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Comparison of formulas for resonant interactions between energetic electrons and oblique whistler-mode waves

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Test particle simulation is a useful method for studying both linear and nonlinear wave-particle interactions in the magnetosphere. The gyro-averaged equations of particle motion for first-order and other cyclotron harmonic resonances with oblique whistler-mode waves were first derived by Bell [J. Geophys. Res. **89**, 905 (1984)] and the most recent relativistic form was given by Ginet and Albert [Phys. Fluids B **3**, 2994 (1991)], and Bortnik [Ph.D. thesis (Stanford University, 2004), p. 40]. However, recently we found there was a $(-1)^{l-1}$ term difference between their formulas of perpendicular motion for the *l*th-order resonance. This article presents the detailed derivation process of the generalized resonance formulas, and suggests a check of the signs for self-consistency, which is independent of the choice of conventions, that is, the energy variation equation resulting from the momentum equations should not contain any wave magnetic components, simply because the magnetic field does not contribute to changes of particle energy. In addition, we show that the wave centripetal force, which was considered small and was neglect in previous studies of nonlinear interactions, has a profound time derivative and can significantly enhance electron phase trapping especially in high frequency waves. This force can also bounce the low pitch angle particles out of the loss cone. We justify both the sign problem and the missing wave centripetal force by demonstrating wave-particle interaction examples, and comparing the gyro-averaged particle motion to the full particle motion under the Lorentz force. *© 2015 AIP Publishing LLC.*
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I. INTRODUCTION

Relativistic electrons trapped in the Earth's magnetic field constitute the outer radiation belt which is highly dynamic.[4] Recent measurements from the Van Allen Probes have shown that the 2.5 MeV electron flux increased by ~4 orders of magnitude within 1 day during a geomagnetic storm, suggesting that these electrons are heated by internal acceleration in the heart of the outer radiation belt, i.e., resonant interactions with various naturally occurring magnetospheric Very Low Frequency waves.[5,6] Quasi-linear theory has been widely used to study the effects of resonant interactions.[7–13] However, the quasilinear approach inherently neglects nonlinear effects arising from the interactions between radiation belt electrons and various magnetospheric waves, and these nonlinear effects have been studied using test-particle simulations.[14–17]

The motion of charged particles in an electromagnetic field is governed by the Lorentz force, and thus can be modeled by integrating the Lorentz equation directly.[18] But it is often more instructive and computationally economical to gyro-average the equations so that numerical integration can proceed on time scales comparable to the gyro-period. The gyro-averaged formulas for Landau resonant interactions between non-relativistic electrons and obliquely propagating whistler-mode waves were first given by Inan and Tkalcevic.[19] Then, this approach was applied by Bell[1] to model first-order cyclotron resonant interactions, and was generalized to arbitrary harmonic resonance in a phase trapping study.[1] The relativistic interaction formulas for arbitrary harmonic resonances with oblique waves were derived by Ginet and Albert[2] and Bortnik,[3] and have been applied to many studies using test particle simulations.[20–22]

Recently, however, we found that a $(-1)^{l-1}$ factor difference in the perpendicular motion for the *l*th order harmonic resonance between the formulas given by Bortnik[3,20] and by Ginet and Albert,[2] and the resultant particle perpendicular motions calculated by their formulas behave differently for Landau and other even-order resonances. We point out that Bortnik's formulas are not self-consistent because they do not meet the simple criteria that the consequent

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energy equation should not contain the contribution of the wave magnetic components.

In addition, we suggest that the interaction phase variation should include a centripetal acceleration force term caused by the wave. Although it is a small term, it possesses a profound time derivative and offers an additional "driving force" for phase trapping motion, causing much more electrons stably trapped in resonance. With a $p_\perp$ factor in the denominator, the centripetal force can also bounce the small pitch angle particles out of the loss cone.

In Sec. II, we describe the whistler-mode wave model and present the detailed derivation process of gyro-averaged motion equations for resonant interactions of a general harmonic between electrons and oblique whistler-mode waves. In Sec. III, we check the generalized resonance formulas developed by Bell[1] and Bortnik[3] by transforming to their conventions, and justify the sign correction of their formulas for test particle simulations. Finally, in Sec. IV, we show that the wave centripetal force plays an important role in particle non-linear motions via enhancing phase trapping and bouncing low pitch angle electrons out of loss cone, and thus should be taken into account in the resonance formulas.

II. GYRO-AVERAGED MOTION FOR RESONANT INTERACTION

A. Whistler-mode wave model

We choose a Cartesian coordinate system in which the z axis is oriented along the background magnetic field line. The magnetic and electric fields of a monochromatic whistler-mode wave that propagates in the x–z plane at an angle ψ with respect to the z axis, as shown in Fig. 1 are written as

$$\mathbf{B}^w = \mathbf{B}_0^w e^{i\Phi}, \qquad \mathbf{E}^w = \mathbf{E}_0^w e^{i\Phi}, \tag{1}$$

where $\Phi = \omega t - \int \mathbf{k}\cdot d\mathbf{r}$ is the wave phase, and the wave normal vector is given as $\mathbf{k} = (k_\perp, 0, k_\parallel)$, where $k_\perp = k\sin\psi$ and $k_\parallel = k\cos\psi$. The relative ratios of the wave components can be obtained according to the cold plasma

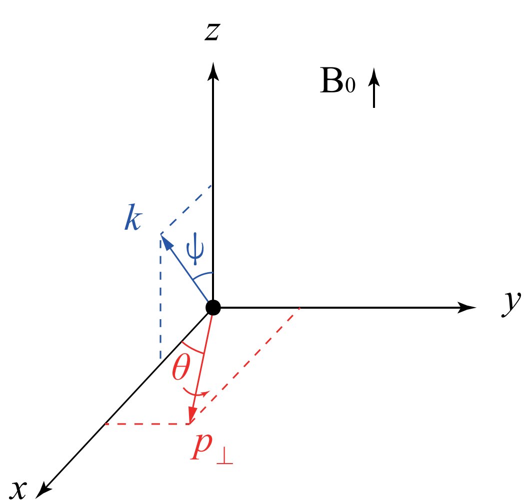

FIG. 1. The geometry of wave propagation and electron perpendicular momentum $\mathbf{p}_\perp$.

