value on the class A of a surface $\Sigma \subset X$ with boundary $\partial\Sigma \subset L$ is the Maslov number of $(TX, TL)|_\Sigma$.

The Maslov class $\mu(\xi, \Lambda)$ of a contact manifold (Y, ξ) with Legendrian $\Lambda \subset Y$ is defined similarly, and we have

$$\mu(\xi, \Lambda) = \mu(TX, TL) \qquad \text{if} \qquad (X, L) = ([0, 1] \times Y, [0, 1] \times \Lambda)$$

We will require a few properties of the Maslov number. First, we must record how the Maslov number changes under change of end trivializations.

Lemma 88 (Trivial Bundle)**.** Consider the a bundle pair with punctures

$$(\mathbb{C}^n, F) \to (\partial\Sigma, \partial_\circ\Sigma)$$

where $F : \partial_\circ\Sigma \to \mathrm{LGr}(2n)$ is a collection of arcs in the Lagrangian Grassmanian with ends on $\mathbb{R}^n$. Then

$$\mu(E, F; \tau) = \mu(F) + \mu(\tau^{-1}(\mathbb{R}^n))$$

Here $\mu(F)$ and $\mu(\tau^{-1}(\mathbb{R}^n))$ are the Maslov indices of the (collections of) loops of Lagrangians F and $\tau^{-1}(\mathbb{R}^n)$.

Proof. As in Proposition 84, the Maslov index $\mu(E, F; \tau)$ is given by

$$\mu(E, F; \tau) = \mu(E, \bar{F})$$

where $\bar{F}$ is the Lagrangian sub-bundle acquired by taking the union of F over $\partial_\circ\Sigma$ and $\tau^{-1}(\mathbb{R}^n)$ over $\partial_\star\Sigma$. The result thus follows immediately from Example 86(b). $\square$

As with the Chern number, the Maslov number can be interpreted as a signed count of zeros.

Lemma 89 (Zero Count)**.** Let $(E, F) \to (\Sigma, \partial\Sigma)$ be a bundle pair with punctures equipped with boundary trivialization τ. We further assume E is two dimensional. Let $\psi : (\Sigma, \partial_\circ\Sigma) \to (E, F)$ be a section with

$$\psi \text{ is transverse to } \Sigma \quad \text{and} \quad \psi(\partial_\star\Sigma) \subset \tau(\mathbb{R} \setminus 0)$$

Then the Maslov number $\mu(E, F; \tau)$ is given by

$$\mu(E, F; \tau) := 2 \cdot \#(\psi \cap \mathrm{int}(\Sigma)) + \#(\psi \cap \partial_\circ\Sigma) \tag{4.2.2}$$

Proof. Let $-\Sigma$ and $-E$ denote the surface and bundle with reversed orientations, respectively. Recall a trivialization τ is achoice of a map

$$\tau : E|_{\partial_*\Sigma} \to \mathbb{C}$$

With this trivialization, as with the case how we defined Maslov indices, we extend F over $\partial_*\Sigma$ as

$$\tau : (E, F)|_{\partial_*\Sigma} \to (\mathbb{C}, \mathbb{R})$$

Furthermore, let $\bar{\tau}$ denote the composition of τ with complex conjugation (which is anti-symplectic).

$$(E, F)|_{\partial_\star \Sigma} \simeq (\mathbb{C}, \mathbb{R}) \xrightarrow{z \to \bar{z}} (\mathbb{C}, \mathbb{R})$$

This composition is a trivialization of $-E$ over $\partial_\star \Sigma$ in $-\Sigma$. Viewing $-E$ as a bundle over $-\Sigma$, we have

$$\mu(-E, F; \bar{\tau}) = \mu(E, F; \tau)$$

Moreover, there is a natural (isotopy class of) symplectic bundle map

$$(-E, F)|_{\partial_\circ \Sigma} \to (E, F)|_{\partial_\circ \Sigma} \qquad \text{covering} \qquad \mathrm{Id} : \partial_\circ \Sigma \to \partial_\circ \Sigma$$

given by complex conjugation with respect to the totally real subspace F. Thus, we can glue E and $-E$ to a bundle DE over the double $D\Sigma = \Sigma \cup_{\partial_\circ \Sigma} -\Sigma$ of Σ along the boundary region $\partial_\circ \Sigma$. Using the composition property of Maslov class (Theorem C.3,5 in [91] and the fact we have fixed trivializations around punctures so that the totally bundle is trivial) gives us that

$$\mu(E, F; \tau) = \frac{1}{2}\mu(DE, \emptyset; \tau) = c_1(DE, \emptyset; \tau)$$

Now take a section ψ as in the lemma statement. We may assume (after a small isotopy leaving the zeros unchanged) that this section doubles to a section ϕ of DE that is transverse to the zero section. The relative Chern class of DE with respect to τ is computed by the zeros of ϕ, and thus

$$\mu(E, F; \tau) = c_1(DE, \emptyset; \tau) = \#(\phi \cap D\Sigma) = 2 \cdot \#(\psi \cap \mathrm{int}(\Sigma)) + \#(\psi \cap \partial_\circ \Sigma) \qquad \square$$

Euler Characteristic

Consider a compact surface Σ with boundary and corners $\partial\Sigma = \partial_\circ \Sigma \cup \partial_\star \Sigma$. In this paper, we will use the following version of the Euler characteristic.

Definition 90. The *orbifold Euler characteristic* $\bar{\chi}(\Sigma)$ is the quantity

$$\chi(\Sigma) - \frac{1}{2}\chi(\partial_\circ \Sigma) \quad \text{or equivalently} \quad \frac{1}{2} \cdot \chi(\Sigma \cup_{\partial_\circ \Sigma} -\Sigma)$$

Like the ordinary Euler characteristic, this quantity is a special case of the Maslov number. Specifically, consider the tangent bundle of Σ. This naturally has the structure of a bundle pair with punctures.

$$(T\Sigma, T(\partial_\circ \Sigma)) \to (\Sigma, \partial\Sigma)$$

Moreover, this bundle pair comes with a canonical isotopy class of trivialization

$$\tau_{\partial\Sigma} : (T\Sigma, T(\partial_\circ\Sigma))|_{\partial_\star\Sigma} \simeq (\mathbb{C}, \mathbb{R}) \quad \text{with} \quad \tau_{\partial\Sigma}^{-1}(\mathbb{R}) \pitchfork T(\partial_\star\Sigma)$$

The Maslov number of the tangent pair in the canonical trivialization is precisely twice the orbifold Euler characteristic.

Lemma 91 (Euler Characteristic)**.** Let Σ be a surface with boundary $\partial\Sigma$. Then

$$\frac{1}{2} \cdot \mu(T\Sigma, T(\partial_o\Sigma); \tau_{\partial\Sigma}) = \bar{\chi}(\Sigma)$$

Proof. First, consider two special cases. If Σ is closed, this is equivalent to the fact that $c_1(T\Sigma) = \chi(\Sigma)$. Likewise, if $\Sigma = \mathbb{D}$ with $\partial_\star\Sigma = \partial\mathbb{D}$, then $T(\partial\mathbb{D}) \subset T\mathbb{D} \simeq \mathbb{D}\times\mathbb{R}^2$ is a loop of Lagrangians with Maslov index 2. Thus by Example 86(c), we have

$$\frac{1}{2} \cdot \mu(T\mathbb{D}, T(\partial\mathbb{D})) = 1 = \chi(\mathbb{D})$$

This verifies these special cases. In the general case we can consider the double of Σ along $\partial_\circ\Sigma$.

$$S = \Sigma \cup_{\partial_\circ\Sigma} \bar{\Sigma}$$

Applying the proof of Lemma 89 and the property of the Euler characteristic under gluing, we deduce

$$\mu(TS, \emptyset; \tau_{\partial S}) = 2 \cdot \mu(T\Sigma, T(\partial_\circ\Sigma); \tau_{\partial\Sigma}) \qquad \text{and} \qquad \chi(S) = 2 \cdot \bar{\chi}(\Sigma)$$

On the other hand, S can be acquired by taking a closed surface and removing a collection of disks to produce puncture boundary. Thus, by Proposition 84(d) and the special cases, we have

$$\mu(TS, \emptyset; \tau_{\partial S}) = \chi(S)$$

□

Setup And Trivializations

Before proceeding, it will be useful for us to fix a common setup for the remainder of this section. For each $\bullet \in \{+, -\}$, we fix

(a) a compact contact 3-manifold $(Y_\bullet, \xi_\bullet)$ with boundary $\partial Y_\bullet$.

(b) a closed Legendrian sub-manifold $\Lambda_\bullet \subset \partial Y_\bullet$ in the boundary of $Y_\bullet$.

(c) a contact form $\alpha_\bullet$ with non-degenerate Reeb orbits and chords.

(d) an orbit-chord set $\Xi_\bullet = \Gamma_\bullet \cup C_\bullet$ consisting of a Reeb orbit set $\Gamma_\bullet = \{(\gamma_i^\bullet, m_i^\bullet)\}$ and Reeb chord set $C_\bullet = \{(c_i^\bullet, n_i^\bullet)\}$.

(e) a symplectic 4-dimensional cobordism $X : Y_+ \to Y_-$ with symplectic form ω. This is a manifold with boundary and corners, where $\partial X = \partial_+ X \cup \partial_\circ X \cup \partial_- X$ such that $\partial_\bullet X \simeq Y_\bullet$.

(d) a Lagrangian cobordism $L : \Lambda_+ \to \Lambda_-$ contained in the horizontal boundary $\partial_\circ X$ of X.

We will also need to refer to a symplectic trivialization of ξ along the orbits and chords in Ξ. We fix the following terminology.

Definition 92 (Trivialization)**.** A *trivialization* τ of ξ over Ξ is a symplectic bundle isomorphism

$$\tau : \xi|\gamma_i \simeq \mathbb{C} \qquad \tau : (\xi, T\Lambda)|c_i \simeq (\mathbb{C}, \mathbb{R})$$

The difference $\sigma - \tau$ between two trivializations over Ξ is defined by

$$\sigma - \tau = \sum_i m_i \cdot \mu(\sigma \circ \tau^{-1}|_{\gamma_i}(\mathbb{R})) + \sum_i n_i \cdot \mu(\sigma \circ \tau^{-1}|_{c_i}(\mathbb{R}))$$

We use $\mathcal{T}(\Xi_+, \Xi_-)$ to denote the space of pairs $\tau = (\tau_+, \tau_-)$ of trivializations τ_+ over Ξ_+ and τ_- over Ξ_-.

Note that a trivialization $\tau : \xi|_\gamma \simeq \mathbb{C}$ of the contact structure over an orbit induces a trivialization over any iterate of γ. By abuse of notation, we also denote this trivialization by τ.

Surface Classes

We next discuss the set of surface classes associated to the pair (X, L).

Definition 93. A *proper surface* $S = (\iota, S)$ in (X, L) from Ξ_+ to Ξ_- consists of a compact surface S with boundary $\partial S = \partial_+ S \cup \partial_- S \cup \partial_\circ S$ and a continuous map $\iota : S \to X$ such that

(a) The restriction of ι to $\partial_\pm S$ is a map

$$\iota : \partial_\pm S \to Y_\pm = \partial_\pm X$$

consisting of a collection of orbits and chords in $Y_\pm$ representing the orbit set $\Xi_\pm$.

(b) The boundary region $\partial_\circ S$ maps to the Lagrangian L, i.e. $\iota(\partial_\circ S) \subset L$

A proper surface (ι, S) will be called *well-immersed* if it also satisfies the following criteria.

(a) The restriction of ι to $S \setminus (\partial_+ S \cup \partial_- S)$ is an immersion with transverse double points. Note that these double points may occur on $\partial_\circ S$.

(b) The map ι is an embedding in a neighborhood of $\partial_+ S \cup \partial_- S$ (except at $\partial_+ S \cup \partial_- S$ itself) and is transverse to $\partial_+ S \cup \partial_- S$.

Definition 94. The set $S(\Xi_+, \Xi_-)$ of *surface class* $A : \Xi_+ \to \Xi_-$ is the set of proper maps $\iota : S \to X$ bounding Ξ_+ and Ξ_- modulo the equivalence relation that

$$\iota \sim \iota' \qquad \text{if} \qquad [S \cup -S'] = 0 \in H_2(X, L)$$

Here $S \cup -S'$ is the 2-chain in X with boundary on L represented by the map $\iota \cup \iota' : S \cup S' \to X$.

The set of surface classes $S(\Xi_+, \Xi_-)$ is (tautologically) a torsor over the relative homology group $H_2(X, L)$ and we use the notation

$$A - B \in H_2(X, L) \quad \text{for the class such that} \quad A = B + (A - B)$$

Example 95. Let (Y, Λ) be a closed contact manifold and a Legendrian and fix an orbit-chord set Ξ. Consider the cobordism

$$X := Y \times [0,1] \qquad L := \Lambda \times [0,1]$$

A *trivial branched covers* of Ξ will refer to a map from a surface C to $Y \times [0,1]$ of the form

$$C \xrightarrow{\kappa} \Xi \times [0,1] \subset Y \times [0,1]$$

Here κ is a branched cover whose covering multiplicity at Ξ_+ and Ξ_- satisfies Definition 94.

Any two collection of orbits and chords representing Ξ (i.e. with the same underlying simple orbits and chords with multiplicity) can be connected by a trivial branched cover. Furthermore, the surface classes of all such maps all agree since $H_2(\Xi, \partial\Xi) = 0$.

There are some important operations on proper surfaces and surface classes that we will use in later parts of this paper.

Definition 96 (Union)**.** The *union* $S \cup T$ of two proper surfaces $(\iota.S)$ and $(\jmath, T)$ in (X, L) is

$$\iota \sqcup \jmath : S \sqcup T \to (X, L)$$

This descends to a map on the space of surface classes of the form

$$S(\Xi_0, \Theta_0) \times S(\Xi_1, \Theta_1) \to S(\Xi_0 \cup \Xi_1, \Theta_0 \cup \Theta_1) \qquad \text{with} \qquad (A, B) \mapsto A \cup B$$

Definition 97. A *composition* $S \circ T$ of two proper surfaces $S : \Xi_0 \to \Xi_1$ in (X, L) and $T : \Xi_1 \to \Xi_2$ in (X', L') is proper surface of the form

$$S \cup Z \cup T \subset X \circ X'$$

where Z is a trivial branched cover of Ξ connecting $\partial_- S$ and $\partial_+ T$. This descends to a map of surface classes of the form

$$S(\Xi_0, \Xi_1) \times S(\Xi_1, \Xi_2) \to S(\Xi_0, \Xi_1) \qquad \text{given by} \qquad (A, B) \to A \circ B$$

Maslov Number Of A Surface Class

Given any proper surface S in (X, L), there is a natural bundle pair with punctures over S induced by pullback.

$$(\iota^* TX, \iota^* TL) \to (S, \partial_\circ S)$$

We extend the trivialization of ξ_{c_i} to trivializations of $\iota^* TX|_{c_i}$. And similarly we extend trivilizations of ξ_{γ_i} to $\iota^* TX|_{\gamma_i}$. We here describe our choice for such an extension (this is essentially the same choice has we have for Lemma 101.) We assume near $Y_\pm$ the cobordism X has collar neighborhoods of the form $(1-\epsilon, 1] \times Y_+$ and $[0, \epsilon) \times Y_-$. We call the direction in the half open interval the symplectization direction. Then ξ and the plane given by the symplectization direction and the Reeb direction on over chords and orbits span TX. We extend τ in this way to TX over both chords and orbits. We further assume that with this choice of trivialization $\tau : \iota^*(TX)|_{c_i} \to \mathbb{C}^2|_{c_i}$, we have $\tau(\iota^*(TX)|_{\partial_\pm S \cap \partial_o S} = \mathbb{R}^2 \subset \mathbb{C}^2$.

Definition 98. The *Maslov number* $\mu_\tau(A, \tau)$ of a surface class A with respect to a trivialization τ over Ξ is given by

$$\mu(A, \tau) := \mu(\iota^* TX, \iota^* TL; \tau) \qquad \text{for any representative} \qquad \iota : S \to X$$

Here $\mu(\iota^*TX, \iota^*TL; \tau)$ is the Maslov index of (ι^*TX, ι^*TL) as a bundle pair (see Proposition 84).

A priori, it is not clear that $\mu(A, \tau)$ is independent of the choice of representative. We now prove that this is the case.

Proposition 99. The Maslov number $\mu(A, \tau)$ is well-defined and has the following properties.

- (Trivialization) Let σ and τ be two trivializations in $\mathcal{T}(\Xi_+, \Xi_-)$. Then

$$\mu(A, \tau) - \mu(A, \sigma) = (\sigma_+ - \tau_+) - (\sigma_- - \tau_-)$$

- (Maslov Class) Let $A : \Xi \to \Theta$ and $B : \Xi \to \Theta$ be a pair of surface classes. Then

$$\mu(A, \tau) - \mu(B, \tau) = \mu(X, L) \cdot (A - B)$$

- (Union) Let $A : \Xi_0 \to \Theta_0$ and $B : \Xi_1 \to \Theta_1$ be a pair of surface classes and let τ be a trivialization that agrees on $\Xi_0 \cap \Xi_1$ and $\Theta_0 \cap \Theta_1$. Then

$$\mu(A \cup B, \tau) = \mu(A, \tau) + \mu(B, \tau)$$

- (Composition) Let $A : \Xi_0 \to \Xi_1$ and $B : \Xi_1 \to \Xi_2$ be composible surface classes, and let τ be a trivialization along $\Xi_0 \cup \Xi_1 \cup \Xi_2$. Then

$$\mu(A \circ B, \tau) = \mu(A, \tau) + \mu(B, \tau)$$

To proceed with the proof, we first consider the case of branched covers as in Example 95.

Lemma 100. Let Ξ be an orbit-chord set in Y and consider a trivial branched cover of Ξ, as in Example 95.

$$\iota : (C, \partial_\circ C) \to (X, L) = ([0,1] \times Y, [0,1] \times \Lambda)$$

Choose a pair of trivializations σ over $\Xi_- = \Xi \times \{0\}$ and τ over $\Xi_+ = \Xi \times \{1\}$. Then

$$\mu(\iota^*TX, \iota^*TL; \iota^*(\sigma \cup \tau)) = \tau - \sigma$$

Here $\iota^*(\sigma \cup \tau)$ is the trivialization of ι^*TX over $\partial_\star C = \partial_+ C \cup \partial_- C$ induced by σ and τ.

Proof. The trivialization σ extends to a trivialization

$$(\xi, T\Lambda) \simeq (\mathbb{C}, \mathbb{R}) \qquad \text{over} \qquad \Xi \times [0,1]$$

This pulls back to a trivialization of $\iota^*\xi$ over C. Then τ and σ are identified, respectively, with

$$\iota^*(\tau \circ \sigma^{-1}) \quad \text{along} \quad \partial_+ C = \iota^{-1}(\partial_+ X) \qquad \text{and} \qquad \text{Id} \quad \text{on} \quad \partial_- C = \iota^{-1}(\partial_- X)$$

Now decompose $\partial_+ C$ into regions $S_i = \iota^{-1}(\gamma_i)$ and $T_i = \iota^{-1}(c_i)$. Then as a simple application of Lemma 88, we find that

$$\mu(\iota^*\xi, \iota^*T\Lambda; \iota^*(\sigma \cup \tau)) = \sum_i \deg(\iota|_{S_i}) \cdot \mu(\tau \circ \sigma^{-1}|_{S_i}) + \deg(\iota|_{T_i}) \cdot \mu(\tau \circ \sigma^{-1}|_{T_i})$$

This is precisely the difference $\tau - \sigma$ of the trivializations over Ξ. The same formula follows for (ι^*TX, ι^*TL) since it is isomorphic to the direct sum of $(\iota^*\xi, \iota^*T\Lambda)$ and a trivial bundle pair $(\mathbb{C}, \mathbb{R})$ (spanned by the $\mathbb{R}$-direction and Reeb direction in $Y \times [0, 1]$). $\square$

Proof. (Proposition 99) The union and composition properties follow immediately from the disjoint union and puncture gluing properties of the Maslov number (Proposition 84). Here we will argue the other two properties.

Well-Definedness and Maslov Class. We prove well-definedness and the Maslov class property together, since the argument is the same. Choose representatives of A and B in $S(\Xi_+, \Xi_-)$ respectively.

$$\iota : S \to X \qquad \text{and} \qquad \jmath : T \to X$$

Since S and T bound the same orbit sets Ξ_+ and Ξ_-, there exist trivial branched covers

$$\kappa_\bullet : C_\bullet \to [0,1] \times \Xi_\bullet \subset [0,1] \times Y_\bullet \qquad \text{for} \qquad \bullet \in \{+,-\}$$

such that κ_+ connects the positive ends of S and T and κ_- connects the negative ends of S and T. We can form a new surface

$$\Sigma = S \cup C_+ \cup C_- \cup -T$$

by gluing S and T to C_+ along their common orbit-chord ends, and likewise at C_- (see Figure 4.7). This surface inherits a continuous map

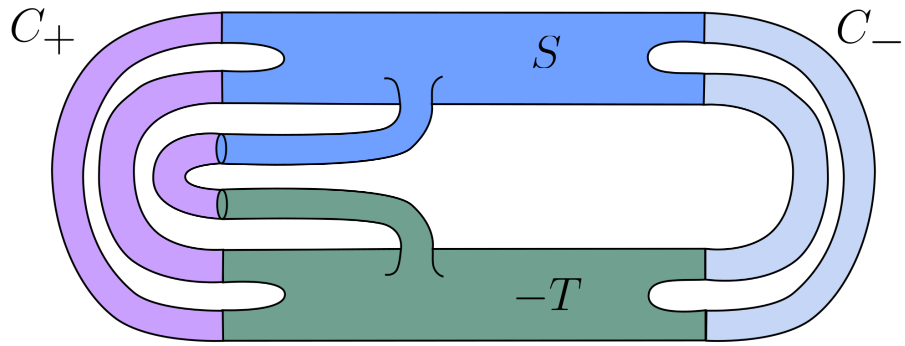


Figure 4.7: The surface Σ

$$\kappa : (\Sigma, \partial\Sigma) \to (X, L)$$

restricting to ι on S, to $\jmath$ on T and to the map $C_\pm \to \Xi_\pm$ on $C_\pm$. By the gluing property of the Maslov number in Proposition 84(d), we have

$$\mu(\kappa^*TX, \kappa^*TL) = \mu(\iota^*TX, \iota^*TL; \tau) - \mu(\jmath^*TX, \jmath^*TL; \tau) + \mu(TX|_{C_+}, F_{C_+}) + \mu(TX|_{C_-}, F_{C_-}) \tag{4.2.3}$$

The left hand side is simply $\mu(X, L) \cdot [\Sigma]$ where $\mu(X, L)$ is the Maslov class (see Example 87) and $[\Sigma] = A - B$. On the right hand side, the cylinder C_+ is equipped with the trivialization $\tau|_{\Xi_+}$ on both ends and likewise for C_-, so by Lemma 100

$$\mu(TX|_{C_+}, F_{C_+}) = 0 \qquad \text{and} \qquad \mu(TX|_{C_+}, F_{C_-}) = 0$$

Therefore, (a) follows from (4.2.3).

Trivialization. Fix a representative S of A as in (a) and let $C_\pm$ be the trivial (unbranched cover) of $\Xi_\pm \times [0, 1]$, as in Example 95. Equip $C_\pm$ with the trivialization $\rho_\pm$ with

$$\rho_\pm|_{\Xi_\pm \times 0} = \sigma|_{\Xi_\pm} \qquad \text{and} \qquad \rho_\pm|_{\Xi_\pm \times 1} = \tau|_{\Xi_\pm}$$

We may form a glued surface $T = S \cup C_+ \cup C_-$ and a map $\jmath : T \to X$ representing the class A by a similar process to (a). By applying the puncture gluing property Proposition 84(c), we have

$$\mu(A, \tau) = \mu(A, \sigma) + \mu(C_+, \rho_+) - \mu(C_-, \rho_-)$$

Therefore, the result follows from 100 since

$$\mu(C_+, \rho_+) = \mu(\sigma \circ \tau^{-1}|_{\Xi_+}\mathbb{R}) \qquad \text{and} \qquad \mu(C_-, \rho_-) = \mu(\sigma \circ \tau^{-1}|_{\Xi_-}\mathbb{R}) \qquad \square$$

We conclude this section by considering the Maslov number of a symplectic, well-immersed proper surface (ι, S) with class $A \in S(\Xi_+, \Xi_-)$. In this case, we have a decomposition

$$(TX, TL) = (T\Sigma, T(\partial_\circ \Sigma)) \oplus (\nu S, \nu(\partial_\circ S)) \tag{4.2.4}$$

Here νS is the symplectic perpendicular to $T\Sigma$ and $\nu(\partial S)$ is any choice of transverse sub-bundle to $T(\partial_\circ S)$ in TL.

Note that νS is canonically identified with $\xi_\pm$ near the positive and negative ends of (ι, S). In particular, a choice of trivialization τ over Ξ induces a puncture trivialization of (TX, TL) (also denoted τ) by direct sum with the canonical puncture trivialization of the tangent pair $(T\Sigma, T(\partial_\circ\Sigma))$. By the direct sum property of the Maslov number, we thus acquire

Lemma 101. Let (ι, S) be a symplectic, well-immersed proper surface. Then

$$\mu(S, \tau) = 2\bar{\chi}(S) + \mu(\nu S, \nu(\partial_\circ S); \tau)$$

Writhe and Linking

We next introduce the notion of the writhe and linking number of a braid, and of an admissible representative of a surface class.

To introduce these concepts, we must first clarify the notion of braid that we will use.

Definition 102. A *braid* ζ around a Reeb chord or orbit η in Y is a disjoint union of arcs (if η is a chord) or of loops (if η is an orbit) in a tubular neighborhood $\text{Nbhd}(\eta)$ of η such that

$$\zeta \text{ is transverse to } \xi \qquad \text{and} \qquad \partial\zeta \in \Lambda \text{ if } \eta \text{ is a chord}$$

Definition 103. Let S be a well immersed surface representative of a surface class $A \in S(\Xi_+, \Xi_-)$. Suppoose for each chord or orbit $\eta_i^\pm$ in the set Ξ_+, there is an associated (isotopy class of) braid

$$\zeta_i^\pm \quad \text{around} \quad \eta_i^\pm$$

constructed as follows. Choose a collar neighborhood of pairs

$$(Y_\pm \times [0,1), \Lambda \times [0,1)) \to (X, L) \qquad \text{near} \qquad \partial_\pm X$$

Then, under this identification, we define $\zeta_i^\pm$ to be the intersection

$$\zeta_i^\pm := S \cap (\text{Nbhd}(\eta) \times \{\epsilon\})$$

Here Nbhd(η) is a small tubular neighborhood disjoint from other orbits and chords in Ξ determined by a choice of trivialization of the contact structure along the orbit or chord:

$$\phi_\tau : \text{Nbhd}(\eta) \simeq [0,1] \times D^2 \qquad \text{or} \qquad \phi_\tau : \text{Nbhd}(\eta) \simeq S^1 \times D^2$$

In the chord case, ϕ_τ maps the boundary $\partial\eta$ to the x-axis $(\mathbb{R} \times 0) \cap D^2$ in $0 \times D^2$ and $1 \times D^2$. We take $\epsilon > 0$ to be small. Suppose the braids constructed as above does not depend on ϵ if it is small enough. Then we say the surface S is an admissible representative. We collectively refer to these braids as the *ends* of S.

Given a trivialization τ along Reeb orbits or chords, as in the above definition, this gives rise to a choice of tubular neighborhood around the chord or Reeb orbit of the form $[0,1] \times D^2$ (resp. $S^1 \times D^2$). Via the projection map $D^2 \to \mathbb{R}$ to the x-axis, we acquire projection maps

$$\pi : [0,1] \times D^2 \to [0,1] \times \mathbb{R} \qquad \text{and} \qquad \pi : S^1 \times D^2 \to S^1 \times \mathbb{R}$$

Definition 104 (Writhe)**.** The *writhe* $w_\tau(\zeta) \in \mathbb{Z}$ of a braid ζ around η with respect to a trivialization τ is the signed count of self-intersections of the curve

$$\pi \circ \phi_\tau(\widetilde{\zeta})$$

Here $\widetilde{\zeta}$ is a perturbation of ζ relative to $\partial\zeta$ such that $\pi \circ \phi_\tau(\zeta')$ has only transverse double points. The sign convention in ECH is that anticlockwise rotations in the disk D^2 contributes a positive crossing. This is described, for instance after Definition 2.7 in [55]

Likewise, the *writhe* $w_\tau(S)$ of an admissible representative S of a surface class A with respect to a trivialization $\tau \in \mathcal{T}(\Xi_+, \Xi_-)$ is the signed sum of writhes of the ends.

$$w_\tau(S) := \sum_i w_\tau(\zeta_i^+) - \sum_j w_\tau(\zeta_j^-) \tag{4.2.5}$$

Definition 105. The *linking number* $l_\tau(\zeta,\zeta') \in \frac{1}{2}\mathbb{Z}$ of a pair of disjoint braids ζ and ζ' around η is half of the signed count of intersections between the pair of curves

$$\pi \circ \phi_\tau(\zeta) \qquad \text{and} \qquad \pi \circ \phi_\tau(\zeta')$$

The *linking number* $l_\tau(S,S')$ of a pair of disjoint, admissible representatives of two surface classes A and B is the analogous signed sum of linking numbers to (4.2.5).

