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Green's Functions on Self-Similar Sets

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ABSTRACT OF THE DISSERTATION

Green's Functions on Self-Similar Sets

by

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University of California, Riverside, June 2019
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In the area of fractal analysis, many details are known about the analytic structure of certain post-critically-finite (p.c.f.) self-similar structures such as the Sierpinski gasket. These include details about its Laplacian, Green's function, and solutions to differential equations. While general techniques have been proposed, many examples have yet been worked out, such as the Hata tree. Here, we work out and discuss said analytic structure for these examples. While the technical details are significantly more advanced, several fascinating patterns appear, some of which are of a completely different nature than the analytic structure of the Sierpinski gasket. We will use this structure to determine its respective Green's function, which are critical to the study of differential equations.

Contents

List of Figures

Introduction

Let $n \in \mathbb{N}$. Given a set of contractions, $\{f_i : \mathbb{R}^n \to \mathbb{R}^n\}_{i=1}^k$, we can define what's called an *iterated function system.* That is, we can define a map on nonempty compact subspaces of $\mathbb{R}^n$, by the formula

$$f(A) = \bigcup_{i=1}^{k} f_i(A).$$

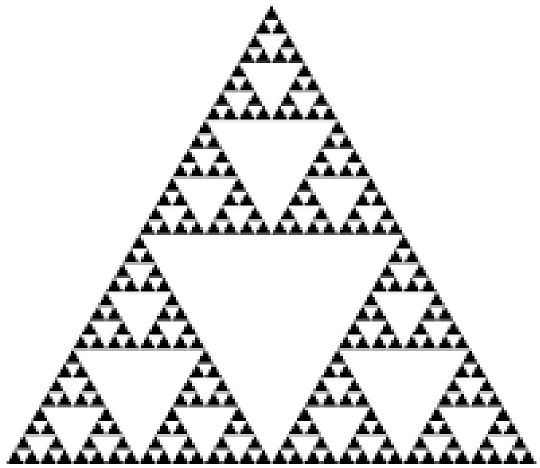
We can show that, in an appropriate metric, specifically the Hausdorff metric, f forms a contraction [10]. By the Banach fixed point theorem, this induces a unique non-empty compact space K, called a self-similar set.

First, we shall be discussing the Sierpinski gasket. There are two reasons for this. First, much is already known about the gasket from other analysts [16], [18], [31]. Second, it is probably the most basic non-trivial fractal. We construct the gasket via the fixed compact space induced by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i, \quad i = 0, 1, 2$$

where $p_0 = (0, 0)$, $p_1 = (\frac{1}{2}, \frac{3}{2})$, and $p_2 = (1, 0)$. The Sierpinski gasket is shown in Figure 0.1.

Figure 0.1: The Sierpinski gasket

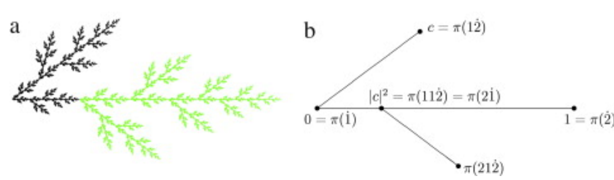


Next, we introduce the Hata tree-like structure (from here on will be simply referred to as the Hata tree). Let $c \in \mathbb{C}$ be such that $|c| < 1$ and $|1-c| < 1$. and consider $F_1, F_2 : \mathbb{C} \to \mathbb{C}$ given by

$$F_1(z) = c\overline{z}, \quad f_2(z) = (1 - |c|^2)\overline{z} + |c|^2.$$

The Hata tree is simply a self-similar structure generated by $\{F_i\}$.

Figure 0.2: The Hata tree-like structure



There are of course many other examples of self-similar structures [16] which we will touch on.

Next, we wish to define some notion of analysis on a self-similar set. Namely, we need two notions.

1. How do we define measures on K?

2. How do we define the notion of derivatives on K?

The first question is answered by the following theorem [18].

Theorem 1. *Let S be a finite set. If $p = (p_i)_{i\in S}$ satisfies $\sum_{i\in S} p_i = 1$ and $0 < p_i < 1$ for all i. Then there exists a unique complete Borel regular measure μ_p on $(\Sigma, \mathcal{M}^p)$, where $\Sigma = S^{\mathbb{N}}$ satisfies $\mu^p(\Sigma_w) = p_{w_1} \dots p_{w_n}$ for any $w = w_1 \dots w_n \in W_*$.*

For the second question, the main part of interest to us will be to define an appropriate notion of a Laplacian. That is, given some $u : K \to \mathbb{R}$, we have a well defined notion of $\Delta u : K \to \mathbb{R}$. Much of this work was introduced by the works of Kigami [16] and and Strichartz [30]. Namely, once we have some appropriate Dirichlet form, $\mathcal{E}(\cdot, \cdot)$, we have a weak notion of the Laplacian

$$\mathcal{E}(u, v) = \int v \Delta u d\mu.$$

On the real line, the appropriate notion of our Dirichlet form is

$$\mathcal{E}(u, v) = \int |\nabla u| \cdot |\nabla v| d\mu, \quad u : \mathbb{R} \to \mathbb{R}$$

where μ is the Lesbesque measure. It turns out that we can construct the Laplacian on a finite set of points via a set of axioms. Then, we can find the Laplacian via an appropriate limit.

In the case of the Sierpinski gasket the formula is given by

$$\begin{aligned}\mathcal{E}(u,v) &= \lim_{m\to\infty}\left(r^{-m}\sum_{x\sim_m y}(u(x)-u(y))(v(x)-v(y))\right)\\ &= \lim_{m\to\infty}\mathcal{E}_m(u,v),\end{aligned}$$

where $r=\frac{3}{5}$ [30]. Many other properties are known about the gasket's Laplacian as well, such as it's spectrum [30], random walks and Brownian motion [16], and solutions to differential equations [31], [12].

With this in mind, we can discuss solutions to differential equations. Namely, we shall be looking at

$$\Delta u = f, \quad u|_{V_0} = 0 \tag{0.1}$$

While several different approaches have been proposed, both inside and outside the field and fractal analysis, we shall be looking at two popular numerical techniques. Namely,

- Green's functions, and
- the finite element method.

The Green's function is simply the impulse response of Laplace's equation. That is, it the the function $G(\cdot,\cdot)$ satisfying

$$u(x) = \int G(x,y)f(y)d\mu(y).$$

This of course assumes that all other variables are fixed. The theory of Green's functions is a common techniques for analyzing differential equations [9]. The theory of Green's functions have also had incredible success in solving differential equations on fractals [30] [18]. We can derive a Green's function similar to how we defined many other analytic concepts, we build up, inductively, successive approximations to our fractal. While much of the work has already been done that applies to an arbitrary p.c.f. fractal, it has been mostly restricted to use in studying the Sierpinski gasket. Here, I will present solutions involving more than just that, such as the Hata tree.

Chapter 1

Introduction to Fractals

We will first try and ask the question: what is a fractal? We will actually not arrive at a definitive answer, only a bunch of suggestions of what it could be. All of these examples are well known in the literature of fractal analysis.

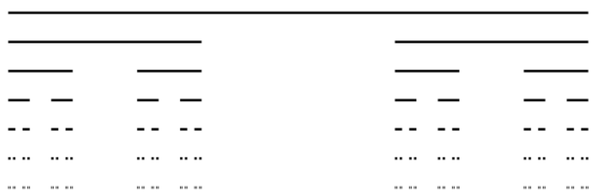
1.1 Examples

First, let us discuss the construction of several popular examples that will be used again and again.

Example 1. The *Cantor set*, which we will denote by $C \subset \mathbb{R}$, is constructed as follows. Let $C_0 = I = [0, 1]$. Define C_n inductively, by removing each interval contained in C_{n-1} by the open middle third. Then define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

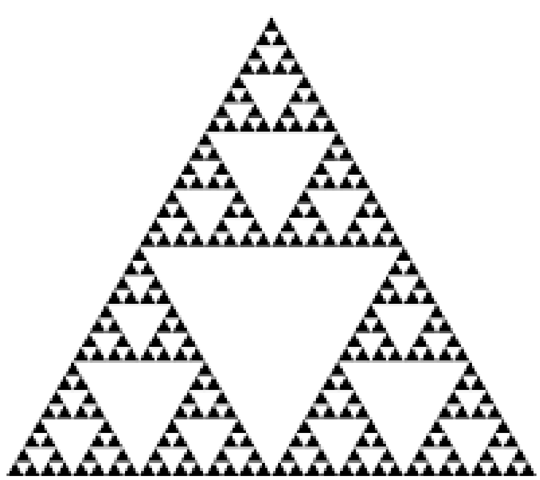
Figure 1.1: A visual construction of the Cantor set



C is than compact, uncountable, and of measure zero [11]. This process is pictorially shown in Figure 1.

Example 2. We take the construction as from [10]. The *Serpinski gasket*, denoted by $K \subset \mathbb{R}^2$. We construct this as follows. We pick some vertices of a equilaterial triangle, K_0, say with vertices at $p_1, p_2, p_3 = (0,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (1,0)$. Construct K_n inductively by considering each equilateral triangle in K_{n-1}, to which we at each edge, place vertex at the mid-point.Proceed to connect all the vertices created in each triangle by three more edges. The process is shown in Figure 2.

Figure 1.2: The Sierpinski gasket



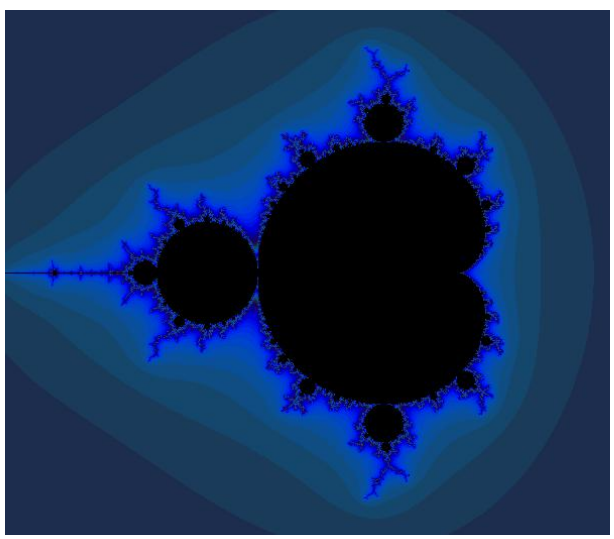
These are two very prominent examples. In fact, ninety percent of this thesis will be simply concerned with examples much like Example 2. A lot of this is due to the self similar nature of these fractals. While one might be tempted to regard this as the definition of a fractal, I will next give a very prominent non-example.

Example 3. Let $c \in \mathbb{C}$ and consider the dynamical system given by

$$z_{n+1} = z_n^2 + c, \quad z_0 = 0. \tag{1.1}$$

If we consider the set of c such that the above dynamical system converges to a non-infinite value, we obtain what is called the Mandelbrot set. The set is graphically shown. A further discussion of this set can be found in [29].

Figure 1.3: The Mandelbrot set



Example 4. If we consider some c in the Mandelbrot set, we have an associated Julia set, where we consider the values z_0 so that $\{z_n\}_{n=0}^{\infty}$ converges to a non-infinite value when (1.1) holds. This forms a so-called Julia set.

1.2 Box Counting Dimension

We begin with a theoretical underpinnings behind the theory of fractal. Taking a remark directly from [10].

Remark 1. We denote $N_\delta(F)$ for some $F \subset \mathbb{R}^n$ to be any one of the following (equivalent) definitions

1. the smallest number of sets of diameter at most δ that cover F,
2. the smallest number of closed balls of radius δ that cover F,
3. the smallest number of cubes of side length δ that cover F,
4. the number of δ-mesh cubes that intersect F,
5. the largest number of discount balls of radius δ with centers in F.

Definition 1. Define the *lower* and *upper box-counting dimensions* of some set $F \subset \mathbb{R}^n$ to be

$$\underline{\dim}_B(F) = \liminf_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$$
$$\overline{\dim}_B(F) = \limsup_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}.$$

If the above are equal, we refer to it as the *box counting* definition of F, denoted by

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}.$$

Proposition 1. *Equivalently, if*

$$F_\delta := \{x \in \mathbb{R}^n : |x - y| \leq \delta \textit{ for some } y \in F\},$$

and μ is the Lebesque measure of $\mathbb{R}^n$, then

$$\underline{\dim}_B F = n - \limsup \frac{\log \mu(F\delta)}{\log \delta}$$
$$\overline{\dim}_B F = n - \liminf \frac{\log \mu(F\delta)}{\log \delta}.$$

Of course, from this

$$\dim_B F = n - \lim \frac{\log \mu(F\delta)}{\log \delta}.$$

We give probably the two most common examples often given, see for example [2] or [10].

Example 5. If F is the Cantor set, then

$$\dim_B(F) = \frac{\log(2)}{\log(3)}$$

Example 6. Let F be the Sierpinski triangle with side length 1. Then

$$\dim_B(F) = \frac{\log(3)}{\log(2)}$$

1.3 Hausdorff Measure

Again, the next series of definitions and results are straight out of [10], but the exact same ideas can be found elsewhere.

Definition 2. A δ*-cover* of F is a cover $\{U_i\}$ such that $0 < |U_i| \leq \delta$ for all i.

Definition 3. We let

$$\mathcal{H}^s_\delta(F) := \inf \left\{ \sum |U_i|^2 : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

From this, define the *s-Hausdorff dimension* of F by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

It is a straightforward argument to prove that this is indeed a measure [11]. The following are a few nice properties proven in [10].

