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On the Spectrum of Laplace Operator and Asymptotic Expansion of Bergman Kernel on Kähler Manifolds

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On the Spectrum of Laplace Operator and Asymptotic Expansion of Bergman Kernel on Kähler Manifolds

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Hang Xu

Dissertation Committee:
Professor Zhiqin Lu, Chair
Professor Jeffrey Streets
Professor Li-Sheng Tseng

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DEDICATION

To Longcan and my parents.

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ABSTRACT OF THE DISSERTATION

On the Spectrum of Laplace Operator and Asymptotic Expansion of Bergman Kernel on Kähler Manifolds

By

Hang Xu

Doctor of Philosophy in Mathematics

University of California, Irvine, 2016

Professor Zhiqin Lu, Chair

This dissertation contains two parts. The first part considers related problems of Laplace operator on Kähler manifolds. Together with my advisor Zhiqin Lu, we generalized the spectrum relation in [5] to any Hermitian manifolds. And we proved the closure of Laplace operator $\square = \delta d$ on the moduli space of polarized Calabi-Yau manifolds is self-adjoint. The second part considers the asymptotic expansion of the Bergman kernel on a polarized Kähler manifold. Together with Hezari, Kelleher and Seto [9], we give an alternative proof of the asymptotic expansion.

Chapter 1

Introduction

In this thesis, we mainly consider two topics in complex geometry. The first one is about the spectrum and self-ajoint extension of Laplace operator. And the second topic is about the asymptotic expansion of Bergman kernel on Kähler manifolds.

In Chapter 2, we assume (M, g) is a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Consider Gaffney extension $\square_{p,q}$ of Hodge Laplacian on the E valued (p, q) forms. We will prove some spectrum relations among the self-adjoint operators $\square_{p,q}$, $\overline{\partial}\bar{\partial}^*_{p,q}$ and $\bar{\partial}^*\overline{\partial}_{p,q}$.

In Chapter 3, we consider the moduli space of polarized Calabi-Yau manifold endowed with the Weil-Petersson metric, denoted as $(\mathcal{M}, \omega_{WP})$. As $(\mathcal{M}, \omega_{WP})$ is generally not complete, the Cauchy boundary $\partial_c\mathcal{M}$ is not empty. However, we can prove that the Cauchy boundary is small in the sense that the capacity of $\partial_c\mathcal{M}$ is zero. Therefore, two Sobolev spaces $W^1_0(\mathcal{M})$ and $W^1(\mathcal{M})$ coincide with each other just like in the case of complete manifolds. As a result, we can prove the closure of the Laplace operator $\square = \delta d$ on functions is self-adjoint.

In Chapter 4, we consider the Bergman kernel of a polarized Kähler manifold (M, L). Methods to analyze the asymptotic expansion of the Bergman Kernel have been worked out over the years. Initially Tian gave leading asymptotics on the diagonal [22]. Extending the result of Fefferman [6], a complete expansion was given by Zelditch [24], and independently by Catlin [4]. In particular, their off-diagonal asymptotic expansions, as $k \to \infty$, are given of the form, with b_l certain Hermitian functions,

$$K(x,y)e^{-k\psi(x,y)} = \frac{k^n}{\pi^n}\left(1 + \sum_{l=1}^{\infty} \frac{b_l(x,y)}{k^l}\right). \tag{1.1}$$

Lu demonstrates that the functions $b_l(x,y)|_{x=y}$ encode geometric information about the underlying manifold M [11]. Based on the joint work with Hezari, Kelleher and Seto [9], we give an alternative proof of the asymptotic expansion.

Chapter 2

The Spectrum Relation of Gaffney Extensions

2.1 Gaffney Extension of Hodge Laplacian

Assume (M, g) is a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Let $m = \dim M$ and p, q be any integers between 0 and m. Consider Hodge Laplacian on E-valued (p, q) forms with compact support. As Hodge Laplacian is symmetric but not self-adjoint, we will instead consider Gaffney extension of Hodge Laplacian via the corresponding closed quadratic form. For more detailed discussions, we recommend references [15, 21].

We begin with the d-bar differential operator

$$\overline{\partial}_{p,q} : L^2(M, \Lambda^{p,q}(E)) \to L^2(M, \Lambda^{p,q+1}(E)),$$

which has the following domain of definition

$$Dom(\overline{\partial}_{p,q}) = \{\varphi \in L^2(M, \Lambda^{p,q}(E)) : \text{the distributional derivative } \overline{\partial}\varphi \in L^2(M, \Lambda^{p,q+1}(E))\}.$$

With the above domain of definition, the operator $\overline{\partial}_{p,q}$ is a densely defined closed operator. We denote the L^2 inner product on $L^2(M, \Lambda^{p,q}(E))$ as $(\cdot,\cdot)_{p,q}$ for any $0 \le p, q \le m$. With respect to the L^2 inner products on $L^2(M, \Lambda^{p,q}(E))$ and $L^2(M, \Lambda^{p,q+1}(E))$, we have the adjoint operator of $\overline{\partial}_{p,q}$ as

$$\bar{\partial}^*_{p,q+1} : L^2(M, \Lambda^{p,q+1}(E)) \to L^2(M, \Lambda^{p,q}(E)),$$

with

$$Dom(\bar{\partial}^*_{p,q+1}) = \{\phi \in L^2(M, \Lambda^{p,q+1}(E)) : \exists\, \varphi \in L^2(M, \Lambda^{p,q}(E)) \text{ such that}$$
$$(\overline{\partial}_{p,q}u, \phi)_{p,q+1} = (u, \varphi)_{p,q} \text{ for any } u \in Dom(\overline{\partial}_{p,q})\}.$$

And in the above notation, $\bar{\partial}^*_{p,q+1}\phi$ is defined to be φ.

In the following, we will suppress the indices p, q in the operators and the inner products for simplicity when there is no confusion from context.

Now let us recall Hodge Laplacian and the associated quadratic form. We use the notation $\mathcal{D}(M, \Lambda^{p,q}(E))$ to denote the set of all smooth E-valued (p, q) forms with compact support.

Definition 2.1.1 (Hodge Laplacian)**.**

i) Let $\Box : \mathcal{D}(M, \Lambda^{p,q}(E)) \to \mathcal{D}(M, \Lambda^{p,q}(E))$ be the Hodge Laplacian defined as

$$\Box = \overline{\partial}\bar{\partial}^* + \bar{\partial}^*\overline{\partial}.$$

ii) Let $Q : \mathcal{D}(M, \Lambda^{p,q}(E)) \times \mathcal{D}(M, \Lambda^{p,q}(E)) \to \mathbb{C}$ be the quadratic form associated to $\Box$ defined as

$$Q(\varphi, \phi) = (\overline{\partial}\varphi, \overline{\partial}\phi) + (\bar{\partial}^*\varphi, \bar{\partial}^*\phi) \text{ for any } \varphi, \phi \in \mathcal{D}(M, \Lambda^{p,q}(E)).$$

Since $\overline{\partial}, \bar{\partial}^*$ are closed operators, if we endow quadratic form Q with $Dom(Q) = Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*)$, then Q is closed. That means, for any sequence $\varphi_n \in Dom(Q)$, if $\varphi_n \xrightarrow{L^2} \varphi$ and $Q(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \to 0$ as $m, n \to \infty$, then $\varphi \in Dom(Q)$ and $Q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$.

We cite the following theorem from [20] in Chapter VIII.6.

Theorem 2.1.1 ([20]). *If q is a closed semibounded quadratic form, then q is the quadratic form of a unique self-adjoint operator.*

By applying this theorem to our quadratic form Q with $Dom(Q) = Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) \subset L^2(M, \Lambda^{p,q}(E))$, we get a self-adjoint extension of $\Box$, which is called Gaffney extension and denoted as $\Box_G$ or still $\Box$ when there is no ambiguity. The domain of $\Box_G$ is

$$\begin{aligned} Dom(\Box_G) = \{\varphi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) : \exists \eta \in L^2(M, \Lambda^{p,q}(E)) \text{ such that} \\ Q(\varphi, \phi) = (\eta, \phi) \text{ for any } \phi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*)\}. \end{aligned} \tag{2.1}$$

And in the same notation as above, $\Box_G\varphi$ is defined to be η.

The following Gaffney's Theorem from Chapter 3 in [15] tells us that Gaffney extension can be viewed as the composition of $\overline{\partial}$ and $\bar{\partial}^*$ as follows.

Theorem 2.1.2 (Gaffney).

$$Dom(\Box_G) = \{\varphi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) : \overline{\partial}\varphi \in Dom(\bar{\partial}^*) \text{ and } \bar{\partial}^*\varphi \in Dom(\overline{\partial})\}. \tag{2.2}$$

And for any $\varphi \in Dom(\Box_G)$, we have

$$\Box_G \varphi = \overline{\partial}\bar{\partial}^* \varphi + \bar{\partial}^* \overline{\partial} \varphi.$$

We recall the following definition of Sobolev spaces. Denote $Q_1(\cdot, \cdot) = Q(\cdot, \cdot) + (\cdot, \cdot)$. It is not hard to see Q_1 is an inner product on $\mathcal{D}(M, \Lambda^{p,q}(E))$.

Definition 2.1.2 (Sobolev Spaces)**.**

$$\begin{aligned} W_0^1(M, \Lambda^{p,q}(E)) =& \text{Completion of } \mathcal{D}(M, \Lambda^{p,q}(E)) \\ & \text{with respect to } Q_1 \text{ inner product}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} W^1(M, \Lambda^{p,q}(E)) =& \text{Completion of } \{\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E)) : Q_1(\varphi, \varphi) < \infty\} \\ & \text{with respect to } Q_1 \text{ inner product.} \end{aligned} \tag{2.4}$$

Remark 2.1.1. Note that φ is not necessarily in $Dom(\bar{\partial}^*_{p,q})$ when $\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E))$. So in the definition of $W^1(M, \Lambda^{p,q}(E))$, to be precise, $Q_1(\varphi, \varphi) < \infty$ means $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the distributional differentials $\overline{\partial}\varphi, \bar{\partial}^*\varphi$ (or pointwise differential since φ is smooth) belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively.

Remark 2.1.2. One can prove $\varphi \in W^1(M, \Lambda^{p,q}(E))$ if and only if $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the distributional differentials $\overline{\partial}\varphi, \bar{\partial}^*\varphi$ belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively.

Remark 2.1.3. Note that $W_0^1 \subset Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) \subset W^1$. But they are generally not equal to each other.

Example 2.1.4. Take the Hermitian manifold $M = \Omega \subset \mathbb{C}^n$ to be a bounded open set with smooth boundary. Let the Hermitian vector bundle E be the trivial line bundle. Assume $u \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q})$. Let us investigate the boundary condition induced from $\Box_G$ in this case.

If $u \in Dom(\bar{\partial}^*)$, then

$$(\overline{\partial}\varphi, u) = (\varphi, \bar{\partial}^* u) \text{ for any } \varphi \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q-1}).$$

Note

$$(\overline{\partial}\varphi, u) = \int_\Omega \overline{\partial}\varphi \wedge *\overline{u} = \int_{\partial\Omega} \varphi \wedge *\overline{u} + (-1)^{p+q} \int_\Omega \varphi \wedge \overline{\partial} * \overline{u} = \int_{\partial\Omega} \varphi \wedge *\overline{u} + (\varphi, \bar{\partial}^* u).$$

Here $*$ is the Hodge star operator. The second equality follows from Stokes' Theorem and the third equality is based on the identity $\bar{\partial}^* = - * \partial *$. Therefore, we have

$$\int_{\partial\Omega} \varphi \wedge *\overline{u} = 0 \text{ for any } \varphi \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q-1}).$$

It implies $*u|_{\partial\Omega}$ (the restriction of $*u$ to $\partial\Omega$) vanishes. So by Theorem 2.1.2, $u \in Dom(\Box_G)$ implies the boundary condition $*u|_{\partial\Omega} = 0$ and $*\overline{\partial}u|_{\partial\Omega} = 0$.

2.2 Spectrums of Gaffney Extension

The main goal of this section is to prove the following spectrum relations of Gaffney extensions.

Theorem 2.2.1. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Consider the Gaffney extension $\Box_{p,q} : L^2(M, \Lambda^{p,q}(E)) \to L^2(M, \Lambda^{p,q}(E))$. We have the following spectrum relations.*

$$Spec(\Box_{p,q}) \cup \{0\} = Spec(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q-1}) \cup \{0\}. \tag{2.5}$$

$$Spec(\Box_{p,q}) \cup \{0\} = Spec(\overline{\partial}\bar{\partial}^*_{p,q}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q}) \cup \{0\}. \tag{2.6}$$

Remark 2.2.1. The above notation $\overline{\partial}\bar{\partial}^*_{p,q}$ means $\overline{\partial}_{p,q-1}\bar{\partial}^*_{p,q}$ and $\bar{\partial}^*\overline{\partial}_{p,q}$ means $\bar{\partial}^*_{p,q+1}\overline{\partial}_{p,q}$. Note that $\overline{\partial}\bar{\partial}^*_{p,q}$ and $\bar{\partial}^*\overline{\partial}_{p,q}$ are self-adjoint operators by Von Neumann's Theorem (see Chapter X in [19]) since both $\overline{\partial}_{p,q}$ and $\bar{\partial}^*_{p,q}$ are densely defined closed operators. In the following, we will omit the subindex p, q when there is not confusion from the context.

This is a generalization of the results in [5], where similar spectrum relations were proved for complete Riemannian manifolds. One main tool we are going to use is the generalized Weyl's Criterion from [5]. The advantage of this generalized Weyl's Criterion is that we do not necessarily pick the test sequence from the domain of an unbounded operator. After proving it, we will mention a well-known relation between the Gaffney extension and L^2 estimates.

We will split the proof for Theorem 2.2.1 into several Lemmas. First, we prove Lemma 2.2.1, which is one containment of identity 2.5.

