Mathematical insights as novel connections: Evidence from expert mathematicians

Shadab Tabatabaeian (stabatabaeian@ucmerced.edu)
Cognitive & Information Sciences, University of California
5200 North Lake Rd., Merced, CA 95343

Artemisia Deluna (artemisiadeluna@gmail.com)

David Landy (dhlandy@gmail.com)
Netflix, Los Gatos, CA & Indiana University Bloomington

Tyler Marghetis (tyler.marghetis@gmail.com)
Cognitive & Information Sciences, University of California Merced

Abstract

Expert mathematicians make discoveries about entirely abstract entities. Where do these insights come from? According to a variety of classic accounts, creativity is a multi-stage process that involves combining ideas in novel ways. Most of the evidence for these accounts, however, is drawn from artificial lab-based settings or is zoomed out from the messy, moment-to-moment details of discovery. Here, we take advantage of a video corpus of expert mathematicians generating proofs in their natural habitat: at the blackboard, chalk in hand. These mathematicians experienced spontaneous insights as they worked on proofs. We find that mathematicians begin by creating a variety of inscriptions, filling the blackboard with equations and diagrams. They then interact with these inscriptions through gaze, speech, gesture, and writing. When they experience an insight, however, their interactions become unpredictable, and they begin to connect inscriptions in novel ways (quantified by an information-theoretic measure, surprisal). Expert mathematical discovery, we conclude, exhibits the stages and combinatorial processing that have been proposed to characterize creativity. Even at the pinnacle of abstraction, at the highest levels of expertise, new ideas are born when the body discovers unexpected affinities among ideas.

Keywords: mathematical insight; expert mathematicians; combinatorial creativity; stage-based models of creativity; mathematical inscriptions; situated reasoning

Introduction

Many discoveries involve perceiving the right thing in the right way. An ornithologist might observe a heretofore unknown bird, or an astrophysicist might use data to infer the shape of a black hole. But other discoveries occur in the absence of new empirical evidence. Mathematics, for instance, involves reasoning about entirely abstract entities, from infinity sets to functions that are impossible to visualize. And yet mathematical practice is replete with moments of sudden insight. Where do these insights come from?

Creativity is at the core of expert mathematics. Mathematics is more than just a static collection of established theorems and techniques. Mathematicians continue to make new discoveries (Davis, Hersh, & Marchisotto, 2012; Mann, 2006; Sriraman, 2004; Sternberg & Ben-Zeev, 1996). To do so, they scribble, draw, erase, sketch. They get stuck. They may give up. But sometimes, they suddenly have an insight that allows them to make progress (Poincaré, 1913; Hadamard, 1954). Given the abstraction and complexity of mathematics, this stands as a paragon of human creativity. How does this happen? What is the process by which expert mathematicians have insights?

Insight as multi-stage and combinatorial

Creative insight has been described as a multi-stage process. One popular model, proposed by Wallas (1926), suggests four stages: preparation, incubation, illumination, and verification. In the preparation phase, individuals begin to understand the problem. In the incubation phase, they unconsciously search for solutions. In the illumination phase, they experience an "aha!" moment that signifies the discovery of a potential solution. And in the verification stage, individuals actively test the solution to confirm its viability. Other stage-based accounts have been proposed (Guilford, 1967; Campbell, 1960; Lubart, 2001).

When a creative insight arrives, it often appears to combine old ideas in new ways. In the lead-up to insight, many potential combinations may need to be considered before a successful combination is discovered (Hadamard, 1954; Simonton, 2009, 2012, 2021). Most of these combinations are common and therefore uninformative or of no use. However, a rare combination may turn out to be highly informative, revealing connections between concepts previously thought unrelated. These new connections can even give rise to a change in how one thinks about or represents the problem (Duncker, 1945; Hélie & Sun, 2010; Knoblich, Öllinger, & Spivey, 2005; Ohlsson, 1992). Such combinations thus provide novel and fruitful insights. For example, Koestler (1964) defined creativity not as creating something out of nothing, but as a process that "uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas, faculties, skills" (p. 323).

We illustrate this process in the top panels of Figure 1. Before an insight, a reasoner might explore a variety of connections (edges) between ideas (nodes) (Fig. 1A). The moment of insight arrives when two previously disconnected ideas are recognized as related, indicated here by the blue edges (Fig. 1B).