We will require a few elementary properties of the linking number and the writhe.

Lemma 106. The linking number and the writhe satisfy the following properties.

- (Union) The linking number and writhe are related by the formula

$$w_\tau(\zeta \cup \zeta') = w_\tau(\zeta) + w_\tau(\zeta') + 2 \cdot l_\tau(\zeta,\zeta') \tag{4.2.6}$$

- (Trivialization) If σ and τ are two trivializations of ξ along η then

$$w_\tau(\zeta) - w_\sigma(\zeta) = m(m-1)\cdot(\tau - \sigma)$$

  where m is the number of strands in the braid ζ.

Classical Intersection Pairing

We next consider a half-integer valued intersection pairing associated to a pair of a 4-manifold with a 2-manifold in its boundary. This is analogous to the intersection pairing of closed 4-manifolds.

Definition 107 (Intersection)**.** Let M be a compact 4-manifold with corners and let $N \subset \partial M$ be an oriented embedded 2-manifold with boundary. The *intersection pairing*

$$Q : H_2(M,N) \otimes H_2(M,N) \to \frac{1}{2}\mathbb{Z}$$

is defined as follows. Given classes $A, B \in H_2(M,N)$, choose immersed representatives

$$\iota : (S,\partial S) \to (M,N) \qquad \text{and} \qquad \jmath : (T,\partial T) \to (M,N)$$

of A and B, respectively, that intersect transversely (including along N). Then each intersection point $p \in S \cap T$ is isolated and has a well-defined sign $\iota(M,S,T,p) = \pm 1$ (cf. [121]). We let

$$\#_M(\text{int}(S)\cap\text{int}(T)) = \sum_{p\in\text{int}(S)\cap\text{int}(T)} \iota(M,S,T,p) \qquad \text{and} \qquad \#_M(\partial S\cap\partial T) = \sum_{p\in\partial S\cap\partial T} \iota(M,S,T,p)$$

The boundaries ∂S and ∂T are also transverse as sub-manifolds of N. Thus, we have well-defined intersect numbers in N for each $p \in \partial S \cap \partial T$, denoted

$$\iota(N, \partial S, \partial T, p) = \pm 1 \qquad \text{and} \qquad \#_N(\partial S \cap \partial T) = \sum_{p \in \partial S \cap \partial T} \iota(N, \partial S, \partial T, p)$$

After an isotopy of S (or T) that fixes ∂S, we may assume that

$$\iota(M, S, T, p) = \iota(N, \partial S, \partial T, p) \qquad \text{for each} \qquad p \in S \cap T \tag{4.2.7}$$

Note that this isotopy may introduce intersections in the interior of S and T, but we may perturb S and T to be transverse to each other. We now define the intersection pairing Q as follows.

$$Q(A, B) := \#_M(\text{int}(S) \cap \text{int}(T)) + \frac{1}{2} \cdot \#_M(\partial S \cap \partial T) \tag{4.2.8}$$

In general, the intersection numbers $\#_N(\partial S \cap \partial T)$ and $\#_M(\partial S \cap \partial T)$ may not agree. However, we can always isotope S so that this is the case.

Lemma 108. Let $p \in \partial S \cap \partial T$ be a boundary intersection and assume that

$$\iota(M, S, T, p) = -\iota(N, \partial S, \partial T, p)$$

Then there is a small neighborhood U of p and a surface S' isotopic to S such that

$$\partial S' = \partial S \qquad S = S' \text{ on } M \setminus U \qquad S' \cap T \cap U = \{p, q\}$$

Here the boundary intersection point p has intersection numbers

$$\iota(N, \partial S', \partial T, p) = -\iota(N, \partial S, \partial T, p)$$

and q is an interior intersection point such that

$$\iota(M, S', T, q) = \iota(M, S, T, p)$$

Proof. By choosing coordinates and (possibly) reversing the orientation of T, we can reduce to the local picture where M and N are given by

$$M := \{(z_1, z_2) \,:\, \text{Im}(z_1 + z_2) \geq 0\} \subset \mathbb{C}^2 \quad \text{and} \quad N = \mathbb{R}^2$$

Moreover, we may assume that S and T take the form

$$S := (\mathbb{C} \oplus 0) \cap M \qquad \text{and} \qquad T := (0 \oplus \mathbb{C}) \cap M$$

Now consider the sub-space

$$S'' := \mathbb{R} \oplus i\mathbb{R} \cap M \qquad \text{and} \qquad T'' := i\mathbb{R} \oplus \mathbb{R} \cap M$$

Note that $S'' \cap T'' = S \cap T = (0,0)$ and we have

$$\#_M(S \cap T) = -\#_M(S'' \cap T'') = 1 \qquad \text{and} \qquad \#_N(S \cap T) = \#_N(S'' \cap T'') = 1$$

Now note that there is a pair of braids in the hemisphere $S^3 \cap M$ given by

$$B = (S \cup T) \cap S^3 \qquad \text{and} \qquad B'' = (S'' \cup T'') \cap S^3.$$

We can define homotopies of the components S_t from S to S'' and T_t from T to T'' as follows.

$$S_t = \text{span}_{\mathbb{R}}(1 \oplus 0, v_t) \cap S^3 \cap M \qquad \text{where} \qquad v_t = \cos(\frac{\pi t}{2}) \cdot (i \oplus 0) + \sin(\frac{\pi t}{2}) \cdot (0 \oplus i)$$

$$T_t = \text{span}_{\mathbb{R}}(0 \oplus 1, w_t) \cap S^3 \cap M \qquad \text{where} \qquad w_t = \cos(\frac{\pi t}{2}) \cdot (0 \oplus i) + \sin(\frac{\pi t}{2}) \cdot (i \oplus 0)$$

Note that S_t and T_t are disjoint, except at $t = \frac{1}{2}$. We can use S_t and T_t to form a pair of surfaces

$$\Sigma(S) = \{(2-t) \cdot z \ : \ 0 \le t \le 1 \text{ and } z \in S_t\}$$
$$\Sigma(T) = \{(2-t) \cdot z \ : \ 0 \le t \le 1 \text{ and } z \in T_t\}$$

These surfaces intersect at one point with sign -1. We now let S' and T' be, respectively, smoothings of the C^0 embedded surfaces

$$(S \cap B^3(1)) \cup \Sigma(S) \cup (S'' \setminus B^3(2)) \qquad \text{and} \qquad (T \cap B^3(1)) \cup \Sigma(S) \cup (T'' \setminus B^3(2))$$

These surfaces are smooth except along some curves that are disjoint from their intersections, so the intersections of S' and T' are given by

$$\text{int}(S') \cap \text{int}(T') = S'' \cap T'' = \{0\} \qquad \partial S' \cap \partial T' = \Sigma(S) \cap \Sigma(T) = \{\frac{3i}{2} \cdot (1,1)\}$$

This is precisely the local picture required in the lemma, so we are done. □

Corollary 109. There are isotopic surfaces S' and T' to S and T respectively such that

$$Q(S',T') = Q(S,T) \qquad \text{and} \qquad \#_M(\partial S \cap \partial T) = \#_N(\partial S \cap T\partial T)$$

Proposition 110. The intersection pairing Q in Definition 107 is well-defined.

Proof. Fix classes $A, B \in H_2(M,N)$. Choose immersed representatives

$$\iota : (S, \partial S) \to (M,N) \qquad \iota' : (S', \partial S') \to (M,N) \quad \text{and} \quad \jmath : (T, \partial T) \to (M,N)$$

of the classes A, A and B, respectively, such that S and S' are both transverse to T. Moreover, we may assume by Corollary 109 that

$$\#_M(S \cap T) = \#_N(S \cap T) \qquad \text{and} \qquad \#_M(S' \cap T) = \#_N(S' \cap T)$$

To prove well-definedness, it now suffices to show that

$$Q(S,T) - Q(S',T) = 0 \tag{4.2.9}$$

To prove (4.2.9), we must consider the standard short exact sequence of the pair (M,N).

$$H_2(N) \to H_2(M) \to H_2(M,N) \to H_1(N) \tag{4.2.10}$$

By hypothesis, we have an immersion of $S \cup -S'$ in the homology class $\iota_*[S] - \jmath_*[S'] = 0$ in $H_2(M,N)$. Therefore, $\partial S \cup -\partial S'$ is an immersed null-homologous 1-manifold in N, and

$$\#((\partial S \cup -\partial S') \cap \partial T) = 0 \tag{4.2.11}$$

Furthermore, we may choose a map $\kappa : R \to N$ from a compact surface R bounding $\partial S \cup -\partial S'$, and acquire an immersion of a closed surface.

$$f : \Sigma = S \cup R \cup -S' \to X$$

The class $[\Sigma] \in H_2(M)$ maps to $[S \cup -S] = 0 \in H_2(M,N)$ under the map $H_2(M) \to H_2(M,N)$ in (4.2.10). Thus by the exactness of (4.2.10), we know that $[\Sigma \cup Z] = 0$ for some closed immersed surface $Z \to N$. We may assume that $[\Sigma] = 0$ by absorbing Z into the choice of bounding surface R.

Now choose a collar neighborhood $N \times (-1,1)_t$ of N in ∂M and extend this to a collar into M as $U = N \times (-1,1)_t \times [0,1)_s$. By choosing this collar to be very small, we can assume that

$$U \cap (f(\Sigma) \cap \jmath(T)) = \jmath(T) \cap (R \times 0 \times 0) \subset N \times 0 \times 0$$

We can choose a vector-field $v = \phi(s) \cdot \partial_t$ where $\phi : [0,1) \to [0,1)$ is a non-negative function with $\phi = 0$ near $s = 1$. By flowing Σ along v for a small amount of time, we perturb f to a new map f' such that

$$f'(\Sigma) \cap T \cap U = \emptyset$$

Finally, we can smooth f' to a smooth immersion $\phi : \Sigma \to \text{int}(X)$ agreeing with f' away from a neighborhood of ∂X that satisfies

$$\#(\Sigma \cap T) = \#(S \cap T) - \#(S' \cap T) = Q(S,T) - Q(S',T)$$

We now simply observe that since $[\Sigma] = [S] - [S'] = 0$, we have

$$Q(S,T) - Q(S',T) = \#(\Sigma \cap T) = \text{PD}[T] \cdot [\Sigma] = 0 \text{ where } [T] \in H_2(X, \partial X) \qquad \square$$

There is also an analogue of the intersection pairing for a 3-manifold with boundary, equipped with a 1-manifold in the boundary.

Definition 111. Let Y be a compact 3-manifold with boundary and corners, and let $Z \subset \partial Y$ be a closed 1-manifold in ∂Y. The *intersection pairing*

$$Q : H_1(Y,Z) \otimes H_2(Y,Z) \to \frac{1}{2}\mathbb{Z}$$

is defined as follows. Consider the 4-manifold

$$M = [0,1] \times Y \quad \text{with sub-manifold} \quad N = [0,1] \times Z \subset \partial X$$

Let $S = [0,1] \times \Gamma$ where $\Gamma \subset Y$ is a 1-manifold representing a class $A \in H_1(Y,Z)$ and let T be a 2-manifold representing B in $H_2(M,N) \simeq H_2(Y,Z)$ contained in $s \times Y$ for $s \in (0,1)$, and intersecting Γ transversely.

$$Q(A,B) := \#_M(\text{int}(S) \cap \text{int}(T)) + \frac{1}{2} \cdot \#(\partial S \cap \partial T)$$

The proof of well-definedness is analogous to Proposition 110.

Relative Intersection Number

Next, we generalize the classical intersection number to an intersection number for surface classes.

Definition 112. Let (X, L) be a pair of a symplectic cobordism with boundary and a Lagrangian cobordism $L \subset X$, and let $A \in S(\Xi, \Theta)$ be a surface class. The associated *intersection map*

$$q_A : H_2(X, L) \to \frac{1}{2}\mathbb{Z} \qquad \text{given by} \qquad B \mapsto q_A(B)$$

is defined as the same count of intersections in (4.2.8) where S is a representative of the surface class A and T is a compact surface with boundary in the interior of X representing B.

The proof of well-definedness is analogous to Proposition 110. Moreover, we can prove the following lemma.

Lemma 113. Consider a trivial symplectic cobordism pair (X, L) of the form

$$([0, 1] \times Y, [0, 1] \times \Lambda) \qquad \text{for a contact manifold } Y \text{ with Legendrian } \Lambda \subset \partial Y$$

Let $A \in S(\Xi, \Theta)$ be any surface class between orbit-chord sets in 1-homology class $\Gamma = [\Xi] = [\Theta]$. Then

$$q_A(B) = Q(\Gamma, B) \qquad \text{for any} \qquad B \in H_2(X, L) \simeq H_2(Y, \Lambda)$$

Proof. Choose a representative surface $S \subset [0, 1] \times Y$ of the surface class A that is a cylinder $[0, \epsilon) \times \eta$ over a 1-manifold η representing Γ in $[0, \epsilon) \times Y$. We may choose an immersed 2-manifold $T \subset Y$ with boundary $\partial T \subset \Lambda$ transverse to η. Then the corresponding count of intersections computes both $q_A(B)$ and $Q(\Gamma, B)$. □

More generally, we define the intersection number between two surface classes as follows.

Definition 114 (Relative Intersection)**.** Fix a pair of surface classes

$$A : \Xi \to \Theta \qquad \text{and} \qquad B : \Xi' \to \Theta'$$

and a trivialization τ of ξ along $\Xi \cup \Xi'$ and $\Theta \cup \Theta'$. The *relative intersection pairing* of A and B with respect to τ is the half-integer

$$Q_\tau(A, B) \in \frac{1}{2}\mathbb{Z}$$

is defined as follows. Pick admissible surfaces S and T representing A and B, respectively.

$$\iota : S \to X \qquad \text{and} \qquad \jmath : T \to X$$

Assume that S and T are disjoint near $\partial_\pm X$ (except along $\partial_\pm X$) and transversely intersecting away from $\partial_+ X \cup \partial_- X$. Then we let

$$Q_\tau(A, B) := \#(\mathrm{int}(S) \cap \mathrm{int}(T)) + \frac{1}{2} \cdot \#(\partial_\circ S \cap \partial_\circ T) - l_\tau(S, T) \tag{4.2.12}$$

The relative intersection number satisfies a number of very useful axioms. We now prove these axioms in detail.

Proposition 115. The relative intersection pairing Q_τ is well-defined and has the following properties.

- (Trivialization) Let $A : \Xi_+ \to \Xi_-$ and $B : \Theta_+ \to \Theta_-$ be two surface classes and let σ and τ be two trivializations that differ only along one orbit or chord

  $$\eta \quad \text{of multiplicity } m \text{ in } \Xi \text{ and } n \text{ in } \Theta$$

  Then the self-intersection numbers of σ and τ differ as follows.

  $$Q_\tau(A, B) - Q_\sigma(A, B) = m \cdot n \cdot (\tau - \sigma)$$

- (Difference) Let $A, A' : \Xi_+ \to \Xi_-$ and $B \in S(\Psi_+, \Psi_-)$ be surface classes between the same orbit-chord sets. Then

  $$Q_\tau(A, B) - Q_\tau(A', B) = q_B(A - A')$$

- (Union) Let A, A' and B be surface classes. Then

  $$Q_\tau(A \cup A', B) = Q_\tau(A, B) + Q_\tau(A', B)$$

- (Composition) Let $A, A' : \Xi_0 \to \Xi_1$ and $B, B' : \Xi_1 \to \Xi_2$ be two pairs of composible surface classes in (X, L) and (X', L') respectively. Then

  $$Q_\tau(A \circ B, A' \circ B') = Q_\tau(A, A') + Q_\tau(B, B')$$

In order to prove these axioms, we will need the following lemma.

Lemma 116. Let η be a Reeb chord or orbit. Let ζ^+, ζ^- and β be disjoint braids such that ζ^+ and ζ^- have the same degree. Then there exists a symplectically immersed cobordism Z from ζ^+ to ζ^- such that

$$\#([0,1] \times \beta \cap Z) + \frac{1}{2}\#([0,1] \times \partial\beta \cap \partial_\circ Z) = l_\tau(\zeta^+, \beta) - l_\tau(\zeta^-, \beta)$$

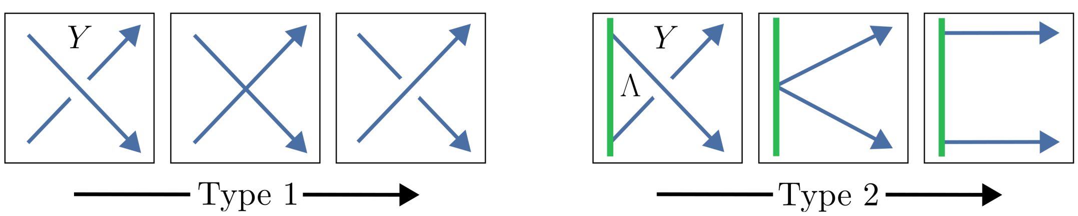
Proof. We prove the result for braids near a Reeb chord, as the Reeb orbit case can be treated identically. It suffices to consider the following setup. Let

$$Y := [0,1]_t \times D^2 \qquad \text{and} \qquad \Lambda := 0 \times I \cup 1 \times I$$

Here $I := (\mathbb{R} \times 0) \cap D^2$ is the segment of the x-axis on the disk. We consider the braid diagram given by the projection onto the (x,t)-plane.

$$\pi : Y \subset \mathbb{R}^3 \to \mathbb{R}^2$$

Recall that any two braids of the same degree can be related by a series of braid isotopies and the following crossing changes (and the reverse moves).



We call these Type 1 and Type 2 moves. A sequence of braid isotopies, Type 1 and Type 2 moves can be viewed as a regular homotopy and thus lifted to an immersed cobordism

$$\iota : \Sigma \to [0,1]_s \times Y$$

from the initial braid at $s = 0$ to the final braid at $s = 1$. This immersed cobordism is symplectic as in Lemma 2.4 of Golla-Etnyre [37]. A Type 1 move yields a transverse interior self-intersection of Σ with sign -1 and a Type 2 move yields a transverse boundary self-intersection of Σ with sign -1. The reverse moves yield sign $+1$.

Now choose a sequence of isotopies and Type 1/2 moves from $\zeta^+ \cup \beta$ to $\zeta^- \cup \beta$. We may assume that these moves leave β fixed, so that the corresponding cobordism Σ decomposes as

$$\Sigma = Z \cup [0,1] \times \beta$$

Here Z is an immersed cobordism from ζ^+ to ζ^- induced by a regular homotopy of braids ζ_t and intersections between Z and $[0,1] \times \beta$ correspond to Type 1 and 2 moves between the braids ζ_t and β. A Type 1 move adds -1 intersections and changes the linking number by -1. A Type 2 move adds $-1/2$ intersections and changes the linking number by $-1/2$. This yields the result. □

Proof. (Proposition 115) We demonstrate each property individually. Note that the argument for the difference property suffices to prove well-definedness, since the latter follows from the same argument by taking $A = A'$.

Trivialization. This follows immediately from the corresponding transformation law for the linking number.

Well-Definedness And Difference. Let T be an admissible representative of a class B in $S(\Psi_+, \Psi_-)$ and let

$$\iota : S \to X \qquad \text{and} \qquad \iota' : S' \to X$$

be two admissible representatives of classes A and A' in $S(\Xi_+, \Xi_-)$, respectively, that satisfy the transversality conditions in Definition 114 with respect to T. We construct an immersion $\phi : (\Sigma, \partial\Sigma) \to (X, L)$ transverse to T such that

$$[\Sigma] = A - A' \in H_2(X, L) \quad \text{and} \quad \#(\Sigma \cap T) + \frac{1}{2}\#(\partial\Sigma \cap \partial T) = Q_\tau(S, T) - Q_\tau(S', T) \tag{4.2.13}$$

The result will then follow since $q_B([\Sigma])$ is precisely given by $\#(\Sigma \cap T) + \frac{1}{2}\#(\partial\Sigma \cap \partial T)$.

To begin the construction, fix the following notation for a collar near $\partial_\pm X$.

$$V_\pm := \partial_\pm X \times [0, \mp\epsilon] \subset X \quad \text{and} \quad U = \overline{X \setminus (V_+ \cup V_-)} \text{ with } \partial_\pm U := \partial_\pm X \times \{\mp\epsilon\}$$

Here ϵ is an arbitrarily small parameter. Note that $\iota^{-1}(U) \subset S$ and $[\iota']^{-1}(U) \subset S'$ are equal to S and S' minus collars near $\partial_\pm S$ and $\partial_\pm S'$, respectively. The underlying surface Σ is of the form

$$\Sigma = \iota^{-1}(U) \cup Z_+ \cup Z_- \cup -[\iota']^{-1}(U)$$

Here $Z_\pm$ is a surface with boundary and corners that we will specify shortly.

To define the immersion ϕ (and in the process, $Z_\pm$), we proceed as follows. First, we let

$$\phi = \iota \text{ on } \iota^{-1}(U) \qquad \text{and} \qquad \phi = \iota' \text{ on } [\iota']^{-1}(U) \tag{4.2.14}$$

Note that the image of ϕ along $\partial_\pm U$ consists of the braids at the positive and negative ends of S and S', and the remaining boundary lies in L. Next, we assert that $\phi(Z_\pm)$ is contained in a union of disjoint tubular neighborhoods of the orbits and chords in $\Xi_\pm$.

$$\phi(Z_\pm) \subset \bigsqcup_i \text{Nbhd}(\eta_i^\pm) \times [0, \mp\epsilon] \subset V_\pm \tag{4.2.15}$$

To describe ϕ in each neighborhood, fix an chord or orbit $\eta = \eta_i^\pm$ in $\Xi_\pm$ and a tubular neighborhood $N = \text{Nbhd}(\eta_i^\pm)$ of η in $Y_\pm$. Denote the braids of S, S' and T around $\eta_i^\pm$ by ζ, ζ' and β respectively. Note that the braid β will be empty if η is not in the orbit set Ψ. For definiteness let's focus on what happens on V_+. By Lemma 116, we may choose a cobordism Z in $N \times [0, -\epsilon)$ from ζ to ζ' with

$$\#(\text{int}(Z) \cap \beta \times [0, -\epsilon]) + \frac{1}{2} \cdot \#(\partial_\circ Z \cap \beta \times [0, \epsilon]) = l_\tau(\zeta, \beta) - l_\tau(\zeta', \beta)$$

The definition for V_- is similar. We thus may associate a surface $Z_i^\pm$ to the end chord or orbit $\eta_i^\pm$, namely

$$Z_i^\pm = Z \cup \zeta'_\pm \times [0, \epsilon]$$

We may view $Z_i^\pm$ as a smooth surface with boundary and corners that is topologically embedded into $V_\pm$ and meets $\iota^{-1}(U)$ and $[\iota']^{-1}(U)$ along $\partial_\pm U$. We then let

$$Z_\pm := \bigsqcup_i Z_i^\pm \tag{4.2.16}$$

and we let $\phi|_{Z_\pm}$ be given by a smoothing of the tautological map $Z_\pm \to V_\pm$. Note that since the asymptotic braids β and ζ' are disjoint, we have

$$\#((Z_+ \cup Z_-) \cap T) = l_\tau(\zeta, \beta) - l_\tau(\zeta', \beta) \tag{4.2.17}$$

The map $\phi : \Sigma \to \mathbb{Z}$ is smoothly immersed and satisfies the criteria in (4.2.13) by construction. This concludes the proof of well-definedness and the difference property.

Union. We can choose well-immersed representatives S, S' and T of A, A' and B respectively so that S, S' and $S \cup S'$ are transverse to T away from $\partial_\pm X$, disjoint from T near (but not along) $\partial_\pm X$ and transverse to $\partial_\pm X$. Then

$$Q_\tau(A \cup A', B) = \#(\text{int}(S \cup S') \cap \text{int}(T)) + \frac{1}{2} \cdot \#(\partial_\circ(S \cup S') \cap \partial_\circ T) - l_\tau(S \cup S', T)$$

$$= Q_\tau(A, B) + Q_\tau(A', B)$$

Composition. Choose representatives S, S', T and T' of A, A', B and B', respectively. Via Lemma 116, we may assume that the braid of S at the positive end agrees with the braid of S' at the negative end, and likewise for T and T'. Then

$$S \cup T \qquad \text{and} \qquad S' \cup T'$$

are representatives in $(X \circ X', L \circ L')$ of $A \circ B$ and $A' \circ B'$, respectively. We can smooth both to surfaces U and V so that near $\partial_- X = \partial_+ X'$, they agree with the negative braid of $S \cup S'$ (or equivalently, the positive braid of $T \cup T'$). Then we have

$$\#(\text{int}(U) \cap \text{int}(V) = \#(\text{int}(S) \cap \text{int}(S')) + \#(\text{int}(T) \cap \text{int}(T'))$$

$$\#(\partial_\circ(U) \cap \partial_\circ(V) = \#(\partial_\circ(S) \cap \partial_\circ(S')) + \#(\partial_\circ(T) \cap \partial_\circ(T'))$$

$$l_\tau(U, V) = l_\tau(S, S') + l_\tau(T, T')$$

Note that the last formula above involves the cancelation of the linking of the negative ends of S and S' with the linking at the positive ends of T and T'. This proves the result. □

Topological Adjunction

We are now ready to prove a topological version of Legendrian adjunction. The holomorphic curve version will be proven in Section 4.3 after the appropriate discussion of background.

Theorem 117 (Topological Adjunction)**.** *Let $S : \Xi \to \Theta$ be a symplectic, well-immersed surface in a symplectic cobordism (X, L) with a trivialization τ along ∂S. Then S satisfies the adjunction formula*

$$\mu(S, \tau) = 2(\bar{\chi}(A) + Q_\tau(S) + w_\tau(S) - 2\delta(S) - \epsilon(S))$$

If S is simply smooth (and not necessarily symplectic) then the adjunction formula holds mod 2.

Proof. By Lemma 101, we simply need to show that

$$\mu_\tau(\nu S, \nu(\partial_\circ S)) = 2(Q_\tau(S, S) + w_\tau(S) - 2\delta(S) - \epsilon(S)) \tag{4.2.18}$$

where $\nu(\partial_\circ S)$ is the normal bundle of $\partial_\circ S \subset L$ as before. To prove (4.2.18), choose a section

$$\phi : (S, \partial S) \to (\nu S, \nu(\partial_\circ S)) \qquad \text{satisfying} \qquad \phi|_{\partial_\pm S} \neq 0 \qquad \phi \pitchfork \partial_\circ \Sigma \quad \text{and} \quad \phi \pitchfork \text{int}(\Sigma)$$

Also assume that, near the Reeb chords and orbits in ∂S, the section ϕ is constant with respect to our chosen trivialization τ. By Lemma 89, the Maslov index is given by the following count of intersections

$$\mu_\tau(\nu S, \nu(\partial_\circ S)) = 2 \cdot \#(\phi \cap \text{int}(S)) + \#(\phi \cap \partial_\circ S). \tag{4.2.19}$$

On the other hand, let S' be the perturbation of S by the section ϕ in a neighborhood of S. In both $\text{int}(S)$ and $\partial_\circ S$, there is a single transverse intersection in $S \cap S'$ for each 0 of ϕ and two intersections in $S \cap S'$ each transverse double point. Thus

$$\#(\phi \cap \text{int}(S)) = \#(\text{int}(S') \cap \text{int}(S)) - 2\delta(S) \tag{4.2.20}$$

$$\#(\phi \cap \partial_\circ S) = \#(\partial_\circ S' \cap \partial_\circ S) - 2\epsilon(S) \tag{4.2.21}$$

Finally, we note that by Definition 114, we know that

$$Q_\tau(S, S) + w_\tau(S) = Q_\tau(S, S) + l_\tau(S, S') = \#(\phi \cap \text{int}(S)) + \frac{1}{2} \cdot \#(\partial_\circ S' \cap \partial_\circ S) \tag{4.2.22}$$

Combining (4.2.19)-(4.2.22) proves (4.2.18), and thus the proposition.

If S is simply a smooth, well-immersed surface then we still have a decomposition of oriented, real bundle pairs

$$(TX, TL) = (TS, T(\partial_\circ S)) \oplus (\nu S, \nu(\partial_\circ S))$$

After isotopy of S near the ends, we may assume that this splitting respects the symplectic structure near ∂S. Modifying the symplectic stricture on $TX|_S$ away from ∂S is equivalent to direct summing with a symplectic bundle on S^2, and thus changes $\mu(TX, TL)$ by an even integer. Therefore

$$\mu_\tau(TX|_S, TL|_{\partial S}) = \mu_\tau(TS, T(\partial_\circ S)) + \mu(\nu S, \nu(\partial_\circ S))$$

The proof now reduces to (4.2.18), and proceeds by essentially the same argument. □

4.3 J-Holomorphic Curves with Boundary

In this section, we examine holomorphic curves with Legendrian boundary conditions in convex sutured contact manifolds.

Sutured Contact Manifolds

We stary by reviewing contact manifolds with sutured boundary, and the appropriate classes of contact forms and complex structures.

Remark 118. Our setting is most similar to [28]. We will only state our definitions for 3- manifolds, but we note many of our definitions have higher dimensional versions, as found in [28].

