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GENERATING THE ENVELOPE OF A SWEPT TRIVARIATE SOLID

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Abstract

We present a method for calculating the envelope of the swept surface of a solid along a path in three-dimensional space. The generator of the swept surface is a trivariate tensor-product Bèzier solid and the path is a non-uniform rational B-spline curve. The boundary surface of the solid is the combination of parametric surfaces and an implicit surface where the determinant of the Jacobian of the defining function is zero. We define methods to calculate the envelope for each type of boundary surface, defining characteristic curves on the envelope and connecting them with triangle strips. The envelope of the swept solid is then calculated by taking the union of the envelopes of the family of boundary surfaces, defined by the surface of the solid in motion along the path.

Keywords: trivariate splines; sweeping; envelopes; solid modeling.

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1. Introduction

Geometric modeling systems construct complex models from objects based on simple geometric primitives. As the needs of these systems become more complex, it is necessary to expand the inventory of geometric primitives and design operations. In this paper, we address the generation of envelopes and the sweeping of free-form solid models.

The initial work on envelopes is that of Boltyanskii [1]. Flaquer *et al.* [3] analyze the envelope generated by sweeping a quadric surface. A more general study of sweeping solids using envelopes is given by Wang and Wang [7]. Here, it is assumed that the boundaries of the solid are known and can be represented parametrically. Joy [4] has discussed the sweeping of a trivariate solid, and uses a subdivision strategy to generate the image of the solid.

In this paper, we present a new algorithm to generate the envelope of a swept trivariate free-form solid. We draw on the results of Joy [4] and Duchaineau and Joy [2] to characterize the free-form solid, generating two types of boundary surfaces, each of which must be swept independently.

In Section 2, we describe the generation of envelopes and their mathematical properties. In Section 3, we describe the trivariate tensor-product Bézier solid and the calculation of points on the swept surface of the solid. Section 4 presents the implementation of the envelope generation algorithm and Section 5 shows some results of this work. Finally, Section 6 summarizes the contents of the paper.

2. Envelopes

In two-dimensions, the envelope of a family of curves $f(x, y, \alpha)$ is defined to be a curve (sometimes more than one) that is tangent at each of its points to a curve of the family, see Boltyanskii [1]. Every point on the envelope is the intersection of two members of the family that are infinitely close. For example, take two fairly close curves from a one-parameter family, say, $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \epsilon) = 0$ where ϵ is some constant not equal to zero. The intersection of these two curves is a point given by the simultaneous solution of the equations

$$\left.\begin{aligned} f(x,y,\alpha) &= 0 \\ \tfrac{f(x,y,\alpha+\epsilon)-f(x,y,\alpha)}{\epsilon} &= 0 \end{aligned}\right\} \quad . \tag{1}$$

To achieve the desired "infinitely close" relationship, let ϵ approach zero so that the intersection point approaches the limiting form

$$\left.\begin{aligned} f(x,y,\alpha) &= 0 \\ \frac{\partial}{\partial\alpha} f(x,y,\alpha) &= 0 \end{aligned}\right\} \quad . \tag{2}$$

If we can eliminate α from Equation (2), then we can explicitly write down the mathematical formulation for the envelope. In the case where the family of curves $F(t,\alpha) = (x(t,\alpha), y(t,\alpha))$ is given parametrically, it can be shown (see Stoker [6]) that the envelope is the locus of points in the family at which the determinant of the Jacobian of F vanishes, *i.e.*,

$$\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} \end{vmatrix} \quad = \quad 0 \quad . \tag{3}$$

When dealing with surfaces, the determination of the envelope (in this case, a surface) is similar to that for a curve. The intersection between two close members of the family of surfaces will be a curve instead of a point. The locus of these intersection curves will be the envelope surface. In this case, Equation (1) and Equation (2) generalize to the the following (see Spivak [5], Stoker [6], and Boltyanskii [1]): Suppose that a family of surfaces is defined by the equations

$$\left.\begin{aligned} f(x,y,z,\alpha_1,\alpha_2,...,\alpha_m) &= 0 \\ g_1(\alpha_1,\alpha_2,...,\alpha_m) &= 0 \\ &\vdots \\ g_{m-1}(\alpha_1,\alpha_2,...,\alpha_m) &= 0 \end{aligned}\right\} \quad . \tag{4}$$

