Title

Degrees of Freedom of 2-user and 3-user Rank-Deficient MIMO Interference Channels

Authors

Krishnamurthy, Sundar Rajan
Jafar, Syed Ali

Publication Date

2012-04-02

Degrees of Freedom of 2-user and 3-user Rank-Deficient MIMO Interference Channels

Sundar R. Krishnamurthy and Syed A. Jafar
Electrical Engineering and Computer Science
University of California Irvine

Abstract

We study the degrees of freedom (DoF) of 2-user and 3-user multiple input multiple output (MIMO) interference channels with rank deficient channel matrices. Only achievable DoF results and trivial outer bounds were previously available for these problems, restricted to symmetric settings. For the 2-user rank deficient MIMO interference channel we prove the optimality of previously known achievable DoF in the symmetric case and generalize the result to fully asymmetric settings. For the 3-user rank deficient MIMO interference channel, we improve the achievable DoF and provide a tight outer bound to establish optimality. Linear precoding based achievable schemes are found to be DoF optimal in both cases.

1 Introduction

Rank deficiency of channel matrices is an important aspect of MIMO wireless systems. Poor scattering and presence of single or very few direct paths are some reasons for rank deficiency in wireless channels. While the implications of rank deficient channel matrices are well understood for the single user point to point setting, much less is known for MIMO interference networks. In particular, the interplay between the number of signal dimensions (degrees of freedom) available through interference management schemes and channel rank-deficiencies is largely unexplored.

For full rank channels, the DoF of the 2-user MIMO interference channel are characterized in [2], and those of the 3-user MIMO interference channel are characterized in [1]. A study of the DoF of rank-deficient channels is initiated in [3] by Chae et al., who present an achievable scheme for the K user rank deficient MIMO channels. However, in the absence of outer bounds, the optimality of the achieved DoF is neither established, nor conjectured. Further, Chae et al. consider only the symmetric setting where all transmitters have M nodes, all receivers have N nodes and all channels are of rank D. In this paper, our focus is on *optimal* DoF results of 2-user and 3-user rank deficient channels with less restrictive symmetry assumptions.

For 2-user rank deficient channels, Chae et al. present an achievable scheme specifically for the symmetric (M, N, D) setting, which achieves $\min(2D, M + N - D)$ total DoF. In this paper, we show that this DoF is optimal using a genie-based outer bound and also present an achievable scheme and outer bound for the generic setting with arbitrary number of transmitter and receiver antennas and arbitrary channel ranks. For 3-user rank-deficient channels, our results show that the achievable DoF result of [3] is not optimal even for the symmetric (M, M, D) channel. While Chae et al. achieve DoF equal to $\min(D, \max(\frac{2M-D}{3}, \frac{DL}{L+1}))$ per user, where $L = \lfloor \frac{M}{D} \rfloor$, we present an improved achievable scheme and a tight information theoretic outer bound, establishing the

DoF value of $\min(D, \frac{M}{2})$ per user for same (M, M, D) channel. We also characterize the DoF of less symmetric settings where direct and cross channels have different ranks. Symbol or spatial extensions can be considered when the achievable DoF per user is not an integer.

Notation: When dealing with $H_{k(k+1)}$ and $H_{k(k-1)}$, indexing is interpreted in a circular wrap-around manner, modulo the number of users. We use the notation $o(x)$ to represent any function $f(x)$ such that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. $(x)^+$ indicates $\max(0, x)$.

2 Two User Interference Channel

Consider the $(M_1, N_1; D_{11}, D_{21})$, $(M_2, N_2; D_{22}, D_{12})$ rank-deficient MIMO interference channel where transmitter T_1 has a message for receiver R_1 only and transmitter T_2 has a message for receiver R_2 only. Rank of channel matrix H_{ji} is denoted by D_{ji}. This interference channel is characterized by the following input-output relations:

$$\begin{aligned} Y_1 &= H_{11}X_1 + H_{12}X_2 + W_1 \\ Y_2 &= H_{22}X_2 + H_{21}X_1 + W_2 \end{aligned}$$

where H_{11}, H_{22} are the direct channel matrices of size $N_1 \times M_1$ and $N_2 \times M_2$, respectively and H_{12}, H_{21} are the cross (interfering) channel matrices of size $N_1 \times M_2$ and $N_2 \times M_1$, respectively. $X_1; X_2$ are the M_1, M_2 dimensional input vectors, $Y_1; Y_2$ are the N_1, N_2 dimensional output vectors, and $W_1; W_2$ are the N_1, N_2 dimensional additive white gaussian noise (AWGN) vectors, respectively.

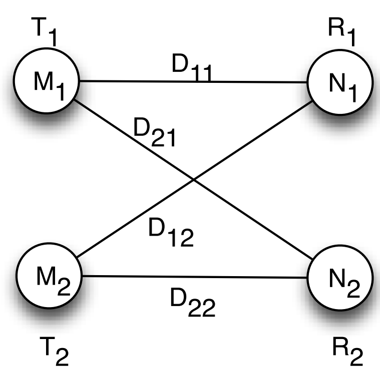

Figure 1: 2-user rank deficient interference channel

We assume that the channels are generic. A generic rank-deficient matrix of size $M \times M$ with rank D, can be seen without loss of generality, as a product of two full rank matrices of size $M \times D$ and $D \times M$. Coefficients of these two matrices are generic, e.g., chosen i.i.d. from a continuous distribution and their absolute values are bounded between a nonzero minimum value and a finite maximum value.

2.1 Achievability: Inner Bound on DoF

Lemma 1 *For the* $(M_1, N_1; D_{11}, D_{21})$*,* $(M_2, N_2; D_{22}, D_{12})$ *rank deficient interference channel, following total degrees of freedom are achievable.*

$$\eta_s(K2) \geq \min\{D_{11} + D_{22}, M_1 + N_2 - D_{21}, M_2 + N_1 - D_{12}\} \quad (1)$$

Proof: Since the proof is similar to that of the 2-user full rank interference channel [2], we do not repeat all the details. Fig 2 illustrates the proof setting with an example where $M_1 = 5$, $M_2 = 4$, $N_1 = 4$, $N_2 = 4$, $D_{11} = 3$, $D_{22} = 3$, $D_{12} = 2$ and $D_{21} = 4$, where a total of 5 DoF are achieved.