Definition 119. A *sutured 3-manifold* Y is a compact 3-manifold Y with boundary and corners, a closed sub-manifold $\Gamma \subset \partial Y$ of codimension 2 and a neighborhood of Γ of the form

$$V(\Gamma) \simeq [-1,1]_t \times (-\epsilon, 0]_\tau \times \Gamma \qquad \text{with} \qquad \Gamma \simeq 0 \times 0 \times \Gamma \tag{4.3.1}$$

The boundary ∂Y must divide into smooth strata $\partial_- Y, \partial_\sigma Y$ and $\partial_+ Y$ such that, in the chart (4.3.1)

$$\partial_\sigma Y \simeq [-1,1] \times 0 \times \Gamma \qquad \text{and} \qquad \partial_\pm Y \cap V(\Gamma) \simeq \pm 1 \times (-\epsilon, 0] \times \Gamma$$

Definition 120. [28, §2] A contact form α on a sutured manifold Y is *adapted* to Y if

(a) $\alpha|_{\partial_\pm Y}$ is a Liouville form on $\partial_\pm Y$.

(b) In the neighborhood $V(\Gamma)$ in (4.3.1), α is given by

$$\alpha = C \cdot dt + e^\tau \cdot \beta$$

where $C > 0$ is a constant and β is a one-form on Γ independent of t and has no dt-term. Consequently in this neighborhood the Reeb vector field is given by $\frac{1}{C}\partial_t$.

Remark 121. When Y is equipped with an adapted contact form, we may extend t to a function

$$t : \mathrm{Nbhd}(\partial Y) \to \mathbb{R}$$

on a neighborhood of ∂Y by setting $t(\partial_\pm Y) = \pm 1$ and then extending t to be Reeb invariant near $\partial_\pm Y$. We will regularly use this extension without further comment.

Definition 122. A *(convex) sutured contact manifold* (Y, ξ) is a sutured manifold Y and a contact structure ξ on Y that is the kernel of an adapted contact form.

Definition 123. A collection of Legendrians Λ in the horizontal boundary of the convex sutured convex contact manifold is called *exact* if the Liouville form $\beta_\pm$ vanishes on Λ.

Consequently on the exact Legendrians the contact distribution ξ is tangent to the horizontal boundary.

Remark 124. We give some examples where one can find exact Legendrians. It suffices to find a two dimensional Liouville manifold (Σ, β) where β is the Liouville form, and a collection of Lagrangians $\{L_i\}$ on which β vanishes. The simplest possible example is (D^*S^1, β), the codisk bundle of S^1; and β is the canonical one form. The Lagrangian in question is then the zero section.

More generally speaking, let $\{L_i\}$ denote a collection of circles. Consider the Liouville manifolds (D^*L_i, β_i) as above. We attach one handles between this collection of Liouville manifolds to form the Livioulle manifold (S, β). Then the collection of curves $\{L_i\}$ can be taken to be the required collection of Lagrangians.

Definition 125. [28, §3.1] A complex structure J on ξ that is *tailored* to (Y, Λ) if

(a) J is Reeb invariant near ∂Y.

(b) Consider the completion of $\partial_\pm Y$ near $V(\Gamma)$ given by $\partial_\pm Y \cup [0, \infty) \times \Gamma$, with Liouville form $e^\tau \beta$ in a neighborhood of the form $(-\epsilon, \infty) \times \{\pm 1\} \times \Gamma$. We require that in both $V(\Gamma)$ and small tubular neighborhoods $\partial_+ Y \times [1, 1-\epsilon]$ and $\partial_- Y \times [-1, -1+\epsilon]$, the almost complex structure J is the pullback via the natural projection $\pi : \xi \to T(\partial_\pm Y)$ of a complex structure J_0 that is β compatible on $(-\epsilon, \infty) \times \Gamma$, and compatible with $\alpha|_{\partial_\pm Y}$ on $\partial_\pm Y$.

We let $\widehat{J}$ denote the complex structure induced on $\mathbb{R} \times Y = \mathbb{R}_s \times Y$, given by

$$\widehat{J}(\partial_s) = R \qquad \widehat{J}|_\xi \text{ is a tailored complex structure } J.$$

Remark 126. Given a sutured contact manifold, it is also helpful to think about its completion as in Section 2.4 in [28]. First, "vertically" complete $V(\Gamma)$ by gluing $[1,\infty) \times \partial_+ Y$ and $(-\infty,-1] \times \partial_- Y$ with the forms $Cdt + \alpha|_{\partial_+ Y}$ and $Cdt + \alpha|_{\partial_- Y}$ respectively. Now the boundary is $\{0\} \times \mathbb{R} \times \Gamma$. Second, "horizontally" complete by gluing $[0,\infty) \times \mathbb{R} \times \Gamma$ with the form $Cdt + e^\tau \beta$. We denote the completion as (M^*, α^*).

Holomorphic Maps

We now recall the basic definitions regarding J-holomorphic maps in SFT, as required in this paper.

Let (Σ, j) be a Riemann surface with boundary and corners, and assume that Σ is equipped with a boundary decomposition into smooth components

$$\partial\Sigma = \partial_+\Sigma \cup \partial_\circ\Sigma \cup \partial_-\Sigma$$

We think of of $\partial_\circ\Sigma$ as boundaries of the surface. For $\partial_\pm\Sigma$, we think of Σ as surface with boundary/interior punctures, and each near boundary puncture we equip the puncture with a semi-infinite strip-like (resp. cylindrical) neighborhood $[0,\pm\infty] \times [0,1]$ (resp. $[0,\infty] \times S^1$, and we think of $\partial_\pm\Sigma$ as the components of the boundary at infinity, of the form $\pm\infty \times [0,1]$ (resp. $\pm\infty \times S^1$).

Let (Y,ξ) be a convex sutured contact manifold with closed Legendrians $\Lambda \subset \partial Y_\pm$ and fix an adapted contact form α and tailored complex structure J. A *J-holomorphic map*

$$u : (\Sigma, \partial_\circ\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

is a smooth map u from $\Sigma \setminus (\partial_+\Sigma \cup \partial_-\Sigma)$ to the symplectization $\mathbb{R} \times Y = \mathbb{R} \times Y$ that maps $\partial_\circ\Sigma$ to $\mathbb{R} \times \Lambda = \mathbb{R} \times \Lambda$, and that satisfies the non-linear Cauchy-Riemann equations

$$du \circ j = \widehat{J} \circ du. \tag{4.3.2}$$

A connected component C of $\partial_+\Sigma \cup \partial_-\Sigma$ is called a *puncture* of u. A puncture is *positive* if it is in $\partial_+\Sigma$ and *negative* if it is in $\partial_-\Sigma$. Likewise, a closed component is an *interior* puncture and a component with boundary is a *boundary* puncture. We require positive punctures under the map u are at $+\infty$ of the symplectization direction, and negative punctures map to $-\infty$ in the symplectization direction.

The *energy* of a J-holomorphic curve u is given by

$$E(u) = \sup_\phi \Big(\int_\Sigma u^* d(\phi(s)\alpha) \Big) \in [0,\infty)$$

Here the supremum is over all non-decreasing smooth maps $\phi : \mathbb{R} \to [0,1]$.

Local Maximum Principles

Tailored complex structures satisfy two harmonicity results (which may be viewed as maximum principles) that will be used throughout this paper.

Lemma 127. [28, Lem. 5.6] Let $u : \Sigma \to \mathbb{R} \times Y$ be a J-holomorphic map (possibly with boundary) with respect to a tailored J. Then

$$t \circ u : \Sigma \to \mathbb{R} \qquad \text{defined on} \qquad u^{-1}(\text{Nbhd}(\partial_+ Y \cup \partial_- Y))$$

is harmonic in a neighborhood of $\partial_+ Y$ and $\partial_- Y$.

Proof. We follow the proof of Lemma 5.6 in [28]. Without loss of generality, we focus on a tubular neighborhood of $\partial_+ Y$ of the form $\partial_+ Y \times [1+\epsilon, 1-\epsilon)$ (we imagine extending the t coordinate slightly in the upwards direction). The key observation is that since $\partial_+ Y$ is two dimensional, J_0 is Stein, which means there exists $f : \partial_+ Y \to \mathbb{R}$ with the boundary of $\partial_+ Y$ its level set. Further, the 1-form $\beta' := -df \circ J_0$ gives $\partial_+ Y$ the structure of a Liouville manifold, and the symplectic form $d\beta'$ is J_0 compatible. If we take the contact form $\alpha' = dt + \beta'$ on $(1-\epsilon, +\infty)_t \times \partial_+ Y$ (which we think of the upwards completion of Y, see [28]), then the same proof as Lemma 5.6 in [28] tells us that $t \circ u$ is harmonic in this region. □

Lemma 128. [28] Let J be a tailored almost complex structure as above. Restricted to the Liouville manifold

$$(-\epsilon, 0]_\tau \times \Gamma \quad \text{with Liouville form} \quad e^\tau \alpha|_\Gamma,$$

the almost complex structure J is compatible with $\alpha|_\Gamma$. Then the τ coordinate $\tau : (-\epsilon, 0]_\tau \times \Gamma \to (-\epsilon, 0]$ is pluri-subharmonic.

Proof. See lemma 5.5 in [28]. □

Local Properties Of Boundary Singularities

We will also need a number of local results governing the singularities of holomorphic curves with boundary. To state these results, we fix the following notation. Let $\mathbb{U}$ denote the upper half-disk

$$\mathbb{U} := \mathbb{D} \cap \mathbb{H} \qquad \text{with} \qquad \partial_\circ \mathbb{U} := \mathbb{U} \cap \mathbb{R}$$

We adopt coordinates $z = s + it$ on $\mathbb{U} \subset \mathbb{C}$. We also denote the upper half-ball in $\mathbb{C}^n$ as follows.

$$U^{2n} := B^{2n} \cap (\mathbb{H} \times \mathbb{C}^{n-1}) \qquad \text{with} \qquad \partial_\circ U^{2n} = U^{2n} \cap \mathbb{R} \times \mathbb{C}^{n-1}$$

Finally, we let $R^n := B^{2n} \cap \mathbb{R}^n$ denote the Lagrangian in $\partial_\circ U^{2n}$ given by the real unit disk.

Lemma 129. Let J be an almost complex structure on U^4 such that $J(TR^n) \cap TR^n = 0$ and consider a J-holomorpic map

$$u : (\mathbb{U}, \partial_\circ \mathbb{U}) \to (U^{2n}, R^n) \qquad \text{with} \qquad u(0) = 0 \quad \text{and} \quad du(0) \neq 0$$

Then there is an open neighborhood $\Omega \subset U^{2n}$ of 0 and a local diffeomorphism $\phi : \Omega \to U^{2n}$ such that

$$\phi \circ u(z) = (z, 0, \dots, 0) \qquad \text{and} \qquad J = \phi^* J_0 \quad \text{on} \quad \mathbb{U} \times 0 \times \dots \times 0.$$

Here J_0 is the standard almost complex structure on $\mathbb{C}$.

Proof. Choose a complex bundle isomorphism

$$\tau : (\mathbb{C}^n, J_0) \simeq u^*(\mathbb{C}^n, J) \qquad \text{with} \qquad \tau(\mathbb{R}^n) = \mathbb{R}^n \text{ along } \partial_\circ \mathbb{U}.$$

Also assume that τ satisfies the following constraint

$$\tau_{u(z)}(1, 0, \dots, 0) = \frac{\partial u}{\partial s} \qquad \text{in coordinates } z = s + it \text{ on } \mathbb{U}.$$

Then we may define ϕ in terms of the exponential map as follows.

$$\phi^{-1}(z, z_2, \dots, z_n) := \exp_{u(z)}(\tau_{u(z)}(0, z_2, \dots, z_n))$$

Verifying the claimed properties of ϕ is standard (see [91, Lem. 2.4.2]). □

Next we prove that boundary intersections of (locally) distinct curves are isolated. The analogue for interior intersections is proven in [91, Lem. 2.4.3] and the proof is directly analogous.

Lemma 130. Fix an almost complex structure J on U^{2n} with $J(TR^n) \cap TR^n = 0$ and consider a pair of J-holomorphic maps

$$u, v : (\mathbb{U}, \partial_\circ \mathbb{U}) \to (U^{2n}, R^n) \qquad \text{with} \qquad u(0) = v(0) \quad \text{and} \quad du(0) \neq 0$$

Suppose that there are sequences $\{z_k\}$ and $\{w_k\} \in \mathbb{U}$ such that

$$u(z_k) = v(w_k) \qquad \lim_{k\to\infty} z_k = \lim_{k\to\infty} w_k = 0 \quad \text{and} \quad z_k \neq 0$$

Then there exists a neighborhood $\Omega \subset \mathbb{U}$ of 0 and a map $\phi : \Omega \to \mathbb{U}$ such that

$$\phi(0) = 0 \qquad \text{and} \qquad v = u \circ \phi$$

Proof. By Lemma 129, we can assume that $u(z) = (z, 0, \dots, 0)$ and that $J = J_0$ along u. Write

$$v(z) = (v_1(z), \widetilde{v}(z)) \qquad \text{for maps} \qquad v_1 : \mathbb{U} \to \mathbb{U} \quad \text{and} \quad \widetilde{v} : \mathbb{U} \to \mathbb{C}^{n-1}$$

We claim that $\widetilde{v}$ vanishes to infinite order at $z = 0$. Indeed, suppose that $\widetilde{v}$ is order $l \geq 1$, i.e. that $\widetilde{v}(z) = O(|z|^l)$ and $\widetilde{v}(z) \neq O(|z|^{l+1})$. Since $J = J_0$ along u, we know that

$$J(v(z)) = J_0 + O(|z|^l)$$

and therefore that the order l Taylor series of v is holomorphic, i.e. that

$$v_1(z) = p(z) + O(|z|^{l+1}) \qquad \text{and} \qquad \widetilde{v}(z) = \widetilde{a}z^l + O(|z|^{l+1})$$

for a polynomial p of order l and a non-zero constant $\widetilde{a}$. In particular, $\widetilde{v}(z) \neq 0$ in some neighborhood of 0 and the intersections of v and u are isolated away from 0, contradicting the hypotheses. Finally, note that since $J = J_0$ along u, we have (for all $w = (w_1, \widetilde{w})$)

$$\frac{\partial J(w_1, 0)}{\partial x_1} = \frac{\partial J(w_1, 0)}{\partial y_1} = 0 \quad \text{and thus} \quad |\frac{\partial J(w)}{\partial x_1}| + |\frac{\partial J(w)}{\partial y_1}| \leq C|\widetilde{w}|$$

Since v is J-holomorphic, this implies that

$$|\Delta \widetilde{v}| \leq C(|\widetilde{v}| + |\partial_s \widetilde{v}| + |\partial_t \widetilde{v}|) \tag{4.3.3}$$

and we can apply Aronszajn's theorem (cf. [91, Thm 2.3.4]). In particular we extend $\widetilde{v}$ smoothly past the boundary, then Equation 4.3.3 continues to hold in a small neighborhood of the origin, and the origin is still a zero of infinite order. Then we can apply Aronszajn's theorem in its original form to conclude that $\widetilde{v} = 0$ identically. Since $\mathbb{C} \times 0 \times \cdots \times 0 \cap U^{2n}$ is precisely the (embedded) image of u, this proves the result. See also the results in [80]. □

Singularities

We now apply the local maximum principles of §4.3 to study the singularities of holomorphic curves with boundary in the symplectization of Y.

For the rest of this sub-section, fix a convex sutured contact 3-manifold (Y, ξ) and a closed (possibly disconnected) Legendrian $\Lambda \subset \partial_+ Y \cup \partial_- Y$. Also fix a non-degenerate, adapted contact form α and a tailored complex structure J.

Lemma 131 (Boundary Immersion)**.** Let $u : (\Sigma, \partial_\circ \Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a finite energy $\widehat{J}$-holomorphic map with boundary. Then

$$u^{-1}(\mathbb{R} \times \partial Y) = \partial_\circ \Sigma \qquad \text{and} \qquad u \text{ is transverse to } \mathbb{R} \times \partial Y$$

Proof. Since u has finite energy, it is asymptotic to $\Xi_\pm$ at $\pm\infty$, in the sense that u exponentially approaches the trivial cylinders and strips over $\Xi_\pm$ at $\pm\infty$. Recall by our notation $\Xi_\pm$ denotes a collection of orbits and chords at $\pm\infty$ in the symplectization direction. The specific manner in which they approach the orbits and chords are detailed in [108] and [2]. Since the Reeb chords and orbits are transverse to $\partial_\pm Y$ and disjoint from $\partial_\sigma Y$ (we assume implicitly the Legendrians are in the interior of $\partial_\pm Y$) , our theorem holds on

$$u^{-1}((-\infty, -T] \cup [T, \infty) \times Y) \subset \Sigma \qquad \text{for any sufficiently large} \qquad T > 0$$

To show that the result holds elsewhere, consider

$$A := u^{-1}([-T, T] \times \text{Nbhd}(\partial_\sigma Y)) \subset \Sigma.$$

For sufficiently large T and sufficiently small $\text{Nbhd}(\partial_\sigma Y)$, this is a smooth surface, whose boundary is disjoint from $\mathbb{R} \times \partial_\sigma Y$. Thus by Lemma 128, $u|_A$ (and thus u) is disjoint from $\partial_\sigma Y$. Finally, consider the compact surface with boundary

$$B := u^{-1}([-T, T] \times [-1, -1 + \epsilon] \times \partial_- Y) \subset \Sigma$$

The restriction $t \circ u|_B$ is harmonic by Lemma 127. Thus, the minimum -1 is achieved only on the boundary of B, which is the inverse image of

$$\{-T, +T\}_s \times [-1, -1 + \epsilon]_t \times \partial_\circ Y \cup [-T, T]_s \times \{-1, -1 + \epsilon\}_t \times \partial_\circ Y \subset [-T, T] \times Y$$

For large T the minimum cannot be achieved on

$$\{-T, +T\} \times (0, \epsilon] \times \partial_\circ Y \qquad \text{or} \qquad [-T, T] \times \epsilon \times \partial_\circ Y$$

Thus $t \circ u$ only takes the value of -1 on $\partial\Sigma \cap B = u^{-1}([-T, T] \times \{-1\} \times \partial_\circ Y)$. A similar argument holds near $\partial_+ Y$, and this proves that

$$u^{-1}(\mathbb{R} \times \partial Y) = \partial\Sigma$$

Finally, we argue that u is an immersion near $\mathbb{R} \times \partial Y$. This is evident away from $[-T, T] \times Y$ by the normal forms of the curve near punctures, as in [108] and [2]. Thus it suffices to show that

$$t \circ u : u^{-1}([-T, T] \times \text{Nbhd}(\partial_- Y)) \to \mathbb{R}$$

has no critical points along $\partial_\circ\Sigma$ (and similarly for $\partial_+ Y$). Thus, pick any $p \in \partial_\circ\Sigma$ with $u(p) \in \partial_- Y$. In a neighborhood of p, since $t \circ u$ is harmonic, it is modelled on the real part of a map holomorphic map with a 0 of order k, i.e.

$$z \mapsto z^k \qquad \text{for} \qquad k \geq 1$$

In particular, the number of connected components of the set

$$(t \circ u)^{-1}((0, \infty)) \cap \text{Nbhd}(p)$$

is equal to the order k for small enough $\text{Nbhd}(p)$. Since $t \circ u > 0$ in $\text{Nbhd}(\partial_- Y)$, we thus know that $k = 1$. This concludes the proof. □

Lemma 132 (Boundary Submersion)**.** Let $u : (\Sigma, \partial_\circ\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a finite energy $\widehat{J}$-holomorphic map with boundary and let Γ be a connected component of $\partial_\circ\Sigma$. Then the map

$$\pi_\mathbb{R} \circ u : \Gamma \xrightarrow{u} \mathbb{R} \times \Lambda \xrightarrow{\pi_\mathbb{R}} \mathbb{R}$$

is a proper submersion (and therefore a diffeomorphism).

Proof. By Lemma 131, u is transverse to $\mathbb{R} \times \partial Y$. On the otherhand, suppose that

$$d(\pi_\mathbb{R} \circ u|_\Gamma) = d\pi_\mathbb{R} \circ du|_\Gamma = 0 \qquad \text{at some } p \in \Gamma$$

Then du maps $T\Gamma$ to $0 \oplus TY$ at p, since the kernel of $d\pi_\mathbb{R}$ is precisely TY. Since $u(\partial_\circ\Sigma) \subset \mathbb{R} \times \Lambda$, we must then have

$$du_p(T\Sigma) \subset (0 \oplus T\Lambda) \oplus (0 \oplus J(T\Lambda)) = 0 \oplus \xi$$

Moreover, $\xi_p = T(\partial Y)_p$ at p since Λ is an exact Legendrian. Thus u is tangent to $\mathbb{R} \times \partial Y$ at p. This contradicts Lemma 131, so this proves that $\pi_R \circ u|_\Gamma$ is a submersion. Since $\pi_\mathbb{R}$ and u are proper, $\pi_\mathbb{R} \circ u|_\Gamma$ is also proper. □

As a consequence of the boundary immersion property, we can easily prove that there are a finite number of critical points.

Lemma 133 (Finite Critical Points)**.** Let $u : (\Sigma, \partial_\circ \Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a finite energy, non-constant J-holomorphic curve. Then the set of critical points $\mathrm{Crit}(u) \subset \Sigma$ is finite.

Proof. By Lemma 131, the set $\mathrm{Crit}(u) \subset \Sigma$ is a closed (and thus compact) subset of the interior of $u^{-1}([-T, T] \times Y)$ for some large $T > 0$. By [91, §2.4, p. 26], interior critical points of a non-constant holomorphic curve are isolated. Therefore, $\mathrm{Crit}(u)$ is finite. □

We next state a key proposition that will be essential to the proof of $d^2 = 0$.

Proposition 134. (No closed boundary components) Let $u : (\Sigma, \partial_\circ \Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a finite energy J-holomorphic map, then no component of $\partial_o \Sigma$ is closed unless u is constant.

Proof. Suppose not, let S^1 denote a closed component of $\partial_o \Sigma$ with coordinate $q \in S^1$. It is mapped by u to $\mathbb{R} \times L$, where L is an exact Legendrian. Let $\pi_{\mathbb{R}}$ denote the projection to the symplectization direction, and assume $\pi_{\mathbb{R}} u|_{S^1}$ achieves its maximum at p. Then $\frac{d}{dq} \pi_{\mathbb{R}} u(p) = 0$. Thus at p the tangent space of u coincides with the contact distribution ξ, which at p is tangent to the horizontal boundary. But this implies the curve u is not transverse to the boundary of $\mathbb{R} \times Y$ at p, which contradicts the local form in Lemma 131, 129. □

Simple And Somewhere Injective Curves

We can now prove an equivalence between the class of simple curves and somewhere injective curves, generalizing the analogous statement in the case of curves without boundary.

Definition 135. A point p in the domain of a J-holomorphic map $u : (\Sigma, j) \to (X, J)$ is *injective* if

$$u^{-1}(u(p)) = \{p\} \qquad \text{and} \qquad du_p \neq 0 \tag{4.3.4}$$

The J-holomorphic map u is called *somewhere injective* if it has an injective point.

Definition 136. A J-holomorphic map $u : (\Sigma, j) \to (X, J)$ is *simple* if it does not factor as

$$(\Sigma, j) \xrightarrow{\phi} (S, i) \xrightarrow{v} (X, J)$$

where ϕ is a branched cover of degree 2 or more, and v is J-holomorphic.

Lemma 137 (Factorization)**.** Any finite energy, proper J-holomorphic map $u : (\Sigma, \partial_\circ\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ factors as

$$(\Sigma, j) \xrightarrow{\phi} (S, i) \xrightarrow{v} (X, J)$$

where ϕ is a branched cover and v is a proper J-holomorphic map whose injective points form a dense and open set.

Proof. Let Cr denote the set of critical values of u and let $B \subset u(\Sigma) \setminus Cr$ be the set of non-critical values where multiple branches of u meet. That is, $y \in B$ if and only if there are points p, q such that

$$y = u(p) = u(q) \qquad p \neq q \quad \text{and} \quad u(\text{Nbhd}(p)) \neq u(\text{Nbhd}(q))$$

for any neighborhoods Nbhd(p) and Nbhd(q). The set Cr is finite by Lemma 133 and the set B is discrete in $u(\Sigma) \setminus Cr$ by [91, Lem. 2.4.3] and Lemma 130. Therefore, $S' = u(\Sigma) \setminus (B \cup Cr) \subset \mathbb{R} \times Y$ is a Riemann surface with (interior and boundary) punctures, equipped with a tautological map $v : S' \to \mathbb{R} \times Y$. The punctures of S' mapping to B and C are removable, and so we can choose extensions of S' and v

$$S' \subset S \qquad \text{and} \qquad v : S \to \mathbb{R} \times Y$$

so that the resulting map is proper (the extension on the interior works the same was as Proposition 2.5.1 in [91], the extension near the boundary comes from the boundary immersion property in Lemma 131). The map $\Sigma \setminus u^{-1}(B \cup Cr) \to S'$ extends across $u^{-1}(B \cup Cr)$ to a map $\phi : \Sigma \to S$ of Riemann surfaces with punctures. The maps $\phi : \Sigma \to S$ and $v : S \to \mathbb{R} \times Y$ are precisely the desired maps as we observe that v is simple by construction, and its injective points form an open and dense set. □

Remark 138. We emphasize that the results of this lemma *do not* hold for general holomorphic curves with Lagrangian (or totally real) boundary conditions. Indeed, simple curves with Lagrangian boundary need not have finite self-intersections and may not be determined by their image. An example given in Remark 2.5.6 in [91]: a disc mapping in S^2 with boundary on the equator can wrap one and half times around the sphere. Here we are able to obtain nice results because we restricted the Lagrangian to lie at the boundary of our manifold, and $t \circ u$ is harmonic as in Lemma 131. This allows us to conclude that B is *finite* near the boundary (which is not the case for the partially wrapped disk around S^2 in the example we just gave), and ensures the factorization Lemma 137 holds.

Lemma 139 (Simple Is Mostly Injective)**.** Fix a non-constant, simple, finite energy J-holomorphic map

$$u : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

Then the set of injective points is cofinite.

Proof. Let $S(u) \subset \Sigma \times \Sigma$ denote the set of pairs (p, q) that satisfy $u(p) = u(q)$ and $p \neq q$. Then the set of non-injective points in Σ is precisely

$$\pi_1(S(u)) \cup \mathrm{Crit}(u) \qquad \text{where} \qquad \pi_1(p, q) = p$$

$\mathrm{Crit}(u)$ is finite by Lemma 133, and so it suffices to show that $S(u)$ is finite. Note that by Lemma 131, we can decompose $S(u)$ as

$$S(u) = S_\partial(u) \cup S_{\mathrm{int}}(u)$$

where $S_\partial(u)$ consists of pairs in $\partial\Sigma \times \partial\Sigma$ and $S_{\mathrm{int}}(u)$ consists of pairs $\mathrm{int}(\Sigma) \times \mathrm{int}(\Sigma)$.

Now, note that u is locally injective, i.e. for every $z \in \Sigma$ there exists a neighborhood U of z such that $u|_U$ is injective. If $z \in \partial\Sigma$, then this follows from the fact that $du_z \neq 0$ by Lemma 131. If $z \in \mathrm{int}(\Sigma)$, see [91, Rmk. E.1.3]. This implies that $S(u)$ is separated from an open neighborhood of the diagonal.

Next, note that $S(u)$ is a discrete set. Indeed, Lemma 130 implies that any pair $(z, w) \in S_\partial(u)$ is isolated from other pairs in $S(u)$. Moreover, pairs in $S_{\mathrm{int}}(u)$ are isolated by [91, Thm. E.1.2, Claim (ii)].

Finally, note that the image of pairs of points in $S(u)$ must be contained in a compact region $[-T, T] \times Y$ because near the punctures of u, we have asymptotic formulas (see [108] and the Appendix). Since u is proper, this implies that $S(u)$ is bounded in $\Sigma \times \Sigma$.

Thus, $S(u)$ is a closed, bounded, isolated subset of $(\Sigma \times \Sigma) \setminus \Delta$ and is therefore finite. This proves the result. $\square$

Corollary 140. Fix a pair of connected, simple, finite energy J-holomorphic maps

$$u : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \qquad \text{and} \qquad v : (\Sigma', \partial\Sigma') \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

Suppose that $u(\Sigma) = v(\Sigma')$. Then there is a biholomorphism $\varphi : \Sigma \simeq \Sigma'$ such that $u = v \circ \varphi$.

Proof. The same proof as Corollary 2.5.4 in [91] follows through. □

Counts Of Singularities And Legendrian Adjunction

We can now associate a count of singularities to the J-holomorphic maps of interest. We first must explain the local situation, starting with the interior case.

Proposition 141. (cf. [121, Lemma 2.6]) Let J be an almost complex structure on $B^4 \subset \mathbb{C}^2$ and let $\Omega = \mathbb{D}_1 \cup \cdots \cup \mathbb{D}_m$ where $\mathbb{D}_i = \mathbb{D}$ is a copy of the disk with origin z_i. Consider a J-holomorphic map

$$v : (\Omega, i) \to (B^4, J) \qquad \text{with} \qquad v(z_i) = 0 \quad \text{and} \quad v(\partial\Omega) \subset \partial B^4$$

Assume that the only non-injective points of v are z_i. Then there is a unique positive integer

$$\delta(v) = \delta(v; 0) \in \mathbb{Z}_+$$

with the following property: there exists a symplectic immersion

$$\widetilde{v} : \Omega \to B^4 \qquad \text{with} \qquad \widetilde{v} = v \text{ on } \operatorname{Nbhd}(\partial\Omega)$$

with precisely $\delta(v)$ positive, transverse double points .