dispersion relation,[23] which is a reasonable assumption for whistler mode waves in the Earth's inner magnetosphere.[24] In the rational form, the magnetic and electric fields of a whistler-mode plasma wave are given by

$$\mathbf{B}^w = \mathbf{e}_x B_x^w \cos\Phi + \mathbf{e}_y B_y^w \sin\Phi + \mathbf{e}_z B_z^w \cos\Phi, \tag{2a}$$

$$\mathbf{E}^w = \mathbf{e}_x E_x^w \sin\Phi + \mathbf{e}_y E_y^w \cos\Phi + \mathbf{e}_z E_z^w \sin\Phi. \tag{2b}$$

Given the total magnetic wave amplitude B_{tot}^w, the components of $\mathbf{E}_0^w$ and $\mathbf{B}_0^w$ are as follows:

$$E_x^w = I^w(S-\mu^2)(P-\mu^2\sin^2\psi), \tag{3a}$$

$$E_y^w = I^w D(P-\mu^2\sin^2\psi), \tag{3b}$$

$$E_z^w = -I^w\mu^2\cos\psi\sin\psi(S-\mu^2), \tag{3c}$$

$$B_x^w = -I^w D\cos\psi(P-\mu^2\sin^2\psi)\mu/c, \tag{3d}$$

$$B_y^w = I^w P\cos\psi(S-\mu^2)\mu/c, \tag{3e}$$

$$B_z^w = I^w D\sin\psi(P-\mu^2\sin^2\psi)\mu/c, \tag{3f}$$

where

$$I^w = \frac{B_{tot}^w}{\mu\sqrt{D^2\left(P-\mu^2\sin^2\psi\right)^2 + P^2\cos^2\psi(S-\mu^2)^2}}. \tag{4}$$

Here P, S, and D are the usual Stix parameters,[23] and $\mu = kc/\omega$ is the refractive index. The wave model described above takes a most conventional form, which is different from related literature,[1–3,19,25,26] where various kinds of conventions of ψ and Φ, and positive direction of z axis, parallel wave vector number $k_\parallel$, and parallel particle momentum $p_\parallel$ were defined.

B. Particle adiabatic motion in ($\boldsymbol{e_z}$, $\boldsymbol{e_\perp}$, $\boldsymbol{e_\theta}$) coordinates

The force experienced by an electron moving in an electromagnetic field is described by the Lorentz equation,

$$\dot{\mathbf{p}} = -e\left[\mathbf{E}^w + \frac{\mathbf{p}}{\gamma m}\times(\mathbf{B}+\mathbf{B}^w)\right]. \tag{5}$$

Here, $\mathbf{p}$ represents the momentum of the particle, m its rest mass, e the elementary charge, $\gamma \equiv \sqrt{1+p^2/m^2c^2}$ the relativistic factor, $\mathbf{B}$ the ambient magnetic field, and the electric and magnetic waves are expressed in Eq. (2). Although we can directly integrate the Lorentz equation using standard ordinary differential equation integrators, it is often more convenient to solve the particle motion in a gyro-averaged sense so that numerical integration can proceed on a time scale comparable to the gyro-period. In the rotating coordinate system, the Lorentz force (5) can be decomposed in three orthogonal directions as

$$\dot{p}_\parallel = -e\left[\mathbf{E}^w + \frac{\mathbf{p}}{\gamma m}\times(\mathbf{B}^w+\mathbf{B})\right]\cdot\mathbf{e}_z, \tag{6a}$$

$$\dot{p}_\perp = -e\left[\mathbf{E}^w + \frac{\mathbf{p}}{\gamma m}\times(\mathbf{B}^w+\mathbf{B})\right]\cdot\mathbf{e}_\perp, \tag{6b}$$

$$\dot{p}_\theta = -e\left[\mathbf{E}^w + \frac{\mathbf{p}}{\gamma m} \times (\mathbf{B}^w + \mathbf{B})\right] \cdot \mathbf{e}_\theta. \tag{6c}$$

Here, $p_\parallel$ and $p_\perp$ represents the particle parallel and perpendicular momentum, respectively, and $\dot{p}_\theta$ represents the centripetal force that causes the particle gyro-motion. The elementary vector $\mathbf{e}_\perp$ points towards the particle momentum, and $\mathbf{e}_\theta$ points to the guiding-center, as illustrated in Figure 2. Applying the identities $\mathbf{e}_z \times \mathbf{p} = p_\perp \mathbf{e}_\theta$, $\mathbf{e}_\perp \times \mathbf{p} = -p_\parallel \mathbf{e}_\theta$, and $\mathbf{e}_\theta \times \mathbf{p} = p_\parallel \mathbf{e}_\perp - p_\perp \mathbf{e}_z$, together with $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$, the electron motion in each direction can be expressed as

$$\dot{p}_\parallel = -e\mathbf{E}^w \cdot \mathbf{e}_z - \frac{ep_\perp}{\gamma m}(\mathbf{B}^w + \mathbf{B}) \cdot \mathbf{e}_\theta, \tag{7a}$$

$$\dot{p}_\perp = -e\mathbf{E}^w \cdot \mathbf{e}_\perp + \frac{ep_\parallel}{\gamma m}(\mathbf{B}^w + \mathbf{B}) \cdot \mathbf{e}_\theta, \tag{7b}$$

$$\dot{p}_\theta = -e\mathbf{E}^w \cdot \mathbf{e}_\theta - \frac{ep_\parallel}{\gamma m}(\mathbf{B}^w + \mathbf{B}) \cdot \mathbf{e}_\perp + \frac{ep_\perp}{\gamma m}(\mathbf{B}^w + \mathbf{B}) \cdot \mathbf{e}_z. \tag{7c}$$