Proposition 2. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^n$ be a mapping such that*

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

for all x, y for some constant $c > 0$. Then for each s

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F)$$

Proposition 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity transformation of scale factor $\lambda > 0$. If $F \subset \mathbb{R}^n$, then*

$$H^s(f(F)) = \lambda^s \mathcal{H}^s(F).$$

1.4 Hausdorff Dimension

Note that from (2) we see that

$$\sum |U_i|^t \leq \sum |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum |U_i|^s,$$

so that

$$\mathcal{H}^t_\delta \leq \delta^{t-s}\mathcal{H}^s_\delta(F),$$

ad thus we see that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for all $t > s$. Thus, there can only be one critical value s which $\mathcal{H}^s(F)$ cannot be 0 or ∞. We refer to this as the *Hausdorff dimension* of F. This is denoted by $\dim_H(F)$. We state analogs of (2) and (3).

Proposition 4. *Let* $F \subset \mathbb{R}^n$ *and suppose* $f : F \to \mathbb{R}^m$ *satisfies the Holder condition*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}$$

for all $x, y \in F$. *Then* $\dim_H f(F) \leq \frac{1}{\alpha} \dim_H(F)$.

Proposition 5. *If* $f : F \to \mathbb{R}^m$ *is bi-Lipschitz, that is*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|$$

for all $x, y \in F$, *and fixed* c_1, c_2 *are nonzero and finite, then* $\dim_H f(F) = \dim_H F$.

Again, proofs are discussed in [10]. Of course (5) will be used significantly to extend calculations of the Hausdorff dimension from one space to many others.

1.5 Definition of Fractals

We discuss briefly, and largely informally, a proper definition towards fractals. We start with a definition originally given by Mandelbrot. [23]

Definition 4. Let $S \subset \mathbb{R}^n$ with topological dimension m. Then S is a *fractal* if $\mathcal{H}(S) > m$.

There is namely one issue with this definition, namely 1 provides a simple counter example. The next definition was proposed by Lapidus [21]

Definition 5. Let $S \subset \mathbb{R}^n$, then S is a *fractal* if S has non-trivial complex dimensions.

While going into this definition further is beyond the scope of this thesis, Alexander Henderson showed if we endow $\mathbb{R}^n$ with the p-adic metric, a singleton point is a fractal. [**?**]

1.6 Iterated Function Systems

We now present some tools to allow for easy calculation of the Hausdorff measure for various spaces. These are common tools found inside fractal analysis. While I won't need these tools outside of $\mathbb{R}^2$ inside the standard L^2 metric, I will be developing everything in full generality. [10] and [30] develop everything in the specific case, whereas [16] goes through everything in full generality, with much greater depth.

Definition 6. Let $D \subset X$ for some metric space (X, d). A mapping $S : D \to D$ is called a contraction on D if there exists some $c \in (0, 1)$ such that

$$d(S(x), S(y)) < c \cdot d(x, y)$$

for all $x, y \in D$

It is straight forward to show that contractions have a unique fixed point if D is complete. That is, there exists a unique $x \in D$ such that $S(x) = x$ (see [26], [22]). In particular, defining some $x_0 \in D$, and let $x_n = S(x_{n-1})$, then $x = \lim x_n$ exists, and is the unique fixed point.

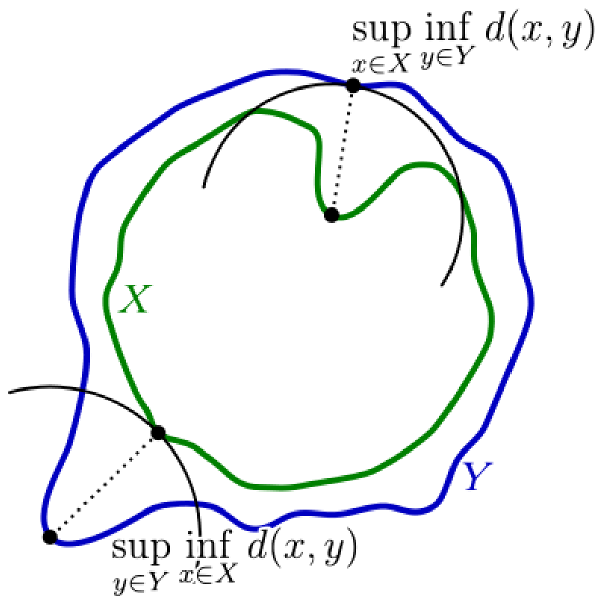
With this, we shall define a notion of distance between compact sets in $\mathbb{R}^n$.

Definition 7. Let $D \subset \mathbb{R}^n$ be closed, and let $\mathcal{S}$ be the set of all non-empty compact sets in D. We define the *Hausdorff metric* or distance by

$$d(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y),\ \sup_{y \in Y} \inf_{x \in X} (x, y) \right\}$$

Graphically, this is nicely demonstrated by Figure 1.6. It is straightforward to show that the Hausdorff metric satisfies the axioms of a metric space [10]. One can also

Figure 1.4: Graphical representation of Hausdorff distance



show that the Hausdorff metric is complete. [10]

Theorem 2. *Assume $D \subset \mathbb{R}^n$ be a closed set. Let $\{S_1, \ldots, S_m\}$, where $S_i : D \to D$ for all i is a contraction, that is, there exists a c_i such that*

$$|S_i(x) - S_i(y)| \leq c_i |x - y|$$

for all $x, y \in D$. If $\mathcal{S}$ is the set of all compact subsets of D, then $S : D \to D$ defined by

$$S(X) = \bigcup_1^m S_i(X)$$

for all X is a contraction in the Hausdorff metric.

The statement is proved in [10]. In particular, systems of contractions induce naturally a compact set. Now define $\mathcal{C}(X)$ to be the set of nonempty compact subsets of X.

Corollary 1. *Let (X, d) be a complete metric space and let $S_i : X \to X$ be a contraction* *for $i = 1, 2, \ldots, n$. Define $S : \mathcal{C}(X) \to \mathcal{C}(X)$ via the formula*

$$S(A) = \bigcup S_i(A).$$

Then S has a unique fixed point, and in particular $S^n(A)$ converges to said unique fixed point in the Hausdorff metric.

This formular will be the basis for the so-called self-similar structure.

Example 7. Let $D = \mathbb{R}^2$, and let

$$F_j(z) = \frac{1}{2}(z - p_j) + p_j$$

where $\{p_1, p_2, p_3\}$ are already given in Example 2. This shows that the Sierpinski gasket is a self-similar structure.

Example 8. Let $D = I$, and $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then the fixed point is the cantor set C. In particular, starting with $X_0 = I$, you will obtain the same sequence as shown in 1.

Now we present the major means of computing the Hausdorff dimension of various spaces.

Theorem 3. *Suppose that we have some IFS $\{S_1, \ldots, S_m\}$, whose contraction factors are $c_1, \ldots, c_m$ respectively. Then $\dim_H F = \dim_B F = s$, and furthermore*

$$\sum c_i^s = 1,$$

and $\mathcal{H}^s(F)$ is nonzero and finite.

Example 9. Using Theorem 3, we can obtain the results of Example 5 and Example 6.

While there are many examples of fractals that do not exhibit this so called self-similar structure, it is nonetheless a very important class of fractals. Self-similarity gives us an incredibly rich structure for us to work with. In particular, most of this thesis will be focused on what are so called p.c.f. self-similar structures. While the technical details will be provided later, p.c.f. self-similar structures give us the property that our self-similar sets, when we apply our self-similar contractions, will intersect at only a finite number of points. The Sierpinski gasket is an important, and probably most well known example of a p.c.f. self similar fractal, with points of intersection occurring only at $\frac{1}{2}(p_i + p_j)$ for $i, j = 1, 2, 3$ and $i \neq j$. Many other examples will be provided later.

Chapter 2

Analysis on Fractals

We will first give an exact definition of a self-similar structure, which general the notion given by Corollary 1.

2.1 Self-Similarity

A large portion of these results are related ideas in analysis, extended to the domain of fractals. Such extensions are definitely non-trivial, however most of the ideas are fairly standardized. Most of the following definitions and theorem in this section are of the result of [30], with a few abstractions used from [16].

A lot of this is done to discuss analysis on fractals, and in fact this is the primary motivation for this material.

Definition 8. Let N be a natural number.

1. For $m \geq 1$, we define

$$W_m^N = \{1, 2, \ldots, N\}^m = \{w_1 w_2 \ldots w_m : w_i \in \{1, 2, \ldots, N\}\}.$$

   In this case we call $w \in W_m^N$ a *word* of length m with symbols $\{1, \ldots, N\}$. We call $W_0^N = \emptyset$ the empty word. Also, let $W_*^N = \bigcup_{m \geq 0} W_m^N$ and denote the length of $w \in W_*^N$ by $|w|$.

2. Define Σ^N, which is called the shift space with N symbols, as

$$\Sigma^N = \{1, 2, \ldots, N\}^{\mathbb{N}} = \{w_1 w_2 \ldots w_i \in \{1, \ldots, N\}\}.$$

If $1 \leq k \leq N$, we may define the map $\sigma_k : \Sigma^N \to \Sigma^N$ by $\sigma_k(w_1 w_2 \ldots) = k w_1 w_2 \ldots$. Define $\sigma : \Sigma^N \to \Sigma^N$ by $\sigma(w_1 w_2 \ldots) = w_2 w_3 \ldots$. For ease of notation, we shall often omit the mention of N. That is, we shall write $W_m = W_m^N$, $W_* = W_*^N$, $\Sigma = \Sigma^N$.

Remark 2. Note that although elements in W_* may be of arbitrary lengths, we do not have notation of infinite length words.

The following lemma is from [24].

Definition 9. We say that a space X is sequentially compact if every sequence in X, $\{x_i\}$ has a subsequence which converges.

Lemma 1. *Let (X, d) be a metric space. Then X is compact iff X is sequentially compact.*

This is used to prove the following, from [16].

Theorem 4. *For $\omega, \tau \in \Sigma$, let $0 < r < 1$ and define $\sigma_r(\omega, \tau) = r^{s(\omega,\tau)}$. Where $s(\omega, \tau) = n$ iff $\omega_k = \tau_k$ for all $k \leq n$ and $\omega_{n+1} \neq \tau_{n+1}$. Then δ_r defines a metric on Σ and (Σ, δ_r) is a* *compact metric space. Furthermore σ_k defines a contraction on this metric space, and Σ is the self-similar metric set with respect to $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$.*

Proof. Let us first show that δ_r is an metric space. It is straight forward to see that $\delta_r(\omega, \tau) \geq 0$ for all ω, τ, and that $\delta_r(\omega, \tau) = 0$ iff $\omega = \tau$. Next, notice that $\min\{s(\omega, \tau), s(\tau, \kappa)\} \leq s(\omega, \kappa)$.

$$\delta_r(\omega, \tau) = r^{s(\omega,\tau)} \leq r^{\min\{s(\omega,\tau),s(\tau,\kappa)\}}.$$

In particular, δ_r is an ultra metric space.

Next, let us show (Σ, δ_r) is a compact space. Now, for $w = w_1 \ldots w_m \in W_*$, we can define

$$\Sigma_\omega = \{\omega = \omega_1\omega_2 \cdots \in \Sigma : \omega_1 \ldots \omega_m = w_1 \ldots w_m\}.$$

Now let $\{\omega^n\}$. Pick some $\tau \in \Omega$, such that $\{n \geq 1 : (\omega^n)_j$ for $j = 1, 2, \ldots, m\}$ is an infinite set. From this, we can define a subsequence of $\{\omega^n\}$ that converges to τ.

Finally, let $\omega = \omega_1\omega_2 \ldots$ and $\tau = \tau_1\tau_2 \ldots$. It follows that

$$\begin{aligned} d(\sigma(\omega), \sigma(\tau)) &= d(k\omega_1\omega_2 \ldots, k\tau_1\tau_2 \ldots) \\ &= r^{s(\omega,\tau)+1} \\ &= rd(\omega, \tau), \end{aligned}$$

as desired. □

Σ is called the topological Cantor set with N symbols. Assume for the rest of this section that (X, d) is a complete metric space, $f_i : X \to X$ is a contraction with respect to (X, d) for all $i \in \{1, 2, \ldots, N\}$ and K is the self-similar set with respect to $\{f_1, \ldots, f_N\}$.

Theorem 5. *Let $w = w_1w_2 \dots w_m \in W_*$, set $f_w = f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_n}$, and $K_w = f_w(K)$. Then for any $w = w_1w_2 \cdots \in \Sigma$, $\bigcap K_{w_1 \dots w_n}$ contains only one point. If we define $\pi : \Sigma \to K$ by $\{\pi(w)\} = \bigcap K_{w_1 w_2 \dots}$, then π is a continuous surjective map. Moreover for any $i \in \{1, 2, \dots, N\}$, $\pi \circ \sigma_i = f_i \circ \pi$.*

The definition is given as follows.

Another way to summarize this result is by saying the following diagram commutes.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\pi} & K \\ \big\downarrow{\scriptstyle \sigma_i} & & \big\downarrow{\scriptstyle f_i} \\ \Sigma & \xrightarrow{\pi} & K \end{array}$$

The above theorem will provide the backdrop for the rest of this chapter, being the basis of what we will call a self-similar structure. The definition is given as follows.

Definition 10. Let K be a compact metrizable topological space and S be a finite set. Also, let F_i be a continuous injection from K to itself for any $i \in S$. Then, $(K, S, \{F_i\}_{i \in S})$ is called a *self-similar structure* if there exists a continuous surjective $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for every $i \in S$, where $\Sigma = S^{\mathbb{N}}$. We often refer to K as a *self-similar set.*

In this case, Theorem 5 essentially provides us with the existence of self-similar structures, and ties them to Corollary 1.

Example 10. If we let $K = [0, 1]$, $S = \{1, 2\}$, and $F_i(x) = \frac{1}{2}x + \frac{1}{2}\delta_{i2}$, then $\mathcal{L} = (K, S, \{F_i\})$ is a self-similar structure.