In J. Culbertson, A. Perfors, H. Rabagliati & V. Ramenzoni (Eds.), *Proceedings of the 44th Annual Conference of the Cognitive Science Society.*

This combinatorial vision of creativity has been widely invoked (Hummel & Holyoak, 2002; Simonton, 2009; Thagard & Stewart, 2011). Fauconnier and Turner (2002), for instance, coined the term "conceptual blending" to refer to the general cognitive ability that allows humans to integrate ideas from different domains to create new concepts and meanings. They considered innovation and creativity as one manifestation of conceptual blending. In yet another account, Mednick (1962) described creativity as "the forming of associative elements into new combinations which either meet specified requirements or are in some way useful" (p. 221). In a similar vein, studies of analogical reasoning provide evidence that discovering connections between remote concepts can prompt creative insights (Dunbar, 1995; English, 2004; Gick & Holyoak, 1980; Holyoak & Thagard, 1996; Pólya, 1990). Finally, in the mathematician Poincaré's poetic description (1913), creativity involves recognizing an "unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another" (p. 386). In other words, creative insight doesn't necessarily require a *new* fact. Instead, ideas that were previously treated separately are suddenly brought into conversation, combined, seen as connected.

But what about genuine mathematical discovery?

As evident from the brief review above, many scholars — and self-reflective creative individuals such as Poincaré — have pointed to the multi-stage and combinatorial nature of insight. Empirical evidence for these proposals has typically come from simple tasks in artificial, lab-based settings (Stephen, Boncoddo, Magnuson, & Dixon, 2009; Duncker, 1945; Knoblich, Ohlsson, Haider, & Rhenius, 1999; Chesebrough, 2021; Bieth et al., 2021). This raises the question of whether this account actually extends to more complex, real-world feats of discovery.

When scholars *have* examined insight among scientific and mathematical experts, they have mostly relied on anecdote (Poincaré, 1913), self-report (Hadamard, 1954), or a zoomed out analysis of the long sweep of history (Simonton, 2021, 1999).

The moment-to-moment work of mathematics is material and messy. Mathematicians write, scribble, draw, sketch. Then they elaborate, erase, gesture toward, and interpret those inscriptions (Barany & MacKenzie, 2014; Greiffenhagen, 2014; Marghetis, Samson, & Landy, 2019; Goldstone, Marghetis, Weitnauer, Ottmar, & Landy, 2017). This embodied activity—in which mathematicians literally use their bodies to advance their thinking—is a core part of doing mathematics at the highest levels. In a study of mathematical practice, one mathematician described a near-total dependence on paper and pen: "Of course you could train yourself to do it mentally [...] but this is not how you work. This is not how I work. When I work, I write things down on a piece of paper" (Johansen & Misfeldt, 2020, p. 3726). Another dismissed entirely the possibility of solving problems in his mind alone: "No, no. You write... You write... You write" (p. 3726). For the working mathematician, therefore, writing is integral to the process of discovery.

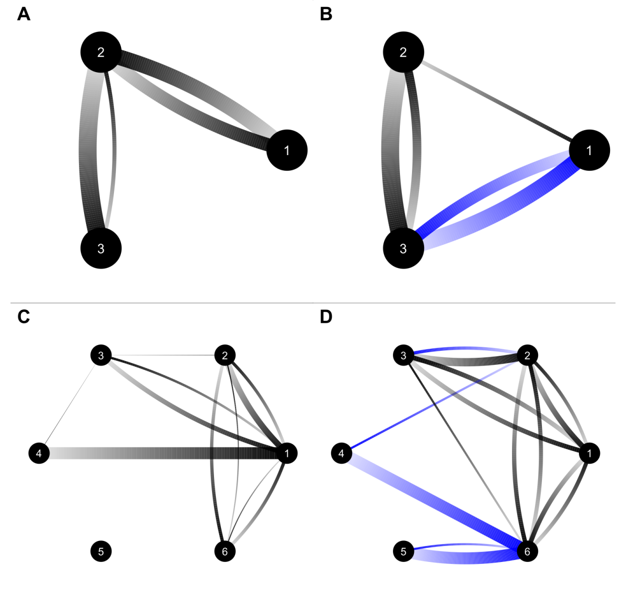

Figure 1: **Insight as a combinatorial process.** (A, B) Schematic illustration of combinatorial insight. Nodes represent ideas. Edges represent directed connections among ideas, with edge width indicating how frequently the connection is made, and shading indicating the direction of the connection (from light to dark). Prior to the insight (left panel), the same connections are explored, while others are not considered at all. After the insight (right panel), a novel connection is discovered (blue edges), and this productive combination of ideas is explored while older combinations are abandoned. (C, D) Combinatorial insight by a mathematician in the video corpus. Nodes represent inscriptions on the blackboard. Edges indicate shifts of attention between inscriptions, with edge width indicating empirical frequency. Prior to the insight (left panel), some inscriptions were never connected by the mathematician. After the mathematician experienced an insight (right panel), they more frequently linked inscriptions that before were only weakly connected (e.g., 2 and 3), and created new links between inscriptions that had been totally disconnected.

So, what does discovery look like in the moment-to-moment dynamics of the mathematicians' struggle at the blackboard?

The present study

To study the process of creative insight in expert mathematicians, we relied on the centrality of *inscription* in mathematical practice. Expert mathematicians write extensively, sometimes with pen or pencil, but often with chalk at a blackboard (Barany & MacKenzie, 2014; Greiffenhagen, 2014; Wynne, 2021).