We will use the following analogue of Proposition 141 for boundary singularities. The proof is significantly simpler than analogous results (e.g. [121, Lemma 2.6]) in the closed case because we may assume that there are no critical points near the boundary.

Proposition 142. Let J be an almost complex structure on $U^4 = B^4 \cap \mathbb{H} \times \mathbb{C}$ and let $\Omega = \mathbb{U}_1 \cup \cdots \cup \mathbb{U}_m$ where $\mathbb{U}_i = \mathbb{D} \cap \mathbb{H}$ is a copy of the upper half-disk with origin z_i. Consider a J-holomorphic immersion

$$v : (\Omega, i) \to (U^4, J) \qquad \text{with} \qquad v(z_i) = 0 \quad v(\partial_\circ \mathrm{U}_i) \subset R^2 \quad \text{and} \quad v(\partial\Omega) \subset \partial U^4$$

Assume that the only non-injective points of v are z_i. Then there is a unique positive integer

$$\epsilon(v) = \epsilon(v; 0) \in \mathbb{Z}_+$$

with the following property: there exists a symplectic immersion

$$\widetilde{v} : \Omega \to U^4 \qquad \text{with} \qquad \widetilde{v} = v \text{ on } \operatorname{Nbhd}(\partial\Omega \setminus \partial_\circ\Omega)$$

with precisely $\epsilon(v)$ positive, transverse boundary double points.

Proof. We first consider the simple case where $v|_{\partial\mathbb{U}_i}$ and $v|_{\partial\mathbb{U}_j}$ intersect transversely on the Lagrangian boundary $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}$. Then the intersection viewed as an intersection between two surfaces in $\mathbb{C} \times \mathbb{C}$ is positive, since it is modelled on the intersection of two complex planes in $\mathbb{C}^2$.

In the next case consider a collection of $v|_{\partial\mathbb{U}_i}, i = 1, \cdots k$ intersecting at a single point $p \in \mathbb{R}^2$, but such that pairwise their intersections at p are transverse. Then by perturbing the $v|_{\mathbb{U}_i}$ slightly we recover the conclusion of the theorem.

The most difficult case is when $v|_{\partial\mathbb{U}_i}$ and $v|_{\partial\mathbb{U}_j}$ are tangent to each other at $p \in \mathbb{R}^2$. We turn to this situation next.

Let $U^4(\epsilon)$ be the half-ball of radius ϵ. Note that the image of v is disjoint from $0 \times \mathbb{C}$ except at 0. Therefore, by Lemma 130, we may acquire a (transverse) braid given by

$$L = \partial B^4(\epsilon) \cap v(\Omega) \quad \text{with components} \quad L_i = \partial B^4(\epsilon) \cap v(\mathbb{U}_i)$$

To prove this, fix $i \neq j$ and consider the maps $v|_{\mathbb{U}_i}$ and $v|_{\mathbb{U}_j}$. By Lemma 129, we can (after a change of coordinates) assume that

$$v|_{\mathbb{U}_i}(z) = (z, 0) \qquad \text{and} \qquad J|_{\mathbb{U}\times 0} = J_0$$

In these coordinates, we may write v restricted to $\mathbb{U}_j$ as follows.

$$v|_{\mathbb{U}_j}(z) = (z, a \cdot z^l) + O(|z|^{l+1})$$

Write the second factor of $v|_{\mathbb{U}_j}$ as $w(z) = a \cdot z^l + O(|z|^{l+1})$. After an orientation check, we find that the linking number of $v|_{\mathbb{U}_i}$ and $v|_{\mathbb{U}_j}$ is now precisely the (half-integer) winding number of the path

$$[0,1] \to \mathbb{C} \setminus 0 \qquad \text{given by} \qquad \theta \mapsto w(e^{\pi i \theta})$$

Since $w(z) = a \cdot z^l + O(|z|^{l+1})$, this winding number is simply $l/2 > 0$. This concludes the proof. $\square$

The most important application of Proposition 142 for our purposes is the following resolution of singularities result. It states that we can replace any somewhere injective J-curve in our setting with a well-behaved surface in the same surface class.

Lemma 143. Fix a non-constant, simple, finite energy J-holomorphic map

$$u : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

Then there exists an symplectic, well-immersed surface

$$\iota : (S, \partial_\circ S) \to ([0,1] \times Y, [0,1] \times \Lambda)$$

representing the surface class A corresponding to u such that

(a) S has $\delta(C)$ transverse, positive double points in the interior $\mathrm{int}(S)$.

(b) S has $\epsilon(C)$ transverse, positive double points in the boundary $\partial_\circ\Sigma$.

(c) The domain S is the same as the domain of u.

(d) ι agrees with C in a neighborhood of $\partial_\pm\Sigma$.

Proof. By Lemma 139, we know that u has a finite number of isolated non-injective points $\mathrm{Sing}(u) \subset \Sigma$. To acquire a map $\iota : S \to \mathbb{R} \times Y$ satisfying (c) and (d), we simply modify u in a neighborhood the points $\mathrm{Sing}(u)$. To ensure that this map satisfies (a) and (b), we perform this modification by applying Proposition 141 at the interior singularities and Proposition 142 to boundary singularities. □

As a corollary, we can prove the Legendrian adjunction result stated in the introduction.

Corollary 144 (Theorem 8)**.** Let C be an finite energy, simple holomorphic curve in $\mathbb{R} \times Y$ with boundary on $\mathbb{R} \times \Lambda$. Then

$$\mu(C, \tau) = 2(\bar{\chi}(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C) - \epsilon(C))$$

Proof. By Lemma We 143, we can find a symplectic, well-immersed surface S representing the same surface class of C that has the same domain, writhe and count of singularities, so this follows from topological adjunction (Theorem 117). □

4.4 J-Holomorphic Currents With Boundary

We can now formally introduce J-holomorphic currents and describe their basic properties.

Definition 145. A *J-holomorphic current with boundary C* in an almost complex manifold with boundary (X, J) is a finite set of pairs

$$(C_i, m_i) \qquad \text{for} \qquad i = 1, \ldots, k$$

where C_i is a distinct, simple, somewhere injective J-holomorphic curves $C \subset X$ with boundary on ∂X and m_i is a positive integer called the *multiplicity* of C_i.

We will utilize a specific topology on the space of currents, introduced by Taubes [52, §2.4].

Definition 146. A sequence of J-holomorphic currents $\mathcal{C}^\nu = \{(C_i^\nu, m_i^\nu)\}$ for $\nu \in \mathbb{N}$ in an almost complex manifold (X, J) *converges* to an J-holomorphic current $\mathcal{C} = \{(C_i, m_i)\}$ if

- $\mathcal{C}^\nu$ converges to $\mathcal{C}$ as a point set. More precisely, the point sets $S^\nu = \cup_i C_i^\nu$ converge to the set $S = \cup_i C_i$ in the Hausdorff metric, on each compact subset of X.
- $\mathcal{C}^\nu$ converges to $\mathcal{C}$ as a current, i.e. for every compactly supported 2-form σ on X, we have
$$\lim_{\nu\to\infty} \sum_i \Big(m_i^\nu \cdot \int_{C_i^\nu} \sigma\Big) \to \sum_i \Big(m_i \cdot \int_{C_i} \sigma\Big)$$

A key property of holomorphic currents, which does not hold for curves, is a general compactness property given a uniform action bound.

Theorem 147. *(Taubes, cf. [52]) Let (X, ω) be a symplectic manifold with boundary and compatible almost complex structure J. Let C_i be a sequence of J-holomorphic currents with boundary on ∂X such that*

$$\int_{C_i} \omega < C \qquad \textit{for all } i \textit{ and some } C > 0$$

Then there is a subsequence that converges to a J-holomorphic current $\mathcal{C}$.

Currents With Boundary In Symplectizations

We are interested in currents in the symplectization of a pair (Y, Λ) of a convex sutured contact 3-manifold Y and a pair of closed Legendrians $\Lambda_\pm \subset \partial_\pm Y$.

In this setting, J-holomorphic maps, curves and currents admit a natural $\mathbb{R}$-action, denoted by

$$\mathcal{C} \mapsto \mathcal{C} + s \qquad \text{for any} \qquad s \in \mathbb{R}$$

On curves, this action is given by composing the map $u : \Sigma \to \mathbb{R} \times Y$ with translation $\mathbb{R} \times Y \to \mathbb{R} \times Y$ by s. Moreover, the translation invariant J-holomorphic currents coincide with cylinders over sets of orbits and chords, as follows.

Example 148. The *trivial current* $\mathbb{R} \times \Xi$ over an orbit-chord set Ξ consisting of simple orbits and chords γ_i of multiplicity m_i is the current of pairs

$$(\mathbb{R} \times \gamma_i, m_i)$$

Every tame enough current in $\mathbb{R} \times Y$ is asymptotic to an orbit-chord sets near infinity, as follows.

Lemma 149. Let $\mathcal{C}$ be a J-holomorphic current in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ such that each curve C_i in $\mathcal{C}$ is proper and finite energy. Then there are orbit sets Ξ_+ and Ξ_- such that

$$\mathcal{C} + s_i \xrightarrow{\text{Def 146}} \mathbb{R} \times \Xi_\pm \qquad \text{for any sequence } s_i \in \mathbb{R} \text{ with } s_i \to \pm\infty \tag{4.4.1}$$

Definition 150. A J-holomorphic current $\mathcal{C}$ from Ξ_+ to Ξ_- is a current satisfying (4.4.1). We denote the space of currents from Ξ_+ to Ξ_- by

$$\mathcal{M}(\Xi_+, \Xi_-)$$

We denote the quotient by the $\mathbb{R}$-action on currents by $\mathcal{M}(\Xi_+, \Xi_-)/\mathbb{R}$. More generally, a *broken J-holomorphic current* $\bar{\mathcal{C}}$ from Ξ_+ to Ξ_- is a sequence

$$\mathcal{C}_i \in \mathcal{M}(\Xi_i, \Xi_{i+1})/\mathbb{R} \qquad \text{for } i = 1, \dots, m \qquad \text{where} \qquad \Xi_1 = \Xi_+ \text{ and } \Xi_{m+1} = \Xi_-$$

The space of broken J-holomorphic currents from Ξ_+ to Ξ_- is denoted by $\bar{\mathcal{M}}(\Xi_+, \Xi_-)$.

Remark 151. Any J-holomorphic current in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ appearing in the rest of this paper will be assumed to be finite energy and proper, so that

$$\mathcal{C} \in \mathcal{M}(\Xi, \Theta) \qquad \text{for some orbit-chord sets } \Xi \text{ and } \Theta$$

There is an tautological map sending a finite energy, proper J-holomorphic curve C from orbit-chord set ξ to orbit-chord set Θ to a corresponding surface class $[C]$. This extends to a map

$$\bar{\mathcal{M}}(\Xi, \Theta) \to S(\Xi, \Theta) \qquad \text{denoted by} \qquad \bar{\mathcal{C}} \mapsto [\bar{\mathcal{C}}] \tag{4.4.2}$$

that is compatible with union and composition in the sense that

$$[\mathcal{C} \cup \mathcal{D}] = [\mathcal{C}] \cup [\mathcal{D}] \qquad \text{for any pair of currents } \mathcal{C} \text{ and } \mathcal{D}$$
$$[\bar{\mathcal{C}} \circ \bar{\mathcal{D}}] = [\bar{\mathcal{C}}] \circ [\bar{\mathcal{D}}] \qquad \text{for any pair of broken currents } \bar{\mathcal{C}} \text{ and } \bar{\mathcal{D}}$$

Gromov Compactness

We next introduce the notion of Gromov compactness of broken currents. This compactness result resembles SFT compactness of Bourgeouis-Eliashberg-Hofer-Wyosocki-Zehnder, but does not depend on it.

Definition 152. A sequence of (equivalence classes of) J-holomorphic currents

$$\mathcal{C}^\nu \in \mathcal{M}(\Xi_+, \Xi_-)/\mathbb{R} \qquad \text{for} \qquad \nu \in \mathbb{N}$$

converges in the Gromov topology to a broken J-holomorphic current $\bar{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_m)$ if there are representatives of $\mathcal{C}^\nu$ and $\mathcal{C}_i$, denoted respectively by

$$\mathcal{S}^\nu \in \mathcal{M}(\Xi_+, \Xi_-) \qquad \text{and} \qquad \mathcal{S}_i \in \mathcal{M}(\Xi_i, \Xi_{i+1})$$

and a set of sequences $s_i^\nu \in \mathbb{R}$ for each $i = 1, \ldots, m$ such that

$$\mathcal{S}^\nu + s_i^\nu \xrightarrow{\text{Def 146}} \mathcal{S}_i \qquad \text{as } \nu \to \infty$$

The using the same argument as [52], we have the following compactness result for currents.

Proposition 153 (Gromov Compactness)**.** Let $\mathcal{C}^\nu \in \mathcal{M}(\Xi_+, \Xi_-)$ be a sequence of currents. Then, after passing to a subsequence, there is a broken J-holomorphic current $\bar{\mathcal{C}}$ from Ξ_+ to Ξ_- such that

$$\mathcal{C}^\nu \to \bar{\mathcal{C}} \qquad \text{and} \qquad [\bar{\mathcal{C}}] = [\mathcal{C}^\nu]$$

Intersection Numbers And Linking Of Currents

It will be convenient to extend the intersection number and linking number to currents under a certain disjointness hypothesis.

Definition 154. A pair of J-holomorphic currents $\mathcal{C}$ and $\mathcal{D}$ have *distinct components* if the component curves (C_i, m_i) and (D_j, n_j) satisfy

$$C_i \neq D_j \qquad \text{for all } i, j$$

Currents with disjoint components have a well-defined geometric intersection number that counts actual intersections (with multiplicity).

Definition 155. The *geometric intersection number* of J-holomorphic currents $\mathcal{C}$ and $\mathcal{D}$ with distinct components is the non-negative half-integer given by

$$\mathcal{C} \cdot \mathcal{D} \qquad \text{is given by} \qquad \mathcal{C} \cdot \mathcal{D} := \sum_{i,j} m_i \cdot n_j \cdot (C_i \cdot C_j)$$

where the geometric intersection $C \cdot D$ of two distinct, somewhere injective J-holomorphic curves C and D is defined as the following count of singularities.

$$C \cdot D := (\epsilon(C \cup D) - \epsilon(C) - \epsilon(D)) + \frac{1}{2}(\delta(C \cup D) - \delta(C) - \delta(D))$$

Remark 156. It is possible to generalize the geometric intersection number to a self-intersection number of curves and currents (see [55]). However, we will not carry this out in this paper.

The finiteness (and thus well-definedness) of the geometric intersection number follows from the analysis of singularities of J-holomorphic curves in our setting (see Section 4.3).

Definition 157. The *linking number* of two currents $\mathcal{C}$ and $\mathcal{D}$ with disjoint components with respect to a trivialization τ, denoted by

$$l_\tau(\mathcal{C}, \mathcal{D})$$

is defined as follows. Let (C_i, m_i) and (D_j, n_j) denote the component curves of $\mathcal{C}$ and $\mathcal{D}$. Then

$$l_\tau(\mathcal{C}, \mathcal{D}) = \sum_{i,j} m_i \cdot n_j \cdot l_\tau(C_i, D_j)$$

Here we adopt the following conventions for the linking number of curves.

- If C is non-trivial curve with an end on η and $D = \mathbb{R} \times \eta$ is a trivial curve, then
$$l_\tau(C, D) = \text{wind}_\tau(\zeta_+) - \text{wind}_\tau(\zeta_-)$$
where $\zeta_\pm$ re the asymptotic braids of C at $\pm\infty$.

- If C and D are both non-trivial, then
$$l_\tau(C, D) = l_\tau(\zeta_+, \xi_+) - l_\tau(\zeta_-, \xi_-)$$
where $\zeta_\pm$ and $\xi_\pm$ are the asymtptotic braids of C and D, respectively.

Note that the asymptotic braids can be empty. In this case, the winding number and linking number with any other braid are zero by convention.

This extended linking number transforms in precisely the same way as its topological counterpart under change of trivialization.

Lemma 158. Let $\mathcal{C} \in \mathcal{M}(\Xi_+, \Xi_-)$ and $\mathcal{D} \in \mathcal{M}(\Theta_+, \Theta_-)$ be J-holomorphic currents with distinct components.
$$\eta \quad \text{of multiplicity } m \text{ in } \Xi_\pm \text{ and } n \text{ in } \Theta$$
Then the linking number of $\mathcal{C}$ and $\mathcal{D}$ differ as follows.
$$l_\tau(\mathcal{C}, \mathcal{D}) - l_\sigma(\mathcal{C}, \mathcal{D}) = m \cdot n \cdot (\sigma - \tau)$$

The proof is straightforward so we omit it. The linking number is also related to the geometric intersection number as follows.

Lemma 159. Let $\mathcal{C}$ and $\mathcal{D}$ be J-holomorphic currents with distinct components. Then
$$\mathcal{C} \cdot \mathcal{D} = Q_\tau(\mathcal{C}, \mathcal{D}) + l_\tau(\mathcal{C}, \mathcal{D})$$

Proof. By bilinearity with respect to union, we assume that $\mathcal{C}$ and $\mathcal{D}$ are somewhere injective (and even connected) curves, and that C is non-trivial. Then $C \cup D$ is somewhere injective and we can use Lemma 143 to replace C and D with well-immersed symplectic surfaces S and T in the same surface class that satisfy
$$C \cdot D = \#(\text{int}(S) \cap \text{int}(T)) + \frac{1}{2} \cdot \#(\partial_\circ S \cap \partial_\circ T) \quad \text{and} \quad l_\tau(C, D) = l_\tau(S, T)$$
Note that we are using the fact that, when D is a trivial cylinder over an orbit or chord
$$l_\tau(C, D) = \text{wind}_\tau(\zeta_+) - \text{wind}_\tau(\zeta_-)$$
The result follows from Definition 114. □

4.5 The Conley-Zehnder And Fredholm Indices

In this section, we review several versions of the Conley-Zehnder index. In particular, we introduce the Conley-Zehnder index of a Lagrangian path.

Paths Of Lagrangians and the Conley-Zehnder Index

Recall that given a path of symplectic matrices, $\Phi_t, t \in [0,1]$ satisfying $\Phi_0 = Id$ and Φ_1 does not have 1 as an eigenvalue by nondegeneracy, as in [122] we can associate an integer valued Conley-Zehnder index to this path of symplectic matrices, which we call the Conley-Zehnder index. We refer the reader to lecture 3 of [122] for properties of this index.

We next provide a definition for the Conley-Zehnder index of a path of Lagrangian sub-spaces. We adopt the perspective that this is naturally a *half-integer*.

Definition 160. A path $L : [0,1] \to \mathrm{LGr}(2n)$ is *non-degenerate* if the ends are transverse.

$$L_1 \pitchfork L_0$$

Definition 161. The *Conley-Zehnder index* of a non-degenerate path of Lagrangians $L : [0,1] \to \mathrm{LGr}(2n)$ is the half-integer

$$\mathrm{CZ}(L) := \mu(\bar{L}) + \frac{n}{2} \in \frac{1}{2}\mathbb{Z}$$

Here $\bar{L} : S^1 \to \mathrm{LGr}(2n)$ is the loop of Lagrangians constructed as follows. Choose a complex structure J on $\mathbb{C}^n$ such that

$$J \text{ is compatible with } \omega_0 \qquad \text{and} \qquad JL_1 = L_0$$

Then $\bar{L}$ is the loop acquired by joining L to the path $\exp(-J \cdot \frac{\pi t}{2})$ for $t \in [0,1]$ from L_1 to L_0.

Remark 162. By abuse of terminology and notation, we will refer to $\mu(\bar{L})$ as the Maslov index of the path L and denote it by $\mu(L)$.

We can also define the Conley-Zehnder index of a Lagrangian path as half of the ordinary Conley-Zehnder index of an associated path of symplectic matrices. See [43] for an exposition of how various flavours of Conley-Zehnder indices are related to each other.

Bundles Over 1-Manifolds

Next, we introduce the notion of an asymptotic operators and discuss the Conley-Zehnder index of an asymptotic operator.

Let S be an oriented 1-manifold with boundary. Let $(E, F) \to (S, \partial S)$ be a symplectic bundle pair and fix a symplectic connection

$$\nabla : C^\infty(S, \partial S; E, F) \to C^\infty(S; \Omega^1 S \otimes E)$$

Definition 163. The connection ∇ is *non-degenerate* if the kernel of ∇ is trivial.

Given a choice of trivialization $\tau : (E, F) \simeq (\mathbb{C}^n, \mathbb{R}^n)$, we can assign a Conley-Zehnder index to the above data. To explain how, let $\eta \subset S$ be a component of S and choose an oriented identification

$$\phi : \mathbb{R}/\mathbb{Z} \simeq \eta \qquad \text{or} \qquad \phi : [0,1] \simeq \eta$$

In either case, parallel transport via the connection ∇ determines a family of symplectic maps

$$\mathrm{PT}_t : E_{\phi(0)} \to E_{\phi(t)} \quad \text{for each} \quad t \in [0,1]$$

Given the choice of trivialization τ, this then determines a 1-parameter family of matrices

$$\Phi^\tau : [0,1] \to \mathrm{Sp}(2n) \quad \text{with} \quad \Phi^\tau_t := \tau_{\phi(t)} \circ \mathrm{PT}_t \circ \tau^{-1}_{\phi(0)}$$

and a family of Lagrangians $L^\tau := \Phi^\tau(\mathbb{R}^n)$.

Definition 164. The *Conley-Zehnder index* $\mathrm{CZ}(\eta, \tau)$ of a component $\eta \subset S$ with respect to a trivialization τ of (E, F) is

$$\mathrm{CZ}(\Phi^\tau) \text{ if } \eta \simeq \mathbb{R}/\mathbb{Z} \qquad \text{and} \qquad \mathrm{CZ}(L^\tau) \text{ if } \eta \simeq [0,1]$$

The Conley-Zehnder index $\mathrm{CZ}(S, \tau)$ of (E, F, ∇) with respect to τ is simply the sum

$$\mathrm{CZ}(S, \tau) := \sum_\eta \mathrm{CZ}(\eta, \tau)$$

Let S be a compact 1-manifold diffeomorphic to the interval $[0,1]$ and let $(E, F) \to (S, \partial S)$ be a symplectic bundle pair. From this point onward assume E is two-dimensional. Given a nowhere vanishing vector field R on S, a compatible complex structure J on E and a symplectic connection ∇, we define the associated asymptotic operator to be

$$A : C^\infty(S, \partial S; E, F) \to C^\infty(S, E) \quad AX := -J\nabla_R X$$

Lemma 165. The asymptotic operator A satisfies the following basic properties:

(i) A has trivial kernel if and only if ∇ is non-degenerate.

(ii) All eigenvalues of A are simple.

(iii) All eigenfunctions of A are nowhere vanishing.

Proof. The first assertion is immediate because the operators A and ∇ have the same kernel. Let λ be an eigenvalue of A. Since $A-\lambda$ is a linear first order ordinary differential operator, the initial value homomorphism

$$\ker(A-\lambda) \to F_p \quad X \mapsto X(p)$$

is injective. Here $p \in \partial S$ is a boundary point of S. Thus the eigenspace of λ has dimension 1. This proves the second assertion. Let $X \in \ker(A-\lambda) \setminus \{0\}$ be an eigenvector. Again because $A-\lambda$ is a linear first order ordinary differential operator, X vanishes at one single point $q \in S$ if and only if it is identically equal to zero. The third assertion follows. $\square$

In particular, given an asymptotic operator A, a symplectic trivialization τ of (E, F) and an eigenvalue λ of A, there is a well-defined half-integer-valued winding number $w(\tau, A; \lambda) \in \frac{1}{2}\mathbb{Z}$ associated to the eigenfunction of A with eigenvalue λ. To be specific, under our choice of trivialization τ, if $e(t) : [0,1] \to \mathbb{C}$ is such an eigenfunction that maps the end points $\{0,1\}$ of $[0,1]$ to $\mathbb{R} \subset \mathbb{C}$, using the fact that $e(t)$ is never vanishing, the winding number is $\frac{1}{2}$ times the number of half turns $e(t)$ makes about the origin.

Lemma 166. The function $w(\tau, A; \cdot) : \operatorname{Spec} A \to \frac{1}{2}\mathbb{Z}$ is monotonic and a bijection.

Proof. We solve this problem by deforming to a simplified operator A_0 with the same Conley-Zehnder index (see below for description). Recall the asymptotic operator A is of the form $-J_0\partial_t - S(t) : C^\infty([0,1], \{0,1\}; \mathbb{R}^2, \mathbb{R} \times 0) \mapsto C^\infty([0,1], \mathbb{R}^2)$. Its Conley-Zehnder index is computed by considering the path of symplectic matrices $\Phi(t)$ satisfying

- $\Phi(0) = Id$
- $-J_0\partial_t\Phi(t) - S(t)\Phi(t) = 0$

and considering the loop of Lagrangians $\Phi_t(\mathbb{R}) \subset \mathbb{C}$. In particular, for general operators of this form, the path of Lagrangians $\Phi_t(\mathbb{R})$ being nondegenerate implies that A has trivial kernel.

Now suppose we have an asymptotic operator A_0 satisfying the conclusion of the lemma (we will specify what this operator is directly in a bit) such that A and A_0 have the same Conley-Zehnder index as described above, then we can choose a path $(A_s)_{s\in[0,1]}$ of nondegenerate asymptotic operators connecting A_0 to $A_1 = A$. Let $\lambda_0^{k/2}$ for $k \in \mathbb{Z}$ denote the eigenvalues of A_0 labelled by the winding number $k/2$. Since the spectrum of A_s is simple for all $s \in [0,1]$, we may continuously extend $\lambda_0^{k/2}$ to a family $\lambda_s^{k/2}$ such that the spectrum of A_s is given by $\lambda_s^{k/2}$ for $k \in \mathbb{Z}$. By assumption, the function $k \mapsto \lambda_0^{k/2}$ is monotonic. Thus the same must continue to hold for $s > 0$. We may choose families $X_s^{k/2}$ of associated eigenvectors. Since $X_s^{k/2}$ is nowhere vanishing for all s, the winding number of $X_1^{k/2}$ agrees with the winding number of $X_0^{k/2}$, which is equal to $k/2$. Thus the winding number of $\lambda_1^{k/2}$ is actually equal to $k/2$. This proves that the lemma holds for $A = A_1$.

Let us prove the lemma for the following explicit asymptotic operator A_0.

$$A_0 : C^\infty([0,1], \{0,1\}; \mathbb{R}^2, \mathbb{R}\times 0) \mapsto C^\infty([0,1], \mathbb{R}^2) \quad X \mapsto -J_0\partial_t X - \pi/2(1+2l)X$$

This can be solved explicitly as follows. Consider $(x(t), y(t)) \in \mathbb{R}^2$, this satisfies the equation

$$-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} - \pi/2(1+2l)\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

This is the system of equations

$$y' = (\lambda + \pi/2(1+2l))x, \quad x' = -(\lambda + \pi/2(1+2l))y$$

differentiating twice to get

$$y'' = -(\lambda + \pi/2(1+2l))^2 y, \quad x'' = -(\lambda + \pi/2(1+2l))^2 x$$

with boundary conditions:

$$y(0) = y(1) = 0$$

Hence the eigenvalues of the asymptotic operator are $\lambda = \pi/2(2n+1)$ ($n \in \mathbb{Z}$), and the corresponding eigenvectors are

$$y_n(t) = a\sin((\lambda + \pi/2(2l+1))t), \quad x_n(t) = a\cos((\lambda + \pi/2(2l+1))t)$$

For each choice of eigenvalue λ of A_0, we may compute the winding number w from these explicit eigenvectors and see that the theorem is satisfied.