Lemma 2.2.1. *Under the same assumption as in Theorem 2.2.1, we have*

$$Spec(\square_{p,q}) \subset Spec(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q-1}) \cup \{0\}. \tag{2.7}$$

Proof. In this proof, we will use $\square$ to represent $\square_{p,q}$ for simplicity. Take $\lambda_0 \in \mathrm{Spec}(\square)$ and $\lambda_0 > 0$. By Weyl's criterion, there exists a sequence $u_j \in Dom(\square)$ with $(u_j, u_j) = 1$ such that

$$(\square - \lambda_0)u_j \to 0 \text{ as } j \to \infty.$$

Since $\square$ is nonnegative and self-adjoint, $(1+\square)^{-1} : L^2(M, \Lambda^{p,q}(E)) \to Dom(\square_{p,q}) \subset L^2(M, \Lambda^{p,q}(E))$ is a bounded operator. By identity 2.1, we have

$$Q((1+\square)^{-2}u_j, (1+\square)^{-2}u_j) = (\square(1+\square)^{-2}u_j, (1+\square)^{-2}u_j). \tag{2.8}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\Box$. Then

$$(\Box(1+\Box)^{-2}u_j,(1+\Box)^{-2}u_j)=\int_0^\infty \frac{\lambda}{(1+\lambda)^4}d(P_\lambda u_j,u_j). \tag{2.9}$$

Take $C(\lambda_0)=\min_{\lambda\in[\frac{\lambda_0}{2},\frac{3\lambda_0}{2}]}\frac{\lambda}{(1+\lambda)^4}>0$. Then

$$\int_0^\infty \frac{\lambda}{(1+\lambda)^4}d(P_\lambda u_j,u_j)\geq C(\lambda_0)\int_{\frac{1}{2}\lambda_0}^{\frac{3}{2}\lambda_0}d(P_\lambda u_j,u_j)\geq C(\lambda_0)\|P_{(\frac{1}{2}\lambda_0,\frac{3}{2}\lambda_0)}u_j\|^2. \tag{2.10}$$

We denote $u_j^{(1)}=P_{(\frac{1}{2}\lambda_0,\frac{3}{2}\lambda_0)}u_j$ and $u_j^{(2)}=u_j-u_j^{(1)}$. By using the Projection Valued Measure again, we have

$$((\Box-\lambda_0)u_j,(\Box-\lambda_0)u_j)=\int_0^\infty(\lambda-\lambda_0)^2d(P_\lambda u_j,u_j)\geq\frac{\lambda_0^2}{4}\|u_j^{(2)}\|^2.$$

Since we know $(\Box-\lambda_0)u_j\to 0$ as j goes to infinity, we have

$$\|u_j^{(2)}\|\to 0 \text{ as } j\to\infty,$$

whence

$$\|u_j^{(1)}\|\to 1 \text{ as } j\to\infty. \tag{2.11}$$

Take 2.11 together with 2.8, 2.9 and 2.10. For sufficiently large j, we have

$$\|\overline{\partial}(1+\Box)^{-2}u_j\|^2+\|\bar{\partial}^*(1+\Box)^{-2}u_j\|^2\geq\frac{C(\lambda_0)}{2}>0. \tag{2.12}$$

On the other hand, we have

$$
\begin{aligned}
&\|(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 + \|(\bar{\partial}^*\overline{\partial} - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 \\
=&\|\overline{\partial}(\Box - \lambda_0)(1+\Box)^{-2}u_j\|^2 + \|\bar{\partial}^*(\Box - \lambda_0)(1+\Box)^{-2}u_j\|^2 \\
=&\left(\Box(1+\Box)^{-2}(\Box-\lambda_0)u_j, (1+\Box)^{-2}(\Box-\lambda_0)u_j\right) \\
\leq&\|(\Box-\lambda_0)u_j\|^2.
\end{aligned}
$$

The first equality is obtained because $\overline{\partial}\circ\overline{\partial} = 0$ on $Dom(\overline{\partial})$ and $\bar{\partial}^*\circ\bar{\partial}^* = 0$ on $Dom(\bar{\partial}^*)$. The second equality follows from identity 2.1 and the commutativity of $\Box$ and $(1+\Box)^{-1}$ on $Dom(\Box)$. And the last inequality follows from $\|(1+\Box)^{-1}\|_{L^2\to L^2} \leq 1$ and $\|\Box(1+\Box)^{-1}\|_{L^2\to L^2} \leq 1$. Therefore,

$$
\|(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 + \|(\bar{\partial}^*\overline{\partial} - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 \to 0. \tag{2.13}
$$

Combining 2.12 and 2.13, we have $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup \mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1})$ by Weyl's Criterion. So the result follows. $\Box$

Now we will prove Lemma 2.2.2, the other containment of identity 2.5 in Theorem 2.2.1.

Lemma 2.2.2. *Under the same assumption as in Theorem 2.2.1, we have*

$$
Spec(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q-1}) \subset Spec(\Box_{p,q}) \cup \{0\}.
$$

In order to prove this lemma, we will use one generalized Weyl's Criterion from [5].

Theorem 2.2.2 (Charalambous-Lu)**.** *Let H be a nonnegative self-adjoint operator on Hilbert space $\mathcal{H}$. A positive real number λ_0 is contained in Spec(H) if there exists a sequence $u_j \in \mathcal{H}$ such that*

(1) For any j, $\|u_j\| = 1$.

(2) $((H-\lambda_0)(1+H)^{-m}u_j, u_j) \to 0$ *for* $m = 1, 2$.

Note that compared to the classical Weyl's Criterion, the above theorem does not require $u_j \in Dom(H)$. We give a proof of this theorem here for the completeness.

Proof. Note that

$$(H-\lambda_0)^2(1+H)^{-2} = (H-\lambda_0)(1+H)^{-1} - (\lambda_0+1)(H-\lambda_0)(1+H)^{-2}.$$

The assumption (2) implies that

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \to 0. \tag{2.14}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of H. Then

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) = \int_0^\infty \frac{(\lambda-\lambda_0)^2}{(1+\lambda)^2} d(P_\lambda u_j, u_j). \tag{2.15}$$

Define $u_j^{(1)} = P_{(\lambda_0-\varepsilon_j, \lambda_0+\varepsilon_j)}u_j$ and $u_j^{(2)} = u_j - u_j^{(1)}$. The sequence of constants $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ is to be selected later.

$$\frac{(\lambda-\lambda_0)^2}{(1+\lambda)^2} \geq \min\left(\frac{\varepsilon_j^2}{(1+\lambda_0-\varepsilon_j)^2}, \frac{\varepsilon_j^2}{(1+\lambda_0+\varepsilon_j)^2}\right) \geq \frac{\varepsilon_j^2}{(1+\frac{3}{2}\lambda_0)^2}. \tag{2.16}$$

Therefore,

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \geq \frac{\varepsilon_j^2}{(1+\frac{3}{2}\lambda_0)^2}\|u_j^{(2)}\|^2. \tag{2.17}$$

Choose a sequence $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ such that

i) $\varepsilon_j \to 0$.

ii) $((H-\lambda_0)^2(1+H)^{-2}u_j, u_j)/\varepsilon_j^2 \to 0$.

For example, we can take $\varepsilon_j = ((H-\lambda_0)^2(1+H)^{-2}u_j, u_j)^{\frac{1}{3}}$. Therefore, 2.17 implies

$$\|u_j^{(2)}\| \to 0 \text{ as } j \to \infty,$$

whence

$$\|u_j^{(1)}\| \to 1 \text{ as } j \to \infty. \tag{2.18}$$

On the other hand, as

$$\int_0^\infty \lambda^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \le (\lambda_0 + \varepsilon_j)^2 \|u_j\|^2 < \infty,$$

we have the sequence $u_j^{(1)} \in Dom(H)$. So we can apply Weyl's Criterion to the sequence $u_j^{(1)}$. By Projection Valued Measure again, we have

$$\|(H-\lambda_0)u_j^{(1)}\|^2 = \int_0^\infty (\lambda - \lambda_0)^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \le \varepsilon_j^2 \to 0, \tag{2.19}$$

which implies $\lambda_0 \in \mathrm{Spec}(H)$. So the result follows. □

Remark 2.2.2. Note that by the proof, the condition (2) in Theorem 2.2.2 can be weaken to

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \to 0.$$

Remark 2.2.3. Theorem 2.2.2 also holds for $\lambda_0 = 0$. In fact, we can prove that conditions (1) and (2) are also necessary for $\lambda_0 \in \mathrm{Spec}(H)$. More details can be found in [5].

With the generalized Weyl's Criterion (Theorem 2.2.2), we are ready to prove Lemma 2.2.2.

Proof. Here we prove $\mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \subset \mathrm{Spec}(\Box_{p,q}) \cup \{0\}$. The other containment $\mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1}) \subset \mathrm{Spec}(\Box_{p,q}) \cup \{0\}$ can be proved similarly.

Take $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*)$ and $\lambda_0 > 0$. By Weyl's Criterion, there exists a sequence $u_j \in Dom(\overline{\partial}\bar{\partial}^*)$ with $(u_j, u_j) = 1$ such that

$$((\overline{\partial}\bar{\partial}^* - \lambda_0)u_j, (\overline{\partial}\bar{\partial}^* - \lambda_0)u_j) \to 0. \tag{2.20}$$

We will verify that the sequence $\bar{\partial}^* u_j$ satisfies the conditions in Theorem 2.2.2. For $m = 1, 2$,

$$\begin{aligned}
&\left((\Box - \lambda_0)(1+\Box)^{-m}\bar{\partial}^* u_j, \bar{\partial}^* u_j\right) \\
= &\left((\Box - \lambda_0)(1+\Box)^{-m} u_j, \overline{\partial}\bar{\partial}^* u_j\right) \\
= &\left((\overline{\partial}\bar{\partial}^* - \lambda_0)(1+\Box)^{-m} u_j, \overline{\partial}\bar{\partial}^* u_j\right) \\
= &\left(\overline{\partial}\bar{\partial}^*(1+\Box)^{-m} u_j, (\overline{\partial}\bar{\partial}^* - \lambda_0) u_j\right).
\end{aligned}$$

The first equality is a result of $(1+\Box)^{-1}\bar{\partial}^* = \bar{\partial}^*(1+\Box)^{-1}$ on $Dom(\bar{\partial}^*)$, which follows from Theorem 2.1.2. The second equality follows from $\overline{\partial} \circ \overline{\partial} = 0$ on $Dom(\overline{\partial})$. The third one comes from the self-adjointness of $\overline{\partial}\bar{\partial}^*$ and straightforward calculations. Since

$$\|\overline{\partial}\bar{\partial}^*(1+\Box)^{-m}u_j\| \leq \|\Box(1+\Box)^{-m}u_j\| \leq \|u_j\| = 1, \tag{2.21}$$

2.20 implies

$$\left((\Box - \lambda_0)(1+\Box)^{-m}\bar{\partial}^* u_j, \bar{\partial}^* u_j\right) \to 0 \text{ for } m = 1, 2. \tag{2.22}$$

We also need to verify that $\|\bar{\partial}^* u_j\|$ has a positive lower bound uniformly for all j, which is from the following calculations.

$$(\bar{\partial}^* u_j, \bar{\partial}^* u_j) = ((\overline{\partial}\bar{\partial}^* - \lambda_0)u_j, u_j) + \lambda_0 \to \lambda_0 > 0. \tag{2.23}$$

So 2.22 and 2.23 imply $\lambda_0 \in \mathrm{Spec}(\Box)$ by Theorem 2.2.2, and therefore the result follows. $\Box$

Now we will prove Lemma 2.2.3, which is one containment in identity 2.6.

Lemma 2.2.3. *Under the same assumption as in Theorem 2.2.1, we have*

$$Spec(\overline{\partial}\bar{\partial}^*_{p,q}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q}) \subset Spec(\Box_{p,q}) \cup \{0\}.$$

Proof. We only prove $\mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q}) \subset \mathrm{Spec}(\Box_{p,q}) \cup \{0\}$ here. And $\mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q}) \subset \mathrm{Spec}(\Box_{p,q}) \cup \{0\}$ can be proved in a similar way.

Take $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q})$ and $\lambda_0 > 0$. By Weyl's Criterion, there exists a sequence $u_j \in Dom(\overline{\partial}\bar{\partial}^*)$ with $\|u_j\| = 1$ such that

$$(\overline{\partial}\bar{\partial}^* - \lambda_0)u_j \to 0. \tag{2.24}$$

We will verify that $\overline{\partial}\bar{\partial}^* u_j$ satisfies all the conditions in Theorem 2.2.2 for $\Box$. First, 2.24 directly implies

$$\|\overline{\partial}\bar{\partial}^* u_j\| \to \lambda_0 > 0. \tag{2.25}$$

Secondly, by similar calculations as in the proof of Lemma 2.2.2, for $m = 1, 2$, we have

$$\left((\Box - \lambda_0)(1+\Box)^{-m}\overline{\partial}\bar{\partial}^* u_j, \overline{\partial}\bar{\partial}^* u_j\right) = \left(\overline{\partial}\bar{\partial}^*(1+\Box)^{-m}(\overline{\partial}\bar{\partial}^* - \lambda_0)u_j, \overline{\partial}\bar{\partial}^* u_j\right). \tag{2.26}$$

Since

$$\|\overline{\partial}\bar{\partial}^*(1+\square)^{-m}(\overline{\partial}\bar{\partial}^* - \lambda_0)u_j\| \leq \|(\overline{\partial}\bar{\partial}^* - \lambda_0)u_j\| \to 0, \tag{2.27}$$

we have

$$\big((\square - \lambda_0)(1+\square)^{-m}\overline{\partial}\bar{\partial}^* u_j, \overline{\partial}\bar{\partial}^* u_j\big) \to 0. \tag{2.28}$$

So 2.25 and 2.28 imply $\lambda_0 \in \mathrm{Spec}(\square)$ by Theorem 2.2.2 and therefore the result follows. $\square$

Now we are going to finish the proof of Theorem 2.2.1 by proving the next lemma.

Lemma 2.2.4. *Under the same assumption as in Theorem 2.2.1, we have*

$$Spec(\square_{p,q}) \subset Spec(\overline{\partial}\bar{\partial}^*_{p,q}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q}) \cup \{0\}.$$

Proof. Take $\lambda_0 \in \mathrm{Spec}(\square)$ and $\lambda_0 > 0$. Then by Weyl's Criterion, there exists a sequence $u_j \in Dom(\square)$ with $\|u_j\| = 1$ such that

$$(\square - \lambda_0)u_j \to 0. \tag{2.29}$$

We will use $\overline{\partial}\bar{\partial}^*(1+\square)^{-2}u_j$ and $\bar{\partial}^*\overline{\partial}(1+\square)^{-2}u_j$ as the test sequences. By the fact that $\overline{\partial} \circ \overline{\partial} = 0$ on $Dom(\overline{\partial})$ and $(1+\square)^{-1}\square = \square(1+\square)^{-1}$ on $Dom(\square)$, we have

$$(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}\bar{\partial}^*(1+\square)^{-2}u_j = \overline{\partial}\bar{\partial}^*(1+\square)^{-2}(\square - \lambda_0)u_j. \tag{2.30}$$

Since $\|\overline{\partial}\bar{\partial}^*(1+\square)^{-2}\|_{L^2 \to L^2} \leq 1$, we have

$$\|(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}\bar{\partial}^*(1+\square)^{-2}u_j\| \leq \|(\square - \lambda_0)u_j\| \to 0. \tag{2.31}$$

Similarly, we also have

$$\|(\bar{\partial}^*\overline{\partial} - \lambda_0)\bar{\partial}^*\overline{\partial}(1+\Box)^{-2}u_j\| \leq \|(\Box - \lambda_0)u_j\| \to 0. \tag{2.32}$$

Now we need to verify that either $\|\overline{\partial}\bar{\partial}^*(1+\Box)^{-2}u_j\|$ or $\|\bar{\partial}^*\overline{\partial}(1+\Box)^{-2}u_j\|$ has a positive lower bound which is uniform for j. Note

$$\|\overline{\partial}\bar{\partial}^*(1+\Box)^{-2}u_j\|^2 + \|\bar{\partial}^*\overline{\partial}(1+\Box)^{-2}u_j\|^2 = \|\Box(1+\Box)^{-2}u_j\|^2. \tag{2.33}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\Box$. Then

$$\|\Box(1+\Box)^{-2}u_j\|^2 = \int_0^\infty \frac{\lambda^2}{(1+\lambda)^4} d(P_\lambda u_j, u_j) \geq C(\lambda_0)\|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)}u_j\|^2, \tag{2.34}$$

where $C(\lambda_0) = \min_{\lambda\in[\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0]} \frac{\lambda^2}{(1+\lambda)^4}$. Note $(\Box - \lambda_0)u_j \to 0$ implies

$$\|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)}u_j\| \to 1. \tag{2.35}$$

Therefore for sufficiently large j,

$$\|\overline{\partial}\bar{\partial}^*(1+\Box)^{-2}u_j\|^2 + \|\bar{\partial}^*\overline{\partial}(1+\Box)^{-2}u_j\|^2 \geq \frac{C(\lambda_0)}{2} > 0. \tag{2.36}$$

So $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q}) \cup \mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q})$ by Weyl's Criterion and the result follows. $\Box$

One direct corollary from Theorem 2.2.1 is the following spectrum relations of Gaffney extensions.

Corollary 2.2.1. *Under the same assumption as Theorem 2.2.1, we have*

$$\mathit{Spec}(\Box_{p,q}) \subset \mathit{Spec}(\Box_{p,q+1}) \cup \mathit{Spec}(\Box_{p,q-1}) \cup \{0\}. \tag{2.37}$$

At the end of this section, let us recall the well-known relation between the spectrum of Gaffney extension and L^2 estimates.