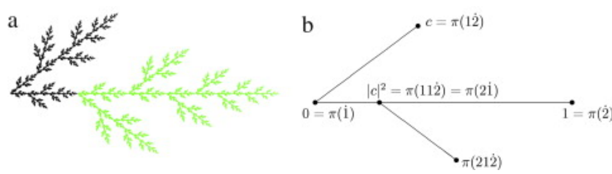
Example 11. If we let $S = \{1, 2, 3\}$, and p_1, p_2, p_3 be the elements of V_0 on the Sierpinski gasket and $F_i(x) = \frac{1}{2}(x - p_i)x + p_i$. If K is the unique fixed compact set corresponding to $\{F_i\}$, then $\mathcal{L} = (K, S, \{F_i\})$ is a self-similar structure.

Example 12. Let $S = \{1, 2\}$, let $c \in \mathbb{C}$ be such that $|c| < 1$ and $|1 - c| < 1$, and define $F_i : \mathbb{C} \to \mathbb{C}$ by

$$F_1(z) = c\overline{z}, \quad f_2(z) = (1 - |c|^2)\overline{z} + |c|^2.$$

We define $K \subset \mathbb{C}$ to be the compact set associated to $\{F_i\}_{i=1,2}$ by Corollary 1. We note the associated self-similar structure $(K, S, \{F_i\}_{i=1,2})$.

Figure 2.1: The Hata tree-like structure



While this definition comes directly from [16], not much is known about the Hata tree. Much of my work has been to provide more insight into this fascinating structure.

Example 13. Let $\{p_1, p_2, p_3\}$ be the vertices embedding in $\mathbb{C}$. Define

$$p_4 = \frac{1}{2}p_2p_3, \quad p_5 = \frac{1}{2}p_1p_3, \quad p_6 = \frac{1}{2}p_1p_2$$

Let $\frac{1}{3} < \alpha < \frac{1}{2}$. We define a set of contractions by

$$F_i(z) = \begin{cases} \alpha(z - p_i) + p_i & i = 1, 2, 3 \\ \\ (a - 2\alpha)(z - p_i) + p_i & i = 4, 5, 6. \end{cases}$$

The self-similar set generated by $\{F_i\}_{i=1}^{6}$ is referred to as the *modified Sierpinski gasket.*

The above definition is from [17].

Now that we have given several examples, much of my attention will now turn to defining properties pertaining to self-similar structures.

Proposition 6. *Define $\dot{w} = www\ldots$. If $w \in W_*$ and $w \neq \emptyset$. then $\pi(\dot{w})$ is the unique fixed point of f_w.*

The above proposition gives us some intuition behind exactly how π behaves.

Definition 11. Let $\mathcal{L}_j = (K_j, S_j, F_i^{(j)})$ be a self-similar structure for $j = 1, 2$. Also let $\pi_j : \Sigma(S_j) \to K_j$ be the continuous surjective associated with $\mathcal{L}_j$ for $j = 1, 2$. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic if there exists a bijective map $\rho : S_1 \to S_2$ such that $\pi_2 \circ \iota_\rho \circ \pi_1^{-1}$ is a well defined homeomorphism between K_2 and K_1 where $\iota_\rho(w_1 w_2 \ldots) = \rho(w_1)\rho(w_2)\ldots$

The next proposition is given in [16].

Proposition 7. *If $(K, S, \{F_i\}_{i \in S})$ is a self similar structure, then π is unique. In fact,*

$$\{\pi(w)\} = \bigcap_{m \geq 0} F_{w_1 w_2 \ldots w_m}(K)$$

for any $w = w_1 w_2 \cdots \in \Sigma$.

2.2 General Theory

While now we laid the cards out on the table, we are in a position to lay out some further details and structure about self-similar structures.

2.2.1 Self-Similar Measures

Proposition 8. *Let S be a finite set. If $p = (p_i)_{i \in S}$ satisfies $\sum_{i \in S} p_i = 1$ and $0 < p_i < 1$ for all i. Then there exists a unique complete Borel regular measure μ_p on $(\Sigma, \mathcal{M}^p)$, where*

$\Sigma = S^{\mathbb{N}}$ *satisfies* $\mu^p(\Sigma_w) = p_{w_1} \dots p_{w_n}$ *for any* $w = w_1 \dots w_n \in W_*$.

This measure is called the Bernoulli measure on Σ with weight p.

Proposition 9. *Let* $\mathcal{L} = (K, S, \{F_i\})$ *be a self-similar structure and let* $\pi : \Sigma \to K$ *be the natural map associated with* $\mathcal{L}$. *If* $p = (p_i)_{i \in S} \in \mathbb{R}^S$ *satisfies* $\sum_{i \in S} p_i = 1$ *and* $0 < p_i < 1$ *for any* $i \in S$, *then define* v^p *by* $\nu^p(A) = \mu^p(\pi^{-1}(A))$ *for any* $A \in \mathcal{N}^P = \{A : A \subset K, \pi^{-1}(A) \in \mathcal{M}^p\}$. *Then* ν^p *is a Borel regular measure on* (K, ν^p). ν^p *is called the self-similar measure on* K *with weight* p.

2.2.2 Post-Critically Finite Sets

So now, we wish to discuss the issue of post-critically finite structures (p.c.f.). As mentioned earlier, the issue of post-critically finite comes from a desire for our self-similar structures to intersect at only a finite number of points under their self-similar transformations.

Analytically, if we wish to define the notion of a derivative, we need to know what happens when we make a small change. If we're working in $\mathbb{R}^m$, the answer is clear. However, when working in the Sierpinski gasket, what directions we can take depends wildly from point to point. A p.c.f. structure allows us to gain some control of the chaos resulting. First, we need to define the notion of critical and post-critical sets.

Definition 12. We define

$$\mathcal{C}_{\mathcal{L},K} = \bigcup_{i \neq j} (F_i(K) \cap F_j(K)),$$

$$\mathcal{C}_{\mathcal{L}} = \pi^{-1}(\mathcal{C}_{\mathcal{L},K}),$$

$$\mathcal{P}_{\mathcal{L}} = \bigcup_{n=1}^{\infty} \sigma^n(\mathcal{C}_{\mathcal{L}}).$$

$\mathcal{C}$ is called the *critical set* and $\mathcal{L}$ and $\mathcal{P}_{\mathcal{L}}$ is called the *post critical set* of $\mathcal{L}$. Also we define $V_0(\mathcal{L}) = \pi(\mathcal{P}_{\mathcal{L}})$.

We will very be be briefly discuss the intuition behind each of these definitions. First, C_K is the points of intersection as discussed before, but in the space our self-similar set is embedded in. Next, C discusses what each of these points are in terms of a word. Next, P tells us exactly how we should start building our self-similar set up from. Specifically, we shall use

$$V_0 = \pi(P),$$

so that we can think of our self-similar set as a graph, whose 'boundary' is V_0. We shall denote

$$V_n = \bigcup_{|w| \leq n} F_w(V_0).$$

We of course define the notion of *post-critically finite set* (or p.c.f.) to be a self-similar strucutre whose post-critical set is finite.

Example 14. Let $\mathcal{L}$ be the usual self-similar structure of the Cantor set, then $C = P = \emptyset$. Hence, the Cantor set is a p.c.f. self-similar set.

Example 15. Let $\mathcal{L} = (K, S, \{f_i\})$ be the usual self-similar structure for the Sierpinski gasket. Then $C_K = \left\{\frac{1}{2}(p_2 + p_3), \frac{1}{2}(p_2 + p_3), \frac{1}{2}(p_2 + p_3)\right\}$, hence the critical set is given by $C = \{1\dot{2}, 2\dot{1}, 2\dot{3}, 3\dot{2}, 3\dot{1}, 1\dot{3}\}$, hence the post-critically finite set is given by $P = \{\dot{1}, \dot{2}, \dot{3}\}$. Hence, the Sierpinski gasket is a post-critically finite set.

Example 16. Consider the Hata tree as discussed before. In fact, $C_K = \{0\}$, so that $C = \{11\dot{2}, 2\dot{1}\}$, while yields that finally $P = \{\dot{1}, \dot{2}\}$.

Proposition 10. *Let $\mathcal{L} = (K, S, \{F_i\}$ be a self-similar structure. For any $x \in K$ and $m \geq 0$, define*

$$K_{m,x} = \bigcup_{w \in W_m,\ x \in K_w} K_w.$$

Then $\{K_{m,x}\}$ os a fundamental system of neighborhoods of x.

Proposition 11. *Let $\mathcal{L} = (K, S, \{F_i\})$ be a self-similar structure. Then*

1. *$\pi^{-1}(V_0) = \mathcal{P}$.*

2. *If $\Sigma_w \cap \Sigma_v = \emptyset$, for $w, v \in W_*$, then $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$, where $K_w = F_w(K)$.*

3. *$\mathcal{C} = \emptyset$ iff π is injective.*

Definition 13. Let K be a compact metrizable topology space and S be a finite set. Also, let F_i be a continuous injection from K to itself for any $i \in S$. Then, $(K, S, \{F_i\}_{i \in S})$ is called a self-similar structure if there exists a continuous surjective $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for every $i \in S$, where $\Sigma = S^{\mathbb{N}}$ is the one sided shift space and $\sigma_i : \Sigma \to \Sigma$ is defined by $\sigma_i(w_1 \ldots) = iw_1 \ldots$

Definition 14. Let $\mathcal{L}_j = (K_j, S_j, F_i^{(j)})$ be a self-similar structure for $j = 1, 2$. Also let $\pi_j : \Sigma(S_j) \to K_j$ be the continuous surjective map associated with $\mathcal{L}_j$ for $j = 1, 2$. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic if there exists a bijective map $\rho : S_1 \to S_2$ such that $\pi_2 \circ \iota_\rho \circ \pi_1^{-1}$ is a well defined homeomorphism between K_2 and K_1, where $\iota_\rho(w_1 w_2 \dots) = \rho(w_1)\rho(w_2)\dots$

Proposition 12. *If $(K, S, \{F_i\}_{i \in S})$ is a self similar structure, then π is unique. In fact,*

$$\{\pi(w)\} = \bigcap_{m \geq 0} F_{w_1 w_2 \dots w_m}(K)$$

for any $w = w_1 w_2 \dots \in \Sigma$.

We shall use the notation for $\sigma : \Sigma \to \Sigma$ by the formula $\sigma(w_1 w_2 \dots) = w_2 \dots$, for $w \in S$.

2.2.3 Fundamental Operators

Remark 3. Let V be a finite set. We define $\ell(V) = \{f : V \to \mathbb{R}\}$. We define a canonical vector space structure on $\ell(V)$, as well as an inner product

$$\langle u, v \rangle = \sum_{p \in V} u(p)v(p).$$

Thus $\ell(V)$ is a Hilbert space.

Definition 15. Let V be a finite set. A symmetric bilinear form on $\ell(V)$, $\mathcal{E}$ is called a *Dirichlet form* on V if it satisfies

(D1) $\mathcal{E}(u, u) \geq 0$ for all $u \in \ell(V)$,

(D2) $\mathcal{E}(u, u) = 0$ iff u is constant, (D3) and for any $u \in \ell(V)$, $\mathcal{E}(u,u) \geq \mathcal{E}(\overline{u},\overline{u})$ where

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

We use $\mathcal{DF}(V)$ to denote the collection of all Dirichlet forms on V, and $\widetilde{\mathcal{DF}}$ to denote all symmetric bilinear forms satisfying (D1) and (D2). We refer to (D3) as the *Markov property.*

Definition 16. A symmetric linear operator $H : \ell(V) \to \ell(V)$ is called a *Laplacian* on V if it satisfies

(L1) $-H$ is positive definite.

(L2) $Hu = 0$ iff u is constant.

(L3) $H_{pq} \geq 0$ for all $p \neq q$.

We use $\mathcal{LA}(V)$ to denote the collection of Laplacians on V. Denote again $\widetilde{LA}(V)$ by symmetric linear operator satisfying (L1) and (L2).

Now, we define a natural function between $\mathcal{DF}(V)$ and $\mathcal{LA}(V)$.

For $H : \ell(V) \to \ell(V)$, we can define a quadratic form $\mathcal{E}_H(\cdot,\cdot)$ on $\ell(V)$ by $\mathcal{E}_H(u,v) = -\langle u, Hv\rangle$ for $u, v \in \ell(V)$. Write $\tau(H) = \mathcal{E}_H$.

Theorem 6. *τ is a bijective mapping between $\widetilde{LA}(V)$ and $\widetilde{DF}(V)$. Moreover, $\pi(LA(V)) = DF(V)$. For any symmetric linear operator $H : \ell(V) \to \ell(V)$, we may define $\mathcal{E}_H(\cdot,\cdot)$ via $\mathcal{E}_H(u,v) = -\langle u, Hv\rangle$*

Proof. Showing $\mathcal{E}_H(\cdot,\cdot)$ is a symmetric bilinear form is straight forward. Now $\pi(\widetilde{\mathcal{LA}}(V)) = \widetilde{\mathcal{DF}}(V)$, as (L1) is equivalent to (D1) and (L2) is equivalent to (D2) under the correspondence π. Now observe that

$$\begin{aligned}\mathcal{E}_H(u,u) &= -\langle u, Hu\rangle \\ &= \sum_{pq} H_{pq}u(p)u(q) \\ &= \frac{1}{2}\sum_{pq}\left[u(p)-u(q)\right]^2,\end{aligned}$$

and hence $\pi(\mathcal{LA}(V)) \subset \mathcal{DF}(V)$. Now pick some $H \in \widetilde{\mathcal{LA}}(V)$ but $H \notin \mathcal{LA}(V)$. Pick some $p \neq q$ such that $H_{pq} < 0$. □

Example 17. Definition 15 and 16 is an abstraction of the following. Let μ be a measure on some set X (that isn't necessarily finite). Assuming u and v are two functions belonging to $C^2(\mu)$,

$$\mathcal{E}(u,v) = \int (\nabla u \cdot \nabla v)\, d\mu. \tag{2.1}$$

While

$$Hu = \nabla^2 u. \tag{2.2}$$

Note that Theorem 6 follows from integration by parts. While X doesn't necessarily have to be finite, we often take limits with respect to Definition 15 and 16.