Step 1: We consider SVD of the interference channels $H_{12} = U_1 \Lambda_{12} V_1^{\dagger}$ and $H_{21} = U_2 \Lambda_{21} V_2^{\dagger}$. Λ_{12} and Λ_{21} are diagonal matrices with singular values of H_{12}, H_{21} respectively on the main diagonal and zeros elsewhere. Using the standard MIMO SVD diagonalization approach as in [2], we absorb the unitary matrices into the corresponding input and output vectors as:

$$\begin{aligned} Y'_1 &= H'_{11}X'_1 + \Lambda_{12}X'_2 + W'_1 \\ Y'_2 &= H'_{22}X'_2 + \Lambda_{21}X'_1 + W'_2 \end{aligned}$$

where $Y'_1 = U_1^{\dagger}Y_1$, $Y'_2 = U_2^{\dagger}Y_2$, $X'_1 = V_2^{\dagger}X_1$, $X'_2 = V_1^{\dagger}X_2$, $W'_1 = U_1^{\dagger}W_1$, $W'_2 = U_2^{\dagger}W_2$, $H'_{11} = U_1^{\dagger}H_{11}V_2$ and $H'_{22} = U_2^{\dagger}H_{22}V_1$. Since first D_{12} columns of Λ_{12} have nonzero values on the diagonal and other columns are zeros, only $X_1^{(2)'}, X_2^{(2)'}, ..., X_{D_{12}}^{(2)'}$ present interference from T_2 at R_1. Similarly only $X_1^{(2)'}, X_2^{(2)'}, ..., X_{D_{21}}^{(2)'}$ present interference from T_1 at R_2. Bold channels in Fig 2 represent interference links after diagonalization, and there are 2 parallel paths from T_2 to R_1 and 4 parallel paths from T_1 to R_2.

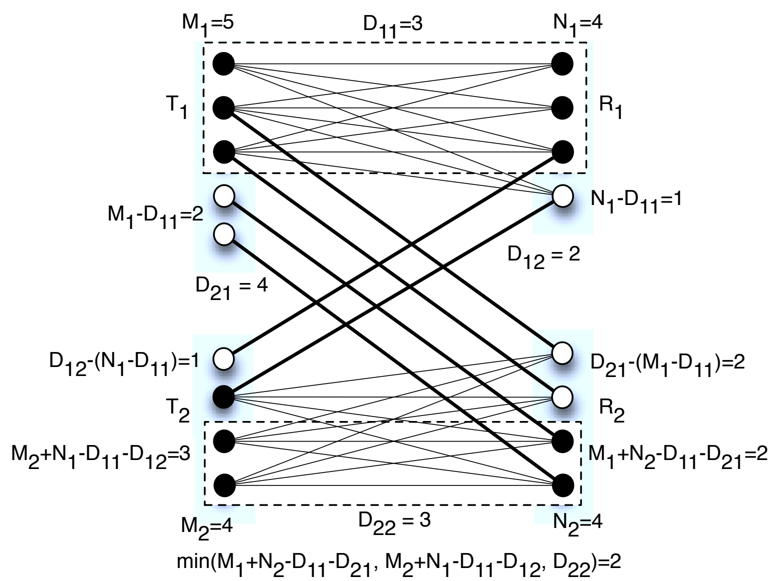

Figure 2: Achievability for 2-user Rank deficient channel

Step 2: At transmitter T_1, inputs $X_1^{(1)'}, X_2^{(1)'}, ..., X_{M_1-D_{11}}^{(1)'}$ are set to zero, i.e., we do not transmit on these inputs. This leaves D_{11} available inputs, $X^{(1)'}_{M_1-D_{11}+1}, ..., X^{(1)'}_{M_1}$ at T_1. In Fig 2, 2 transmit antennas have inputs set to zero (white circles) and remaining 3 dark circles indicate the available inputs at T_1.

Step 3: At receiver R_1, D_{11}=3 is the dimension of desired signal received from T_1. Hence we consider only outputs $Y^{(1)'}_1, Y^{(1)'}_2, ..., Y^{(1)'}_{D_{11}}$ and discard remaining outputs $Y^{(1)'}_{D_{11}+1}, ..., Y^{(1)'}_{N_1}$ marked in white circles. Receiver R_1 sees D_{12} dimensional interference from T_2, and since N_1-D_{11} outputs are already discarded at receiver, transmitter T_2 need to avoid transmitting in $(D_{12}-(N_1-D_{11}))^+$ inputs. In Fig 2, one output is discarded at receiver R_1, hence transmitter T_2 does not transmit on the remaining 1 dimension that could contribute to interference. After discarding some inputs, T_2 transmits its message using $M_2 - (D_{11} + D_{12} - N_1)^+$ inputs.

Step 4: Discarding $(D_{12} - (N_1 - D_{11}))^+$ inputs at T_2 ensures that at receiver R_1, interference is eliminated and it can decode the message from transmitter T_1 to achieve D_{11} DoF.

Step 5: Receiver R_2 receives interference from transmitter T_1 over channel of rank D_{21}. In step 2, M_1-D_{11} inputs have been set to zero, hence remaining $(D_{21}-(M_1-D_{11}))^+$ inputs cause interference at R_2. In order to eliminate interference from T_1, receiver R_2 discards $(D_{21}-(M_1-D_{11}))^+$ outputs. Therefore, R_2 receives signal from T_2 only on its $N_2 - (D_{11} + D_{21} - M_1)^+$ remaining outputs. In Fig. 2, transmitter T_1 sets 2 of its inputs to zero, and receiver R_2 discards remaining 2 outputs. R_2 decodes its signal using remaining 2 outputs.

Step 6: From step 3, we have $M_2-(D_{11}+D_{12}-N_1)^+$ inputs available at T_2 so that no interference is caused at R_1. From step 5, we have $N_2 - (D_{11} + D_{21} - M_1)^+$ outputs available at R_2 that are interference-free. Channel between T_2 and R_2 is of rank D_{22}. Hence communication between T_2 and R_2 takes place with DoF of $\min(M_2 - (D_{11} + D_{12} - N_1)^+, N_2 - (D_{11} + D_{21} - M_1)^+, D_{22})$.