We examined spontaneous insights experienced by expert mathematicians as they worked on non-trivial problems in their natural habitat—that is, with chalk in hand at the blackboard. We identified moments of insight expressed by the mathematicians (e.g., a mathematician exclaims, “Aha!”). We also tracked where they were attending as they worked, based on their gaze, gesture, writing, and speech. We could thus examine how mathematicians changed their interactions with their inscriptions around the moment of insight. This design allows us to investigate expert mathematical creativity as it occurs in an ecologically valid setting.

If multi-stage models of creativity are correct, mathematicians should engage in different patterns of activity. During the early “preparation” stage, mathematicians might create new inscriptions, externalizing their ideas and creating an “ecosystem” of blackboard inscriptions (Marghetis et al., 2019). If the “insight” stage involves the discovery of novel combinations, mathematicians experiencing an insight might shift their pattern of interactions, the way they shift their attention among inscriptions (Fig. 1, A and B). Previously unrelated inscriptions might suddenly become linked for the mathematician, with attention shifting back and forth. Previously linked inscriptions might no longer seem related at all. A period of insight, therefore, might manifest in the mathematicians' embodied[1] activity at the blackboard.

Methods

Video Corpus

We used a subset of a video corpus of PhD-level mathematical experts working on non-trivial problems. This corpus (total corpus length: 4 hours and 40 minutes) was collected by Marghetis et al. (2019) to study mathematical reasoning as it occurs in its natural context: in the Math department, working at a blackboard. Participants (hereafter, “mathematicians”) were completing their PhD in mathematics at an R1 research university, were recruited via email, and were compensated for their participation.

They were presented with up to three conjectures that they were asked to prove. The conjectures were selected from William Lowell Putnam Mathematics competition and included various topics, such as set theory, geometry, and analysis:

1. Find an uncountable subset, S, of the power set of a countable set, such that the intersection of each pair of elements in S is finite.
2. Let f : R2→R be a function such that f(x, y) + f(y, z) + f(z, x) = 0 for all real numbers x, y, and z. Prove that there exists a function g: R→R such that f(x, y) = g(x) - g(y) for all real numbers x and y.
3. Let d1, d2, ... d12 be real numbers in the interval (1, 12). Show that there exist distinct indices i, j, k such that di, dj, dk are the side lengths of an acute triangle.

These were chosen to be non-trivial but approachable for PhD-level mathematical experts. The mathematicians were encouraged to verbalize their reasoning while working on the proof, but otherwise were left to their own devices. Each mathematician worked for up to an hour, moving on to a new conjecture when they either felt they had proved the current conjecture or did not know how to make further progress.

For the analyses here, we included only those mathematicians who attempted both conjectures 2 and 3 ($N = 6$). This subset of the corpus consisted of twelve proof sessions lasting a total of 4 hours and 5 minutes.

Video Analysis

One researcher, blind to our hypotheses about mathematical insight, coded every time a mathematician directed their attention toward a blackboard inscription, whether by creating a new inscription or by shifting their attention to an existing inscription. Shifts of attention were inferred from mathematicians' gaze,[2] gesture, speech, writing, or the act of erasing an inscription (Fig. 2, A and B). This generated a time series of “events” in which mathematicians shifted their attention among inscriptions. Distinct inscriptions were identified based on two criteria: semantic relatedness and spatial proximity. For example, the different parts of a graph—e.g., two axes, axes labels, the line representing the function—would be treated as a single inscription.

A second coder identified, from the video, every time that a mathematician verbally expressed that they were having an insight (e.g., saying “aha!” or “ohhhhhh, I see”). We identified a total of 24 insights in the corpus.

Previous work has suggested that the insight process can begin prior to its verbal expression (the “flash” of insight) and can continue for some time after (Wallas, 1926; Sadler-Smith, 2015). We, therefore, analyzed the periods immediately before and after the moment of the verbal expression of the insight. Inspection of the video corpus suggested that one-minute periods (before and after) would capture the pace at which mathematicians progressed through their reasoning. A sensitivity analysis confirmed that our results do not depend on the choice of this specific duration.

Quantifying novelty of connections

When mathematicians shifted their attention from one inscription to another, this event could range in novelty. Mundane events involved shifts of attention between inscriptions that had been connected previously. Novel events connected inscriptions that had not previously been connected. To quantify the novelty of these connections, we used a measure from information theory, *surprisal*, that quantifies how unexpected or informative an event is.

We calculated the surprisal of each shift of attention relative to the shifts that had occurred previously in the proof

[1] Here, we use the word “embodied” to refer to the actual use of the body—to gesture, write, etc.—rather than to the activation of sensorimotor brain areas.

[2] Shifts of attention through gaze were inferred by the coder from the video. To maximize ecological validity, we did not use eye tracking technology.