Theorem 2.2.3. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Assume the Gaffney extension of Hodge Laplacian $\Box_{p,q+1} : L^2(M, \Lambda^{p,q+1}(E)) \to L^2(M, \Lambda^{p,q+1}(E))$ satisfies Spec$(\Box_{p,q+1}) \subset [a, \infty)$ for some positive number a. Then for any $f \in \ker \overline{\partial}_{p,q+1} \subset L^2(M, \Lambda^{p,q+1}(E))$, there exists $u \in L^2(M, \Lambda^{p,q}(E))$ such that $\overline{\partial} u = f$ with the following estimate*

$$(u, u) \leq \frac{1}{a}(f, f). \tag{2.38}$$

Proof. In the proof, we will use $\Box$ to represent $\Box_{p,q+1}$ for simplicity. By the condition Spec$\Box \subset [a, \infty)$, we have that $\Box^{-1} : L^2(M, \Lambda^{p,q+1}(E)) \to Dom(\Box) \subset L^2(M, \Lambda^{p,q+1}(E))$ is a bounded operator with

$$\|\Box^{-1}\|_{L^2 \to L^2} \leq \frac{1}{a}. \tag{2.39}$$

Take $u = \bar{\partial}^* \Box^{-1} f$ and we will verify u satisfies all the conclusions. First, since the Gaffney extension satisfies $\Box = \overline{\partial}\bar{\partial}^* + \bar{\partial}^*\overline{\partial}$ by Theorem 2.1.2, we have

$$\overline{\partial} u = \overline{\partial}\bar{\partial}^* \Box^{-1} f = f - \bar{\partial}^* \overline{\partial} \Box^{-1} f. \tag{2.40}$$

Therefore, $f \in \ker \overline{\partial}$ implies $\bar{\partial}^* \overline{\partial} \Box^{-1} f \in \ker \overline{\partial}$. By taking the following inner product

$$0 = \left(\overline{\partial}\bar{\partial}^* \overline{\partial} \Box^{-1} f, \overline{\partial} \Box^{-1} f\right) = \left(\bar{\partial}^* \overline{\partial} \Box^{-1} f, \bar{\partial}^* \overline{\partial} \Box^{-1} f\right), \tag{2.41}$$

we have

$$\bar{\partial}^* \overline{\partial} \Box^{-1} f = 0. \tag{2.42}$$

Again, by taking the following inner product with $\Box^{-1} f$

$$0 = \left(\bar{\partial}^* \overline{\partial} \Box^{-1} f, \Box^{-1} f\right) = \left(\overline{\partial} \Box^{-1} f, \overline{\partial} \Box^{-1} f\right), \tag{2.43}$$

we have

$$\overline{\partial} \Box^{-1} f = 0. \tag{2.44}$$

Combining with 2.40, we have

$$\overline{\partial} u = f.$$

Second, we will verify the estimate 2.38. By 2.44 and straightforward calculations, we have

$$(u, u) = (\bar{\partial}^* \Box^{-1} f, \bar{\partial}^* \Box^{-1} f) = (\overline{\partial} \bar{\partial}^* \Box^{-1} f, \Box^{-1} f) = (f, \Box^{-1} f).$$

Therefore 2.39 implies the result. $\Box$

Remark 2.2.4. Note we cannot directly use $\overline{\partial}_{p,q+2} \Box^{-1}_{p,q+1} f = \Box^{-1}_{p,q+2} \overline{\partial}_{p.q+1} f = 0$ in the proof as we do not know the existence of $\Box^{-1}_{p,q+2}$.

Chapter 3

Self-adjointness of the Laplace Operator on Calabi-Yau Moduli Space

3.1 Manifolds with almost polar boundary

Let (M, g) be a Riemannian manifold. First, we recall the following notations. Let $\mathcal{D}(M)$ be the set of smooth functions on M with compact support. We use $Q_1(\cdot,\cdot) = (\cdot,\cdot) + (d\cdot, d\cdot)$ to denote the quadratic form on functions. And recall the definition of Sobolev spaces, as shown below.

Definition 3.1.1.

$$W_0^1(M) = \text{Completion of } \mathcal{D}(M) \text{ with respect to } Q_1 \text{ inner product,} \tag{3.1}$$

$$W^1(M) = \text{Completion of } \{\varphi \in \mathcal{C}^\infty(M) : Q_1(\varphi,\varphi) < \infty\} \text{ with respect to } Q_1 \text{ inner product.} \tag{3.2}$$

Generally, we know $W^1(M) = W_0^1(M)$ for complete Riemannian manifolds. In [16, 17], Masamune proved $W^1(M) = W_0^1(M)$ for Riemannian manifolds with almost polar boundary. We will repeat the proof here for the sake of completeness, and because there is a gap in Masamune's proof.

We need to introduce some more definitions and notations. Let d be the distance function induced by the length of piecewise curves on M. Then (M, d) is a metric space. We use $(\overline{M}_c, d)$ to denote the Cauchy completion of (M, d). We define the Cauchy boundary $\partial_c M = \overline{M}_c - M$.

Definition 3.1.2. We define the capacity of an open set $O \subset \overline{M}_c$ by

$$\text{cap}(O) = \inf\{Q_1(u, u) : u \in W^1(M), 0 \leq u \leq 1 \text{ and } u|_{O\cap M} = 1\}. \tag{3.3}$$

We also define the capacity of an arbitrary set $\Sigma \subset \overline{M}_c$ by

$$\text{cap}(\Sigma) = \inf\{\text{cap}(O), \Sigma \subset O, O \subset \overline{M}_c \text{ is open}\}. \tag{3.4}$$

A set Σ is said to be almost polar if $\text{cap}(\Sigma) = 0$.

Remark 3.1.1. For any open set $O \subset \overline{M}_c$, $e \in W^1(M)$ is called the equilibrium potential of O if it satisfies the following conditions.

1. $Q_1(e, e) = \text{cap}(O)$.

2. $e|_O = 1$.

3. $0 \leq e \leq 1$.

It is known that the equilibrium potential exists for any open set $O \subset \overline{M}_c$. See [7] for more details.

Here is the main theorem we are going to prove.

Theorem 3.1.1. *Let* (M, g) *be a Riemannian manifold. If cap*$(\partial_c M) = 0$*, then*

$$W^1(M) = W^1_0(M). \tag{3.5}$$

It might be a good idea to go through the main idea of the proof beforehand. First we show that $L^\infty(M) \cap W^1(M) \subset W^1(M)$ is dense. Then without loss of generality, we only need to consider $f \in L^\infty(M) \cap W^1(M)$. After choosing a sequence of open sets $\{V_n\}$ decreasing to $\partial_c M$, by using the equilibrium potential of V_n, say e_n, we can approximate f by $(1 - e_n)f$ whose support is contained in $M - V_n$. In the end, we want to modify the function $(1 - e_n)f$ to be compactly supported. As $(\overline{M}_c, d)$ is only a complete metric space, the closed metric ball excluding an open set containing $\partial_c M$ might not be a compact set even if $\text{cap}(\partial_c M) = 0$ (see Section 3.4 for more details). So we will use the intrinsic distance on $M - V_n$ instead. By Hopf-Rinow-Cohn-Vossen Theorem (see Theorem 2.5.28 in [3]), we know the closed metric ball with respect to the intrinsic distance is compact . And we will use some cut-off function to finish the modification on the support.

We begin the proof with the following lemma.

Lemma 3.1.1. *For any Riemannian manifold* (M, g)*,* $L^\infty(M) \cap W^1(M)$ *is dense in* $W^1(M)$*.*

Proof. Take $f \in W^1(M)$. Define a cut-off function $\rho \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\rho(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x \geq 2 \end{cases},$$

and

$$0 \leq \rho \leq 1, \quad -C \leq \rho' \leq 0.$$

Then define $f_m = \rho_m(|f|)f$. Note $f_m \in L^\infty(M) \cap W^1(M)$ and we will prove $f_m \to f$ in $W^1(M)$. By the dominated convergence theorem, we directly get $f_m \to f$ in $L^2(M)$.

As to df_m, we have

$$df_m - df = \big(\rho(\frac{|f|}{m}) - 1\big)df + \frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f|. \tag{3.6}$$

The first term on the right hand side converges to 0 in $L^2(M, \Lambda^1)$ as $|\rho(\frac{|f|}{m}) - 1| \leq \chi_{\{|f| \geq m\}}$. For the second term, since

$$\Big|\frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f|\Big| \leq 2C\chi_{\{m \leq |f| \leq 2m\}}|df|, \tag{3.7}$$

it follows that $\frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f| \to 0$ in $L^2(M, \Lambda^1)$. So we have $f_m \to f$ in $W^1(M)$ and the result follows. □

In the next two lemmas, we will construct open sets containing $\partial_c M$ with smooth boundary.

Lemma 3.1.2. *$\partial_c M \subset \overline{M}_c$ is a closed subset.*

Proof. Since M is the complement of $\partial_c M$ in $\overline{M}_c$, it is equivalent to verify that $M \subset \overline{M}_c$ is an open subset. For any $x \in M$, let i_x be the injectivity radius at x. Then for any $r \in (0, i_x)$, by considering the exponential map at x, we know $\overline{B_M(x,r)} = \{y \in M, d(x,y) \leq r\}$ is compact, whence complete. Therefore $B_M(x,r) = B_{\overline{M}_c}(x,r) = \{y \in \overline{M}_c, d(x,y) < r\}$ since we will not add any new point to $B_M(x,r)$ during the Cauchy completion of M. So $B_{\overline{M}_c}(x,r) \subset M$ and the result follows. □

Lemma 3.1.3. *For any open set $U \subset \overline{M}_c$ containing $\partial_c M$, there exists an open set $V \subset \overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} \subset U$ and $\partial(\overline{M}_c \setminus V) \subset M$ is a smooth submanifold of codimension 1.*

Proof. Let U^C be the complement of U in $\overline{M}_c$. Since $\partial_c M$ and U^C are both closed in $(\overline{M}_c, d)$. By Urysohn's Lemma, there exists a function $f \in \mathcal{C}(\overline{M}_c)$ such that $0 \le f \le 1$, $f^{-1}(\{0\}) = \partial_c M$ and $f^{-1}(\{1\}) = U^C$. Take $S = f^{-1}([0, \frac{1}{2}))$. Then S is an open subset of $\overline{M}_c$ such that $\partial_c M \subset S \subset \overline{S} \subset U$.

Note that $\overline{S} \setminus \partial_c M = \overline{S} \cap M$ and U^C are both closed in M. By the Smooth Urysohn's Lemma in [18], there exists a function $g \in \mathcal{C}^\infty(M)$ such that $0 \le g \le 1$, $g^{-1}(\{0\}) = \overline{S} \setminus \partial_c M$ and $g^{-1}(\{1\}) = U^C$. By Sard's Theorem, without loss of generality, we can assume $\frac{1}{2}$ is a regular value of g. Take $V = g^{-1}([0, \frac{1}{2})) \cup \partial_c M \subset \overline{M}_c$. Then it is easy to see $V = g^{-1}([0, \frac{1}{2})) \cup S$. Therefore, V is open in $\overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} = \overline{g^{-1}([0, \frac{1}{2}))} \cup \overline{S} \subset U$. Because $\partial(\overline{M}_c \setminus V) = g^{-1}(\{\frac{1}{2}\})$ and $\frac{1}{2}$ is a regular value of g, the remaining part of the lemma follows. □

Let V be an open subset satisfying the conclusion in the above lemma. Denote $V^C = \overline{M}_c \setminus V$ as the complement of V in $\overline{M}_c$. Then $V^C = \cup_{\lambda \in \Lambda} A_\lambda$, where each A_λ is a connected component of V^C and Λ is the index set. Since V^C is locally path-connected, each A_λ is both open and closed in V^C. We define the intrinsic distance function d_{A_λ} on A_λ as follows.

Definition 3.1.3. Define the intrinsic distance on A_λ as $d_{A_\lambda} : A_\lambda \times A_\lambda \to [0, \infty)$,

$$d_{A_\lambda}(x, y) = \inf_{l \in L_{A_\lambda}} \|l\| \tag{3.8}$$

where $L_{A_\lambda} = \{$all piecewise smooth curves contained in A_λ from x to $y\}$ and $\|l\|$ denotes the length of curve l.

Remark 3.1.2. $d(x, y) \le d_{A_\lambda}(x, y)$ for any $x, y \in A_\lambda$ as d is the infimum over a larger set.

In general, d and d_{A_λ} are not globally equivalent to each other on A_λ. The next lemma shows that they are locally equivalent on A_λ.

Lemma 3.1.4. *For any $x \in A_\lambda$, there exists $r = r(x) > 0$ such that*

$$d_{A_\lambda}(x, y) \le 4d(x, y) \quad \text{for any } y \in B_{A_\lambda}(x, r). \tag{3.9}$$

where $B_{A_\lambda}(x, r) = \{y \in A_\lambda, d(x, y) < r\}$.

Proof. For any $x \in A_\lambda \subset V^C \subset M$, either x is in the interior of V^C or $x \in \partial V^C$. In the first case, take $r < i_x$ (i_x denotes the injectivity radius at x) small enough such that $B_M(x, r) \subset A_\lambda$. Then for any $y \in B_M(x, r)$, there exists a minimizing geodesic $l \subset B_M(x, r)$ such that $\|l\| = d(x, y)$. Therefore, $d_{A_\lambda}(x, y) = d(x, y)$ for any $y \in B_{A_\lambda}(x, r) = B_M(x, r)$.

In the second case, i.e. $x \in \partial V^C$, take $r < i_x$. We can identify $B_{\mathbb{R}^m}(o, r)$(w.r.t the Euclidean metric g_x for fixed x) with $B_M(x, r)$ by the exponential map Exp_x at x. By shrinking r, we can assume the Riemannian metric on $B_M(x, r)$ is equivalent to the metric at x, say $\frac{1}{2}g_x \le g \le 2g_x$. Let $\{e_i\}_{i=1}^m$ be the standard orthonormal basis of $\mathbb{R}^m$. Up to an orthonormal linear transformation, we can assume $\{e_i\}_{i=1}^{m-1} \subset T_x(\partial V^C)$ and e_m is the normal direction of ∂V^C at x. According to Lemma 3.1.3, possibly by shrinking r again, we can assume $\partial V^C = \{(x_1, x_2, \cdots, x_m) \in B(o, r), x_m = h(x_1, x_2 \cdots, x_{m-1})\}$ where $h \in \mathcal{C}^\infty(\mathbb{R}^{m-1})$ and $h(0, \cdots, 0) = 0$. Since $\{e_i\}_{i=1}^{m-1}$ are tangent vectors of ∂V^C at x, $\nabla h(0, \cdots, 0) = 0$. By shrinking r again, we can assume $|\nabla h| \le 1$ in $B_{\mathbb{R}^{m-1}}(o, r)$.

For any point $y \in B_{\mathbb{R}^m}(o, r)$, consider the curves

$$l_1 = (ty_1, ty_2, \cdots, ty_{m-1}, h(ty_1, \cdots, ty_{m-1})) \text{ for } t \in [0, 1],$$

and

$$l_2 = (y_1, y_2, \cdots, y_{m-1}, ty_m + (1-t)h(y_1, y_2, \cdots, y_{m-1})) \text{ for } t \in [0, 1].$$

Then the concatenation $l_1 \cup l_2 \subset V^C$ is from x to y. The Euclidean length of l_1, l_2 are respectively

$$\begin{aligned}\|l_1\|_{\mathbb{R}^m} =& \int_0^1 \sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2 + |\nabla h(ty_1, ty_2, \cdots, ty_{m-1}) \cdot (y_1, y_2, \cdots, y_{m-1})|^2} dt \\ \leq& 2\sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2}, \\ \|l_2\|_{\mathbb{R}^m} =& |y_m - h(y_1, y_2, \cdots, y_{m-1})| \\ \leq& |y_m| + \sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}d_{A_\lambda}(x, y) \leq& \|l_1\| + \|l_2\| \\ \leq& 2\|l_1\|_{\mathbb{R}^m} + 2\|l_2\|_{\mathbb{R}^m} \\ \leq& 4\sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2 + y_m^2} \\ =& 4d(x, y).\end{aligned}$$

The second inequality is due to $\frac{1}{2}g_x \leq g \leq 2g_x$. So the result follows. $\square$

Based on Remark 3.1.2 and Lemma 3.1.4, we have the following proposition on $(A_\lambda, d_{A_\lambda})$.

Proposition 3.1.3. *$(A_\lambda, d_{A_\lambda})$ satisfies the following properties.*

(a). $(A_\lambda, d_{A_\lambda})$ and (A_λ, d) have the same topology.