Example 18. Let $V_0 = \{p_1, p_2, p_3\}$ from the Sierpinski gasket. If $u((p_1,p_2,p_3)) = (x_1,x_2,x_3)$ and $v((u_1,u_2,u_3)) = (y_1,y_2,y_3)$. We can define our Dirichlet form as

$$\mathcal{E}(u,v) = \sum_{i\neq j}(x_i - x_j)(y_i - y_j) \tag{2.3}$$

and our Laplacian is

$$\Delta u = \sum_{i \neq j} (x_i - x_j). \tag{2.4}$$

We shall see how to extend these results into $V_1, V_2, \ldots.$ I will be using a more theoretical approach seen in [16] that can be generalized. A more practical intuitive approach can be seen in [30].

Lemma 2. *Let V be a finite set and let U be a proper subset of V. If $H \in \widetilde{\mathcal{LA}}(V)$, we may write H as follows. We define $T : \ell(U) \to \ell(U)$, $J : \ell(U) \to \ell(V-U)$ and $X = \ell(V-U) \to \ell(V-U)$ by*

$$H = \begin{pmatrix} T & J^T \\ J & X \end{pmatrix}$$

Let us write $u_0 = u|_U$ and $u_1 = u|_{V-U}$. Observe that

$$\begin{aligned} -\mathcal{E}_H(u,u) &= (u_0, u_1) \cdot \begin{pmatrix} Tu_0 + J^T u_1 \\ Ju_0 + Xu_1 \end{pmatrix} \\ &= \langle u_0, Tu_0 \rangle + \langle u_0, J^T u_1 \rangle + \langle u_1, Ju_0 \rangle + \langle u_1, Xu_1 \rangle \\ &= -\mathcal{E}_X(u_1 + X^{-1}Ju_0, u_1 - X^{-1}Ju_0) + \mathcal{E}_{T - J^T X^{-1} J}(u_0, u_0) \end{aligned}$$

Hence, when we project down, we want to think of the energy on the projection, but the projection minus a factor of $J^T x^{-1} J$. With this is mind, it is appropriate to define $P_{V,U}(H) : \mathcal{LA}(V) \to \mathcal{LA}(U)$ via

$$P_{V,U}(H) = T - VX^{-1}J. \tag{2.5}$$

Theorem 7. *Assume the same situation as in Lemma 2.1.5. For $u \in \ell(U)$, define $h(u) \in \ell(V)$ by $h(u)|_v = u$ and $h(u)|_{V-U} = -X^{-1}Ju$. Then $h(u)$ is the unique element that attains $\min_{v\in\ell(V), v|_U=u} \mathcal{E}_H(u,v)$.*

$$\mathcal{E}_{P_{V,U}}(u,u) = \mathcal{E}_H(h(u),h(u)) = \min \mathcal{E}_H(u,u). \tag{2.6}$$

Moreover, if $H \in \mathcal{LA}(V)$, then $P_{V,U}(H) \in \mathcal{LA}(U)$.

Theorem 7 effectively allows you to project these analytic definitions into a subspace.

Note that when we discuss extending these operations to a superset, there are an infinite number of ways to do this. The above projection allows us to define a natural way of extending our result. This is formalized in the following definitions.

Definition 17. Let V_i be a finite set and $H_i \in \tilde{\mathcal{LA}}(V_i)$ for $i = 1, 2$. We say that $(V_1, H_1) \leq (V_2, H_2)$ if $V_1 \subset V_2$ and $P_{V_2,V_1}(H_2) = H_1$.

Definition 18. Let V_m be a finite set and let $H_m \in \widetilde{L\mathcal{A}}(V)$ for each $m \geq 0$. $\{(V_m, H_m)\}_{m\geq 0}$ is called a *compatible sequence* if $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$ for all $m \geq 0$. Let $S = \{(V_m, H_m)\}_{m\geq 0}$ be a compatible sequence. Set $V_* = \bigcup_{m\geq 0} V_m$ and define

$$\mathfrak{F}(S) = \{u : u \in \ell(V_*), \lim_{m\to\infty} \mathcal{E}_{H_m}(u|_{V_m}, u|_{V_m}) < \infty\}$$

$$\mathcal{E}_S(u,v) = \lim_{m\to\infty} \mathcal{E}_{H_m}(u|_{V_m}, v|_{V_m})$$

Note that we are taking a limit of a monotone sequence (as a result of Theorem 7), so a limit is well defined (although it may be infinite).

2.3 Analysis on P.C.F. Structures

Definition 19. If $D \in \mathcal{LA}(V_0)$ and $r = (r_1, \ldots, r_N)$ where $r_i > 0$ for $i \in S$. We define $\mathcal{E}^{(m)} \in \mathcal{DF}(V_m)$ by

$$\mathcal{E}^{(m)}(u, v) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}_D(u \circ F_w, v \circ F_w) \tag{2.7}$$

for $u, v \in \ell(V_m)$. $H_m \in \mathcal{LA}(V_m)$ is characterized by $\mathcal{E}^{(m)} = \mathcal{E}_{H_m}$. Often we write $\mathcal{E}^{(m)}$ as $\mathcal{E}_m$.

We note that $H_m = \sum_{w \in W_m} \frac{1}{r_i} R_w^T D R_w$ where $R_w : \ell(V_m) \to \ell(V_0)$ is defined by $R_w f = f \circ F_w$.

Definition 20. (D,r) is called a harmonic structure iff $\{(V_m, H_m)\}_{m \geq 0}$ is a compatible sequence of r-networks. This harmonic structure is said to be regular if $0 < r_i < 1$ for all $i \in S$.

Proposition 13. *(D,r) is a harmonic structure iff* $(V_0, D) \leq (V_1, H_1)$.

Proof. We shall prove the claim by induction. The base case is true by assumption. Now assume $(V_{m-1}, H_{m-1}) \leq (V_m, H_m)$. We have that

$$\mathcal{E}_{m-1}(u \circ F_i, u \circ F_i) = \min\{\mathcal{E}_m(v \circ F_i, v \circ F_i) : v \in \ell(V_{m+1}, V|_{V_m} = u\}. \tag{2.8}$$

Hence

$$\mathcal{E}_m(u \circ F_i, u \circ F_i) = \min\{\mathcal{E}_{m+1}(v \circ F_i, v \circ F_i) : v \in \ell(V_{m+1}, V|_{V_m} = u\}. \tag{2.9}$$

Thus $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$. □

This proposition allows one to simply do one check in order to determine that a pair (D, r) relatively easily.

Example 19. The Sierpinski gasket has a harmonic structure for

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

and $r = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5})$. Note that a quick computation yields

$$H_1 = \begin{pmatrix} -\frac{10}{3} & 0 & 0 & \frac{5}{3} & 0 & \frac{5}{3} \\ 0 & -\frac{10}{3} & 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ 0 & 0 & -\frac{10}{3} & 0 & \frac{5}{3} & \frac{5}{3} \\ \frac{5}{3} & \frac{5}{3} & 0 & -\frac{20}{3} & \frac{5}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & -\frac{20}{3} & \frac{5}{3} \\ \frac{5}{3} & 0 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & -\frac{20}{3} \end{pmatrix}.$$

Note this is precisely what we predict given the $2/5 - 1/5$ - rule.

Example 20. For the Hata tree, we obtain a regular harmonic structure for

$$D = \begin{pmatrix} -h & h & 0 \\ h & -(h+1) & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad r = (x, 1 - x^2)$$

(see [16]). From this, we can compute that

$$\mathcal{E}_0(u, v) = \frac{1}{x}(x_0 - y_0)(x_1 - y_1) + (y_0 - z_0)(y_1 - z_1).$$

$$H_1 = \begin{pmatrix} -\frac{1}{x} & \frac{1}{x} & 0 & 0 & 0 \\ \frac{1}{x} & \frac{-1-\frac{1}{x}}{x} & 0 & \frac{1}{x^2} & 0 \\ 0 & 0 & -\frac{1}{-x^2+1} & \frac{1}{-x^2+1} & 0 \\ 0 & \frac{1}{x^2} & \frac{1}{-x^2+1} & \frac{-1-\frac{1}{x}}{-x^2+1} - \frac{1}{x^2} & \frac{1}{x(-x^2+1)} \\ 0 & 0 & 0 & \frac{1}{x(-x^2+1)} & -\frac{1}{x(-x^2+1)} \end{pmatrix}.$$

As a side note, we see here why x has to be strictly between 0 and 1. If $x = 0$ or $x = 1$, then H_1 will be undefined.

Further on, we shall just assume $x = \frac{1}{2}$ for simplicity, although the techniques generalize fairly well. The issue is that everything starts to become so much more complicated, and inferences become hard to make.

Example 21. Moving on to the modified Sierpinski gasket, if we let $0 < t < \frac{-1+\sqrt{21}}{2}$, then let

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad r = (1, 1, 1, t, t, t).$$

(see [17]). It follows than that (D, r) is a regular harmonic structure for the modified Sierpinski gasket.

So far, we have been strictly focusing on what goes on around V_0 and V_1. We need to develop ways to extend these functions to V_*. While there are in theory an uncountable number of extensions from V_0 to V_*, there is one that we shall be focused on in particular. Specifically, the harmonic extension. There are two characterizations of the harmonic extension. First, we desire that the Dirichlet form, given by (8), is minimized. Equivalently, we can define it as a function whose Laplacian is 0. Formally, this is is given as the following proposition.

Proposition 14. *For any* $\rho \in \ell(V_0)$*, there exists a unique* $u \in \mathfrak{F}$ *such that* $u|_{V_0} = \rho$ *and* $\mathcal{E}(u,u) = \min\{\mathcal{E}(v,v) : v \in \mathfrak{F}, v|_{V_0} = \rho\}$*. Furthermore,* u *is the unique solution of*

$$(H_m v)|_{V_m - V_0} = 0, \quad v|_{V_0} = \rho. \tag{2.10}$$

Theorem 8. *Let* u *be a harmonic function. Then there exists a unique* $\tilde{u} \in C(K)$ *such that* $u|_{V_*} = \tilde{u}|_{V_*}$*. Furthermore, if we assume that* u *has boundary values* ρ*, then*

$$A_i \rho = R_i \begin{pmatrix} \rho \\ -X^{-1} J \rho \end{pmatrix}. \tag{2.11}$$

It then follows that

$$u|_{F_w(V_0)} = A_{w_m} \dots A_{w_1} \rho. \tag{2.12}$$

where ρ *indicates the boundary values of* u*.*

Example 22. For the harmonic structure discussed before on the Sierpinski gasket,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1/5 & 1/5 & 2/5 \\ 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \end{pmatrix} \quad A_3 = \begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 1/5 & 2/5 & 2/5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, this reduces to the 2/5-1/5 rule.

Example 23. For the Hata tree, we compute

$$u(p_0, p_1, p_2) = (x_0, y_0, z_0), \quad \tilde{u}(p_0, p_1, p_2, p_3, p_4) = (x_0, y_0, z_0, q_1, q_2)$$

The harmonic extension yields $q = q_1 = q_2 = y_0(1 - x^2) + z_0 x^2$.

$$A_1 = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

Chapter 3

Green's Functions

3.1 Introduction

We begin with a brief overview of Green's functions. This follows the paper [18] however they are studied regularly in the context of differential equations [9], as well as several areas of physics [4].

Let us introduce things on the interval $I = [0, 1]$. Assume we have the differential equation

$$\Delta u = f, \quad u(0) = u(1) = 0,$$

where u is subject the boundary conditions $u(0) = u(1) = 0$ (also typically referred to as Dirichlet boundary conditions). We can write the solution as

$$u(x) = \int_0^1 G(x, y) f(y) dy. \tag{3.1}$$

We refer to $G(x, y)$ as the *Green's function* associated with the interval I. Providing an abuse of notation, we can say that $G = \Delta^{-1}$. In this case, we see that

$$G(x, y) = \begin{cases} x(1-y), & x \leq y \\ y(1-x), & y \leq x \end{cases}$$

satisfies (3.1).

3.2 Green's Functions on Fractals

We now turn to building a theory of how to construct a Greens function on the fractal. The way I will be building this up will be in a way that can be generalized to an arbitrary PCF, along the lines of [18]. A far more intuitive construction can be seen in [30].

As discussed earlier, we are attempting to invert Δ_μ, where μ is the appropriate metric on on our fractal, K. Recall that $\Delta_\mu u$ is defined by

$$\mathcal{E}(u, v) = -\int v \Delta_\mu u d\mu.$$

We are attempting to construct the Green's function associated with that operator. Namely, we are trying to evaluate

$$(G_w)_{jk} = G(F_w v_j, F_w v_k) \tag{3.2}$$

for each word w.

We shall construct this process inductively, by first constructing the case where $|w| = 1$. This is obtained by our matrix G on V_1. This can be done by $G_{pq} = -X^{-1}$ for X

the restriction of H on $V_1 - V_0$. Of course, $G = 0$ on V_0 due to satisfy Dirichlet boundary conditions. We can recursively construct this from

$$(B_i)_{jk} = G_{F_i v_j, F_i, v_k}.$$

Note that B_i is related to, but not the same thing as G. We can iterate this result to obtain

$$G_w = \sum_{k=1}^{m} r_{w_1} \dots r_{w_{k-1}} (A_{w_m} \dots A_{w_{k-1}}) B_{w_k} (A_{w_{k-1}} \dots A_{w_m})^T \tag{3.3}$$

To see a specific case, when $w = j...j$, where $|w| = m$, we see that

$$G_w = \sum_{k=1}^{m} r_j^{k-1} A_j^{m-k} B_j (A_j^T)^{m-k} \tag{3.4}$$

3.3 Examples

We now apply this theory to several different examples.