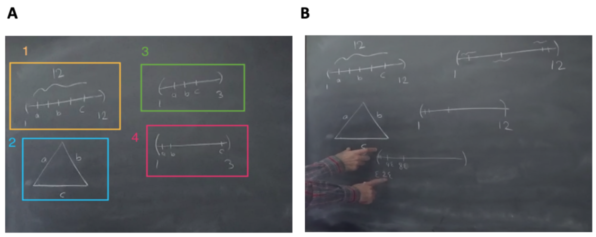

Figure 2: **Illustration of mathematicians' inscription-making behavior and shifts of attention across inscriptions.** (A) Screen capture of a mathematician working at the blackboard. Inscriptions are identified by numbered squares. As the mathematician worked on the proof, he created new inscriptions and shifted his attention between existing ones. (B) Two pictures are blended to illustrate the mathematicians' shift of attention from one inscription to the next. The shifts of attention were identified based on mathematicians' inscription-making behavior, gesture, gaze, and speech.

session.[3] For the i-th shift of attention, s_i, we calculated the probability transition matrix, P_i, of all shifts of attention that had occurred up to and including that moment. We then calculated the surprisal, h, of the shift of attention s_i:

$$h(s_i) = -\log_2 P_i(s_i) \qquad (1)$$

Surprisal thus ranges from 0 to ∞, with 0 indicating that the current shift of attention was identical to all previous shifts, and greater values indicating that the shift was unlikely given all previous shifts.

Results

Decreased introduction of new inscriptions before the flash of insight

Mathematicians were most likely to introduce new inscriptions when they were not experiencing an insight (Fig. 3). We calculated the probability that a blackboard interaction involved the introduction of a new inscription (rather than, say, a shift of attention to an existing inscription). The probability was lowest in the one-minute period before a flash of insight ($M = .035$, $SE = .011$), and highest when they were not near an insight at all (i.e., not in the one-minute periods before or after a flash of insight; $M = .074$, $SE = .009$).

We constructed a mixed effects logistic model of whether each blackboard interaction event involved the creation of a new inscription. The model included a fixed effect for time in minutes. It also included fixed effects for whether the event was in the period immediately before the flash of insight, another for whether the event was in the period immediately after the flash of insight. These fixed effects thus compare the periods immediately before or after the flash of insight,

[3]This approach is similar to analyses of language in which the surprisal of words or phrases is calculated relative to the preceding discourse context (Levy, 2008; Hale, 2003; Kuznetsova, Chen, & Choi, 2013).

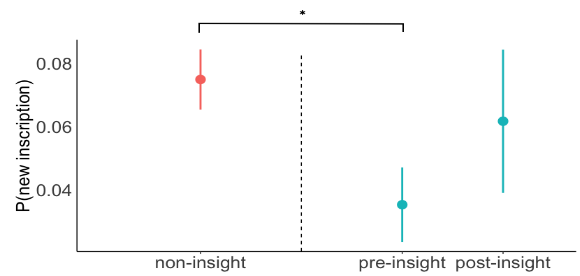

Figure 3: **The probability of creating a new inscription when interacting at the blackboard was lowest in the period immediately before a flash of insight.** The vertical axis represents the proportion of blackboard interactions that introduced a new inscription. *Non-insight* events, in red, did not occur near a flash of insight. Insight, in teal, was divided into *pre-insight* (the period immediately before the flash of insight) and *post-insight* (the period immediately after). Dots indicate means; error bars indicate SEM. (* = $p < .05$)

to all other times during the session. Finally, we included an interaction between these two to account for events that were sandwiched between two back-to-back flashes of insight.[4] To account for variability among individuals and conjectures, we included random intercepts for sessions.

Mathematicians were significantly less likely to create new inscriptions as the session continued ($b = -.053 \pm .009$ SE, $Z = -5.437$, $p < .001$). Critically, they were also less likely to create new inscriptions in the one-minute period before the flash of insight, compared to periods away from the insight ($b = -0.73 \pm 0.298$ SE, $Z = -2.44$, $p = .014$). New inscriptions were numerically but not significantly less likely to occur in the one-minute period after the flash of insight ($b = -0.23 \pm 0.24$ SE, $Z = -0.94$, $p = .34$). The interaction was not significant ($b = 0.33 \pm 0.64$ SE, $Z = .52$, $p = .60$). These results were robust to different choices of the duration of the window before and after the flash of insight, ranging from 45 seconds to 75 seconds. The production of new inscriptions, therefore, dropped off significantly in the period leading up to a flash of insight.

