

UC San Diego

UC San Diego Electronic Theses and Dissertations

Title

3-dimensional topological field theory and Harrison homology

Permalink

<https://escholarship.org/uc/item/3f43m91x>

Author

Cooper, Benjamin

Publication Date

2009

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

3-Dimensional Topological Field Theory and Harrison Homology

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Benjamin Cooper

Committee in charge:

Professor Justin Roberts, Chair
Professor Benjamin Grinstein
Professor Ken Intriligator
Professor James Lin
Professor Daniel Rogalski

2009

Copyright
Benjamin Cooper, 2009
All rights reserved.

The dissertation of Benjamin Cooper is approved,
and it is acceptable in quality and form for publi-
cation on microfilm:

Chair

University of California, San Diego

2009

DEDICATION

To my mother Yvonne who fought and overcame challenges to her health during my time in graduate school, my father David for his love and support, and to my brother Samuel for his persistent and infectious enthusiasm.

EPIGRAPH

“I am not the man from Nantucket!”
—A Man From Nantucket

TABLE OF CONTENTS

	Signature Page	iii
	Dedication	iv
	Epigraph	v
	Table of Contents	vi
	Acknowledgements	viii
	Vita and Publications	ix
	Abstract	ix
1	Introduction	1
1.1	Summary	1
1.2	Motivation	1
1.3	Field Theory With Simple Manifolds	3
1.4	Organization of Document	5
2	Differential Graded Categories	8
2.1	Monoidal Categories	8
2.2	Differential Graded Categories	9
2.3	Modules Over Differential Graded Categories	11
3	Graphs	14
4	Operadics	17
4.1	Operads	17
4.1.1	Operads and Cooperads	17
4.1.2	Cyclic Operads	24
4.2	Relation to Differential Graded Algebra	26
4.3	Bar and Cobar Operators on Operads	31
5	Cobordism Categories	39
5.1	3-dimensional Cobordism Categories	39
5.2	Open, Closed and Open-Closed Subcategories	41

6	Homotopy Equivalence Groups	44
6.1	A Theorem of Hatcher Vogtmann and Wahl	46
6.1.1	Reduction of the Open Category	50
6.1.2	The Open-Closed Category as a Module	51
7	Triangulated spaces	52
7.1	Simplicial Complexes	52
7.2	Cellular Stratifications	53
7.3	Open Simplicial Complexes	54
8	Outer Spaces	56
8.1	Open Outer Space	56
8.2	Reduction To Spine	60
8.3	Cellular Stratification by Cubes	61
8.4	Homology	64
8.4.1	From Operads to Graph Complexes	64
8.4.2	Cubical Chains Compute A Double Dual	65
8.5	Corollaries	69
9	The Open Category	70
10	Extension and Torus	74
10.1	The Boundary Torus	74
10.2	The Complex Associated with Extension	75
10.3	Flatness and Exactness	79
10.4	Deligne's Conjecture	81
10.5	Harrison Homology	82
	Bibliography	84

ACKNOWLEDGEMENTS

Nothing would have been possible without the inspiring conversations and relentless optimism of my advisor Justin Roberts.

Other professors who influenced me, Nitu Kitchloo, Mark Gross, Nolan Wallach. Others, John Franco, Jentai Ding, Allen Hatcher and Jim Conant. Other graduate students with whom I have studied or otherwise mathematically interacted. Amanda Beeson, Maia Avarett, David Clark, Yu Li, Michael Gurvich, Gregg Musiker, Oded Yacobi, Tom Fleming, Kevin Costello, Jiri Lebl, Kevin McGown, Dan Budreau, Mike Slawinski, and many others.

I would like to thank my family and friends for all of their patience and encouragement. My father and mother, David and Yvonne Cooper for taking up yoga. My brother and sister-in-law Sam and Anne Cooper for making it official. Alan Calpe and John Orth for demonstrating quiche. Ramon Osa teaching me how play the violin with a beard. Shimon Rura for feigning intent for the kvetch and periodically winning in Scrabble.

This work is heavily indebted to previous work by Maxim Kontsevich, Yan Soibelman, Kevin Costello, Ezra Getzler, Mikhail Kapranov, Victor Ginzburg and Karen Vogtmann, Allen Hatcher, Marc Culler, Nathalie Wahl, Jim Conant and many others. I am grateful for the helpful correspondence with Nathalie Wahl.

VITA

- 2003 B. S. in Mathematics *magna cum laude*, Cornell University
- 2003-2009 Graduate Teaching Assistant, University of California, San Diego
- 2009 Ph. D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

3-Dimensional Topological Field Theory and Harrison Homology

by

Benjamin Cooper

Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Justin Roberts, Chair

In the following I show that each homotopy commutative algebra yields a certain kind of 3-dimensional topological field theory in a functorial way and that this implies the existence of an action of certain 3-manifold cobordisms on the Harrison homology of the algebra.

1 Introduction

1.1 Summary

Below I outline my work on Topological Field Theory (TFT) in three dimensions, explaining how each homotopy commutative algebra (C_∞ algebra) yields a certain kind of 3-dimensional TFT in a functorial way, how this implies the existence of an action of certain 3-manifold cobordisms on Harrison homology, and how this development leads to new perspectives and a number of interesting questions and conjectures.

1.2 Motivation

In a mathematical context Conformal Field Theory (CFT) is defined as a smooth functor from a category \mathcal{M}_2 , whose objects are parametrized 1-manifolds and whose morphisms are Riemann surfaces with fixed boundary circles labelled by the objects, to the category of Hilbert spaces [Seg04]. Inspired by the potential for interesting global structure and the need for a framework in which to understand Gromov-Witten theory a simplification of CFT, Topological Conformal Field Theory (TCFT), was defined and explored independently by Segal and Getzler [Get94, Seg]. Since $\text{Hom}_{\mathcal{M}_2}(A, B)$ is a topological space, we can define a category \mathcal{C}_2 with the same objects as \mathcal{M}_2 such that

$$\text{Hom}_{\mathcal{C}_2}(A, B) = C_*(\text{Hom}_{\mathcal{M}_2}(A, B); \mathbb{Q})$$

where $C_*(-; \mathbb{Q})$ is singular chains with rational coefficients. A TCFT is a differential graded functor from \mathcal{C}_2 to the category of chain complexes. Algebraically this functor can be viewed as a \mathcal{C}_2 -module, if \mathcal{C}_2 is thought of as an algebra with multiple objects. Topologically the reduction made by passing to chain complexes up to quasi-isomorphism is a passage to a derived category or stable rational homotopy category of spaces.

In what follows we will exchange the spaces $\text{Hom}_{\mathcal{M}_2}(A, B)$ of conformal structures on topological surfaces with the classifying spaces of mapping class groups,

$$B\Gamma(\Sigma, \partial) = B\pi_0 \text{Diff}(\Sigma, \partial)$$

of the associated surfaces, because rationally they are the same. This is because the space of conformal structures on a surface is contractible, the quotient by the action of $\text{Diff}(\Sigma)$ is a rational model for $B\text{Diff}(\Sigma)$, and because the connected components of $\text{Diff}(\Sigma)$ are themselves contractible $B\Gamma(\Sigma) \simeq B\text{Diff}(\Sigma)$. In low dimensions then the categories can be thought of as purely of topological origin and so we will just say Topological Field Theory or TFT instead of TCFT throughout.

An important recent theorem about TFT by Costello illustrates the relationship between homotopy associative algebras (A_∞ algebras¹) and the moduli of Riemann surfaces. It is an exciting addition to a story which has developed since Deligne conjectured that the action of the homology of the little disks operad (associated to the Gerstenhaber structure) on the Hochschild homology of a Frobenius algebra, $HH_*(A, A)$, comes from an action defined at the chain level. That is, an action of chains on the little disks operad acting on a chain complex which computes the Hochschild homology. Deligne's conjecture was shown to be true if the little disks operad is replaced with the framed little disks operad (see [MSS02]), but thinking of framed little disks as genus 0 Riemann surfaces leads to a more general theorem of which it is only one consequence [KS]. The following is Kevin Costello's statement [Cos07]:

¹Ordinary associative algebras are examples of A_∞ .

Theorem 1. *The category of A_∞ algebras with a choice of inner product is equivalent to the category of modules over the category \mathcal{O}_2 of chains on Riemann surfaces with open boundary (intervals).*

Theorem 2. *There exists a differential graded category \mathcal{OC}_2 of Riemann surfaces with open boundary and closed boundary (intervals and circles). Such that*

1. $j : \mathcal{O}_2 \hookrightarrow \mathcal{OC}_2$ is a subcategory.
2. $i : \mathcal{C}_2 \hookrightarrow \mathcal{OC}_2$ is a subcategory.

And the following statements are true,

1. \mathcal{OC}_2 is free as an $\mathcal{OC}_2 - \mathcal{O}_2$ bimodule
2. For any A_∞ algebra M ,

$$H_*(i^*(j_*(M))(S^1)) \cong H_*(i^*(\mathcal{OC}_2 \otimes_{\mathcal{O}_2} M)(S^1)) \cong HH_*(M, M)$$

That is, a complex which computes Hochschild homology is associated to the circle object of the category \mathcal{OC}_2 by extension of coefficients.

Corollary 1. *For any A_∞ algebra M with inner product the Hochschild homology $HH_*(M, M)$ is acted on by the homology of the cobordism category $H_*(\mathcal{C}_2)$.*

The development and the success of TFT in mathematics leads one to ask.

Problem. What happens in three dimensions?

1.3 Field Theory With Simple Manifolds

In this thesis I work with relatively simple kinds of 3-manifolds; Those which come from connected sums and punctures of $S^1 \times S^2$. While this may at first appear to be a limitation which will only allow for uninteresting results, we will see in the

results to follow that the structure of the differential graded cobordism category has a rich structure that can be leveraged to prove interesting theorems.

When working on a new theory it is natural to ask whether there are any structural similarities to the known theory. Costello's structure theorem outlined above states that the operad A_∞ plays a central role in 2-dimensions. My results suggest that in 3-dimensions the C_∞ operad acts in its place.

Theorem 3. *The category of C_∞ algebras with a choice of inner product is equivalent to the category of modules over the category \mathcal{O} of chains on the 3-manifolds below with boundary S^2 .*

The objects of the category \mathcal{O} are disjoint unions of labelled 2-spheres and

$$\mathrm{Hom}_{\mathcal{O}}(-, -) = \left(\coprod_{g \geq 0} C_*(B\pi_0 \mathrm{Diff}(\#^g(S^1 \times S^2) \#^{i+j} D^3, \partial); \mathbb{Q}) \right)$$

The composition in this category is induced from gluing along boundary spheres. The spaces above first appear in the work of Hatcher and Vogtmann on homological stability [HV04]. For applications of this theorem to the literature see the next section.

There is a different category, \mathcal{C} , defined by using tori in place of spheres in the definition of \mathcal{O} . Using this we can state an analogue of the second theorem above.

Theorem 4. *There exists a differential graded category \mathcal{OC} of 3-manifolds with boundary S^2 and T^2 . Such that*

1. $j : \mathcal{O} \hookrightarrow \mathcal{OC}$ is a subcategory.
2. $i : \mathcal{C} \hookrightarrow \mathcal{OC}$ is a subcategory.

And the following statements are true,

1. \mathcal{OC} is free as an $\mathcal{OC} - \mathcal{O}$ bimodule

2. For any C_∞ algebra M ,

$$H_*(i^*(j_*(M))(T^2)) \cong H_*(i^*(\mathcal{OC} \otimes_{\mathcal{O}} M)(T^2)) \cong \text{Harr}_*(M, M)$$

Thus a complex that computes Harrison homology is associated to the torus object of the category \mathcal{OC} upon extension of coefficients.

In order to construct \mathcal{OC} we poke toroidal holes in \mathcal{O} . The objects of \mathcal{OC} are pairs, (X, Y) , in which the X are labelled 2-spheres and the Y are labelled 2-tori. The morphisms are defined by,

$$\text{Hom}_{\mathcal{OC}}(-, -) = \coprod_{g \geq 0} C_*(B\pi_0 \text{Diff}(\#^g(S^1 \times S^2) \#^{i+j} D^3 \#^{n+m}(S^1 \times D^2), \partial); \mathbb{Q})$$

The composition in this category is induced from the gluing of manifolds. The spaces above are used to stabilize mapping class groups of 3-manifolds in the recent work of Hatcher and Wahl [HW05].

The corollary then follows.

Corollary 2. *For any C_∞ algebra M with inner product the Chevalley-Eilenberg homology $\text{Harr}_*(M, M)$ is acted on by the homology of the cobordism category $H_*(\mathcal{C})$.*

1.4 Organization of Document

The thesis is split into ten sections including the introduction. What follows is a brief description of the contents of the material contained in each section.

Differential Graded Categories

We begin with the basic definitions developed by Keller, Getzler, Costello and others [Kel94, Cos07]. This is the language of differential graded categories, their representations and homotopy theory. All of the objects in this paper will be either differential graded categories (cobordism categories) or modules over differential graded categories (topological field theories).

Graphs

Graphs will be essential in everything that comes after this section. Basic definitions are given and common terminology articulated. The definitions are in common use throughout literature see Ch.8 [Igu02].

Operadics

The definition of operad and its many variations variations of definitions of operads developed by many authors are recalled. The operads essential to this paper are then defined in terms of trees and their compositions using the free operad. Operads and their algebras are then shown to be equivalent to functorially defined differential graded categories and their modules. Finally we review the Cobar and Bar functors on cyclic operads and cooperads.

Cobordism Categories

The differential graded cobordism categories of open, closed and open-closed cobordisms will be rigorously defined in terms of the cobordism category and mapping class groups of 3-manifolds.

Homotopy Equivalence Groups

This section contains the definition of a family of groups of self-homotopy equivalences of graphs and the relation to the mapping class groups defined in the previous section as found in the work of Hatcher, Vogtmann and Wahl.

Triangulated Spaces

We review the basic definitions of spaces, open simplicial complexes, simplicial complexes and orbi-cellular stratifications used in the next section.

Outer Spaces

The moduli space of graphs satisfying a particular homotopy type is shown to be a rational classifying space for groups of homotopy equivalences of graphs and thus for the mapping class groups used in the construction of the open, closed and open-closed categories. These moduli spaces are then simplified and stratified by orbi-cells given by graphs paired with contractible subgraphs called forests.

The Open Category

For the theorems outlined above to be true the composition of the open category defined by classifying spaces of mapping class groups of cobordisms must be reducible to a composition which is orbi-cellular. This is proven here. The equivalence between C_∞ algebras with invariant inner product and open topological field theories then follows from this and the reduction of the previous section.

Extension and Torus

Finally we show that the open closed category is quasi-isomorphic as a module to a flat module over \mathcal{O} . Given a C_∞ algebra A together with invariant inner product and thus a homologically split functor $M : \mathcal{O} \rightarrow \text{Ch}_k$ or open topological field theory, it is shown that the extension $\mathcal{OC} \otimes_{\mathcal{O}} M$ defines an open closed topological field theory. After a short discussion of the definition of Harrison homology. It is then shown that the chain complex associated to the torus object, $(\mathcal{OC} \otimes_{\mathcal{O}} M)(T^2)$ by this theory computes the Harrison homology $\text{Harr}_*(A, A)$.

2 Differential Graded Categories

The underlying field $k = \mathbb{Q}$ in all constructions will be fixed to be the rational numbers. We denote by Top the category of topological spaces, by Orbi the category of orbifolds, by Group the category of groups, by Vect_k the category of vector spaces over k and by Ch_k the category of chain complexes of vector spaces over k .

2.1 Monoidal Categories

A category \mathcal{C} is *symmetric monoidal* if it is equipped with a bifunctor

$$- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

an object 1 and isomorphisms,

1. $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$
2. $1 \otimes a \cong a \cong a \otimes 1$
3. $a \otimes b \cong b \otimes a$

satisfying various coherence conditions see [ML98]. Note that the commutativity isomorphisms are not necessarily identity. There are monoidal structures on Top , Group , Vect_k and Ch_k given by disjoint union, product and tensor product in the usual way.

A *monoidal* functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between symmetric monoidal categories equipped with maps, not necessarily isomorphisms, $F(a) \otimes F(b) \rightarrow F(a \otimes b)$ that satisfy,

1. Associativity,

$$\begin{array}{ccc}
 F(a) \otimes F(a') \otimes F(a'') & \longrightarrow & F(a \otimes a') \otimes F(a'') \\
 \downarrow & & \downarrow \\
 F(a) \otimes F(a' \otimes a'') & \longrightarrow & F(a \otimes a') \otimes F(a'')
 \end{array}$$

2. Commutativity,

$$\begin{array}{ccc}
 F(a) \otimes F(a') & \longrightarrow & F(a \otimes a') \\
 \downarrow & & \downarrow \\
 F(a') \otimes F(a) & \longrightarrow & F(a' \otimes a)
 \end{array}$$

Every symmetric monoidal category \mathcal{C} has a subcategory $\text{Ob}(\mathcal{C})$ with the same objects and morphisms generated by identity maps, permutations of tensors $a \otimes a' \cong a' \otimes a$ and their tensor products.

2.2 Differential Graded Categories

All of the categories in this paper will have extra structure in a sense that can be captured by the idea of enrichment. A category \mathcal{C} is *enriched* over a category \mathcal{D} if for all objects $X, Y \in \text{Ob}(\mathcal{C})$,

$$\text{Hom}_{\mathcal{C}}(X, Y) \in \text{Ob}(\mathcal{D})$$

For instance, the sets $\text{Hom}_{\text{Top}}(X, Y)$ may be endowed with the compact open topology showing that Top is enriched over Top . Such a category will be called *topological*. A *linear* category is a category enriched over Vect_k . Ch_k is a linear category.

A *differential graded* or *dg* category is a category enriched over Ch_k . A *differential graded symmetric monoidal* or *dgs* category is a monoidal category which is

differential graded. Most of the categories in this paper will be dgsm categories. The category Ch_k is an example of a dgsm category; the morphisms being chain maps form chain complexes.

Enrichments of categories can be transferred by functors. If \mathcal{X}/\mathcal{D} is the category whose objects are categories \mathcal{C} enriched over \mathcal{D} : a morphism is functor $F : \mathcal{C}/\mathcal{D} \rightarrow \mathcal{B}/\mathcal{D}$ that satisfies,

$$F : \text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{B}}(F(a), F(b)) \in \text{Hom}_{\mathcal{D}}(\text{Hom}_{\mathcal{C}}(a, b), \text{Hom}_{\mathcal{B}}(F(a), F(b)))$$

Any monoidal functor $F : \mathcal{D} \rightarrow \mathcal{E}$ defines the *pushforward*, a functor $F_* : \mathcal{X}/\mathcal{D} \rightarrow \mathcal{X}/\mathcal{E}$ such that $\text{Hom}_{F_*(\mathcal{C})}(x, y) = F(\text{Hom}_{\mathcal{C}}(x, y))$.

For example the functor $B : \text{Group} \rightarrow \text{Top}$ given by taking the geometric realization of a simplicial complex (or set) called the nerve of a group. The functor B satisfies $B(G \times H) = BG \times BH$ so it is monoidal. It induces $B_* : \mathcal{X}/\text{Group} \rightarrow \mathcal{X}/\text{Top}$. See [Seg68].

The most important example is $C_*(-; k)$, the singular chains on X . It induces a functor $\mathcal{X}/\text{Top} \rightarrow \mathcal{X}/\text{Ch}_k$ that is if \mathcal{C} is a topological category then there is a dgsm category $C_*(\mathcal{C}; k)$ in which,

$$\begin{aligned} \text{Ob}(C_*(\mathcal{C}; k)) &= \text{Ob}(\mathcal{C}) \\ \text{Hom}_{C_*(\mathcal{C}; k)}(A, B) &= C_*(\text{Hom}_{\mathcal{D}}(A, B); k) \end{aligned}$$

That the functor C_* is monoidal up to quasi-isomorphism follows from the Eilenberg-Zilber theorem; see [EZ53].

In the same spirit given a dgsm category \mathcal{D} , there is a linear category $H_*(\mathcal{D}; k)$ defined by $\text{Ob}(H_*(\mathcal{D}; k)) = \text{Ob}(\mathcal{D})$ and $\text{Hom}_{H_*(\mathcal{D}; k)}(A, B) = H_*(\text{Hom}_{\mathcal{D}}(A, B); k)$. There is also a category $H_0(\mathcal{D})$; if \mathcal{C} is a topological category and $\mathcal{D} = C_*(\mathcal{C})$ then $H_0(\mathcal{D})$ is the category of connected components of \mathcal{C} .