We first gyro-average the particle adiabatic motions caused by the "mirror" force of the ambient magnetic field $\mathbf{B}$, which can be decomposed as $\mathbf{B} = B\mathbf{e}_z + \mathbf{B}_\perp$ to first order. Here, B represents the magnetic magnitude, and the perpendicular component $\mathbf{B}_\perp$ can be Taylor expanded as

$$\mathbf{B}_\perp = \mathbf{e}_x \frac{\partial B}{\partial x} \rho \sin\theta - \mathbf{e}_y \frac{\partial B}{\partial y} \rho \cos\theta, \tag{8}$$

where the particle gyro-phase θ is defined as the angle between the x axis and the perpendicular momentum $\mathbf{p}_\perp$, and $\rho = p_\perp/eB$ represents the particle gyro-radius. Here, we have dropped the second order term of $\epsilon \sim \rho/R_E$, where R_E is the radius of the Earth. This approach neglects the curvature and perpendicular gradient of the background magnetic field, and therefore the particle drifting motion is also neglected. Using Eq. (8) and the fact that $\mathbf{e}_\theta = (-\sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_y)$, we obtain

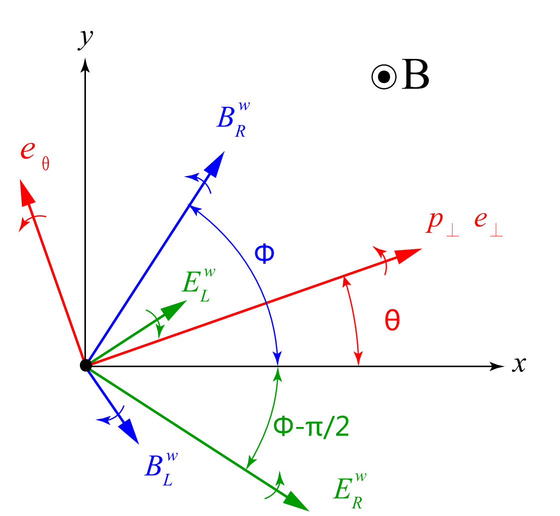

FIG. 2. The geometry of oblique whistler-mode wave phase and electron momentum phase. The wave phase for $\mathbf{B}_R, \mathbf{B}_L, \mathbf{E}_R$, and $\mathbf{E}_L$ are Φ, $-\Phi$, $\Phi - \pi/2$, and $\pi/2 - \Phi$, respectively.

$$\mathbf{B} \cdot \mathbf{e}_\theta = -\frac{\partial B}{\partial x} \rho \sin^2\theta - \frac{\partial B}{\partial y} \rho \cos^2\theta. \tag{9}$$

Gyro-averaging the above expression and applying Gauss's theorem $\nabla \cdot \mathbf{B} = 0$ yields

$$\langle \mathbf{B} \cdot \mathbf{e}_\theta \rangle = \frac{\rho}{2} \frac{\partial B}{\partial z}, \tag{10}$$

where the angle bracket $\langle\rangle$ represents the averaging operation over the gyro-motion. Similarly, the fact that $\mathbf{e}_\perp = (\cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y)$ yields

$$\langle \mathbf{B} \cdot \mathbf{e}_\perp \rangle = 0.$$

The resultant components of the adiabatic motion in three directions are

$$\dot{p}_\parallel^{ad} = -\frac{1}{2B} \frac{p_\perp^2}{\gamma m} \frac{\partial B}{\partial z}, \tag{11a}$$

$$\dot{p}_\perp^{ad} = \frac{1}{2B} \frac{p_\perp p_\parallel}{\gamma m} \frac{\partial B}{\partial z}, \tag{11b}$$

$$\dot{p}_\theta^{ad} = \frac{eB}{\gamma m} p_\perp. \tag{11c}$$

C. Particle motion under resonant interactions

To apply gyro-averaging, the perpendicular electric and magnetic waves are decomposed into two circularly polarized components with opposite senses of rotation, and the wave components are expressed as

$$\mathbf{B}_R = B_R[\mathbf{e}_x \cos\Phi + \mathbf{e}_y \sin\Phi], \tag{12a}$$

$$\mathbf{B}_L = B_L[\mathbf{e}_x \cos\Phi - \mathbf{e}_y \sin\Phi], \tag{12b}$$

$$\mathbf{B}_z = B_z^w \mathbf{e}_z \cos\Phi, \tag{12c}$$

$$\mathbf{E}_R = E_R[\mathbf{e}_x \sin\Phi - \mathbf{e}_y \cos\Phi], \tag{12d}$$

$$\mathbf{E}_L = E_L[\mathbf{e}_x \sin\Phi + \mathbf{e}_y \cos\Phi], \tag{12e}$$

$$\mathbf{E}_z = E_z^w \mathbf{e}_z \sin\Phi, \tag{12f}$$

where $B_R = (B_x^w + B_y^w)/2$, $B_L = (B_x^w - B_y^w)/2$, $E_R = (E_x^w - E_y^w)/2$, and $E_L = (E_x^w + E_y^w)/2$. The particle motion modulated by the wave is

$$\dot{p}_\parallel^w = -e\mathbf{E}_z \cdot \mathbf{e}_z - \frac{ep_\perp}{\gamma m}(\mathbf{B}_R + \mathbf{B}_L) \cdot \mathbf{e}_\theta, \tag{13a}$$

$$\dot{p}_\perp^w = -e(\mathbf{E}_R + \mathbf{E}_L) \cdot \mathbf{e}_\perp + \frac{ep_\parallel}{\gamma m}(\mathbf{B}_R + \mathbf{B}_L) \cdot \mathbf{e}_\theta, \tag{13b}$$

$$\dot{p}_\theta^w = -e(\mathbf{E}_R + \mathbf{E}_L) \cdot \mathbf{e}_\theta - \frac{ep_\parallel}{\gamma m}(\mathbf{B}_R + \mathbf{B}_L) \cdot \mathbf{e}_\perp + \frac{ep_\perp}{\gamma m}\mathbf{B}_z^w \cdot \mathbf{e}_z. \tag{13c}$$