(b). $(A_\lambda, d_{A_\lambda})$ is locally compact.

(c). $(A_\lambda, d_{A_\lambda})$ is complete.

Proof. Part (a) directly follows from Remark 3.1.2 and Lemma 3.1.4.

Now we prove part (b). Since V^C is a closed subset of (M, d) and (M, d) is locally compact, (V^C, d) is locally compact. Because A_λ is a closed subset of (V^C, d), (A_λ, d) is locally compact. The result follows from part (a).

Lastly we prove part (c). Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in $(A_\lambda, d_{A_\lambda})$. By Remark 3.1.2, $\{x_n\}_{n=1}^\infty$ is also a Cauchy sequence in (A_λ, d). Since A_λ is closed in (V^C, d) and V^C is closed in the complete space $(\overline{M}_c, d)$, (A_λ, d) is complete. Then we know there exists some $x \in A_\lambda$ such that $\lim d(x, x_n) = 0$. By Lemma 3.1.4, $\lim d_{A_\lambda}(x_n, x) = 0$ and therefore the result follows. ☐

For any $x_0 \in A_\lambda$, define the function $r_{x_0} : A_\lambda \to [0, \infty)$ as $r_{x_0}(x) = d_{A_\lambda}(x_0, x)$. Then we have the following proposition on r_{x_0}.

Proposition 3.1.4. *For the function r_{x_0} defined as above, we have*

$$|\nabla r|_g \leq 4. \tag{3.10}$$

Proof. Since $|r(x) - r(y)| \leq d_{A_\lambda}(x, y)$, the result follows from Lemma 3.1.4. ☐

The closed metric ball induced from d_{A_λ} is compact though it is not the case for metric d. The following lemma is essentially Hopf-Rinow-Cohn-Vossen Theorem. See Theorem 2.5.28 in [3] for more details.

Lemma 3.1.5. *For any $x \in A_\lambda, r > 0$, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact. Here $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ denotes the set $\{y \in A_\lambda, d_{A_\lambda}(x, y) < r\}$.*

Remark 3.1.5. By part (a) in Proposition 3.1.3, the closures of $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ in (A_λ, d) and in (A_λ, d_λ) are the same. The compactness in (A_λ, d) and that in (A_λ, d_λ) are also the same. So there is no ambiguity in the above lemma.

Proof. By part (b) in Proposition 3.1.3, the set $\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact$\}$ is nonempty. So we can define $r_0 = \sup\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact$\}$. Now it suffices to prove $r_0 = \infty$. Assume not. Then $r_0 \in (0, \infty)$.

First, we prove that $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is compact. Take an arbitrary $\varepsilon > 0$. Since $d_{A_\lambda}(x, y) \leq r_0$ for any $y \in \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$, there exists a piecewise smooth curve $l \subset A_\lambda$ from x to y such that $\|l\| < r_0 + \varepsilon$. Reparametrize the curve l by its arc length. Then the restriction $l|_{[r_0-\varepsilon, \|l\|]}$ is a piecewise smooth curve from a point in $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}$ to y. Since $\|l|_{[r_0-\varepsilon, \|l\|]}\| < 2\varepsilon$, $y \in B_{(A_\lambda, d_{A_\lambda})}(\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}, 2\varepsilon)$. Therefore, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)} \subset B_{(A_\lambda, d_{A_\lambda})}(\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}, 2\varepsilon)$. Since $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}$ is compact by the definition of r_0, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is totally bounded in $(A_\lambda, d_{A_\lambda})$. Therefore, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is compact by part (c) in Proposition 3.1.3.

Second, we prove that $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 + \delta)}$ is compact for some $\delta > 0$, which contradicts the definition of r_0 and therefore we get the result. Since $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is also compact, together with part (b) in Proposition 3.1.3, we know $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ has a finite cover $\{B_{(A_\lambda, d_{A_\lambda})}(y_i, \delta_i)\}_{i=1}^N$, such that $y_i \in \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$, $\delta_i > 0$ and $\overline{B_{(A_\lambda, d_{A_\lambda})}(y_i, 2\delta_i)}$ is compact for each i. Take $\delta = \min_{1 \leq i \leq N} \delta_i$. Then $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 + \delta)} \subset \cup_{i=1}^N \overline{B_{(A_\lambda, d_{A_\lambda})}(y_i, 2\delta_i)}$ is compact, contradicting that r_0 is the supreme of $\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact$\}$. So the result follows. ∎

Now we are ready to prove the Theorem 3.1.1.

Proof. Since $\text{cap}(\partial_c M) = 0$, there exists a sequence of open sets $\{U_n\}_{n=1}^\infty$ such that $\partial_c M \subset U_n$ and $\lim \text{cap}(U_n) = 0$. For U_1, by Lemma 3.1.3, there exists an open set V_1 such that $\partial_c M \subset V_1 \subset \overline{V_1} \subset U_1$ and $\partial(V_1^C)$ is a smooth submanifold. Then for $V_1 \cap U_2$, by Lemma 3.1.3, there exists an open set V_2 such that $\partial_c M \subset V_2 \subset \overline{V_2} \subset V_1 \cap U_2$ and $\partial(V_2^C)$ is a smooth submanifold. Inductively, we construct V_{i+1} by applying Lemma 3.1.3 to $V_i \cap U_{i+1}$. So we get a sequence of decreasing open sets $\{V_n\}_{n=1}^{\infty}$ such that $\partial_c M \subset V_n \subset \overline{V_n} \subset V_{n-1} \cap U_n$ and $\partial(V_n^C)$ is a smooth submanifold. In particular $V_n \subset U_n$, so we have $\lim \mathrm{cap}(V_n) = 0$.

Take $f \in W^1(M) \cap L^\infty$. It suffices to prove $f \in W_0^1(M)$.

First, we approximate f by using functions with the support sets in some V_n^C. Let e_n be the equilibrium potential (see Remark 3.1.1) of V_n, i.e. e_n satisfies the following properties.

1. $e_n \in W$ and $Q_1(e_n, e_n) = \mathrm{cap}(V_n)$.

2. $e_n|_{V_n} = 1$.

3. $0 \leq e_n \leq 1$.

Since $\|e_n\|_W = \mathrm{cap}(V_n) \to 0$, we can assume $e_n \to 0$ a.e. by passing to a subsequence. Let $f_n = (1 - e_{n-1})f$. Then $f_n \to f$ in $W^1(M)$ and $\mathrm{supp}(f_n) \subset V_{n-1}^C \subset \overline{V_n}^C \subset \mathrm{interior}(V_n^C)$.

Secondly, we approximate each f_n with $\mathrm{supp}(f_n) \subset \mathrm{interior}(V_n^C)$ by using functions with compact supports. From now on, we fix f_n and V_n^C. For simplicity, we suppress the index n. Write V^C as the disjoint union of connected components, $V^C = \cup_{\lambda \in \Lambda} A_\lambda$. Since $f \in W^1(M)$ and $\{A_\lambda\}_{\lambda \in \Lambda}$ is pairly disjoint, f vanishes on all but countably many A_λ, say $\{A_{\lambda_j}\}_{j=1}^{\infty}$. Denote $g_j = f\chi_{A_{\lambda_j}}$ where $\chi_{A_{\lambda_j}}$ is the characteristic function of A_{λ_j}. Note $g_j \in W^1(M)$ and $\nabla g_j = (\nabla f)\chi_{A_{\lambda_j}}$, due to the fact that $\partial A_{\lambda_j} \subset \partial(V^C)$[1] and that f vanishes close to $\partial(V^C)$ as $\mathrm{supp}\, f \subset \mathrm{interior}(V^C)$. Then $f = \sum_{j=1}^{\infty} g_j$ and $\|f\|_W^2 = \sum_{j=1}^{\infty} \|g_j\|_W^2$. Therefore, for any $\varepsilon > 0$, there exists $N > 0$ such that $\|f - \sum_{j=1}^{N} g_j\|_W < \varepsilon$.

Now it suffices to approximate each g_j by using compact supported functions. Take $x_j \in A_{\lambda_j}$ and define $r_j : A_{\lambda_j} \to [0, \infty)$ as $r_j(x) = d_{A_{\lambda_j}}(x_j, x)$. Then we have $|\nabla r_j|_g \leq 4$ by Proposition 3.1.4. Let $\varphi \in \mathcal{C}^\infty(R)$ satisfy the following conditions.

1. φ is a decreasing function and $0 \leq \varphi \leq 1$.

[1] This is a fact in locally connected topological space. See [1] for more details.

2. $\varphi|_{(-\infty,0]} = 1$ and $\varphi_{[1,\infty)} = 0$.

3. $|\varphi'| \leq C$ and C is a fixed constant.

Define $\varphi_k(x) = \varphi(\frac{x}{k})$. Then $\varphi_k \circ r_j \to 1$ a.e. on A_{λ_j} as $k \to \infty$ and $|\nabla(\varphi_k \circ r_j)|_g \leq \frac{4C}{k}$. Therefore, we have $(\varphi_k \circ r_j)g_j \to g_j$ in $W^1(M)$. And $\mathrm{supp}((\varphi_k \circ r_j)g_j) \subset \mathrm{supp}(\varphi_k \circ r_j) \subset \overline{B_{(A_{\lambda_j}, d_{A_{\lambda_j}})}(x_j, 2k)}$, which is compact by Lemma 3.1.5. So the result follows. □

3.2 Moduli Space of Polarized Calabi-Yau Manifolds

In this section, we consider the moduli space of polarized Calabi-Yau manifolds $\mathcal{M}$. In [23], Viehweg proved the moduli space $\mathcal{M}$ is a quasi-projective variety. Take $\overline{\mathcal{M}}$ as the compactification of $\mathcal{M}$. By resolution of singularities and normalization, we may assume $\overline{\mathcal{M}}$ is a smooth manifold and the divisor $Y = \overline{\mathcal{M}} \setminus \mathcal{M}$ is a divisor of normal crossings. Let ω_{WP} be the Weil-Petersson metric. From now on, we will work on this quasi-projective Kähler manifold $(\mathcal{M}, \omega_{WP})$ with the compactification $\overline{\mathcal{M}}$ which is a compact Kähler manifold.

Here is the main theorem we are going to prove in this section.

Theorem 3.2.1. *The moduli space of Polarized Calabi-Yau manifolds $(\mathcal{M}, \omega_{WP})$ has an almost polar Cauchy boundary, i.e. cap$(\partial_c\mathcal{M}) = 0$.*

Remark 3.2.1. In general, the Cauchy completion $\overline{\mathcal{M}}_c$ is not necessarily identical to the compactification $\overline{\mathcal{M}}$. Therefore, the Cauchy boundary $\partial_c\mathcal{M}$ is not necessarily identical to the divisor $Y = \overline{\mathcal{M}} \setminus \mathcal{M}$.

It is well-known that there is a complete Kähler metric on $\mathcal{M}$ such that it is asymptotical to the Poincaré metric near infinity. We call it Poincaré metric and denote it by ω_P(See Lemme 3.1 in [14]). The key ingredient in proving Theorem 3.2.1 is the following lemma from [12].

Lemma 3.2.1. *For any $\varepsilon > 0$ small enough, there is a smooth real valued function $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ such that*

(a). $0 \le \rho_\varepsilon \le 1$*;*

(b). There is a constant C, independent of ε, such that $-C\omega_P \le \sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \le C\omega_P$;

(c). In some neighborhood of Y, $\rho_\varepsilon = 0$ and $\rho_\varepsilon(x) = 1$ if the Euclidean distance of $x \in M$ to Y is greater than 2ε.

Proof. As $Y \subset \overline{\mathcal{M}}$ is a divisor of normal crossings, we can find a finite cover $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ such that $Y \subset \cup_{\alpha=1}^s U_\alpha$ and $U_{s+1} \cup \cdots \cup U_t) \cap Y = \emptyset$. Furthermore, we can assume that $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha, w_1^\alpha, \cdots, w_{b_\alpha}^\alpha)$ for any $1 \le \alpha \le s$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that $0 \le \eta \le 1$ and

$$\eta = \begin{cases} 1 & x \le 0 \\ 0 & x \ge 1 \end{cases}.$$

Let

$$\eta_\varepsilon(z) = \begin{cases} 1 & |z| \le e^{-\frac{1}{\varepsilon}} \\ \eta(\frac{(\log\frac{1}{|z|})^{-1} - \varepsilon}{\varepsilon}) & e^{-\frac{1}{\varepsilon}} \le |z| \le e^{-\frac{1}{2\varepsilon}} \\ 0 & |z| \ge e^{-\frac{1}{2\varepsilon}} \end{cases}.$$

And let

$$\eta_\varepsilon^\alpha(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha) = \prod_{j=1}^{a_\alpha}(1 - \eta_\varepsilon(s_j^\alpha)).$$

Then define the function

$$\rho_\varepsilon = \sum_{\alpha=1}^{s} \psi_\alpha \eta_\varepsilon^\alpha + \sum_{\alpha=s+1}^{t} \psi_\alpha,$$

where $\{\psi_\alpha\}$ is a partition of unity subordinated to $\{U_\alpha\}$.

Then $0 \le \rho_\varepsilon \le 1$. By a straightforward calculation, we have

$$\begin{aligned}
\overline{\partial}\eta_\varepsilon &= \frac{1}{2\varepsilon}\eta' \frac{d\bar{z}}{\bar{z}(\log\frac{1}{|z|})^2}, \\
\partial\overline{\partial}\eta_\varepsilon &= \frac{1}{4\varepsilon^2}\eta'' \frac{dz \wedge d\bar{z}}{|z|^2(\log\frac{1}{|z|})^4} + \frac{1}{2\varepsilon}\eta' \frac{dz \wedge d\bar{z}}{|z|^2(\log\frac{1}{|z|})^3}.
\end{aligned}$$

Note that $\eta' = 0$ unless $\varepsilon \le (\log\frac{1}{|z|})^{-1} \le 2\varepsilon$. Therefore,

$$|\overline{\partial}\eta_\varepsilon| \le C|\frac{d\bar{z}}{|z|\log\frac{1}{|z|}}|, \quad |\partial\overline{\partial}\eta_\varepsilon| \le C|\frac{dz \wedge d\bar{z}}{|z|^2(\log\frac{1}{|z|})^2}|,$$

where C is a constant independent of ε. Therefore, we obtain part (b) as ψ_α are fixed smooth functions on $\overline{\mathcal{M}}$.

Let $x \in \mathcal{M}$. When x is sufficiently close to Y, $\psi_\alpha = 0$ for any $\alpha \ge s+1$ and $\eta_\varepsilon^\alpha = 0$ for any $\alpha \le s$. Therefore, we obtain $\rho_\varepsilon = 0$ in a neighborhood of Y. If the distance from x to Y is at least 2ε, then there is a constant $C > 0$ such that $|s_j^\alpha| \ge C\varepsilon$ for any $1 \le j \le a_\alpha$ and $1 \le \alpha \le s$. Since $\varepsilon e^{\frac{1}{2\varepsilon}} \to \infty$ as $\varepsilon \to 0$, we have $\rho_\varepsilon(x) = \sum \psi_\alpha = 1$ when ε is small enough. □

Now we are ready to prove Theorem 3.2.1.