Increased surprisal of connections during insight

Mathematicians made significantly more novel connections during periods of insight, both immediately before and after expressing the flash of insight (Fig. 4). This is illustrated in the bottom panels of Figure 1. Here we see the way an actual mathematician in the corpus shifted their attention between inscriptions both before and during an insight. The right panel, (D), shows the mathematician's activity in a two-minute period centered around a flash of insight. The left panel, (C), shows the two-minute period before that. Notice

[4]This interaction term thus accounted for cases where mathematicians experienced two insights within a brief period of time, such that the one-minute period after the first insight would overlap with the one-minute period before the second.

that, during the insight, some old connections (edges) that were infrequent (thin) are now more frequent (thick), and they are even new connections that did not exist previously (blue edges). In this case, therefore, the flash of insight was accompanied by a period of novel, unexpected connections between inscriptions.

This was true in general. Mathematicians' shifts of attention between inscriptions were most predictable when they were not experiencing an insight (surprisal: $M = 1.98$, $SE = .09$). By contrast, in the one-minute periods both before and after the flash of insight, they made shifts of attention between inscriptions that were more unexpected ($M_{before} = 2.14$, $SE = 0.12$; $M_{after} = 2.19$, $SE = .09$).

We constructed a linear mixed effects model of the surprisal of each shift of attention. Like the model of new inscriptions described above, this model included fixed effects for time, for whether the event was in the period before the flash of insight, for whether the event was in the period after the flash of insight, and for the interaction between these latter two. We added a fixed effect for whether the event involved introducing an entirely new inscription, since these would have higher surprisal by definition. Once again, we included random intercepts for sessions.

Surprisal increased over the course of the session, with a .06 increase in surprisal for every passing minute ($b = .06 \pm .003$ SE, $t = 19.6$, $p < .01$). Unsurprisingly, shifts of attention to entirely new inscriptions had significantly higher surprisal ($b = .67 \pm .08$ SE, $t = 7.89$, $p < .01$). Critically, surprisal was significantly greater in the one-minute periods both immediately before the flash of insight ($b = .19 \pm .08$ SE, $t = 2.34$, $p = .02$) and immediately after the flash of insight ($b = .23 \pm .08$ SE, $t = 2.89$, $p < .01$). The interaction was not significant ($b = -.27 \pm .18$ SE, $t = -1.54$, $p = .12$). These results were robust to different choices of the duration of the window before and after the flash of insight, ranging from 45 seconds to 75 seconds. Mathematical insight, therefore, was associated with making more novel connections among inscriptions.

Discussion

In an analysis of a video corpus of expert mathematicians, we found evidence that preparation and insight were associated with different patterns of blackboard interaction. Mathematicians introduced fewer new inscriptions in the moments before a flash of insight. Insights were associated with shifts of attention between blackboard inscriptions that were more unpredictable, as measured by surprisal. These results are in line with multi-stage and combinatorial models of creativity, in which reasoners initially prepare for insight by introducing elements that might be useful, and then arrive at insights by making novel connections between those elements.

While early accounts of mathematical insight were foundational for the scholarly study of creativity, subsequent empirical work on creativity has tended to focus on artificial lab settings (Stephen, Boncoddo, et al., 2009) or zoomed out analyses of the products of expert inquiry (Simonton, 2021). Here, we leveraged a naturalistic corpus to study insight as it occurs in the chalky, material, furniture-cluttered world of mathematical activity. We see this work as a complement to—and extension of—the anecdotes and introspection of mathematicians (Poincaré, 1913; Hadamard, 1954).

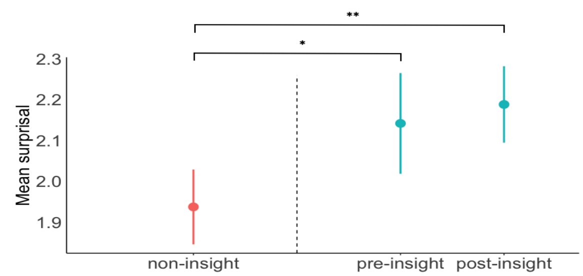

Figure 4: **Mathematicians made more unexpected connections during an insight**. The vertical axis represents the surprisal of shifts of attention between inscriptions. *Non-insight* shifts of attention, in red, did not occur near a flash of insight. Insight, in teal, was divided into *pre-insight* (the period immediately before the flash of insight) and *post-insight* (the period immediately after). Dots indicate means; error bars indicate SEM.(* = $p < .05$, ** = $p < .01$)

Stage-based theories of insight

Our results provide evidence for stage-based theories of creativity. On one influential account, creativity involves four stages: preparation, incubation, illumination, and verification (Wallas, 1926). The illumination stage, in particular, often gets the most attention, since it is associated with the "aha!" moment itself. When mathematicians in our corpus expressed verbally that they'd had an insight, this general period presumably corresponded to their "illumination" stage. Some have argued that illumination begins even before we become aware of it (Sadler-Smith, 2015; Wallas, 1926), and this was borne out in our data: Mathematicians began making more novel connections even *before* they expressed their insight verbally.