Through Figure 2, which shows the phase geometry of wave electric and magnetic components, and of electron perpendicular momentum as well, we can obtain

$$\dot{p}_\parallel^w = -eE_z^w \sin\Phi - \frac{ep_\perp}{\gamma m}\left[B_R \sin(\Phi - \theta) - B_L \sin(\Phi + \theta)\right], \tag{14a}$$

$$\dot{p}_{\perp}^{w} = -eE_R \sin(\Phi-\theta) - eE_L \sin(\Phi+\theta) + \frac{ep_{\parallel}}{\gamma m}\left[B_R \sin(\Phi-\theta) - B_L \sin(\Phi+\theta)\right], \tag{14b}$$

$$\dot{p}_{\theta}^{w} = eE_R \cos(\Phi-\theta) - eE_L \cos(\Phi+\theta) - \frac{ep_{\parallel}}{\gamma m}\left[B_R \cos(\Phi-\theta) + B_L \cos(\Phi+\theta)\right] + \frac{ep_{\perp}}{\gamma m} B_z^w \cos\Phi. \tag{14c}$$

The wave phase at the position of the electron can be expressed as

$$\Phi = \omega t - \int k_{\parallel} v_{\parallel} dt - \beta \sin\theta, \tag{15}$$

where $\beta = k_{\perp} p_{\perp}/eB$, and the $\beta \sin\theta$ term comes from $k_{\perp} x$ in our definition of Φ in Eq. (1). Using the well-known Bessel function identity,

$$e^{i\beta \sin\theta} = \sum_{l=-\infty}^{\infty} J_l(\beta) e^{il\theta}, \tag{16}$$

where J_l represents the Bessel function of the first type with the argument β, we obtain

$$\begin{aligned}\sin\Phi &= \Re\mathfrak{e}\left\{-ie^{i\left(\omega t - \int k_{\parallel} v_{\parallel} dt - \beta\sin\theta\right)}\right\} \\ &= \Re\mathfrak{e}\left\{-ie^{i\left(\omega t - \int k_{\parallel} v_{\parallel} dt\right)} \sum_{l=-\infty}^{\infty} J_l(\beta) e^{-il\theta}\right\} \\ &= \sum_{l=-\infty}^{\infty} J_l(\beta) \sin\eta_l,\end{aligned} \tag{17}$$

where $\Re\mathfrak{e}$ represents the real part, and

$$\eta_l = \omega t - \int k_{\parallel} v_{\parallel} dt - l\theta \tag{18}$$

represent a wave-particle interaction phase. Similarly, we have

$$\sin(\Phi - n\theta) = \sum_{l=-\infty}^{\infty} J_{l-n}(\beta) \sin\eta_l, \tag{19a}$$

$$\cos(\Phi - n\theta) = \sum_{l=-\infty}^{\infty} J_{l-n}(\beta) \cos\eta_l. \tag{19b}$$

Usually, the integration of $\sin\eta_l$ and $\cos\eta_l$ over time averages to zero, except in the case of $d\eta_l/dt = 0$ or

$$\omega - k_{\parallel} v_{\parallel} = l\omega_{ce}/\gamma, \tag{20}$$

where $\omega_{ce} = eB/m$ is the non-relativistic electron gyrofrequency in the background magnetic field, $\gamma = \sqrt{1+(p/mc)^2}$ the relativistic factor. The above equation describes the lth-order cyclotron resonance condition, indicating that the Doppler shifted wave frequency observed by the particle is equal to the lth harmonic of the electron gyrofrequency, where $l = 0$ represents Landau resonance.

The wave modulation on particle motion near the lth-order resonance becomes

$$\dot{p}_{\parallel}^{w} = F_{\parallel}^{w} \sin\eta, \quad \dot{p}_{\perp}^{w} = F_{\perp}^{w} \sin\eta, \quad \dot{p}_{\theta}^{w} = -F_{\theta}^{w} \cos\eta, \tag{21a}$$

where

$$F_{\parallel}^{w} = -e\left[E_z^w J_l + \frac{p_{\perp}}{\gamma m} B_R J_{l-1} - \frac{p_{\perp}}{\gamma m} B_L J_{l+1}\right], \tag{22a}$$

$$F_{\perp}^{w} = -e\left[E_R J_{l-1} + E_L J_{l+1} - \frac{p_{\parallel}}{\gamma m} B_R J_{l-1} + \frac{p_{\parallel}}{\gamma m} B_L J_{l+1}\right], \tag{22b}$$

$$F_{\theta}^{w} = -e\left[E_R J_{l-1} - E_L J_{l+1} - \frac{p_{\parallel}}{\gamma m} B_R J_{l-1} - \frac{p_{\parallel}}{\gamma m} B_L J_{l+1} + \frac{p_{\perp}}{\gamma m} B_z^w J_l\right]. \tag{22c}$$

The relation between $F_{\parallel}^{w}$ and $F_{\perp}^{w}$ is simplified in the Appendix with details, and is expressed as

$$F_{\perp}^{w} = F_{\parallel}^{w} \frac{l\omega_{ce}}{\gamma k_{\parallel} v_{\perp}} = F_{\parallel}^{w} \frac{l\omega_{ce}}{\gamma\omega - l\omega_{ce}} \cot\alpha, \tag{23}$$

where the pitch angle α is defined as $\alpha = \tan^{-1}(p_{\perp}/p_{\parallel})$. This relation is equivalent to the resonant diffusion curve equation in the $(p_{\parallel}, p_{\perp})$ plane.[27]

The interaction phase variation is

$$\dot{\eta} = \omega - k_{\parallel} v_{\parallel} - l\dot{\theta}. \tag{24}$$

The derivative of gyro-phase can be expressed as

$$\dot{\theta} = \frac{\dot{p}_{\theta}}{p_{\perp}} = \frac{\dot{p}_{\theta}^{ad} + \dot{p}_{\theta}^{w}}{p_{\perp}} = \frac{\omega_{ce}}{\gamma} - \frac{F_{\theta}^{w}}{p_{\perp}} \cos\eta. \tag{25}$$