Then every point of the envelope satisfies the equation obtained by adjoining

$$\begin{vmatrix} \frac{\partial f}{\partial \alpha_1} & \frac{\partial f}{\partial \alpha_2} & \cdots & \frac{\partial f}{\partial \alpha_m} \\ \frac{\partial g_1}{\partial \alpha_1} & \frac{\partial g_1}{\partial \alpha_2} & \cdots & \frac{\partial g_1}{\partial \alpha_m} \\ \frac{\partial g_2}{\partial \alpha_1} & \frac{\partial g_2}{\partial \alpha_2} & \cdots & \frac{\partial g_2}{\partial \alpha_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial \alpha_1} & \frac{\partial g_{m-1}}{\partial \alpha_2} & \cdots & \frac{\partial g_{m-1}}{\partial \alpha_m} \end{vmatrix} \quad = \quad 0$$

to the equations (4) and eliminating the parameters α_1, α_2, ... , α_m. In the same way, if the family of surfaces can be expressed in parametric form as $F(u,v,\alpha) = (x(u,v,\alpha), y(u,v,\alpha), z(u,v,\alpha))$,

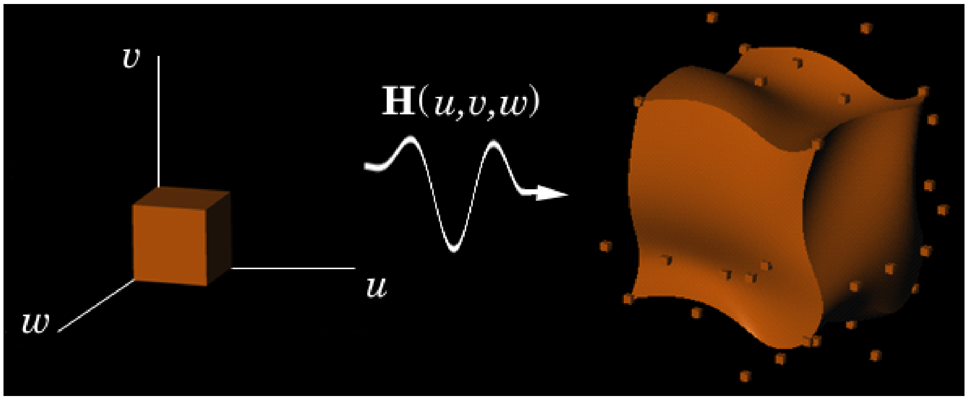


FIGURE 1: A trivariate tensor-product Bézier Solid $\mathbf{H}(u, v, w)$ shown with 64 control points (most of which are inside the solid).

then a point lies on the envelope surface if the determinant of the Jacobian of F vanishes at the point.

3. Sweeping a Trivariate Bézier Solid

The trivariate tensor-product Bézier solid is defined by a set of $(l+1) \times (m+1) \times (n+1)$ control points $\{H_{i,j,k} : 0 \leq i < l, 0 \leq j < m, 0 \leq k < n\}$, where

$$\mathbf{H}(u, v, w) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} H_{i,j,k} B_{i,p}(u) B_{j,q}(v) B_{k,r}(w)$$

where p is the degree of the solid in the u direction, q is the order of the spline in the v direction, and r is the order of the spline in the w direction. $B_{i,n}(t)$, $i = 1, ..., n$, are the nth degree Bernstein polynomials, and the parameters (u, v, w) are restricted to lie in a box, $u \in [u_0, u_1)$, $v \in [v_0, v_1)$, $w \in [w_0, w_1)$, in the domain space. The image of this trivariate solid is the envelope of the set of bivariate Bézier patches defined by treating either u, v or w as a fixed parameter and considering $\mathbf{H}$ as a function of only two independent variables. As the fixed parameter ranges over its domain interval, these patches form the family of surfaces for the envelope.