Combining Steps 4 and 6, we have established achievability of $D_{11} + \min(M_2 - (D_{11} + D_{12} - N_1)^+, N_2-(D_{11}+D_{21}-M_1)^+, D_{22})$ total DoF for 2-user channel. This expression can be evaluated to be equal to $\min\{D_{11} + D_{22}, M_1 + N_2 - D_{21}, M_2 + N_1 - D_{12}\}$. Setting inputs or outputs to zero is equivalent to perfoming zero-forcing at transmitter or receiver.

2.2 Converse: Outer Bound on DoF

For the $(M_1, N_1; D_{11}, D_{21})$, $(M_2, N_2; D_{22}, D_{12})$ 2-user rank-deficient MIMO interference channel, the following is the outer bound on total degrees of freedom.

Lemma 2

$$\eta_s(K2) \leq \min\{D_{11} + D_{22}, M_1 + N_2 - D_{21}, M_2 + N_1 - D_{12}\}$$

Proof: Trivial outer bound on total DoF of $D_{11}+D_{22}$ is known for this channel. Following converse proof is similar to that of full rank channels (refer Theorem 1 in [2]), and so, we only present a proof sketch for rank-deficient channels.

For sum capacity of this channel to be bounded above by 2 constituent MAC channels, each receiver must be able to decode messages from both transmitters. For this, receiver must have access to the full interference signal space, i.e., it does not get zero-forced at the transmitters. Noise can then be reduced at a receiver, say R_1, if needed, so that it sees a better channel than receiver R_2, and message intended for receiver R_2 becomes decodable at receiver R_1.

In the 2-user rank-deficient MIMO interference channel, receiver R_1 can access only a D_{12} dimensional signal space of transmitter T_2 in its M_2 dimensional space. This implies, T_2 can zero-force part of its signal to R_1 and R_1 cannot decode message from T_2 by reducing noise. Hence only through additional antennas at R_1 can it access full signal space of T_2. Additional receiver antennas cannot hurt, so the converse argument is not violated. To this end, we add $M_2 - D_{12}$ antennas at R_1. Since channel coefficients corresponding to new antennas are drawn i.i.d. from a continuous distribution, interference channel between T_2 and R_1, now a matrix of size $(N_1 + M_2 - D_{12}) \times M_2$, will be full rank. Noise at R_1 can be reduced to decode message from T_2. Similarly, additional antennas are added at receiver R_2, so that it can access full signal space of transmitter T_1. Interference channel between T_1 and R_2, a matrix of size $(N_2 + M_1 - D_{21}) \times M_1$, is full rank. Noise at R_2 can be reduced to decode message from T_1.

Now, we argue that the sum capacity is bounded above by corresponding MAC channels $(M_1, M_2, N_1 + M_2 - D_{12})$ and $(M_1, M_2, N_2 + M_1 - D_{21})$ with modified additive noise. Since $(N_2 + M_1 - D_{21}) \geq M_1$ and $(N_1 + M_2 - D_{12}) \geq M_2$, it can be seen that Theorem 1 in [2] holds true for above argument with N_1 modified as $N_1 + M_2 - D_{12}$ and N_2 modified as $N_2 + M_1 - D_{21}$. R_1 can decode its message and subtract from its received signal vector, and we assume a genie provides X_1 to R_2, so that R_2 can subtract out interference from T_1. While initial output vectors Y_1 and Y_2 are of size $(N_1 + M_2 - D_{12}) \times 1$ and $(N_2 + M_1 - D_{21}) \times 1$ respectively, after noise reduction and SVD operations, output vectors Y_{1new} and Y_{2new} are both of size $M_2 \times 1$. With these changes, R_1 and R_2 would be able to decode both messages. Hence, total DoF is upper-bounded as $\eta_s(K2) \leq \min(D_{11} + D_{22}, N_2 + M_1 - D_{21})$ and $\eta_s(K2) \leq \min(D_{11} + D_{22}, N_1 + M_2 - D_{12})$. This is because DoF expressions of 2 rank-deficient MAC channels would have sum of channel ranks instead of that of number of transmit antennas. Combining these 2 bounds, we get the converse result of Lemma 2.

Theorem 1 *For $(M_1, N_1; D_{11}, D_{21})$, $(M_2, N_2; D_{22}, D_{12})$ 2-user rank deficient interference channel, total DoF is*

$$\eta_s(K2) = \min\{D_{11} + D_{22}, M_1 + N_2 - D_{21}, M_2 + N_1 - D_{12}\}$$

Proof of Theorem 1 follows from Lemma 1 and 2.

Reciprocity holds true for rank deficient channels similar to full rank channels, i.e., DoF is unaffected if M_1 and M_2 are switched with N_1 and N_2 respectively.

For the symmetric special case, i.e., the (M, N, D) MIMO interference channel where each transmitter has M antennas, each receiver has N antennas and all channel matrices are of rank D, optimal DoF can be calculated as $\eta_s(K2) = \min(M + N - D, 2D)$. This is same as the achievable DoF value established by Chae et al. [3], now proved to be optimal.

3 Three User Interference Channel

Consider the 3-user rank-deficient MIMO interference channel, as in Fig 3, wherein all direct channel matrices H_{kk} are of rank D_0, cross channel matrices $H_{k(k+1)}$ are of rank D_1 and cross channel matrices $H_{k(k-1)}$ are of rank D_2. In this section, we use nullspace to refer to the right nullspace unless otherwise explicitly mentioned.