Finally, we got the gyro-averaged particle motion for lth-order cyclotron resonance in $(p_{\parallel}, p_{\perp}, \eta)$ coordinates as

$$\dot{p}_{\parallel} = F_{\parallel}^{w} \sin\eta - \frac{1}{2B} \frac{p_{\perp}^2}{\gamma m} \frac{\partial B}{\partial z}, \tag{26a}$$

$$\dot{p}_{\perp} = F_{\perp}^{w} \sin\eta + \frac{1}{2B} \frac{p_{\perp} p_{\parallel}}{\gamma m} \frac{\partial B}{\partial z}, \tag{26b}$$

$$\dot{\eta} = \frac{l F_{\theta}^{w}}{p_{\perp}} \cos\eta + \omega - \frac{k_{\parallel} p_{\parallel}}{\gamma m} - \frac{l\omega_{ce}}{\gamma}. \tag{26c}$$

D. Variation of energy, pitch angle, and magnetic momentum

The gyro-averaged rate of energy change for an electron traveling through a whistler wave-field can be directly obtained by

$$\frac{dE_k}{dt} = \frac{1}{\gamma m}\left(p_{\parallel} \frac{dp_{\parallel}}{dt} + p_{\perp} \frac{dp_{\perp}}{dt}\right), \tag{27}$$

and is expressed as

$$\frac{dE_k}{dt} = -\frac{e}{\gamma m}\left[E_z^w p_{\parallel} J_l(\beta) + E_R p_{\perp} J_{l-1}(\beta) + E_L p_{\perp} J_{l+1}(\beta)\right] \sin\eta. \tag{28}$$

Here, E_k represents the kinetic energy of the electron, the three terms of Eq. (28) in the right hand side represent the energy variation contributed by $\mathbf{E}_z^w$, $\mathbf{E}_R$, and $\mathbf{E}_L$, respectively. All the magnetic wave components vanish in Eq. (28). The energy variation equation (27) can be simplified using the diffusion curve equation (23) as

$$\frac{dE_k}{dt} = \frac{1}{\gamma m}\left(p_\parallel F_\parallel^w \sin\eta + p_\perp F_\parallel^w \frac{\omega - k_\parallel v_\parallel}{k_\parallel v_\perp}\sin\eta\right),$$

which is simply

$$\frac{dE_k}{dt} = F_\parallel^w \frac{\omega}{k_\parallel}\sin\eta. \tag{29}$$

This equation implies that a low frequency ($\omega \ll \omega_{ce}$) wave, such as the magnetospheric electromagnetic ion cyclotron (EMIC) wave, can scatter the electron pitch angles via cyclotron resonance by converting particle energy from parallel to perpendicular direction, or vice versa, but the net gain of the particle energy is quite little.

It is useful to derive the motion of particle magnetic momentum μ, which is the first invariant in a static magnetic field. From the definition $\mu = p_\perp^2/2mB$, we obtain

$$\frac{d\mu}{dt} = \frac{p_\perp \dot{p}_\perp}{mB} - \frac{p_\perp^2}{2mB^2}\frac{\partial B}{\partial z}\frac{p_\parallel}{\gamma m}. \tag{30}$$

Substituting the expression of $\dot{p}_\perp$ to the above equation yields

$$\frac{d\mu}{dt} = \frac{p_\perp}{mB} F_\perp^w \sin\eta, \tag{31}$$

and applying the diffusion curve equation (23) produces

$$\frac{d\mu}{dt} = \frac{lm}{ek_\parallel} F_\parallel^w \sin\eta. \tag{32}$$

The general relations

$$\frac{dp_\parallel^w}{dE_k} = \frac{k_\parallel}{\omega}, \quad \frac{d\mu}{dE_k} = \frac{m}{e}\frac{l}{\omega} \tag{33}$$

apply to any cyclotron harmonic resonant interactions between an electron and a monochromatic wave, and are consistent with the relation given by Walker[28] and Albert *et al.*[29]

Using the relation of $\alpha = \tan^{-1}(p_\perp/p_\parallel)$, the rate of pitch angle change of a particle moving through an oblique whistler wave can be easily obtained as

$$\frac{d\alpha}{dt} = \frac{m}{p^2 k_\parallel}\sin\eta\left(p_\parallel F_\perp^w - p_\perp F_\parallel^w\right) + \frac{p_\perp}{2\gamma mB}\frac{\partial B}{\partial z}. \tag{34}$$

Applying Eqs. (20) and (23) to the above equation yields

$$\frac{d\alpha}{dt} = -\frac{F_\parallel^w}{p_\perp}\left(1 + \frac{\cos^2\alpha}{lY - 1}\right)\sin\eta + \frac{p_\perp}{2\gamma mB}\frac{\partial B}{\partial z}, \tag{35}$$

where $Y = \omega_{ce}/\gamma\omega$. This equation is the same as that given by Bortnik *et al.*[22] in terms of his conventions.

III. COMPARISON WITH BELL'S EQUATION: SIGN PROBLEM

We carefully transformed the gyro-averaged particle motion equations (26) to Bell and Bortnik's conventions (the final relativistic form given by Bortnik *et al.*[3,20]), and found a $(-1)^{l-1}$ term in difference between our results and their formulas in the perpendicular motion equation for the lth-order resonance. Since different conventions have been used in various literature, making the sign checking rather complicated, we suggest a easy test for self-consistency independent of the choice of convention, that is, no matter what convention is used, the energy variation equation given by Eq. (27) should not contain any wave magnetic components, because the magnetic field does not contribute to changes in particle energy. However, the energy variation equation from Bortnik's formulas[3,20] contains contributions from wave magnetic components, which will be canceled after adding $(-1)^{l-1}$ term in the perpendicular motion.

In order to investigate the impact of the sign problem in the motion equation, we simulated the Landau resonance between an electron and a magnetosonic wave in the magnetosphere using both Bortnik's solver and corrected gyro-averaged solver, and compared the results to that by Lorentz solver. Following the work by Bortnik *et al.*,[20] we modeled a whistler-mode magnetosonic wave with frequency $f = 33.3$ Hz, amplitude $B^w = 250$ pT, and wave normal angle 89°. We set the ambient electron density $n_{e0} = 10.3\,\mathrm{cm}^{-3}$, and chose a background magnetic field with intensity $B_0 = 342$ nT to represent the geomagnetic field at L = 4.5, where the McIlwain's L-value describes the set magnetic field lines crossing the Earth's magnetic equator at a geocentric distance L in units of the Earth's radius R_E. The Landau resonance interaction between such a wave and an electron with energy $E_k = 300$ keV and pitch angle $\alpha = 30°$ is then simulated, and the variation of particle energy, parallel momentum, and perpendicular momentum are illustrated in Fig. 3. With the $(-1)^{l-1}$ term missing, Bortnik's formulas produce an opposite behavior of perpendicular motion, while the corrected formulas produce a result that agrees well with full particle simulation using Lorentz solver. Using the corrected formulas, Li *et al.*[30] recently showed that the test particle simulation of interactions between magnetosonic waves and energetic electrons is in good agreement with quasi-linear theory when the waves propagate enough wavelengths along the ambient magnetic field lines.