Example 24. In our case

$$\tilde{A}_1 = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \tilde{B}_1 = \frac{3}{50} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.5}$$

We can obtain by induction

$$\tilde{A}_1^k = \frac{5^{-n}}{2} \begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix} \tag{3.6}$$

Hence,

$$\tilde{A}_1^k \tilde{B}_1^T \tilde{A}_1^k = \frac{3}{50} 5^{-2k} \begin{pmatrix} 2\cdot 3^{2k}+1 & 2\cdot 3^{2k}-1 \\ 2\cdot 3^{2k}-1 & 2\cdot 3^{2k}+1 \end{pmatrix}, \tag{3.7}$$

giving us

$$\tilde{G}_m = \frac{3}{5}\left(2\sum_{k=1}^{m}\left(\frac{3}{5}\right)^{2m-k} \pm \sum_{k=1}^{m}\frac{3^k}{5^{2m-k}}\right)$$

This yields

$$G(F_1^m v_2, F_1^m v_2) = c_1\left(\frac{3}{5}\right)^m - c_2\left(\frac{3}{5}\right)^{2m} - c_3\left(\frac{1}{5}\right)^{2m} \tag{3.8}$$

and

$$G(F_1^m v_2, F_1^m v_3) = c_4\left(\frac{3}{5}\right)^m - c_2\left(\frac{3}{5}\right)^{2m} + c_3\left(\frac{1}{5}\right)^{2m}, \tag{3.9}$$

for

$$c_1 = \frac{51}{140}, \quad c_2 = \frac{3}{10}, \quad c_3 = \frac{9}{140}, \quad c_4 = \frac{33}{140}.$$

Example 25. We now consider the Hata tree. Note this is parameterized by r. We consider the $r = 1/2$ cases for simplicity. We use $V_0 = \{c, 0, 1\}$, and $V_1 = V_0 \cup \{|c|^2, *\}$.

We obtain first some of the critical matrices, which are

$$B_1 = \frac{3}{16}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \frac{3}{16}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and recall that

$$A_1 = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

We then apply (3.3).

Chapter 4

Spline Analysis

Discussing the concept of Green's function, we only developed the tools to look at one differential equation on our self-similar set, K, namely

$$\Delta_\mu u = f, \quad u|_{V_0} = 0.$$

4.0.1 Multiharmonic Functions

Previously, we discussed the topic of harmonic functions. These are functions whose Laplacian is 0. Here, we extend this notion by discussing multi-harmonic functions. Much of this is section is based off of [31].

Definition 21. We define the space of $(j+1)$-*harmonic functions*, denoted by $\mathcal{H}_j$, as

$$\mathcal{H}_j = \{f : \Delta^{j+1} f = 0\}.$$

We shall see that $\dim \mathcal{H}_j = (j+1)N_0$. Furthermore, 1-harmonic functions are equivalent to harmonic functions as defined in [16] or [30]. Next, we will describe a basis for $\mathcal{H}_j$.

Lemma 3. *Denote* f_{jk} *to be the solution to* $\Delta^{j+1} f_{jk} = 0$ *such that*

$$\Delta^m f_{jk}(v_n) = \delta_{mj}\delta_{kn}. \tag{4.1}$$

Then $\{f_{mk}\}_{0\le m\le j, 1\le k\le N_0}$ *is a basis for* $\mathcal{H}_j$ *(recall that* $N_0 = |V_0|$*). Furthermore,*

$$f = \sum_{m=0}^{j} \sum_{k=1}^{N_0} (\Delta^m f(v_k) f_{mk}. \tag{4.2}$$

Proof. Note that both sides of (4.2) evaluate the same value for $\Delta^\ell f(v_n)$ for all $0 \le \ell \le j$ and $\ell \le n \le N_0$. □

Observe that $\Delta^\ell f_{jk} = f_{(j-\ell)k}$. Also note that $\{f_{mk}\}_{0\le m\le j, 1\le k\le N_0}$ not an orthonormal set. In fact, this shall be a useful property as we shall soon see. First, two important identities are as follows. First, observe that $\delta f_{jk} = f_{(j-1)k}$, and f_{jk} vanishes on V_0, hence we have

$$f_{jk}(x) = -\int G(x,y) f_{(j-1)k} d\mu(y). \tag{4.3}$$

for the Green's function of our Laplacian, as described in [16] or [30].

Next, we have from our scaling identity

$$\Delta^m(\phi \circ F_i) = (r_i\mu_i)^m (\Delta^m \phi) \circ F_i. \tag{4.4}$$

Lemma 4. *For all* i, j, k*, we have*

$$f_{jk} \circ F_i = \sum_{\ell=0}^{j} \sum_{n=1}^{N_0} (r_i\mu_i)^\ell f_{(j-l)k}(F_i v_n) f_{ln}. \tag{4.5}$$

Proof. Note that $f_{jk} \circ F_i \in \mathcal{H}_j$ from (4.4). Using (4.1), we obtain

$$
\begin{aligned}
f_{jk} \circ F_i &= \sum_{l=0}^{j} \sum_{n=1}^{N_0} \left(\Delta^m [f_{jn} \circ F_i]\right) f_{ln} \\
&= \sum_{l=0}^{j} \sum_{k=1}^{N_0} \left(\Delta^m [f_{jk} \circ F_i]\right) f_{ln} \\
&= \sum_{l=0}^{j} \sum_{k=1}^{N_0} \left(\Delta^m [f_{(j-l)k}(F_i v_n)\right) f_{ln},
\end{aligned}
$$

as desired. □

Next, let

$$
I(jk, j'k') = \int f_{jk} f_{j'k'} d\mu. \tag{4.6}
$$

We have the following recursive relationship used to help compute (4.6).

Lemma 5. *For all j, k, j', k'*

$$
I(jk, j'k') =
$$

$$
\sum_{i=1}^{N} \sum_{\ell=0}^{j} \sum_{n=1}^{N_0} \sum_{\ell'=0}^{j'} \sum_{n'=1}^{N_0} \mu_i (r_i \mu_i)^{\ell+\ell'} f_{(j-\ell)k}(F_i v_n) f_{(j'-\ell')k'}(F_i v_{n'}) I(\ell n, \ell' n').
$$

Proof. We use a self-similar measure on (4.6).

$$
I(jk, j'k') = \sum_{i=1}^{N} \mu_i \int (f_{jk} \circ F_i)(f_{j'k'} \circ F_i) d\mu.
$$

Applying (4.5) gives the desired result. □

Note that as a result,

$$
I(0k, 0k') = \sum_{n=1}^{N_0} \sum_{n'=1}^{N_0} A(kk', nn') I(0n, 0n'), \tag{4.7}
$$

where

$$A(kk', nn') = \sum_{i=1}^{N} \mu_i f_{0k}(F_i v_n) f_{0k'}(F_i f_{n'}). \tag{4.8}$$

Now, we write

$$G(F_i v_n, F_{i'} y) = \sum_{n'=1}^{N_0} \gamma(i'i', n, n') f_{0n'}(y).$$

Lemma 6. *For any j, k, i, n, we have*

$$f_{jk}(F_i v_n) = \tag{4.9}$$

$$- \sum \sum \sum \sum \mu_{i'} (r_{i'} \mu_{i'})^l \gamma(i, i', n, n') I(lk', 0n') f_{(j-1-l)k}(F_{i'} v_{k'}). \tag{4.10}$$

Proof. We use (4.3).

$$\begin{aligned} f_{jk}(F_i v_n) &= - \int G(F_{i'} v_n, y) f_{(j-1)k} d\mu(y) \\ &= - \sum \mu_{i'} \int G(F_{i'} v_n, F_{i'} y) f_{(j-1)k}(F_{i'} y) d\mu(y). \end{aligned}$$

Using (4.5) to evaluate $f_{(j-1)k}(F_{i'} y)$.

□

We can use this to compute numerically solutions to differential equations. The idea is outlined in [12].

4.1 The Sierpinski Gasket

We shall now describe our algorithm on the Sierpinski gasket. This example has been computed from [31]. In this case, $\mu_i = \frac{1}{3}$ and $r_i = \frac{3}{5}$ for all i. Also, recall that

$$G_{pq} = \begin{cases} \frac{9}{50} & p = q \\ \\ \frac{3}{50} & p \neq q. \end{cases} \tag{4.11}$$

Note there are four quantities we wish to calculate.

$$a_\ell = I(\ell k, 0k), \tag{4.12}$$

$$b_\ell = I(\ell k, 0n) \quad n \neq k, \tag{4.13}$$

$$p_\ell = 5^\ell f_{\ell k}(F_i v_k) \quad i \neq k, \tag{4.14}$$

$$q_\ell = 5^\ell f_{\ell k}(F_i v_n) \quad i, k, n \text{ distinct.} \tag{4.15}$$

The initial values are as follows:

$$a_0 = \frac{7}{45}, \quad b_0 = \frac{4}{45}, \quad p_0 = \frac{2}{5}, \quad q_0 = \frac{1}{5}. \tag{4.16}$$

The recursive relations are given as

$$5^j a_j = \frac{43}{75} a_j + \frac{56}{65} + \sum_{l=0}^{j-1} \frac{2}{15}(4p_{j-l} + q_{j-l})(a_l + 2b_l) \tag{4.17}$$

$$5^j b_j = \frac{16}{75} a_j + \frac{47}{45} + \sum_{l=0}^{j-1} \frac{2}{15}(4p_{j-l} + 2q_{j-l})(a_l + 2b_l) \tag{4.18}$$

Chapter 5

Conclusions

Discussing where to go from here, there are several different answers. First, in [18], Jun Kigami has provided several numerical graphs of Green's functions on the gasket. Many of these numerical techniques can be extended to include the given examples of the Hata tree and the Modified Sierpinski gasket in order to provide a visual aid as to the behavior of these functions asymptotically. Green's functions are very important to the theory of differential equations, and having a better intuition behind them is import to developement of further techniques. Next, one could use the results in Green's functions and apply them to the theory of splines. This will allow one to solve a broader set of differential equations, such as the heat equation and the Schrodinger's equations. Many of these have been worked out in [31] and [12] for the Sierpinski gasket. In particular, it would be at this point a straight forward (albeit tedious) calculation to go through the finite element method for the Hata tree or the modified Sierpinski gasket. However, the calculations now, which were first done in [31] are much more complicated now due to lack of symmetry. Also, much of the work done has strictly been on p.c.f. self-similar sets in this thesis. Many of these results could see replications for non-p.c.f. self-similar sets. For instance, the Sierpinski carpet, as well as the Hanoi attractor. The latter of which is a fascinating example of a fractal, and what is known about it is discussed in the appendix.

Appendix A

Theoretical Results

While much of this material pertains to a lot of my original research, a lot more recent work has pertained to more applied aspects of analysis on fractals. Nonetheless, I find much of this to give a lot of insights into many of these analytic properties of fractals. Much of these properties, while helpful, are not strictly necessary to understanding the rest of this thesis.

A.1 C* Algebras

Many important geometric properties of fractals can actually be described quite nicely using the language of C*-algebras. The subject of C*-algebras comes out naturally in the subject of functional analysis. Given some Hilbert space X, we can define a linear operator $H : X \to X$, such that $H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$. We can define the space of all linear operators by $L(H)$. The abstract study of such spaces is often the start of the study of C*-algebras.

Most of this is presented in [25], [14], and [7]

Definition 22. Let A be an algebra. A norm of A, $||\cdot||$ is said to be submultiplicative if

$$||ab|| \leq ||a|| \cdot ||b||$$

for all $a, b \in A$. In this case, A is called a normed algebra. If A admits a unit 1 such that $||1|| = 1$, we say A is a *unital normed algebra.* A complete normed algebra is called a *Banach Algebra.*

Definition 23. An involution on an algebra A is a conjugate linear map $a \mapsto a^*$ on A such that

1. $a^{**} = a$,
2. $(ab)^* = b^* a^*$,

for all $a, b \in A$. If in addition $||a^*a|| = ||a||^2$ for all $a \in A$ we say that A is a C^* algebra.

A.1.1 Examples

Example 26. $M_n(\mathbb{C})$ is a C^*-algebra with involution given as conjugate transposition.

Example 27. Let H be a Hilbert space, then $B(H)$ is a C^*-algebra under the involution $u \mapsto u^*$, where u^* is the unique map such that $\langle ux, y\rangle = \langle x, u^*y\rangle$ for all $x, y \in H$.

Example 28. If Ω is a compact topological space, the set $C(\Omega)$ of all continuous complex valued function on Ω with involution given as pointwise congugacy, is a C*-Algebra.

Example 29. If D is the unit disk, then the analytic functions on the disk, $A(D)$, under $f(z) \mapsto \overline{f(\overline{z})}$ is an involution but not a C*-algebra.

A.1.2 Some Results

The main reason we wish to use C*-algebras to study fractal geometry (and geometry in general) comes from the Gelfand-Naimark theorem [13].

Theorem 9. *If A is a commutative C*-algebra, there exists a *-isomorphism of A onto $C_0(M(A))$.*

This theorem essentially allows us to draw a comparison between C*-algebras and compact topological spaces.

These are many important properties of C*-algebras. Many of these are well known results, but many of the statements are out of [25].

Theorem 10. *Any C*-Algebra has an isometric representation as a closed subalgebra of the algebra $B(H)$ of bounded operators on some Hilbert space.*

A.2 Spectral Triples

Most of these results are that of [19], [5], as well as [**?**].

A.2.1 Definition

Here, I will define what is called a spectral triple, give an example, and briefly explain why it is a useful tool in fractal geometry. This definition comes directly out of [5].

Definition 24. Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* (H, D) *over* $\mathcal{A}$ consists of a Hilbert space H which carries a unital representation π of $\mathcal{A}$ and an unbounded self-adjoint operator D on H such that

(a) the set $\{a \in \mathcal{A} : [D, \pi(a)]$ is densely defined and extends to a bounded operator on $H\}$ is a dense subset of $\mathcal{A}$,

(b) the operator $(I + D^2)^{-1}$ is compact.

Definition 25. Let $\mathcal{A}$ be a unital C^*-algebra and (H, D) an unbounded Fredholm module over $\mathcal{A}$. If the underlying representation π is faithful, then $(\mathcal{A}, H, D)$ is called a spectral triple.