3.1 Achievability: Inner Bound on DoF

Lemma 3 *For the 3-user rank-deficient MIMO interference channel, following degrees of freedom are achievable per user.*

$$\eta(K3) \geq \min\{D_0, M - \tfrac{\min(M, D_1 + D_2)}{2}\} \quad (2)$$

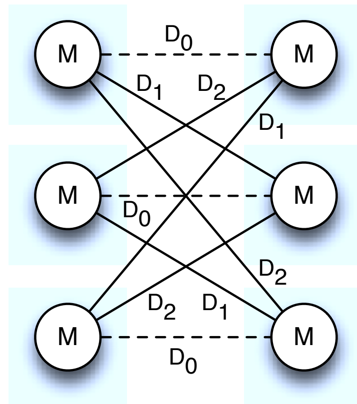

Figure 3: Three user rank-deficient MIMO interference channel

Proof: Achievability proof for 3-user rank deficient interference channel is first presented for cases where direct channels are full rank. Later, achievability with rank deficient direct channels is discussed. We categorize beamforming vectors used at each transmitter $k = 1, 2, 3$, to 4 types:

$\mathbf{V_k^{Za}}$ - Zero-forcing vectors in nullspace of $H_{(k-1)k}$, maximum number of vectors chosen can be $M - D_1$. Vectors used at transmitter k will not cause interference at receiver $k - 1$.

$\mathbf{V_k^{Zb}}$ - Zero-forcing vectors in nullspace of $H_{(k+1)k}$, maximum number of vectors chosen can be $M - D_2$. Vectors used at transmitter k will not cause interference at receiver $k + 1$.

$\mathbf{V_k^{Zc}}$ - Zero-forcing vectors in common nullspace of $H_{(k-1)k}$ and $H_{(k+1)k}$ (overlapping dimensions in 2 nullspaces). Maximum number of vectors chosen can be $M - D_1 - D_2$ since $M - D_1$ and $M - D_2$ dimensional generic nullspaces overlap in a $M - D_1 - D_2$ dimensional space at each transmitter. Vectors chosen in these overlapping dimensions do not cause interference at either of the 2 unintended receivers.

$\mathbf{V_k^{A}}$ - Alignment vectors that align signal at a receiver in the span of interference from other unintended transmitter. Maximum number of vectors chosen can be $D_1 + D_2 - M$ since D_1 and D_2 dimensional generic interference subspaces overlap in $D_1 + D_2 - M$ dimensional space at each receiver.

Different cardinalities are chosen for these 4 types of beamforming vectors to form the transmit beamforming matrix. The beamforming matrix at each transmitter is then of the form $V_k = [V_k^{Za}\ V_k^{Zb}\ V_k^{Zc}\ V_k^{A}]$. We now discuss achievability by analyzing the beamforming vector cardinalities listed in Table I and by using linear dimension counting arguments.

Table I: Achievable DoF in 3-user channel for different D_1, D_2 with $D_0 = M$

Case	$\mathbf{D_1 + D_2}$	$\lvert\mathbf{V_k^{Za}}\rvert + \lvert\mathbf{V_k^{Zb}}\rvert$	$\lvert\mathbf{V_k^{Zc}}\rvert$	$\lvert\mathbf{V_k^{A}}\rvert$	dim(**Int**)	dim(**Des**)	**Total**
1	$0 < D_1 + D_2 \le M$	$\frac{D_1+D_2}{2}$	$M-(D_1+D_2)$	0	$\frac{D_1+D_2}{2}$	$M-\frac{D_1+D_2}{2}$	M
2	$M < D_1 + D_2 \le \frac{3M}{2}$	$\frac{M}{2}$	0	0	$\frac{M}{2}$	$\frac{M}{2}$	M
3	$\frac{3M}{2} < D_1+D_2 \le 2M$	$2M-D_1-D_2$	0	$D_1+D_2-\frac{3M}{2}$	$\frac{M}{2}$	$\frac{M}{2}$	M

Using Table I, we first analyze the setting in which direct channels are full rank and cross channels are rank deficient. First 2 cases correspond to zero-forcing based achievability schemes, and last case involves interference alignment. For convenience, only sum cardinality of the chosen zero-forcing vectors V_k^{Za} and V_k^{Zb} is specified, i.e., $|V_k^{Za}| + |V_k^{Zb}|$. This is because each of these vectors chosen at a transmitter helps in cancelling interference at one receiver but causes interference at another receiver. Since we have 2 unintended transmitters causing interference, these zero-forcing vectors can be treated in same manner. dim(Desired) and dim(Interference) are the number of desired and interference signal dimensions seen at each receiver respectively. Then we have,

$$\begin{aligned} dim(\text{Desired}) &= |V_k^{Za}| + |V_k^{Zb}| + |V_k^{Zc}| + |V_k^{A}| \\ dim(\text{Interference}) &= |V_k^{Za}| + |V_k^{Zb}| + |V_k^{A}| \end{aligned}$$

While the first relation is trivial, the second one can be explained as follows: V_k^{Zc} at transmitter k do not cause interference at both unintended receivers. Therefore dim(Interference) does not contain that term. Further, both zero-forcing (using non-overlapping nullspace) and interference alignment are similar in the sense that, vector chosen for zero-forcing one receiver causes interference at other receiver, and vector chosen for aligning interference at one receiver causes interference at another. Hence at each receiver, dim(Interference) is the sum of the number of zero-forcing vectors (using non-overlapping nullspace) and the number of Interference alignment vectors.

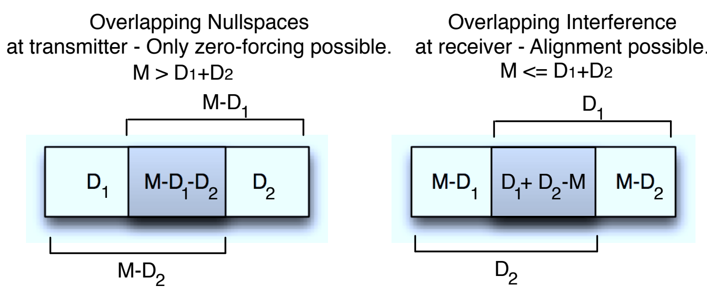

Figure 4: M-dimensional signal space in 3-user channel

For the first case of Table I, $|V_k^A| = 0$ since interference alignment is not possible ($D_1 + D_2 \leq M$). $|V_k^{Zc}|$ is chosen to be the maximum possible overlapping nullspace dimensions. Remaining vectors are chosen from the non-overlapping nullspace and chosen number of vectors $|V_k^{Za}| + |V_k^{Zb}| < D_1 + D_2$, maximum number of non-overlapping nullspace dimensions. At each receiver, interference occupies $|V_k^{Za}| + |V_k^{Zb}|$ dimensions.