Illumination is supposed to follow preparation and incubation, and precede verification. While our data do not allow us to distinguish these stages in the mathematicians' activity, we presume that preparation and incubation corresponded to the period before they began to experience illumination. This is given credence by the pattern of inscription creation. In general, more inscriptions were created toward the start of the session, and there was a drop in new inscriptions right before the flash of insight. This suggests that mathematicians front loaded their inscription-making as they familiarized themselves with the conjecture, thus preparing themselves for insight by crafting a "notational niche" conducive to their proof-making efforts (Marghetis et al., 2019; Menary, 2015).

Inside the skull, or out in the world?

Writing is at the core of mathematical practice. This was impossible to ignore in the ecologically valid setting we investigated here, where the creation and manipulation of inscriptions were ubiquitous. All six mathematicians made abundant inscriptions during proof generation, and their insights were associated with changing interactions with those inscriptions. How should we think of the relationship between these interactions and the insight process itself?

One approach is to consider mathematicians' external interactions with inscriptions as merely a *proxy* for their brain-based mathematical reasoning. On this account, creative insights occur within the skull and are only later projected onto the blackboard. This approach is in line with accounts that attribute creative insight primarily to the unconscious mind—where, perhaps, mathematical concepts bounce around like atoms, in hope of finding a useful connection (Poincaré, 1913).

On the other hand, creating and interacting with inscriptions might be a constitutive part of the reasoning process (Clark, 2008; Goldstone et al., 2017; Hutchins, 1995, 2005; Johansen & Misfeldt, 2020; Marghetis et al., 2019; Menary, 2007). Interactions with the external world of inscriptions might actively contribute to the discovery of new connections.

Our observational data cannot decide between these approaches. But we find the latter, embodied approach more plausible, more fruitful, and a more natural fit to the fleshy, chalky activity that we observe. Past work has found that, when participants are trying to solve a simple lab-based puzzle, the moment of insight is preceded by changes in the entropy of eye movements and gestures (Stephen, Boncoddo, et al., 2009; Stephen, Dixon, & Isenhower, 2009), as if the process of discovery is reflected in the entire embodied organism, not just confined to the brain. Other work has found that even slight modifications of inscriptions can affect mathematical perception, performance, and reasoning, as if the inscriptions themselves were playing an active role (Goldstone, Landy, & Son, 2010; Goldstone et al., 2017; Landy & Goldstone, 2010). In addition, previous research has established a link between embodied mathematical practice and more effective teaching and learning of mathematics (Abrahamson, Dutton, & Bakker, 2022; Abrahamson et al., 2020; Alibali & Nathan, 2012; Cook & Goldin-Meadow, 2006; Edwards, 2009; Flood, Shvarts, & Abrahamson, 2020; Goldin-Meadow, Cook, & Mitchell, 2009; Hall & Nemirovsky, 2011; Lakoff & Núñez, 2000; Nemirovsky & Ferrara, 2009; Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014; Núñez, 2008; Núñez, Edwards, & Matos, 1999; Richland, Zur, & Holyoak, 2007). Here, we showed that mathematical creativity is reflected in the interactions between the mathematician's body and the external world, in line with embodied accounts of mathematical cognition.

Indeed, shifts of attention among external inscriptions may be the cause, not the effect, of insight. While ideas in memory fade when not under the spotlight of attention, ideas on the blackboard retain their luster even if they've been forgotten temporarily by the mathematician. Once externalized, ideas are easy to connect—not by laboriously bringing them together in working memory—but by linking them in the external world with gestures or movements of the eyes (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001; Goldin-Meadow, 2004; Cook, Friedman, Duggan, Cui, & Popescu, 2017). Old ideas can be rediscovered with a single glance. New connections can be discovered by accident (Kirsh, 2014) as one's hands and eyes are drawn across the blackboard. And with ideas strewn across the blackboard, it becomes easier for the mathematician to connect a new thought—not yet externalized as an inscription—to the stable external record of all the other ideas currently in play. It's easier to hold a single idea in working memory, after all, than to hold two while trying to discern their kinship. The web of relations among brain, body, and the broader material world invites an account of mathematical creativity that does not restrict insight to the confines of the mathematician's skull.

Conclusion

More than a century ago, the mathematician Henri Poincaré (1913) proposed an analogy between insight and a cloud of atoms. Just as the atoms dance around, meeting and joining to create new compounds, mathematical facts bounce around in our minds, combined in a variety of ways, until eventually a connection is made that is both novel and illuminating. An evocative metaphor, certainly. But since then, data on the actual process of expert mathematical creativity has been sorely lacking. Here, we studied insight as it emerges from the moment-to-moment activity of expert mathematicians in their natural habitat: generating proofs in their office or a seminar room, using chalk and blackboard. As predicted, the mathematicians' insights were multi-stage and combinatorial. They prepared by adding inscriptions to the blackboard. And then the insight itself involved novel, unexpected connections between inscriptions. Even at the pinnacle of abstraction, at the highest levels of expertise, new ideas emerge when the body discovers "unsuspected kinship" among ideas (Poincaré, 1913).