IV. EFFECT OF WAVE CENTRIPETAL FORCE IN PHASE TRAPPING

The effect of the wave centripetal acceleration force F_θ^w (the first term on the right-hand side of Eq. (26c)) was often neglected in the generalized resonance formulas.[1–3,26] Usually, this centripetal acceleration term is a small term on the right hand side of Eq. (26) except for loss cone particles, which possess very small $p_\perp$.[25,31–33] But in the second-order resonance given by $d^2\eta/dt^2 = 0$, which enables the stable trapping of a resonant electrons,[34] this centripetal acceleration force plays an important role. Although this centripetal

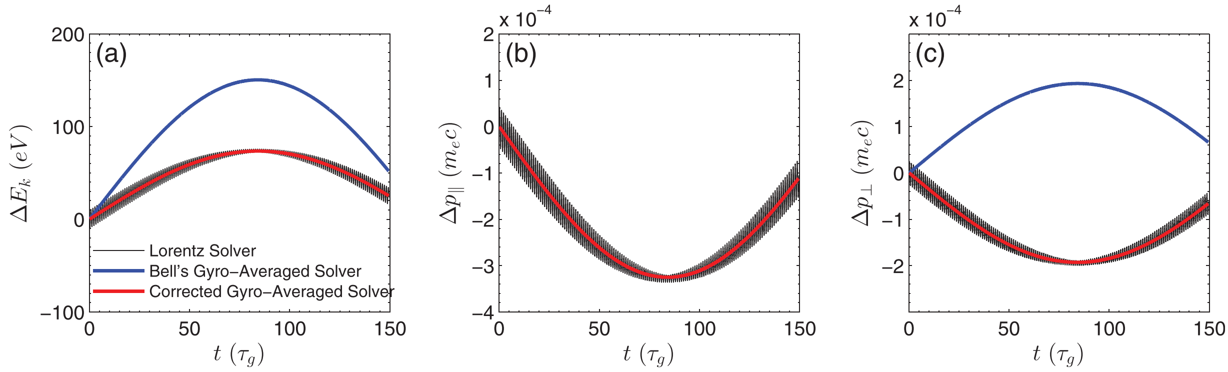

FIG. 3. The test particle simulation of Landau resonance between an electron and a magnetosonic wave using the Lorentz solver, Bell's solver, and corrected solver in this paper. The comparison of (a) the particle energy change, (b) the parallel momentum change, and (c) the perpendicular momentum change clearly reveals the sign mistake of perpendicular motion of Bell's solver, and its consequence in particle energy variation.

term had already been included in some numerical studies,[14,16] but was usually not taken into account in theoretical analysis.[1,26,29,34–38]

The motion of phase trapped electrons is dominated not by the ambient magnetic field, but by the wave field,[26,39] which enables the particle parallel velocity to follow the resonant velocity all the way in the wave field.[36] This nonlinear interaction has been proposed to be accountable for nonlinear excitation of chorus waves in the Earth's radiation belt.[34–38] To make the physics brief, we use a simple example of first-order cyclotron resonant interaction between a non-relativistic electron and a parallel-propagating wave with constant ω and $k_\parallel$ to show the compact of F_θ^w on phase trapping. The gyro-averaged electron motion equations are simplified as

$$\dot{v}_\parallel = -\frac{eB_R}{m} v_\perp \sin\eta - \frac{1}{2B} v_\perp^2 \frac{\partial B}{\partial z}, \tag{36a}$$

$$\dot{v}_\perp = -\frac{eB_R}{m}\frac{\omega_{ce}}{k_\parallel}\sin\eta + \frac{1}{2B} v_\perp v_\parallel \frac{\partial B}{\partial z}, \tag{36b}$$

$$\dot{\eta} = \frac{-eB_R}{m}\frac{\omega_{ce}}{k_\parallel v_\perp}\cos\eta + \omega - k_\parallel v_\parallel - \omega_{ce}. \tag{36c}$$

Here, we have used the Faraday's law and the resonance condition Eq. (20). When F_θ^w is omitted, the phase trapping condition has the form of a driving pendulum,