For the second and third cases, $|V_k^{Zc}| = 0$ since there are no overlapping nullspace dimensions at the transmitters ($D_1 + D_2 > M$). For case 2, though alignment is possible, beamforming matrix can be formed with the zero-forcing vectors only, i.e., $|V_k^{Za}| + |V_k^{Zb}|$ can be chosen as $\frac{M}{2}$. This is because $\frac{M}{2} \leq 2M - D_1 - D_2$, dimensions in the nullspaces of $H_{(k-1)k}$ and $H_{(k+1)k}$.

Case 3 involves both zero forcing and interference alignment. At transmitter $k \in \{1, 2, 3\}$, $M - D_1$ symbols are sent along the $M - D_1$ dimensional null space of the channel to receiver $k - 1$ and $M - D_2$ symbols are sent along the $M - D_2$ dimensional null space of the channel to receiver $k + 1$. This is performed by choosing columns of a full rank linear transformation T_k to be beamforming vectors V_k^{Za} of size $M - D_1$ and V_k^{Zb} of size $M - D_2$.

$$H_{(k-1)k} V_k^{Za} = 0, \quad H_{(k+1)k} V_k^{Zb} = 0 \qquad k \in \{1, 2, 3\}$$

$$T_k = \begin{bmatrix} & 0 & \\ V_k^{Za} & I_{D_1+D_2-M} & V_k^{Zb} \\ & 0 & \end{bmatrix} \quad k \in \{1,2,3\}$$

The remaining $D_1 + D_2 - M$ dimensional space at the transmitter will be used to send the remaining $M/2-(M-D_1)-(M-D_2) = D_1+D_2-3M/2$ symbols that participate in interference alignment. To this end, random entries could be chosen for $M \times (D_1+D_2-M)$ submatrix of T_k. We choose square identity matrix of dimension D_1+D_2-M with $M-D_1$ rows of zeros above and $M-D_2$ rows of zeros below.

Receiver k sees $M-D_1$ dimensional interference from transmitter $k-1$ and $M-D_2$ dimensional interference from transmitter $k+1$. These $(M-D_1)+(M-D_2)$ interference symbols are zero-forced by projecting the M dimensional received space into the $M-(M-D_1)-(M-D_2)$ dimensional space that is orthogonal to the interference symbols. This is performed using a full rank linear transformation R_k of size $(D_1+D_2-M) \times M$ at receiver k.

$$R_k[H_{k(k-1)}V_{k-1}^{Za} \quad H_{k(k+1)}V_{k+1}^{Zb}] = 0, \qquad k \in \{1,2,3\}$$

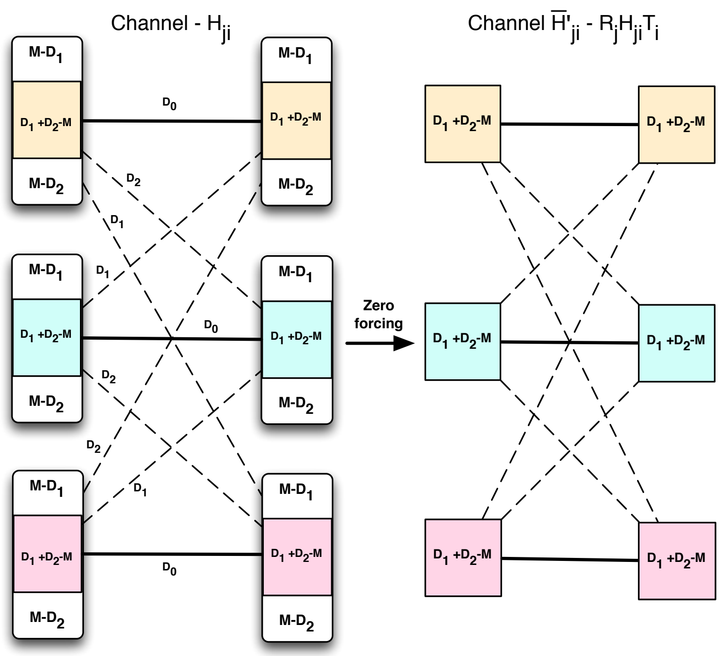

Figure 5: Alignment in 3-user interference channel

With this, residual interference at receiver k due to zero-forcing beamforming vectors chosen at all transmitters would be zero-forced at the receiver. For the remaining symbols, i.e., for the remaining interference alignment problem, the zero forcing operations at the transmitters and receivers, described thus far leave us with a 3 user MIMO interference channel with D_1+D_2-M input dimensions at each transmitter and D_1+D_2-M dimensions at each receiver, with below channel matrices. This is illustrated in Fig 5.

$$\bar{H}_{kj} = R_k H_{kj} T_j$$

We have constructed $\bar{H}'_{ji}$ by considering D_1+D_2-M columns of matrix $\bar{H}_{ji}$ after excluding first $M-D_1$ and last $M-D_2$ columns. Since D_1+D_2-M is not larger than D_1, D_2, these channels are full rank, generic channels over which the eigenvectors-based interference alignment solution of [4] can be directly applied to send the remaining $D_1+D_2-3M/2$ symbols (Note that 2 channel uses are needed for the aligned symbols if M is an odd number, each corresponding to a new set of zero-forcing symbols). Thus, the effective receiver sees a D_1+D_2-M dimensional generic space within which $D_1+D_2-3M/2$ aligned interference dimensions and $(M-D_1)+(M-D_2)+(D_1+D_2-3M/2)$ desired dimensions are resolved.