With this in mind, we may define dimension as

$$\partial_{ST} = \inf\{p > 0 : tr((I + D^2)^{-p/2}). \tag{A.1}$$

And assuming $\mathcal{A} = C(M)$, H a Hilbert space. If d_g is the geodesic distance on (M, g), then under certain mild conditions [5] we obtain

$$d_g(p, q) = \sup_{a \in \mathcal{A}}\{|a(p) - a(q)| : \ ||[D, \pi(a)]|| \leq 1\}. \tag{A.2}$$

A.2.2 Spectral Triple for a Circle

This example is based on the construction from work by Christensen, Ivan, and Lapidus from [5]. C_r denote the circle of radius $r > 0$ and centered at 0. We are going let

1. $\mathcal{A}C_r$ denote the algebra of complex continuous $2\pi r$-periodic functions,

2. $H_r = L^2([-\pi r, \pi r], \frac{1}{2\pi r} m)$, where m is the Lebesgue measure,

3. $\pi_r : \mathcal{A}C_r \to B(\mathcal{H})$, defined by $\pi_r(f)(h) = fh$.

An orthonormal basis for H_r is given by

$$\forall k \in \mathbb{Z}, \ \phi_k^r(x) := e^{\frac{ikx}{r}}.$$

Observe that

$$\frac{1}{i}\frac{d}{dx}\phi_k^r(x) = \frac{k}{r}\phi_k^r(x)$$

and hence these are eigenfunctions for the differential operator $\frac{1}{i}\frac{d}{dx}$ with eigenvalues $\frac{k}{r}$. The choice of Dirac operator D_r is the closure of the above operator restricted to the linear space of the basis $\{\phi_k^r : k \in \mathbb{Z}\}$. D_r is self-adjoint by integration by parts. The domain is given by

$$\forall f \in H_r : \; f \in \mathrm{dom} D_r \iff \sum \frac{k^2}{r^2} \left|\langle f, \phi_k^r\rangle\right|^2 .$$

Next,

$$\begin{aligned}
[D_r, \pi_r(f)](g) &= D_r \pi_r(f)(g) - \pi_r(f) D_r(g) \\
&= g D_r(f) + f D_r(g) - f D_r(g) \\
&= -igf' \\
&= \pi_r(-if')(g),
\end{aligned}$$

which is densely defined. Finally, observe that

$$(1 + D_r^2)^{-1}\phi_k^r = \frac{r^2}{r^2 + k^2}\phi_k^r$$

and hence,

$$(1 + D_r^2)^{-1}\left(\sum a_k \phi_k\right) = \sum a_n \frac{r^2}{r^2 + k^2}\phi_n,$$

which can be represented as a limit of finite rank operators. With this in mind, the natural spectral triple, $ST_n(C_r)$ for the circle algebra $\mathcal{A}C_r$ is defined by $ST_n(C_r) := \mathcal{A}C_r, H_r, D_r)$. Again, the next theorem is due to [5].

Theorem 11. *Let $r > 0$, and let $(\mathcal{A}_r C, H_r, D_r)$ be the $ST_n(C_r)$ circle spectral triple. Then the following two results hold:*

1. *The metric, say d_r, induced by the spectral triple $ST_n(C_r)$ on the circle is the geodesic distance on C_r.*

2. *The spectral triple $ST_n(C_r)$ is summable for any $s > 1$, but not for $s = 1$. Hence it has metric dimension 1.*

Proof. For the first claim, observe that $||[D_r, \pi_r(f)]|| = ||\pi_r(if')|| = ||f'||$, for all $f \in C(C_r)$, so (A.2) becomes

$$d_g(p, q) = \{|f(p) - f(q) : |f'| \leq 1\}$$

which holds.

We shall prove the second claim. Note that

$$\begin{aligned} tr((I + D^2)^{-p/2}) &= tr(D^{-p}) \\ &= \sum \left(\frac{k}{r}\right)^{-p} \end{aligned}$$

which converges for all $p > 1$ and diverges for all $p \leq 1$. □

Next, we shall introduce the translated Dirac operator, which is given as

$$D_r^t := D_r + \frac{I}{2r}.$$

The set of eigenvalues is now $\{(2k + 1)2r : k \in \mathbb{Z}\}$, but the domain remains the same. Also $[D_r^t, \pi_r(f)] = [D_r, \pi_r(f)]$. In other words, this translation does not effect the properties of the spectral triple.

Definition 26. The translated spectral triple, $ST_r(C_r)$, for the circle algebra, $\mathcal{A}C_r$ is defined by $ST_t(C_r) := (\mathcal{A}C_r, H_r, D_r^t)$.

We next introduce exactly which functions $f \in \mathcal{A}C_r$ the commutator $[D_r^t, \pi_r(f)]$ is bounded and densely defined. (See [5])

Lemma 7. *Let $f \in \mathcal{A}C_r$. Then the following conditions are equivalent:*

(i) $[D_r^t, \pi_r(f)]$ *is densely defined and bounded.*

(ii) $f \in dom(D_r)$ *and* $D_r f$ *is essentially bounded.*

(iii) There exists a measurable essentially bounded function g on the interval $[\pi r, \pi r]$ such that

$$\int_{-\pi r}^{\pi r} g(t)dt = 0 \text{ and } \forall x \in [-\pi r, \pi r] : f(x) = f(0) + \int_0^x g(t)dt.$$

If the conditions above are satisfied, then $g(x) = (iD_r f)(x) a.e.$

A.2.3 The Interval Triple

Definition 27. Given any $\alpha > 0$, the α-interval spectral triple $ST_\alpha(\mathcal{A}_\alpha H_\alpha, D_\alpha)$ is defined by

(i) $\mathcal{A}_\alpha = C([0, \alpha])$.

(ii) $H_\alpha = L^2([-\alpha, \alpha], m/2\alpha)$, where $m/2\alpha$ is the normalized Lebesque measure.

(iii) The representation $\pi_\alpha : \mathcal{A}_\alpha \to B(H_\alpha)$ is defined for f in $\mathcal{A}_\alpha$ as the multiplication operator on H_α which multiplies by the function $\Phi_\alpha(f)$.

(iv) An orthonormal basis $\{e_k : k \in \mathbb{Z}\}$ for H_α is given by $e_k(x) : \exp(i\pi kx/\alpha)$ and D_α is the self-adjoint operator on H_α which has all the vectors e_k as eigenvectors and such that $D_\alpha e_k = (\pi k/\alpha)e_k$ for each $k \in \mathbb{Z}$.

The next proposition is from [5].

Proposition 15. *Let f be a continuous real and even function on the interval $[-\alpha, \alpha]$ such that f is boundedly continuously differentiable outside a set of finitely many points. Then f is in the domain of definition of D_α and $D_\alpha f$ is bounded outside a set of finitely many points.*

Theorem 12. *Given $\alpha > 0$, let $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ be the α-interval spectral triple. Then, for any pair of real s, t such that $0 \leq s < t \leq \alpha$, we have*

$$|t - s| = \sup\{|f(t) - f(s)| : |[D_\alpha, \pi_\alpha(f)]| \leq 1\}.$$

Further, the triple is summable for any real $s > 1$ and not summable for $s = 1$. Hence, it has metric dimension 1.

Proof. Follow the same steps as in the proof of theorem 11. □

A.2.4 The r-triple, ST_r

Let $\mathcal{T}$ be a compact Hausdorff space let $r : [0, \alpha] \to \mathcal{T}$ be a continuous and injective mapping. The next proposition is from [5].

Proposition 16. *Let $r : [0, \alpha]$ be a continuous inject mapping and $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ the α-interval spectral triple.*

Consider the triple ST_r defined by $ST_r := (C(\mathcal{T}), H_\alpha, D_\alpha)$ where the representation $\pi_r : C(\mathcal{T}) \to B(H_\alpha)$, where the representation $\pi_r : C(\mathcal{T}) \to B(H_\alpha)$ is defined via a homomorphism ψ_r of $C(\mathcal{T})$ onto A_α as follows:

1. *For all $f \in C(\mathcal{T})$, for all $s \in [0, \alpha] : \psi_r(f)(s) = f(r(s))$;*

2. *For all $f \in C(\mathcal{T})$, $\pi_r(f) := \pi_\alpha(\psi_r(f))$.*

Then ST_r is an unbounded Fredholm module, which is summable for any $s > 1$ and not summable for $s = 1$.

Proof. The nontrivial part of this argument is showing that

$$LC := \{f \in C(\mathcal{T}) : [D_\alpha, \pi_r(f)] \text{ is densely defined and bounded}\}$$

is dense in $C(\mathcal{T})$. To this end, we use the Stone-Weierstrass theorem. LC is an algebra by Liebniz differentiation rules. It is clearly unital and self-adjoint. To see that LC seperates points, use Urysohn's lemma on the sets $\{x_0\}$ and $\{x_1\}$ where $x_0 \neq x_1$. □

Definition 28. Let $r : [0, \alpha] \to \mathcal{T}$ be a continuous injective mapping and $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ the α-interval spectral triple. The unbounded Fredholm module $ST_r := (C(\mathcal{T}), H_\alpha, D_\alpha)$ is then called the unbounded Fredholm module associated to the continuous curve r.

Again, citing [5].

Proposition 17. *Let $r : [0, \alpha] \to \mathcal{T}$ be a continuous injective mapping, and $ST_r = (C(\mathcal{T}), H_\alpha, D_\alpha)$ the unbounded Fredholm module associated to r. The metric induced on*

$\mathcal{T}$ by ST_r is given by

$$d_r(p,q) = \begin{cases} 0 & p = q, \\ \infty & p \neq q \text{ and } (p \notin \mathcal{R} \text{ or } q \notin \mathcal{R}), \\ |r^{-1}(p) - r^{-1}(q)| & \text{if } p \neq q \text{ and } p \in \mathcal{R} \text{ and } q \in \mathcal{R}. \end{cases}$$

Proof. Pick some point p not on our curve, and $p \neq q$. Apply Urysohn's lemma to obtain a function f on $\mathcal{T}$ such that $f(p) = 1$ and $f(q) = 0$, and $f(r(t)) = 0$ for any point $r(t)$ on $\mathcal{R}$. Thus $\pi_r(f) = 0$, so for all N $||D_\alpha, \pi_r(Nf)|| \leq 1$, and so $d_r(p,q) \geq 1$ and thus the second case holds.

Now suppose $p \neq q$ and $p, q \in \mathcal{R}$. □

A.2.5 Sums of Curves of Triples

Again, citing [5].

Proposition 18. *Let $\mathcal{T}$ be a compact and Hausdorff space and for $1 \leq i \leq h$, let $r_i : [0, \alpha_i]$ be a continuous curve. If for each $i \neq j$, $r_i([0,\alpha_i]) \cap r_j[(0,\alpha_j])$ is finite, then $\bigoplus_1^h ST_{r_i}$ is an unbounded Fredholm module for $C(\mathcal{T})$.*

A.2.6 The Sierpinski Gasket

Definition 29. (i) Given $n \in \mathbb{N}_0$, choose numbering of the 3^n triangles of size 2^{-n} which form K_n, and let $\Delta_{n,i}$,$i \in \{1, 2, \ldots, 3^n\}$, denote the numbered triangles.

(ii) Let, for each $n \in \mathbb{N}_0$ for each $i \in \{1, 2, \ldots, 3^n\}$, the mapping $r_{n,i} : [-2^{-n}\pi, 2^{-n}\pi] \to \Delta_{n,i}$ be defined such that $r_{n,i}(0)$ equals the lower right-hand corner of $\Delta_{n,i}$ and the mapping is an isometry, modulo $2^{1-n}\pi$, of this interval onto the triangle $\Delta_{n,i}$ equipped with the geodesic distance as metric, and the counter clockwise orientation. The mapping $r_{n,i}$ induces a surjective homomorphism Φ of $C(K)$ onto $C([-2^{-n}\pi, 2^{-n}\pi])$ by

$$\forall t \in [-2^{-n}\pi, 2^{-n}\pi], \forall f \in C(K): \quad \Phi_{n,i}(f)(t) = f(r_{n,i}(t))$$

Let, for each $n \in \mathbb{N}_0$ and $i \in 1, 2, \ldots, 3^n$, the unbounded Fredholm module $ST_{n,i}(K) = (C(K), H_{n,i}, D_{n,i})$ for K be given by

(1) $H_{n,i} = H_{2^{-n}}$

(2) the representation $\pi_{n,i} : C(K) \to B(H_{n,i})$ is defined for $f \in C(K)$ as the multiplication operator which multiplies by the function $\Phi_{n,i}(f)$.

(3) $D_{n,i} = D^t_{2^{-n}}$.

Theorem 13. *The direct sum of all unbounded Fredholm modules $ST_{n,i}(K)$ for $n \in N_0$, $i \in \{1, 2, \ldots, 3^n\}$ gives a spectral triple for K. This spectral triple is denoted $ST(K) = (C(K), H_K, D_K)$, and it is s-summable iff $s > \frac{\log(3)}{\log(2)}$. Hence, its metric dimension is $\frac{\log(3)}{\log(2)}$.*

A.2.7 Alternative Construction of the Gasket

We begin with an alternative construction of the spectral triple on the gasket. This is a result of [6]. While significantly more advanced, this construction will allow us to derive certain aspects about the gasket such as it's harmonic structure. Define $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Now let the polylogarithm function of order s be

$$Li_s(z) := \sum_{k \in \mathbb{N}} \frac{z^k}{k^s}, \quad |z| < 1.$$

Note that we have an analytic continuation on $[1, \infty)$. Next, define the Clausen cosine function

$$Ci_s(t) = \begin{cases} \sum_{k \in \mathbb{N}} \frac{\cos(kt)}{k^s} & \Re(s) \leq 1 \\ \\ Li_s(t) & \Re(s) > 1 \end{cases}$$

Next, define $\partial_\alpha : \mathfrak{F}_\alpha \to L^2(\mathbb{T} \times \mathbb{T})$ by $\partial_\alpha(f)(z, w) = \varphi_\alpha(z - w)^{1/2}(f(z) - f(w))$, where $\varphi_\alpha = -2\pi Ci_{-2\alpha}$. With that define $D_\alpha := \begin{pmatrix} 0 & \partial_\alpha \\ \\ \partial_\alpha^* & 0 \end{pmatrix}$. Now, let $\mathcal{A}_\alpha = \{f \in C(\mathbb{T}) : ||[D, L_f]|| < \infty\}$ and $\mathcal{K}_\alpha = L^2(\Omega_\alpha^*)$

This theorem [6].