The category of dgsm categories is the subcategory of the category \mathcal{C}/Ch_k of categories enriched over Ch_k which possess a monoidal structure and have monoidal

morphisms as described in 2.1. Specifically, a *morphism* of dgsm categories $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor of categories enriched over Ch_k , a monoidal functor which respects the differential graded structure as described above.

Categories enriched over Ch_k have extra structure that can be used to define a homotopy theoretic notation of equivalence. Every such category \mathcal{C} contains a subcategory $Q(\mathcal{C})$ with the same objects as \mathcal{C} and morphisms $\text{Hom}_{Q(\mathcal{C})}(a, b)$ called quasi-isomorphisms. An element $\varphi : a \rightarrow b \in \text{Hom}_{Q(\mathcal{C})}(a, b)$ is a *quasi-isomorphism* if for all objects $c \in \text{Ob}(\mathcal{C})$ the morphism

$$\bar{\varphi} : \text{Hom}(c, a) \rightarrow \text{Hom}(c, b)$$

induces an isomorphism on homology,

$$\bar{\varphi}_* : H_*(\text{Hom}(c, a); k) \rightarrow H_*(\text{Hom}(c, b); k)$$

The composition of two quasi-isomorphisms is a quasi-isomorphism and as defined $\text{Hom}_{Q(\mathcal{C})}(a, b)$ includes any isomorphisms $a \rightarrow b$ in \mathcal{C} (such as $1 : a \rightarrow a$) so that $Q(\mathcal{C})$ is a subcategory of \mathcal{C} .

A dgsm functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *exact* if it preserves the class $Q(\mathcal{C})$ of quasi-isomorphisms, $F(Q(\mathcal{C})) \subset Q(\mathcal{D})$. In words, for every quasi-isomorphism $\varphi : a \rightarrow b$ in \mathcal{C} we have a quasi-isomorphism $F(\varphi) : F(a) \rightarrow F(b)$ in \mathcal{D} .

Two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are *quasi-isomorphic*, $F \simeq G$, if there are natural transformations $\varphi : F \rightarrow G$ such that $\varphi(c)$ is a quasi-isomorphism for all $c \in \text{Ob}(\mathcal{C})$.

Two dgsm categories \mathcal{C} and \mathcal{D} are *isomorphic* or *quasi-equivalent*, $\mathcal{C} \cong \mathcal{D}$ if there are dgsm functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $FG \simeq 1_{\mathcal{D}}$ and $GF \simeq 1_{\mathcal{C}}$.

2.3 Modules Over Differential Graded Categories

If \mathcal{A} is a dgsm category then a *left \mathcal{A} -module* is a dgsm functor $\mathcal{A} \rightarrow \text{Ch}_k$. A *right \mathcal{A} -module* is a dgsm functor $\mathcal{A}^{op} \rightarrow \text{Ch}_k$.

As functors modules must respect the differential graded structure,

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(a, b) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(a), F(b)) \\
\downarrow d & & \downarrow d \\
\mathrm{Hom}_{\mathcal{A}}(a, b) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(a), F(b))
\end{array}$$

Morphisms between modules M and N are natural transformations $\phi : M \rightarrow N$ of the underlying functors that satisfy,

1. All $\phi(a) \in \mathrm{Hom}_{\mathcal{B}}(M(a), N(a))$ are chain maps.
2. ϕ respects the monoidal structure,

$$\begin{array}{ccc}
M(a) \otimes M(a') & \longrightarrow & N(a) \otimes N(a') \\
\downarrow & & \downarrow \\
M(a \otimes a') & \longrightarrow & N(a \otimes a')
\end{array}$$

The category of left (right) modules over \mathcal{A} will be denoted by $\mathcal{A}\text{-mod}$ ($\text{mod-}\mathcal{A}$). This is not a differential graded category.

For a functor to be monoidal we *only* require the existence of a map

$$F(a) \otimes F(b) \rightarrow F(a \otimes b)$$

satisfying the axioms described in section 2.1. It is often the case that these structure maps satisfy stronger conditions. A module is *split* if the monoidal structure maps $F(a) \otimes F(b) \rightarrow F(a \otimes b)$ are isomorphisms and *h-split* or homologically split if they are quasi-isomorphisms.

If two categories are isomorphic then the pullback maps induced by the isomorphisms between categories of modules will compose to identity.

Theorem 5. *If \mathcal{C} and \mathcal{D} are dgsm categories then $\mathcal{C} \cong \mathcal{D} \Rightarrow \mathcal{C}\text{-mod} \cong \mathcal{D}\text{-mod}$.*

The usual product of categories extends to one which respects the dgsm structure. If \mathcal{A} and \mathcal{B} are categories then there is a category $\mathcal{A} \otimes \mathcal{B}$ defined by

$$\text{Ob}(\mathcal{A} \otimes \mathcal{B}) = \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{B})$$

$$\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(a \times c, b \times d) = \text{Hom}_{\mathcal{A}}(a, c) \otimes_k \text{Hom}_{\mathcal{B}}(b, d)$$

If \mathcal{A}, \mathcal{B} are differential graded then $\mathcal{A} \otimes \mathcal{B}$ is differential graded using the usual tensor product of chain complexes. If \mathcal{A}, \mathcal{B} are monoidal then $\mathcal{A} \otimes \mathcal{B}$ is monoidal using $(a \times c) \otimes (b \times d) = (a \otimes b) \times (c \otimes d)$.

If \mathcal{A} and \mathcal{B} are dgsm categories then an $\mathcal{B} - \mathcal{A}$ bimodule is a dgsm functor from the category $\mathcal{B} \otimes \mathcal{A}^{op}$ to Ch_k . If M is a $\mathcal{B} - \mathcal{A}$ bimodule and N is a left \mathcal{A} -mod then there exists a left \mathcal{B} -mod, $M \otimes_{\mathcal{A}} N$, defined so that for $b \in \text{Ob}(\mathcal{B})$,

$$(M \otimes_{\mathcal{A}} N)(b) = \bigoplus_{a \in \text{Ob}(\mathcal{A})} M(b, a) \otimes_k N(a)$$

modulo relations which allow the diagram below to commute.

$$\begin{array}{ccc} M(b, a) \otimes_k \text{Hom}_{\mathcal{A}}(a', a) \otimes_k N(a') & \longrightarrow & M(b, a) \otimes_k N(a) \\ \downarrow & & \downarrow \\ M(b, a') \otimes_k N(a') & \longrightarrow & (M \otimes_{\mathcal{A}} N)(b) \end{array}$$

Every dgsm category \mathcal{C} yields a $\mathcal{C} - \mathcal{C}$ bimodule, $\mathcal{C} \otimes \mathcal{C}^{op} \rightarrow \text{Ch}_k$ given by

$$\mathcal{C}(x \times y) = \text{Hom}_{\mathcal{C}}(y, x)$$

If $\mathcal{X}, \mathcal{Y} \subset \mathcal{C}$ are subcategories then the action of $\mathcal{C} \otimes \mathcal{C}^{op}$ on \mathcal{C} pulls back to an action of $\mathcal{X} \otimes \mathcal{Y}^{op}$.

A module M is *flat* if the functor $- \otimes M$ is exact. Since most of the constructions to follow will involve considering dgsm categories and their modules up to quasi-isomorphism, strictly speaking, we should be working in a derived category. As such the tensor product $M \otimes N$ of a $\mathcal{B} - \mathcal{A}$ bimodule M and a left \mathcal{A} -module N as above should be defined by $M \otimes_{\mathcal{A}}^{\mathbb{L}} N = M \otimes_{\mathcal{A}} F_{\bullet} N$ where $F_{\bullet} N$ is an acyclic resolution of N . This can be done so that the tensor product exists and satisfies the appropriate universal properties [Cos07].

3 Graphs

By a *graph* G we mean a finite set G with two partitions.

1. Into pairs $e = \{a, b\}$ called *edges*.

$$G = \coprod_e \{a, b\}$$

2. Into sets $H(v) = \{h_1, h_2, \dots, h_n\}$ called *vertices*.

$$G = \coprod_v H(v)$$

Denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. A graph in the sense above can be specified by giving a collection of vertices, edges and specifying the vertices to be found at each end of each edge.

The elements of G will be called *half edges*. Two half edges $a, b \in G$ *meet* if $a, b \in H(v)$ for some vertex v . Given an edge $e \in E(G)$ the set $e = \{x, y\}$ is the set of half edges associated to e in G and for every vertex $v \in V(G)$ the set $H(v)$ is the set of half edges associated to v in G . The *valence* $\text{val}(v)$ of $v \in V(G)$ is the number of half edges $\#H(v)$. All graphs G in this document are required to have vertices v of valence $\text{val}(v) = 1$ or $\text{val}(v) \geq 3$ unless otherwise noted.

A *subgraph* H of G is the set of all vertices of G together with some subset of the set of edges of G . A *cycle* of G based at a half edge $h \in H(v)$ is a subgraph $C_v \subset G$, is an ordered sequence of edges $C_v = (e_1, e_2, \dots, e_n)$, $e_k \in E(G)$ that form a cycle which starts at v , $e_1 = \{h, x_1\}$, $e_2 = \{x_1, x_2\}$, \dots , $e_{n-1} = \{x_{n-2}, x_{n-1}\}$, $e_n = \{x_n, h\}$.

The *boundary* $\partial(G)$ of a graph G is the collection of edges that contain a vertex having valence one. An *internal* edge is an edge not in the boundary while an *external* edge is not internal.

Let $[n]$ be the set $\{1, \dots, n\}$. A graph G is *boundary labelled* if there is a choice of partition $\partial(G) = \text{In}(G) \cup \text{Out}(G)$ of the boundary into a set of *incoming* and *outgoing* edges together with bijections $i_G : [\#\text{In}(G)] \rightarrow \text{In}(G)$ and $o_G : [\#\text{Out}(G)] \rightarrow \text{Out}(G)$.

A *geometric graph* is a 1-dimensional CW complex. Every graph G in the sense given above has an associated geometric graph $|G|$ such that

1. The 0-skeleton $|G|^0 = V(G)$.
2. There is a 1-cell of $|G|$ for each edge $e \in E(G)$ and its boundary is glued to the two vertices containing the half edges $e = \{a, b\}$ of e .

We may refer to graphs a combinatorial as opposed to geometric graphs if it is necessary to draw a distinction between the two.

A graph G is *connected* if $H_0(|G|) \cong \mathbb{Z}$. A graph G has *genus* g if $H_1(|G|) \cong \mathbb{Z}^g$. A *forest* is a graph of genus 0. A *tree* is a connected forest. A *rooted tree* is a tree together with a choice of outgoing edge the rest of the boundary edges being incoming. A tree with a single vertex will be called a *corolla*. An n -Tree is a tree with n incoming edges.

A tree T is *planar* if for all $v \in V(T)$ there is a fixed ordering of $H(v)$ this is equivalent to specifying an immersion of the tree in the plane up to isotopy.

Given an edge $e \in E(G)$, $e = \{x, y\}$ we can form a new graph G/e by removing e and replacing $H(x)$ and $H(y)$ with $H(x) \cup H(y) - \{x, y\}$. This operation called *edge collapse* is a homotopy equivalence of $|G|$ if x and y are not contained in the same $H(v)$. If F is a subgraph of G isomorphic to a forest then all of its edges can be collapsed forming a graph G/F called the *forest collapse*.

Two graphs G and H are *isomorphic* if there is a bijective set map between half edges $\varphi : H \rightarrow G$ that respects the two partitions. Specifically, if $e = \{a, b\}$ is an

edge in H then $\varphi\{a, b\} = \{\varphi(a), \varphi(b)\}$ is an edge in G and if $v = \{h_1, \dots, h_n\}$ is a vertex in H then $\varphi(v)$ is a vertex in G .

4 Operads

One can think of operads as an axiomatization of basic operations on spaces or linearizations of spaces. Formally this is reducible to the more standard approach of algebras and their representations (or actions and modules). But operads also provide a powerful well-developed framework for discussing the case in which the algebras are parametrized by trees. We will use the language from this perspective and make explicit its relation to the differential graded algebra developed in Chapter 2 in 4.3.

4.1 Operads

This section has two parts. We first define operads and their duals (axiomatically opposite objects) cooperads from the perspective of rooted trees. In the second half we discuss the extension of these ideas to unrooted trees in this case the objects are called cyclic operads and cyclic cooperads.

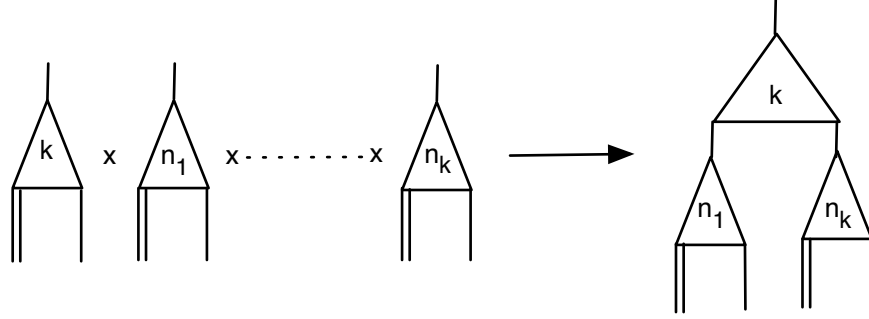
4.1.1 Operads and Cooperads

Basic Definitions

A differential graded (dg-) *operad* \mathcal{O} is a collection of objects $\{\mathcal{O}(n)\}_{n=0}^{\infty}$ in Ch_k with,

1. A composition,

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \rightarrow \mathcal{O}(n_1 + \cdots + n_k)$$



2. An action of the symmetric group Σ_n on $\mathcal{O}(n)$.

3. A unit $1 \in \mathcal{O}(1)$.

These satisfy axioms,

1. The composition is associative. Using the vector notation,

$$\mathcal{O}(\hat{n}_m) = \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m)$$

We have,

$$\begin{array}{ccc} \mathcal{O}(m) \otimes \mathcal{O}(\hat{n}_m) \otimes \mathcal{O}(\hat{k}_{n_1}^1) \otimes \cdots \otimes \mathcal{O}(\hat{k}_{n_m}^{n_m}) & \longrightarrow & \mathcal{O}\left(\sum_{i=1}^m n_i\right) \otimes \mathcal{O}(\hat{k}_{n_1}^1) \otimes \cdots \\ \downarrow & & \downarrow \\ \mathcal{O}(m) \otimes \mathcal{O}\left(\sum_j k_j^1\right) \otimes \cdots \otimes \mathcal{O}\left(\sum_j k_j^{n_m}\right) & \longrightarrow & \mathcal{O}\left(\sum_{i,j} k_j^i\right) \end{array}$$

2. Using $\Sigma_m \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_m} \rightarrow \Sigma_{n_1+\cdots+n_m}$,

$$(\sigma, \tau_1, \dots, \tau_n) \mapsto (\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)})$$

The composition $\mathcal{O}(k) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_k) \rightarrow \mathcal{O}(a_1 + \cdots + a_k)$ is equivariant with respect to the action of $\Sigma_m \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$ on $\mathcal{O}(k) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_k)$ and $\Sigma_{n_1+\cdots+n_m}$ on $\mathcal{O}(a_1 + \cdots + a_k)$.

3. The unit is a unit behaves like a unit.

In all of the cases to follow $\mathcal{O}(1) = k$. Good references for detailed information regarding operads are [May97, MSS02, Vor05].

There is a dual notion, *cooperads* \mathcal{P} are given by a family of Σ_n modules $\{\mathcal{P}(n)\}$ together with (co)compositions,

$$\gamma : \mathcal{P}\left(\sum_{j=1}^k n_j\right) \rightarrow \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)$$

Which satisfy the opposite operad axioms or those obtained by reversing the arrows of all of the above in direct analogy with the relationship between algebras and coalgebras.

Given a chain complex X define the *endomorphism operad*, \mathcal{E}_X , by

$$\mathcal{E}_X(n) = \text{Hom}_{\text{Ch}_k}(X^{\otimes n}, X)$$

This is an operad with $f \in \mathcal{E}_X(n)$, $g_i \in \mathcal{E}_X(m_i)$,

$$\gamma(f, g_1, \dots, g_n) = f(g_1, \dots, g_n)$$

The action of Σ_n is given by permuting the arguments of $f \in \mathcal{E}_X(n)$. A morphism $\varphi : \mathcal{O} \rightarrow \mathcal{O}'$ of operads is given by a collection $\{\varphi_n\}$ of chain maps

$$\varphi_n \in \text{Hom}_{\mathcal{C}}(\mathcal{O}(n), \mathcal{O}'(n))$$

That satisfy,

1. The φ_n commute with operadic composition,

$$\begin{array}{ccc} \mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m) & \longrightarrow & \mathcal{O}\left(\sum_{i=1}^m n_i\right) \\ \varphi_m \otimes \varphi_{n_1} \cdots \otimes \varphi_{n_m} \downarrow & & \varphi_{\sum_i n_i} \downarrow \\ \mathcal{O}'(m) \otimes \mathcal{O}'(n_1) \otimes \cdots \otimes \mathcal{O}'(n_m) & \longrightarrow & \mathcal{O}'\left(\sum_{i=1}^m n_i\right) \end{array}$$

2. The φ_n are morphisms of Σ_n modules.
3. φ takes units to units.

There is a natural notion of quasi-isomorphism between differential graded operads. Two operads $\mathcal{O}_1, \mathcal{O}_2$ are *quasi-isomorphic* if there is a morphism $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, $\varphi = \{\varphi_n\}$ such that the induced maps on homology $(\varphi_n)_*$ are isomorphisms for all n .

$$(\varphi_n)_* : H_*(\mathcal{O}_1(n); k) \rightarrow H_*(\mathcal{O}_2(n); k)$$

A chain complex X is an *algebra* over an operad \mathcal{O} if there is a morphism of operads $\mathcal{O} \rightarrow \mathcal{E}_X$. If we view the object $\mathcal{O}(n)$ as a space of n -fold operations then an algebra structure on X means maps,

$$\mathcal{O}(n) \otimes X^{\otimes n} \rightarrow X$$

Every dg cooperad \mathcal{P} gives rise to a dg operad \mathcal{O} and vice versa by taking the linear dual. $\mathcal{P}(n) \mapsto \mathcal{O}(n)$ where

$$\mathcal{O}(n)_i = \mathcal{P}(n)_i^*$$

Free Operads

Operads can and usually should be described by trees because $\mathcal{O}(n)$ is to be thought of as a moduli space parameterizing some collection of n -fold operations. Finite dimensional dg operad operations are always, though not uniquely, represented by families of trees modulo relations.

Given an operad \mathcal{O} an *ideal* $I \subset \mathcal{O}$ is a collection of Σ_n equivariant subspaces $I(n) \subset \mathcal{O}(n)$ for each n . If γ is the operad composition map then I satisfies,

$$a \in I \Rightarrow \gamma(\dots, a, \dots) \in I$$

If $I \subset \mathcal{O}$ is an ideal define the *quotient operad* \mathcal{O}/I to be $\{\mathcal{O}(n)/I(n)\}$ for $n \geq 1$ with structure maps induced from \mathcal{O} .

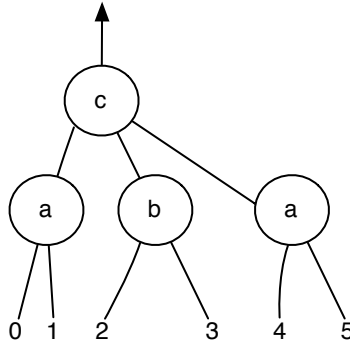
If we let GrSet be the category of graded sets $\{S(n)\}_{n=1}^{\infty}$ and set maps that preserve the grading n . There is a forgetful functor $\text{Forget} : \text{Operad} \rightarrow \text{GrSet}$ that takes any operad $\mathcal{O} = \{\mathcal{O}(n)\}$ in Vect_k to a collection of sets $\mathcal{O}(n)$ in GrSet . The *free operad* on $\{S(n)\}$, $\text{Free}(\{S(n)\})$ is adjoint to the forgetful functor,

$$\text{Hom}_{\text{Operad}}(\text{Free}(\{S(n)\}), \mathcal{P}) = \text{Hom}_{\text{GrSet}}(\{S(n)\}, \text{Forget}(\mathcal{P}))$$

Given a graded collection of sets $\{S(n)\}$ for $n \geq 1$ the free operad is given explicitly by,

$$\text{Free}(S)(n) = \bigoplus_{n\text{-Tree } T} k \cdot \text{Hom}_V(V(T), S)$$

which is the sum over all rooted planar trees with $[n]$ -labelled leaves assigning to each vertex v an element of $S(m)$ if m is the valence $\text{val}(v)$.