The beamforming matrix constructed $\bar{V}_k$ would have $(M-D_1)+(M-D_2)$ columns from the identity matrix, shown on left and right ends in example below. Remaining columns of $\bar{V}_k$ would be eigen-vector based solution of dimension D_1+D_2-M and rows of zeros above and below. Suppose $M=6$ and $D_1=D_2=5$, $\bar{V}_k$ constructed with 2 zero-forcing vectors and 1 alignment vector would be of following form

$$\bar{V}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_{k1}^a & 0 \\ 0 & v_{k2}^a & 0 \\ 0 & v_{k3}^a & 0 \\ 0 & v_{k4}^a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

wherein $\bar{V}_k^A = [v_{k1}^a \; v_{k2}^a \; v_{k3}^a \; v_{k4}^a]^T$ is the interference alignment vector constructed as in [4], which is then extended with $M-D_1$ rows of zeros above and $M-D_2$ rows of zeros below to form V_k^A. The resultant beamforming matrix V_k used at transmitter k is then

$$V_k = T_k \bar{V}_k = [V_k^{Za} \quad V_k^A \quad V_k^{Zb}]$$

Linear transformations at all transmitters and receivers T_k, R_k are full rank matrices based on construction described. It can be noted that matrices $\bar{V}_k$ and V_k are full rank since columns are linearly independent due to orthogonal construction of $\bar{V}_k$. Note that desired channels are not used in the design of precoding vectors, which maintains their generic character and thereby the linear independence of desired signal vectors from the interference. We also note that a similar layered precoding approach is presented in [5] as well.

When direct channels are rank deficient, no more than D_0 vectors can be used for beamforming. For all values of D_0 such that $d_M \le D_0 < M$, same DoF can be obtained as in Table I by choosing specified number of beamforming vectors. When $D_0 < d_M$, we send only D_0 beamforming vectors corresponding to all 3 cases, choosing first the zero-forcing vectors and then the alignment vectors as needed. In all cases, dim(Interference)+dim(Desired)$\le M$ since both desired and interference dimensions reduce with these changes.

Combining DoF results, achievability of $\min(D_0, M - \frac{\min(M, D_1+D_2)}{2})$ DoF per user has been proved.

3.2 Converse: Outer Bound on DoF

For the 3-user rank deficient interference channel, following is the outer bound on the degrees of freedom per user.

Lemma 4

$$\eta(K3) \leq \min\{D_0, M - \frac{\min(M, D_1 + D_2)}{2}\} \tag{3}$$

Proof: Proofs are described separately for two cases: $D_1 + D_2 > M$ and $D_1 + D_2 \leq M$

4a: Outer bound when $D_1 + D_2 > M$:

Change of Basis:

Step 1: For each receiver, a linear transformation R_k is designed such that the first $M - D_2$ antennas of receiver k do not hear transmitter $k-1$ (left nullspace of $H_{k(k-1)}$) and the last $M - D_1$ antennas of receiver k do not hear transmitter $k+1$ (left nullspace of $H_{k(k+1)}$). This is possible since rank($H_{k(k+1)}$)=D_1 and rank($H_{k(k-1)}$)=D_2.

Step 2: In M-dimensional space at transmitter k, there is a D_1-dimensional subspace orthogonal to $M - D_1$ receiver antennas $(k-1)a$ and D_2-dimensional subspace orthogonal to $M - D_2$ receiver antennas $(k+1)c$. These two subspaces overlap in $I = D_1 + D_2 - M$ dimensions within the M-dimensional space seen by the transmitter, and these I columns are chosen for matrix T_k at the transmitter. Other columns of T_k are chosen such that the first $M - D_2$ antennas of transmitter k are not heard by receiver $k+1$ (right nullspace of $H_{k(k-1)}$) and the last $M - D_1$ antennas of transmitter k are not heard by receiver $k-1$ (right nullspace of $H_{k(k+1)}$)

Step 3: Remaining $D_1 + D_2 - M$ rows for receiver R_k are chosen so that they are linearly independent of other rows. Resulting network connectivity is shown in Fig 6.

$\lvert X_{1a}\rvert = M - D_2$	$\circ$	$\circ$	$S_{1a}(X_{2a})$	$\lvert S_{1a}\rvert = M - D_2$
$\lvert X_{1b}\rvert = D_1 + D_2 - M > 0$	$\circ$	$\circ$	$S_{1b}(X_{2a}, X_{2b}, X_{3b}, X_{3c})$	$\lvert S_{1b}\rvert = D_1 + D_2 - M > 0$
$\lvert X_{1c}\rvert = M - D_1$	$\circ$	$\circ$	$S_{1c}(X_{3c})$	$\lvert S_{1c}\rvert = M - D_1$
$\lvert X_{2a}\rvert = M - D_2$	$\circ$	$\circ$	$S_{2a}(X_{3a})$	$\lvert S_{2a}\rvert = M - D_2$
$\lvert X_{2b}\rvert = D_1 + D_2 - M > 0$	$\circ$	$\circ$	$S_{2b}(X_{3a}, X_{3b}, X_{1b}, X_{1c})$	$\lvert S_{2b}\rvert = D_1 + D_2 - M > 0$
$\lvert X_{2c}\rvert = M - D_1$	$\circ$	$\circ$	$S_{2c}(X_{1c})$	$\lvert S_{2c}\rvert = M - D_1$
$\lvert X_{3a}\rvert = M - D_2$	$\circ$	$\circ$	$S_{3a}(X_{1a})$	$\lvert S_{3a}\rvert = M - D_2$
$\lvert X_{3b}\rvert = D_1 + D_2 - M > 0$	$\circ$	$\circ$	$S_{3b}(X_{1a}, X_{1b}, X_{2b}, X_{2c})$	$\lvert S_{3b}\rvert = D_1 + D_2 - M > 0$
$\lvert X_{3c}\rvert = M - D_1$	$\circ$	$\circ$	$S_{3c}(X_{2c})$	$\lvert S_{3c}\rvert = M - D_1$

Figure 6: Basis change for 3-user channel: $D_1 + D_2 > M$

Outer bound proof:

Desired signal is assumed to be decodable and can be removed. Genie information to be given to receiver 1 should include $2M - (D_1 + D_2)$ dimensions - X_{2c}^n, X_{3a}^n which are not heard by receiver 1. Receiver 1 has M equations with $D_1 + D_2$ unknowns. Hence only if genie information includes another $D_1 + D_2 - M$ dimensions, then at receiver 1, there will be M equations resolvable using M unknowns.