In addition, with this choice of A_0 we also compute the Conley-Zehnder index associated to this choice of trivialization. We consider the matrix satisfying

$$\partial_t \Phi = \pi/2(1+2l)J\Phi$$

Then identifying $\mathbb{R}^2$ with $\mathbb{C}$, the matrix Φ can be identified with $e^{i\pi/2(1+2l)t}$. We consider the path of Lagrangians $\Phi(t)(\mathbb{R})$ and append the path $e^{-i\frac{\pi}{2}t}(i\mathbb{R})$ to make this a loop of Lagrangians. We see this loop has Maslov index l, and hence the Conley-Zehnder index of the operator is $l+\frac{1}{2}$. The fact that $\{l+\frac{1}{2}|l \in \mathbb{Z}\}$ enumerates all possible Conley-Zehnder indices that an asymptotic operator A can have, proves that the path $(A_s)_{s\in[0,1]}$ we chose exists. □

Lemma 167. Suppose that ∇ is non-degenerate. We have

$$\mathrm{CZ}(S,\tau) = \min\{w(\tau, A; \lambda) \mid \lambda \in \mathrm{Spec}(A)\cap\mathbb{R}_{>0}\}+\max\{w(\tau, A; \lambda) \mid \lambda \in \mathrm{Spec}(A)\cap\mathbb{R}_{<0}\}$$

Proof. We assume the Conley-Zehnder index of A is $l+\frac{1}{2}$ ($l \in \mathbb{Z}$). Same as what we did in the proof of Lemma 166, we begin by choosing a path of non-degenerate asymptotic operators $(A_s)_{s\in[0,1]}$ such that $A_1 = A$ and $A_0 = -J_0\partial_t - \pi/2(1+2l)$. As in the proof of Lemma 166, let $\lambda_s^{k/2}$ denote the unique eigenvalue of A_s with winding number $k/2$. Since for every s, the asymptotic operator A_s is non-degenerate, we observe that 0 is never an eigenvalue of A_s, and hence the paths of eigenvalues $(\lambda_s^{k/2})_{s\in[0,1]}$ do not cross 0. It follows that

$$\min\{w(\tau, A_s; \lambda) \mid \lambda \in \mathrm{Spec}(A_s) \cap \mathbb{R}_{>0}\} + \max\{w(\tau, A_s; \lambda) \mid \lambda \in \mathrm{Spec}(A_s) \cap \mathbb{R}_{<0}\}$$

is constant for $s \in [0,1]$, and hence it suffices to verify the claim for A_0.

By our explicit calculation in the proof of Lemma 166, the smallest positive eigenvalue of A_0 is $\frac{\pi}{2}$ with winding number $\frac{l+1}{2}$, and the largest negative eigenvalue of A_0 is $-\frac{\pi}{2}$ with winding number $\frac{l}{2}$, so the claim holds true for A_0. □

Fredholm Index

We the previous section on Conley-Zehnder indices we are now prepared to discuss the Fredholm index for pseudo-holomorphic curves in our setting.

Theorem 168. *Let Y be a sutured contact manifold of dimension $2n-1$. Let $u : (\Sigma, \partial_\circ \Sigma) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a pseudoholomorphic curve. Suppose that u is positively asymptotic to orbits and chords $\gamma_1^+, \dots, \gamma_k^+$ and negatively asymptotic to orbits and chords $\gamma_1^-, \dots, \gamma_\ell^-$. Then the Fredholm index of u is given by*

$$\operatorname{ind}(u) = (n-3)\bar{\chi}(\Sigma) + \mu(u,\tau) + \sum_{i=1}^{k} \mathrm{CZ}_\tau(\gamma_i^+) - \sum_{i=1}^{\ell} \mathrm{CZ}_\tau(\gamma_i^-)$$

The way to interpret the above formula is as follows: if u is somewhere injective, and J is generic, then u lives in a moduli space that has dimension given by the index formula.

Let Σ be a compact connected Riemann surface, possibly with boundary. Let $\Gamma \subset \Sigma$ be the finite set of punctures (previously we denoted this by $\partial_o \Sigma$), possibly on the boundary. Let $\Gamma = \Gamma_i \cup \Gamma_b$ be the partition of Γ into interior and boundary punctures. Moreover, let $\Gamma = \Gamma^+ \cup \Gamma^-$ be a partition of Γ into positive and negative punctures. Let $\dot{\Sigma} = \Sigma \setminus \Gamma$ denote the punctured surface. Let $(E,F) \to \dot{\Sigma}$ be a bundle pair and let τ denote a trivialization near the punctures.

Lemma 169. Let

$$D : W^{k,p}(E,F) \to W^{k-1,p}(\Lambda^{0,1}T^*\dot{\Sigma} \otimes E)$$

be a Cauchy-Riemann type operator with non-degenerate asymptotic operators A_z^τ at the punctures $z \in \Gamma$. Then D is Fredholm and its index is given by

$$\operatorname{ind}(D) = n \cdot \bar{\chi}(\dot{\Sigma}) + \mu^\tau(E,F) + \sum_{z \in \Gamma^+} \mathrm{CZ}(A_z^\tau) - \sum_{z \in \Gamma^-} \mathrm{CZ}(A_z^\tau).$$

Proof. The form of the asymptotic operators A_z^τ is described in [14], and using our Definition 164 we can associate to it a Conley-Zehnder index. The lemma is exactly the statement of the Fredholm index formula proved in [14]. Note that [14] uses different definitions for the Euler characteristic of a punctured surface with boundary and for the Conley-Zehnder index. The Euler characteristic $X(\Sigma, \Gamma^\pm)$ appearing in [14] is related to the orbifold Euler characteristc $\bar{\chi}(\dot{\Sigma})$ via

$$X(\Sigma, \Gamma^\pm) = \bar{\chi}(\dot{\Sigma}) + \frac{1}{2}\#\Gamma_b^+ - \frac{1}{2}\#\Gamma_b^-.$$

At interior punctures the Conley-Zehnder index $\mu_{\mathrm{CZ}}(A_z^\tau)$ showing up in [14] agrees with our Conley-Zehnder index $\mathrm{CZ}(A_z^\tau)$. At boundary punctures, the relationship is given by

$$\mu_{\mathrm{CZ}}(A_z^\tau) = \mathrm{CZ}(A_z^\tau) - \frac{n}{2}$$

where n denotes the complex rank of E. Plugging these identities into the index formula given in [14] readily yields the lemma. □

Lemma 170 (Section 3, [120])**.** The real dimension of the moduli space of punctured Riemann surfaces with boundary of topological type $(\Sigma, \Gamma = \Gamma_i \cup \Gamma_b)$ is given by

$$-3\chi(\Sigma) + 2\#\Gamma_i + \#\Gamma_b = -3\bar{\chi}(\dot{\Sigma}) - \#\Gamma_i - \frac{1}{2}\#\Gamma_b$$

Proof of Theorem 168. The moduli space containing u is the zero set of a certain Fredholm section. For a general description of this setup see Section 3.3 of [120] and Section 5 of [34]. The linearization of this Fredholm section at u is a Fredholm operator of the form

$$D : W^{k,p,\delta}(u^*T\mathbb{R} \times Y, u^*T\mathbb{R} \times \Lambda) \oplus V \oplus T \to W^{k-1,p,\delta}(\Lambda^{0,1}T^*\dot{\Sigma} \otimes u^*T\mathbb{R} \times Y).$$

Here δ is a small positive exponential weight. The space $W^{k,p,\delta}(u^*T\mathbb{R}\times Y, u^*T\mathbb{R}\times\Lambda)$ means we use a weighted Sobolev space with weight of the form $e^{\delta|s|}$ near each puncture of $\dot{\Sigma}$ around which we have already chosen cylindrical coordinates of the form $(s,t) \in [0, \pm\infty) \times S^1$ or $[0, \pm\infty) \times [0,1]$ The dimension of the tangent space V of the space of asymptotic markers is given by $2\#\Gamma_i + \#\Gamma_b$. By Lemma 170 the dimension of the Teichmüller slice T is given by $-3\bar{\chi}(\dot{\Sigma}) - \#\Gamma_i - \frac{1}{2}\#\Gamma_b$. The restriction D' of D to $W^{k,p,\delta}(u^*T\mathbb{R} \times Y, u^*T\mathbb{R} \times \Lambda)$ is a Cauchy-Riemann type operator. Let $z \in \Gamma$ be a puncture and let γ denote the corresponding Reeb chord or orbit. Then the asymptotic operator of D' at z takes the form $A^\tau_z \oplus -i\partial_t$ where A^τ_z acts on sections of $\gamma^*\xi$ and $-i\partial_t$ acts on sections of the trivial bundle spanned by ∂_s and R. It follows from Lemma 169 that if the exponential weight δ is sufficiently small, the index of D' is given by

$$\mathrm{ind}(D') = n \cdot \bar{\chi}(\dot{\Sigma}) + \mu(u,\tau) + \sum_{z\in\Gamma^+} \mathrm{CZ}(A^\tau_z \oplus (-i\partial_t + \delta)) - \sum_{z\in\Gamma^-} \mathrm{CZ}(A^\tau_z \oplus (-i\partial_t - \delta)).$$

The Conley-Zehnder index is additive under direct sums, i.e.

$$\mathrm{CZ}(A^\tau_z \oplus (-i\partial_t \pm \delta)) = \mathrm{CZ}(A^\tau_z) + \mathrm{CZ}(-i\partial_t \pm \delta)$$

A direct computation shows that if we regard $-i\partial_t \pm \delta$ as an asymptotic operator over the circle, then

$$\mathrm{CZ}(-i\partial_t \pm \delta) = \mp 1$$

for $\delta > 0$ sufficiently small. If we regard it as an asymptotic operator over the interval, then

$$\mathrm{CZ}(-i\partial_t \pm \delta) = \mp 1/2.$$

Combining these identities we compute

$$\mathrm{ind}(D) = \mathrm{ind}(D') + \dim V + \dim T = (n-3)\bar{\chi}(\Sigma) + \mu(u,\tau) + \sum_{i=1}^{k} \mathrm{CZ}_\tau(\gamma_i^+) - \sum_{i=1}^{\ell} \mathrm{CZ}_\tau(\gamma_i^-)$$

□

Regular Almost Complex Structures

We conclude this section by reviewing the notion of regular almost complex structures.

Definition 171. A tailored almost complex structure J on (Y, Λ) is *regular* if every somewhere injective, finite energy J-holomorphic curve in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ is transversely cut out.

A standard argument shows that regular, tailored almost complex structures are generic. In particular, we have the following proposition.

Proposition 172. The set $\mathcal{J}_{\mathrm{reg}}(Y, \Lambda)$ of regular, tailored almost complex structures on (Y, Λ) is comeager in the space of all tailored almost complex structures on (Y, Λ).

The proof follows the analogous arguments in the case of J-holomorphic curves without boundary in symplectizations (cf. [122, §7, 8] and more specifically [122, Thm 7.2]).

Proposition 173. If J is a regular, tailored almost complex structure on (Y, Λ), then the set of simple, finite energy, J-holomorphic curves with boundary is a smooth manifold and the dimension at a curve u is given by $\mathrm{ind}(u)$.

This also follows from analogues of the arguments in [122, §7, 8] for somewhere injective curves in symplectizations, and the fact that simple implies somewhere injective in our setting (Lemma 139).

Remark 174. In general, for a curve u with Lagrangian boundary condition, there is no guarantee that u can be factored into a branched cover and a somewhere injective (and therefore regular) curve. It is key that we can use Lemma 139 to replace somewhere injective with simple in the statement of Proposition 173. It implies that the underlying J-holomorphic curves in any J-holomorphic current are transversely cut out when J is regular.

4.6 Legendrian ECH Index And Its Properties

In this section, we introduce the Legendrian ECH index and establish its basic properties.

ECH Conley-Zehnder Term

We begin by introducing the Conley-Zehnder term that appears in the Legendrian ECH index.

Definition 175. The *ECH Conley-Zehnder term* of an orbit-chord set Ξ with respect to a trivialization τ along Ξ is the unique half-integer

$$\mathrm{CZ}_\tau^{ECH}(\Xi) \in \frac{1}{2}\mathbb{Z}$$

that is additive with respect to disjoint union, in that

$$\mathrm{CZ}_\tau^{ECH}(\Xi \sqcup \Theta) = \mathrm{CZ}_\tau^{ECH}(\Xi) + \mathrm{CZ}_\tau^{ECH}(\Theta)$$

and that is given by the following formula for a simple Reeb orbit γ and Reeb chord c

$$\mathrm{CZ}_\tau^{ECH}((\gamma, m)) = \sum_{i=1}^{m} \mathrm{CZ}_\tau(\gamma^i) \quad \text{and} \quad \mathrm{CZ}_\tau^{ECH}((c, m)) = \frac{m}{2} + \frac{m(m+1)}{2} \cdot (\mathrm{CZ}_\tau(c) - \frac{1}{2})$$

The ECH Conley-Zehnder term of a surface class $A \in S(\Xi_+, \Xi_-)$ is defined to be

$$\mathrm{CZ}_\tau^{ECH}(A) := \mathrm{CZ}_\tau^{ECH}(\Xi_+) - \mathrm{CZ}_\tau^{ECH}(\Xi_-)$$

We adopt the same definition for J-holomorphic curves and currents.

We will make heavy use of the following change of trivialization rule.

Lemma 176 (Trivialization)**.** Let σ, τ be two trivializations of ξ along an simple orbit or chord η. Then

$$\mathrm{CZ}_\tau^{ECH}((\eta, m)) - \mathrm{CZ}_\sigma^{ECH}((\eta, m)) = \frac{m(m+1)}{2} \cdot (\sigma - \tau) \in \frac{1}{2}\mathbb{Z} \tag{4.6.1}$$

Proof. Recall that the Conley-Zehnder index itself transforms as

$$\mathrm{CZ}_\tau(\eta) - \mathrm{CZ}_\sigma(\eta) = 2(\sigma - \tau) \in \mathbb{Z} \qquad \text{by Lemma 176}$$

Note that this formula holds in the Reeb orbit and chord case (and in the former case, the difference $\sigma - \tau$ is a whole integer). In the case of a Reeb orbit, we thus have

$$\mathrm{CZ}^{ECH}_\tau((\eta, m)) - \mathrm{CZ}^{ECH}_\sigma((\eta, m)) = \sum_{i=1}^{m} i \cdot (\sigma - \tau) = \frac{m(m+1)}{2} \cdot (\sigma - \tau)$$

In the case of a chord, the difference formula follows directly from the definition of CZ^{ECH}_τ. □

Partition Conditions

We next briefly review the ECH partition conditions. Let $P(m)$ denote the set of partitions of an integer $m \in \mathbb{N}$.

Definition 177. The *positive partition* $p^+_\gamma(m)$ and *negative partition* $p^-_\gamma(m)$ associated to a non-degenerate Reeb orbit γ in (Y, α) are partitions in $P(m)$ defined as follows.

Fix a trivialization of ξ along γ and let L be the period of γ. Consider the linearized flow

$$\phi_\tau = \tau_{\gamma(0)} \circ d\phi_L(\gamma(0))|_\xi \circ \tau^{-1}_{\gamma(0)} \in \mathrm{Sp}(2)$$

- If γ is *positive hyperbolic* (i.e. ϕ_τ has positive real eigenvalues) then

$$p^+_\gamma(m) = p^-_\gamma(m) = (1, 1, \dots, 1)$$

- If γ is *negative hyperbolic* (i.e. ϕ_τ has negative real eigenvalues) then

$$p^+_\gamma(m) = p^-_\gamma(m) = \begin{cases} (2, 2, \dots, 2) & \text{if } m \text{ is even} \\ (2, 2, \dots, 2, 1) & \text{if } m \text{ is odd} \end{cases}$$

- If γ is *elliptic* (i.e. ϕ_τ has unit complex eigenvalues) then ϕ_τ is conjugate to a rotation by angle $2\pi\theta$ for $\theta \in \mathbb{R}/\mathbb{Z}$. Then

$$p^+_\gamma(m) = p^+_\theta(m) \qquad \text{and} \qquad p^-_\gamma(m) = p^-_\theta(m)$$

Here $p^{\pm}_{\theta}(m)$ are partitions defined using only the rotation angle θ as follows. Let $\Lambda^{+}_{\theta}(m) \subset \mathbb{R}^2$ be the maximal concave polygonal graph with vertices at lattice points in $\mathbb{Z}^2$ that starts on $(0,0)$, ends on $(m, \lfloor m\theta \rfloor)$ and lies below the line $y = \theta x$ where $\theta \in (0,1)$. Then $p^{+}_{\theta}(m)$ is the sequence of horizontal displacements of the consecutive vertices of $\Lambda^{+}_{\theta}(m)$. We can define $p^{-}_{\theta}(m) = p^{+}_{-\theta}(m)$.

Definition 178. A somewhere injective J-holomorphic curve C in (Y, Λ) from orbit set Ξ_+ to orbit set Ξ_- *satisfies the ECH partition conditions* if

- For each orbit γ of multiplicity $m \geq 1$ in Γ_+, the partition of γ determined by the positive end of C at γ is $p_+(m, \gamma)$.
- For each orbit γ of multiplicity $m \geq 1$ in Γ_-, the partition of γ determined by the negative end of C at γ is $p_-(m, \gamma)$.

Given a simple, elliptic Reeb orbit γ of Y and two partitions $p, q \in P(m)$, we write

$$p \prec_{\gamma} q$$

if there is a degree m J-holomorphic branched cover

$$u : \Sigma \to \mathbb{R} \times \gamma \subset \mathbb{R} \times Y$$

with Fredholm index 0 and whose ends yield the partition q on the positive end and p at the negative end.

Lemma 179. [52, Exercise 3.1 and A.1.4] The relation $\prec_{\gamma}$ is a partial order on $P(m)$. Moreover, $p^{+}_{\gamma}(m)$ and $p^{-}_{\gamma}(m)$ are respectively minimal and maximal with respect to this partial order.

A helpful consequence of Lemma 179 is the following lemma controlling the triviality of trivial cylinders over closed Reeb orbits.

Lemma 180. Let γ be a simple closed Reeb orbit in Y. Fix a sequence of partitions $p_1, \ldots, p_n$ of m and let Z_i be a branched cover of $\mathbb{R} \times \gamma$ with positive partition p_i and negative partition p_{i+1}. Assume that

$$p_1 = p_n = p^{+}_{\gamma}(m) \qquad \text{or} \qquad p_1 = p_n = p^{-}_{\gamma}(m) \tag{4.6.2}$$

Then Z_i is a trivial cover and $p_i = p^{\pm}_{\gamma}(m)$ for each i.

Proof. We may view $\mathbf{Z} = (Z_1, \dots, Z_n)$ as an SFT building (cf. [1]) with Fredholm index

$$\mathrm{ind}(\mathbf{Z}) = \sum_i \mathrm{ind}(Z_i) = 0$$

By [60, Lem. 1.7(a)], we know that $\mathrm{ind}(Z_i) \geq 0$ for each i. Therefore $\mathrm{ind}(Z_i) = 0$ for each i. Moreover, [60, Lem. 1.7(b)] states that the cover is unbranched if γ is hyperbolic. If γ is elliptic, then

$$p^{\pm}_{\gamma}(m) \prec_{\gamma} p_i \prec_{\gamma} p^{\pm}_{\gamma}(m)$$

for each i. It follows from the fact that $\prec_{\gamma}$ is a partial order that $p_i = p^{\pm}_{\gamma}(m)$ for each i. The Fredholm index is then given by

$$\mathrm{ind}(Z_i) = -\chi(\Sigma_i) = 0$$

where Σ_i is the domain of the cover Z_i. Riemann-Hurwitz then implies that Z_i is unbranched. □

There is also an analogous lemma for chords, which we include here for completeness.

Lemma 181. Let c be a Reeb chord in (Y, Λ) and let Z be an m-fold branched cover of $\mathbb{R} \times c$. Then $\mathrm{ind}(Z) \geq 0$ with equality if and only if Z is unbranched.

Proof. Let Σ be the domain of Z. We may double $\mathbb{R} \times c$ to a cylinder $\mathbb{R} \times S^1$ and double Σ to a surface S. Then the curve Z is an equivalence class of branched cover

$$u : S \to \mathbb{R} \times S^1$$

By choosing a trivialization of ξ along c and extending it along Z, we find that

$$\mathrm{ind}(Z) = -\bar{\chi}(\Sigma) = \frac{-\chi(S)}{2}$$

Note that S has at least 2 punctures, and thus has at non-negative Euler characteristic. Thus $\mathrm{ind}(Z) = 0$. If $\mathrm{ind}(Z) = -\frac{\chi(S)}{2} = 0$, then Riemann-Hurwitz implies that the cover u is unbranched. This implies that Z is unbranched. □

Writhe And Linking Bounds

We are now ready to prove the Legendrian version of the writhe bound, and a separate linking bound that will be used to prove the ECH index inequality.

Lemma 182 (Writhe-Linking Bound At Chord)**.** Let C be a somewhere injective, J-holomorphic curve with respect to a tailored J on (Y, Λ), asymptotic to the orbit-chord sets $\Xi_\pm$ at $\pm\infty$. Then

- For any chord η in the orbit-chord set Ξ_+, the asymptotic braid ζ of C in $\text{Nbhd}(\eta)$ satisfies

$$w_\sigma(\zeta) \le 0 \quad \text{in the unique trivialization } \sigma \text{ with } \text{CZ}_\sigma(\eta) = \frac{1}{2}$$

  and the component braids ζ_i of ζ satisfy

$$l_\sigma(\zeta_i, \zeta_j) \le 0 \qquad \text{and} \qquad \text{wind}_\sigma(\zeta_i) \le 0$$

- For any chord η in the orbit-chord set Ξ_-, the asymptotic braid ζ of C in $\text{Nbhd}(\eta)$ satisfies

$$w_\sigma(\zeta) \ge 0 \quad \text{in the unique trivialization } \sigma \text{ with } \text{CZ}_\sigma(\eta) = -\frac{1}{2}$$

  and the component braids ζ_i of ζ satisfy

$$l_\sigma(\zeta_i, \zeta_j) \ge 0 \qquad \text{and} \qquad \text{wind}_\sigma(\zeta_i) \ge 0$$

Moreover, equality occurs if and only if ζ is the trivial braid in the trivialization σ.

Proof. We prove the result for a chord in Ξ_+. The proof is entirely analogous in the other case.

Consider a strip-like end of the holomorphic curve C which is positively asymptotic to the chord η. For $s_0 \gg 0$ sufficiently large, the intersection of C with the half infinite cylinder $[s_0, \infty) \times Y$ is given by the graph of a function

$$[s_0, \infty) \times [0, 1] \to [s_0, \infty) \times Y \quad \text{given by} \quad (s, t) \mapsto (s, \exp_{\eta(t)} U(s, t))$$

Here exp denotes the exponential map of an auxiliary Riemannian metric on Y with the property that the Legendrian Λ consists of closed geodesics. Moreover, U is a family of sections

$$U : ([s_0, \infty) \times [0, 1], [0, \infty) \times \{0, 1\}) \to \eta^*(\xi, T\Lambda)$$

of the bundle pair $\eta^*(\xi, T\Lambda)$. The following result of Abbas states that U decays exponentially as $s \to \infty$.

Theorem 183. *([2]) There exists an eigenvector e of the asymptotic operator of η with negative eigenvalue λ and a family $r(s,t)$ of sections of $\eta^*\xi$ exponentially decaying as s tends to ∞ such that*

$$U(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

We require two ingredients from other parts of this paper for this proof. First, we have the following refinement of Theorem 183. Consider two strip-like ends of C positively asymptotic to η. Let U and V denote the associated families of sections of $\eta^*\xi$.

Theorem 184. *Assume that U and V do not agree identically. Then there exist an eigenvector e of the asymptotic operator of η with negative eigenvalue λ and a family $r(s,t)$ of sections of $\eta^*\xi$ exponentially decaying as s tends to ∞ such that*

$$U(s,t) - V(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

The proof of Theorem 184 requires a rather long analytical digression and is deferred to the Appendix. Second, we require the following result on the winding number of e in Theorem 184.

Lemma 185. Let σ be a representative of the unique homotopy class of trivializations of $\eta^*\xi$ such that

$$\mathrm{CZ}_\sigma(\eta) = \frac{1}{2}$$

Then the winding number of e with respect to σ is strictly positive if $\lambda > 0$ and non-positive if $\lambda < 0$.

This is an immediate corollary of Lemma 166 and Lemma 167 in Section 4.5.

We now proceed with the proof of Lemma 182. Let ζ_i denote the components of the braid ζ. By Theorem 184 and Lemma 185, the winding numbers and pairwise linking numbers satisfy

$$\mathrm{wind}_\sigma(\zeta_i) \leq 0 \qquad \text{and} \qquad l_\sigma(\zeta_i, \zeta_j) \leq 0 \qquad \text{for all } i \neq j$$

Hence it follows from (4.2.6) that the writhe $w_\sigma(\zeta)$ is non-positive, proving the inequality.

Finally, we claim that the braid ζ must be trivial with respect to σ if $w_\sigma(\zeta) = 0$. The asymptotic operator of η has a 1-dimensional eigenspace whose elements have zero winding number with respect to σ. Let e be a non-zero vector in this eigenspace, $\lambda < 0$ be the associated eigenvalue and $U_i(s,t)$ be the section of $\eta^*\xi$ associated to the braid ζ_i. Since $l_\sigma(\zeta_i, \zeta_j)$ is non-positive for all $i \neq j$ and the writhe $w_\sigma(\zeta)$ vanishes by assumption, we can deduce that $l_\sigma(\zeta_i, \zeta_j) = 0$ for all $i \neq j$. Thus, for every tuple $i \neq j$, there exist a non-zero real number a_{ij} and an exponentially decaying family of sections $r_{ij}(s,t)$ such that

$$U_i(s,t) - U_j(s,t) = e^{\lambda s}(a_{ij}e(t) + r_{ij}(s,t)) \tag{4.6.3}$$

We may replace the trivialization σ by a homotopic one with the property that $e(t)$ is constant with respect to the new trivialization. Then it follows from (4.6.3) that the braid ζ is trivial. □

We can now prove Proposition 186. It largely follows from the local version for chords.

Proposition 186 (Writhe Bound)**.** Let C be a somewhere injective J-holomorphic curve in (Y, Λ) asymptotic to $\Gamma_\pm = (\gamma_i^\pm)$ at $\pm\infty$. Then

$$w_\tau(C) \leq \mathrm{CZ}_\tau^{ECH}(C) - \mathrm{CZ}_\tau(\Gamma_+) + \mathrm{CZ}_\tau(\Gamma_-)$$

Moreover, there is equality only if following conditions are satisfied.

- The orbit parts of Γ_+ and Γ_- satisfy the ECH partition conditions.
- The chords in Γ_- are multiplicity one.

Proof. The writhe bound is additive under disjoint union of simple orbits and chords. Therefore, it suffices to consider each orbit and chord in $\Xi_\pm$ independently. For Reeb orbits, the result is proven in [52, §5.1]. For Reeb chords, we may write

$$\mathrm{CZ}_\tau^{ECH}(c,m) - m \cdot \mathrm{CZ}_\tau(\eta) = \frac{m(m-1)}{2} \cdot (\mathrm{CZ}_\tau(c) - \frac{1}{2})$$

Thus, it is sufficicient to prove to the following pair of statements.

- If η is a Reeb chord that appears in Ξ_+ with multiplicity m and ζ is the corresponding asymptotic braid of C, then

$$w_\tau(\zeta) \leq \frac{m(m-1)}{2} \cdot (\mathrm{CZ}_\tau(\eta) - \frac{1}{2}) \tag{4.6.4}$$

- If η is a Reeb chord that appears in Ξ_- with multiplicity n and ζ is the corresponding asymptotic braid of C, then

$$w_\tau(\zeta) \geq \frac{m(m-1)}{2} \cdot (\mathrm{CZ}_\tau(\eta) - \frac{1}{2}), \tag{4.6.5}$$

  Moreover, equality holds if only if $m = 1$.

Also, observe that the above inequalities are equivalent for different choices of the trivialization τ on c. If σ and τ are two trivializations of ξ along c with difference $\tau - \sigma \in \frac{1}{2}\mathbb{Z}$, then

$$w_\tau(\zeta) - w_\sigma(\zeta) = m(m-1) \cdot (\sigma - \tau) \qquad \text{by Lemma 106} \tag{4.6.6}$$

$$CZ_\tau(\eta) - CZ_\sigma(\eta) = 2 \cdot (\sigma - \tau)) \qquad \text{by Lemma 176} \tag{4.6.7}$$

So it suffices to prove the inequalities for any choice of trivialization.

For the first claim, we choose the trivialization τ so that $\mathrm{CZ}_\tau(\eta) = \frac{1}{2}$. Then Proposition 182 implies that

$$w_\tau(\zeta) \leq 0 = \frac{m(m-1)}{2} \cdot (\mathrm{CZ}_\tau(\eta) - \frac{1}{2}). \tag{4.6.8}$$

For the second claim, we choose τ so that $\mathrm{CZ}_\tau(\eta) = -\frac{1}{2}$. Proposition 182 implies that

$$w_\tau(\zeta) \geq 0 \geq -\frac{m(m-1)}{2} = \frac{m(m-1)}{2} \cdot (\mathrm{CZ}_\tau(\eta) - \frac{1}{2}) \tag{4.6.9}$$

with equality holds only if the multiplicity $n = 1$. □

Remark 187 (Writhe Asymmetry)**.** The observant reader will notice that there is an asymmetry in the chord condition required for the writhe bound to yield an equality. This is an artifact of our convention for the Legendrian ECH index. We could have alternatively defined it as

$$-\frac{m}{2} + \frac{m(m+1)}{2} \cdot (\mathrm{CZ}_\tau(c) + \frac{1}{2})$$

Then the partition conditions imposed on the positive chords and negative chords would have been reversed. An asymmetry of this type is more or less unavoidable.

There is also an analogous statement to the writhe inequality about the linking number of holomorphic currents.