Since this solid can be described by its envelope surface, we can describe the surface of the solid by the six "boundary face patches"

$$
\begin{aligned}
B_1 &= \{\mathbf{H}(u_0, v, w) : v \in [v_0, v_1), w \in [w_0, w_1)\}, \\
B_2 &= \{\mathbf{H}(u_1, v, w) : v \in [v_0, v_1), w \in [w_0, w_1)\}, \\
B_3 &= \{\mathbf{H}(u, v_0, w) : u \in [u_0, u_1), w \in [w_0, w_1)\}, \\
B_4 &= \{\mathbf{H}(u, v_1, w) : u \in [u_0, u_1), w \in [w_0, w_1)\}, \\
B_5 &= \{\mathbf{H}(u, v, w_0) : u \in [u_0, u_1), v \in [v_0, v_1)\}, \\
B_6 &= \{\mathbf{H}(u, v, w_1) : u \in [u_0, u_1), v \in [v_0, v_1)\}
\end{aligned}
\tag{5}
$$

and the set of points on the trivariate solid at which the determinant of the Jacobian vanishes. Here, the determinant of the Jacobian is equal to

$$
\mathcal{J}(\mathbf{H}(u, v, w)) = \begin{vmatrix} \frac{\partial x}{\partial u}(u, v, w) & \frac{\partial y}{\partial u}(u, v, w) & \frac{\partial z}{\partial u}(u, v, w) \\ \frac{\partial x}{\partial v}(u, v, w) & \frac{\partial y}{\partial v}(u, v, w) & \frac{\partial z}{\partial v}(u, v, w) \\ \frac{\partial x}{\partial w}(u, v, w) & \frac{\partial y}{\partial w}(u, v, w) & \frac{\partial z}{\partial w}(u, v, w) \end{vmatrix} \tag{6}
$$

which is the *scalar triple product* of the vectors $\frac{\partial}{\partial u}\mathbf{H}$, $\frac{\partial}{\partial v}\mathbf{H}$, $\frac{\partial}{\partial w}\mathbf{H}$. That is, equation (6) can be rewritten as

$$
\mathcal{J}(\mathbf{H}(u, v, w)) = \frac{\partial}{\partial u}\mathbf{H} \cdot (\frac{\partial}{\partial v}H \times \frac{\partial}{\partial w}\mathbf{H}).
$$

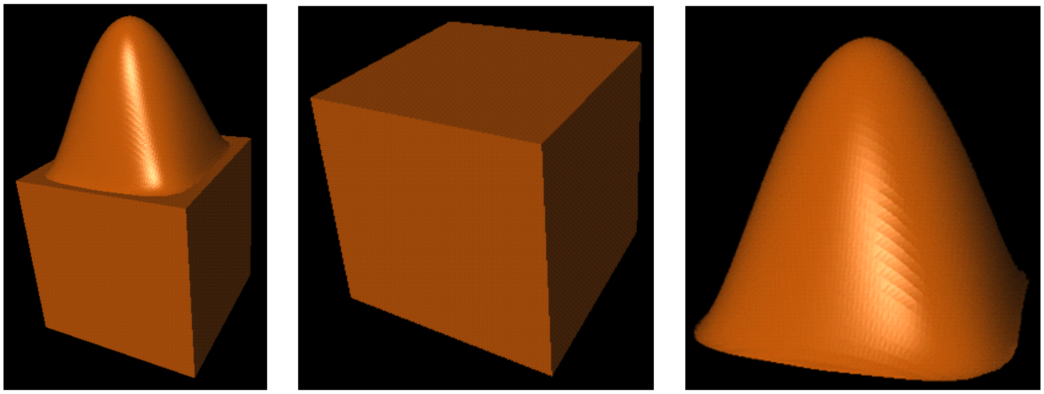
Thus, the Jacobian determinant vanishes if, and only if, the scalar triple product vanishes. Since three vectors are lineary independent if, and only if, their scalar triple product is not zero, we have that the vectors $\frac{\partial}{\partial u}\mathbf{H}$, $\frac{\partial}{\partial v}\mathbf{H}$, $\frac{\partial}{\partial w}\mathbf{H}$ are coplanar on the surface of the trivariate solid.