An element of $\text{Free}(S)(5)$ if $S(2) = \{a, b\}$ and $S(3) = \{c\}$

The composition of trees is given by grafting boundary edges. The symmetric group Σ_n acts on the $[n]$ -labelling and there is a trivial tree consisting of a vertex which acts as identity.

Essential Operads

The key examples of operads to follow are now defined using the material of the previous section. Let $S(2)$ be a set with one element and $\#S(n) = 0$ if $n \neq 2$. If R be the ideal of $\text{Free}(\{S(n)\})$ generated by the Σ_n orbits of the following relations,

From the first we obtain the commutative (C) and from the second the associative (A) operads. The Jacobi (IHX) and anti-symmetry relations,

give the Lie operad. An algebra over C is an associative commutative algebra, over A is an associative algebra and an algebra over L is a Lie algebra.

In what follows we will need operations on chain complexes generated by differential graded operads. The most important examples of these operads are the homotopy versions of the operads above.

If $S(n) = \{m_n\}$ for $n \geq 1$ then,

$$A_\infty = \text{Free}(\{S(n)\})$$

Where the degree of the corolla labelled by m_n is set in $n - 2$. We specify the boundary as follows,

$$\partial m_n(1, \dots, n) = \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} m_i(1, \dots, m_j(s+h+1, \dots, s+h+j+1), \dots, n)$$

This can be visualized as a signed sum over all ways to compose two generators of lower order. It is the cellular boundary of a Stasheff associahedron K_n [MSS02].

If we view A as an operad of chain complexes situated in degree 0 then there is a morphism of operads $\alpha : A_\infty \rightarrow A$, given by $\alpha(m_2) = m_2$ and $\alpha(m_j) = 0$ if $j \neq 2$ which is a quasi-equivalence. Every operad has an associated homotopy version which can be determined by building minimal model for the operad in question [Mar04].

The L_∞ operad is the free operad on antisymmetric corolla. If $S(n) = \{l_n\}$ for $n \geq 1$ then,

$$L_\infty = \text{Free}(\{S(n)\})/I$$

Where the degree of l_n is $n-2$ and $I = \{l_n(\dots, i, i+1, \dots) = -l_n(\dots, i+1, i, \dots)\}$ is the antisymmetry relation.

A (p, q) shuffle $\sigma \in \text{Sh}(p, q)$ is a permutation $\sigma \in \Sigma_{p+q}$ which satisfies,

$$\sigma(1) < \sigma(2) < \dots < \sigma(p) \quad \sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$$

We specify the boundary as follows,

$$\partial l_n(1, \dots, n) = \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{\sigma \in \text{Sh}(i,j)} \text{sgn}(\sigma) (-1)^{j(i-1)} l_i(l_j(\sigma(1), \dots, \sigma(j)), \sigma(j+1), \dots, \sigma(n))$$

If \mathfrak{g} is a semi-simple Lie algebra, setting $\partial = 0$, $l_2 = [-, -]$ and $l_n = 0$ for all $n \geq 2$ an example of an L_∞ algebra. The Quillen algebra associated to a manifold, $\pi_*(X) \otimes \mathbb{Q}$ is an L_∞ algebra with bracket the Whitehead product and higher products related to higher Whitehead products.

The homotopy commutative associative or C_∞ operad is given by a quotient of the A_∞ operad by relations generated by shuffles.

If we denote by $m_n(1, \dots, n)$ the corolla m_n labelled by $[n]$ then we add the relations,

$$\sum_{\sigma \in \text{Sh}(i, n-i)} \text{sgn}(\sigma) m_n(\sigma(1), \dots, \sigma(n))$$

for all $1 < i < n$ where $\text{sgn}(\sigma)$ is the sign of a permutation. For instance when $k = 2$ the relation becomes,

$$0 = m_2(a, a') - m_2(a', a)$$

The operad C_∞ is the kernel of the natural map $A_\infty \rightarrow L_\infty$ obtained by extending the map $\varphi : A \rightarrow L$ defined by $[a, b] = ab - ba$ to the map,

$$l_n(1, 2, \dots, n) = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) m_n(\sigma(1), \dots, \sigma(n))$$

The operads introduced above fit together into an exact sequence,

$$0 \rightarrow C_\infty \rightarrow A_\infty \rightarrow L_\infty \rightarrow 0$$

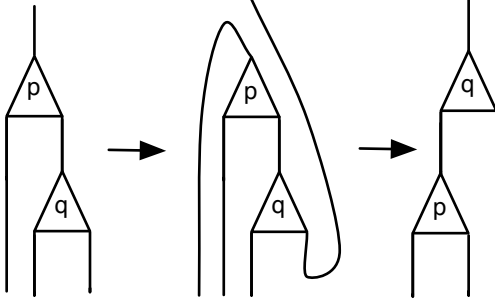
Given a manifold the differential commutative algebra given by the de Rham complex $\Omega^*(M)$ is an example of a C_∞ algebra with trivial m_n for $n \geq 3$.

4.1.2 Cyclic Operads

Cyclic operads are to unrooted trees what operads are to rooted trees. A differential graded *cyclic operad* \mathcal{O} is a dg-operad \mathcal{O} as above together with an extension of the action of Σ_n on $\mathcal{O}(n)$ to an action of Σ_{n+1} on $\mathcal{O}(n)$. This must satisfy the condition that if $\tau_{n+1} \in \Sigma_{n+1}$ is the cycle $(0, 1, 2, \dots, n)$ then,

$$\tau_{m+n-1}(p(1, \dots, 1, q)) = (-1)^{|p||q|} (\tau_n q)(\tau_m p), 1, \dots, 1$$

Pictorially this means we can turn operadic operations about in the plane,



The definition of free operad found in 4.1.1 extends to use labelled n trees which are not rooted so that the extension of the Σ_n action is given by permuting all boundary edges. In the same manner all important examples of operads described in terms of trees have straightforward interpretations as cyclic operads. The extension of the Σ_n action for the free operad is compatible with the relations used to define the commutative, associative and Lie operads.

There is an analogue of the *endomorphism* operad \mathcal{E}_X in the context of cyclic operads. Given a chain complex X choose a non-degenerate inner product $\langle -, - \rangle$ on X and use this to identify $\text{Hom}(X^{\otimes n}, X)$ with $\text{Hom}(X^{\otimes(n+1)}, k)$. The symmetric group Σ_{n+1} then acts by permuting the tensors of $X^{\otimes(n+1)}$.

If \mathcal{O} is a cyclic operad and X is an algebra over the underlying operad then an inner product

$$\langle -, - \rangle : X \otimes X \rightarrow k$$

is *invariant* if the maps $\langle - \rangle_n : \mathcal{O}(n) \otimes X^{\otimes(n+1)} \rightarrow k$ given by,

$$\langle p \otimes x_0 \otimes \cdots \otimes x_n \rangle_n = (-1)^{|x_0||p|} \langle x_0, p(x_1 \otimes \cdots \otimes x_n) \rangle$$

are invariant under the diagonal action of Σ_{n+1} on $\mathcal{O}(n) \otimes A^{\otimes(n+1)}$. For example in the case of L above the inner product $\langle -, - \rangle$ is invariant if $\langle [a, b], c \rangle = \langle a, [b, c] \rangle$ assuming the elements a, b, c are in degree 0. For C_∞ and A_∞ algebras,

$$\langle m_n(x_0, \dots, m_{n-1}), x_n \rangle = (-1)^{(n+1)|x_0| \sum_{i=1}^{n-1} |x_i|} \langle m_n(x_1, \dots, m_n), x_0 \rangle$$

A *morphism* of cyclic operads is the same as a morphism of the underlying operads whose Σ_n module maps respect the extension to Σ_{n+1} . There is an analogous notion of *quasi-isomorphism* of differential graded cyclic operads. An *algebra* X over a cyclic operad \mathcal{O} is an algebra over \mathcal{O} together with a choice of invariant inner product.

For more information concerning cyclic operads and the analogous notion for cyclic cooperads see [GK95, MSS02].

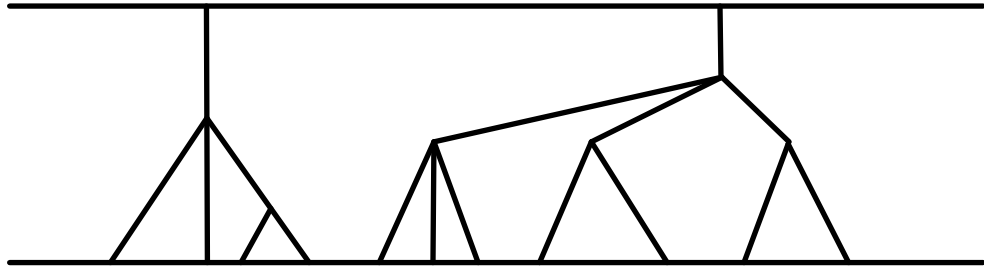
4.2 Relation to Differential Graded Algebra

As previously mentioned the language of differential graded operads is contained in the language of differential graded categories and their representations or modules. We would like then to define a dgsd category \mathcal{O}^b so that the category of h-split \mathcal{O}^b modules is quasi-equivalent to the category of \mathcal{O} algebras. This could be considered the dgsd “enveloping category” for the operad \mathcal{O} .

Given an operad \mathcal{O} in Ch_k with generators $\{S(n)\}$ and relations R as in 4.1. We can define a dgsd category \mathcal{O}^b generated by one object X and morphisms generated by the symbols,

$$s \in \text{Hom}_{\mathcal{O}^b}(X^{\otimes n}, X) \quad \text{for all } s \in S(n)$$

subject to the relations of R and possessing a differential structure the same as that of \mathcal{O} .



An element of $\text{Hom}_{\mathcal{O}^b}(X^{\otimes 11}, X^{\otimes 2})$

By construction the category \mathcal{O}^b includes factorization isomorphisms,

$$\theta_{n,m} = \text{Hom}_{\mathcal{O}^b}(X^{\otimes n}, X^{\otimes m}) \cong \bigotimes_{i=1}^m \text{Hom}_{\mathcal{O}^b}(X^{\otimes n_i}, X) \quad \text{such that} \quad \sum_i n_i = n$$

This construction is functorial. Any morphism $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ of dg operads induces a functor $\varphi^b : \mathcal{O}_1^b \rightarrow \mathcal{O}_2^b$. Given $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ we have $\varphi = \{\varphi_n\}$ that is chain maps $\varphi_n : \mathcal{O}_1(n) \rightarrow \mathcal{O}_2(n)$ so define $\varphi^b(X) = X$ and given $c \in \text{Hom}_{\mathcal{O}_1}(X^{\otimes n}, X)$, $\varphi^b(c) = \varphi_n(c)$. Our φ^b then uniquely extends to a functor via $\theta_{n,m}$.

Since different presentations yield isomorphic operads our construction is observably independent of presentation.

Observation. *The quasi-equivalence class of \mathcal{O}^b as a differential graded symmetric monoidal category in the category of differential graded symmetric monoidal categories is independent of the choice of presentation.*

It then follows from the observation together with Theorem 1 in Chapter 2 that the category of h-split modules of \mathcal{O}^b is a well defined notion.

Lemma 1. *The category of \mathcal{O} -algebras is equivalent to the category of split left \mathcal{O}^b -modules.*

Proof. Any functor $F : \mathcal{O}^b \rightarrow \text{Ch}_k$ identifies the object X with a chain complex $F(X)$ and by split monoidality identifies the object $X^{\otimes m}$ with $F(X)^{\otimes m}$. Consider the action of \mathcal{O}^b on $F(X)$. Using $\theta_{n,m}$, $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_m$ such that $\varphi_i \in \text{Hom}_{\mathcal{O}^b}(X^{\otimes n_i}, X)$ and $\sum_{i=1}^m n_i = n$. Each φ_i is also an element of $\mathcal{O}(n_i)$ and this identification,

$$\begin{array}{ccc} \text{Hom}_{\mathcal{O}^b}(X^{\otimes n}, X) \otimes F(X)^{\otimes n} & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ \mathcal{O}(n) \otimes X^{\otimes n} & \longrightarrow & X \end{array}$$

commutes with the composition in the category \mathcal{O}^b and the operadic composition \mathcal{O} respectively.

□

Later in the paper a certain category of topological field theories will be shown to be quasi-equivalent to the category of h-split left \mathcal{O}^b modules. In order to use the above theorem we require,

Lemma 2. *There is an equivalence of categories between the category of h-split left \mathcal{O}^b -modules and the category of split left \mathcal{O}^b -modules.*

Proof. The equivalence will come from a functor η from h-split to split modules. If F is an h-split \mathcal{O}^b -module define a split module $\eta(F)$ by,

$$\eta(F)(X^{\otimes n}) = F(X)^{\otimes n}$$

Note that since F is h-split there are quasi-isomorphisms

$$\varphi_{X^j} : \eta(F)(X^{\otimes j}) \rightarrow F(X^{\otimes j})$$

By definition $\eta(F)$ is split we need to show that it can be extended to a functor. Each $m_j \in \mathcal{O}(j)$ induces a map,

$$(m_j)_* : F(X)^{\otimes j} \rightarrow F(X)$$

These are natural with respect to the φ_{X^j} and given any $f \in \text{Hom}_{\mathcal{O}^b}(X^{\otimes m}, X^{\otimes n})$ using $\theta_{n,m}$ isomorphisms $f = \theta_{n,m}^{-1}(m_{n_1} \otimes \cdots \otimes m_{n_k})$. So the action of \mathcal{O} can be extended to an action of \mathcal{O}^b giving a unique split \mathcal{O}^b module $\eta(F)$ quasi-equivalent to the h-split \mathcal{O}^b module F via $\{\varphi\}$. \square

In order to reduce some rather complicated complexes to simpler ones later we will need,

Lemma 3. *If \mathcal{O}_1 and \mathcal{O}_2 are quasi-isomorphic operads then the enveloping categories \mathcal{O}_1^b and \mathcal{O}_2^b are quasi-equivalent.*

$$\mathcal{O}_1^b \cong \mathcal{O}_2^b$$

Proof. If $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, $\varphi = \{\varphi_n\}$ is a quasi-isomorphism then φ induces a quasi-isomorphism in the sense previously considered between the dgsms categories since the induced maps on chain complexes of morphisms,

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_1^b}(X^{\otimes n}, X^{\otimes m}) & \xrightarrow{\varphi_*} & \mathrm{Hom}_{\mathcal{O}_2^b}(X^{\otimes n}, X^{\otimes m}) \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}_{\mathcal{O}_1^b}(X^{\otimes n_i}, X) & \xrightarrow{\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k}} & \bigoplus_i \mathrm{Hom}_{\mathcal{O}_2^b}(X^{\otimes n_i}, X) \\
\cong \downarrow & & \cong \downarrow \\
\bigoplus_i \mathcal{O}_1^b(n_i) & \xrightarrow{\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k}} & \bigoplus_i \mathcal{O}_2^b(n_i)
\end{array}$$

induce isomorphisms on homology. □

It follows from Theorem 1 of Chapter 2 that the categories of modules of quasi-equivalent operads are quasi-equivalent.

Constructions For Cyclic Operads

Cyclic operads \mathcal{O} in Ch_k given by generators $\{S(n)\}$ and relations R as above also yield dgsms categories \mathcal{O}^b with one object X and morphisms generated by the symbols,

$$s \in \mathrm{Hom}_{\mathcal{O}^b}(X^{\otimes n}, X) \quad \text{for all } s \in S(n)$$

together with cap and cup morphisms corresponding to an invariant inner product and its dual,

$$\langle -, - \rangle \in \mathrm{Hom}_{\mathcal{O}^b}(X \otimes X, k) \quad \text{and} \quad \langle -, - \rangle^* \in \mathrm{Hom}_{\mathcal{O}^b}(k, X \otimes X)$$

Or pictorially,



subject to the relations generated by R and relations involving the caps and cups.



and possessing a differential structure the same as that of \mathcal{O} . Notice that the addition of a cap and a cup gives us much larger morphism spaces. $\text{Hom}_{\mathcal{O}^b}(X^{\otimes n}, X^{\otimes m})$ is now the space of graphs labelled by the generators in S .

Theorem 6. *The category of \mathcal{O} algebras with invariant inner product is equivalent to the category of split left \mathcal{O}^b -modules.*

The proof of this is essentially the same as the one for operads the inner product and its dual satisfy the graphical relations above. (If the operad \mathcal{O} is the cobar construction on a differential graded cooperad \mathcal{P} then the morphisms of the category \mathcal{O}^b are those of the Feynmann transformation or graph complex $\mathcal{G}(\mathcal{P}^*)$ – see later).

It follows from the theorem that,

Observation. *If \mathcal{O}_1 and \mathcal{O}_2 are isomorphic operads then the category of modules over enveloping categories \mathcal{O}_1^b and \mathcal{O}_2^b are isomorphic,*

$$\mathcal{O}_1^b\text{-mod} \cong \mathcal{O}_2^b\text{-mod}$$

4.3 Bar and Cobar Operators on Operads

Following Kontsevich's seminal work [Kon94], Ginzburg and Kapranov developed a duality theory for operads [GK94, Gin]. This was later extended to cyclic operads by Getzler and Kapranov and reduced to a generalization of the Bar and Cobar construction for algebras first defined by Eilenberg and MacLane by Getzler and Jones [GK95, GJ]. As above we won't give a complete exposition of their theory, but only mention what is necessary for later. The reader is warned that the functors Bar and Cobar are formulated in slightly different ways in different places. What comes later here will be consistent with the definitions given below.

The Bar construction is a functor which takes a dg operad \mathcal{P} to a dg cooperad $\text{Bar}(\mathcal{P})$ while the Cobar construction is a functor taking a dg cooperad \mathcal{O} to a dg operad $\text{Cobar}(\mathcal{O})$.

When we defined the free operad $\text{Free}(S)$ on a system of sets $S = \{S(n)\}$ above each vertex of valence k the tree was labelled by a set element $x \in S(k)$. However, if one is careful then beginning with a cyclic dg operad or cyclic dg cooperad instead of S one can again produce a cyclic dg cooperad or cyclic dg operad. These (co)operads will be free on a set of corolla with vertices labelled by (co)operad elements. Geometrically this simple idea can be interpreted as an instance of Verdier duality [GK94, LV08].

Orientations and Preliminaries

If V_* is a graded vector space then we define a different graded vector space, the j -fold (de)suspension $V[j]_*$ by,

$$V[j]_i = V_{i+j}$$

An *orientation* of a graded vector space W of dimension $n = \dim(W)$ is a non-zero vector in the alternating algebra

$$\det(W) = \Lambda^n(W)[-n]$$

We also define the inverse,

$$\det(W)^* = \Lambda^n(W)[n]$$

If S is a set then we orient S using $\det(S) = \det(k\langle S \rangle)$. Two orientations are equivalent if they are positive scalar multiples of each other. An *orientation of a graph G* is defined as follows,

$$\det(G) = \det(E(G)) \otimes \det(k^O) \otimes \det(H_0(G)) \otimes \det(H_1(G))^*[O - \chi]$$

where,

1. $E(G)$ are the internal edges of G . Those edges which are not part of the incoming or outgoing boundary ∂G .
2. O is the number of outgoing boundary edges.
3. $\chi = \chi(G)$ is the Euler characteristic of G .

This amounts to an ordering of the internal and outgoing boundary edges together with an ordering of the connected components of G and the cycles. Everything is packaged together and then placed in the degree $\#E(G)$.

Observation. *A short exact sequence of vector spaces,*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

yields a canonical isomorphism of orientations,

$$\det(B) \cong \det(A) \otimes \det(C)$$

Given two graphs G_0 and G_1 then grafting one of the incoming edges of G_1 to an outgoing edge of G_0 forming $G_0\#G_1$ there is an isomorphism,

$$\det(G_0) \otimes \det(G_1) \rightarrow \det(G_0\#G_1)$$

This is given by splitting the Mayer-Vietoris sequence below into short exact sequences any applying the observation above together with the additivity of the set of internal plus outgoing boundary edges under grafting.

$$\begin{aligned} 0 \rightarrow H_1(G_1) \otimes H_1(G_0) \rightarrow H_1(G_0\#G_1) \rightarrow H_0(G_0 \cap G_1) \rightarrow \\ H_0(G_1) \otimes H_0(G_0) \rightarrow H_0(G_0\#G_1) \rightarrow 0 \end{aligned}$$

This definition also behaves well under disjoint union of graphs,

$$\det(G_0 \amalg G_1) \cong \det(G_0) \otimes \det(G_1)$$

This orientation convention for graphs agrees with the standard graph complex definitions when restricted to connected graphs without boundary. It differs slightly from the differential graded dual construction of Ginzburg and Kapranov inasmuch as trees in the dg dual have oriented internal and incoming edges and in our case trees have oriented internal and outgoing edges.