Theorem 14. *Let $\alpha \in (0, 1]$. The triple $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$ described above is a densely defined Spectral Triple on the algebra $C(\mathbb{T})$, in the sense of Connes. In particular,*

(i) D_α^{-1} has compact resolvent, and the function $\zeta_D(s) = tr(|D_\alpha|^{-s}) = 4\zeta(\alpha s)$.

(ii) The dimension of the triple is α^{-1}, and $Res_{s=\alpha^{-1}} tr(|D_\alpha|^{-s} = \frac{4}{\alpha} \int f(t)dt$

(iii) The distance d_D induced on $\mathbb{T}$ by the spectral triple satisfies, for any $\epsilon > 0$, $d_D(x, y) \geq \frac{1}{c_\epsilon}|x - y|^{\alpha+\epsilon}$, $x, y \in \mathbb{T}$. Moreover, if $\alpha \geq \frac{1}{2}$, $d_D(x, y) \leq \frac{1}{\tilde{C}_\alpha}|x - y|^\alpha$, $x, y \in \mathbb{T}$,

$$c_\epsilon = \frac{1}{\epsilon}\left(\frac{4}{\epsilon}\right)^\epsilon (4 + 23(4^{2\epsilon} - 1))^{1/2}$$

$$\tilde{c}_\alpha = \frac{\sqrt{3\sin(\pi\alpha)}}{16\sqrt{2}}$$

(iv) The Dirichlet form $\mathcal{E}_\alpha$ can be recovered, for any $f \in H^\alpha(\mathbb{T})$ via the formula

$$\mathcal{E}_\alpha = \frac{2}{\alpha} \lim_{s \to 1} (s - 1) tr(|D|^{s/2} |[D, f]|^2 |D|^{s/2}).$$

A.2.8 Further Concepts

Let K denote the Sierpinkski gasket. If $p_0, p_1, p_2 \in \mathbb{R}^2$ are vertices of an equilateral triangle of unit length and consider contractions $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane $w_i(x) = p_i + \frac{1}{2}(x - p_i)$. Note that K is the fixed point with respect to the contraction $E \mapsto w_0(E) \cup w_1(E) \cup w_2(E)$ under the Hausdorff metric. For any $\sigma = \sigma_1 \dots \sigma_m \in \Sigma$, denote $w_\sigma = w_{\sigma_1} \circ \dots w_{\sigma_m}$. As discussed earlier, will use the Dirichlet form

$$\mathcal{E}[f] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e \in E_m} |f(e_+) - f(e_-)|^2$$

Definition 30. For any word $\sigma \in \Sigma_m$, define a corresponding cell in K as follows

$$C_\sigma := w_\sigma(K).$$

We also define the lacuna $\ell_\emptyset$ as the boundary of the first removed triangle. For any $\sigma \in \Sigma$, define $\ell_\sigma = w_\sigma(\ell_\emptyset)$.

Now choose $\alpha \in (0, 1]$ and construct a triple on K according to the prescriptions given before. Let S be the main lacuna ℓ_0 of the gasket, identified isometrically with $\mathbb{T}$. Consider the triple $\mathcal{T} = (\pi, \mathcal{H}, D_\alpha)$ constructed before. Define $\mathcal{T}_\emptyset = (\pi_\emptyset, \mathcal{H}_\emptyset, D_\emptyset)$, where $\pi_\emptyset(f) = \pi(f|_\emptyset)$, $\mathcal{H}_\emptyset = \mathcal{H}$ and $D_\emptyset = D_\alpha$. For any $\sigma \in \Sigma$ consider the triple $(\pi_\sigma, \mathcal{H}_\sigma, D_\sigma)$ where $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$, $\mathcal{H}_\sigma = \mathcal{H}_\emptyset$, and $D_\sigma = 2^{|\sigma|} D_\emptyset$.

Appendix B

Hanoi Attractors

B.1 The Hanoi Attractor

B.1.1 The Construction

Much like the modified Sierpinski gasket, there are many ways to extend the idea of the Sierpinski gasket. This one is significantly easier to understand in concept, but there are several complexities theoretically which we will go into. Let us start by defining $p_0, \ldots, p_5$ as follows.

$$p_0 = (0,0), \qquad p_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_2 = (1,0)$$

$$p_3 = \frac{1}{2}(p_2 + p_3), \qquad p_4 = \frac{1}{2}(p_1 + p_3), \qquad p_5 = \frac{1}{2}(p_1 + p_2),$$

$p_0, \ldots, p_2$ are the exact same vertices as is the gasket. The remaining points are simply the midpoints of said vertices. Now, with these in mind, define

$$A_4 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_6 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

Then let

$$G_{\alpha,i}(x) = \begin{cases} \frac{1-\alpha}{2}(x - p_i) + p_i \text{ for } i = 1, 2, 3, \\ A_i\alpha(x - p_i) + p_i \text{ for } i = 4, 5, 6. \end{cases}$$

Now, we are ready to define the Hanoi attractor.

Definition 31. Let $0 \leq \alpha < \frac{1}{3}$, we define the *Hanoi attractor of index* α, denoted by K_α, by the unique non-empty compact set such that

$$K_\alpha = \bigcup_{i=0}^{5} G_{\alpha,i}(K_\alpha).$$

The case of $\alpha = 1/2$ is shown in Figure B.1 (Originally from [28]). So, the first three maps simply do the same thing as the Gasket's contractions; they provide three separate copies of our fractal, although this time contracted by a factor of $\frac{1-\alpha}{2}$ each. The last three simply connect each contraction by a line.

Of course, this definition still makes sense if $\alpha > \frac{1}{3}$, however some important results don't hold. Instead of stating where $\alpha < \frac{1}{3}$, but for simplicity I'll be just assuming this throughout here. Let us introduce some notation. First, we note that the set of vertices here is

$$W_{\alpha,n} = \bigcup_{\omega \in \mathcal{A}^n} G_{\alpha,\omega}(W_{\alpha,0}),$$

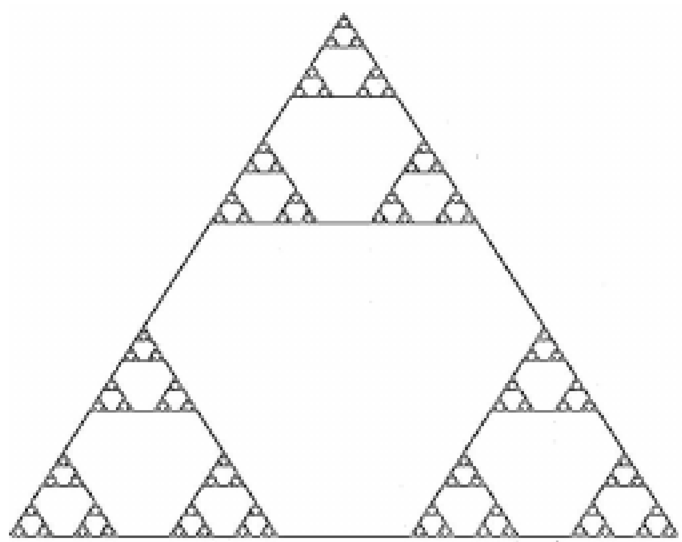


Figure B.1: The Hanoi attactor for $\alpha = 1/2$

where $W_{\alpha,0} = V_0 = \{p_0, p_1, p_2\}$. Of course,

$$W_{\alpha,*} = \bigcup W_{\alpha,n}.$$

Furthermore, let $B = \{(0,1),(0,2),(1,2)\}$ and e_b be the edge connecting the component containing p_i and p_j if $b = (i,j) \in B$. From this, we can define the edges

$$J_{\alpha,n} = \bigcup_{m=0}^{n-1} \bigcup_{\omega \in A^m} G_{\alpha,\omega} \left(\bigcup_{i=1}^{n-1} e_i \right)$$

and

$$J_{\alpha,*} = \bigcup J_{\alpha,n},$$

We note that

$$K_\alpha = \overline{W_{\alpha,*} \cup J_{\alpha,*}}.$$

B.1.2 Comparison to the Gasket

Of course, $K_0 = K$. However, something stronger holds. Indeed, we can say that $K_\alpha \to K$ as $\alpha \to 0$, which is our next result. This is due to [28], and will be outlined below. First, we prove several technical lemmas. For the remainder of this subsection, we use d to refer to Hausdorff distance.

Lemma 8. *It holds that*

$$d(V_*, W_{\alpha,*}) \to 0 \text{ as } \alpha \to 0$$

Proof. Fix some $\alpha \in (0, \frac{1}{3})$, we shall show that

$$V_m \subset (W_{\alpha,*})_\alpha. \tag{B.1}$$

We shall prove this by induction on m. The $m = 0$ case is clear, as $V_0 = W_{\alpha,0}$.

Now assume the $V_m \subset (W_{\alpha,*})_\alpha$, and pick $x \in V_{m+1} - V_m$. Then there exists some $x' \in V_m$ and symbol $k \in \mathcal{A}$ such that

$$x = S_k(x').$$

Now pick $y' \in W_{\alpha,*}$ such that

$$|x' - y'| \leq \alpha.$$

Now consider the point $y = G_{\alpha,k}(y') \in W_{\alpha,*}$. Observe now

$$\begin{aligned} |x-y| &= |S_k(x') - G_{\alpha,k}(y')| \\ &= \left|\frac{1}{2}x' + \frac{1}{2}p_k - \frac{1-\alpha}{2}y' - \frac{1+\alpha}{2}p_k\right| \\ &= \frac{1}{2}|x' - y' + \alpha y' - \alpha p_k| \\ &\leq \frac{1}{2}\left|x' - y'\right| + \frac{\alpha}{2}\left|y' - p_k\right| \\ &= \alpha. \end{aligned}$$

Thus we have shown (B.1) by induction.

By repeating the above argument, interchanging the roles of V_* and $W_{\alpha,*}$, we can prove that

$$V_m \subset (W_{\alpha,*})_\alpha.$$

The desired result holds as a result. □

Lemma 9. *Let $0 < \alpha < \frac{1}{3}$, and consider the sets HG_α and F_α. Then,*

$$F_\alpha \subset HG_\alpha$$

Proof. Define the map

$$T : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2)n \qquad T(B) = \bigcup_{i=1}^{3} G_{\alpha,i}(B).$$

Note the unique fixed point of T is F_α. Furthermore, if we pick some $B_0 \in \mathcal{H}(\mathbb{R}^2)$, and define

$$B_n = T(B_{n-1}), \qquad n \geq 1$$

inductively, we obtain $F_\alpha = \bigcap_{n=1}^{\infty} B_n$ by [10].

On the other hand, if we define

$$T' : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2) \qquad T'(B) = \bigcup_{i=1}^{6} G_{\alpha,i}(B).$$

Similarly, HG_α is the fixed point of T'. We may similarly define a sequence in $C_n \in \mathcal{H}^2(\mathbb{R}^2)$ recursively via T'. We set

$$\Delta = B_0 = C_0,$$

where Δ is the triangle with vertices p_1, p_2, p_3. It follows

$$F_\alpha = \bigcap B_n = \bigcap T^n(\Delta) \subset T'(\Delta) = HG_\alpha.$$

□

Lemma 10. *It holds that*

$$d(HG_\alpha, F_\alpha) \to 0 \qquad \textit{as } \alpha \to 0.$$

Proof. Note that $F_\alpha \subset HG_\alpha$ if $\alpha < \frac{1}{3}$, and thus

$$F_\alpha \subset (HG_\alpha)_\epsilon$$

for all $\epsilon > 0$. Now, we shall prove that

$$HG_\alpha = (F_\alpha)_{\frac{\alpha}{2}}.$$

Pick some $x \in HG_\alpha - F_\alpha$. There exists some word $\omega = \omega_1 \ldots \omega_n \in \{1, \ldots, 6\}^n$ so that

$$G_{\alpha,\omega}(HG_\alpha).$$

Let ω_k be the first letter in $\{4, 5, 6\}$, and define $\omega' = \omega_1 \ldots \omega_{k-1}$. Note that

$$x \in G_{\alpha,\omega'\omega_k}(HG_\alpha).$$

In other words, there exists some $z \in G_{\alpha,\omega_k}(HG_\alpha)$ such that $x = G_{\alpha,\omega'}(z)$. We may find a point $y\{G_{\alpha,i}(p_j), G_{\alpha,j}(p_i)\}$ for $i, j \in \mathcal{A}$, $i + j + \omega_k = 9$ such that

$$|x - y| \leq \frac{\alpha}{2}.$$

Define $y' = G_{\alpha,\omega'}(y) \in F_\alpha$. Since $G_{\alpha,1}$, $G_{\alpha,2}$, and $G_{\alpha,3}$ are similitudes of ratio $\frac{1-\alpha}{2}$, $G_{\alpha,\omega'}$ is similitude $\left(\frac{1-\alpha}{2}\right)^{k-1}$. In other words

$$|G_{\alpha,\omega'}(z) - G_{\alpha,\omega'(y)}| \leq \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y|.$$

Hence

$$\begin{aligned} |x - y'| &= |G_{\alpha,\omega'(z) - G_{\alpha,\omega'}(y)}| \\ &\leq \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y| \\ &\leq \left(\frac{1-\alpha}{2}\right)^{k-1} \cdot \frac{\alpha}{2} \\ &\leq \frac{\alpha}{2}, \end{aligned}$$

as desired. □

Now we define the set F_α as being the unique nonempty compact set such that

$$F_\alpha = \bigcup_{i=1}^{3} G_{\alpha,i} F_\alpha.$$

This next result is due to [1].