$$\ddot{\eta} + R\sin\eta = D, \tag{37}$$

where

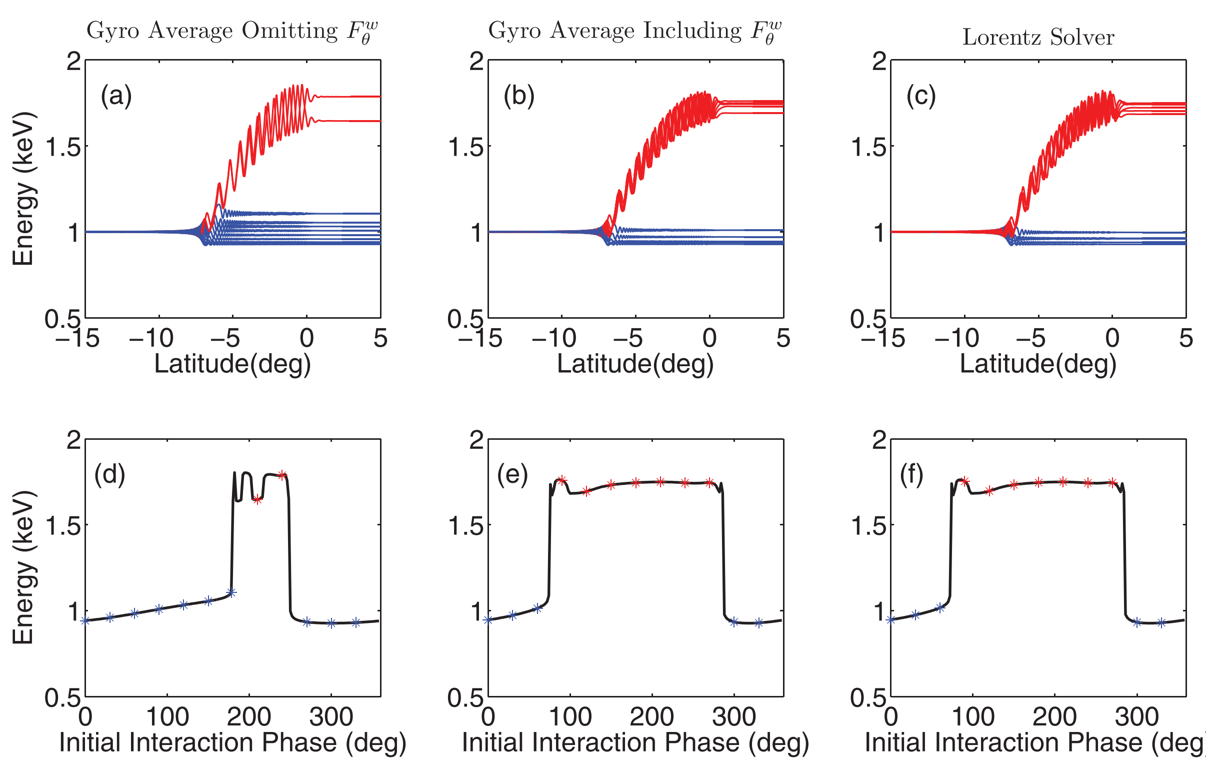

FIG. 4. (a)–(c) The energy variation of 12 representative electrons with $E_k = 1$ keV and $\alpha_{eq} = 20^\circ$ undergoing a chorus wave, calculated by gyro-averaged solver omitting the wave centripetal force, that including it, and the Lorentz solver, respectively. (d)–(f) The final electron energy as a function of initial interaction phases by three methods.

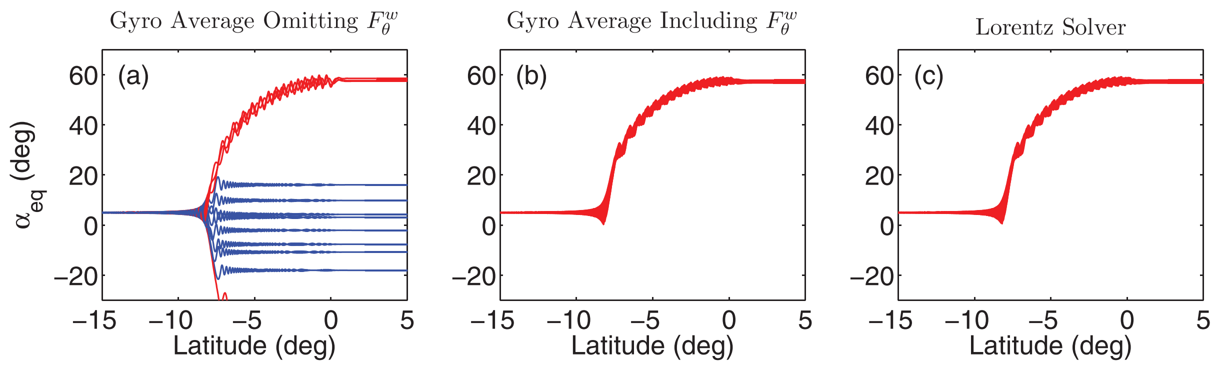

FIG. 5. (a)–(c) The pitch angle variation of 12 representative electrons with $E_k = 1$ keV and $\alpha_{eq} = 5°$ undergoing a chorus wave calculated by three methods.

$$R = -\frac{eB_R}{m}\cdot k_\parallel v_\perp,$$
$$D = \left(k_\parallel \frac{1}{2B}v_\perp^2 - \frac{ev_\parallel}{m}\right)\frac{\partial B}{\partial z} \tag{38}$$

represent the "restoring force" and "driving force," respectively. However, considering the F_θ^w effect would bring a couple of extra terms in the "restoring force," including a major term

$$R_1 = \frac{-eB_R}{m}\frac{\omega_{ce}}{k_\parallel v_\perp}\omega. \tag{39}$$

When $R_1 > R$ or equivalently

$$\omega_{ce}\omega > k_\parallel^2 v_\perp^2, \tag{40}$$

this extra "restoring force" can greatly enhance the phase trapping, making more electrons being stably phase-trapped. The F_θ^w effect also yields extra "driving forces," which are small compared to D.

Equation (40) implies that the F_θ^w effect is most evident for electrons with small $v_\perp$ undergoing phase trapping in a high frequency wave. Here, we demonstrate this fact by simulations of the interactions between a chorus wave with $\omega = 0.7\,\omega_{ce}$ and a bunch of electrons in the dipole geomagnetosphere at L = 5. The monochromatic chorus wave was assumed to be generated at the magnetic equator and propagates towards the southern hemisphere along the magnetic field line with an amplitude of 70 pT. The ambient electron density was set to be $n_{e0} = 10\,\mathrm{cm}^{-3}$. A group of 360 electrons were simulated traveling northward with an identical initial energy $E_k = 1$ keV, an equatorial pitch angle $\alpha_{eq} = 20°$, and initial interaction phases η_0 from 0° to 359°, respectively. The gyro-averaged energy motion along latitude of 12 representative electrons ($\eta_0 = 0°, 30°, \ldots 330°$) calculated omitting the F_θ^w effect (Figure 4(a)), that including the F_θ^w effect (Figure 4(b)) is compared to the motion under Lorentz force (Figure 4(c)). The full information of electron final energy after resonance as a function of η_0 calculated by three methods is plotted in Figures 4(d)–4(f), respectively. Only 68 out of 360 electrons undergo phase trapping if the F_θ^w effect is omitted (Figure 4(d)), while a simulation including this effect (Figure 4(e)) produces 212 phase trapped electrons ($\eta_0 = 76° - 287°$), in excellent agreement with that obtained by Lorentz solver (Figure 4(f)).