This orientation convention is equivalent one obtained by ordering the connected components, ordering the vertices of the graph and ordering the half-edges of each edge. This later condition can be thought of as placing an arrow on each edge so that reversing the direction of any arrow changes the sign of the orientation.

Given a set S and a cyclic dg (co)operad \mathcal{O} then we can define a *labelling* of S by \mathcal{O} using the coinvariants trick,

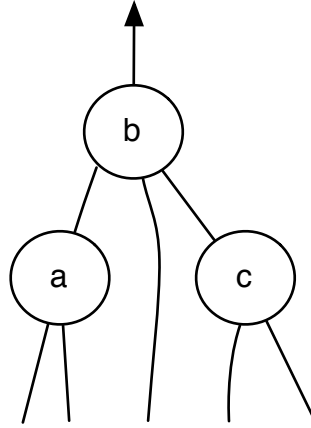
$$\mathcal{O}(S) = (\mathcal{O}(n) \times \text{Bij}([n], S))_{\Sigma_{n+1}}$$

where $\text{Bij}([n], S)$ is the set of bijections from S to $[n] = \{0, 1, \dots, n\}$ and Σ_{n+1} acts diagonally.

If T is a tree then we can define a *labelling* of T by \mathcal{O} assigning each vertex v an element of $\mathcal{O}(H(v))$,

$$\mathcal{O}(T) = \bigotimes_{v \in V(T)} \mathcal{O}(H(v))$$

An example of a tree with two quadrivalent vertices and one trivalent vertex labelled by a choice of $a \otimes b \otimes c \in \mathcal{O}(4)^{\otimes 2} \otimes \mathcal{O}(3)$ is pictured below,



Every edge contraction $c : T \rightarrow T/e$ induces a map of labellings. If we denote by e the vertex obtained by the edge collapse and by v and w the two identified end points then there is an operad map,

$$\mathcal{O}(\text{val}(v)) \otimes \mathcal{O}(\text{val}(w)) \rightarrow \mathcal{O}(\text{val}(e))$$

and a cooperad map,

$$\mathcal{P}(\text{val}(e)) \rightarrow \mathcal{P}(\text{val}(v)) \otimes \mathcal{P}(\text{val}(w))$$

these can be extended to maps $c_* : \mathcal{O}(T) \rightarrow \mathcal{O}(T/e)$ and $c^* : \mathcal{P}(T/e) \rightarrow \mathcal{P}(T)$ by tensoring the above with the identity map.

The Bar and Cobar constructions

The *Bar construction* $\text{Bar}(\mathcal{O})$ of a cyclic differential graded operad \mathcal{O} is the dg cooperad which in degree n is given by the complex of labelled unrooted n -trees with contracting differential.

$$\text{Bar}(\mathcal{O})(n) = \bigoplus_{n\text{-Tree } T} \mathcal{O}(T) \otimes \det(T)$$

Where the degree of a labelled tree is determined by the orientation convention given above. Concretely this means,

$$\text{Bar}(\mathcal{O})(n) = \bigoplus_{\substack{n\text{-Tree } T \\ |T|=1}} \mathcal{O}(T) \otimes \det(T) \leftarrow \cdots \leftarrow \bigoplus_{\substack{n\text{-Tree } T \\ |T|=n-1}} \mathcal{O}(T) \otimes \det(T)$$

The *Cobar construction* $\text{Cobar}(\mathcal{P})$ of a cyclic differential graded cooperad \mathcal{P} is the dg operad which in degree n is given by the complex of labelled unrooted n -trees with expanding differential.

$$\text{Cobar}(\mathcal{P})(n) = \bigoplus_{n\text{-Tree } T} \mathcal{P}(T) \otimes \det(T)$$

Where the degree of a labelled tree is determined by the orientation convention given above. Concretely,

$$\text{Cobar}(\mathcal{P})(n) = \bigoplus_{\substack{n\text{-Tree } T \\ |T|=1}} \mathcal{P}(T) \otimes \det(T) \rightarrow \cdots \rightarrow \bigoplus_{\substack{n\text{-Tree } T \\ |T|=n-1}} \mathcal{P}(T) \otimes \det(T)$$

In the formulas above $|T|$ is the number of internal vertices of T and the complex is graded so that the term spanned by trees with 1 internal vertex is situated in degree 0.

The differential δ in either case can be described by its matrix elements,

$$(\delta)_{T,T'} : \mathcal{O}(T) \otimes \det(T) \rightarrow \mathcal{O}(T') \otimes \det(T')$$

$$(\delta)_{T',T} : \mathcal{P}(T') \otimes \det(T') \rightarrow \mathcal{P}(T) \otimes \det(T)$$

If T' is not isomorphic to T/e for some edge $e \in T$ then we define $(\delta)_{T',T} = 0$. Otherwise let $c : T \rightarrow T' \cong T/e$ so that if $c_* : \mathcal{O}(T) \rightarrow \mathcal{O}(T')$ or $c^* : \mathcal{P}(T') \rightarrow \mathcal{P}(T)$ are the induced maps, δ is given by

$$(\delta)_{T',T} = c_* \otimes p_e$$

$$(\delta)_{T,T'} = c^* \otimes p^e$$

If collapsing the edge e then the map of orientations $p_e : \det(T') \rightarrow$

$$p_e(y_0 \wedge \cdots \wedge e \wedge \cdots \wedge y_n) = y_0 \wedge \cdots \wedge \hat{e} \wedge \cdots \wedge y_n$$

and orientation p^e for the expanding differential is defined analogously.

In either case if the operad \mathcal{O} or cooperad \mathcal{P} has a non-trivial differential then the total differential is the sum of the differential defined above together with the original internal differential.

The cocomposition for the cooperad $\text{Bar}(\mathcal{O})$ is given by cutting a given \mathcal{O} labelled tree apart into compositions of \mathcal{O} labelled trees. The composition for the operad $\text{Cobar}(\mathcal{P})$ is given by grafting the incoming boundary edge of an \mathcal{P} labelled tree to an outgoing boundary edge of an \mathcal{P} labelled tree and eliminating the resulting bivalent vertex. If the (co)composition is defined in this manner the differential defined above satisfies the (co)Leibniz rule. Notice that $\text{Cobar}(\mathcal{P})$ is generated by \mathcal{P} -labelled corolla.

If we denote by $\text{Operad}(\text{Ch}_k)$ the category of differential graded operads and by $\text{Cooperad}(\text{Ch}_k)$ the category of differential graded cooperads

$$\text{Hom}_{\text{Cooperad}(\text{Ch}_k)}(\text{Bar}(\mathcal{O}), \mathcal{P}) \cong \text{Hom}_{\text{Operad}(\text{Ch}_k)}(\mathcal{O}, \text{Cobar}(\mathcal{P}))$$

See Chapter 3 [GK94]. The maps given by the inclusions $\mathcal{O} \rightarrow \text{Bar}(\text{Cobar}(\mathcal{O}))$ and $\mathcal{P} \rightarrow \text{Cobar}(\text{Bar}(\mathcal{P}))$ are quasi-isomorphisms of cooperads and operads respectively.

Ginzburg and Kapranov originally defined the differential graded dual $D(\mathcal{O})$ of a dg operad \mathcal{O} as $D(\mathcal{O}) = \text{Bar}(\mathcal{O})^*$. They proved that it is an exact endofunctor on the category $\text{Operad}(\text{Ch}_k)$ and that $D(D(\mathcal{O})) \simeq \mathcal{O}$ which is equivalent to the adjunctions above.

Observation.

$$\text{Bar}(C)^* = L_\infty$$

Since the commutative operad C has a unique n -fold composition for each n vertices of trees are labelled uniquely. So that the chain complexes that make up $\text{Bar}(C)^*$ are spanned by trees that are compositions of n -corolla which satisfy the antisymmetry relation due to the orientation convention of the Bar construction. As noted before and explained in Chapter 8 our convention agrees with orienting by an ordering of the edges of the trees. It is easy then to see that changing the order of adjacent edges changes the sign by minus one.

Observation.

$$\text{Cobar}(\text{Bar}(C)) \simeq \text{Cobar}(L^*)$$

This follows from the previous observation because $L_\infty \simeq L$ by construction.

Observation. $C_\infty \simeq \text{Bar}(L)^*$

This follows from the first observation by dualizing and using the adjunction.

Cobar and Enveloping Categories

It will be useful to transform enveloping categories by quasi-isomorphisms later. This can be done sometimes,

Lemma 4. *If \mathcal{P} and \mathcal{Q} are quasi-isomorphic cooperads then $\text{Cobar}(\mathcal{P})^b \cong \text{Cobar}(\mathcal{Q})^b$.*

Proof. The morphisms of $\text{Cobar}(\mathcal{P})^b$ and $\text{Cobar}(\mathcal{Q})^b$ are graph complexes so this follows from the usual argument. Filter the cone complex of the induced map by the number of edges. The E^1 page of the spectral sequence associated to the filtration is the induced map which was a quasi-isomorphism by assumption showing that the cone complex is contractible and thus that the induced map is a quasi-isomorphism. See [GK98, LV08]. \square

The same argument works for the C_∞ operad.

Corollary 3.

$$C_\infty^b \cong \text{Cobar}(\text{Bar}(C))^b$$

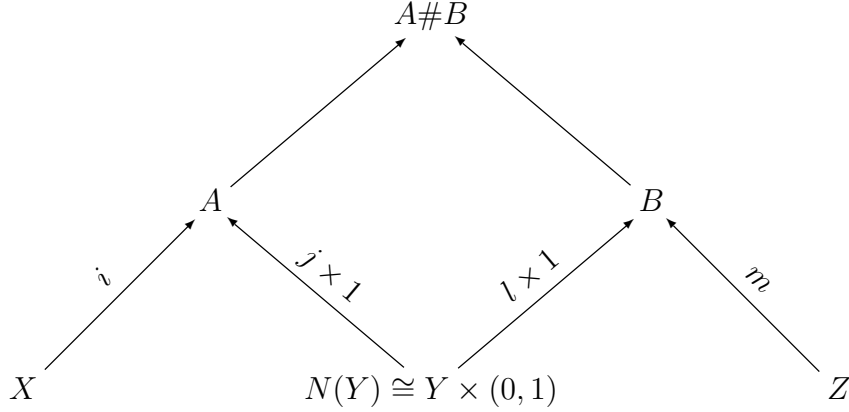
5 Cobordism Categories

Let M be a smooth manifold with boundary and $\text{Diff}(M)$ the group of orientation preserving diffeomorphisms of M . Let $\text{Diff}(M, \partial) \subset \text{Diff}(M)$ be the subgroup of diffeomorphisms which fix a regular neighborhood of the boundary ∂M . The *mapping class group* $\Gamma(M, \partial)$ of M is $\pi_0 \text{Diff}(M, \partial)$, that is the group of connected components of $\text{Diff}(M, \partial)$.

5.1 3-dimensional Cobordism Categories

In this section we will define a dgsM category \mathcal{M} called the *differential graded cobordism* category or dg cobordism category.

We begin with the definition of a symmetric monoidal category with objects disjoint unions of orientable labelled surfaces called the *cobordism category* which will be denoted \mathcal{N} . A morphism $M' \in \text{Hom}_{\mathcal{N}}(X, Y)$ is a triple $M' = (M, i, j)$ where M is a diffeomorphism class (rel ∂) of smooth oriented 3-manifold whose boundary $\partial M = I \amalg J$ splits into disjoint union of incoming I and outgoing J surfaces the orientation of which is induced by that of M and $i : X \rightarrow I$ and $j : Y \rightarrow J$ are choices of orientation preserving and orientation reversing embeddings from objects $X, Y \in \text{Ob}(\mathcal{N})$ into M which restrict to diffeomorphisms on I, J respectively. Given $A' = (A, i, j) \in \text{Hom}(X, Y)$ and $B' = (B, l, m) \in \text{Hom}(Y, Z)$ define $C' = B' \circ A' \in \text{Hom}(X, Z)$ as follows, if $A \# B = A \amalg B / \sim$ where $x \sim y$ if $j(y) = l(y)$ for $y \in N(Y)$, a regular neighborhood of Y in A or B , then $C' = (A \# B, i, m)$. Associativity follows from the local nature of the gluing composition.



We would like to define \mathcal{M} as the category with objects and morphisms given by,

$$\begin{aligned} \text{Ob}(\mathcal{M}) &= \text{Ob}(\mathcal{N}) \\ \text{Hom}_{\mathcal{M}}(X, Y) &= C_*(B\Gamma(\text{Hom}_{\mathcal{N}}(X, Y), \partial); k) \end{aligned}$$

The category of singular chains on the classifying space of the mapping class groups of \mathcal{N} .

We apply these functors to the triplets above in the most straightforward way that is, if $M' = (M, i, j)$ is a morphism in \mathcal{N} then $\Gamma(M', \partial) = (\Gamma(M, \partial), i, j)$ and gluing of triples in \mathcal{N} as defined above induces a composition. Specifically, if $A' = (A, i, j) \in \text{Hom}_{\mathcal{N}}(X, Y)$, $B' = (B, l, m) \in \text{Hom}_{\mathcal{N}}(Y, Z)$ then given $(\phi, i, j) \in \Gamma(A', \partial)$ and $(\psi, l, m) \in \Gamma(B', \partial)$, by requiring that group elements fix a neighborhood of the boundary it follows that there exists a map $\psi\#\phi : A\#B \rightarrow A\#B$ induced by $(\psi, \phi) : A \amalg B \rightarrow A \amalg B$ so that $(\psi\#\phi, i, m)$ is a morphism in $\text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Z)$. The local nature of the gluing in this construction also implies associativity of the composition in the category.

Let $\Gamma(M', \partial) \in \text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Y)$ then we say that $g \in \Gamma(M', \partial) = (\Gamma(M, \partial), i, j)$ if $g \in \Gamma(M, \partial)$. Such elements form a group so that the functor B can be applied to $\text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Y)$. We then apply $C_*(-; k)$ to these classifying spaces. As discussed in section 2.2 both B and $C_*(-; k)$ are monoidal.

Notice that $\mathcal{N} = H_0(\mathcal{M}; k)$, so that we may think of \mathcal{M} as a choice of chain level representative for \mathcal{N} .

Definition. A *3-dimensional topological field theory* or TFT is a h-split left \mathcal{M} -module.

The category \mathcal{M} will be explored in the future. This paper will illustrate the relationship between several subcategories of \mathcal{M} : the *open* category \mathcal{O} , the *closed* category \mathcal{C} , and the *open-closed* category \mathcal{OC} related to homotopy algebras and their homology.

5.2 Open, Closed and Open-Closed Subcategories

A subcategory $\langle \mathcal{S} \rangle$ of \mathcal{M} is *generated by* a collection \mathcal{S} of compact orientable 3-manifolds with boundary if $\langle \mathcal{S} \rangle$ is the subcategory of \mathcal{M} with the objects only those surfaces found as boundaries in \mathcal{S} and morphisms equal to $C_*(B\Gamma(X, \partial); k)$ where X is any possible composition of manifolds from \mathcal{S} in \mathcal{N} .

The categories below will use *doubled handle bodies* with sphere and torus boundary as generating manifolds. Let,

$$M_{(g,e,t)} = \#^g S^1 \times S^2 \#^e D^3 \#^t S^1 \times D^2$$

be the connected sum of g copies of $S^1 \times S^2$, e copies of D^3 and t copies of $S^1 \times D^2$. Notice that each D^3 summand introduces a boundary 2-sphere and each $S^1 \times D^2$ introduces a boundary torus. The boundary of $M_{(g,e,t)}$ consists of e 2-sphere and t tori. This is the same as starting with a doubled handle body of genus g , $\#^g S^1 \times S^2$ then removing e solid three balls and t solid tori.

The *open-closed category* \mathcal{OC} is the subcategory of \mathcal{M} that has morphisms consisting of $\mathcal{S} = \{M_{(g,e,t)}\}$. Such that there is always incoming and outgoing boundary. If $t = 0$ then $e \geq 2$ and if $e = 0$ then $t \geq 2$. In particular, there is no morphism from the empty set to a sphere or torus.

The set \mathcal{S} is closed under composition. If we write $M_{(g,e,t)} = M_{(g,i+j,n+m)}$ for a manifold $M \in \mathcal{S}$ of genus g with i incoming spheres, n incoming tori, j outgoing spheres and m outgoing tori. Then

$$M_{(g,i+j,n+m)} \# M_{(g',j+l,m+r)} = M_{(g+g'+j-1+m,i+l,n+r)}$$

Since gluing the j spheres together adds $j - 1$ factors of $S^1 \times S^2$ and gluing the m tori together adds an additional m factors of $S^1 \times S^2$.

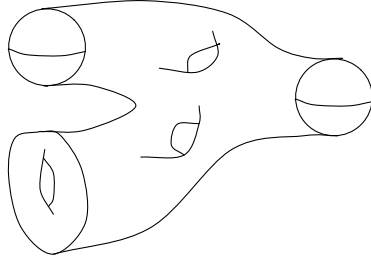
The objects of \mathcal{OC} are spheres A, A' and tori B, B' . The morphisms,

$$\text{Hom}_{\mathcal{OC}}(A \amalg B, A' \amalg B') \subset \text{Hom}_{\mathcal{M}}(A \amalg B, A' \amalg B')$$

are generated by the manifolds,

$$\coprod_{\substack{g \geq 0 \\ i = \#A, j = \#A' \\ n = \#B, m = \#B'}} M_{(g,i+j,n+m)}$$

For instance,



$M_{(2,2,1)}$

The *open category* \mathcal{O} is defined to be the subcategory of \mathcal{OC} whose objects are spheres and whose morphisms are generated by the spaces $M_{(g,i+j,0)}$. Similarly the *closed category* \mathcal{C} is the subcategory of \mathcal{OC} whose objects are tori and whose morphisms are generated by the spaces $M_{(g,0,n+m)}$.

In each case, the composition is induced from gluing along boundary and identity morphisms are added as above.

Definition. In this paper an *open-closed* topological field theory is a h-split left \mathcal{OC} -module. An *open* topological field theory is a h-split left \mathcal{O} -module. A *closed* topological field theory is a h-split left \mathcal{C} -module.

Units

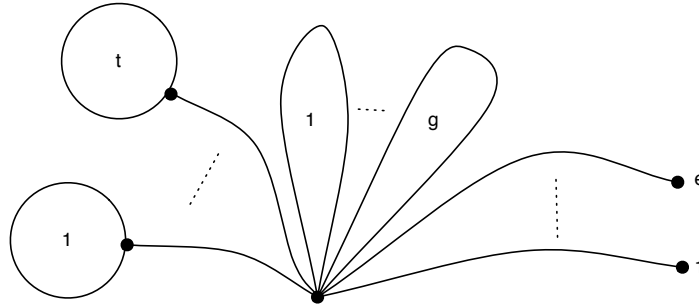
The unit in the open category is represented by $S^2 \times I$. The classifying space of the mapping class group of this manifold relative to the boundary is a 0-cell which can be glued onto the ends of boundary labelled graphs in the obvious way.

Note also that by construction we have forbidden the morphism from the empty manifold to the 2-sphere and from the 2-sphere to the empty manifold.

6 Homotopy Equivalence Groups

Recall from Chapter 3 that by a *geometric graph* G we mean a 1-dimensional CW complex consisting of vertices $V(G)$ and edges $E(G)$.

Define a *boundary torus* or *balloon* to be the graph formed from two edges with both ends of one edge glued to one end of the other. If $g, e, t \in \mathbb{Z}_+$ then define a *spaghetti* to be the graph $G_{(g,e,t)}$ consisting of a wedge of g circles with e edges and t tori glued to the one base vertex along the end of their free edge.