Proof. Take the function ρ_ε constructed in Lemma 3.2.1. As $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ and $0 \le \rho_\varepsilon \le 1$, we have

$$\operatorname{cap}(\partial_c\mathcal{M}) \leq \int_{\mathcal{M}} |1-\rho_\varepsilon|^2 \frac{\omega_{WP}^n}{n!} + \int_{\mathcal{M}} |d(1-\rho_\varepsilon)|^2 \frac{\omega_{WP}^n}{n!}, \text{ for any } \varepsilon > 0. \tag{3.11}$$

Since $\rho_\varepsilon \to 1$ pointwise on $\mathcal{M}$ and the volume of Weil-Petersson metric is finite by Theorem 1.1 in [14],

$$\lim_{\varepsilon\to 0} \int_{\mathcal{M}} |1-\rho_\varepsilon|^2 \frac{\omega_{WP}^n}{n!} = 0. \tag{3.12}$$

It suffices to prove that $\int_{\mathcal{M}} |d\rho|^2 \to 0$. Note

$$\begin{aligned}\int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n =& 2\int_{\mathcal{M}} |\overline{\partial}\rho_\varepsilon|^2 \omega_{WP}^n \\ =& 2n \int_{\mathcal{M}} \sqrt{-1}\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1} \\ =& -2n \int_{\mathcal{M}} \sqrt{-1}\rho_\varepsilon \partial\overline{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1}.\end{aligned}$$

Since $-C\omega_P \leq \sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \leq C\omega_P$ and $\omega_{WP} \leq C\omega_P$(see Proposition 3.1 in [14]), we have

$$\int_{\mathcal{M}} |\overline{\partial}\rho_\varepsilon|^2 \omega_{WP}^n \leq C \int_{\operatorname{supp}(\overline{\partial}\rho_\varepsilon)} \omega_P^n.$$

Use the same cover of $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ as in Lemma 3.2.1. Then $Y \subset \cup_{\alpha=1}^s U_\alpha$, $U_{s+1} \cup \cdots \cup U_t) \cap Y = \emptyset$ and $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha, w_1^\alpha, \cdots, w_{b_\alpha}^\alpha)$ for any $1 \leq \alpha \leq s$. When ε is small enough, we can assume that $\operatorname{supp}(\overline{\partial}\rho_\varepsilon) \cap U_\alpha \subset \{|s_j^\alpha| \leq \frac{1}{2}, |w_j^\alpha| \leq \frac{1}{2}\}$ for any $1 \leq \alpha \leq s$. In $U_\alpha - Y$ for any $1 \leq \alpha \leq s$, the Poincaré metric ω_P is asymptotic to

$$\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{a_\alpha} \frac{ds_j^\alpha \wedge d\bar{s}_j^\alpha}{|s_j^\alpha|^2 (\log\frac{1}{|s_j^\alpha|})^2} + \sum_{j=1}^{b_\alpha} dw_j^\alpha \wedge d\bar{w}_j^\alpha\right), \tag{3.13}$$

so we have

$$\int_{\mathrm{supp}(\overline{\partial}\rho_\varepsilon)} \omega_P^n \leq C \sum_{\alpha=1}^{s} \prod_{j=1}^{a_\alpha} \int_{e^{-\frac{1}{2\varepsilon}}}^{e^{-\frac{1}{\varepsilon}}} \frac{1}{|s_j^\alpha|(\log \frac{1}{|s_j^\alpha|})^2} d|s_j^\alpha| \prod_{j=1}^{b_\alpha} \int_0^{\frac{1}{2}} |w_j^\alpha| d|w_j^\alpha| \leq C\varepsilon. \tag{3.14}$$

Therefore,

$$\lim_{\varepsilon \to 0} \int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n = 0 \tag{3.15}$$

and the result follows. □

3.3 Self-Adjointness of the Laplacian on Moduli Space

In this section, we will consider the self-adjointness of Laplacian on $(\mathcal{M}, \omega_{WP})$. Let us consider the differential operators d and δ defined respectively on C^1 functions and C^1 forms on $\mathcal{M}$. We define the domain $Dom(d)$ of d to be the set of C^1 functions f defined on $\mathcal{M}$ such that both f and df are in L^2. Similarly, we define the domain $Dom(\delta)$ of δ to be the set of C^1 1-forms w such that both w and δw are in L^2. We then define the Laplacian Δ with respect to ω_{WP} as δd. And the domain $Dom(\Delta)$ is given by the set of C^2 functions f such that $f \in Dom(d)$ and $df \in Dom(\delta)$. In this section, we will prove the closure $\overline{\Delta}$ of Δ is self-adjoint.

Theorem 3.3.1. *On $(\mathcal{M}, \omega_{WP})$, the closure $\overline{\Delta}$ of Laplacian on functions is self-adjoint.*

It is proved in [10] that $\overline{\Delta}$ is self-adjoint on $M \setminus \Sigma_M$ when M is an algebraic variety with the induced Fubini-Study metric and Σ_M is the singular set of at least real codimension 2. Here our result is different as we are considering the Weil-Petersson metric.

Proof. By the theorem of Gaffney in [8], in order to show $\overline{\Delta}$ is self-adjoint, it is sufficient to prove

$$(df, w) = (f, \delta w) \text{ for any } f \in Dom(d) \text{ and } w \in Dom(\delta). \tag{3.16}$$

By Theorem 3.2.1 and 3.1.1, we have $W^1(\mathcal{M}) = W_0^1(\mathcal{M})$. Since $Dom(d) \subset W^1(\mathcal{M})$, there exists a sequence $f_n \in \mathcal{D}(\mathcal{M})$ such that $f_n \to f$ in $W^1(\mathcal{M})$. As each f_n has a compact support, through integration by parts, we have

$$(df_n, w) = (f_n, \delta w). \tag{3.17}$$

The result follows by taking $n \to \infty$. □

3.4 An Example

Let (M, g) be a Riemannian manifold. As we have mentioned in Section 3.1, a closed metric ball in $(\overline{M}_c, d)$ excluding an open set containing $\partial_c M$ might not be compact even if $\mathrm{cap}(\partial_c M) = 0$. And that is why we use the intrinsic distance instead in the proof of Theorem 3.1.1. In this section, we will give a concrete example to demonstrate.

Consider the Riemannian manfold (M, g) as follows. Suppose $M = \mathbb{R}^3$ and

$$g = e^{2z}(dr^2 + f^2(r)d\theta^2 + dz^2), \tag{3.18}$$

where (r, θ, z) is the cylindrical coordinates. Here the function $f \in \mathcal{C}^\infty([0, \infty))$ satisfies the following properties.

1. $f(r) = r$ for $r \in [0, \frac{1}{2}]$.
2. f is increasing on $[0, 1]$ and $f(1) = 1$.

3. f is decreasing on $[1, \infty)$.

4. $f(r) = e^{-r}$ for $f \in [2, \infty)$.

For any piecewise smooth curve $l : [a, b] \to M$, we denote the lenth of l by $\|l\|$, i.e.

$$\|l\| = \int_a^b e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t))\dot{\theta}^2(t) + \dot{z}^2(t)} dt. \tag{3.19}$$

Define the distance function d as

$$d(p, q) = \inf_{l \in L} \|l\|,$$

where $L = \{\text{all piecewise smooth curves from } p \text{ to } q\}$. Then we know (M, d) is a metric space. Let us first obtain some bounds on the distance function d.

Lemma 3.4.1. *For any $P_1, P_2 \in M$, denote the coordinates of P_i as (r_i, θ_i, z_i) for $i = 1, 2$. Then*

$$d(P_1, P_2) \leq e^{z_1} + e^{z_2}. \tag{3.20}$$

Proof. For any $t_0 < \min(z_1, z_2)$. Define the following three smooth curves:

$l_1 : (r_1, \theta_1, t)$ for $t \in [t_0, z_1]$ oriented from z_1 to t_0,

$l_2 : (r_1 + (r_2 - r_1)t, \theta_1 + (\theta_2 - \theta_1)t, t_0)$ for $t \in [0, 1]$,

$l_3 : (r_2, \theta_2, t)$ for $t \in [t_0, z_2]$.

Then $l_1 \cup l_2 \cup l_3$ is a piecewise smooth curve conecting P_1 and P_2. We can calculate the lenth of these curves straightfowardly.

$$\begin{aligned}
\|l_1\| &= \int_{t_0}^{z_1} e^t dt = e^{z_1} - e^{t_0}, \\
\|l_3\| &= \int_{t_0}^{z_2} e^t dt = e^{z_2} - e^{t_0}, \\
\|l_2\| &= \int_0^1 e^{t_0}\sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2 f^2(r_1 + (r_2 - r_1)t)}dt \\
&\leq e^{t_0}\sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}.
\end{aligned}$$

Therefore,

$$d(P_1, P_2) \leq e^{z_1} + e^{z_2} - 2e^{t_0} + e^{t_0}\sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}.$$

Taking $t_0 \to -\infty$, the result follows. □

Define $H_I = \mathbb{R}^2 \times I = \{(r, \theta, z) : z \in I\}$ for any $I \subset \mathbb{R}$. And we will use *diam* S to denote the diameter of set $S \subset M$.

Corollary 3.4.1. *diam* $H_{(-\infty,0]} \leq 2$.

Proof. For any $P_1, P_2 \in H_{(-\infty,0]}$, we have $d(P_1, P_2) \leq e^{z_1} + e^{z_2} \leq 2$. □

Lemma 3.4.2. *For any* $P_1, P_2 \in M$,

$$d(P_1, P_2) \geq |e^{z_1} - e^{z_2}|. \tag{3.21}$$

Proof. For any piecewise smooth curve $l : [0,1] \to M$ from P_1 to P_2, we have

$$\begin{aligned}\|l\| &= \int_0^1 e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t))\dot{\theta}^2(t) + \dot{z}^2(t)}dt \\ &\geq \int_0^1 e^{z} |\dot{z}(t)| dt \\ &\geq |e^{z_1} - e^{z_2}|.\end{aligned}$$

□

Note that the metric space (M,d) is not complete. $\{(0,0,-n)\}_{n=1}^{\infty}$ is a Cauchy sequence since $d((0,0,-m),(0,0,-n)) \leq e^{-m} + e^{-n}$. But it is not convergent in M.

Theorem 3.4.1. *Let $\overline{M}_c$ be the Cauchy completion of M with respect to metric d. Then $\overline{M}_c = M \cup \{\infty\}$ where $\{\infty\}$ is defined as the Cauchy sequence $\{(0,0,-n)\}_{n=1}^{\infty}$.*

We want show that for any Cauchy sequence $\{P_n\}_{n=1}^{\infty}$, either it is convergent in M or it is equivalent to the Cauchy sequence $\{(0,0,-n)\}_{n=1}^{\infty}$. We split the proof into the following lemmas.

Lemma 3.4.3. *Let $\{P_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. Then either $\{z_n\}_{n=1}^{\infty}$ is convergent in $\mathbb{R}$ or $\lim_{n\to\infty} z_n = -\infty$.*

Proof. By inequality 3.21, we have $d(P_m, P_n) \geq |e^{z_m} - e^{z_n}|$. Therefore, $\{e^{z_n}\}$ is a Cauchy sequence in $\mathbb{R}$. So the result follows. □

Lemma 3.4.4. *Let $\{P_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. If $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, then $\{r_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let $z_0 = \lim z_n$. By dropping finitely many initial terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Let $\delta = \delta(z_0) = e^{z_0 - 1} - e^{z_0 - 2}$. Since $\{P_n\}$ is Cauchy, by dropping more initial terms, we can assume further that $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}^+$. By the definition of distance d, there exists a piecewise smooth curve $l_{mn} : [0, 1] \to M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. We claim

$$\min_{t\in[0,1]} z(t) > z_0 - 2. \tag{3.22}$$

Assume not. Take $t = t_0 \in [0, 1]$ to be the first time such that $z(t) = z_0 - 2$, which implies that $z(t) \geq z_0 - 2$ for $t \in [0, t_0]$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)}|\dot{z}(t)|dt \geq e^{z_m} - e^{z(t_0)} \geq e^{z_0-1} - e^{z_0-2} = \delta.$$

However, according to our assumption on l_{mn}, we have

$$\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2}, \tag{3.23}$$

which is a contradiction and therefore the claim follows. Thus we have

$$\frac{3}{2}d(P_m, P_n) \geq l_{mn} \geq \int_0^1 e^{z(t)}|\dot{r}(t)|dt \geq e^{z_0-2}|r_m - r_n|.$$

Therefore, $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. □

Lemma 3.4.5. *Let $\{P_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. If $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and $\lim r_n > 0$, then $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let $z_0 = \lim z_n$ and $r_0 = \lim r_n$. By dropping finitely many initial terms, we can assume that $z_n \in [z_0 - 1, z_0 + 1]$ and $r_n \in [\frac{1}{2}r_0, \frac{3}{2}r_0]$ for any $n \in \mathbb{Z}$. Define $\delta(z_0) = e^{z_0-1} - e^{z_0-2}$ and $\delta(r_0, z_0) = \frac{1}{4}r_0e^{z_0-2}$. And take $\delta = \min\{\delta(z_0), \delta(r_0, z_0)\}$. By dropping more initial terms, we can further assume $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}$. Again, we take a piecewise smooth curve $l_{mn} : [0, 1] \to M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. By the proof of Lemma

3.4.4, we have $\min z(t) \geq z_0 - 2$. Here we claim

$$r(t) \in [\frac{1}{4}r_0, \frac{7}{4}r_0] \text{ for any } t \in [0,1]. \tag{3.24}$$

Assume not. Let $t = t_0$ be the first time such that $r(t_0) = \frac{1}{4}r_0$ or $\frac{7}{4}r_0$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)}|\dot{r}(t)|dt \geq e^{z_0-2}|r(t_0) - r_m| \geq \frac{1}{4}r_0 e^{z_0-2} = \delta(r_0, z_0).$$

But we also have

$$\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2},$$

which is a contradiction. So the claim follows. Therefore,

$$\begin{aligned}\frac{3}{2}d(P_m, P_n) \geq \|l_{mn}\| &\geq \int_0^1 e^{z(t)}f(r(t))|\dot{\theta}(t)|dt \\ &\geq e^{z_0-2}\min\{f(\frac{1}{4}r_0), f(\frac{7}{4}r_0)\}|\theta_m - \theta_n|.\end{aligned}$$

It follows that $\{\theta_n\}$ is a Cauchy sequence. □

Lemma 3.4.6. *Let $P_n = (r_n, \theta_n, z_n)$ be a sequence in M. If $r_n \to r_0, \theta_n \to r_0, z_n \to z_0$ in $\mathbb{R}$, then P_n converges to $P_0 = (r_0, \theta_0, z_0)$ with respect to metric d.*

Proof. Since $z_n \to z_0$ in $\mathbb{R}$. By dropping finitely many initial terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Define a smooth curve from P_0 to P_n as

$$l(t) = (r_0 + (r_n - r_0)t, \theta_0 + (\theta_n - \theta_0)t, z_0 + (z_n - z_0)t). \tag{3.25}$$

Then

$$
\begin{aligned}
d(P_0, P_n) \leq \|l\| = &\int_0^1 e^{z(t)} \sqrt{(r_n - r_0)^2 + f^2(r_0 + (r_n - r_0)t)(\theta_n - \theta_0)^2 + (z_n - z_0)^2} dt \\
\leq & e^{z_0+1} \sqrt{(r_n - r_0)^2 + (\theta_n - \theta_0)^2 + (z_n - z_0)^2}.
\end{aligned}
$$

So the result follows. □

Now we are ready to prove Theorem 3.4.1.

Proof. Let $\{P_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. By Lemma 3.4.3, we have either $\lim z_n = -\infty$ or $\lim z_n = z_0$ for some $z_0 \in \mathbb{R}$. In the first case, we have

$$
d(P_n, (0, 0, -n)) \leq e^{z_n} + e^{-n} \to 0.
$$

Therefore Cauchy sequence $\{P_n\}$ and $\{(0, 0, -n)\}$ are equivalent to each other.

In the second case that $z_0 = \lim z_n \in \mathbb{R}$, we can assume $z_n \in [z_0 - 1, z_0 + 1]$ for any $n \in \mathbb{Z}^+$. By Lemma 3.4.4, we know that $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. Let $r_0 = \lim r_n$. We have two sub-cases, either $r_0 = 0$ or $r_0 > 0$. When $r_0 = 0$, define a smooth curve l from $(0, 0, z_0)$ to P_n as $l(t) = (r_n t, \theta_n t, z_0 + (z_n - z_0)t)$. Then

$$
\begin{aligned}
d((0, 0, z_0), P_n) \leq \|l\| = &\int_0^1 e^{z(t)} \sqrt{r_n^2 + f^2(r_n t)\theta_n^2 + (z_n - z_0)^2} dt \\
\leq & e^{z_0+1} \int_0^1 \sqrt{r_n^2 + 4\pi^2 f^2(r_n t) + (z_n - z_0)^2} dt \\
\to & 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, $P_n \to (0, 0, z_0)$ in M.