References

Abrahamson, D., Dutton, E., & Bakker, A. (2022). Towards an enactivist mathematics pedagogy. In S. A. Stolz (Ed.), *The body, embodiment, and education: An interdisciplinary approach* (pp. 156–182). Routledge.

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers in Education*, 1–29.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the learning sciences*, *21*(2), 247–286.

Barany, M. J., & MacKenzie, D. (2014). Chalk: Materials and concepts in mathematics research. In C. Coopmans,

J. Vertesi, M. Lynch, & S. Woolgar (Eds.), *Representation in scientific practice revisited* (pp. 107–130). Inside Technology, MIT Press.

Bieth, T., Kenett, Y., Ovando-Tellez, M., Lopez-Persem, A., Lacaux, C., Oudiette, D., & Volle, E. (2021). Dynamic changes in semantic memory structure support successful problem-solving. *PsyArXiv*.

Campbell, D. T. (1960). Blind variation and selective retentions in creative thought as in other knowledge processes. *Psychological review*, *67*(6), 380–400.

Chesebrough, C. B. (2021). *Conceptual change induced by analogical reasoning sparks "aha!" moments* (Unpublished doctoral dissertation). Drexel University.

Clark, A. (2008). *Supersizing the mind: Embodiment, action, and cognitive extension*. Oxford University Press.

Cook, S. W., Friedman, H. S., Duggan, K. A., Cui, J., & Popescu, V. (2017). Hand gesture and mathematics learning: lessons from an avatar. *Cognitive science*, *41*(2), 518–535.

Cook, S. W., & Goldin-Meadow, S. (2006). The role of gesture in learning: Do children use their hands to change their minds? *Journal of cognition and development*, *7*(2), 211–232.

Davis, P. J., Hersh, R., & Marchisotto, E. A. (2012). *The mathematical experience, study edition*. Springer.

Dunbar, K. (1995). How scientists really reason: Scientific reasoning in real-world laboratories. In R. J. Sternberg & J. E. Davidson (Eds.), *The nature of insight* (pp. 365–395). The MIT Press.

Duncker, K. (1945). On problem-solving. *Psychological monographs*, *58*(5), 1–113.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics*, *70*(2), 127–141.

English, L. D. (Ed.). (2004). *Mathematical and analogical reasoning of young learners*. Routledge.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. Basic Books.

Flood, V. J., Shvarts, A., & Abrahamson, D. (2020). Teaching with embodied learning technologies for mathematics: responsive teaching for embodied learning. *ZDM*, *52*, 1307–1331.

Gick, M. L., & Holyoak, K. J. (1980). Analogical problem solving. *Cognitive Psychology*, *12*(3), 306–355.

Goldin-Meadow, S. (2004). Gesture's role in the learning process. *Theory into practice*, *43*(4), 314–321.

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological science*, *20*(3), 267–272.

Goldin-Meadow, S., Nusbaum, H., Kelly, S. D., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological science*, *12*(6), 516–522.

Goldstone, R. L., Landy, D., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*(2), 265–284.

Goldstone, R. L., Marghetis, T., Weitnauer, E., Ottmar, E. R., & Landy, D. (2017). Adapting perception, action, and technology for mathematical reasoning. *Current Directions in Psychological Science*, *26*(5), 434–441.

Greiffenhagen, C. (2014). The materiality of mathematics: presenting mathematics at the blackboard. *The British Journal of Sociology*, *65*(3), 502–528.

Guilford, J. P. (1967). *The nature of human intelligence*. McGraw-Hill.

Hadamard, J. (1954). *An essay on the psychology of invention in the mathematical field*. Dover Publications.

Hale, J. (2003). The information conveyed by words in sentences. *Journal of Psycholinguistic Research*, *32*(2), 101–123.

Hall, R., & Nemirovsky, R. (2011). Introduction to the special issue: Modalities of body engagement in mathematical activity and learning. *Journal of The Learning Sciences*, *21*, 1–9.

Hélie, S., & Sun, R. (2010). Incubation, insight, and creative problem solving: a unified theory and a connectionist model. *Psychological review*, *117*(3), 994–1024.

Holyoak, K. J., & Thagard, P. (1996). *Mental leaps: Analogy in creative thought*. MIT press.

Hummel, J. E., & Holyoak, K. J. (2002). Analogy and creativity: Schema induction in a structure-sensitive connectionist model. In T. Dartnall (Ed.), *Creativity, cognition, and knowledge: An interaction* (pp. 181–210). Praeger Publishers/Greenwood Publishing Group Inc.

Hutchins, E. (1995). *Cognition in the wild*. MIT press.

Hutchins, E. (2005). Material anchors for conceptual blends. *Journal of pragmatics*, *37*(10), 1555–1577.