The *base vertex* v of $G_{(g,e,t)}$ is the 0-cell on which the first edge is attached. Let $\text{Htpy}(G_{(g,e,t)}, \partial)$ be the space self-homotopy equivalences of $G_{(g,e,t)}$ that,

1. Fix the e edges
2. Fix the t loops of the boundary tori pointwise.
3. Does not identify the base vertices of any two boundary tori.

The compatibility between condition 3 and the Cobar construction will become apparent in the last chapter.

This is given the compact open topology. Let $H_{(g,e,t)} = \pi_0 \text{Htpy}(G_{(g,e,t)}, \partial)$ be the group of path components of the space of self-homotopy equivalences described above. This is a group see Proposition 0.19 [Hat02].

When we write $G_{(g,e,t)}$ as $G_{(g,i+o,a+b)}$ we mean that the number of entering and exiting edges $i = \# \text{In}(G)$, $o = \# \text{Out}(G)$ and $i + o = e$. The number of entering and exiting tori is $a = \# \text{Tin}(G)$, $b = \# \text{Tout}(G)$ and $a + b = t$.

A graph $G_{(g,i+o,a+b)}$ can be labelled as above. If $[n]$ is the set $\{1, \dots, n\}$ then a *boundary labelling* is a choice of homeomorphisms, $i_H : [\# \text{In}(G)] \times I \rightarrow \text{In}(G)$ and $o_H : [\# \text{Out}(G)] \times I \rightarrow \text{Out}(G)$. So that the $i \times [0, 1]$ is mapped homeomorphically onto the i th incoming or outgoing edge with 0 sent to the boundary vertex. For the balloons we use

$$a_H : [\# \text{Tin}(G)] \times S^1 \rightarrow \text{Tin}(G) \quad \text{and} \quad b_H : [\# \text{Tout}(G)] \times S^1 \rightarrow \text{Tout}(G)$$

If $S^1 = [0, 2\pi)$ we require that the $a_H(i, 0)$ and $b_H(i, 0)$ are the base vertices of the boundary torus.

Since the homotopy equivalences are required to fix the edges and boundary tori they are compatible with our definition of labelling and we can build a symmetric monoidal category \mathcal{H} enriched over Group. The objects $\text{Ob}(\mathcal{H})$ of \mathcal{H} are generated by two objects e and t the object e representing the labelled edges and the object t representing the boundary tori. The morphisms self-homotopy equivalences of boundary labelled graphs fixing boundary elements,

$$\text{Hom}_{\mathcal{H}}(e^{\otimes i} \otimes t^{\otimes j}, e^{\otimes k} \otimes t^{\otimes l}) = \coprod_g H_{(g,i+k,j+l)}$$

There are no morphisms between empty objects of any kind. The composition in \mathcal{H} can be described as follows. Given $\varphi \in \text{Hom}_{\mathcal{H}}(e^{\otimes i} \otimes t^{\otimes j}, e^{\otimes k} \otimes t^{\otimes l})$ and $\psi \in \text{Hom}_{\mathcal{H}}(e^{\otimes k} \otimes t^{\otimes l}, e^{\otimes r} \otimes t^{\otimes n})$ we've $\varphi \in H_{(g,i+k,j+l)}$ and $\psi \in H_{(g',k+r,l+n)}$. Making a choice of equivalence classes we have maps $\varphi : G_{(g,i+k,j+l)} \rightarrow G_{(g,i+k,j+l)}$ and $\psi : G_{(g',k+r,l+n)} \rightarrow G_{(g',k+r,l+n)}$ which preserve the boundary labelling. Gluing the

outgoing edges and tori of $G_{(g,i+k,j+l)}$ to the incoming edges and tori of $G_{(g',k+r,l+n)}$ gives the graph $G_{(g+g'+k-1+l,i+r,j+n)}$ and our homotopy equivalences can be glued to give an equivalence $\varphi\#\psi : G_{(g+g'+k-1+l,i+r,j+n)} \rightarrow G_{(g+g'+k-1+l,i+r,j+n)}$.

With any continuous variation of φ or ψ within their respective path components the graph $\varphi\#\psi$ varies continuously within the corresponding path component of $\text{Htpy}(G_{(g+g'+k-1+l,i+r,j+n)}, \partial)$. So that $(\varphi, \psi) \mapsto \varphi\#\psi$ yields a well-defined composition law,

$$H_{(g,i+k,j+l)} \times H_{(g',k+r,l+n)} \rightarrow H_{(g+g'+k-1+l,i+r,j+n)}$$

Allowing the genus to vary these maps define the composition law for \mathcal{H} ,

$$\text{Hom}_{\mathcal{H}}(e^{\otimes i} \otimes t^{\otimes j}, e^{\otimes k} \otimes t^{\otimes l}) \times \text{Hom}_{\mathcal{H}}(e^{\otimes k} \otimes t^{\otimes l}, e^{\otimes r} \otimes t^{\otimes n}) \rightarrow \text{Hom}_{\mathcal{H}}(e^{\otimes i} \otimes t^{\otimes j}, e^{\otimes r} \otimes t^{\otimes n})$$

It follows that there is a monoidal category $B\mathcal{H}$ enriched over Top with the same objects and morphism spaces equal to classifying spaces of the groups defined above.

$$\text{Hom}_{B\mathcal{H}}(e^{\otimes i} \otimes t^{\otimes j}, e^{\otimes k} \otimes t^{\otimes l}) = \coprod_g BH_{(g,i+k,j+l)}$$

6.1 A Theorem of Hatcher Vogtmann and Wahl

The theorem below appears in the papers of Hatcher, Vogtmann and Wahl stemming from Hatcher's work on the homotopy type of the diffeomorphism group of $S^1 \times S^2$ and Vogtmann's study of Outer Space [CV86]. The synthesis of these ideas has recently led to homological stability results for 3-manifolds [HV04, HW05]. It can be seen as a generalization earlier work by Laudenbach [Lau74].

The mapping class groups in our construction will differ from those considered in the references above by requiring that group elements fix a regular neighborhood of the boundary as in the construction of the cobordism categories of Chapter 5. As such they will be subgroups $\Gamma(M_{(g,e,t)}, \partial) \subset \Gamma(M_{(g,e,t)})$ generated by the same

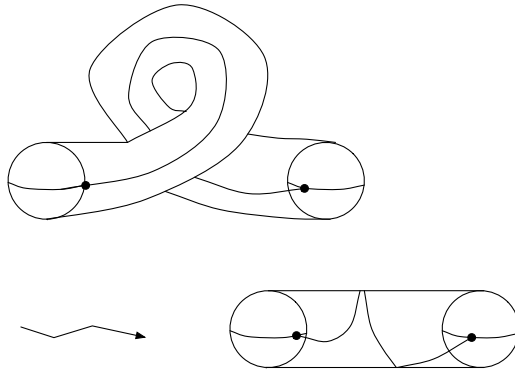
generators given by Wahl and Jensen minus those which contain Dehn twists of the boundary torus [JW04]. Differences will be noted along the way.

Define the group,

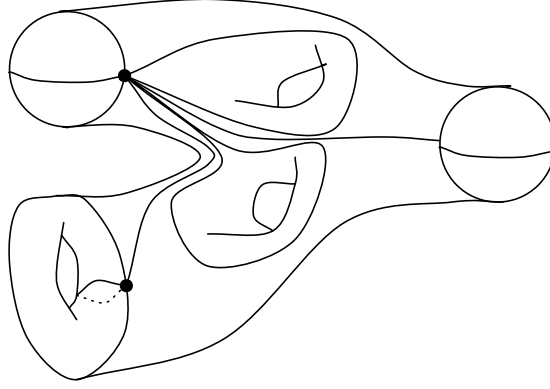
$$\Gamma_{(g,e,t)} = \Gamma(M_{(g,e,t)}, \partial)$$

to be the mapping class group of the space $M_{(g,e,t)}$ considered in 5.2.

Since $\pi_1(SO(3)) \cong \mathbb{Z}/2$, the inclusion $SO(3) \hookrightarrow \text{Diff}(S^2)$ yields a 1-parameter family of diffeomorphisms $\varphi : S^2 \times I \rightarrow S^2$ such that composition along the second parameter yields a homotopically trivial map. A *Dehn twist* along a 2-sphere in a 3-manifold is obtained by deleting the sphere and gluing the two boundary components back together along a copy of $S^2 \times I$ using φ . This amounts to twisting one of the boundary spheres formed by the deletion by a full revolution.



Up to isotopy we fix a standard embedding $i : G_{(g,e,t)} \hookrightarrow M_{(g,e,t)}$ by mapping the end of each boundary edge e to a boundary sphere, each boundary torus or balloon of the graph must map to the loop on the longitude of boundary torus of $M_{(g,e,t)}$ and each of the g loops is sent to the S^1 component of a corresponding $S^1 \times S^2$ summand. The inclusion i induces an isomorphism on fundamental groups. Let $r : M_{(g,e,t)} \rightarrow G_{(g,e,t)}$ be the retraction onto $i(G_{(g,e,t)})$. These maps are canonical up to isotopy with respect to the decomposition of $M_{(g,e,t)}$ into punctured handle bodies.



$$G_{(2,2,1)} \hookrightarrow M_{(2,2,1)}$$

There is a map from $h : \Gamma_{(g,e,t)} \rightarrow H_{(g,e,t)}$ defined as follows. If $l : M_{(g,e,t)} \rightarrow M_{(g,e,t)}$ is a diffeomorphism then we obtain a homotopy equivalence $h(l) = r \circ l \circ i$.

Theorem 7. (*Hatcher-Vogtmann-Wahl*) *The map $h : \Gamma_{(g,e,t)} \rightarrow H_{(g,e,t)}$ is an epimorphism. Its kernel is isomorphic to a finite direct sum of \mathbb{Z}_2 generated by Dehn twists along spheres.*

$$1 \longrightarrow \bigoplus_k \mathbb{Z}_2 \longrightarrow \Gamma_{(g,e,t)} \xrightarrow{h} H_{(g,e,t)} \longrightarrow 1$$

Proof. In their work Hatcher Vogtmann and Wahl allow the mapping class groups above to move the boundary while we do not. In our discussion of the difference we will simply matters slightly only discussing the tori since fixing a neighborhood of the spherical boundary components prevents only Dehn twists which are not relevant in what follows.

If the number of edges $e = 0$ then the full group of graph automorphisms associated to this is generated by,

1. $P_{i,j}$ exchanges x_i and x_j
2. I_i exchanges x_i and x_i^{-1}
3. $(x_i; x_j)$ $x_i \rightarrow x_i x_j$
4. $(x_i; y_j)$ $x_i \rightarrow x_i y_j$
5. $(x_i^{-1}; y_j)$ $x_i \rightarrow y_j^{-1} x_i$
6. $(y_i^\pm; x_j)$ $y_i \rightarrow x_j^{-1} y_i x_j$
7. $(y_i^\pm; y_j)$ $y_i \rightarrow y_j^{-1} y_i y_j$

Where the x_i represent generators of $\pi_1(G_{(g,0,t)})$ associated to factors of $S^1 \times S^2$ and y_i represent generators of $\pi_1(G_{(g,0,t)})$ associated to factors of $S^1 \times D^2$.

If we view our 3-manifold as the boundary of a punctured handle body then generators 3-7 above can be represented by handle slides along the curves x_i and y_j . Handle slides are associated to generators of the automorphism group as follows,

3. The handle x_i slides over x_j .
4. The handle x_i slides over y_j .
5. The handle x_i^{-1} slides over y_j .
6. The torus y_i slides over the handle x_j .
7. The torus y_i slides over the torus y_j .

In order to slide a handle or a torus (thought of as a connected sum of $S^1 \times D^2$) over a torus a Dehn twist must be performed fixing the boundary then kills generators 4, 5 and 7. Since our homotopy groups are defined to completely fix the loop contained in the torus the correspondence is preserved. \square

Corollary 4. *The spaces $B\Gamma_{(g,e,t)}$ and $BH_{(g,e,t)}$ are k -equivalent.*

Proof. This follows from $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty \simeq_{\mathbb{Q}} S^\infty$. S^∞ is contractible. So that in the k -homotopy category of spaces Bh is an equivalence. \square

6.1.1 Reduction of the Open Category

Observation. *The epimorphism h as defined above is compatible with gluing the spherical boundary components,*

$$\begin{array}{ccc}
 \Gamma_{(g,i+k,0)} \times \Gamma_{(g',k+r,0)} & \xrightarrow{\#} & \Gamma_{(g+g'+k-1,i+r,0)} \\
 \downarrow h & & \downarrow h \\
 H_{(g,i+k,0)} \times H_{(g',k+r,0)} & \xrightarrow{\#} & H_{(g+g'+k-1,i+r,0)}
 \end{array}$$

Proof. This follows from the definition of h above. Given two diffeomorphisms $\varphi \in \text{Diff}(M_{(g,i+k,0)}, \partial)$ and $\psi \in \text{Diff}(M_{(g',k+r,0)}, \partial)$, the action of $\varphi \# \psi$ on $i(G_{(g,i+k,0)}) \# i(G_{(g',k+r,0)}) \subset M_{(g+g'+k-1,i+r,0)} = M_{(g,i+k,0)} \# M_{(g',k+r,0)}$ is the same as the action of φ on $i(G_{(g,i+k,0)})$ glued to the incoming edges of ψ on $i(G_{(g',k+r,0)})$ as φ and ψ are required to fix a regular neighborhood of the boundary. \square

The same observation holds with groups involving torus boundary as long as either no gluing is performed or a factor of $\#S^1 \times S^2$ is introduced by the gluing of the underlying 3-manifolds.

Let \mathcal{OH} be the subcategory of the category of homotopy equivalences of graphs \mathcal{H} that consists of only equivalences of graphs with open edges. Then we have,

Theorem 8. *In the category of dgsm categories the open category \mathcal{O} is isomorphic to,*

$$\mathcal{O}' = C_*(B\mathcal{OH}; k)$$

Proof. The maps h induce a functor $\mathcal{O} \rightarrow \mathcal{O}'$. One can choose sections of i of h , $i : H_{(g,e,t)} \rightarrow \Gamma_{(g,e,t)}$. So that there is a functor $i : \mathcal{O}' \rightarrow \mathcal{O}$. We have $hi = 1$ and $ih \simeq_k 1$. \square

6.1.2 The Open-Closed Category as a Module

As discussed in Chapter 2, the category \mathcal{OC} defines an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule or a functor

$$\mathcal{OC} : \text{Ob}(\mathcal{OC}) \otimes \mathcal{O}^{op} \rightarrow \text{Ch}_k$$

in a natural way. If $(e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k} \in \text{Ob}(\text{Ob}(\mathcal{OC}) \otimes \mathcal{O}^{op})$ then

$$\mathcal{OC}((e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k}) = \text{Hom}_{\mathcal{OC}}(o^{\otimes k}, e^{\otimes n} \otimes t^{\otimes m})$$

The morphisms of category $\text{Ob}(\mathcal{OC})$ consisting of permutations of tensors act on the left and the open morphisms $c \in \text{Hom}_{\mathcal{O}}(o^{\otimes i}, o^{\otimes j})$ acts on the right by post-composition.

The category $\mathcal{OC}' = C_*(B\mathcal{H}; k)$ is also an $\text{Ob}(\mathcal{OC}') - \mathcal{O}'^{op}$ bimodule. Since $\text{Ob}(\mathcal{OC}') = \text{Ob}(\mathcal{OC})$ and by the previous theorem $\mathcal{O}' \cong \mathcal{O}$, \mathcal{OC}' is quasi-equivalent to an $\text{Ob}(\mathcal{OC}) - \mathcal{O}^{op}$ bimodule. It follows from the corollary that,

Theorem 9. *As an $\text{Ob}(\mathcal{OC}) - \mathcal{O}^{op}$ bimodules the categories \mathcal{OC} and \mathcal{OC}' are quasi-equivalent.*

7 Triangulated spaces

Here we review the definitions of triangulated spaces and spaces k -homotopy equivalent to triangulated spaces.

7.1 Simplicial Complexes

A *simplicial complex* A is a collection of sets such that if $\sigma \in A$ and $\tau \subset \sigma$ then $\tau \in A$. An n -*simplex* σ of a simplicial complex A is an element $\sigma \in A$ that satisfies $\#\sigma = n + 1$. The set of n -simplices of A will be denoted by A_n . A *vertex* is a 0 simplex. A collection of vertices $\{v_0, \dots, v_k\}$, $v_i \in A$, are said to *span* a simplex in A if $\{v_0, \dots, v_k\} \in A_k$.

A map of simplicial complexes $f : A \rightarrow B$ is a function which satisfies the property,

$$\{f(v_0), \dots, f(v_k)\} \text{ spans a simplex if } \{v_0, \dots, v_k\} \text{ spans a simplex}$$

We will denote the category of simplicial complexes by SC.

A *simplicial subset* X of a simplicial complex Y is an inclusion $X \hookrightarrow Y$.

For every simplicial complex A there is a category $F(A)$ with objects the elements of X and morphism sets generated by $\text{Hom}_{F(A)}(\sigma^k, \rho^{k-1})$ a unique non-trivial arrow if ρ is a face of σ . A morphism of simplicial complex $A \rightarrow B$ induces a functor of categories $F(A) \rightarrow F(B)$.

The *geometric* simplices are defined by,

$$\Delta^n = \{(t_0, \dots, t_n) \in [0, 1]^n \mid \sum_k t_k = 1\} \subset \mathbb{R}^{n+1}$$

and these combine to give a functor $|\cdot| : \text{SC} \rightarrow \text{Top}$ called *geometric realization*. If X is a simplicial complex then there is a functor $F : F(A) \rightarrow \text{Top}$ defined by, $F(\sigma) = \Delta^{|\sigma|}$. The geometric realization $|A|$ of A is given as the colimit of the functor F .

$$|A| = \text{colim } F = \coprod_n A_n \times \Delta^n / \sim$$

If $f : A \rightarrow B$ is a simplicial map then there is an induced map $|f| : |A| \rightarrow |B|$ taking simplices to simplices.

7.2 Cellular Stratifications

A *cellular stratification* of a space X is a choice of *CW* structure.

$$X = \bigcup_j X^j$$

Where X^j is obtained from X^{j-1} by attaching j dimensional disks D^j along their boundary spheres, $\partial D^j \rightarrow X^{j-1}$.

A *cellular map* $\varphi : X \rightarrow Y$ between cellularly stratified spaces X and Y is a continuous map which is cellular with respect to the chosen stratifications on X and Y . A cellular map $f : A \rightarrow B$ is cellular *on the nose* if for every k cell σ^k of A there is a k cell τ^k of B and $f(\sigma)$ is homeomorphic to τ .

If X and Y are cellularly stratified simplicial complexes then $|X| \times |Y|$ inherits a cellular stratification using the product of the *CW* complexes associated to $|X|$ and $|Y|$.

A space X is said to possess a *k-cellular stratification* or *orbi-stratification* if there exists a cellularly stratified space Y together with a discrete group G so that G acts cellularly, properly discontinuously with finite stabilizers on Y , and X is

homeomorphic to Y/G . For such a space X there is a filtration $\{\mathcal{F}^k X\}$ induced from the filtration by skeleta of Y so that the k -skeleton is obtained from the $k - 1$ skeleton by attaching a family of orbi-cells.

$$X^k = X^{k-1} \cup \coprod_{\sigma \subset Y^k} \sigma / \text{Stab}_G(\sigma)$$

and there is a k -homotopy equivalence between the associated graded and a wedge of spheres.

$$X^n / X^{n-1} \simeq_k \bigvee_{j=0}^m S^n$$

7.3 Open Simplicial Complexes

An *open simplicial set* is a simplicial complex X together with a simplicial subset $\bar{X} \subset X$. Morphisms between open simplicial complexes $\varphi : (X, \bar{X}) \rightarrow (Y, \bar{Y})$ satisfy $\varphi^{-1}(\bar{Y}) \subset \bar{X}$. The *geometric realization* of (X, \bar{X}) is the topological space $|X| - |\bar{X}|$.