Theorem 15. *If $\alpha < 1/3$, then it holds that*

$$HG_\alpha \to SG$$

as $\alpha \to 0$ in the Hausdorff metric.

Proof. We know that

$$d(F_\alpha, SG) \leq d(F_\alpha, W_{\alpha,*}) + d(W_{\alpha,*}, V_*) + d(V_*, SG)$$

$$= d(W_{\alpha,*}, V_*).$$

But $d(W_{\alpha,*}, V_*) \to 0$ as $\alpha \to 0$. □

Theorem 16. *Let $0 < \alpha < \frac{1}{3}$, then*

$$\dim_H(HG_\alpha) = \frac{\log(3)}{\log(2) - \log(1-\alpha)}.$$

Proof. From [10], the desired quantity is the unique number $s > 0$ such that

$$\sum_{i=1}^{3} \left(\frac{1-\alpha}{2}\right)^s.$$

□

Observe that we can easily recover the dimension of the the Sierpinski gasket.

Corollary 2.

$$\dim_H(HG_\alpha) \to \dim_H(SG)$$

as $\alpha \to 0$.

B.1.3 Energy Forms

Next, we shall discuss energy forms of the Hanoi attractor. As discussed earlier, the Hanoi attractor is not a self-similar p.c.f., hence we cannot construct a harmonic structure. However, it can be thought of as a 'collection' of self-similar p.c.f.'s each can be endowed with their own natural Harmonic structure.

Definition 32. Denote $D_0 = \ell(V)$ and

$$\mathscr{D}_n = \{u : V_n \to \mathbb{R} : u|_e \in H^1(e, dx) \text{ for all } e \in \mathcal{J}_e\}$$

for each $n \in \mathbb{N}$. We then define $E_n : \mathscr{D}_n \to \mathbb{R}$ by

$$E_n(u) = \sum (u(x) - u(y))^2 + \int_{J_n} |\nabla u|^2 dx$$

Here, H^1 is simply the Sobolev space associated with the Hanoi attractor. The analogy should be clear. The first part of our energy is simply the graph energy associated with out collection of vertices, the second is the standard interval energy associated with each continuous connection.

The following is Hanoi attractor's the 2/5-1/5 rule, originally proved in [28].

Proposition 19. *For any function $u \in \mathscr{D}_0$, a harmonic function $\tilde{u}$ is uniquely obtained by*

$$\tilde{u}_1(G_i(p_j)) = \frac{2+3\alpha}{5+3\alpha} u(p_i) + \frac{2}{5+3\alpha} p(p_j) + \frac{1}{5+3\alpha} u(p_k)$$

Proof. Without loss of generality, assign

$$u(p_0) = 1$$

$$u(p_1) = 0$$

$$u(p_2) = 0.$$

Let us first attempt to derive an expression for $u|_e$. Note that this is a well known problem in calculus of variation, and the solution is simply an affine function. That is,

$$\tilde{u}|_e(x) = \frac{u(b_e) - u(a_e)}{b_e - a_e} x + \frac{u(a_e)b_e - u(b_e)a_e}{b_e - a_e}.$$

Integrating this, we obtain

$$\begin{aligned}\int_e |\Delta \tilde{u}| dx &= \frac{(u(b_e) - u(a_e))^2}{|b_e - a_e|} \\ &= \alpha^{-1}(u(b_e) - u(a_e))^2.\end{aligned}$$

We hence define an analog to conductance here,

$$c^1_{pq} = \begin{cases} 1, & p \sim_1 q \\ \alpha^{-1}, & (p,q) \in \mathcal{J}_1 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$E(\tilde{u}) = \sum c^1_{pq}(u(p) - u(q))^2.$$

We attempt to minimize this in the same manner as the gasket case, and we obtain

$$x = \frac{2+3\alpha}{5+3\alpha}, \qquad y = \frac{2}{5+3\alpha}, \qquad z = \frac{1}{5+3\alpha}.$$

□

This next proposition is from [1] and [28], which follows from the previous proposition.

Proposition 20. *Let $n \in N$, and define $d_n = \alpha\left(\frac{1-\alpha}{2}\right)^{n-1}$ for $n \neq 0$, and $d_0 = 0$. For any function $u \in \mathscr{D}_n$,*

$$\inf\{E_{n+1}(v)|v \in \mathscr{D}_1 \text{ and } v|_{V_n} = u\}$$

is the uniquely obtained by at each $p_{wij} = G_{wi}(p_j) \in W_{n+1}$ by

$$\tilde{u}_1(p_{ijw}) = \frac{2+3d_n}{5+3d_n}u(p_{wii}) + \frac{2}{5+3d_n}p(p_{wjj}) + \frac{1}{5+3\alpha}u(p_{wkk})$$

Now, we discuss the concept of renormalized energy. Now, ideally we wish to find a sequence ρ_n such that $\mathcal{E}(u) = \rho_n E(u)$ so that $\mathcal{E}$ is invariant under harmonic extensions. However, such a sequence does not exist. A similar result does hold, though. Let $n \geq 1$, and define

$$r_n^d = \frac{3}{5+3d_n} \qquad r_n^c = \frac{3d_n}{5+3d_n},$$

where d_n is defined as before. Now define

$$\rho_n^d = \prod_{i=1}^n r_i^d, \qquad \rho_n^c = \rho_{n-1}^d r_n^c.$$

Now, we can define

$$\mathcal{E}_n(u) = \frac{1}{\rho_n^d}E_n^d(u) + \sum_{k=1}^n \frac{1}{\rho_k^c}E_{k^-}^c(u),$$

where

$$E_{k^-}^c(u) = \sum_{e \in J_n - J_{n-1}} \int_0^1 |(u \circ \phi)'|^2 dt$$

Proposition 21. *Let $u_n : V_n \to \mathbb{R}$ by a harmonic extension of $u_0 : V_0 \to \mathbb{R}$, then*

$$\mathcal{E}(u_n) = \mathcal{E}(u_0) = E(u_0).$$

B.1.4 Measures

Now, we discuss how to define a measure on the Hanoi attractor. Now, unlike the Sierpinski gasket case, there is no unique measure we can naturally define up to normalization constant. The issue is that the there are two components now, our 0-dimensional vertices much like what we see in the gasket. But also a 1-dimensional continuous component. Now each piece has a natural measure we can place. The issue is there is no single answer to how heavy each piece is total. To this end, we choose someβ such that

$$0 < \beta < \left(\frac{2}{3(1-\alpha)} \right)^2 .$$

We define

$$\mu_\alpha^d(A) = \frac{1}{2\mathcal{H}^{\delta_\alpha}(F_\alpha)} \mathcal{H}_{F_\alpha}^{\delta_\alpha}$$

Here,

$$\delta_\alpha = \dim_H(K_\alpha) = \frac{\log(3)}{\log(2) - \log(1-\alpha)}$$

and $\mathcal{H}^{\delta_\alpha}$ is the δ_α dimensional Hausdorff measure. Also, we define

$$\mu_{\alpha,\beta}^c(A) = \frac{1}{2\tilde{\mu}_{\alpha,\beta}} \tilde{\mu}_{\alpha,\beta}^c(A),$$

where

$$\tilde{\mu}_{\alpha,\beta}^c = \sum_{e \in J_n} \beta_e \lambda(A \cap e).$$

Hence, our measure on K_α is given by

$$\mu_{\alpha,\beta}(A) = \mu_\alpha^d(A \cap F_\alpha) + \mu_{\alpha,\beta}^c(A \cap J_\alpha).$$

Note that $\mu_{\alpha,\beta}(K_\alpha) = 1$. We now introduce a pointwise formula for our Laplacian.

$$\Delta_m u(x) = \sum_{x \sim_m y} (u(x) - u(y)). \tag{B.2}$$

With this,

Proposition 22. *Assume $u \in dom(\mathcal{E})$, then the pointwise formula*

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x).$$

B.1.5 Measure

For $i = 0, 1, 2$ define the self-similar functions

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i \tag{B.3}$$

where $\{q_0, q_1, q_2\}$ are the vertices of some equilateral triangle. We then define the *Sierpinski gasket*, denoted K, as the unique nonempty compact space such that

$$K = \bigcup_{i=0}^{2} F_i(K).$$

For existence and uniqueness of K, see [10]. If $\omega = \omega_1 \ldots \omega_n$ is a word in $\{0, 1, 2\}$, and $\{\mu_0, \mu_1, \mu_2\}$ are a sequence of complex numbers, we define

$$F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}, \qquad \mu_\omega = \mu_{\omega_1} \cdot \cdots \cdot \mu_{\omega_n}$$

Associated with it, we have a regular probability measure μ such that

$$\mu(F_\omega K) = \mu_\omega. \tag{B.4}$$

This forms a probability measure, provided $\sum \mu_i = 1$. Conventionally $\mu_i = \frac{1}{3}$ for all i. Note in this case, we have the renormalized Lebesgue measure.

Now, define $V_0 = \{q_0, q_1, q_2\}$, with this define

$$V_m = \bigcup_{|\omega|=m} F_\omega V_0.$$

Next, let E_0 consist of our three lines connecting V_0, and define

$$E_m = \bigcup_{|\omega|=m} F_\omega E_0.$$

From this, we can define a graph Γ_m whose vertices are V_m and edges are E_n. For $x, y \in V_m$, define $x \sim_m y$ iff x, y are connected by an edge. Note this is not an equivalence relationship.

Bibliography

[1] Patricia Alonso-Ruiz and Uta R Freiberg. Hanoi attractors and the sierpiński gasket. *International Journal of Mathematical Modelling and Numerical Optimisation*, 3(4):251–265, 2012.

[2] Michael F Barnsley. *Fractals everywhere.* Academic Press, 2014.

[3] Bruce Blackadar. *K-theory for operator algebras*, volume 5. Cambridge University Press, 1998.

[4] Frederick W Byron and Robert W Fuller. *Mathematics of classical and quantum physics.* Courier Corporation, 2012.

[5] Erik Christensen, Cristina Ivan, and Michel L Lapidus. Dirac operators and spectral triples for some fractal sets built on curves. *Advances in Mathematics*, 217(1):42–78, 2008.

[6] Fabio Cipriani, Daniele Guido, Tommaso Isola, and Jean-Luc Sauvageot. Spectral triples for the sierpinski gasket. *Journal of Functional Analysis*, 266(8):4809–4869, 2014.

[7] John B Conway. *A course in operator theory.* American Mathematical Soc., 2000.

[8] John B Conway. *A course in functional analysis*, volume 96. Springer Science & Business Media, 2013.

[9] Lawrence C Evans. *Partial Differential Equations: (Graduate Studies in Mathematics), volume 19 of Graduate Series in Mathematics.* American Mathematical Society, Providence, Rhode Island,, 2010.

[10] Kenneth Falconer. *Fractal geometry: mathematical foundations and applications.* John Wiley & Sons, 2004.

[11] Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

[12] Michael Gibbons, Arjun Raj, and Robert S Strichartz. The finite element method on the sierpinski gasket. *Constructive approximation*, 17(4):561–588, 2001.

[13] José M Gracia-Bondía, Joseph C Várilly, and Héctor Figueroa. *Elements of noncommutative geometry.* Springer Science & Business Media, 2013.

[14] Richard V Kadison and John R Ringrose. *Fundamentals of the Theory of Operator Algebras Volume II: Advanced Theory*, volume 2. American Mathematical Soc., 2015.

[15] Jun Kigami. Harmonic calculus on pcf self-similar sets. *Transactions of the American Mathematical Society*, 335(2):721–755, 1993.

[16] Jun Kigami. *Analysis on fractals*, volume 143. Cambridge University Press, 2001.

[17] Jun Kigami and Michel L Lapidus. Weyl's problem for the spectral distribution of laplacians on pcf self-similar fractals. *Communications in mathematical physics*, 158(1):93–125, 1993.

[18] Jun Kigami, Daniel R Sheldon, and Robert S Strichartz. Green's functions on fractals. *Fractals*, 8(04):385–402, 2000.

[19] Michel L Lapidus et al. Analysis on fractals, laplacians on self-similar sets, noncommutative geometry and spectral dimensions. *Topological Methods in Nonlinear Analysis*, 4(1):137–195, 1994.

[20] Michel L Lapidus and Jonathan J Sarhad. Dirac operators and geodesic metric on the harmonic sierpinski gasket and other fractal sets. *Journal of Noncommutative Geometry*, 8(4):947–986, 2014.

[21] Michel L Lapidus and Machiel Van Frankenhuijsen. *Fractal geometry, complex dimensions and zeta functions: geometry and spectra of fractal strings (second edition).* Springer Monographs in Mathematics, Springer, New York, 2016.

[22] John M Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–29. Springer, New York, 2003.

[23] Benoit B Mandelbrot. *The fractal geometry of nature*, volume 1. WH freeman New York, 1982.

[24] James R Munkres. *Topology.* Prentice Hall, 2000.

[25] Gerald J Murphy. *C*-algebras and operator theory.* Academic Press, 2014.

[26] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-Hill New York, 1964.

[27] Patricia Alonso Ruiz, Uta Freiberg, and Jun Kigami. Completely symmetric resistance forms on the stretched sierpinski gasket. *arXiv preprint arXiv:1606.08582*, 2016.

[28] Patricia Alonso Ruiz and Uta R Freiberg. Weyl asymptotics for Hanoi attractors. In *Forum Mathematicum*, volume 29, pages 1003–1021. De Gruyter, 2017.

[29] Edward R Scheinerman. *Invitation to dynamical systems.* Courier Corporation, 2012.

[30] Robert S Strichartz. *Differential equations on fractals: a tutorial*. Princeton University Press, 2006.

[31] Robert S Strichartz and Michael Usher. Splines on fractals. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 129, pages 331–360. Cambridge University Press, 2000.