Johansen, M. W., & Misfeldt, M. (2020). Material representations in mathematical research practice. *Synthese*, *197*(9), 3721–3741.

Kirsh, D. (2014). The importance of chance and interactivity in creativity. *Pragmatics & Cognition*, *22*(1), 5–26.

Knoblich, G., Ohlsson, S., Haider, H., & Rhenius, D. (1999). Constraint relaxation and chunk decomposition in insight problem solving. *Journal of Experimental Psychology: Learning, memory, and cognition*, *25*(6), 1534–1555.

Knoblich, G., Öllinger, M., & Spivey, M. J. (2005). Cognitive processes in eye guidance. In G. Underwood (Ed.), *The nature of insight* (pp. 355–375). Springer.

Koestler, A. (1964). *The act of creation*. Macmillan.

Kuznetsova, P., Chen, J., & Choi, Y. (2013). Understanding and quantifying creativity in lexical composition. In *Proceedings of the 2013 conference on empirical methods in natural language processing* (pp. 1246–1258).

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.

Landy, D., & Goldstone, R. L. (2010). Proximity and precedence in arithmetic. *The Quarterly Journal of Experimental Psychology*, *63*(10), 1953–1968.

Levy, R. (2008). Expectation-based syntactic comprehension. *Cognition*, *106*(3), 1126–1177.

Lubart, T. I. (2001). Models of the creative process: Past, present and future. *Creativity research journal*, *13*(3-4), 295–308.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236–260.

Marghetis, T., Samson, K., & Landy, D. (2019). The complex system of mathematical creativity: Modularity, burstiness, and the network structure of how experts use inscriptions. *CogSci*, 763–769.

Mednick, S. (1962). The associative basis of the creative process. *Psychological Review*, *69*(3), 220–232.

Menary, R. (2007). *Cognitive integration*. Palgrave Macmillan.

Menary, R. (2015). Mathematical cognition: A case of enculturation. In T. K. Metzinger & J. M. Windt (Eds.), *Open MIND* (pp. 1–20). MIND Group.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174.

Novack, M. A., Congdon, E. L., Hemani-Lopez, N., & Goldin-Meadow, S. (2014). From action to abstraction: Using the hands to learn math. *Psychological science*, *25*(4), 903–910.

Núñez, R. (2008). A fresh look at the foundations of mathematics: Gesture and the psychological reality of conceptual metaphor. In A. Cienki & C. Müller (Eds.), *Metaphor and gesture* (pp. 93–114).

Núñez, R., Edwards, L., & Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45–65.

Ohlsson, S. (1992). Information-processing explanations of insight and related phenomena. In M. Keane & K. Gilhooly (Eds.), *Advances in the psychology of thinking* (pp. 1–44). Harvester-Wheatsheaf.

Poincaré, H. (1913). *The foundations of science: Science and hypothesis, the value of science, science and method*. The Science Press.

Pólya, G. (1990). *Mathematics and plausible reasoning.* Princeton University Press.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science*, *316*(5828), 1128–1129.

Sadler-Smith, E. (2015). Wallas' four-stage model of the creative process: More than meets the eye? *Creativity Research Journal*, *27*(4), 342–352.

Simonton, D. K. (1999). Creativity as blind variation and selective retention: Is the creative process Darwinian? *Psychological Inquiry*, *10*(4), 309–328.

Simonton, D. K. (2009). Scientific creativity as a combinatorial process: The chance baseline. In P. Meusburger, J. Funke, & E. Wunder (Eds.), *Milieus of creativity: An interdisciplinary approach to spatiality of creativity* (pp. 39–51). Springer Netherlands.

Simonton, D. K. (2012). Combinational creativity and sightedness: Monte carlo simulations using three-criterion definitions. *The International Journal of Creativity & Problem Solving*, *22*(2), 5–17.

Simonton, D. K. (2021). Scientific creativity: Discovery and invention as combinatorial. *Frontiers in Psychology*, *12*, 1–11.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The mathematics educator*, *14*(1), 19–34.

Stephen, D. G., Boncoddo, R. A., Magnuson, J. S., & Dixon, J. A. (2009). The dynamics of insight: Mathematical discovery as a phase transition. *Memory & Cognition*, *37*(8), 1132–1149.

Stephen, D. G., Dixon, J. A., & Isenhower, R. W. (2009). Dynamics of representational change: entropy, action, and cognition. *Journal of Experimental Psychology: Human Perception and Performance*, *35*(6), 1811–1832.

Sternberg, R. J., & Ben-Zeev, T. (1996). *The nature of mathematical thinking*. Routledge.

Thagard, P., & Stewart, T. C. (2011). The aha! experience: Creativity through emergent binding in neural networks. *Cognitive science*, *35*(1), 1–33.

Wallas, G. (1926). *The art of thought*. Brace and Co.

Wynne, J. (2021). *Do not erase: Mathematicians and their chalkboards*. Princeton University Press.