Figure 2a illustrates a trivariate solid $\mathbf{H}$. This solid was constructed by generating 64-control-point trivariate cubic Bèzier "cube" and translating four of its inner control points outside of the cube. Figure 2b shows the boundary face patches of $\mathbf{H}$, while Figure 2c shows the implicit surface. The implicit boundary surface corresponds to the points where the Jacobian determinant vanishes.

4. Sweeping the Bézier Hyperpatch

To calculate the envelope of a swept trivariate solid, we must independently calculate the envelope for the swept boundary face patches and the envelope for the implicit boundary surface.

Considering the implicit surface, the three vectors, $\frac{\partial}{\partial u}\mathbf{H}$, $\frac{\partial}{\partial v}\mathbf{H}$, and $\frac{\partial}{\partial w}\mathbf{H}$, represent the tangent vectors with respect to u, v, w, respectively . For a point (x, y, z) on the implicit



(a) (b) (c)

FIGURE 2: A trivariate solid: The complete solid is shown in (a); The boundary face patches are shown in (b); and the implicit boundary surface is shown in (c).

boundary surface, the tangent vectors form a linearly dependent set, which means they are coplanar[1] and lie in the plane tangent to the surface at that point. The points on the surface that lie on the envelope are those at which the tangent plane is parallel to the direction in which the trivariate solid is being swept, in this case the direction of the path.

The input to the algorithm is the trivariate solid generator, the B-spline trajectory over which the generator is swept, and a set of frames defined on the path that determine the orientation of the generator at each point of the path.

To find the envelope of the swept implicit surface, we step along the path, and at each step calculate a characteristic curve (a curve on the envelope). Since the tangent vectors are coplanar on the surface, the normal at a point on the surface can be calculated by taking the cross product of two independent tangent vectors.

Using a sampling of points from the implicit boundary surface (see [2]), and for each step along the path, sample two adjacent points, p_1 and p_2, from the implicit boundary surface and find the normals, n_1 and n_2, of the surface at p_1 and p_2. Using the input frames, calculate the sweep directions D_1 and D_2 for each point. Then if $d_1 = D_1 \cdot n_1$ and $d_2 = D_2 \cdot n_2$, and if $d_1 \leq 0 \leq d_2$ or $d_2 \leq 0 \leq d_1$, then we define a point on the characteristic curve to be an interpolated value of these two points depending on the two dot products.

For two consecutive characteristic curves, triangles are generated by connecting the two curves in a ”zipper fashion” as a triangle strip.

[1]In this paper, we do not consider the degenerate case where the vectors are collinear.

To generate the image of the swept boundary face patches, B_1, B_2, B_3, B_4, B_5, and B_6, defined in Equation (5), we find the characteristic curve of each boundary, as was done above for the implicit boundary surface. Here, the normal is found by taking the cross product of the linearly independent tangent vectors defined on the surface patch. The twelve edges of the boundary are then swept. The complete envelope is the union of these surfaces.

Thus, the trivariate solid is swept in the following manner:

- The surface of the trivariate solid is rendered at the starting and ending positions on the path.
- Each of the six Bèzier boundary face patches are swept individually. To do this, the edges are swept and the envelope of the patch along the curve is found.
- The set of points are found which lie on the implicit boundary surface. Then, the envelope of the approximated surface defined by these points is found.
- The above surfaces are combined to obtain the final result.