In the second sub-case that $r_0 > 0$, by Lemma 3.4.5, we have that $\lim \theta_n = \theta_0$ for some $\theta_0 \in \mathbb{R}$. Then by Lemma 3.4.6, we have that P_n converges to $P_0 = (r_0, \theta_0, z_0)$ in M. So the result follows. ∎

Theorem 3.4.2. *The capacity of $\partial_c M = \{\infty\} \subset \overline{M}_c$ is zero.*

Proof. Define a decreasing function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\varphi(z) = \begin{cases} 1 & z \leq 0 \\ 0 & z \geq 1. \end{cases}$$

For any $a \in \mathbb{R}$, define $\varphi_a \in \mathcal{C}^\infty(M)$ as $\varphi_a(P) = \varphi(z-a)$ for any $P = (r, \theta, z) \in M$. Then $\varphi = 1$ on $H_{(-\infty,a)} = B(\infty, e^a)$ and $\varphi = 0$ outside $H_{(-\infty,a+1)} = B(\infty, e^{a+1})$. Then

$$\begin{aligned}
\int_M \varphi_a^2 dV_g &\leq \int_{H_{(-\infty,a+1)}} dV_g \\
&= \int_{-\infty}^{a+1} \int_0^{2\pi} \int_0^\infty e^{3z} f(r) dr d\theta dz \\
&= 2\pi e^{3a+3} \int_0^\infty f(r) dr \\
&\to 0, \quad \text{as } a \to -\infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_M |\nabla \varphi_a|_g^2 dV_g &= \int_{H_{(a,a+1)}} |\varphi'(z-a)|^2 e^{-2z} dV_g \\
&= \int_a^{a+1} \int_0^{2\pi} \int_0^\infty |\varphi'(z-a)|^2 e^z f(r) dr d\theta dz \\
&\leq 2\pi (e^{a+1} - e^a) \sup_{\mathbb{R}} |\varphi'| \int_0^\infty f(r) dr \\
&\to 0, \quad \text{as } a \to -\infty.
\end{aligned}$$

Therefore the result follows. ∎

Proposition 3.4.1. *Let $o = (0,0,0)$. Then $\overline{B(o,2)} \setminus B(\infty, e^{-1})$ is not compact in M.*

Proof. By Corollary 3.4.1, we have

$$\overline{B(o,2)} - B(\infty, e^{-1}) \supset H_{(-\infty,0]} - H_{(-\infty,-1)} = H_{(-1,0]}.$$

Consider the sequence $P_n = (n, 0, 0)$ in $H_{(-1,0]}$. We claim

$$d(P_m, P_n) \geq \min(e^{-1}, 1 - e^{-1}) \quad \text{for any } m \neq n. \tag{3.26}$$

Let $l : [0,1] \to M$ be an arbitrary smooth curve from P_m to P_n. Then either $l \subset H_{(-1,+\infty)}$ or l will hit the plane $z = -1$. In the first case, we have

$$\|l\| \geq \int_0^1 e^{z(t)} |\dot{r}(t)| dt \geq e^{-1} |r_m - r_n| \geq e^{-1}$$

In the second case, take $t = t_0$ be the first time l hit the plane $z = -1$. Then

$$\|l\| \geq \int_0^{t_0} e^{z(t)} |\dot{z}(t)| dt \geq e^{z(0)} - e^{z(t_0)} = 1 - e^{-1}.$$

Combining these two cases, we have $\|l\| \geq \min(e^{-1}, 1 - e^{-1})$ for any piecewise smooth curve from P_m to P_n. So the claim follows. Therefore, there is no convergent subsequence of $\{P_n\}$ and thus the result follows. □

Chapter 4

Bergman Kernel

4.1 Introduction

4.1.1 Background

Let M be a compact complex manifold with $\dim_{\mathbb{C}}(M) = n$. Let L be a positive Hermitian holomorphic line bundle over M with Hermitian metric h. Suppose that the Kähler form ω is defined by $\mathrm{Ric}(h)$ i.e.,

$$\omega = -\tfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h). \tag{4.1}$$

Set $H^0(M, L)$ to denote the subspace of holomorphic global sections of L within $L^2(M, L)$, the space of all square integrable sections of L over M. The L^2-inner product for $f, g \in H^0(M, L)$ is given by the formula

$$\langle f, g\rangle_{L^2} := \int_M (f, g)_h \frac{\omega^n}{n!}, \tag{4.2}$$

where $\frac{\omega^n}{n!}$ is the volume form on M. The orthogonal projection $\mathcal{P}_{H^0} : L^2(M, L) \to H^0(M, L)$ is called the *Bergman projection*, and its kernel with respect to the scalar product mentioned above is K, the *Bergman kernel of* $H^0(X, L)$, a section of $L \otimes \bar{L}$. We naturally extend the metric h on L to $L^{\otimes k} := \overbrace{L \otimes \cdots \otimes L}^{(k \text{ times})}$, the Kähler form becomes $k\omega$ and consider the corresponding kernel. For the remainder of this chapter, both the metric, kernel and corresponding quantities will be denoted by h and K. Given some point $x \in M$, that sufficiently small coordinate neighborhoods U_x admit a local trivialization with corresponding local frame $\{e_L^k\}$. For the case $k = 1$, this trivialization may be chosen so that a real valued smooth plurisubharmonic function φ may represent h in the sense that

$$(e_L, e_L)_h = e^{-\varphi(x)}. \tag{4.3}$$

Set $\psi(x, x) := \varphi(x)$ be the polarization of φ. We will call this a *standard local frame* with the choice of Bochner coordinates, i.e. coordinates so that

$$\varphi(x) = |z|^2 + R(z),\ R = O(|z|^4). \tag{4.4}$$

Note that in the Kähler setting, these coordinates may always be chosen in some neighborhood of p. For general k, given a local trivialization, any local section f of $L^{\otimes k}$, can be written $f = \tilde{f} e_L^k \in H^0(U_x, L^{\otimes k})$, where $\tilde{f}$ is a local holomorphic function. After rescaling the variable $z \mapsto \frac{v}{\sqrt{k}}$, we can view the space of weighted local holomorphic L^2 sections with finite norm

$$L^2(U_x, L^{\otimes k}, k\varphi) := \left\{ f \in H^0(U_x, L) : \int_{U_x} \left| \tilde{f} \right|_h^2 e^{-k\varphi} \frac{\omega^n}{n!} < \infty \right\}. \tag{4.5}$$

as a *perturbed Bargmann-Fock space*. We denote the above space as $L^2(U_x, k\varphi)$ for convenience. Given local holomorphic coordinates $\{z_i\}_{i=1}^n$, the volume form is given by

$$dV := \left(\frac{\sqrt{-1}}{2\pi}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n, \tag{4.6}$$

while the Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} dz^i \wedge d\overline{z}^j = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\overline{z}^j. \tag{4.7}$$

The components of the curvature tensor are given by

$$\mathrm{Rm}_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \overline{z}^j} + g^{s\bar{t}} \frac{\partial g_{i\bar{t}}}{\partial z^k} \frac{\partial g_{s\bar{j}}}{\partial \overline{z}^l}. \tag{4.8}$$

Furthermore set

$$\Omega(z) := k^n \det\left(\partial\overline{\partial}\varphi(z)\right), \tag{4.9}$$

where the determinant is taken over each matrix coefficient of the matrix valued two-form $\partial\overline{\partial}\varphi(z)$ and consequently

$$\frac{\omega^n}{n!} := \Omega dV. \tag{4.10}$$

4.1.2 Bergman kernel

Methods to compute and analyze the coefficients of the Bergman Kernel have been worked out over the last 25 years. Initially Tian gave leading asymptotics on the diagonal [22]. Extending the result of Fefferman [6], a complete expansion was given by Zelditch [24] by regarding it as a Szegö kernel via lifts to the unit circle bundle, and independently by

Catlin using the Boutet de Monvel-Sjöstrand parametrix [4]. In particular, their off-diagonal asymptotic expansions, as $k \to \infty$, are given of the form, with b_l certain Hermitian functions,

$$K(x,y)e^{-k\psi(x,y)} = \frac{k^n}{\pi^n}\left(1 + \sum_{l=1}^{\infty} \frac{b_l(x,y)}{k^l}\right). \tag{4.11}$$

Lu demonstrates that the functions $b_l(x,y)|_{x=y}$ encode geometric information about the underlying manifold M [11].

The purpose of this chapter is to provide an alternative proof of the near diagonal asymptotic expansion of the Bergman Kernel as shown in the following theorem.

Theorem 4.1.1. *Let* $(L,h) \to (M,\omega)$ *be a positive line bundle over a compact Kähler manifold with* $\dim_{\mathbb{C}}(M) = n$*. Consider* $H^0(M, L^{\otimes k})$ *the space of holomorphic sections of* k *tensor powers of* L*. Then the scaled off-diagonal Bergman kernel admits an asymptotic expansion. In Bochner coordinates, the expansion takes the form*

$$K\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) = e^{u\cdot\bar{v}}\left(\sum_{j=0}^{\infty} \frac{c_j(u,\bar{v})}{\sqrt{k^j}}\right). \tag{4.12}$$

The coefficients of $c_j(u,\bar{v}) = \sum_{p,q} c_j^{p,q} u^p \bar{v}^q$ *further satisfy*

$$\begin{cases} c_j^{p,q} = 0 & \text{for } p+q > 2j, \\ c_j^{p,q} = 0 & \text{for } p+q \not\equiv_2 j. \end{cases} \tag{4.13}$$

4.1.3 Sketch of the proof

The proof of the Theorem 4.1.1 is subdivided into two components: construction and analysis of a local reproducing kernel (§4.2, §4.3, §4.4), and demonstration of this construction coinciding with the global reproducing kernel (§4.5).

Local construction and analysis

Initially in §4.2 we focus on proving the following proposition.

Proposition 4.1.1. *Let $k \in \mathbb{N}$ and $f \in H^0(B)$, the set of holomorphic functions on the unit ball centered at the origin, and fix $\chi \in C_c^\infty(\mathbb{C}^n)$ such that*

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{4.14}$$

Set $\chi_k(x) := \chi(k^{\frac{1}{4}}x)$. Then there exists polynomials $c_j(u, \bar{v})$ such that it satisfies the scaled local reproducing property:

$$\begin{aligned} f\left(\frac{u}{\sqrt{k}}\right) = &\left\langle \chi_k\left(\frac{\cdot}{\sqrt{k}}\right) f\left(\frac{\cdot}{\sqrt{k}}\right), e^{\bar{u}\cdot(\cdot)}\left(\sum_j^N \frac{c_j\left(\cdot, \bar{u}\right)}{\sqrt{k^j}}\right)\right\rangle_{L^2\left(B(\sqrt{k}), k\varphi\left(\frac{\cdot}{\sqrt{k}}\right)\right)} \\ &+ O\left(\frac{1}{\sqrt{k}^{N+1-2n}}\right)\|f\|_{L^2(B,k\varphi)}. \end{aligned} \tag{4.15}$$

The strategy is as follows. Given an arbitrary point $p \in M$ we choose an appropriate standard chart. Our choice for the scaling of the cut-off function is to localize the inner product to the "near-diagonal" and to ensure that the expansion of the potential has the asymptotics

$$k\varphi(\tfrac{v}{\sqrt{k}}) \sim |v|^2 + O(1) \tag{4.16}$$

In these coordinates one may model this setting as a perturbed Bargmann-Fock space $L^2(\mathbb{C}^n, k\varphi)$.

Given the c_j, Hermitian functions for all $j \in \mathbb{N}$, we expand them in the form

$$c_j(u, \bar{v}) = \sum_{p,q} c_j^{p,q} u^p \bar{v}^q, \tag{4.17}$$

So that we have the *local reproducing kernel* given by

$$K^{loc}(u, \bar{v}) := e^{u\bar{v}} \sum_{j,p,q} \frac{c_j^{p,q}(u, \bar{v})}{\sqrt{k^j}} = e^{u\bar{v}} \sum_j c_j \left(\tfrac{u}{\sqrt{k}}, \tfrac{\bar{v}}{\sqrt{k}} \right). \tag{4.18}$$

Additionally we group together the quantity $e^{-kR}\Omega$ and expand this as a bivariate power series.

We utilize the expected reproducing property on monomials over $\mathbb{C}^n$ to solve directly for the coefficients $c_j^{p,q}$. This allows us to first conclude that, (provided existence) $c_0^{p,q}$ vanishes. (Lemma 4.2.1). With this we demonstrate that in fact all coefficients c_j which reproduce polynomials truly exist (Lemma 4.2.2) via a formal iterative construction.

To demonstrate that these c_j are truly polynomials, we introduce the notion of the *parity property* (definition 4.2.3), an algebraic characteristic of bivariate power series which is admitted by and yields the vanishing of higher order powers of c_j. Specifically, $\deg c_j \leq 2j$ (Theorem 4.2.1). The theorem relies on two key combinatorial identities and a strategy of embedding quantities as coefficients of a polynomial to demonstrate the desired identity.

Provided the existence, in §4.3 we focus on demonstrating that up to small error, the constructed local kernel K^{loc} reproduces any holomorphic function f. To do so, portions of the integrand are expanded as series and truncated to evaluate remainders.

We then provide an explicit computation in Bochner coordinates of the coefficient c_2 (the coefficients c_0 and c_1 are computed in Lemmas 4.2.5 and 4.4.1 respectively), yielding that

$$\begin{cases} c_0 &= 1, \\ c_1 &= 0, \\ c_2 &= \frac{\varrho}{2} - \frac{1}{4}\operatorname{Rm}_{i\bar{j}k\bar{l}}(0)u^i u^k \overline{v}^j \overline{v}^l. \end{cases} \tag{4.19}$$

This iterative computation captures the essential strategy one may utilize to compute (if they feel so inclined) any desired c_j.

Global construction

The final focus of our work in §4.5 lies in demonstrating that our construction coincides with the globally reproducing kernel. The strategy is essentially the same as [2] and is as follows. We first compute a uniform upper bound for the Bergman function (Lemma 4.5.1)

$$B(x) = \sup_{||s||_{L^2} \leq 1} |s(x)|_h^2. \tag{4.20}$$

This estimate is key in the comparison of the constructed local reproducing kernel K^{loc} and global Bergman kernel K. We apply the local reproducing property to the global Bergman kernel, then consider the difference between the local kernel with the Bergman projection of the local kernel. Since the difference is the orthogonal to the holomorphic functions, it is in fact the L^2 minimal solution to a certain $\overline{\partial}$-equation. By using standard $\overline{\partial}$-estimates, we show the difference is small. Hence the difference between the global and local kernel is of sufficiently small enough order.

4.2 Local construction

The first step of our analysis is to establish the existence of the reproducing kernel of $\mathbb{C}^n$. The strategy is to first establish a formal construction during which we demonstrate the existence of the Hermitian coefficients c_j by considering their formal local expansion (4.17), where p, q are multiindices and $c_j^{p,q} \in \mathbb{C}$. Similarly we discuss the existence of the coefficients $a_j^{p,q}$ generated from the formal expansion of the exponential part of the Bergmen kernel with the Jacobian of the volume form, given by $e^{-kR}\Omega$. The convergence of the coefficients $a_j^{p,q}$ and $c_j^{p,q}$ is not given in this section and will be verified in §4.3 (cf. Lemma 4.3.1).

4.2.1 Notation and Conventions

We set the following conventions which will be used extensively throughout the paper.