Proposition 188. Let $\mathcal{C}$ and $\mathcal{D}$ be a pair of J-holomorphic currents with boundary that have disjoint components. Then

$$\mathrm{CZ}_\tau^{ECH}(\mathcal{C} \cup \mathcal{D}) \geq \mathrm{CZ}_\tau^{ECH}(\mathcal{C}) + \mathrm{CZ}_\tau^{ECH}(\mathcal{D}) + 2 \cdot l_\tau(\mathcal{C}, \mathcal{D})$$

Proof. Let (C_i, m_i) and (D_j, n_j) denote the components of $\mathcal{C}$ and $\mathcal{D}$. Fix an orbit or chord η and let

$$\zeta_i^\pm \qquad \text{and} \qquad \xi_j^\pm$$

denote the braids of C_i and D_j asymptotic to η at the positive and negative ends respectively. We fix the shorthand notation

$$l_\tau(\zeta^+, \xi^+) = \sum_{i,j} m_i \cdot n_j \cdot l_\tau(\zeta_i^+, \xi_j^+) \qquad \text{and} \qquad l_\tau(\zeta^-, \xi^-) = \sum_{i,j} m_i \cdot n_j \cdot l_\tau(\zeta_i^-, \xi_j^-)$$

Finally, let $m_\pm$ and $n_\pm$ denote the multiplicity of η at the positive and negative ends of $\mathcal{C}$ and $\mathcal{D}$, respectively. It suffices to prove that

$$\mathrm{CZ}_\tau^{ECH}((\eta, m_+ + n_+)) - \mathrm{CZ}_\tau^{ECH}((\eta, m_+)) - \mathrm{CZ}_\tau^{ECH}((\eta, n_+)) - 2 \cdot l_\tau(\zeta^+, \xi^+) \geq 0 \tag{4.6.10}$$

$$\mathrm{CZ}_\tau^{ECH}((\eta, m_- + n_-)) - \mathrm{CZ}_\tau^{ECH}((\eta, m_-)) - \mathrm{CZ}_\tau^{ECH}((\eta, n_-)) - 2 \cdot l_\tau(\zeta^-, \xi^-) \leq 0 \tag{4.6.11}$$

for any choice of η. This is proven by Hutchings [55, Eq. 5.6 on p. 47] if η is a Reeb orbit. Thus, we assume that η is a Reeb chord. Finally, as in the proof of Proposition 186, Lemma 158 and Lemma 176 imply that the claim is independent of trivialization.

Next choose the trivialization τ so that $\mathrm{CZ}_\tau(\eta) = \frac{1}{2}$. Then the ECH Conley-Zehnder term is

$$\mathrm{CZ}_\tau^{ECH}((\eta, k)) = 0 \qquad \text{for any integer } k$$

Moreover, by Lemma 182, the linking numbers of ζ_i^+ and ξ_j^+ satisfies

$$l_\tau(\zeta_i^+, \xi_j^+) \leq 0 \qquad \text{for all } i, j$$

Note that the linking number is the winding number, by convention, if C_i or D_j is a trivial cylinder over η. These two inequalities imply (4.6.10). Finally, choose the trivialization σ so that $\mathrm{CZ}_\tau(\eta) = -\frac{1}{2}$. Then

$$\mathrm{CZ}_\tau^{ECH}((\eta, k)) = -\frac{k(k-1)}{2} \qquad \text{for any integer } k$$

In particular, the ECH Conley-Zehnder term in (4.6.11) satisfies

$$\mathrm{CZ}_\tau^{ECH}((\eta, m_- + n_-)) - \mathrm{CZ}_\tau^{ECH}((\eta, m_-)) - \mathrm{CZ}_\tau^{ECH}((\eta, n_-)) < 0$$

Moreover, by Lemma 182, the linking numbers of ζ_i^- and ξ_j^- satisfies

$$l_\tau(\zeta_i^-, \xi_j^-) \geq 0 \qquad \text{for all } i, j$$

This proves (4.6.11) and completes the proof. □

ECH Index And Basic Properties

We are now ready to introduce the Legendrian ECH index in detail, and prove its basic properties.

Definition 189. The *Legendrian ECH index* $I(A)$ of a surface class $A \in S(\Theta, \Xi)$ between orbit-chord sets Θ and Ξ is given by the following formula.

$$I(A) := Q_\tau(A) + \frac{1}{2}\mu_\tau(A) + \mathrm{CZ}_\tau^{ECH}(\Theta) - \mathrm{CZ}_\tau^{ECH}(\Xi) \tag{4.6.12}$$

The terms in the formula above are as follows.

- τ is any trivialization of the bundle pair $(\xi, T\Lambda)$ over Θ and Ξ (see Definition 92)
- μ_τ is the (relative) Maslov number (see Definition 98)
- Q_τ is the relative self intersection number with respect to τ (see Definition 114)
- CZ_τ^{ECH} is the ECH Conley-Zehnder term (see Definition 175).

The Legendrian ECH index of a J-holomorphic current $\mathcal{C} \in \mathcal{M}(\Theta, \Xi)$ is simply the index of its corresponding surface class $A \in S(\Theta, \Xi)$.

$$I(\mathcal{C}) := I(A)$$

Lemma 190. The Legendrian ECH index $I(A)$ of a surface class $A : \Theta \to \Xi$ is an integer that is independent of the trivialization in (4.6.12).

Proof. Since $I(A)$ is a sum of half-integer valued terms, it is a half integer. To show that $I(A)$ is an integer, fix a well-immersed surface S representing A. Let $m(S)$ denote the number of open boundary components of $\partial_+ S \cup \partial_- S$, and note that

$$\bar{\chi}(S) = \frac{1}{2} \cdot m(S) \mod 1$$

By topological adjunction (Theorem 117), we have

$$\frac{1}{2}\mu_\tau(S) + Q_\tau(S) = \bar{\chi}(S) = \frac{1}{2} \cdot m(S) \qquad \mod 1$$

On the other hand, the ECH Conley-Zehnder term is a sum of the (integer) CZ-indices of closed orbits and the CZ-indices of chords. The Conley-Zehnder index of a chord in dimension three is automatically a strict half-integer. Since the total number of chords in Θ and Ξ (with multiplicity) is $m(S)$, we find that

$$\mathrm{CZ}_\tau^{ECH}(\Xi) - \mathrm{CZ}_\tau^{ECH}(\Theta) = \frac{1}{2} \cdot m(S) \mod 1$$

Thus we find that

$$I(A) = \frac{1}{2}\mu_\tau(S) + Q_\tau(S) + \mathrm{CZ}_\tau^{ECH}(\Xi) - \mathrm{CZ}_\tau^{ECH}(\Theta) = m(S) = 0 \mod 1$$

To see that $I(A)$ is independent of trivialization, fix two trivializations σ and τ of ξ along $\Xi_\pm$. We write the orbit sets Θ and Ξ as follows.

$$\Xi = \{(\gamma_i, m_i)\} \qquad \text{and} \qquad \Theta = \{(\eta_j, n_j)\}$$

Fix an orbit or chord $\eta \subset \Xi$ with multiplicity m. Let σ and τ be two trivializations of ξ along Ξ and Θ that agree everywhere except along η. We compute that

$$Q_\tau(A) - Q_\sigma(A) = m^2 \cdot (\tau - \sigma) \qquad \text{by Lemma 115}$$

$$\mu_\tau(A) - \mu_\sigma(B) = \sigma - \tau \qquad \text{by Proposition 99}$$

$$\mathrm{CZ}_\tau^{ECH}(\Xi) - \mathrm{CZ}_\sigma^{ECH}(\Xi) = m(m-1) \cdot (\sigma - \tau) \qquad \text{by Lemma 176}$$

The sum of these terms is zero, thus yielding the desired invariance property. An identical proof works if τ and σ only differ along an orbit or chord of Θ. Any two trivializations are related by a sequence of changes supported on one orbit or chord, so this proves the result. □

The Legendrian ECH index has the following basic properties, generalizing the corresponding properties of the ordinary ECH index (cf. [50, Proposition 1.6]).

Proposition 191 (Basic Properties)**.** The Legendrian ECH index satisfies the following axioms.

- (Index Ambiguity) If $A, B : \Xi_+ \to \Xi_-$ are two classes with difference $B - A \in H_2(Y, \Lambda)$, and let $\Gamma \in H_1(Y, \Lambda)$ denote the homology class of $\Xi_\pm$. Then
$$I(B) - I(A) = \frac{1}{2}\langle \mu(\xi, \Lambda), B - A\rangle + 2 \cdot Q(\Gamma, B - A)$$
where $\mu(\xi, \Lambda)$ is the Maslov class of (ξ, Λ) and Q is the intersection pairing of (Y, Λ).

- (Composition) If $A : \Xi_0 \to \Xi_1$ and $B : \Xi_1 \to \Xi_2$ are composible surface classe then
$$I(A \circ B) = I(A) + I(B)$$

Proof. We prove each of these properties separately.

Index Ambiguity. Let $A, B : \Theta \to \Xi$ be two surface classes with the same ends and fix a trivialization τ over Θ and Ξ. We compute the difference of each term in $I(A)$ and $I(B)$. Starting with the Maslov number, we have

$$\mu_\tau(B) - \mu_\tau(A) = \mu(\xi, \Lambda) \cdot (B - A) \qquad \text{by Proposition 99}$$

To compute the difference between the self-intersection numbers, we note that

$$Q_\tau(B) - Q_\tau(A) = Q_\tau(B, B) - Q_\tau(A, B) + Q_\tau(B, A) - Q_\tau(A, A) = q_B(B - A) + q_A(B - A)$$

Here q_A and q_B are the homomorphisms $H_2(Y, \Lambda) \to \frac{1}{2}\mathbb{Z}$ in Definition 112. By Lemma 113
$$q_A(B - A) + q_B(B - A) = 2 \cdot Q(\Gamma, B - A)$$
Finally, since A and B have the same ends, the ECH Conley-Zehnder terms coincide. This proves the desired formula.

Composition. This follows from the corresponding composition property for Q_τ and μ_τ. □

Index Inequality And Union Property

The Legendrian ECH index has a number of properties that hold for currents with boundary due to the writhe and linking inequalities. We are now ready to demonstrate these properties.

We begin with the the Legendrian ECH index inequality as stated in the introduction.

Theorem 192 (Theorem 7)**.** *Let C be a somewhere injective, J-holomorphic curve with boundary in $(\mathbb{R}\times Y, \mathbb{R}\times\Lambda)$ for tailored J. Then*

$$\operatorname{ind}(C) \leq I(C) - 2\delta(C) - \epsilon(C). \tag{4.6.13}$$

Proof. Let C be a somewhere injective J-holomorphic curve asymptotic to Reeb chords and orbits $\Gamma_\pm$ at $\pm\infty$. Then the difference between ECH index and Fredholm index is

$$I(C) - \operatorname{ind}(C) = Q_\tau(C) - \frac{1}{2}\mu_\tau(C) + \mathrm{CZ}^{ECH}_\tau(\Gamma_+) - \mathrm{CZ}^{ECH}_\tau(\Gamma_-) + \bar{\chi}(C) - \mathrm{CZ}_\tau(\Gamma_+) + \mathrm{CZ}_\tau(\Gamma_-)$$

Then by Legendrian adjunction, Theorem 144, the above formula becomes

$$Q_\tau(C) + \mathrm{CZ}^{ECH}_\tau(\Gamma_+) - \mathrm{CZ}^{ECH}_\tau(\Gamma_-) - \mathrm{CZ}_\tau(\Gamma_+) + \mathrm{CZ}_\tau(\Gamma_-) - Q_\tau(C) - w_\tau(C) + 2\delta(C) + \epsilon(C)$$

Here the writhe $w_\tau(C)$ is the sum of the writhes of the positive asymptotic braid ζ_+ and negative asymptotic braid ζ_- of C. Thus we may write the above formula as

$$= 2\delta(C) + \epsilon(C) + (\mathrm{CZ}^{ECH}_\tau(\zeta_+) - \mathrm{CZ}_\tau(\Gamma_+) - w_\tau(\zeta_+)) - (\mathrm{CZ}^{ECH}_\tau(\zeta_-) - \mathrm{CZ}_\tau(\Gamma_-) - w_\tau(\zeta_-))$$

The writhe bound, Proposition 186, implies that the middle and right terms are non-negative. Thus we have proven that

$$I(C) - \operatorname{ind}(C) \geq 2\delta(C) + \epsilon(C)$$ □

Next, we prove a fundamental sub-additivity property of the ECH index under union. This generalizes [55, Thm. 5.1] to currents with boundary.

Theorem 193 (Union)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be J-holomorphic currents for a tailored J on (Y, Λ), with distinct components. Then the ECH index satisfies*

$$I(\mathcal{C} \cup \mathcal{D}) \geq I(\mathcal{C}) + I(\mathcal{D}) + 2(\mathcal{C} \cdot \mathcal{D})$$

Proof. We compute that the difference of ECH indices is given by

$$I(\mathcal{C} \cup \mathcal{D}) - I(\mathcal{C}) - I(\mathcal{D}) = 2 \cdot Q_\tau(\mathcal{C}, \mathcal{D}) + \mathrm{CZ}_\tau^{ECH}(\mathcal{C} \cup \mathcal{D}) - \mathrm{CZ}_\tau^{ECH}(\mathcal{C}) - \mathrm{CZ}_\tau^{ECH}(\mathcal{D})$$

By Lemma 159, this can be rewritten using the geometric intersection number as

$$2(\mathcal{C} \cdot \mathcal{D}) + \big(\mathrm{CZ}_\tau^{ECH}(\mathcal{C} \cup \mathcal{D}) - \mathrm{CZ}_\tau^{ECH}(\mathcal{C}) - \mathrm{CZ}_\tau^{ECH}(\mathcal{D}) - 2 \cdot l_\tau(\mathcal{C}, \mathcal{D})\big)$$

The second term above is non-negative by the linking bound for currents (Proposition 188). □

4.7 Legendrian ECH

We are now ready to construct Legendrian embedded contact homology in detail. Specifically, we prove the two key technical results Theorem 10 and Theorem 11 from the introduction.

Setup 194. To proceed with our construction, we fix the following setup for the rest of this section.

(a) (Y, ξ) is a contact 3-manifold with convex sutured boundary ∂Y.

(b) $\Lambda = \Lambda_+ \cup \Lambda_-$ is a union of exact Legendrians $\Lambda_\pm \subset \partial_\pm Y$.

(c) α is a non-degenerate, adapted contact form on Y.

(d) J is a tailored complex structure on ξ

We start by recalling the definition of an ECH generator.

Definition 195. An *ECH generator* of (Y, Λ) is an orbit-chord set $\Theta = \{(\gamma_i, m_i)\} \cup \{(c_i, n_i)\}$ where

- Every hyperbolic orbit γ_i has multiplicity 1.
- Every chord c_i is multiplicity 1.
- There is at most one Reeb chord incident to L in Θ for each connected component L of Λ

Classification Of Low ECH Index Currents

We begin by classifying low ECH index currents with boundary in Lemmas 196 and Proposition 197 and 198 below. These results, together, yield Theorem 10 in the introduction.

Lemma 196. Let $\mathcal{C}$ be an J-holomorphic current for a regular, tailored almost complex structure J on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$. Then the following are equivalent.

- $\mathcal{C}$ is a trivial current $\mathbb{R} \times \Xi$ over an orbit-chord set Ξ.
- $\mathcal{C}$ has ECH index less than or equal to zero.

Proof. By the Legendrian ECH index inequality (Theorem 192) and the sub-additivity of the ECH index

$$\operatorname{ind}(C_i) \le I(C_i) - 2\delta(C_i) - \epsilon(C_i) \le I(\mathcal{C}) \le 0 \qquad \text{where} \qquad \mathcal{C} = \{(C_i, m_i)\}$$

For any regular almost complex structure J, the Fredholm index of any somewhere injective curve is non-negative, and zero if and only if the curve is a trivial cylinder or strip. Therefore

$$C_i = \mathbb{R} \times \gamma_i \qquad \text{for an simple orbit or chord } \gamma_i \subset Y$$

It follows that $\mathcal{C}$ is the trivial current $\mathbb{R} \times \Xi$ over the orbit-chord set $\Xi = \{(\gamma_i, m_i)\}$. □

Proposition 197. Let $\mathcal{C}$ be a J-holomorphic current for a regular, tailored almost complex structure on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ of ECH index one. Then

$$\mathcal{C} = C \sqcup \mathcal{T}$$

where $\mathcal{T}$ is trivial and C is an embedded, connected curve with $\operatorname{ind}(C) = I(C) = 1$.

Proof. We may write $\mathcal{C} = \mathcal{S} \cup \mathcal{T}$ where $\mathcal{T}$ is a trivial current $\mathbb{R} \times \Xi$ and $\mathcal{S}$ is a current with no trivial components. By Lemma 196 and the sub-additivity of the ECH index (Theorem 193), we have

$$0 < I(\mathcal{S}) \qquad \text{and} \qquad I(\mathcal{S}) \le I(\mathcal{S}) + I(\mathcal{T}) + 2(\mathcal{S} \cdot \mathcal{T}) \le I(\mathcal{C}) \le 1$$

Thus, $\mathcal{S}$ is non-empty and $\mathcal{T}$ is disjoint from $\mathcal{S}$. We can then write

$$\mathcal{S} = \{(C_i, m_i)\} \qquad \text{for simple, non-trivial curves } C_i \text{ and multiplicities } m_i$$

If (C_i, m_i) has multiplicity $m_i > 1$, we may translate m_i of copies of C_i along the symplectization direction to obtain m_i somewhere injective curves whose union forms a new J-holomorphic curve C'.

We now apply the ECH index inequality (Theorem 192) to C' and obtain:

$$\sum_i m_i \operatorname{ind}(C_i) \leq \sum_i I(\mathcal{C}) - 2\delta(C') - \epsilon(C') \tag{4.7.1}$$

since the Fredholm index is additive under taking unions and ECH index is only dependent on the relative homology class. Since J is regular, $\operatorname{ind}(C_i) \geq 0$ and in fact $\operatorname{ind}(C_i) > 0$ by our assumption of $\mathcal{S}$ and Lemma 196. This contradicts (4.7.1). Furthermore,

$$\mathcal{S} = \{(C_1, 1)\} \qquad \text{for } C_1 \text{ an embedded, non-trivial curve}$$

since $\delta(C_1) = \epsilon(C_1) = 0$ by (4.7.1) again. Therefore, $\operatorname{ind}(C_1) = I(C_1) = 1$.

□

Proposition 198. Let $\mathcal{C}$ be a J-holomorphic current for a regular, compatible almost complex structure on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ of ECH index two. Also assume that the ends Ξ_+ and Ξ_- of $\mathcal{C}$ are ECH generators. Then

$$\mathcal{C} = \mathcal{S} \sqcup \mathcal{T}$$

where $\mathcal{T}$ is trivial and $\mathcal{S}$ is a current of one of the following types.

- A pair of disjoint, embedded curves $C_1 \sqcup C_2$ with $I(C_i) = \operatorname{ind}(C_i) = 1$. Note here that C_1 and C_2 are distinct in the sense they are not $\mathbb{R}$ translates of each other.
- A single embedded curve C of multiplicity 1 with $I(C) = \operatorname{ind}(C) = 2$.

To prove Proposition 198, the following lemma will be helpful. It severely limits the type of boundary singularities that can occur in currents connecting ECH generator.

Lemma 199 (Singularity Censorship)**.** Let $\mathcal{C}$ be a (finite energy, proper) J-holomorphic current with boundary between ECH generators Ξ_+ and Ξ_-. Then

- $\mathcal{S} \cdot \mathcal{T}$ is an integer for any pair of currents $\mathcal{S}$ and $\mathcal{T}$ with disjoint components such that $\mathcal{S} \cup \mathcal{T} \subset \mathcal{C}$.

- Every component curve C of $\mathcal{C}$ is non-singular near the boundary. That is $\epsilon(C) = 0$.

Proof. For the first claim, it suffices to show that any pair of component curves C and D satisfy

$$C \cap D \cap (\mathbb{R} \times \Lambda) = \emptyset$$

Thus suppose otherwise. Then there are components Γ and Γ' of the boundary of C and D, respectively, such that $\Gamma \cap \Gamma'$ is non-empty and lie on $\mathbb{R} \times L$ where $L \subset \Lambda$ is a component.

By Lemma 131, both components Γ and Γ' are diffeomorphic to $\mathbb{R}$. Thus C and D both have punctures asymptotic at $+\infty$ to a Reeb chord of L. Thus Ξ_+ has either a Reeb chord of multiplicity two or two Reeb chords incident to a single component L of Λ. This is a contradiction, since $\Xi_\pm$ is an ECH generator. The second claim follows immediately from Lemma 131. □

proof of Proposition 198. We again write $\mathcal{C} = \mathcal{S} \cup \mathcal{T}$ where $\mathcal{T}$ is a trivial current $\mathbb{R} \times \Xi$ and $\mathcal{S}$ is a current with no trivial components. Applying Lemma 196 and Theorem 193 now yields

$$0 < I(\mathcal{S}) \qquad \text{and} \qquad I(\mathcal{S}) \leq I(\mathcal{S}) + I(\mathcal{T}) + 2(\mathcal{S} \cdot \mathcal{T}) \leq I(\mathcal{C}) \leq 2$$

Since $\mathcal{C}$ has ends on ECH generators, $\mathcal{S}$ and $\mathcal{T}$ are disjoint on $\mathbb{R} \times \Lambda$. By Lemma 199, we must have $\mathcal{S} \cdot \mathcal{T} = 0$ or 1. In the later case, $I(\mathcal{S}) = 0$. Thus we must have

$$I(\mathcal{S}) = 2 \quad \text{and} \quad \mathcal{S} \cdot \mathcal{T} = \emptyset$$

As in Proposition 197, we can assume that $\mathcal{S}$ consists of connected, somewhere injective components C_i of multiplicity 1. We apply the ECH index inequality (Theorem 192) and the sub-additivity of the ECH index (Theorem 193) to find that

$$\sum_i \operatorname{ind}(C_i) \leq \sum_i I(C_i) - 2\delta(C_i) - \epsilon(C_i) \qquad \text{and} \qquad \sum_i I(C_i) + 2\sum_{i<j} C_i \cdot C_j \leq I(\mathcal{S}) = 2$$

Since each C_i is non-trivial, we must have $\operatorname{ind}(C_i) > 0$ for each i. Thus we infer from Lemma 199 that $C_i \cdot C_j = 0$ for each i and j and $\delta(C_i) = \epsilon(C_j) = 0$.

Finally we rule out the case of $\mathcal{C}$ containing a nontrivial curve C with multiplicity two. By assumption on ECH generators, C cannot contain chords or be asymptotic to hyperbolic Reeb orbits. Then the same proof as in regular ECH shows $I(C) - \operatorname{ind}(C)$ is even. Hence we cannot have nontrivial curves of multiplicity ≥ 2. □

Compactness Of Low ECH Index Moduli Spaces

Next, we prove the requisite compactness properties for the low ECH index moduli spaces required in ECH.

We begin by proving a crude topological lower bound for the (corrrected) Euler characteristic of J-holomorphic curves between ECH generators.

Lemma 200 (Topological Index Bound)**.** Let C be a (proper, finite energy) J-holomorphic curve in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ between ECH generators Ξ_+ and Ξ_- in the surface class A. Then

$$-\bar{\chi}(C) \leq -\frac{1}{2} \cdot \mu_\tau(A) + Q_\tau(A) + \mathrm{CZ}_\tau^{ECH}(\check{\Xi}_+) - \mathrm{CZ}_\tau^{ECH}(\check{\Xi}_-)$$

Here $\check{\Xi}_+$ and $\check{\Xi}_-$ are the orbit-chord sets made from Ξ_+ and Ξ_- by reducing the multiplicity of each orbit and chord by one (and removing everything of multiplicity one).

Proof. We apply Legendrian adjunction (Corollary 144) to see that

$$-\bar{\chi}(C) = -\frac{1}{2} \cdot \mu_\tau(C) + Q_\tau(C) + w_\tau(C) - 2\delta(C) - \epsilon(C) \leq -\frac{1}{2} \cdot \mu_\tau(C) + Q_\tau(C) + w_\tau(C)$$

We may decompose the writhe of C into a sum of writhes

$$w_\tau(C) = \sum_i w_\tau(\zeta_i^+) - \sum_j w_\tau(\zeta_j^-)$$

where the sum is over all braids over each orbit and chord appearing as a positive and negative end of C. The writhe is automatically 0 for any chord and orbit of multiplicity 1. Since Ξ_+ and Ξ_- are ECH generators, this includes all chords and hyperbolic orbits. This reduces the desired result to the inequality

$$w_\tau(\zeta^+) \leq \mathrm{CZ}_\tau^{ECH}((\eta, m-1)) \tag{4.7.2}$$

for any simple elliptic orbit of multiplicity $m > 1$ in Ξ_+ and the corresponding inequality

$$w_\tau(\zeta^-) \geq \mathrm{CZ}_\tau^{ECH}((\gamma, n-1)) \tag{4.7.3}$$

for any simple elliptic orbit of multiplicity $n > 1$ in Ξ_-. This is proven in [55, Prop. 6.9]. In particular, [55, Eq. 6.2] states that

$$\mathrm{CZ}^{ECH}((\eta, m-1)) - w_\tau(\zeta^+) \geq n - 1$$

where η is elliptic and n denotes the number of components of the braid ζ^+. This implies (4.7.2), and (4.7.3) can be argued analogously. □

Remark 201. In [55], Hutchings also develops a general theory of the J_0-index, a replacement for the ECH index that provides a topological filtration on any type of contact homology. Proposition 200 is essentially using this theory in the relative case. We will not develop a relative J_0-invariant in this paper.

Next, we demonstrate the compactness of the space of J-holomorphic currents of ECH index 1. This is the first part of Theorem 11 in the introduction.

Lemma 202. The moduli space of J-holomorphic currents $\mathcal{M}_1(\Xi_+, \Xi_-)/\mathbb{R}$ of ECH index 1 is finite.

This is a virtual repeat of the same argument in ECH using Gromov compactness and bounds on topological complexity of J holomorphic curves. We sketch it here briefly for completeness.

Proof. Fix a sequence of distinct currents with boundary of index 1, denoted by

$$\mathcal{C}^\nu \in \mathcal{M}(\Xi_+, \Xi_-)/\mathbb{R}$$

By Proposition 197, we have $\mathcal{C}^\nu = C^\nu \sqcup \mathcal{T}^\nu$ where C^ν is a connected, embedded J-holomorphic curve with $I(C^\nu) = 1$ and $\mathcal{T}^\nu$ is a trivial current over an orbit-chord subset Θ^ν of Ξ_+ and Ξ_-. There are only finitely many orbit-subsets of Ξ_+, so after passing to a subsequence we may assume that

$$\mathcal{T}^\nu = \mathcal{T} = \mathbb{R} \times \Theta \qquad \text{for all } \nu$$

Moreover, let Θ_+ and Θ_- be the unique orbit sets such that $\Xi_\pm = \Theta_\pm \cup \Theta$. Then

$$C^\nu \in \mathcal{M}(\Theta_+, \Theta_-)/\mathbb{R}$$

By Gromov compactness for currents (Theorem 153), we can choose a subsequence of C^ν that lie in the same surface class A in $S(\Theta_+, \Theta_-)$. By Lemma 200, we have

$$-\bar{\chi}(C) \leq -\frac{1}{2} \cdot \mu_\tau(A) + Q_\tau(A) + \mathrm{CZ}_\tau^{ECH}(\check{\Theta}_+) - \mathrm{CZ}_\tau^{ECH}(\check{\Theta}_-)$$

Thus the genus and number of boundary components of C is uniformly bounded. We can thus apply SFT compactness for curves with boundary (cf. Abbas [1, Thm. 3.6]) to acquire a J-holomorphic building

$$\mathbf{C} = (C_1, C_2, \ldots, C_m)$$

Claim 203. The building $\mathbf{C}$ consists of a single level C of ECH index 1.

Proof. Since the ECH index is additive under composition and the surface class of $\mathbf{C}$ agrees with that of C^ν for large ν, we know that

$$\sum_j I(C_j) = I(C^\nu) = 1$$

Moreover, $I(C_j) \geq 0$ for each j since J is regular. Therefore, for some fixed i we must have

$$I(C_i) = \text{ind}(C_i) = 1$$

and all other levels C_j are ECH index 0. By Lemma 196, we have the following equality of currents

$$C_j = \mathbb{R} \times \Xi_+ \quad \text{for } j < i \qquad \text{and} \qquad C_j = \mathbb{R} \times \Xi_- \quad \text{for } j > i$$

and each C_j is (as a curve) an explicit branched cover of the trivial cylinders over the simple orbits and chords of $\Xi_\pm$.

Now fix an orbit or chord γ of multiplicity m in Ξ_+. Let Z_j be the sub-curve C_j that is a cover of $\mathbb{R}\times\gamma$ for $1 \leq j < i$ and consider the building $\mathbf{Z} = (Z_1, \dots, Z_{i-1}, Z_i)$. Then Lemma 180 implies that $\mathbf{Z}$ consists of unbranched covers if γ is an orbit, and Lemma 181 implies the same if γ is a chord. Such components cannot appear in a building, so we must have $i = 1$ and $C_1 = C_i$ is non-trivial.

The same argument shows that there are no trivial levels C_j for $j > i$. □

Returning to the proof of Lemma 202, we now find that the embedded curves C^ν converge to the curve C in the moduli space of somewhere injective Fredholm index 1 curves. Since J is regular, this moduli space is discrete. Thus $C^\nu = C$ for sufficiently large ν, and

$$\mathcal{C}^\nu = C^\nu \sqcup \mathcal{T} = C \sqcup \mathcal{T} \qquad \text{for large } \nu$$

This contradicts the assumption that the currents $\mathcal{C}^\nu$ are distinct. This concludes the proof. □

Next, we demonstrate the compactness of the space of J-holomorphic currents of ECH index 2, under the assumption that the positive and negative ends are on ECH generators.