Lemma 5. *If X is an open simplicial complex there exists a homotopy equivalent simplicial complex NX . That is,*

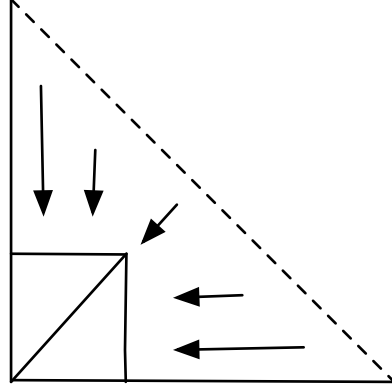
$$|X| \simeq |NX|$$

Proof. Analogous to the category $F(A)$ associated to any simplicial complex A (section 7.1) define the category \mathcal{X} with objects $x \in \text{Ob}(\mathcal{X})$ if $x \in X_k$ and $x \notin \bar{X}_k$ and morphisms generated by identity and $\partial_j : x \rightarrow x'$ if $x' = \partial_j x$ in X . Let $NX = N\mathcal{X}$ be the *nerve* of \mathcal{X} . The simplicial complex whose n simplices are given by n -fold compositions,

$$(N\mathcal{X})_n = \{\{x_0, x_1, \dots, x_n\} | x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n \text{ exists in } \mathcal{X}\}$$

For $\tau, \sigma \in N\mathcal{X}$, $\tau \subset \sigma$ if τ is obtained by composing one or more of the arrows in σ . The geometric realization of the nerve is called the *spine*.

There is a canonical inclusion of the space $i : |X| \hookrightarrow |NX|$ defined by observing that $|NX|$ is precisely the subset of the barycentric subdivision of $|X|$ which excludes the simplices of \bar{X} .



On each simplex of $s \subset |X|$, $s \cong \Delta^k$ there is a retraction onto the image of i defined as follows. Let $b \subset s$ be the portion of s contained in $|\bar{X}|$. If s^b is the barycentric subdivision of s and $y \subset s^b$ are the simplices disjoint from b then y is the image of $|NX|$ in $s \cap |X|$ and there is a collapse map $r_s : s \rightarrow y$.

These maps can be chosen to agree on the faces of the simplices of $|X|$ and so glue together to a map $r : |X| \rightarrow |NX|$. We have $r \circ i = 1$ and $i \circ r \simeq 1$ can be obtained by collapsing the complementary simplices continuously. \square

This construction gives a functor from the category of open simplicial complexes to simplicial complexes. If $\varphi : (X, \bar{X}) \rightarrow (Y, \bar{Y})$ is a map of open simplicial complexes then there is a map of simplicial complexes $N\varphi : NX \rightarrow NY$ where $(N\varphi)_k : (NX)_k \rightarrow (NY)_k$ is defined by,

$$\{x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n\} \mapsto \{\varphi(x_0) \rightarrow \varphi(x_1) \rightarrow \varphi(x_2) \rightarrow \cdots \rightarrow \varphi(x_n)\}$$

The continuous maps between realizations $|\varphi| : |X| \rightarrow |Y|$ and $|N\varphi| : |NX| \rightarrow |NY|$ are homotopic.

8 Outer Spaces

The goal of this and the next section is to construct a combinatorial model for a complex which computes the homology $H_*(BH_{(g,e,t)}; k)$. We begin by constructing an open simplicial complex $L_{(g,e,t)}$ whose geometric realization is a space equipped with a properly discontinuous action of $H_{(g,e,t)}$ that has finite stabilizers. After taking the nerve we obtain a simplicial complex that has simplices which can be grouped into cells yielding a k -cellular stratification of the space $BH_{(g,e,t)}$.

8.1 Open Outer Space

In what follows all graphs will be boundary labelled in sense defined above. However we will consistently write $L_{(g,e,t)}$ instead of $L_{(g,i+o,a+b)}$ because it is simpler.

We begin with a definition of an open simplicial complex $L_{(g,e,t)}$ or a pair of simplicial complexes $(L_{(g,e,t)}, \bar{L}_{(g,e,t)})$ with $\bar{L}_{(g,e,t)} \subset L_{(g,e,t)}$.

A graph G is *labelled* when paired with a map $\varphi : G_{(g,e,t)} \rightarrow G$ that satisfies

1. The function φ preserves the e incoming and outgoing edges and identifies the ends of each of the t boundary tori of $G_{(g,e,t)}$ with circles having t distinct base vertices in G . By circle we mean cycles with one edge and one vertex.
2. If v is the vertex of $G_{(g,e,t)}$. The induced map,

$$\varphi_* : \pi_1(G_{(g,e,t)}, v) \rightarrow \pi_1(G, \varphi(v))$$

is an isomorphism.

Two labelled graphs (G, φ) and (G', ψ) are *equivalent* if there is a graph isomorphism $\rho : G \rightarrow G'$ so that the diagram below commutes,

$$\begin{array}{ccc} \pi_1(G, \varphi(v)) & \xrightarrow{\quad} & \pi_1(G', \psi(v)) \\ & \swarrow \quad \searrow & \\ & \pi_1(G_{(g,e,t)}, v) & \end{array}$$

Definition. Let $L_{(g,e,t)}$ denote the set of equivalence classes (G, φ) of labelled graphs G that have a labelling $\varphi : G_{(g',e,t)} \rightarrow G$ such that $g' \leq g$. The k simplices of the simplicial complex $L_{(g,e,t)}$ are given by graphs with $k + 1$ edges.

If $(G, \varphi) \in L_{(g,e,t)}$ is a k simplex then $(H, \psi) \subset (G, \varphi)$ if H is obtained from G by collapsing edges and $\psi = \bar{\varphi}$ is the labelling induced on H by φ and the quotient.

While some edge collapses preserve the homotopy type of the underlying graph some do not. The additional data necessary to define the spaces in which we are interested $\bar{L}_{(g,e,t)} \subset L_{(g,e,t)}$ is given by the collection of simplices obtained by an edge collapse which isn't an isomorphism on fundamental groups or an edge collapse that identifies the base vertices of two distinct tori. If $t = 0$ then,

$$(\bar{L}_{(g,e,t)})_k = \{(G, \varphi) \in (L_{(g,e,t)})_k \mid \varphi : G_{(g',e,t)} \rightarrow G \text{ and } G_{(g',e,t)} \not\cong G_{(g,e,t)}\}$$

then $L_{(g,e,t)}$ is an open simplicial complex.

Definition. The geometric realization of $L_{(g,e,t)}$ will be denoted by $Y_{(g,e,t)}$ and its quotient by $X_{(g,e,t)} = Y_{(g,e,t)}/H_{(g,e,t)}$.

If $t = 0$ and $e = 0$ this is called Outer space since the construction is a model for the classifying space of the group $\text{Out}(F_n)$ see [CV86]. If $t = 0$ and $e = 1$ this is known as ‘‘Auter space.’’ Other generalizations not involving diffeomorphisms that

fix the boundary can be found in [HV04, JW04, HW05]. These recent results were used to prove a stabilization theorem for the homology of free groups analogous to previous results for mapping class groups of surfaces.

Theorem 10. *There is an action of the group $H_{(g,e,t)} = \pi_0 \text{Htpy}(G_{(g,e,t)}, \partial)$ on the open simplicial complex $L_{(g,e,t)}$. The action is properly discontinuous. The stabilizer of any given simplex is a finite group.*

Proof. The group $H_{(g,e,t)}$ acts on the simplicial complex $L_{(g,e,t)}$ by changing the labelling. That is, if (G, φ) so that $\varphi : G_{(g,e,t)} \rightarrow G$ is a simplex and $f \in H_{(g,e,t)}$ then $f(G, \varphi) = (G, \varphi \circ f)$. By changing the labelling the action of $H_{(g,e,t)}$ preserves the genus g of the graphs so that if $f \in H_{(g,e,t)}$, $f : L_{(g,e,t)} \rightarrow L_{(g,e,t)}$ then $f^{-1}(\bar{L}_{(g,e,t)}) \subset \bar{L}_{(g,e,t)}$ implies that $H_{(g,e,t)}$ acts on $(L_{(g,e,t)}, \bar{L}_{(g,e,t)})$ in the category of open simplicial complexes defined above.

This action is almost free. If (G, φ) is a simplex and $f \in H_{(g,e,t)}$ then $f(G, \varphi) = (G, f \circ \varphi) = (G, \varphi)$ if and only if f is an isomorphism as above of the graph G . Since each graph isomorphism is determined by the manner in which it permutes the edges, the size of the group of graph isomorphisms is bounded above by set of all permutations on edges. So $\text{Stab}(G) = \text{Aut}(G)$ and $\#\text{Aut}(G) \leq \#E(G)!$. \square

That the space $Y_{(g,e,t)}$ is contractible is a standard fact about this aspect of the theory. The proof of contractibility of $Y_{(g,e,t)}$ is essentially equivalent to the proof which appears in Wahl and Jensen's article. Their definition differs from ours by disallowing separating edges of graphs and allowing the boundary cycles to move throughout the graph. Collapsing separating edges in our construction is a homotopy equivalence. Removing the generators from the groups associated to the boundary is a special case of the contractibility argument given by Wahl and Jensen be contractible [JW04].

Corollary 5. *The quotient space $X_{(g,e,t)} = Y_{(g,e,t)}/H_{(g,e,t)}$ is a k -homotopy model for the space $BH_{(g,e,t)}$. In particular,*

$$C_*(BH_{(g,e,t)}; k) \simeq C_*(X_{(g,e,t)}; k)$$

There is a geometric interpretation of $X_{(g,e,t)}$. A *metric graph* is a graph together with a fixed length $l(e) \in \mathbb{R}_+$ assigned to each edge $e \in E(G)$. A metric graph is *balanced* if $\sum_{e \in E(G)} l(e) = 1$.

The space $X_{(g,e,t)}$ is homeomorphic to the space of balanced metric graphs homotopy equivalent to the graph $G_{(g,e,t)}$. For any balanced metric graph G let, e_0, \dots, e_k , be the edges of G then G is uniquely represented by the point given by the barycentric coordinates $(l(e_0), \dots, l(e_k))$ of a k simplex Δ associated to the topological type of G . If we allow the lengths of the edges of G to vary they approach a codimension 1 face of Δ which has either been removed or not depending upon whether the homotopy type of the graph obtained by collapsing the edge changes or is preserved. So that for each topological type of graph G there is an open simplex in the construction above and quotienting by the automorphism group gives the space of balanced metric graphs. The colimit over all appropriate topological types of graphs is the space $X_{(g,e,t)}$.

When the genus is zero and there are no tori there are no automorphisms and quotient is the space of metric trees which is in itself contractible. These spaces were explored in [BHV01] and used to establish a sheaf theoretic interpretation operadic algebra [GK94].

The t boundary tori are represented by balloons attached to the graphs representing points in the moduli space $X_{(g,e,t)}$. The length of the edge at the end of each balloon is fixed. The length of the edge along which the balloon is attached to the rest of the graph is allowed to vary. As this edge length approaches zero, in the space of metric graphs, we approach either an open face or a face depending upon whether collapsing the edge identifies two base vertices or not respectively.

We metrize the graphs in this way because the edge of the balloon corresponding to a torus in a manifold $M_{(g,e,t)}$ is completely fixed by the action of any $b \in \Gamma(M_{(g,e,t)}, \partial)$. The edge about the torus of the balloon in the graph $G_{(g,e,t)}$ thought of as embedded in $M_{(g,e,t)}$ does not vary with respect to the action of the mapping class group. The edge that is used to attach the balloon to the rest of the graph is allowed to vary since b may move the boundary torus about inside of $M_{(g,e,t)}$ as discussed in

Chapter 6. Since there are disjoint regular neighborhoods of the boundary tori in the construction of the cobordism category we can ask for the base vertices of the balloons representing them not to touch.

In contrast, the open edges are given fixed length or equivalently not given length in the moduli space. If taken to have a fixed positive length then when represented as a graph within $M_{(g,e,t)}$ this length reflects the disjointness of the regular neighborhoods of 2-spheres in the construction of the cobordism category. Allowing these lengths to vary is not necessary and would not add anything interesting to what follows, but if we did then it would be necessary for us to consider the scenario in which the collapse of an edge represented a boundary collision as we have done with the tori above.

8.2 Reduction To Spine

As described in 7.3 the open simplicial complex $L_{(g,e,t)}$ gives rise to an honest simplicial complex $L'_{(g,e,t)} = NL_{(g,e,t)}$ in a functorial manner. The geometric realization of $L'_{(g,e,t)}$ satisfies $|L'_{(g,e,t)}| \simeq |L_{(g,e,t)}|$.

An n -simplex in $L'_{(g,e,t)}$ is given by a sequence

$$(G_0, \varphi_0) \subset (G_1, \varphi_1) \subset \cdots \subset (G_n, \varphi_n)$$

where (G_i, φ_i) is obtained from (G_{i+1}, φ_{i+1}) by collapsing one or more edges while preserving the homotopy type.

An equivalent way to specify simplices of the space $L'_{(g,e,t)}$ is to fix a forest $F_0 \subset G$ and a nested sequence of subforests $F_n \subset F_{n-1} \subset \cdots \subset F_0 \subset G$. This gives the simplex,

$$(G/F_0, \bar{\varphi}) \subset (G/F_1, \bar{\varphi}_1) \subset \cdots \subset (G_n/F_n, \varphi)$$

if φ is a labelling of $G = G_n$.

In what follows we will require all forests $F \subset G$ of graphs G to

1. Include all of the internal vertices of G .
2. Include *none* of the incoming or outgoing open boundary edges.
3. Include no two of the base vertices of the tori.

By functoriality the space $L'_{(g,e,t)}$ inherits an action of $H_{(g,e,t)}$. If $f \in H_{(g,e,t)}$ then $f : L_{(g,e,t)} \rightarrow L_{(g,e,t)}$ is defined by $f(G, \varphi) = (G, \varphi \circ f)$ so that $Nf : L'_{(g,e,t)} \rightarrow L'_{(g,e,t)}$ acts by

$$\begin{aligned} (G/F_0, \varphi_0) &\subset (G/F_1, \varphi_1) \subset \cdots \subset (G/F_n, \varphi_n) \\ &\mapsto (G/F_0, \varphi_0 \circ f) \subset (G/F_1, \varphi_1 \circ f) \subset \cdots \subset (G/F_n, \varphi_n \circ f) \end{aligned}$$

It follows from the definition and discussion in the previous section that the stabilizer of an n simplex is the subgroup of the automorphism group $\text{Aut}(G)$ given by those automorphisms which preserve the forest F_0 and its associated filtration and that the quotient is a k -homotopy model for $BH_{(g,e,t)}$.

Definition. We will denote by $L'_{(g,e,t)}$ the spine or the simplicial complex obtained from the nerve construction applied to the open simplicial complex $L_{(g,e,t)}$ defined in the previous section. The geometric realization of $L'_{(g,e,t)}$ will be denoted by $Y'_{(g,e,t)}$ and $X'_{(g,e,t)} = Y'_{(g,e,t)}/H_{(g,e,t)}$ the quotient by the action of $H_{(g,e,t)}$.

8.3 Cellular Stratification by Cubes

Applying the functor N gives us compact spaces $X'_{(g,e,t)}$ and while the given simplicial structure gives a k -cellular stratification of $X'_{(g,e,t)}$ this triangulation is too fine for our purposes. Following Hatcher-Vogtmann, Kontsevich, Conant-Vogtmann and others to compute the homology we group together simplices that can be obtained from the same forest into a single cell [HV98, Kon94, CV03]. The cells and orbi-cells obtained from this construction will be called cubes.

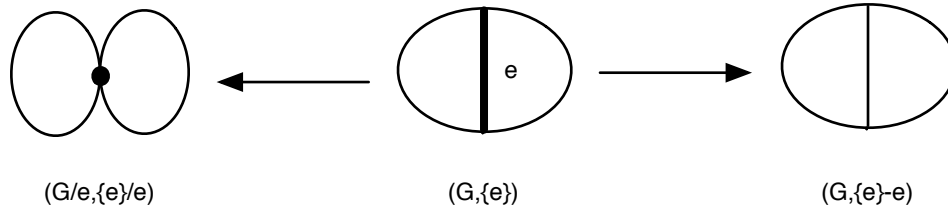
Fix a graph and a labelling (G, φ) defining an open simplex $\sigma \subset Y_{(g,e,t)}$. The portion of the subdivision that lies in $Y'_{(g,e,t)}$, that is, the portion not involving any simplices contained in $\bar{Y}_{(g,e,t)}$ is parameterized by the paths from the barycenter σ to a labelled graph isomorphic to $G_{(g,e,t)}$. Each of these paths gives rise to a simplex, a sequence of edge collapses or, as discussed above, to a forest $F \subset G$ together with a filtration $0 = F_n \subset F_{n-1} \subset \cdots \subset F_0 \subset G$.

A cube $[G, F, \varphi] \subset Y'_{(g,e,t)}$ is obtained by gluing together all the simplices arising from different filtrations of the same choice of forest $F \subset G$ for a given open simplex $\sigma = (G, \varphi) \subset Y_{(g,e,t)}$.

$$[G, F, \varphi] = \coprod_{F_0 \subset \cdots \subset F_m \subset F} (G/F_0 \subset \cdots \subset G/F_{m-1} \subset G/F_m) \times \Delta^m$$

Or in fancy language, applying the forgetful functor on a simplicial (cellular) space indexed by labelled graphs with filtrations of forests to a cellular space indexed by labelled graphs with forests yields a coarser stratification. This is called the *forested graph stratification*.

Each cube $[G, F, \varphi]$ in $Y'_{(g,e,t)}$ can be homeomorphically identified with a k -ball $[0, 1]^k$, where $k = \#E(F)$, by a map defined by assigning to each edge an axis. If $B = \text{star}(G/F, \bar{\varphi}) \cap (G, \varphi)$ is the portion of the star of G/F contained in the open simplex associated to (G, φ) then $[G, F, \varphi] = B \cap (G/(G - F), \hat{\varphi})$.



The codimension 1 faces of a cube $[G, F, \varphi]$ are given by two operations on graphs

1. Removing an edge from the forest. $[G, F, \varphi] \mapsto [G, F - e, \varphi]$ for some edge $e \in E(F)$.

2. Collapsing an edge $e \in F$. $[G, F, \varphi] \mapsto [G/e, F/e, \bar{\varphi}]$ for some edge $e \in E(F)$.

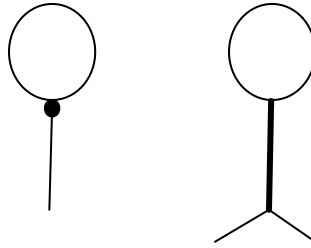
The group $H_{(g,e,t)}$ now acts cellularly. The stabilizer of the cube $[G, F, \varphi]$ under the action of $H_{(g,e,t)}$ is the subgroup of $H_{(g,e,t)}$ consisting of automorphisms of G that send the forest $F \subset G$ to itself.

Each cube $[G, F, \varphi]$ in $Y'_{(g,e,t)}$ then descends to a cube $[G, F]$ in the quotient $X'_{(g,e,t)}$. This cube is not necessarily a cell but an orbi-cell. This follows from identifying the cube in $Y'_{(g,e,t)}$ with a cube $C = [0, 1]^k$ where each edge of F contributes to an axis as above and the graph G/F is situated at the origin. The portion of the cube that descends to $X'_{(g,e,t)}$ is the quotient of C by the stabilizer $\text{Aut}(G, F, \varphi)$. The action of $\text{Aut}(G, F, \varphi)$ on C fixes the origin and permutes the axes so that $C/\text{Aut}(G, F, \varphi)$ is a cone on the quotient of the boundary ∂C .

Lemma 6. *The quotient of an n -sphere by a finite linear group $G \subset GL_n(\mathbb{R})$ is k -homotopic to either a n -sphere or a n -ball. The latter case holds only when the action includes reflections.*

For proof and discussion see [HV98]. So those cubes which have symmetries that do not include reflections survive the quotient.

In X' the tori are represented by trees containing the base vertex of the balloon.



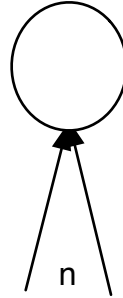
Forests representing tori with one and two outgoing boundaries. Cells of dimensions zero and one respectively.

8.4 Homology

In this section we complete the program of describing the homology of the mapping class groups of the manifolds which determine the morphism spaces of \mathcal{OC} and \mathcal{O} . We begin by defining for each (g, e, t) , a generalized Cobar construction, an exact functor $\mathcal{G}_{(g,e,t)}$ from the category of differential graded cooperads to chain complexes. The complexes \mathcal{G} will be those that generate the morphism spaces of the enveloping functor $\text{Cobar}(\mathcal{O})^b$ defined in Chapter 4. In the second subsection we show that this corresponds to the chain complex obtained from the stratification of $X'_{(g,e,t)}$ by cubes defined in the previous section.