Let $\mathbb{Z}_+$ denote the collection of all nonnegative integers. Let $\ell \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{Z}_+^\ell$ such that $\alpha := (\alpha_1, ..., \alpha_\ell)$ and $\beta := (\beta_1, ..., \beta_\ell)$. Then define a *multiindex factorial* to be given by

$$\alpha! := \prod_{i=1}^{\ell} \alpha_i!.$$

Additionally, set the *multiindex length* to be given by

$$|\alpha| := \sum_{i=1}^{\ell} \alpha_i,$$

and a *multiindex binomial coefficient* is defined by the following

$$\binom{\alpha}{\beta} := \prod_{i=1}^{\ell} \binom{\alpha_i}{\beta_i}. \tag{4.21}$$

Note we establish the convention that if $q > p$ then $\binom{p}{q} := 0$. Lastly, *multiindex inequalities* will be defined as follows. We have

$$\alpha \leq \beta \Longleftrightarrow (\alpha_i \leq \beta_i \text{ for all } i \in [1, \ell] \cap \mathbb{Z}_+), \tag{4.22}$$

and furthermore

$$\alpha < \eta \Longleftrightarrow (\alpha \leq \beta \text{ and } |\eta - r| > 0). \tag{4.23}$$

Given $x \in M$ and $r > 0$ we set

$$B_x(r) := \{y \in M : \operatorname{dist}(y, x) < r\}. \tag{4.24}$$

In the setting of $M = \mathbb{C}^n$ we set $B(r) := B_0(r)$ and $B := B(1)$. We set

$$B_x^c(r) := \{y \in M : \operatorname{dist}(y, x) \geq r\}. \tag{4.25}$$

For any summations if the ranges are not specified one will assume summation indices range over multiindex values $\mathbb{Z}_+^\ell$ with ℓ determined by the free variable.

4.2.2 Formal construction of coefficients

We denote the power series expansion for the product

$$e^{-kR\left(\frac{v}{\sqrt{k}}\right)}\Omega\left(\frac{v}{\sqrt{k}}\right) = \sum_m \sum_{p,q} \frac{a_m^{p,q} v^p \overline{v}^q}{\sqrt{k^m}}. \tag{4.26}$$

We will formally construct coefficients c_j such that for any polynomial F,

$$F\left(\frac{u}{\sqrt{k}}\right) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{C}^n} F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \sum_{j,p,q} \frac{c_j^{p,q}}{\sqrt{k^j}} u^p \bar{v}^q\right) \left(\sum_{m\le d} \sum_{p,q} \frac{a_m^{p,q} v^p \overline{v}^q}{\sqrt{k^m}}\right) dV. \tag{4.27}$$

It is sufficient to consider arbitrary degree l monomials in u, and due to the homogeneity this simplifies (4.27) to

$$\begin{aligned} u^l &= \int_{\mathbb{C}^n} v^l e^{u\cdot\bar{v}-|v|^2} \sum_{t=0}^{N} \sum_{j+m=t} \left(\sum_{p,q} \sum_{r,s} \frac{c_j^{p,q} a_m^{r,s}}{\sqrt{k^t}} u^p \bar{v}^q v^r \overline{v}^s\right) dV \\ &= \sum_{t=0}^{N} \sum_{m+j=t} \sum_{q+s\le l+r} \sum_{p} \frac{c_j^{p,q} a_m^{r,s}}{\sqrt{k^t}} u^{p+l+r-q-s} \frac{(l+r)!}{(l+r-q-s)!}. \end{aligned} \tag{4.28}$$

We can immediately determine the c_0 coefficients, as seen in the following lemma.

Lemma 4.2.1. *For multiindices $p, q \in \mathbb{Z}_+^n$ the following property holds*

$$c_0^{p,q} = \begin{cases} 1 & \text{if } p_i = q_i = 0 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

Proof. The proof proceeds by induction on the length of the multiindex q. First, we consider $|q| = 0$. Then taking (4.28) for $l = (0, \ldots, 0)$ and comparing the coefficient of k coefficient yields

$$1 = \sum_{\substack{p \\ q+s\le r}} c_0^{p,q} a_0^{r,s} u^{p+r-q-s} \frac{r!}{(r-q-s)!}. \tag{4.30}$$

By the vanishing of $a_0^{r,s}$ from Proposition 4.4.1, we have

$$1 = \sum_{p} c_0^{p,0} u^p. \tag{4.31}$$

Immediately we compare the coefficients of u and conclude result (4.29) for this case.

Now we assume the induction hypothesis holds for $|q| \leq \lambda - 1$ and consider the case $|q| = \lambda$, and take $|l| = \lambda$. Then applying the induction hypothesis to (4.28) and parsing apart the right hand summation yields

$$\begin{aligned} u^l &= \sum_p \sum_{q \leq l} u^p c_0^{p,q} \frac{l!}{(l-q)!} u^{l-q} \\ &= \sum_p \sum_{\substack{|q|=\lambda \\ q \leq l}} u^p c_0^{p,q} \frac{l!}{(l-q)!} u^{l-q} + \sum_p \sum_{\substack{q \leq l \\ |q| \leq \lambda - 1}} u^p c_0^{p,q} \frac{l!}{(l-q)!} u^{l-q}. \end{aligned}$$

Note in particular that the requirements in the first right side term that $q \leq l$ and $|q| = \lambda$ immediately imply that $q = l$. Additionally, by the induction hypothesis, the second right side term reduces to simply u^l. Subtracting this from both sides yields

$$0 = \sum_p u^p c_0^{p,l}. \tag{4.32}$$

The coefficients vanish accordingly and we conclude (4.29). The desired result follows. □

Lemma 4.2.2 (Existence of coefficients)**.** *There exist coefficients $c_j^{p,q} \in \mathbb{C}$ depending only on the Kähler potential φ such that for any polynomial F,*

$$F\left(\frac{u}{\sqrt{k}}\right) = \int_{\mathbb{C}^n} F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v} - |v|^2} \left(\sum_{t=0}^{N} \sum_{m+j=t} \sum_{p,q} \sum_{r,s} \frac{c_j^{p,q} a_m^{r,s}}{\sqrt{k^t}} u^p \bar{v}^q v^r \overline{v}^s \right) dV. \tag{4.33}$$

for any $N \geq 0$.

Remark 4.2.1. When $N = 0$ the equation reduces to the reproducing property of the Bargmann-Fock kernel.

Proof. By (4.28), from comparing the coefficients of k, we get

$$\sum_{m+j=t}\sum_{s+q\leq l+r}\sum_{p} c_j^{p,q} a_m^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!} = \begin{cases} u^l & t=0, \\ 0 & t\geq 1. \end{cases} \tag{4.34}$$

In order to determine the coefficients $c_j^{p,q}$, we induct on j. The base case $j=0$ is demonstrated by Lemma 4.2.1. Now assume the induction hypothesis is satisfied for $j \leq \tau - 1$, which implies that the coefficients $c_j^{p,q}$ have been determined for all multiindices p and q and for all such values of j. Take (4.34) for $t = \tau$,

$$\sum_{j=0}^{\tau}\sum_{s+q\leq l+r}\sum_{p} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!} = 0. \tag{4.35}$$

By moving all the terms with $j \leq \tau - 1$ to the other side, we have

$$\begin{aligned} &\sum_{s+q\leq l+r}\sum_{p} c_\tau^{p,q} a_0^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!} \\ &= -\sum_{j=0}^{\tau-1}\sum_{s+q\leq l+r}\sum_{p} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!}. \end{aligned} \tag{4.36}$$

Since we know $a_0^{r,s}$ all vanish except that $a_0^{0,0} = 1$ (c.f. Remark 4.4.1),

$$\sum_{q\leq l}\sum_{p} c_\tau^{p,q} u^{p+l-q}\frac{l!}{(l-q)!} = -\sum_{j=0}^{\tau-1}\sum_{s+q\leq l+r}\sum_{p} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!}. \tag{4.37}$$

We consider various lengths of l to determine the values of the coefficients. When $|l| = 0$, (4.37) reduces to

$$\sum_{p} c_\tau^{p,0} u^p = -\sum_{j=0}^{\tau-1}\sum_{s+q\leq r}\sum_{p} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r-q-s}\frac{r!}{(r-q-s)!}. \tag{4.38}$$

Since, the values of $a_m^{r,s}$ are determined (c.f. Lemma 4.4.1) and $c_j^{p,q}$ are known due to the inductive hypothesis for $j \leq \tau - 1$, we obtain all $c_\tau^{p,0}$ by comparing the coefficients of u in

(4.38). We begin a subinduction argument on the values of $|q|$ such that $c_\tau^{p,q}$ is known for any $|q| \leq \lambda - 1$. The case $\lambda = 0$ is determined via our analysis of (4.38) discussed above. Consider multiindices l such that $|l| = \lambda$ within the (4.39). As in the Lemma 4.2.1, we decompose the left hand summation of (4.37) and rearrange the equality to obtain

$$\begin{aligned}\sum_p c_\tau^{p,l} u^p = &-\sum_{j=0}^{\tau-1} \sum_{s+q \leq l+r} \sum_p c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s} \frac{(l+r)!}{(l+r-q-s)!} \\ &-\sum_p \sum_{\substack{q \leq l \\ |q| \leq \lambda-1}} c_\tau^{p,q} u^{p+l-q} \frac{l!}{(l-q)!}.\end{aligned} \tag{4.39}$$

Due to the induction hypothesis on τ, the first quantity of the right hand side is completely determined. Furthermore, by comparing the coefficients on u^p we can solve for $c_\tau^{p,l}$. This concludes the induction on $|q|$ which implies that $c_\tau^{p,q}$ are completely determined, and thus the induction on τ is also completed. The result follows. □

4.2.3 Parity of the coefficients

We establish the following property of bivariate power series.

Definition 4.2.1 (Parity property of power series)**.** We say the coefficients of the bivariate power series

$$B(x, y) := \sum_{m,p,q} \frac{B_m^{p,q}}{\sqrt{k^m}} x^p y^q, \tag{4.40}$$

has the *parity property* if given $p, q \in \mathbb{Z}_+^n$ with $|p| + |q| \not\equiv_2 m$, then $B_m^{p,q} = 0$.

By demonstrating that this property is preserved under standard algebraic manipulations, we will conclude the vanishing of particular coefficients $c_j^{p,q}$ and $a_j^{p,q}$ of (4.17) and (4.26) respectively. This is key in demonstrating the finiteness and bounds on the degrees of each c_j.

Lemma 4.2.3. *If A and B are bivariate power series with the parity property, then so are $A+B$ and AB.*

Proof. The additive closure is immediate. For the multiplication, we have

$$A(x,y)B(x,y) = \sum_{m=0}^{\infty} \frac{1}{\sqrt{k^m}} \sum_{n=0}^{m} \sum_{p,q,r,s} A_n^{p,q} B_{m-n}^{r,s} x^{p+r} \overline{y}^{q+s}.$$

The term is nonzero when $|p|+|q| \equiv_2 n$ and $|r|+|s| \equiv_2 m-n$, that is, when

$$|p|+|r|+|q|+|s| \equiv_2 m. \tag{4.41}$$

The result follows. $\square$

Lemma 4.2.4. *The bivariate expansion of $e^{-kR}\Omega$ has the parity property.*

Proof. Consider the expansion

$$\begin{aligned} e^{-kR\left(\frac{v}{\sqrt{k}}\right)} &= \sum_n \left((-1)^n (\sqrt{k})^2 \sum \frac{1}{\sqrt{k^m}} R_m^{p,q} v^p \overline{v}^q \right)^n \\ &= (-1)^n (\sqrt{k})^{2n} \sum_n \left(\sum \frac{1}{\sqrt{k^m}} R_m^{p,q} v^p \overline{v}^q \right)^n . \end{aligned}$$

The factors of $k^{-1/2}$ come from evaluating the expansion of the exponential at $(\frac{v}{\sqrt{k}})$, the $R_m^{p,q}$ terms have the parity property. Since the parity property is closed under addition and multiplication by Lemma 4.2.3, and multiplication by k^{2n} preserves the parity property, the entire power series admits the property. Furthermore since $\Omega(\frac{v}{\sqrt{k}})$ also has the parity property, the product also has the parity property. The result follows. $\square$

Lemma 4.2.5. *For all $m \in \mathbb{N}$, given $p,q \in \mathbb{Z}_+^n$ such that $|p|+|q| \not\equiv_2 m$, we have $c_m^{p,q} = 0$.*

Proof. The proof proceeds by induction on m satisfying the Lemma statement. First, the case when $m=0$ is an immediate consequence of (4.29). Next assume that the parity property holds for $m \leq t-1$ and we do a sub induction on $|q|$. From (4.38) we obtain

$$-\sum c_t^{\alpha,0} u^\alpha = \sum_{j=0}^{t-1} \sum_{r \geq q+s} c_j^{p,q} a_{t-j}^{r,s} u^{p+r-q-s} \frac{r!}{(r-q-s)!}. \tag{4.42}$$

On the right hand side, the coefficients are nonzero only when $|p|+|q| \equiv_2 j$ and $|r|+|s| \equiv_2 t-j$. Combining these two equalities yields

$$|p|+|q|+|r|+|s| \equiv_2 t.$$

which implies that $c_t^{\alpha,0} = 0$ for $\alpha \not\equiv_2 t$. Now we assume the induction hypothesis holds for $|q| \leq \lambda - 1$. For $|q| = \lambda$, we have from (4.39)

$$\begin{aligned} \sum_p c_t^{p,l} u^p = &-\sum_{j=0}^{t-1} \sum_{s+q \leq l+r} \sum_p c_j^{p,q} a_{t-j}^{r,s} u^{p+r+l-q-s} \frac{(l+r)!}{(l+r-q-s)!} \\ &-\sum_p \sum_{\substack{q \leq l \\ |q| \leq \lambda-1}} c_t^{p,q} u^{p+l-q} \frac{l!}{(l-q)!}. \end{aligned} \tag{4.43}$$

On the right hand side, in the first summation, when $|p|+|q|+|r|+|s| \not\equiv_2 t$, the terms are zero by the induction on j, hence the exponent must be

$$|p|+|r|+|q|+|s|+|l| \equiv_2 t+|l|.$$

In the second summation, when $|p|+|q| \not\equiv_2 t$, the terms are zero by the induction on $|q|$, hence the exponent must be

$$|p|+|q|+|l| \equiv_2 t+|l|.$$

Then compare the exponent on both sides, we get on the left hand side,

$$|p| \equiv_2 t+|l|.$$

Hence

$$|p| + |l| \equiv_2 t.$$

The subinduction on $|q|$ has been proven, therefore we may determine all $c_t^{p,q}$ for the given t. Consequently the induction on t is complete, and the result follows. ☐

We next establish two combinatorial identities in preparation for the proof of Theorem 4.2.1.