Hence a genie provides $\mathcal{G}_1 = \{X_{2b}^n, X_{2c}^n, X_{3a}^n\}$ to receiver 1. Number of dimensions available to receiver 1 is $M + |\mathcal{G}_1| = 2M$. With $2M$ dimensions, receiver 1 will be able to resolve both interfering signals and can decode all three messages.

$$\begin{aligned} nR_\Sigma &\leq Mn\log\rho + h(X_{2b}^n, X_{2c}^n, X_{3a}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) \quad (4)\\ &\leq Mn\log\rho + h(X_{3a}^n|\bar{Y}_1^n) + h(X_{2b}^n|\bar{Y}_1^n) + h(X_{2c}^n|\bar{Y}_1^n, X_{2b}^n, X_{3a}^n) + n\ o(\log\rho) + o(n) \quad (5)\\ &\leq Mn\log\rho + h(X_{3a}^n) + h(X_{2b}^n|X_{2a}^n) + h(X_{2c}^n|X_{2a}^n, X_{2b}^n) + n\ o(\log\rho) + o(n) \quad (6)\\ &= Mn\log\rho + h(X_{3a}^n) + nR_2 - h(X_{2a}^n) + n\ o(\log\rho) + o(n) \quad (7) \end{aligned}$$

where (4) follows from Fano's inequality and Lemma 3 in [1]. (5) follows from applying the chain rule. (6) follows since dropping condition terms cannot decrease differential entropy. Thus, we only keep S_{1a}^n as the condition term which is X_{2a}^n. (7) is obtained because from the observations of $(X_{2a}^n, X_{2b}^n, X_{2c}^n)$ we can decode W_2 subject to the noise distortion. By advancing user indices, we have:

$$3nR \leq Mn\log\rho + nR + n\ o(\log\rho) + o(n) \quad (8)$$

which implies that $d \leq \frac{M}{2}$. Since D_0 is a known outer bound, we get $\eta(K3) \leq \min(D_0, \frac{M}{2})$.

4b: Outer bound when $D_1 + D_2 \leq M$:

Change of Basis:

Step 1: For each receiver, a linear transformation R_k is designed such that the first D_1 antennas of Receiver k do not hear transmitter $k-1$ (left nullspace of $H_{k(k-1)}$) and the last D_2 antennas of Receiver k do not hear transmitter $k+1$ (left nullspace of $H_{k(k+1)}$). This is possible since rank($H_{k(k+1)}$)=D_1 and rank($H_{k(k-1)}$)=D_2.

Step 2: In M-dimensional space at transmitter k, there is a $M - D_1$ dimensional subspace orthogonal to D_1 receiver antennas $(k-1)a$ and another $M - D_2$ dimensional subspace orthogonal to D_2 receiver antennas $(k+1)c$. These two subspaces have $I = M - (D_1 + D_2)$ dimensional intersection at the transmitter, wherein I columns are chosen for matrix T_k. Then, we choose other columns of T_k such that D_1 antennas of transmitter k are not heard by receiver $k+1$ (right nullspace of $H_{k(k-1)}$) and D_2 antennas of transmitter k are not heard by receiver $k-1$ (right nullspace of $H_{k(k+1)}$)

Step 3: We consider only $D_1 + D_2$ antennas at each receiver, remaining antennas are discarded since no signal is received. Resulting network connectivity is shown in Fig 7.

Outer bound proof:

Desired signal is assumed to be decodable and can be removed. Genie information to be given to receiver 1 should include $2M - (D_1 + D_2)$ dimensions - $X_{2b}^n, X_{2c}^n, X_{3a}^n, X_{3b}^n$ which are not heard by receiver 1. Receiver 1 has M equations with $D_1 + D_2$ unknowns. Since $D_1 + D_2 < M$, choosing signal from only $D_1 + D_2$ antennas would result in $D_1 + D_2$ equations becoming resolvable.

Hence a genie provides $\mathcal{G}_1 = \{X_{2b}^n, X_{2c}^n, X_{3a}^n, X_{3b}^n\}$ to receiver 1. Since receiver 1 considers only $D_1 + D_2$ antennas, number of dimensions available to receiver 1 is $D_1 + D_2 + |\mathcal{G}_1| = 2M$. With $2M$ dimensions, receiver 1 will be able to resolve both interfering signals and can decode all three

$\lvert X_{1a}\rvert = D_1$	$\circ$		$\circ$	$S_{1a}(X_{2a})$	$\lvert S_{1a}\rvert = D_1$
$\lvert X_{1b}\rvert = M - (D_1 + D_2) \geq 0$	$\circ$		$\circ$	$S_{1b}()$	$\lvert S_{1b}\rvert = M - (D_1 + D_2) \geq 0$
$\lvert X_{1c}\rvert = D_2$	$\circ$		$\circ$	$S_{1c}(X_{3c})$	$\lvert S_{1c}\rvert = D_2$

$\lvert X_{2a}\rvert = D_1$	$\circ$		$\circ$	$S_{2a}(X_{3a})$	$\lvert S_{2a}\rvert = D_1$
$\lvert X_{2b}\rvert = M - (D_1 + D_2) \geq 0$	$\circ$		$\circ$	$S_{2b}()$	$\lvert S_{2b}\rvert = M - (D_1 + D_2) \geq 0$
$\lvert X_{2c}\rvert = D_2$	$\circ$		$\circ$	$S_{2c}(X_{1c})$	$\lvert S_{2c}\rvert = D_2$

$\lvert X_{3a}\rvert = D_1$	$\circ$		$\circ$	$S_{3a}(X_{1a})$	$\lvert S_{3a}\rvert = D_1$
$\lvert X_{3b}\rvert = M - (D_1 + D_2) \geq 0$	$\circ$		$\circ$	$S_{3b}()$	$\lvert S_{3b}\rvert = M - (D_1 + D_2) \geq 0$
$\lvert X_{3c}\rvert = D_2$	$\circ$		$\circ$	$S_{3c}(X_{2c})$	$\lvert S_{3c}\rvert = D_2$

Figure 7: Basis change for 3-user channel: $D_1 + D_2 \leq M$

messages.