The gyro-averaging solver omitting F_θ^w may even produce negative pitch angles, as shown in Figure 5(a), which present the dynamics of electrons with an initial pitch angle of 5° and an energy of 1 keV. However, with the inclusion of F_θ^w, the gyro-averaged solver produce a result in excellent agreement with that obtained by Lorentz solver (Figures 5(b) and 5(c)). Here, we point out that a $p_\perp$ factor in the denominator of wave centripetal acceleration term in Eq. (26c) ensures the $p_\perp$ and the pitch angle $\alpha = a\tan(p_\perp/p_\parallel)$ always be positive. This wave centripetal force not only "reflects" the particles when they hit the loss cone but also cause all electrons undergoing phase trapping in the present case.

V. CONCLUSIONS

The gyro-averaged motion formulas for interactions between whistler-mode waves and electrons for any resonance harmonic are derived with general definitions of signs of particle and wave parameters. It is efficient to use these formulas for test particle simulations, and convenient as well to analyze particle energy and pitch angle change rates. By applying the conventions of Bell and Bortnik,[1,3] we find a factor of $(-1)^{l-1}$ missing in their perpendicular motion equation. By simulating the gyro-averaged motion of an electron undergoing Landau resonance with a magnetosonic wave, and comparing to the full particle motion under the Lorentz force, we show that the missing sign causes the particle perpendicular motion behave oppositely. We propose a convention free criteria to justify the various forms of gyro-averaged equations, i.e., the energy variation resulting from the momentum equation should not contain the contribution of wave magnetic components, if Maxwell's relations are not applied. The particle motion formulas derived by Ginet and Albert[2] and Albert *et al.*[29] are correct and agree with that criteria.

Furthermore, we point out that the wave centripetal acceleration term should definitely be included in nonlinear interaction studies. Although it is a small term, its profound time derivative provide an extra "driving force" and can significantly enhance phase trapping, making particle acceleration process much more effectively. This term may also have a potential impact on nonlinear wave excitation and damping process. Besides, this term also numerically keeps $p_\perp$ and α_{eq} being positive, and bounces the low pitch angle electrons out of the loss cone. Gyro-averaged resonant particle motions simulated including the effect of this term are in excellent agreement with the full particle motions under Lorentz force, as shown by two examples.

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APPENDIX: DERIVATION OF DIFFUSION CURVE EQUATION

The wave electric and magnetic components applying Faraday's Law to Eq. (2) yield

$$B_x = -k_\parallel E_y/\omega, \tag{A1}$$

$$B_y = k_\parallel E_x - k_\perp E_z, \tag{A2}$$

$$B_z = k_\perp E_y/\omega. \tag{A3}$$

From the definitions of $B_R=(B_x^w+B_y^w)/2$, $B_L=(B_x^w-B_y^w)/2$, $E_R=(E_x^w-E_y^w)/2$, and $E_L=(E_x^w+E_y^w)/2$, we have

$$B_R = \frac{k_\parallel}{\omega}E_R - \frac{k_\perp}{2\omega}E_z, \tag{A4}$$

$$B_L = -\frac{k_\parallel}{\omega}E_L + \frac{k_\perp}{2\omega}E_z, \tag{A5}$$

$$B_z = \frac{k_\perp}{\omega}E_y = \frac{k_\perp}{\omega}(E_L - E_R). \tag{A6}$$

Inserting the above expressions to Eq. (A10) we obtain

$$\begin{aligned} -\frac{F_\parallel^w}{e} &= E_z J_l + v_\perp\left(\frac{k_\parallel}{\omega}E_R - \frac{k_\perp}{2\omega}E_z\right)J_{l-1} \\ &\quad - v_\perp\left(-\frac{k_\parallel}{\omega}E_L + \frac{k_\perp}{2\omega}E_z\right)J_{l+1}. \end{aligned} \tag{A7}$$

$$\begin{aligned} -\frac{F_\perp^w}{e} &= E_R J_{l-1} - v_\parallel\left(\frac{k_\parallel E_R}{\omega} - \frac{k_\perp E_z}{2\omega}\right)J_{l-1} \\ &\quad + E_L J_{l+1} + v_\parallel\left(-\frac{k_\parallel E_L}{\omega} + \frac{k_\perp E_z}{2\omega}\right)J_{l+1}. \end{aligned} \tag{A8}$$

Using the Bessel equation's identity

$$J_{l-1} + J_{l+1} = \frac{2l}{\beta}J_l = \frac{2l\omega_{ce}/\gamma}{k_\perp v_\perp}J_l, \tag{A9}$$

as well as the resonance condition Eq. (20) we get

$$-\frac{F_\parallel^w}{e} = \frac{k_\parallel v_\parallel}{\omega}E_z J_l + \frac{k_\parallel v_\perp}{\omega}E_R J_{l-1} + \frac{k_\parallel v_\perp}{\omega}E_L J_{l+1}, \tag{A10}$$

$$-\frac{F_\perp^w}{e} = \frac{l\omega_{ce}}{\gamma\omega}\frac{v_\parallel}{v_\perp}E_z J_l + \frac{l\omega_{ce}}{\gamma\omega}E_R J_{l-1} + \frac{l\omega_{ce}}{\gamma\omega}E_L J_{l+1}. \tag{A11}$$

Finally, by comparing the above two equations, we obtain the following relation:

$$F_\perp^w = F_\parallel^w \frac{l\omega_{ce}}{\gamma k_\parallel v_\perp} = F_\parallel^w \frac{l\omega_{ce}}{\gamma\omega - l\omega_{ce}}\cot\alpha, \tag{A12}$$

which is equivalent to

$$p_\perp dp_\perp^w = \frac{l\omega_{ce}}{\gamma\omega - l\omega_{ce}}p_\parallel dp_\parallel^w. \tag{A13}$$

This equation represents the resonant diffusion curve in the $(p_\parallel, p_\perp)$ space, along which particles should diffuse under the influence of an electromagnetic wave.[27]

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