8.4.1 From Operads to Graph Complexes

Fix a differential graded cooperad \mathcal{P} . Let $S_{(g,e,t)}$ be the set of boundary labelled combinatorial graphs of genus $g + t$ with e boundary edges and containing t marked vertices. Special subgraphs isomorphic to a corolla with two edges identified we will call bonnets,



Using the orientation convention described in 4.3 we define a chain complex $\mathcal{G}_{(g,e,t)}(\mathcal{P})$ by,

$$\mathcal{G}_{(g,e,t)}(\mathcal{P}) = \bigoplus_{G \in S_{(g,e,t)}} \mathcal{P}(G) \otimes \det(G)$$

The graphs are labelled by \mathcal{P} and oriented using vertices and half edges as described in section 4.3. The differential is the sum of the differential on \mathcal{P} together with the expanding differential on graphs the definition of which is identical to the one for trees found in 4.3. By construction we have

$$\mathcal{G}_{(0,e,0)}(\mathcal{P}) = \text{Cobar}(\mathcal{P})(e)$$

and,

$$\text{Hom}_{\text{Cobar}(\mathcal{P})^\flat}(x^{\otimes n}, x^{\otimes m}) = \bigoplus_g \mathcal{G}_{(g,n+m,0)}(\mathcal{P})$$

8.4.2 Cubical Chains Compute A Double Dual

Recall from 8.4 that the complex of cubical chains on $X'_{(g,e,t)}$ is spanned by cubes $[G, F]$ where G is a boundary labelled graph with t cycles representing boundary tori and $F \subset G$ is a forest containing all of the internal vertices of G , none of the boundary edges and no two vertices of the boundary tori.

The cubes are oriented by an ordering of the edges of F . Lemma 6 in the same section implies that the antisymmetry relation $[G, -F] = -[G, F]$ holds.

The differential is given by the sum over ways to remove an edge from a forest together with the sum over ways to contract an edge contained in the forest. In either case the cube is oriented by the induced orientation.

$$\partial[G, F] = \sum_{e \in F} [G/e, F/e] + \sum_{e \in F} [G, F - e]$$

The cooperad $\text{Bar}(\mathbb{C})$ is the free cooperad on n -corolla satisfying the antisymmetry relation. The trees are edge oriented with contracting differential. The composition is given by degrafting trees. This is dual to the L_∞ operad described in Chapter 4.

Since $\text{Cobar}(\text{Bar}(\mathbb{C}))$ is a double complex while $C_*^{cell}(X')$ is merely a complex we must collapse the double grading of barbar to make sense of the equivalence to

follow. Let,

$$\text{Cobar}(\text{Bar}(\mathcal{C}))(n)'_i = \bigoplus_j \text{Cobar}(\text{Bar}(\mathcal{C}))(n)_{j,i}$$

The differential d remains the sum of the internal differential contracting the edges of $\text{Bar}(\mathcal{C})$ and the external differential expanding the compositions in Cobar .

Theorem 11. *The k -homology of the mapping class groups can be computed by the generalized Cobar functor \mathcal{G} ,*

$$C_*^{\text{cell}}(X'_{(g,e,t)}; k) = \mathcal{G}_{(g,e,t)}(\text{Bar}(\mathcal{C}))$$

Proof. We first begin by assuming that $t = 0$ that is that there are no tori. It suffices to show that the operad $\text{Cobar}(\text{Bar}(\mathcal{C}))$ is quasi-equivalent to the operad \mathcal{O} with $\mathcal{O}(n) = C_*^{\text{cell}}(X'_{(0,n+1,0)}; k)$. This forms an operad because the cellular composition defined in Chapter 9 exists independently of the identification here. We will see that as complexes the two are plainly isomorphic. That is,

$$\text{Cobar}(\text{Bar}(\mathcal{C}))(n)' = C_*^{\text{cell}}(X'_{(0,n+1,0)}; k)$$

In degree j the complex $C_j^{\text{cell}}(X'_{(0,n+1,0)}; k)$ is spanned by forested trees (T, F) where the forest F contains j edges and a connected component associated to each internal vertex of T . The cell (T, F) is oriented by an ordering of the edges in F .

In bidegree (j, i) the complex $\text{Cobar}(\text{Bar}(\mathcal{C}))(n)_{j,i}$ is spanned by unrooted n trees T containing $j = |T|$ internal vertices each of which is in turn labelled by a tree $F_l \in \text{Bar}(\mathcal{C})(H(v))$. The bidegree $(j, i) = (|T|, \sum_{m=1}^{|T|} (|F_m| - 1))$; the second coordinate being the total number of internal edges. The labelled tree $T \otimes F_1 \otimes \cdots \otimes F_j$ is oriented by the convention described in 4.3 and discussed below.

Thus for the flattened complex $T \otimes F_1 \otimes \cdots \otimes F_j \in \text{Cobar}(\text{Bar}(\mathcal{C}))(n)'_i$ if T is an unrooted n tree labelled by trees F_l whose internal edges total to i .

Our isomorphism associates to a forested tree (T, F) with $F = F_1 \cup \cdots \cup F_j$ the tree T/F with internal vertices labelled by the F_l . The inverse map is obtained by doing the opposite.

The two differentials in either complex are the same. Removing an edge from a tree creates two separate trees which corresponds to a labelling of a vertex by a tree expanding into a labelling of two vertices by two trees which is the Cobar differential. Collapsing the edge of a tree corresponds to collapsing an edge within the labelling of a vertex which is the differential of $\text{Bar}(\mathbb{C})$.

The orientations agree. Recall that the convention for the generalized Cobar functor \mathcal{G} is the same as that described in 4.3. A graph G is oriented by,

$$\det(G) = \det(E(G)) \otimes \det(k^O) \otimes \det(H_0(G)) \otimes \det(H_1(G))^*[O - \chi]$$

where,

1. $E(G)$ are the internal edges of G . Those edges which are not part of the incoming or outgoing boundary ∂G .
2. O is the number of outgoing boundary edges.
3. $\chi = \chi(G)$ is the Euler characteristic of G .

If the graph G is a tree T then this amounts to,

$$\det(T) = \det(E(T)) \otimes \det(k^{O_T})[O_T]$$

That is an ordering of the internal edge and outgoing edges of T situated in degree $\#E(T)$ the number of internal edges of T . A tree T colored by a j component forest $F_1 \otimes \cdots \otimes F_j$ is oriented by,

$$\det(T \otimes F_1 \otimes \cdots \otimes F_j) = \det(E(T)) \otimes \det(k^{O_T}) \bigotimes_{i=1}^j \det(E(F_i)) \otimes \det(k^{O_{F_i}})$$

In our case the number of outgoing edges of T is one. The incoming edges of T are not oriented. The internal edges of T join the labellings of two separate vertices by forest components F_i . One end of each edge of T is an incoming edge of some forest component and the other end is an outgoing edge of some forest component.

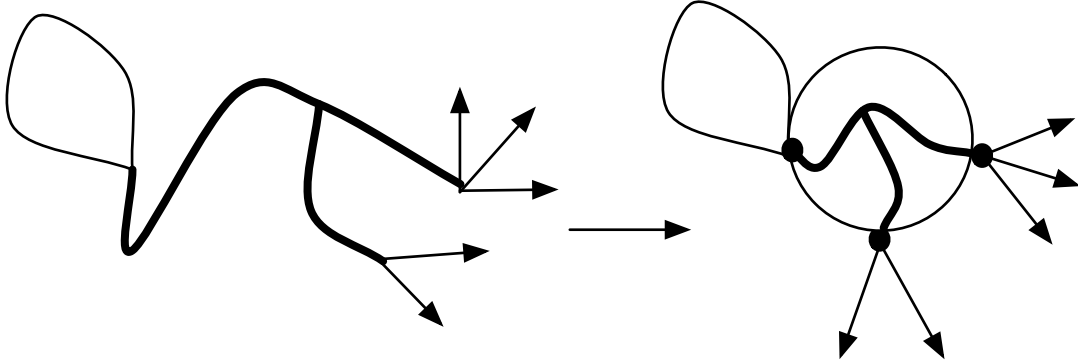
The outgoing components of each forest must correspond to internal edges of T except for the one outgoing edge corresponding to the outgoing edge of T . Thus there is a bijection between the set $E(T) \coprod \text{Out}(T)$ and $\coprod_i \text{Out}(F_i)$. Taking graded determinants yields the isomorphism,

$$\det(E(T)) \cong \det(E(T)) \otimes k \cong \bigotimes_{i=1}^j \det(k^{O_{F_i}})$$

The ordering of the internal edges of T cancels with the orientations of the outgoing edges of the components of the forest and,

$$\det(T \otimes F_1 \otimes \cdots \otimes F_j) \cong \bigotimes_{i=1}^j \det(E(F_i)) \cong \det(G, F)$$

In X' the cells associated to the boundary tori are the trees contain the base vertex of the balloon about the torus.



We represent these graphically by a loop resembling a bonnet attached to the vertex of an n corolla which has been labelled by a tree. This is the extension of \mathcal{G} described above. \square

8.5 Corollaries

The following reduction is computed using a spectral sequence argument [Kon94, CV03]. Since $\text{Bar}(C)^* = L_\infty \simeq L$,

Corollary 6. (*Kontsevich and Conant-Vogtmann*) *The k -homology of the mapping class groups can be computed by the generalized Cobar functor \mathcal{G} ,*

$$C_*(X'_{(g,e,t)}; k) \simeq \mathcal{G}_{(g,e,t)}(L^*)$$

The morphisms of the open category and open closed category coming directly from k -chains on the spaces $X'_{(g,e,t)}$ can be identified with a slightly smaller operad, because $C_\infty \simeq \text{Cobar}(\text{Bar}(C))$,

Corollary 7.

$$\text{Hom}_{\mathcal{O}}(e^{\otimes i}, e^{\otimes j}) \simeq \text{Hom}_{C_\infty^b}(e^{\otimes i}, e^{\otimes j})$$

we also have,

Corollary 8. *The $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule \mathcal{OC} is quasi-isomorphic to the $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule defined by,*

$$(e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k} \mapsto \coprod_g \mathcal{G}_{(g,n+k,m)}(L^*)$$

Proof. Recall that,

$$\mathcal{OC}((e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k}) = \text{Hom}_{\mathcal{OC}}(e^{\otimes n} \otimes t^{\otimes m}, o^{\otimes k})$$

Theorem 9 stated that as bimodules \mathcal{OC} was quasi-isomorphic to \mathcal{OC}' ,

$$\mathcal{OC}'((e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k}) = \coprod_g C_*(BH_{(g,n+k,m)}; k)$$

The corollary follows from the identification,

$$\coprod_g C_*(BH_{(g,n+k,m)}; k) \simeq \coprod_g C_*(X'_{(g,n+k,m)}; k) \simeq \coprod_g \mathcal{G}_{(g,n+k,m)}(L^*)$$

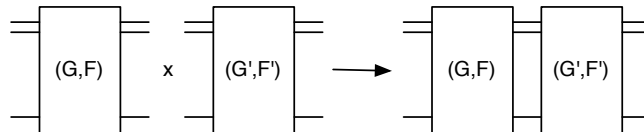
□

9 The Open Category

We concluded the last section with an equivalence relating the chain complexes that comprise the morphisms of the categories we are most interested in: \mathcal{OC} and its subcategories. These chain complexes are quasi-isomorphic to ones defined in an essentially combinatorial way by graphs labelled with the L operad.

Unfortunately, this is not an equivalence of *categories* in all cases. While the complexes that make up the morphisms of the categories can be reduced to combinatorics the gluing maps that define the composition at the level of mapping class groups discussed in section 7 do not descend to a cellular map.

The composition along boundary spheres (or boundary edges) does descend to a cellular map. Given two boundary labelled composable forested graphs (G, F) and (G', F') . Form the graph $G\#G'$ by gluing the relevant ends together and eliminating the resulting bivalent vertices. The forests F and F' together form a forest $F \cup F'$ of $G\#G'$ because forests are not permitted to contain boundary edges. This can be pictured by,



The composition map is defined by first observing that there is an equivariant composition on the total spaces $Y_{(g,e,0)}$ that is cellular on the nose. In the sense that a composition of cells is precisely a product of cells. This then descends to a composition on the k -cellularly stratified orbi-spaces $X'_{(g,e,0)}$ from which our complexes

were defined. Lastly we observe that the composition above agrees with both the composition in 6.1.1 necessitated by k -equivalence in the Hatcher-Vogtmann-Wahl theorem and the composition used by the Cobar construction in 4.3 and so in the dgsm categories of 4.2.

Theorem 12. *The quasi-isomorphisms of 8.6,*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}'}(e^{\otimes i}, e^{\otimes j}) \otimes \mathrm{Hom}_{\mathcal{O}'}(e^{\otimes j}, e^{\otimes k}) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{O}'}(e^{\otimes i}, e^{\otimes k}) \\
\downarrow \varphi_{ij} \otimes \varphi_{jk} & & \downarrow \varphi_{ik} \\
\mathrm{Hom}_{\mathrm{Cobar}(\mathrm{Bar}(\mathcal{C}))^{\flat}}(e^{\otimes i}, e^{\otimes j}) \otimes \mathrm{Hom}_{\mathrm{Cobar}(\mathrm{Bar}(\mathcal{C}))^{\flat}}(e^{\otimes j}, e^{\otimes k}) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathrm{Cobar}(\mathrm{Bar}(\mathcal{C}))^{\flat}}(e^{\otimes i}, e^{\otimes k})
\end{array}$$

respect the composition on the nose.

Proof. We show that the gluing conducted for composition is orbi-cellular with respect to the cube decomposition of the classifying spaces. The composition on \mathcal{O}' ,

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}'}(e^{\otimes i}, e^{\otimes j}) \otimes \mathrm{Hom}_{\mathcal{O}'}(e^{\otimes j}, e^{\otimes k}) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{O}'}(e^{\otimes i}, e^{\otimes k}) \\
\cong \downarrow & & \cong \downarrow \\
\coprod_g C_*(BH_{(g,i+j,0)}; k) \otimes \coprod_g C_*(BH_{(g',j+k,0)}; k) & \xrightarrow{\circ} & \coprod_g C_*(BH_{(g,i+k,0)}; k)
\end{array}$$

is constructed using maps,

$$\circ : C_*(BH_{(g,i+j,0)}; k) \otimes C_*(BH_{(g',j+k,0)}; k) \rightarrow C_*(BH_{(g+g'+j-1,i+k,0)}; k)$$

which are in turn induced by maps

$$\circ : BH_{(g,i+j,0)} \times BH_{(g',j+k,0)} \rightarrow BH_{(g+g'+j-1,i+k,0)}$$

There are k -homotopy equivalences

$$\begin{array}{ccc}
BH_{(g,i+j,0)} \times BH_{(g',j+k,0)} & \xrightarrow{\circ} & BH_{(g+g'+j-1,i+k,0)} \\
\cong \downarrow & & \cong \downarrow \\
X'_{(g,i+j,0)} \times X'_{(g',j+k,0)} & \xrightarrow{\circ} & X'_{(g+g'+j-1,i+k,0)}
\end{array}$$

from the space $BH_{(g,i+j,0)}$ to $X'_{(g,i+j,0)}$. The spaces $X'_{(g,i+j,0)}$ are stratified by orbi-cells $[G, F]$ indexed by forested graphs (G, F) having dimension determined by the number of edges in F .

Residing above each orbi-cell is a collection of cells $[G, F, \varphi]$ in $Y'_{(g,i+j,0)}$ indexed in the orbit of the action of $H_{(g,e,t)}$ by their labellings φ . There is a group action,

$$\begin{array}{ccc}
H_{(g,i+j,0)} & \longrightarrow & Y'_{(g,i+j,0)} \\
& & \downarrow \\
& & X'_{(g,i+j,0)}
\end{array}$$

Given a cell $[G, F, \varphi]$ of dimension n in $Y'_{(g,i+j,0)}$ and a cell $[G', F', \varphi']$ of dimension m in $Y'_{(g',j+k,0)}$ there is a composite $[G\#G', F \cup F', \varphi\#\varphi']$ of dimension $n + m$ and a homeomorphism,

$$[G, F, \varphi] \times [G', F', \varphi'] \rightarrow [G\#G', F \cup F', \varphi\#\varphi']$$

defined by identifying each cell with a cube in $\mathbb{R}^{\#E(F)}$ as described in 8.4. These homeomorphisms together produce the composition,

$$\circ : Y'_{(g,i+j,0)} \times Y'_{(g',j+k,0)} \rightarrow Y'_{(g+g'+j-1,i+k,0)}$$

this is equivariant with respect to the action of $H_{(g,i+j,0)} \times H_{(g',j+k,0)}$ on the left and $H_{(g+g'+j-1,i+k,0)}$ on the right using the map

$$\circ' : H_{(g,i+j,0)} \times H_{(g',j+k,0)} \rightarrow H_{(g+g'+j-1,i+k,0)}$$

described in 6.1.1. That is, if $a \in H_{(g,i+j,0)}$, $b \in H_{(g',j+k,0)}$ and $a \circ' b \in H_{(g+g'+j-1,i+k,0)}$ then,

$$\begin{array}{ccc}
 [G, F, \varphi] \times [G', F', \varphi'] & \xrightarrow{\circ} & [G\#G', F \cup F', \varphi\#\varphi'] \\
 \downarrow a \times b & & \downarrow a \circ' b \\
 [G, F, \varphi \circ a] \times [G', F', \varphi' \circ b] & \xrightarrow{\circ} & [G\#G', F \cup F', \varphi\#\varphi' \circ (a \circ' b)]
 \end{array}$$

and so defines the desired composition on the quotient. For two cubes (G, F) and (G', F') choose honest cells $[G, F, \varphi]$ and $[G', F', \varphi']$ in the fiber above each in the total space. Then the composition \circ is orbi-”on the nose,”

$$\begin{array}{ccc}
 [G, F, \varphi] \times [G', F', \varphi'] & \xrightarrow{\cong} & [G\#G', F \cup F', \varphi\#\varphi'] \\
 \downarrow & & \downarrow \\
 (G, F) \times (G', F') & \longrightarrow & (G\#G', F \cup F')
 \end{array}$$

since the vertical arrows below are k -homotopy equivalences. Taking chains and repeating the construction above to obtain the reduced complex it can be seen that the differential obtained acts as a derivation with respect to this composition law. \square

Given the above following holds,

Corollary 9. *The category of h -split \mathcal{O} -modules is equivalent to the category of $\text{Cobar}(\text{Bar}(\mathcal{C}))$ algebras with a choice of invariant inner product.*

Given the above and the discussion in 4.3,

Corollary 10. *The category of h -split \mathcal{O} modules is equivalent to the category of C_∞ algebras with a choice of invariant inner product.*

10 Extension and Torus

Recall that every dgsm category \mathcal{C} yields a $\mathcal{C} - \mathcal{C}$ bimodule, $\mathcal{C} \otimes \mathcal{C}^{op} \rightarrow \text{Ch}_k$ given by

$$\mathcal{C}(x \times y) = \text{Hom}_{\mathcal{C}}(y, x)$$

If $\mathcal{X}, \mathcal{Y} \subset \mathcal{C}$ are subcategories then the action of $\mathcal{C} \otimes \mathcal{C}^{op}$ on \mathcal{C} pulls back to an action of $\mathcal{X} \otimes \mathcal{Y}^{op}$. The categories $\text{Ob}(\mathcal{OC})$ and \mathcal{O} are both subcategories of \mathcal{OC} . In this section we will study the open-closed category \mathcal{OC} as a $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule. We will see that it is flat and that given an \mathcal{O} -module it determines the complex associated to the torus object,

$$(\mathcal{OC} \otimes_{\mathcal{O}} A)(t)$$

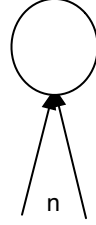
by the extension $\mathcal{OC} \otimes_{\mathcal{O}} - : \mathcal{O}\text{-mod} \rightarrow \mathcal{OC}\text{-mod}$. We will give an explicit description of this complex.

10.1 The Boundary Torus

As noted in 8.4 the boundary tori in the forested graph stratification of the space $X'_{(g,e,t)}$ are represented by trees containing the base vertex in addition to other parts of the cycle representing the boundary torus. This was illustrated in 8.5.2. Using the identification Theorem of 8.5,

$$C_*(X'_{(g,e,t)}; k) = \mathcal{G}_{(g,e,t)}(\text{Bar}(\mathbf{C})) \simeq \mathcal{G}_{(g,e,t)}(\mathbf{L}^*)$$

We can think of the complex computing the relevant homology as trivalent graphs containing forests, the trees of which satisfy the Jacobi (or IHX) relation - that is graphs labelled by $L(n)^*$ or C_∞ graphs, which by abuse of notation, will be denoted $\mathcal{G}_{(g,e,t)}(C_\infty)$. We will represent the boundary tori with subgraphs which for better or worse we will call bonnets. The bonnet with n incoming legs will be denoted $T(n)$. Pedantically, the graph is an $n+2$ corolla with 2 edges sewn together labelled by the generator m_{n+2} .