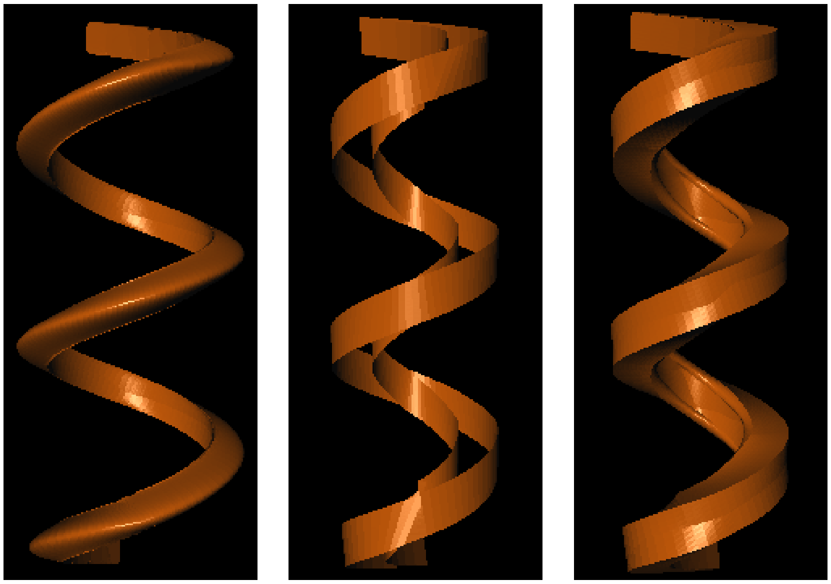
5. Results

To illustrate the results of this method we utilize the generator shown in Figure 2. Figure 3 shows an illustration of this generator swept along a helical path. The frame used to orient the generator is the Frenet frame of the path. Figure 3 shows the swept surface of the implicit boundary, the locus of the characteristic curves of the image boundary patches (B_1 through B_6), and the swept edges of the image boundary patches, respectively. The resulting swept solid is the union of these surfaces.

There are three types of motion illustrated in the Figures 4–8 below.

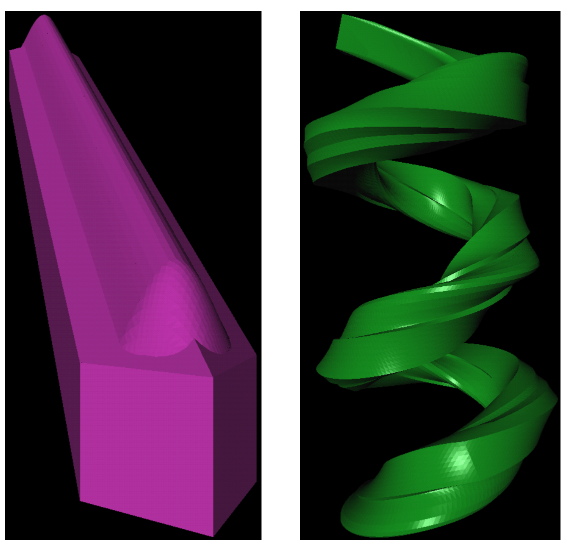
- An upright sweep – The orientation of the generator does not change throughout the sweep, regardless of the path. This is illustrated in Figure 4a.
- A local sweep – The orientation of the generator is dependent on the Frenet frame of the path. This is illustrated in Figure 5.
- A rotating sweep – Like the local sweep, the generator's orientation is dependent on the frame of the path, except here, there is rotation about the tangent of the path, as well. This is illustrated in Figure 4b, Figure 6, and Figure 8.

The trivariate solid generators used in these sweeps were chosen specifically so that the envelope surfaces cannot be described by the six boundary patches alone.