Lemma 204. Let Ξ_1, Ξ_2 be ECH generators. Fix a regular, tailored J on $(\mathbb{R}\times Y, \mathbb{R}\times \Lambda)$. Then the moduli space of ECH index two holomorphic currents

$$\mathcal{M}_2(\Xi_1, \Xi_2)/\mathbb{R}$$

is a topological 1-manifold that admits a (possibly singular) compactification

$$\bar{\mathcal{M}}_2(\Xi_1, \Xi_2)/\mathbb{R} := \mathcal{M}_2(\Xi_1, \Xi_2)/\mathbb{R} \cup \Big(\bigcup_{\Theta} \mathcal{M}_1(\Xi_1, \Theta)/\mathbb{R} \times \mathcal{M}_1(\Theta, \Xi_2)/\mathbb{R}\Big)$$

Note that the above compactification is topologized with the Gromov topology on currents. It is *not* necessarily a manifold itself. There could be many connected components of $\mathcal{M}_2(\Xi_1, \Xi_2)/\mathbb{R}$ that have the same boundary point in the compactification. The number of such components is given by a obstruction bundle count that we describe in more detail below (Section 4.7).

Proof. By Gromov compactness for currents, we know that C^ν converges in the Gromov topology to a broken current

$$\bar{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_m)$$

The ECH index is additive (Proposition 191) and non-negative (by Lemma 196), so the broken current must have $m \leq 2$ component currents. It suffices to show that if $m = 1$ and $\bar{\mathcal{C}} = \mathcal{C}$, then

$$\mathcal{C} = C$$

where C is a connected embedded curve of ECH index two. This is the content of Proposition 198. □

Moduli Spaces Truncation

We now formally describe the truncated moduli space of ECH index two currents in Theorem 11.

In the case of standard ECH, the construction of this moduli space and the proof of its key properties is spread over several papers [52, 60, 62]. Thus, although the construction is exactly analogous to that case, we collect several details here for the reader.

We will require a version of gluing pairs in the sense of [60, Definition 1.9]. We adopt the following definition.

Definition 205. An *ECH gluing pair* in $(\mathbb{R}\times Y, \mathbb{R}\times\Lambda)$ is a pair of J-holomorphic curves

$$(u_+, u_-)$$

satisfying the following properties.

- u_+ and u_- have ECH index one.
- u_+ and u_- are embedded, except for unbranched covers of cylinders of orbits and chords.
- The orbit-chord sets at the negative end of u_+ and the positive end of u_- are the same.

Moreover, let γ be a simple elliptic orbit of total multiplicity m at the negative end of u_- (or equivalently, at the positive end of u_+).

- The partition $p_-(u_+;\gamma)$ of m determined by the negative end of u_+ is the negative ECH partition $p_-(m)$.
- The partition $p_+(u_-;\gamma)$ of m determined by the positive end of u_- is the positive ECH partition $p_+(m)$

There is a space of ECH index two curves that are *nearly broken* at (u_+, u_-), denoted by

$$\mathcal{G}_\delta(u_+, u_-) \subset \mathcal{M}_2(\Xi_+, \Xi_-)/\mathbb{R}$$

consisting of all connected, embedded curves C in $\mathcal{M}_2(\Xi_1, \Xi_2)$ that admit constants R_+, R_- and a decomposition

$$C = C'_+ \cup C_0 \cup C'_-$$

that satisfy the following conditions.

- C_0 is contained in the radius δ neighborhood of $\mathbb{R}\times\Theta$, where Θ is the orbit-chord set at the negative end of u_+.
- $R_+ - R_- > \frac{2}{\delta}$.
- There is a section ψ_+ of the normal bundle of u_+ such that

$$|\psi_+| < \delta \quad \text{and} \quad C'_+ = \big(\exp_{C_+}(\psi_+) + R_+\big) \cap [-1/\delta, \infty) \times Y$$

- There is a section ψ_- of the normal bundle of u_- such that

$$|\psi_-| < \delta \quad \text{and} \quad C'_- = \big(\exp_{C_-}(\psi_-) + R_-\big) \cap (-\infty, 1/\delta] \times Y$$

Lemma 206. Fix an ECH gluing pair of the form

$$(u_+, u_-)$$

Then there is an $\epsilon > 0$ such that any sequence of curves $C_i \in \mathcal{G}_\epsilon(\mathcal{C}_+, \mathcal{C}_-)/\mathbb{R}$ has a subsequence that converges in the SFT topology (see [1, Thm 3.6]) to one of the following types of J-holomorphic buildings.

- A connected, embedded curve C of ECH index two in $\mathcal{G}_\delta(\mathcal{C}_+, \mathcal{C}_-)$
- A building $(u_+, v_1, \dots, v_k, u_-)$ where v_i is a union of branched covers of trivial cylinders representing $\mathbb{R} \times \Theta$.

Proof. Let $C_i \in \mathcal{G}_\epsilon(u_+, u_-)/\mathbb{R}$ be a sequence. By SFT compactness in the relative case [1, Thm 3.6] and the topological bounds in Lemma 200, we know that there is a limit SFT building of the form

$$\mathbf{C} = (w_1, \dots, w_m)$$

The sum of the ECH indices of the levels must be 2 by addivity of the ECH index. Therefore, there are two cases.

Case 1. In the first case, there is a single level of ECH index two and every other level is ECH index zero. In this case, we can argue that there is only one level $C = \mathbf{C}$, by an identical as in Claim 203. It follows that $C_i \to C$ in the SFT topology.

Case 2. In the second case, there are two levels w_a and w_b of $\mathbf{C}$ of ECH index one, and the remaining levels are ECH index zero. The levels w_a and w_b must be equal to u_+ and u_- due to the construction of $\mathcal{G}_\delta$ and the definition of SFT convergence. Moreover, since the positive and negative ends of the curves C_i satisfy the ECH partition conditions, by [60, Lemma 1.7] and [52, Exercise 3.14], the levels w_j must be trivial for $j < a$ and $j > b$. □

Definition 207. The *count of gluings* $\#G(u_+, u_-)$ of an ECH gluing pair (u_+, u_-) is defined as follows. Let ϵ be as in Lemma 206 and choose an open subset $U \subset \mathcal{G}_\epsilon(u_+, u_-)/\mathbb{R}$ such that

- $\bar{U}$ has finitely many boundary points.
- U contains $\mathcal{G}_\delta(u_+, u_-)/\mathbb{R}$ for some $\delta < \epsilon$.

Then we define $\#G(u_+, u_-)$ to be the number of boundary points in the closure $\bar{U}$ of U.

We are now ready to construct the truncated moduli space of ECH index two curves. For each ECH gluing pair (u_+, u_-) between ECH generators Ξ_+ and Ξ_-, fix for the remainder of the section an open set

$$U(u_+, u_-) \subset \mathcal{G}_\epsilon(u_+, u_-)$$

as in Definition 207. For any ECH generators Θ_+ and Θ_-, we adopt the notation

$$W(\Theta_+, \Theta_-) := \big\{C \sqcup \mathcal{T} \,:\, C \in U(u_+, u_-) \text{ for some ECH gluing pair } (u_+, u_-) \text{ and } \mathcal{T} \text{ is trivial}\big\}$$

Definition 208 (Truncated Moduli Space)**.** Let Θ_+ and Θ_- be ECH generators and fix a regular, tailored J on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$. The *truncated moduli space*

$$\mathcal{M}'_2(\Theta_+, \Theta_-)/\mathbb{R} \subset \mathcal{M}_2(\Theta_+, \Theta_-)/\mathbb{R}$$

is defined as follows. By Proposition 198, we may divide $\mathcal{M}_2(\Theta_+, \Theta_-)$ into disjoint pieces

$$\mathcal{A}_2(\Theta_+, \Theta_-) = \big\{C \sqcup \mathcal{T} \,:\, C \text{ is connected with } I(C) = 2 \text{ and } \mathcal{T} \text{ trivial.}\big\}$$

$$\mathcal{B}_2(\Theta_+, \Theta_-) = \big\{C \sqcup D \sqcup \mathcal{T} \,:\, C, D \text{ are connected with } I(C) = I(D) = 1 \text{ and } \mathcal{T} \text{ trivial.}\big\}$$

We define $\mathcal{M}'_2(\Theta_+, \Theta_-)/\mathbb{R}$ as a union of pieces

$$\mathcal{A}'_2(\Theta_+, \Theta_-) \subset \mathcal{A}_2(\Theta_+, \Theta_-) \qquad \text{and} \qquad \mathcal{B}'_2(\Theta_+, \Theta_-) \subset \mathcal{B}_2(\Theta_+, \Theta_-)$$

Truncation Of $\mathcal{A}$. To truncate $\mathcal{A}_2(\Theta_+, \Theta_-)$, we note that by Gromov compactness for currents (Theorem 153), any sequence of currents $\mathcal{C}_i \in \mathcal{A}_2(\Theta_+, \Theta_-)/\mathbb{R}$ has a subsequence of the form

$$\mathcal{C}_i = C_i \sqcup \mathcal{T}$$

where C_i is a connected curve with $I(C_i) = 2$ from Θ_+ to Θ_- converging to

- a connected ECH index two curve $C \in \mathcal{M}_2(\Theta_+, \Theta_-)/\mathbb{R}$ or

- a building $(u_+, v_1, \ldots, v_k, u_-)$ where (u_+, u_-) are an ECH gluing pair and v_i are covers of trivial cylinders and chords.

In the latter case, it follows that $C_i \notin W(\Theta_+, \Theta_-)$ for sufficiently large i. We thus let

$$\mathcal{A}_2'(\Theta_+, \Theta_-) = \mathcal{A}_2(\Theta_+, \Theta_-) \setminus W(\Theta_+, \Theta_-)$$

There is a natural projection map

$$\Pi : \partial\mathcal{A}_2'(\Theta_+, \Theta_-)/\mathbb{R} \to \bigsqcup_{\Theta'} \mathcal{M}_1(\Theta_+, \Xi)/\mathbb{R} \times \mathcal{M}_1(\Xi, \Theta_-)/\mathbb{R}$$

defined by mapping $C \sqcup \mathcal{T} \in \partial\mathcal{A}_2'(\Theta_+, \Theta_-)/\mathbb{R}$ to the broken J-holomorphic current

$$(\mathcal{C}_+, \mathcal{C}_-) = (u_+ \sqcup \mathcal{T}, u_- \sqcup \mathcal{T})$$

where (u_+, u_-) is the unique ECH gluing pair such that $C \in \mathcal{G}_\delta(\mathcal{C}_+, \mathcal{C}_-)$. Moreover, we have

$$\#\Pi^{-1}(\mathcal{C}_+, \mathcal{C}_-) = \#\bar{U}(u_+, u_-) = \#G(u_+, u_-)$$

Truncation Of $\mathcal{B}$. To truncate $\mathcal{B}_2(\Theta_+, \Theta_-)$, note that a component $S \subset \mathcal{B}_2(\Theta_+, \Theta_-)/\mathbb{R}$ is 1-dimensional, consisting of curves of the form

$$\mathcal{C}_s = C \sqcup (D + s) \sqcup \mathcal{T} \qquad \text{for } s \in \mathbb{R}$$

Here C and D are connected, embedded curves of ECH index 1 and $\mathcal{T}$ is a trivial current. Let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the trivial currents over the positive and negative ends of C, and let $\mathcal{T}_+$ and $\mathcal{T}_-$ be the analogous currents for D. Then

$$\lim_{s \to -\infty} \mathcal{C}_s = (C \sqcup \mathcal{T}_+ \sqcup \mathcal{T}, D \sqcup \mathcal{S}_- \sqcup \mathcal{T})$$

$$\lim_{s \to +\infty} \mathcal{C}_s = (D \sqcup \mathcal{S}_+ \sqcup \mathcal{T}, C \sqcup \mathcal{T}_+ \sqcup \mathcal{T})$$

where the limit is taken in the Gromov topology on broken currents. We now truncate S by setting

$$S' = \{\mathcal{C}_s \ : \ s \in [-1, 1]\} \subset S$$

and let $\mathcal{B}_2'$ be the union of these pieces over all components S. There is a natural projection map

$$\Pi : \partial\mathcal{B}_2'(\Theta_+, \Theta_-)/\mathbb{R} \to \bigsqcup_{\Theta'} \mathcal{M}_1(\Theta_+, \Xi)/\mathbb{R} \times \mathcal{M}_1(\Xi, \Theta_-)/\mathbb{R}$$

sending C_{-1} to $\lim_{s \to -\infty} \mathcal{C}_s$ and C_1 to the broken current $\lim_{s \to \infty} \mathcal{C}_s$. Note that

$$\#\Pi^{-1}(\mathcal{C}_+, \mathcal{C}_-) = 1 \qquad \text{for any} \qquad (\mathcal{C}_+, \mathcal{C}_-) \in \Pi(\partial\mathcal{B}_2'(\Theta_+, \Theta_-)/\mathbb{R})$$

Gluing Counts

We conclude this section by explaining the applications of the obstruction bundle gluing results of Hutchings-Taubes [60, 62] to our setting. In particular, we prove the last item of Theorem 11 as Corollary 211. This completes the proofs of all the main results of the paper.

Let γ be a simple Reeb orbit and fix two partitions p and q of an integer m. In [60, §1.5, 1.6], Hutchings-Taubes define *gluing coefficients*

$$c_\gamma(p, q) \in \mathbb{Z}$$

that count (roughly) the number of ways to glue two curves asymptotic to γ with multiplicity m, and corresponding partitions p and q. For chords, we adopt the following definition.

Definition 209. The *gluing coefficient* of a Reeb chord η of (Y, Λ) and of multiplicity one is given by

$$c_\gamma(\gamma, 1) = 1.$$

Given a J-holomorphic curve v, we let $m_\pm(v, \eta)$ be the multiplicity of the orbit or chord η at the $\pm$-end of v, and $p_\pm(v, \gamma)$ be the partition of $m_\pm(v, \gamma)$ determined by the $\pm$-end of a finite energy curve v at any orbit γ. Finally, if γ is a simple orbit and η is a chord, then

$$c_\gamma(u_+, u_-) := c_\gamma(p_-(u_+, \gamma), p_+(u_-, \gamma)) \qquad \text{and} \qquad c_\eta(u_+, u_-) := 1$$

Proposition 210. Fix a tailored, regular almost complex structure J on $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ and let (u_+, u_-) be an ECH gluing pair. Then

$$\#G(u_+, u_-) = \prod_\gamma c_\gamma(u_+, u_-)$$

The product is over all simple closed orbits and chords appearing as a negative end of u_+.

Proof. The essential ingredients to this formula appear already in Hutchings-Taubes [60, 62]. Here we briefly sketch the argument.

Let u_1 and u_2 be a ECH gluing pair. Throughout we assume we have chosen a generic J. For simplicity of exposition we first assume none of the u_1 and u_2 have boundary, and that all negative ends of u_1 are asymptotic to one (simple) Reeb orbit with multiplicities (therefore the same is true for all positive punctures of u_2). We illustrate the general methodology in this case and explain the more general case later.

Then recall from Section 5.2 [62] in order to glue u_1 and u_2 together, we must first preglue a branched cover of trivial cylinder v of Fredholm index zero between them, so that the partition conditions at positive (resp. negative ends) of v match the partition conditions of negative ends of u_1 (resp. positive ends of u_2). Then as in Section 5.4 that there is a gluing is translated into three equations as we describe below.

To borrow the notation of Section 5.3 in [62], let N_* denote the normal bundle of u_i, v. Let $\mathcal{H}_1(N_*)$ denote a suitable completion of sections of N_i. Let $\psi_i \in \mathcal{H}_1(N_i)$ and $\phi \in \mathcal{H}_2(N_v)$ denote sections of the normal bundle. We consider deforming the preglued curve consisting of u_1, u_2 and v using the sections ψ_i and ϕ (patched together using cutoff functions, as in Section 5.3 in [62]). The condition that gluing exists is translated into a system of three equations

$$\Theta_1(\psi_1, \phi) = 0, \quad \Theta_v(\psi_1, \psi_2, \phi) = 0, \quad \Theta_2(\psi_2, \phi) = 0$$

so that given fixed ϕ, the equations Θ_1, Θ_2 can always be solved essentially uniquely (with ψ_i expressed as functions of ϕ). See Section 5.6 of [62].

The number of gluings is equivalent to the number of solutions of Θ_v with independent variable ϕ; and ψ_1 and ψ_2 are functions of ϕ. This is the content of Theorem 7.3 (b) in [62].

The number of solutions of $\Theta_v = 0$ is counted by the number of zeroes of a section $\mathfrak{s}$ of an obstruction bundle $\mathcal{O}(\Sigma) \to \mathcal{M}_\Sigma$. The base $\mathcal{M}_\Sigma$ is the space of branched covers of the trivial cylinder satisfying the same multiplicity and parition conditions as v; and the fiber is the (dual of) cokernel of a linearized Cauchy Riemann operator D_Σ acting on the normal bundle of the branched cover. This obstruction bundle is defined carefully in Definition 2.16 of [60]. See Section 2.1 of [60] for definitions and properties of the base $\mathcal{M}_\Sigma$ and Section 2.2 and 2.3 for the linearized Cauchy Riemann D_Σ. This obstruction section $\mathfrak{s}$ whose zero corresponds to the number of solutions of $\Theta_v = 0$ is described carefully in Equation 5.43 of [62]. The count of zeroes can be viewed as computing some version of a relative Euler class of this obstruction bundle. The zeros of $\mathfrak{s}$ are transverse (proved in Section 10 of [62]), and the number of zeroes counted with sign is equal to c_γ. The definition of c_γ is given in equation

1.7 of [60], and described more carefully in Sections 1.5 and 1.6 of [60]. That the number of zeros is equal to c_γ is the content of Theorem 1.13 in [60].

From this way of gluing, we see readily that we can count the number of gluings as a local count of zeroes of obstructions sections over spaces of branched covers of trivial cylinders. If u_1 and u_2 meet along multiple Reeb orbits, the count of total number of gluings is the product of the count of zeroes of such obstruction sections over all the Reeb orbits. This is the content of Theorem 1.13 in [60].

In the case of ECH gluing pairs for Legendrian ECH, we can again reduce the count of gluing to count of zeroes of obstruction sections over branched covers of trivial cylinders and trivial strips. We here use the observation that by partition conditions all chords between u_+ and u_- must have mulitplicity one. Over branched covers of trivial cylinders such counts over closed Reeb orbits is given by the obstruction bundle counts given in [60, Theorem 1.13], and the gluing over Reeb chords is just standard gluing. □

Corollary 211 (Theorem 11, Truncation)**.** The inverse image $\Pi^{-1}(\mathcal{C}_+, \mathcal{C}_-)$ of a pair of currents in $\mathcal{M}_1(\Theta, \Theta')/\mathbb{R} \times \mathcal{M}_1(\Theta', \Xi)/\mathbb{R}$ under the map

$$\Pi : \mathcal{M}_2'(\Theta_+, \Theta_-) \to \bigsqcup_{\Theta'} \mathcal{M}_1(\Theta_+, \Xi)/\mathbb{R} \times \mathcal{M}_1(\Xi, \Theta_-)/\mathbb{R}$$

has an odd number of points if and only if the orbit set Θ' is an ECH generator.

Proof. For any ECH gluing pair, the gluing coefficients $c_\gamma(u_+, u_-)$ are always odd when γ is is an elliptic orbit [60, Prop. 7.26]. They are odd if and only if the multipliciy of γ in Ξ_+ or Ξ_- is one when γ is a hyperbolic orbit [60, Def. 1.14] or chord (see Definition 209). Thus this follows from Proposition 210. □

4.8 Decay Estimates

In this appendix, we prove asymptotic formulas for ends of J holomorphic curves converging to a Reeb chord in the same style of Siefring [108]. The asymptotic formula for a single end is already established in [2]. Our goal is to explain the general case, where multiple ends approach the same chord. This will be essential in our proof of the general writhe bound.

Remark 212. The general writhe bound is not strictly necessary for our formulation of Legendrian ECH in Section 4.1. This is due to our combination of maximum principles and the definition of ECH generators prevents breaking along multiply covered chords in the ECH setting.

However, the general asymptotic formula and writhe bound may play a role in other theories that count curves asymptotic to chords with higher multiplicities.

The presented proof is an amalgamation of techniques and ideas found in [47], [108], and [2]. The proof has the following outline.

(i) First, we describe the local geometric setup as in [2]. In particular, we follow the choices of metrics and connections in [2].

(ii) Suppose we have two ends approaching the same chord. Following [108], we show the difference between the two ends satisfy a particular PDE (see equation 4.8.3).

(iii) We show that the bounded solutions to Equation 4.8.3 have an asymptotic representation formula. We do this using the setup of [2], and essentially the same techniques there that proved asymptotic formula holds for a single end.

(iv) Finally we translate the asymptotic formula of the difference between two ends into behaviour of two ends of the holomorphic curve, following [108].

Local differential geometry

We start by recalling the geometric setup of [2]. By rescaling the contact form, we may assume that the Reeb chord has action 1. We choose a neighborhood of the Reeb chord and local coordinates x, y, z such that the following holds.

- The Reeb chord η is the line segment $\{(0,0,t) : t \in [0,1]\}$
- The two Legendrians consist of the lines

$$L_1 := \{(t,0,0), t \in \mathbb{R}\} \qquad \text{and} \qquad L_2 := \{(0,t,1), t \in \mathbb{R}\}$$

- The contact form λ is given by

$$\lambda = f dz + c_1 dx + c_2 dy$$

  where $f = 1$ along η, and c_1 and c_2 are smooth functions satisfying $c_1 = c_2 = 0$ on the z-axis.

- The contact form λ satisfies

$$d\lambda(0,0,z) = adx \wedge dy$$

where $a(x,y,z)$ is positive in a neighborhood of the Reeb chord.

- The Reeb vector field is given by

$$X(x,y,z) = f(x,y,z) \cdot (0,0,1)$$

where $f(0,0,z) = 1$.

Given any λ-adapted almost complex structure J on the symplectization $\mathbb{R} \times \mathbb{R}^3$, we write $M(t)$ to denote the 2 by 2 matrix that is the restriction of J to the contact plane $\ker \lambda_{0,0,z}$. It satisfies

$$M^T J_0 M = J_0 \qquad \text{and} \qquad -J_0 M > 0$$

where J_0 is the standard 2 by 2 almost complex structure.

We now consider two J-holomorphic curves in the symplectization $\mathbb{R} \times \mathbb{R}^3$ that have boundary punctures that asymptotic to the Reeb chord η.

$$U, V : [0,\infty) \times [0,1] \to \mathbb{R} \times \mathbb{R}^3 \qquad \text{with} \qquad \lim_{s\to\infty} U(s,-) = \lim_{s\to\infty} V(s,-) = \eta$$

In particular we consider how boundary punctures approach Reeb chords. We consider the case where there are two strip-like ends approaching the same chord. Assume that in the local conformal coordinates (s,t) of the domain, the first end is parametrized as

$$U(s,t) = (b(s,t), x(s,t), y(s,t), z(s,t)).$$

Using this parametrization, we define the local embedding $E : [R,\infty) \times D_\epsilon \times [0,1] \to \mathbb{R} \times \mathbb{R}^3$ to be

$$E(s,h_1,h_2,t) = (b(s,t), x(x,t) + h_1, y(s,t) + h_2, z(s,t) - c_1 f h_1 - c_2 f h_2) \qquad (4.8.1)$$

where $R > 0$ is sufficiently large, $\epsilon > 0$ is sufficiently small, and D_ϵ is the ϵ-neighborhood of the origin. With the embedding E understood, any nearby end could be viewed as a graph $E(s, \eta_1(s,t), \eta_2(s,t), t)$. In particular, we now parametrize the second end as

$$\begin{aligned} V(s,t) &= E(s, \eta_1(s,t), \eta_2(s,t), t) \\ &= (b(s,t), x(s,t) + \eta_1(s,t), y(s,t) + \eta_2(s,t), z(s,t) - fc_1\eta_1(s,t) - fc_2\eta_2(s,t)). \end{aligned} \qquad (4.8.2)$$

Note that by abuse of notation we have used (s,t) to denote the coordinates for the domains of both ends. We also note here that $u(s,t) = (x(s,t), y(s,t), z(s,t))$ is the projection of U onto the last three coordinates, and we shorthand the functions $f(u(s,t))$, $c_1(u(s,t))$ and $c_2(u(s,t))$ as f, c_1 and c_2 respectively. Under these parametrizations, the map $(\eta_1(s,t), \eta_2(s,t))$ satisfies the following PDE:

$$\partial_s \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + M(u(s,t))\partial_t \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \Delta(s,t) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \tag{4.8.3}$$

where the matrix $\Delta(s,t)$ decays exponentially, i.e. there are positive constants d and M_β such that for any multi-index β, $|\partial^\beta \Delta| \leq M_\beta e^{-ds}$.

The asymptotic formula

We now use the estimates from [2] to derive an asymptotic formula for (η_1, η_2). The main difference in Abbas' approach from that of Hofer or Siefring is choosing a 1-parameter family of norms dependent on s which are equivalent to the ordinary L^2 norm on $[0,1]$. We first review this part of his construction, then we will state and prove the analogous lemmas in [2] that will culminate in the asymptotic formula.

We first recall properties of the matrix $M(u(s,t))$, which we will often abbreviate $M(s,t)$ for convenience. It converges to a matrix $M_\infty(t)$ as $s \to \infty$ at an exponential rate. Further, since $M(s,t)$ comes from restricting the ambient λ-adapted almost complex structure on the contact distribution, it satisfies the following algebraic properties:

(i) $M^2 = -1$

(ii) $M^T J_0 M = J_0$

(iii) $-J_0 M > 0$.

We consider the following family of inner products on $L^2([0,1], \mathbb{R}^2)$, let $v_1, v_2 \in L^2([0,1], \mathbb{R}^2)$:

$$(v_1, v_2)_s := \int_0^1 \langle v_1, -J_0 M(s,t) v_2 \rangle dt \tag{4.8.4}$$

and it follows from the fact that $M(s,t)$ exponentially decays to $M_\infty(t)$ that for large values of s this norm is uniformly equivalent to the ordinary L^2 norm.

We define

$$W^{1,2}_\Gamma([0,1], \mathbb{R}^2) := \{v \in W^{1,2}([0,1], \mathbb{R}^2) | v(s,0) \in \mathbb{R}\cdot(1,0), v(s,1) \in \mathbb{R}\cdot(0,1)\} \tag{4.8.5}$$

hence we can further define an inner product

$$(v_1, v_2)_{s,1,2} := (v_1, v_2)_s + (v_1', v_2')_s \tag{4.8.6}$$

where $'$ denotes the derivative with respect to s.

We consider the trivial vector bundle $E := [s_0, \infty) \times [0,1] \times \mathbb{R}^2$ with the fiber $\mathbb{R}^2$. We view maps $\zeta(s,t) : [s_0, \infty) \times [0,1] \to \mathbb{R}^2$ as sections of E. With the above choice of metric, we choose a connection on E by defining the following covariant derivatives:

$$\nabla_s \zeta := \partial_s \zeta - \frac{1}{2} M(s,t) \partial_s M(s,t) \cdot \zeta(s,t)$$

$$\nabla_t \zeta := \partial_t \zeta - \frac{1}{2} M(s,t) \partial_t M(s,t) \cdot \zeta(s,t).$$

We abbreviate:

$$\Gamma_1 := -\frac{1}{2} M(s,t) \partial_s M(s,t)$$

$$\Gamma_2 := -\frac{1}{2} M(s,t) \partial_t M(s,t).$$

The following properties of this connection are proved in [2]:

Proposition 213. Let $X = a_1 \partial_s + a_2 \partial_t$, then we define $\nabla_X := a_1 \nabla_s + a_2 \nabla_t$.

(i) For $u_1, u_2 \in W^{1,2}([s_0, \infty) \times [0,1], \mathbb{R}^2)$, we have:

$$\frac{d}{ds}(u_1, u_2)_s = (\nabla_s u_1, u_2)_s + (u_1, \nabla_s u_2)_s. \tag{4.8.7}$$

(ii) If B is a section in the endomorphism bundle of E, i.e. $B \in \Gamma(End(E)) = \Gamma([s_0, \infty) \times [0,1] \times End(\mathbb{R}^2))$, then we define:

$$\nabla_s B := \partial_s B + \frac{1}{2}[B \cdot M(s,t) \partial_s M(s,t) - M(s,t) \partial_s M(s,t) \cdot B]. \tag{4.8.8}$$

This satisfies:

$$\nabla_s (B \cdot \zeta) = \nabla_s B \cdot \zeta + B \cdot \nabla_s \zeta. \tag{4.8.9}$$

In particular we have

$$\nabla_s M(s,t) = 0 \tag{4.8.10}$$

and

$$\partial_t \nabla_s \zeta - \nabla_s \partial_t \zeta = \partial_t \Gamma_1 \cdot \zeta. \tag{4.8.11}$$

We also define the following family of unbounded self-adjoint operators:

$$A(s) : W^{1,2}_\Gamma([0,1],\mathbb{R}^2) \subset L^2([0,1],\mathbb{R}^2) \longrightarrow L^2([0,1],\mathbb{R}^2) \tag{4.8.12}$$

as

$$A(s) := -M(s,t)\frac{d}{dt}. \tag{4.8.13}$$

We also use A_∞ to denote the operator $-M_\infty(t)\frac{d}{dt}$. The following properties are established in Proposition 3.4 in [2].