The boundary of the trivial bonnet, $T(0)$ is zero. While the boundary of the cell associated to the tori derives from the differential in the Cobar construction of 4.3, the sum over all edge expansions, which at the cellular level was shown to be the same as the sum over all edge deletions in 8.5.2. With C_∞ labellings this amounts to all possible ways to expand a collection of k incoming edges into a product.

We will see in a moment that these generate \mathcal{OC} as a $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule.

10.2 The Complex Associated with Extension

Theorem 13. *The category \mathcal{OC} when considered as an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule is freely generated by the bonnets $T(n)$.*

Proof. It follows from Corollary 5 in 8.6 that,

$$\mathcal{OC}((e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k}) = \coprod_g \mathcal{G}_{(g,n+k,m)}(C_\infty)$$

If $G \in \mathcal{G}_{(g,n+k,m)}(C_\infty)$ is a basis element then G is a C_∞ labelled graph from k open edges to i open edges and j bonnets. We can inductively absorb any of the

graph G that doesn't involve the bonnets into an orbit of a torus under the action of \mathcal{O} .

We need only consider $\text{Hom}_{\mathcal{OC}}(o^{\otimes k}, o^{\otimes i} \otimes t^{\otimes j})$ with $i = 0$ and $j = 1$. This is because we can change incoming edges into outgoing edges and vice versa using the inner product contained in the \mathcal{O} action and multiple bonnets must be composites of tori with respect to the open composition provided by the Cobar construction.

We now show that excluding the torus $T(n)$ the remainder of the graph can be absorbed into the action of \mathcal{O} . In what follows the labellings of the graphs are drawn on tiny disks taken to be small enough not to interfere.

Choose an embedding of G into \mathbb{R}^3 so that

1. The bonnet lies in the plane

$$\mathbb{R}^2 = \{(x, y, 0) | (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

2. The outgoing edges end in some fixed vertical translate $\mathcal{R}^2 + (0, 0, n)$ with $n \in \mathbb{Z}_+$.
3. Every internal vertex $v \in G$ is at a distinct height $h(v) \in \mathbb{Z}$, i.e. is the only vertex contained in the plane $\mathbb{R}^2 + (0, 0, h(v))$.

After choosing an embedding that satisfies the first two conditions, the third can be obtained by perturbing the graph if any two vertices are aligned and then stretching the graph that results (there is no metric structure on the diagram) gives three.

Then if $n = 0$ there are no vertices besides those contained in $T(n)$. If $n > 0$ then the portion of the graph contained in the region

$$\mathbb{R}^2 \times [n - 3/2, n] \subset \mathbb{R}^3$$

is a subgraph of G that can be identified as a composition of a single corolla with one or more inner product operations. The rest follows by induction.

□

Given a C_∞ algebra A together with invariant inner product we know that A defines uniquely an h-split \mathcal{O} module which, by abuse of notation, we will also call A . For each A the extension given by the tensor product allows us to define a functor $\mathcal{OC} \rightarrow \text{Ch}_k$.

The two theorems above tell us that A defines an open-closed topological field theory that is an h-split \mathcal{OC} module: $i_*(A)$. While the open category as a subcategory in \mathcal{OC} must act according to the C_∞^b module structure determined by the operad structure the chain complex associated to the torus $t \in \text{Ob}(\mathcal{OC})$ is new structure that is functorially determined by the geometry underlying this construction.

In what follows we unwind the definitions in order to determine the complex associated to the torus object.

Observation. *Let A be a homotopy commutative algebra with invariant inner product then the complex $i_*(A)$ associated to the boundary torus,*

$$i_*(A)(T) = (\mathcal{OC} \otimes_{\mathcal{O}} A)(T)$$

is isomorphic to the complex $\text{Torus}(A)$ defined below.

Recall that,

$$(\mathcal{OC} \otimes_{\mathcal{O}} A)(t) = \bigoplus_j \mathcal{OC}(t, e^{\otimes j}) \otimes_k A(e^{\otimes j}) = \bigoplus_j \text{Hom}_{\mathcal{OC}}(e^{\otimes j}, t) \otimes_k A^{\otimes j}$$

Modulo the action of \mathcal{O} given by the diagram,

$$\begin{array}{ccc} \mathcal{OC}(t, e^{\otimes k}) \otimes \mathcal{O}(e^{\otimes j}, e^{\otimes k}) \otimes A(e^{\otimes j}) & \longrightarrow & \mathcal{OC}(t, e^{\otimes j}) \otimes A(e^{\otimes j}) \\ \downarrow & & \downarrow \\ \mathcal{OC}(t, e^{\otimes k}) \otimes A(e^{\otimes k}) & \longrightarrow & (\mathcal{OC} \otimes_{\mathcal{O}} A)(t) \end{array}$$

As a left \mathcal{O} -mod each $f \in \text{Hom}_{\mathcal{O}}(e^{\otimes j}, e^{\otimes k})$ induces a map $f_* : A^{\otimes j} \rightarrow A^{\otimes k}$ and as a right \mathcal{O} -mod each such f induces a map,

$$f^* : \text{Hom}_{\mathcal{OC}}(e^{\otimes k}, t) \rightarrow \text{Hom}_{\mathcal{OC}}(e^{\otimes j}, t)$$

given by post-composition. If $g \otimes e^{\otimes k} \in \text{Hom}_{\mathcal{OC}}(e^{\otimes k}, t) \otimes A^{\otimes k}$, the diagram above amounts to the relation,

$$f^*(g) \otimes e^{\otimes k} \sim g \otimes f_*(e^{\otimes k})$$

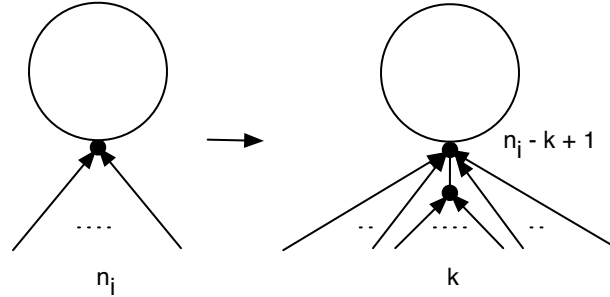
Now each complex $\text{Hom}_{\mathcal{OC}}(e^{\otimes j}, t)$ is quasi-isomorphic to a chain complex of graphs,

$$\text{Hom}_{\mathcal{OC}}(e^{\otimes j}, t) \simeq \bigoplus_g \mathcal{G}_{(g, e, t)}(\mathbb{C}_\infty)$$

containing one boundary torus and j outgoing edges which, by Theorem 13, under the action of \mathcal{O} is generated by the bonnets $T(n)$. So each equivalence class of $(\mathcal{OC} \otimes_{\mathcal{O}} A)(t)$ under the relation \sim has a unique representative of the form,

$$k \langle T(n) \rangle \otimes_k A^{\otimes n}$$

The differential is determined by the internal differential δ of A and the sum of all possible ways to add an edge to a collection of incoming edges at a vertex of the boundary torus. The latter amounts to the sum over pictures,



The orientation of the graphs on the right hand side is taken to be the one induced by the left hand side as described in the Cobar construction of section 4.3.

In algebraic form, let

$$p\text{Torus}(A) = \bigoplus_{j=1}^{\infty} A^{\otimes j}$$

Because the C_{∞} operad's generators m_n vanish on shuffle products and we've labelled the bonnets by elements of it we must quotient $p\text{Torus}(A)$ by the shuffle relations. Define the shuffle product of tensors by,

$$(a_1 \otimes \cdots \otimes a_i) * (a_{i+1} \otimes \cdots \otimes a_n) = \sum_{\sigma \in \text{Sh}(i, n-i)} \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

and let I be the ideal of $p\text{Torus}(A)$ generated by the images of the shuffle products and define,

$$\text{Torus}(A) = p\text{Torus}(A)/I$$

Then given the A_{∞} relation

$$d(a_1 \otimes \cdots \otimes a_n) = \sum_{\substack{i+j=n+1 \\ i, j \geq 2}} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} a_1 \otimes \cdots \otimes m_j(a_{s+h+1} \otimes \cdots \otimes a_{s+h+j+1}) \otimes \cdots \otimes a_n$$

An internal differential from A ,

$$\delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^n a_1 \otimes \cdots \otimes \partial(a_i) \otimes \cdots \otimes a_n$$

The differential on $\text{Torus}(A)$ is given by the sum of the two above.

10.3 Flatness and Exactness

Given a homologically split \mathcal{O} module or a C_{∞} algebra with invariant inner product the inclusion $i : \mathcal{O} \hookrightarrow \mathcal{OC}$ induces an extension,

$$i_*(A) = \mathcal{OC} \otimes_{\mathcal{O}}^{\mathbb{L}} A$$

In order for the extension to be an open-closed field theory in the sense discussed in section 5.2 we must show that $i_*(A)$ is h-split and in order to describe the complex $i_*(A)(t)$ a dramatic simplification can be made,

$$\mathcal{OC} \otimes_{\mathcal{O}}^{\mathbb{L}} A \simeq \mathcal{OC} \otimes_{\mathcal{O}} A$$

by observing that as an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule the category \mathcal{OC} is flat. The following theorems set out to accomplish the latter and the former.

In both cases the theorem follows because there is a natural filtration given on the bimodule \mathcal{OC} given by the degree of the bonnets. A bonnet with vertex labelled by m_n ultimately must come from a cell of underlying dimension $n - 2$. For instance the bonnet in degree 0 represented by a trivalent graph must come from the trivial forest (or zero dimensional cube) covering only the base point of the relevant cycle.

Define a filtration \mathcal{F} of \mathcal{OC} then so that $\mathcal{F}^0 \mathcal{OC}$ contains the identity elements of the open category $\mathcal{OC}(e^{\otimes i}, e^{\otimes i})$ for all i and the associated graded $\text{Gr}^n \mathcal{OC}$ is precisely the n th bonnet $T(n)$. Since $dT(n)$ is a sum of bonnets of lower degree this is a filtration of complexes. This induces a filtration \mathcal{F} on $\mathcal{OC} \otimes_{\mathcal{O}} A$ such that,

$$\text{Gr}^n(\mathcal{OC} \otimes_{\mathcal{O}} A)(e^{\otimes i} \otimes t^{\otimes j})$$

consists of placing the identity factors on the i edges and labelling the j bonnets by elements of $A^{\otimes n}$. Showing that this is true is a computation nearly identical to that of the previous section.

We will exploit the following familiar lemma,

Lemma 7. *If $\varphi : A \rightarrow A'$ is a map of filtered complexes such that $\varphi_0 : \mathcal{F}^0 A \rightarrow \mathcal{F}^0 A'$ is a quasi-isomorphism and $\varphi_* : \text{Gr}^n A \rightarrow \text{Gr}^n A'$ is a quasi-isomorphism then $\varphi_n : \mathcal{F}^n A \rightarrow \mathcal{F}^n A'$ is a quasi-isomorphism for all n . In particular φ is a quasi-isomorphism.*

Theorem 14. *If A is an h-split \mathcal{O} -module then $\mathcal{OC} \otimes_{\mathcal{O}} A$ is an h-split $\text{Ob}(\mathcal{OC})$ -module.*

Proof. We must check that the maps,

$$(\mathcal{OC} \otimes_{\mathcal{O}} A)(x) \otimes (\mathcal{OC} \otimes_{\mathcal{O}} A)(y) \rightarrow (\mathcal{OC} \otimes_{\mathcal{O}} A)(x \otimes y)$$

Are quasi-isomorphisms. Since this is true in filtration degree 0 it follows by induction if it holds for the associated graded. A collection of i bonnets labelled by A tensored with a collection of j bonnets labelled by A is quasi-isomorphic to a collection of $i + j$ bonnets labelled by A . \square

Theorem 15. *As an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule \mathcal{OC} is \mathcal{O} flat. That is, the functor $i_* : \mathcal{O}\text{-mod} \rightarrow \text{Ob}(\mathcal{OC})\text{-mod}$ given by*

$$i_*(A) = \mathcal{OC} \otimes_{\mathcal{O}} A$$

is exact.

Proof. Given a quasi-isomorphism of C_{∞} algebras $\varphi : A \rightarrow A'$. We must check that the induced map,

$$\mathcal{OC} \otimes_{\mathcal{O}} A \rightarrow \mathcal{OC} \otimes_{\mathcal{O}} A'$$

Is a quasi-isomorphism. Since this is true in filtration degree 0 it follows by induction if it holds for the associated graded.

$$\text{Gr}^n(\mathcal{OC} \otimes_{\mathcal{O}} A) \rightarrow \text{Gr}^n(\mathcal{OC} \otimes_{\mathcal{O}} A')$$

Is the map between bonnets labelled by tensor powers of A and A' induced by φ and so a quasi-isomorphism. \square

10.4 Deligne's Conjecture

Corollary 11. *There is an action of the category \mathcal{C} on the complex $\text{Torus}(A)$ specifically,*

$$\mathrm{Hom}_{\mathcal{C}}(t^{\otimes i}, t^{\otimes j}) \otimes \mathrm{Torus}_*(A)^{\otimes i} \rightarrow \mathrm{Torus}_*(A)^{\otimes i}$$

Proof. Notice if we consider A as a \mathcal{O} -mod and \mathcal{OC} as an $\mathcal{OC} - \mathcal{O}$ bimodule then we can define an \mathcal{OC} module associated to A by $\mathcal{OC} \otimes_{\mathcal{O}} A$. Given the inclusion $i : \mathcal{C} \hookrightarrow \mathcal{OC}$ then $i^*(\mathcal{OC} \otimes_{\mathcal{O}} A)$ is a \mathcal{C} -mod. If $X = i^*(\mathcal{OC} \otimes_{\mathcal{O}} A)(t)$ is the chain complex associated to the torus there is an action,

$$\mathrm{Hom}_{\mathcal{C}}(t^{\otimes i}, t^{\otimes j}) \otimes X^{\otimes i} \rightarrow X^{\otimes j}$$

which gives rise to the homology action,

$$H_*(\mathrm{Hom}_{\mathcal{C}}(t^{\otimes i}, t^{\otimes j})) \otimes H_*(X)^{\otimes i} \rightarrow H_*(X)^{\otimes j}$$

Earlier we considered \mathcal{OC} as an $\mathrm{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule and saw that $\mathrm{Torus}(A) = j^*(\mathcal{OC} \otimes_{\mathcal{O}} A)$. On the other hand the complex associated to the torus is independent of the choice of $\mathrm{Ob}(\mathcal{OC})$ verses \mathcal{OC} in considering \mathcal{OC} as a bimodule. In either case the definition is the same,

$$j^*(\mathcal{OC} \otimes_{\mathcal{O}} A)(t) = \bigoplus_k \mathrm{Hom}_{\mathcal{OC}}(e^{\otimes k}, t) \otimes A(e^{\otimes k}) = \bigoplus_k \left(\bigoplus_g \mathcal{G}_{(g, k+0, 1)}(\mathbb{C}_{\infty}) \right) \otimes A(e^{\otimes k})$$

So X is the same as $\mathrm{Torus}(A)$.

□

10.5 Harrison Homology

The operad \mathbb{C}_{∞} being quasi-isomorphic to the operad \mathbb{C} implies that the categories of differential graded homotopy commutative algebras and differential graded commutative algebras are equivalent. In what follows we define the Harrison complex for a dg commutative algebra.

Barr defined the Harrison complex of an associative commutative algebra to be the subcomplex $CH_*(A, A) \subset C_*(A, A)$ of the Hochschild complex spanned by the cokernel of the shuffle product on the free tensor algebra of A [Bar68].

Given the Barr complex of a differential graded associative algebra,

$$C_*(A, A) = \bigoplus_{n=0} A^{\otimes n+1}$$

and its differential

$$\begin{aligned} d(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} \pm a_0 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n \\ &\quad \pm m_2(a_0, a_n) \otimes a_1 \otimes \cdots \otimes a_{n-1} \\ &\quad + \sum_{i=0}^{n-2} \pm a_0 \otimes \cdots \otimes m_2(a_i, a_{i+1}) \otimes \cdots \otimes a_{n-1} \end{aligned}$$

The subcomplex of shuffle products is spanned by the set,

$$\text{Sh}_*(A, A) = \{(a_1 \otimes \cdots \otimes a_i) * (a_{i+1} \otimes \cdots \otimes a_n) \mid 1 < i < n\}$$

Then the Harrison complex is defined to be the quotient of the Barr complex by the subcomplex of shuffles.

$$CH_*(A, A) = C_*(A, A) / \text{Sh}_*(A, A)$$

Observation. *Given an associative commutative algebra A . The Harrison complex of A can be identified with the chain complex associated to the torus: $\text{Torus}(A)$.*

The observation is supported by the agreement between the internal differential in both cases. For a differential graded commutative algebra $m_n = 0$ for $n > 2$ and the second differential involving higher A_∞ terms becomes the second part of the sum above.

Bibliography

- [Bar68] Michael Barr. Harrison homology, Hochschild homology and triples. *J. Algebra*, 8:314–323, 1968.
- [BHV01] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann. Geometry of the space of phylogenetic trees. *Adv. in Appl. Math.*, 27(4):733–767, 2001.
- [Cos07] Kevin Costello. Topological conformal field theories and Calabi-Yau categories. *Adv. Math.*, 210(1):165–214, 2007.
- [CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.
- [CV03] James Conant and Karen Vogtmann. On a theorem of Kontsevich. *Algebr. Geom. Topol.*, 3:1167–1224 (electronic), 2003.
- [EZ53] Samuel Eilenberg and J. A. Zilber. On products of complexes. *Amer. J. Math.*, 75:200–204, 1953.
- [Get94] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.*, 159(2):265–285, 1994.
- [Gin] V. Ginzburg. Lectures on noncommutative geometry. *arXiv:math/0506603*.
- [GJ] Ezra Getzler and J.D.S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. *Unpublished Preprint*.
- [GK94] Victor Ginzburg and Mikhail Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.
- [GK95] E. Getzler and M. M. Kapranov. Cyclic operads and cyclic homology. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 167–201. Int. Press, Cambridge, MA, 1995.

- [GK98] E. Getzler and M.M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [HV98] Allen Hatcher and Karen Vogtmann. Rational homology of $\text{Aut}(F_n)$. *Math. Res. Lett.*, 5(6):759–780, 1998.
- [HV04] Allen Hatcher and Karen Vogtmann. Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.*, 4:1253–1272 (electronic), 2004.
- [HW05] Allen Hatcher and Nathalie Wahl. Stabilization for the automorphisms of free groups with boundaries. *Geom. Topol.*, 9:1295–1336 (electronic), 2005.
- [Igu02] Kiyoshi Igusa. *Higher Franz-Reidemeister torsion*, volume 31 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [JW04] Craig A. Jensen and Nathalie Wahl. Automorphisms of free groups with boundaries. *Algebr. Geom. Topol.*, 4:543–569, 2004.
- [Kel94] Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.
- [Kon94] Maxim Kontsevich. Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 97–121. Birkhäuser, Basel, 1994.
- [KS] Maxim Kontsevich and Yan Soibelman. Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I. *arxiv math/0606241*.
- [Lau74] François Laudenbach. *Topologie de la dimension trois: homotopie et isotopie*. Société Mathématique de France, Paris, 1974. With an English summary and table of contents, Astérisque, No. 12.
- [LV08] A. Lazarev and A. A. Voronov. Graph homology: Koszul and Verdier duality. *Adv. Math.*, 218(6):1878–1894, 2008.
- [Mar04] Martin Markl. Homotopy algebras are homotopy algebras. *Forum Math.*, 16(1):129–160, 2004.
- [May97] J. P. May. Operads, algebras and modules. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 15–31. Amer. Math. Soc., Providence, RI, 1997.

- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [MSS02] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [Seg] Graeme Segal. Lectures at Duke university. <http://www.cgtp.duke.edu/ITP99/segal/>.
- [Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.
- [Seg04] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.
- [Vor05] Alexander A. Voronov. Notes on universal algebra. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 81–103. Amer. Math. Soc., Providence, RI, 2005.