$$\begin{aligned}
nR_\Sigma &\leq Mn\log\rho + h(X_{2b}^n, X_{2c}^n, X_{3a}^n, X_{3b}^n|\bar{Y}_1^n) + no(\log\rho) + o(n) &(9)\\
&\leq Mn\log\rho + h(X_{3a}^n|\bar{Y}_1^n) + h(X_{3b}^n|\bar{Y}_1^n) + h(X_{2b}^n, X_{2c}^n|\bar{Y}_1^n, X_{3a}^n, X_{3b}^n) + no(\log\rho) + o(n) &(10)\\
&\leq Mn\log\rho + h(X_{3a}^n) + h(X_{3b}^n) + h(X_{2b}^n, X_{2c}^n|X_{2a}^n) + no(\log\rho) + o(n) &(11)\\
&= Mn\log\rho + h(X_{3a}^n) + h(X_{3b}^n) + nR_2 - h(X_{2a}^n) + no(\log\rho) + o(n) &(12)\\
&\leq Mn\log\rho + h(X_{3a}^n) + (M - (D_1 + D_2))n\log\rho + nR_2 - h(X_{2a}^n) + no(\log\rho) + o(n) &(13)
\end{aligned}$$

where (9) follows from Fano's inequality and Lemma 3 in [1]. (10) follows from applying the chain rule. (11) follows since dropping condition terms cannot decrease differential entropy. Thus, we only keep S_{1a}^n as the condition term which is X_{2a}^n. (12) is obtained because from the observations of $(X_{2a}^n, X_{2b}^n, X_{2c}^n)$ we can decode W_2 subject to the noise distortion, (13) follows since the entropy of X_{3b}^n is constrained by $M - (D_1 + D_2)$ antennas. By advancing user indices:

$$3nR \leq (2M - (D_1 + D_2))n\log\rho + nR + no(\log\rho) + o(n)$$

which implies that $d \leq \frac{2M-(D_1+D_2)}{2}$. Since D_0 is known outer bound, we get $\eta(K3) \leq \min(D_0, M - \frac{D_1+D_2}{2})$. Result of Lemma 4 follows from converse results of cases 4a and 4b.

Theorem 2 *For the 3-user rank deficient interference channel considered, optimal DoF value per user is*

$$\eta(K3) = \min\{D_0, M - \frac{\min(M, D_1 + D_2)}{2}\} \tag{14}$$

Proof follows from Lemma 3 and 4.

In optimal DoF expressions of both 2-user and 3-user channels, direct channel rank and cross channel rank appear in separate terms in above DoF expression. Intuitively, this is because rank deficiency of direct channels only limits the ability to fill the interference-free space while that of cross channels impact the extent to which interference cancellation or alignment can be performed.

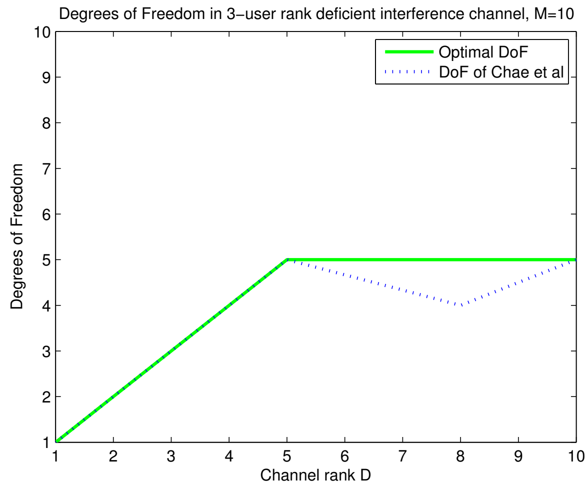

Figure 8: DoF Comparison

Also, from Theorem 2:

When all direct channels are full rank M, and all cross channels are of rank D, optimal DoF is $\max(\frac{M}{2}, M - D)$

When all direct channels are of rank D, and all cross channels are of rank D (or M), optimal DoF is $\min(D, \frac{M}{2})$ which is better (when $D > \frac{M}{2}$, as shown in Fig 8) than (M,M,D) result of Chae et al. in [3], i.e., $\min(D, \max(\frac{2M-D}{3}, \frac{DL}{L+1}))$ where $L = \lfloor \frac{M}{D} \rfloor$.

4 Conclusions

Optimal degrees of freedom results are presented for 2- and 3-user rank deficient interference channels with different channel ranks. For three-user interference channel, achievability was shown using Interference Alignment based on linear beamforming and zero-forcing. Information theoretic outer bound proof was described proving that achievable DoF is also tight. Impact of direct and cross channel rank deficiency were investigated. These results would be helpful in finding optimal DoF results for K-user rank deficient interference channels, which are being studied.

5 Acknowledgments

This work was supported in part by ONR N00014-12-10067, NSF CCF-1161418 and by a gift from Broadcom.

References

[1] C. Wang, T. Gou, S. Jafar, “Subspace Alignment Chains and the Degrees of Freedom of the Three-User MIMO Interference Channel”, arXiv:1109.4350.

[2] S. Jafar, M.J. Fakhereddin, "Degrees of Freedom for the MIMO Interference Channel", IEEE Transactions on Information Theory, July 2007, vol. 53, no.7, Pages: 2637-2642

[3] S.H. Chae; S.Y. Chung, "On the degrees of freedom of rank deficient interference channels", IEEE ISIT 2011

[4] V. Cadambe; S. Jafar, "Interference Alignment and Degrees of Freedom of the K-User Interference Channel", IEEE Transactions on Information Theory, vol.54, no.8, pp.3425-3441, Aug. 2008

[5] X. Xu; Y. Zeng; Y.L. Guan, "On the degrees of freedom of the 3-User Rank-Deficient MIMO Interference channels", submitted to IEEE International Conference on Communications, ICC 2013

[6] B. Nosrat-Makouei; J.G. Andrews; R.W. Heath, "MIMO Interference Alignment Over Correlated Channels With Imperfect CSI", IEEE Transactions on Signal Processing, vol.59, no.6, pp.2783-2794, June 2011

[7] T. Gou; S. Jafar, "Degrees of Freedom of the K User $M \times N$ MIMO Interference Channel", IEEE Transactions on Information Theory, vol.56, no.12, pp.6040-6057, Dec. 2010