Proposition 214. (i) $A(s)$ is self-adjoint on $(L^2([0,1],\mathbb{R}^2),(\cdot,\cdot)_s)$.

(ii) Ker $A(s)=\{0\}$.

(iii) There exists $\delta>0$, independent of s, so that for all $s\in[s_0,\infty)$ (s_0 some fixed number), and for all $\gamma\in W^{1,2}_\Gamma([0,1],\mathbb{R}^2)$ we have

$$||A(s)\gamma||_s \geq \delta||\gamma||_s$$

where here and in what follows, we use $||\cdot||_s$ to denote the norm defined by $(\cdot,\cdot)_s$.

We also need the following proposition on Hilbert spaces, also stated and proved in [2, Theorem 3.7].

Proposition 215. Let $T : D(T)\in H\to H$ be a self-adjoint operator in a Hilbert space H. Let $A_0 : H\to H$ be a linear symmetric, bounded operator. Let $\sigma(\cdot)$ denote the spectrum of an operator. Then

$$\text{dist}(\sigma(T),\sigma(T+A_0)) := max\{\sup_{\lambda\in\sigma(T)}\text{dist}(\lambda,\sigma(T+A_0)),\sup_{\lambda\in\sigma(T+A_0)}\text{dist}(\lambda,\sigma(T))\} \tag{4.8.14}$$

$$\leq ||A_0||_{\mathcal{L}(H)} \tag{4.8.15}$$

Assume further that the resolvent $(T-\lambda_0)^{-1}$ of T exists and is compact for some $\lambda_0\notin\sigma(T)$, then $(T-\lambda)^{-1}$ exists and is compact for every $\lambda\notin\sigma(T)$, and $\sigma(T)$ consists of isolated eigenvalues $\{\mu_k\}_{k\in\mathbb{Z}}$ with finite multiplicities $\{m_k\}_{k\in\mathbb{Z}}$. If we assume $\sup_{k\in\mathbb{Z}} m_k\leq M<\infty$, and that for each $L>0$ there is a number $m_T(L)\in\mathbb{N}$ so that every interval $I\subset\mathbb{R}$ of length L contains at most $m_T(L)$ points of $\sigma(T)$ (counted with multiplicity), then for each $L>0$ there is also a number $m_{T+A_0}(L)$ so that every interval $I\in\mathbb{R}$ of length L contains at most $m_{T+A_0}(L)$ points of $\sigma(T+A_0)$.

To lighten the exposition we also isolate the following lemma from [2], whose proof depends on the above proposition.

Lemma 216. Consider the operators $A_\infty : W^{1,2}_\Gamma([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$ and $A(s) : W^{1,2}_\Gamma([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$. We have:

(i) Given any $L > 0$, there is some positive integer m so that any interval in $\mathbb{R}$ of length L contains at most m eigenvalues of A_∞ (up to multiplicity).

(ii) $\mathrm{dist}(\sigma(A(s)), \sigma(A_\infty)) \to 0$ as $s \to \infty$.

(iii) Let $I_n := [-(n+1)L, -nL]$, then each I_n contains at most m points of $\sigma(A_\infty)$. There is a closed interval $J_n \subset I_n$ of length $2d > 0$ so that for $s > s_0$, the interval J_n does not contain any element of $\sigma(A(s))$. In the following, we will write J_n to be of the form $[r_n - d, r_n + d]$.

Let $\zeta(s,t) : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ be a solution to equation 4.8.3 that exponentially decays to zero as $s \to \infty$. Then we have (see [2] Theorem 3.6):

Proposition 217. If ζ does not vanish identically, then we have the following asymptotic formula:

$$\zeta(s,t) = e^{\int_{s_0}^{s} \alpha(\tau) d\tau}(e(s,t) + r(s,t)),$$

where

(i) $e(s,t)$ is an eigenvector of A_∞ in $W^{1,2}_\Gamma([0,1],\mathbb{R}^2)$ with eigenvalue $\lambda < 0$.

(ii) $\alpha(s) : [s_0, \infty) \times \mathbb{R}$ is a smooth function satisfying $\alpha(s) \to \lambda$ as $s \to \infty$.

(iii) $r(s,t) : [s_0,\infty) \times [0,1] \to \mathbb{R}$ is a smooth function with

$$|\partial^\beta r(s,t)| \to 0$$

as $s \to \infty$. Here $\beta \in \mathbb{N}^2$ is some multi-index.

We break down the proof into a series of lemmas, as was done analogously in [2].

Lemma 218 (Lemma 3.9 in [2])**.** If the assumptions of the above proposition are satisfied, then $\alpha(s) \to \lambda$, where λ is a negative eigenvalue of A_∞.

Proof. We first assume $\zeta(s,t) \neq 0$ for all $s \geq s_0$. The alternative case is handled later. We start with some preliminary manipulations.

Step 0 Define

$$\alpha(s) := \frac{d/ds||\zeta(s,t)||_s^2}{2||\zeta(s,t)||_s^2} \tag{4.8.16}$$

Then trivially we can write

$$\frac{d}{ds}||\zeta(s,t)||_s^2 = 2\alpha||\zeta(s,t)||_s^2$$

from which we deduce

$$||\zeta||_s^2 = e^{2\int_{s_0}^s \alpha ds'}||\zeta(s_0)||_s^2.$$

Hence we need to establish the asymptotics of α. We define

$$\xi(s,t) := \frac{\zeta(s,t)}{||\zeta(s,t)||_s}.$$

Consequently

$$\partial_t\xi = \frac{\partial_t\zeta}{||\zeta||_s}$$

and

$$\partial_s\xi = \frac{\partial_s\zeta}{||\zeta||_s} - \frac{\zeta}{2||\zeta||_s^3}\frac{d}{ds}||\zeta||_s^2.$$

Recall that ζ satisfies the equation 4.8.3, we have

$$\partial_s\xi + \alpha\xi + M(s,t)\partial_t\xi + \Delta(s,t)\xi = 0$$

which we can further rewrite as

$$\nabla_s\xi - \Gamma_1\xi + \alpha\xi + M(s,t)\partial_t\xi + \Delta(s,t)\xi = 0.$$

Taking the $(\cdot,\cdot)_s$ innter product of the above with ξ, we obtain

$$\alpha(s) = (\Gamma_1\xi,\xi)_s - (M\partial_t\xi,\xi)_s - (\Delta\xi,\xi)_s$$

where we used $(\nabla_s\xi,\xi)_s = 0$. Taking the s derivative of both sides, we obtain

$$\begin{aligned}
\alpha'(s) =& -(\nabla_s(M(s,t)\partial_t\xi),\xi)_s - (M\partial_t\xi,\nabla_s\xi)_s + (\nabla_s(\Gamma_1\xi),\xi)_s + (\Gamma_1\xi,\nabla_s\xi)\\
&-(\nabla_s(\Delta(s,t)\xi,\xi)_s - (\Delta(s,t)\xi,\nabla_s\xi)_s\\
=& -(M\partial_t\nabla_s\xi,\xi)_s + (M\partial_t\Gamma_1\xi,\xi)_s + (A\xi,\nabla_s\xi)_s + ((\nabla_s\Gamma_1)\xi,\xi)_s + 2(\Gamma_1\xi,\nabla_s\xi)_s\\
&-((\nabla_s\Delta(s,t)\cdot\xi,\xi)_s + (\Delta(s,t)\cdot\nabla_s\xi,\xi)_s + (\Delta(s,t)\xi,\nabla_s\xi)_s)\\
=&2(\nabla_s\xi,A\xi) + 2(\Gamma_1\xi,\nabla_s\xi)_s + (M\partial_t\Gamma_1\xi,\xi)_s + ((\nabla_s\Gamma_1)\xi,\xi)_s\\
&-((\nabla_s\Delta(s,t)\cdot\xi,\xi)_s + (\Delta(s,t)\cdot\nabla_s\xi,\xi)_s + (\Delta(s,t)\xi,\nabla_s\xi)_s)
\end{aligned}$$

where in the passage from first line to second line we used properties of ∇, as well as the fact that

$$(\Gamma_1 \nabla_s \xi, \xi)_s = (\nabla_s \xi, \Gamma_1 \xi)_s$$

which follows from the definition of $(\cdot, \cdot)_s$ as follows:

$$\begin{aligned}
(\Gamma_1 \nabla_s \xi, \xi)_s &= \int_0^1 \langle -\frac{1}{2} M \partial_s M \nabla_s \xi, -J_0 M \xi \rangle dt \\
&= -\frac{1}{2} \int_0^1 \langle \partial_s M \nabla_s \xi, -J_0 \xi \rangle dt \\
&= \frac{1}{2} \int_0^1 \langle \nabla_s \xi, \partial_s M^T J_0 \xi \rangle dt \\
&= \frac{1}{2} \int_0^1 \langle \nabla_s \xi, -\partial_s (M^T J_0 M M) \xi \rangle dt \\
&= -\frac{1}{2} \int_0^1 \langle \nabla_s \xi, J_0 \partial_s M \xi \rangle dt \\
&= \int_0^1 \langle \nabla_s \xi, -J_0 M (-\frac{1}{2} M \partial_s M \xi) \rangle dt.
\end{aligned}$$

Now inserting

$$A\xi = \nabla_s \xi - \Gamma_1 \xi + \alpha \xi + \Delta(s,t)\xi$$

into the expression for α', we obtain:

$$\begin{aligned}
\alpha'(s) =& 2(\nabla_s \xi, \nabla_s \xi)_s - 2(\nabla_s \xi, \Gamma_1 \xi) + 2(\nabla_s \xi, \Delta(s,t)\xi) + 2(\nabla_s \xi, \Gamma_1 \xi)_s \\
&+ (M \partial_t \Gamma_1 \xi, \xi)_s + ((\nabla_s \Gamma_1)\xi, \xi)_s \\
&- ((\nabla_s \Delta(s,t) \cdot \xi, \xi)_s + (\Delta(s,t) \cdot \nabla_s \xi, \xi)_s + (\Delta(s,t)\xi, \nabla_s \xi)_s) \\
=& 2(\nabla_s \xi, \nabla_s \xi)_s + (\nabla_s \xi, \Delta(s,t)\xi)_s + (M \partial_t \Gamma_1 \xi, \xi)_s + ((\nabla_s \Gamma_1)\xi, \xi)_s \\
&- ((\nabla_s \Delta(s,t) \cdot \xi, \xi)_s + (\Delta(s,t) \cdot \nabla_s \xi, \xi)_s).
\end{aligned}$$

We observe the following bounds

$$(\nabla_s \xi, \Delta(s,t)\xi)_s \leq \epsilon(s) \{ (\nabla_s \xi, \nabla_s \xi)_s + (\xi, \xi)_s \}.$$

where $\epsilon(s)$ is a function that decays exponentially to zero. Likewise

$$|(\Delta(s,t) \cdot \nabla_s \xi, \xi)_s| \leq \epsilon(s) \{ (\nabla_s \xi, \nabla_s \xi)_s + (\xi, \xi)_s \}.$$

Similarly the terms $(M\partial_t\Gamma_1\xi,\xi)_s$, $((\nabla_s\Gamma_1)\xi,\xi)_s$, and $((\nabla_s\Delta(s,t)\cdot\xi,\xi)_s$ have absolute value uniformly upper bounded by $\epsilon(s)$ as well. Hence we arrive at the following inequality for $\alpha'(s)$:

$$\alpha'(s) \geq (2-\epsilon(s))(\nabla_s\xi,\nabla_s\xi)_s - \epsilon(s). \tag{4.8.17}$$

We note our precise definition of $\epsilon(s)$ may change from line to line, but it will always denote a function that exponentially decays to zero. With the above inequality established, from this point onward we can then repeat the arguments in [2].

Step 1. We first show that $\alpha(s)$ is bounded above. Suppose not, then we can find a sequence

$$\{s_k\}_k \to \infty \qquad \text{satisfying} \qquad \alpha(s_k)\to\infty$$

If we had $\alpha(s)\geq\eta$ for some $\eta>0$ for all s large enough, then by definition of α this implies:

$$\frac{d}{ds}||\zeta||_s^2 \geq 2\eta||\zeta||_s^2$$

which would imply (assuming $||\zeta||_s$ is nonzero for large enough s) $||\zeta||_s\to\infty$ as $s\to\infty$, a contradiction.

The above argument gives us that for any $\eta>0$, we can find $s'_k\to\infty$ so that $\alpha(s'_k)<\eta$. Now choose $\eta\in(0,\delta)$, where δ is given by Proposition 214. Let $\widehat{s}_k$ be the smallest number satisfying $\widehat{s}_k > s_k$ and $\alpha(\widehat{s}_k)=\eta$. Observe this η cannot be an eigenvalue of any $A(s)$, because for any $\gamma\in W^{1,2}_\Gamma([0,1],\mathbb{R}^2)$, we have

$$||A(s)\gamma-\eta\gamma|| \geq ||A(s)\gamma|| - \eta||\gamma|| \geq (\delta-\eta)||\gamma|| > 0.$$

Thus

$$\begin{aligned}||\nabla_s\xi(\widehat{s}_k)||_{\widehat{s}_k} &\geq ||A(\widehat{s}_k)\xi(\widehat{s}_k)-\eta\xi(\widehat{s}_k)||_{\widehat{s}_k} - \epsilon(s)||\xi(\widehat{s}_k)||_{\widehat{s}_k}\\ &\geq (\delta-\eta)-\epsilon(s)\\ &\geq \tau\end{aligned}$$

for some $\tau>0$. Consequently

$$\alpha'(\widehat{s}_k) \geq (2-\epsilon(\widehat{s}_k))||\nabla_s\xi(\widehat{s}_k)||^2_{\widehat{s}_k} - \epsilon(\widehat{s}_k) \geq \tau^2$$

for large enough k. This is a contradiction because we picked $\widehat{s}_k$ to be the smallest number greater than s_k such that $\alpha(\widehat{s}_k)=\eta$, and that $\alpha(s_k)>\eta$. The fact $\alpha'(\widehat{s}_k)>0$ implies there is some number in $[s_k,\widehat{s}_k)$ such that $\alpha<\eta$, which is a contradiction.

Step 2. We next proceed to show α is also bounded from below. Assume not, then we can find a sequence $s_n \to \infty$ so that $\alpha(s_n) = r_n$ and $\alpha'(s_n) \leq 0$, where r_n and d are given in Lemma 216. Then we consider

$$A(s_n)\xi(s_n) = \nabla_s \xi(s_n) - \Gamma_1 \xi(s_n) + \alpha\xi(s_n) + \Delta(s_n, t)\xi(s_n)$$

where both sides are functions of t. We have

$$\begin{aligned} ||\nabla_s \xi(s_n)||_{s_n} &= ||A\xi(s_n) - r_n \xi(s_n) + \Gamma_1 \xi(s_n) - \Delta(s_n, t)\xi(s_n)||_{s_n} \\ &\geq d - \epsilon(s_n) \\ &\geq d/2 \end{aligned}$$

for large enough n. Combining this with

$$\alpha'(s) \geq (2 - \epsilon(s))(\nabla_s \xi, \nabla_s \xi)_s - \epsilon(s) \tag{4.8.18}$$

we conclude $\alpha'(s_n) \geq d^2/4$, which is contrary to our assumptions. Hence α is bounded from below as well.

Step 3. We observe we can always find a sequence $\{s_k\}$ so that $||\nabla_s \xi(s_k)||_{s_k} \to 0$. Suppose not, then we can find $\eta > 0$ so that

$$||\nabla_s \xi||_s \geq \eta$$

for large enough s, which implies

$$\alpha'(s) \geq \eta^2$$

contradicting the claim $\alpha(s)$ is bounded.

Step 4. Because α is bounded, we can find a subsequence of $\{s_k\}$, which we also denote by $\{s_k\}$ so that

$$\alpha(s_k) \to \lambda.$$

As in [2], we first show $\lambda \in \sigma(A_\infty)$. Suppose not, let $\epsilon > 0$ be defined by

$$\epsilon := \text{dist}(\lambda, \sigma(A_\infty)).$$

Then for also large enough s we have:

$$\text{dist}(\lambda, \sigma(A(s))) \geq \epsilon/2.$$

Then as before we can compute:

$$\begin{aligned}||\nabla_s\xi(s_k)||_{s_k} &\geq ||A(s_k)\xi(s_k) - \alpha(s_k)\xi(s_k)||_{s_k} - \epsilon(s_k)||\xi(_k,\cdot)||_{s_k}\\ &\geq \epsilon/4 - \epsilon(s_k)\\ &\geq \epsilon/8\end{aligned}$$

for large enough k. This contradicts $||\nabla_s\xi(s_k)||_s \to 0$, and hence $\lambda \in \sigma(A_\infty)$.

Step 5. Finally we show $\lim_{s\to\infty}\alpha(s) = \lambda$. Suppose not, then we can find $\{s'_k\}$ so that

$$\lim_{k\to\infty}\alpha(s'_k) \to \mu$$

Without loss of generality, let us assume $\mu < \lambda$. Then we can find some $\nu \in (\mu, \lambda)$ and $d > 0$ so that

$$\mathrm{dist}(\nu, \sigma(A(s))) \geq d$$

for all s sufficiently large. Then for any $\widehat{s}$ so that $\alpha(\widehat{s}) = \nu$, we have:

$$||\nabla_s\xi(\widehat{s})||_{\widehat{s}} \geq d/2$$

as before, but this implies $\alpha'(\widehat{s}) \geq d^2/4 > 0$, which implies for large enough s we have $\alpha(s) \geq \nu$, which is a contradiction. A very similar argument can be applied in case $\mu > \lambda$. We have therefore $\alpha(s) \to \lambda$. We claim $\lambda < 0$ because the norm of ζ exponentially decays to zero as $s \to \infty$.

Step 6 Finally we return to the case where for some s^* we have $||\zeta(s^*,\cdot)||_{s^*} = 0$, this would then imply $\zeta(s^*, t^*) = 0$ for all t^*, and the Carleman similarity principle then implies ξ is zero everywhere. □

Lemma 219. With the same notation as above. Let $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, and $j \in \mathbb{N}$. We have

$$sup_{(s,t)\in[s_0,\infty)\times[0,1]}|\partial^\beta\xi(s,t)| < \infty$$

$$sup_{s\in[s_0,\infty)}|\frac{d^j\alpha}{ds^j}(s)| < \infty.$$

Proof. The proof is almost verbatim the proof of lemma 3.10 in [2]. The only difference in our case is the appearance of the term $\Delta(s,t)\xi$, but in the notation of lemma 3.10 of Abbas this can be absorbed into $\widehat{\alpha}\xi$, and using the fact $\Delta(s,t)$ and its derivatives decay exponentially, we can recover bounds of derivatives of ξ and α. □

Proposition 220. Let E denote the eigenspace of the operator A_∞ in $W^{1,2}_\Gamma([0,1],\mathbb{R}^2)$ corresponding to eigenvalue λ, then

$$\operatorname{dist}(\xi(s), E) \to 0$$

as we take $s \to \infty$. The notion of distance is defined with respect to $W^{1,2}_\Gamma([0,1],\mathbb{R}^2)$.

Proof. The same proof as in Lemma 3.6 in [47]. □

Lemma 221. There exists $e \in E$ so that $\xi(s) \to e$ as $s \to \infty$.

Proof. The proof is the same as the proof of Lemma 3.12 in [2], as such we will only sketch the notable differences between our proof and theirs. As in their case for every sequence $\{s_n\}$ that converges to infinity, we can extract a subsequence $\{s'_n\}$ so that $\xi(s'_n) \to e \in E$. Hence it remains to show if we have sequences $\{s_n\}$ and $\{\tau_n\}$ so that

- $\xi(s_n) \to e \in E$,
- $\xi(\tau_n) \to e' \in E$

then we have $e = e'$. To this end, consider the inner product on $L^2([0,1],\mathbb{R}^2)$ given by:

$$(u_1, u_2) := \int_0^1 \langle u_1(t), -J_0 M_\infty u_2(t)\rangle dt$$

Let P denote the orthogonal projection to E with respect to the above inner product, and we define

$$\widehat{\xi} := P\xi.$$

Then it follows as in the proof in [2] that:

- $A_\infty P\xi = PA_\infty\xi$.
- If we define $\epsilon'(s) := A(s) - A_\infty$, we have the equation

$$\partial_s\widehat{\xi} = (\lambda - \alpha(s))\widehat{\xi} + P\epsilon'(s)\xi - P\Delta\xi(s)$$

- We have that both $||P\epsilon'(s)\xi||$ and $||P\Delta(s,t)\xi(s)||$ are bounded above by $Ce^{-\rho s}$.

- We have the norm bounds:

$$||\xi(s)|| \to 1,$$
$$||\widehat{\xi} - \xi(s)|| \to 0,$$

which in particular implies for large enough s we have

$$||\widehat{\xi}|| \geq 1/2.$$

With the above we define

$$\eta := \frac{\widehat{\xi}(s,t)}{||\widehat{\xi}(s,t)||}$$

Then we have the equation

$$\begin{aligned}\partial_s \eta &= \frac{\partial_s \widehat{\xi}}{||\widehat{\xi}||} - \frac{(\widehat{\xi}, \partial_s \widehat{\xi(s)})}{||\widehat{\xi}||^3}\widehat{\xi} \\ &= \frac{(\lambda - \alpha(s))\widehat{\xi} + P\epsilon'\xi - P\Delta\xi}{||\widehat{\xi}||} - \frac{(\widehat{\xi}, (\lambda - \alpha(s))\widehat{\xi} + P\epsilon'\xi - P\Delta\xi)}{||\widehat{\xi}||^3}\widehat{\xi} \\ &= \frac{P\epsilon'\xi - P\Delta\xi}{||\widehat{\xi}||} - \frac{(\widehat{\xi}, P\epsilon'\xi - P\Delta\xi)}{||\widehat{\xi}||^3}\widehat{\xi}\end{aligned}$$

From the above norm estimates we obtain

$$|\partial_s \eta| \leq e^{-\rho s}$$

where the above is the genuine absolute value. Then we have

$$\begin{aligned}|\eta(s_n) - \eta(\tau_n)| &\leq \left|\int_{\tau_n}^{s_n} \partial_s \eta ds\right| \\ &\leq \left|\int_{\tau_n}^{s_n} e^{-\rho s} ds\right| \\ &\to 0\end{aligned}$$

as $n \to \infty$. This concludes the proof of our lemma. □

Proof of Proposition 217. With the previous lemmas assembled, the proof of Proposition 217 follows verbatim to that of Proposition 3.6 in [2]. We use Sobolev embedding and induction to show that all derivatives of $r(s,t)$ converge uniformly to zero. We use Arzela-Ascoli to show that $\alpha(s)$ approaches λ in the C^∞ norm. □

We are now ready to apply the above estimates to analyze the difference of two ends asymptotic to the same Reeb orbit. Namely, we prove the following:

Proposition 222. For two different ends asymptotic to the same Reeb orbit with local coordinates:

$$U(s,t) = (b(s,t), x(s,t), y(s,t), z(s,t))$$

and

$$V(s,t) = (b(s,t), x(s,t) + \eta_1(s,t), y(s,t) + \eta_2(s,t), z(s,t) - fc_1\eta_1(s,t) - fc_2\eta_2(s,t)).$$

There are reparametrizations $\phi_u, \phi_v : [R,\infty) \times [0,1] \to [R,\infty) \times [0,1]$ for sufficiently large R, such that

$$U(\phi_u(s,t)) = (s, U_0(s,t), t),$$
$$V(\phi_v(s,t)) = (s, V_0(s,t), t),$$

and

$$V_0(s,t) - U_0(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

where $e(t)$ is an eigenvector of the asymptotic operator with eigenvalue λ, and $r(s,t)$ decays exponentially to zero as $s \to \infty$.

Proof of Proposition 222. Fix a sufficiently large positive number R, and $\epsilon > 0$ sufficiently small. Let D_ϵ be the ϵ-disk around the origin in $\mathbb{R}^2$, and $E : [R,\infty) \times D_\epsilon \times [0,1] \to \mathbb{R} \times \mathbb{R}^3$ be the embedding:

$$E(s,h_1,h_2,t) = (b(s,t), x(x,t) + h_1, y(s,t) + h_2, z(s,t) - c_1 fh_1 - c_2 fh_2) \quad (4.8.19)$$

where, as before, we shorthand $c_1(u(s,t))$, $c_2(u(s,t))$ and $f(u(s,t))$ as c_1, c_2 and f respectively. Now the first end U is parameterized as:

$$U(s,t) = E(s,0,0,t)$$

and since any different end that is asymptotic to the same Reeb chord can be viewed as an embedded surface in the image of E, V can be parametrized as

$$V(s,t) = E(s, \eta_1(s,t), \eta_2(s,t), t).$$

Since both ends limit to the Reeb chord exponentially, there are reparametrizations $\phi_u, \phi_v : [R,\infty) \times [0,1] \to [R,\infty) \times S^1$ such that

$$U(\phi_u(s,t)) = (s, U_0(s,t), t),$$

$$V(\phi_v(s,t)) = (s, V_0(s,t), t).$$

Let λ be the negative eigenvalue that appears in the asymptotic expansion of

$$\eta(s,t) = (\eta_1(s,t), \eta_2(s,t))$$

provided by Proposition 217, and $e(t)$ be the corresponding eigenvector. We have:

$$\phi_u(s,t) - \phi_v(s,t) = o_\infty(\lambda)$$

Here we use the same notation as in [108]. Namely, a function $f : [R, \infty) \times [0,1] \to \mathbb{R}^2 = o_\infty(\lambda)$ if and only if there are constants $\delta > 0$ and $M_\beta > 0$ for every multi-index β, such that

$$|\partial^\beta(e^{\lambda s} f)| \leq M_\beta e^{-\delta s}$$

The reason is that, by definition

$$\phi_u^{-1}(s,t) - \phi_v^{-1}(s,t) = (0, -c_1(u(s,t))f(u(s,t))\eta_1(s,t) - c_2(u(s,t))f(u(s,t))\eta_2(s,t)),$$

therefore $\phi_u^{-1}(s,t) - \phi_v^{-1}(s,t) = o_\infty(\lambda)$, and hence $\phi_u(s,t) - \phi_v(s,t) = o_\infty(\lambda)$. Now let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection to the second and third coordinates. We have:

$$\begin{aligned}
V_0(s,t) - U_0(s,t) &= \pi(V(\phi_v(s,t)) - U(\phi_v(s,t))) \\
&= \pi(E(\phi_v(s,t), \eta(\phi_v(s,t))) - E(\phi_u(s,t), 0, 0)) \\
&= \pi(E(\phi_u(s,t), \eta(\phi_u(s,t))) - E(\phi_u(s,t), 0, 0)) + o_\infty(\lambda) \\
&= \eta(\phi_u(s,t)) + o_\infty(\lambda) \\
&= e^{\lambda s} e(t) + o_\infty(\lambda)
\end{aligned}$$

□

The above discussion finishes the proof of Theorem 184. To see this fact, choose an Riemannian metric on Y so that near the Reeb chord η with the chosen local coordinates, the metric agrees with the standard Riemannian metric on $\mathbb{R}^3$. Extend this metric to an $\mathbb{R}$-invariant metric on $\mathbb{R} \times Y$, and we see by definition that the functions U_0 and V_0 are asymptotic representatives of the two ends (see Definition 2.1 of [108]). The above Proposition shows that there is a section $r(s,t)$ of $\eta^*\xi$ that exponentially decays to zero as s tends to ∞, so that under the chosen local coordinates the two asymptotic representatives satisfy

$$U_0(s,t) - V_0(s,t) = e^{\lambda s}(e(t) + r(s,t)).$$

Having established the above proposition, the exact same argument as in Section 4 in [108] proves the following:

Proposition 223. (Compare Theorem 2.4 in [108]) Let γ be a Reeb chord connecting two Legendrians Γ_0 and Γ_1, with action 1, and fix a trivialization τ of $\gamma^*\xi$. Let $\{u_i\}_{i=1}^n$ be a collection of pseudo-holomorphic curves with γ as a positive (resp. negative) end. Then there exist a neighborhood U of γ, a smooth embedding $\Phi : \mathbb{R} \times U \to \mathbb{R} \times \mathbb{R}^2 \times [0,1]$ with the property

$$\Phi(s, \gamma(t)) = (s, 0, 0, t),$$

proper reparametrizations $\psi_i : [R, \infty) \times [0,1] \to \mathbb{R} \times [0,1]$ asymptotic to the identity, and positive integers N_i, such that near the asymptotic ends

$$\Phi \circ u_i \circ \psi_i(s,t) = (s, t, \sum_{k=1}^{N_i} e^{\lambda_{i,j} s} e_{i,j}(t))$$

where $\lambda_{i,j}$ are negative (resp. positive) eigenvalues of the asymptotic operator associated to γ and τ, and $e_{i,j}$ are eigenvectors with eigenvalue $\lambda_{i,j}$.

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