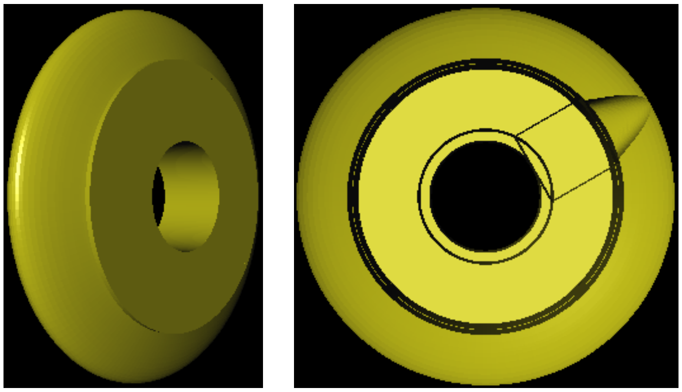
(a) (b) (c)

FIGURE 3: Image (a) shows the implicit boundary surface swept along the helical path. Image (b) is the locus of characteristic curves of the boundary face patches along the path. Image (c) shows the swept edges of the image boundary patches. The resulting envelope is the union of these surfaces.



(a) (b)

FIGURE 4: Image (a) is the swept surface of the trivariate solid from Figure 2.1 along a diagonal. This is an example of an upright sweep. In image (b), the same trivariate solid is swept along the helix from Figure 3, except in a rotating sweep. The total amount of rotation is $360°$.



(a) (b)

FIGURE 5: Image (a) shows the local sweep of the trivariate solid swept along a plane circle. Image (b) is a clipped version of image (a). The generator is seen here in its starting (and finishing) position.

FIGURE 6: The same sweep as was shown in Figure 5 except with a rotating sweep.

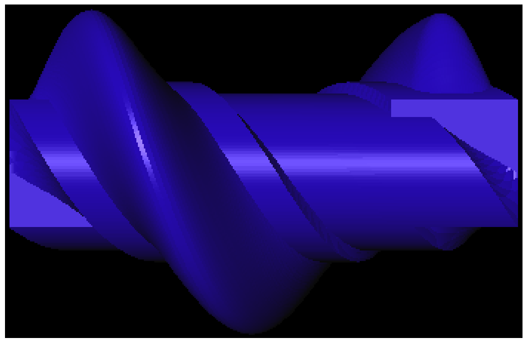


FIGURE 7: Image is the swept surface of the trivariate solid rotating a total of $360°$ along a horizontal path.

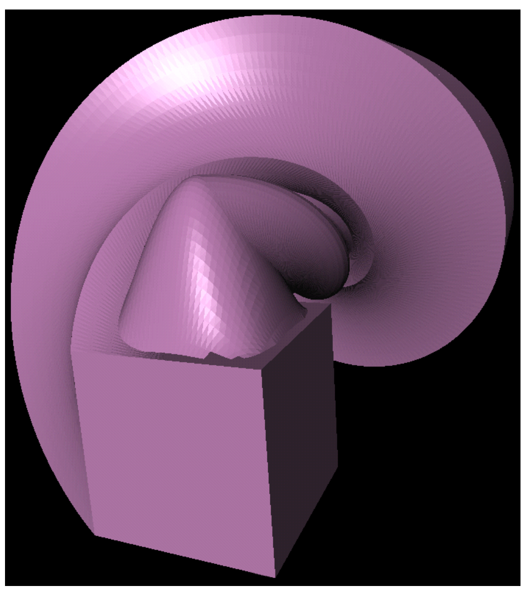


FIGURE 8: This was the same sweep as was shown in Figure 7 where the center of rotation has been shifted.

The pictures in this paper were generated on an SGI Indigo^2 with a 150 MHZ R4400 with 64 MB of RAM.

6. Conclusions

In this paper, we have discussed a method for calcuating the envelope generated by sweeping a trivariate tensor-product Bézier solid along an arbirary path. We developed the mathematics of the envelope surfaces and exhibited a method for calculating these envelopes by separating the boundary surfaces of the trivariate solids into two sets – the parametric boundary face patches, and the implicit boundary surface. We generated characteristic curves on the envelope for each step and generated a triangular mesh by connecting the characteristic curves.

This method works is a large number of cases, but it is not robust, and can fail on very complex generators. Our future work is to generate a robust method based on isosurface techniques that handles even the most complex cases.

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