Lemma 4.2.6. *Given $l \in \mathbb{Z}_+$, for all $s \in [1, l] \cap \mathbb{Z}_+$ the following equality holds*

$$\sum_{w=0}^{s} (-1)^s \binom{l}{w} \binom{l-w}{l-s} = 0. \tag{4.44}$$

Proof. We assign the terms of (4.44) as polynomial coefficients and rearrange terms appropriately,

$$\begin{aligned}
\sum_{w=0}^{l} \sum_{s=0}^{w} (-1)^s \binom{l}{w} \binom{l-w}{l-s} x^{l-s} &= \sum_{w=0}^{l} \left(\sum_{s=w}^{l} \binom{l-w}{l-s} x^{l-s} \right) \binom{l}{w} (-1)^s \\
&= \sum_{w=0}^{l} \left((x+1)^{l-w} \right) \binom{l}{w} (-1)^w \\
&= \sum_{w=0}^{l} \left((x+1)^{l-w} \binom{l}{w} (-1)^w \right) \\
&= x^l.
\end{aligned}$$

The result follows. ☐

Corollary 4.2.1. *Given a multiindex $l \in \mathbb{Z}_+^n$, for all nonzero multiindices $s \leq l$ the following equality holds*

$$\sum_{w \leq s} (-1)^{|w|} \binom{l}{w} \binom{l-w}{l-s} = 0. \tag{4.45}$$

Proof. Taking the left hand side of (4.2.1) and decomposing it as a product of binomial coefficients we obtain

$$\sum_{w\le s}(-1)^{|w|}\binom{l}{w}\binom{l-w}{l-s} = \prod_{i=1}^{n}\left(\sum_{w_i\le s_i}(-1)^{w_i}\binom{l_i}{w_i}\binom{l_i-w_i}{l_i-s_i}\right).$$

We observe that since s is nonzero then there is at least one $i \in [1,n] \in \mathbb{Z}_+$ such that s_i is nonzero. Applying Lemma 4.2.6 to this index within the product yields the desired result. □

Lemma 4.2.7. *For all $\eta \in [1,l] \cap \mathbb{Z}_+$, and $r \in [0,\eta-1] \cap \mathbb{Z}_+$,*

$$\sum_{w=0}^{\eta}(-1)^{w}\binom{l}{w}\binom{r+l-w}{\eta-w} = 0. \tag{4.46}$$

Proof. To verify the lemma we apply the following combinatorial identity. For $0 \le w \le 2$, we have

$$\binom{r+l-w}{\eta-w} = \left(\binom{l-w}{\eta-w}\binom{r}{0} + \binom{l-w}{\eta-w-1}\binom{r}{1} + \cdots + \binom{\eta+1-w}{0}\binom{r}{\eta-w}\right). \tag{4.47}$$

We expand left hand side of (4.49) by the above identity and obtain

$$\sum_{w=0}^{\eta}(-1)^{w}\binom{l}{w}\sum_{v=0}^{\eta-w}\binom{l-w}{\eta-w-v}\binom{r}{v} = \sum_{v=0}^{\eta}\underbrace{\left(\sum_{w=0}^{\eta-v}(-1)^{w}\binom{l}{w}\binom{l-w}{\eta-w-v}\right)}_{L}\binom{r}{v}.$$

It suffices to prove that the labeled quantity $L = 0$, that is

$$\sum_{w=0}^{\eta-v}(-1)^{w}\binom{l}{w}\binom{l-w}{\eta-j-v} = 0. \tag{4.48}$$

This is equivalent to demonstrating that for any $\eta \in \mathbb{Z}_+$ such that $1 \le \eta \le l+1$,

$$r_\eta = \sum_{w=0}^{\eta}(-1)^{w}\binom{l}{w}\binom{l-w}{\eta-w} = 0. \tag{4.49}$$

We again embed (4.49) as coefficients of a polynomial in x and with careful manipulation obtain,

$$\begin{aligned}\sum_{\eta=0}^{l} r_\eta x^\eta &= \sum_{\eta=0}^{l}\sum_{w=0}^{\eta}(-1)^w \binom{l}{w}\binom{l-w}{\eta-w}x^\eta \\ &= \sum_{w=0}^{l}\left(\sum_{\eta=w}^{l}\binom{l-w}{\eta-w}x^{\eta-w}\right)\binom{l}{w}(-x)^w \\ &= \sum_{w=0}^{l}(1+x)^{l-w}\binom{l}{w}(-x)^w \\ &= 1.\end{aligned}$$

The result follows. □

Corollary 4.2.2. *For all multiindices $\eta, r \in \mathbb{Z}_+^n$ with $r < \eta$,*

$$\sum_{w\leq\eta}(-1)^{|w|+1}\binom{l}{w}\binom{r+l-w}{\eta-w} = 0. \tag{4.50}$$

Proof. Taking the left hand side of (4.2.2) and decomposing it as a product of binomial coefficients we obtain

$$\sum_{w\leq\eta}(-1)^{|w|+1}\binom{l}{w}\binom{r+l-w}{\eta-w} = -\prod_{i=1}^{n}\left(\sum_{w_i\leq\eta_i}(-1)^{w_i}\binom{l_i}{w_i}\binom{r_i+l_i-w_i}{\eta_i-w_i}\right).$$

We observe that since s is nonzero then there is at least one $i \in [1,n] \in \mathbb{Z}_+$ such that $r_i < \eta_i$. Applying Lemma 4.2.7 to this index within the product yields the desired result. □

Theorem 4.2.1. *Given $m \in \mathbb{Z}_+$ and multiindices $p, q \in \mathbb{Z}_+^n$ such that $|p+q| > 2m$, then $c_m^{p,q} = 0$.*

Proof. Recall the identity (4.37) established in Lemma 4.2.2, we have

$$\sum_{q\le l}\sum_{p} c_\tau^{p,q} u^{p+l-q}\frac{l!}{(l-q)!} = -\sum_{j=0}^{\tau-1}\sum_{s+q\le l+r}\sum_{p} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s}\frac{(l+r)!}{(l+r-q-s)!}. \quad (4.51)$$

With the theorem statement as the induction hypothesis, for each fixed $\tau \in \mathbb{N}$, we induct on appropriate values of m. For $m = 0$ this is true by (4.29). Next assume the hypothesis holds when $m \le \tau - 1$. Then we consider the case $m = \tau$. Define the quantities (given by the left and right hand sides of (4.51) respectively),

$$\mathcal{P}_l := \sum_{q\le l}\sum_{p} c_\tau^{p,q} u^{p+l-q}\frac{l!}{(l-q)!}, \quad (4.52)$$

and

$$\mathcal{Q}_l := -\sum_{r+l\ge q+s}\sum_{j=0}^{\tau-1} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s}\frac{(r+l)!}{(r+l-q-s)!}. \quad (4.53)$$

Then we have that by (4.51) that $\mathcal{P}_l = \mathcal{Q}_l$ for all multiindices $l \in \mathbb{Z}_+^n$. We prove the identity $c_m^{p,q} = 0$ by embedding the two families of coefficients $\{\mathcal{P}_w\}_{w\le l}$ and $\{\mathcal{Q}_w\}_{w\le l}$ into a polynomial. Set, for $B \in \{\mathcal{P}, \mathcal{Q}\}$,

$$\Psi_l(B) := \sum_{w\le l} (-1)^{|w|}\binom{l}{w} u^w B_{l-w}. \quad (4.54)$$

First we compute $\Psi_l(\mathcal{P})$ by inserting (4.53) into (4.54), carefully rearranging terms with respect to powers of u:

$$\begin{aligned}
\Psi_l(\mathcal{P}) &= \sum_{w\leq l} (-1)^{|w|} \binom{l}{w} u^w \mathcal{P}_{l-w} \\
&= \sum_{w\leq l} (-1)^{|w|} \sum_{p} \sum_{q\leq l-w} \binom{l}{w} c_\tau^{p,q} u^{l+p-q} \frac{(l-w)!}{(l-w-q)!} \\
&= \sum_{p} \sum_{w\leq l} (-1)^{|w|} \left(\sum_{w\leq s\leq l} c_\tau^{p,(l-s)} u^{p+s} \binom{l}{w} \frac{(l-w)!}{(s-w)!} \right) \\
&= \sum_{p} \sum_{s\leq l} c_\tau^{p,(l-s)} u^{p+s} \left(\sum_{w\leq s} (-1)^{|w|} \binom{l}{w} \frac{(l-w)!}{(s-w)!} \right).
\end{aligned}$$

Note the substitution used to obtain the second to last line is through the identification $s = l - q$, and then interchanging the order of summation yields the final line. We decompose the above summation into two pieces to conclude that

$$\Psi_l(\mathcal{P}) = \sum_{p} c_\tau^{p,l} u^p l! + \sum_{p} \sum_{0\neq s\leq l} c_\tau^{p,(l-s)} u^{p+s} \left(\sum_{w\leq s} (-1)^{|w|} \binom{l}{w} \frac{(l-w)!}{(s-w)!} \right). \tag{4.55}$$

Applying Corollary 4.2.1, we have that the quantity above reduces to simply

$$\Psi_l(\mathcal{P}) = \sum_{p} c_\tau^{p,l} u^p l!. \tag{4.56}$$

We next compute $\Psi_l(\mathcal{Q})$. We collect up terms with respect to the powers of u.

$$
\begin{aligned}
\Psi_l(\mathcal{Q}) &= \sum_{w\leq l} (-1)^{|w|}\binom{l}{w} u^w \mathcal{Q}_{l-w} \\
&= -\sum_{w\leq l} (-1)^{|w|}\binom{l}{w} u^w \sum_{r+l\geq q+s+w} \sum_{j=0}^{\tau-1} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-w-q-s} \frac{(r+l-w)!}{(r+l-w-q-s)!} \\
&= \sum_{j=0}^{\tau-1} \sum_{w\leq l} \sum_{r+l\geq q+s+w} (-1)^{|w|+1}\binom{l}{w} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s} \frac{(r+l-w)!}{(r+l-w-q-s)!} \\
&= \sum_{j=0}^{\tau-1} \sum_{q+s=r}^{r+l} \sum_{w\leq r+l-q-s} (-1)^{|w|+1}\binom{l}{w} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+r+l-q-s} \frac{(r+l-w)!}{(r+l-w-q-s)!}.
\end{aligned}
\tag{4.57}
$$

For simplicity, set $\eta := r+l-q-s$ and allow it to range $0 \leq \eta \leq l$. Updating the index of (4.57) we obtain

$$
\begin{aligned}
\Psi_l(\mathcal{Q}) &= \sum_{j=0}^{\tau-1} \sum_{q+s=r}^{r+l} \sum_{w\leq \eta} (-1)^{|w|+1}\binom{l}{w} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+\eta} \frac{(r+l-w)!}{(\eta-w)!} \\
&= \sum_{j=0}^{\tau-1} \sum_{q+s=r}^{r+l} c_j^{p,q} a_{\tau-j}^{r,s} u^{p+\eta} \sum_{w\leq \eta} (-1)^{|w|+1}\binom{l}{w} \frac{(r+l-w)!}{(\eta-w)!}.
\end{aligned}
\tag{4.58}
$$

As a result of Corollary 4.2.2 we then only need to consider $r \geq \eta$, otherwise the term vanishes. As a result of the induction hypothesis combined with the fact that $a_j^{r,s} = 0$ when $r+s > 2j$ (cf. Remark 4.4.1) we apply these facts to $\Psi_l(\mathcal{Q})$,

$$2\tau \geq |p+q+r+s| = |p+2r+l-\eta|.$$

Manipulating the above expression yields

$$|p+\eta| \leq 2\tau - 2|r| - |l| + 2|\eta| \leq 2\tau - 2|(r-\eta)| - |l| \leq 2\tau - |l|.$$

Combining this fact with (4.58) we conclude

$$\deg \Psi_l(\mathcal{Q}) \leq 2\tau - |l|.$$

Recall that since $\mathcal{P}_l = \mathcal{Q}_l$ for all l by (4.51), so that $\Psi_l(\mathcal{P}) = \Psi_l(\mathcal{Q})$. Noting that as a result of (4.56), we have that

$$|p| \leq \deg \Psi_l(\mathcal{Q}) \leq 2\tau - |l|,$$

therefore if $2\tau < |p| + |l|$, then $c_\tau^{p,l} = 0$, demonstrating the desired induction step. The result follows. □

4.3 Remainder Estimates

4.3.1 On the choice of the shrinking radius

We begin choosing a point $p \in M$ and choosing some local neighborhood U_x which admits a local trivialization as well as a plurisubharmonic $k\varphi$ corresponding to the metric h on M. Due to the rescaling property of $k\varphi$, we rescale coordinates via the identifications

$$\begin{cases} x & = p + \frac{u}{\sqrt{k}}, \\ y & = p + \frac{v}{\sqrt{k}}. \end{cases} \tag{4.59}$$

Without loss of generality we additionally take coordinates so that p is identified with the origin. Thus the potential scales as follows:

$$k\varphi\left(\frac{v}{\sqrt{k}}\right) = |v|^2 + kR\left(\frac{v}{\sqrt{k}}\right). \tag{4.60}$$

Consequently, for large k, one may compute in the setting of a *perturbed Bargmann-Fock space*. Under Bochner coordinates (cf. Proposition 4.4.1) we have $R(z) = O(|z|^4)$, so we further stipulate that v will be chosen from within $B(k^{\frac{1}{4}})$ to ensure that $R\left(\frac{v}{\sqrt{k}}\right)$ is bounded above by a constant. Thus we choose some cutoff function $\chi \in C_c^\infty(\mathbb{C}^n)$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{4.61}$$

and set $\chi_k(x) := \chi(k^{\frac{1}{4}}x)$. We then incorporate χ_k into the domain of integration, yielding the local reproducing property of Proposition 4.3.1. Note that

$$\text{supp}(d\chi_k(x)) \subset \{z \mid \tfrac{1}{2}k^{\frac{1}{4}} \leq |z| \leq k^{\frac{1}{4}}\} \tag{4.62}$$

So for some $|u| \leq 1$ and $v \in \text{supp}(d\chi_k(x))$, their distance has a lower bound $|u - v| \geq \frac{1}{4}k^{\frac{1}{4}}$, which is crucial in obtaining an estimate for the exponential decay outside the near-diagonal neighborhood. We then analyze the orders of the remainders of the truncations of the locally defined pieces to show that their contribution to the local reproducing property is of negligible order.

4.3.2 Local Reproducing Kernel

We demonstrate that we have constructed a local reproducing kernel with the coefficients chosen in §4.2 up to small error, and give measure of such error. For the following section we use the following notation: given a domain $U \subset \mathbb{C}^n$ and $f \in H^0(U)$ we expand f via the Taylor series expansion

$$f(x) = \sum_{j=0}^{\infty} \left(\frac{(D^j f)(0)}{j!} x^j \right). \tag{4.63}$$

Then we set

$$f_N(x) := \sum_{j=0}^{N} \left(\frac{(D^j f)(0)}{j!} x^j \right). \tag{4.64}$$

Proposition 4.3.1 (Local reproducing property)**.** *Let $f \in H^0(B)$, and c_j quantities as constructed above in §4.2. Then for $u \in B$, the following equality holds*

$$\begin{aligned} f\left(\frac{u}{\sqrt{k}}\right) = &\left\langle \chi_k\left(\frac{\cdot}{\sqrt{k}}\right) f\left(\frac{\cdot}{\sqrt{k}}\right), e^{\bar{u}\cdot(\cdot)}\left(\sum_j^N \frac{c_j(\cdot, \bar{u})}{\sqrt{k^j}}\right)\right\rangle_{L^2\left(B(\sqrt{k}), k\varphi\left(\frac{\cdot}{\sqrt{k}}\right)\right)} \\ &+ O\left(\frac{1}{\sqrt{k}^{N+1-2n}}\right)\|f\|_{L^2(B,k\varphi)}. \end{aligned} \tag{4.65}$$

To prepare to verify Proposition 4.3.1, we prove a series of estimates on the truncations and Taylor series remainders of the volume form. We also show that the integral outside of the disk rapidly decays.

Lemma 4.3.1 (Remainder of exponential term)**.** *For $N \geq 0$ and any $f \in H^0(B)$,*

$$\begin{aligned} &\int_{B(\sqrt{k})} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2} \left(\sum_{j=0}^N \tfrac{c_j(u,\bar{v})}{\sqrt{k^j}}\right)\left(e^{-kR\left(\frac{u}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{u}{\sqrt{k}}\right)}\right)_{2N+1}\right)\Omega\left(\tfrac{v}{\sqrt{k}}\right) dV \\ &= \|f\|_{L^2(B,k\varphi)} O\left(k^{-\frac{N+1}{2}}\right). \end{aligned}$$

Proof. First note that since $|v| \leq k^{\frac{1}{4}}$ we have

$$kR\left(\frac{v}{\sqrt{k}}\right) = O(1). \tag{4.66}$$

We regroup the quantity

$$e^{-kR} - e^{-kR_{2N+5}} = e^{-kR}\left(1 - e^{k(R-R_{2N+5})}\right). \tag{4.67}$$

By Taylor expansion

$$
\begin{aligned}
k\left|(R - R_{2N+5})\left(\frac{v}{\sqrt{k}}\right)\right| &\leq k \sup_{\substack{|\alpha|=2N+6 \\ |\xi|\leq \frac{|v|}{\sqrt{k}}}} \left|\frac{D^\alpha R(\xi)}{(\alpha)!}\right| \left|\frac{v}{\sqrt{k}}\right|^\alpha \\
&\leq C_N k \left(\frac{|v|}{\sqrt{k}}\right)^{2N+6} \\
&\leq C_N k^{-\frac{N+1}{2}}.
\end{aligned}
$$

Applying the above to (4.67), we have

$$
\left|e^{-kR\left(\frac{v}{\sqrt{k}}\right)} - e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right| \leq C_N k^{-\frac{N+1}{2}}. \tag{4.68}
$$

Next we consider the difference

$$
\begin{aligned}
\left|e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}\right| &\leq \sup_{\substack{|\alpha|=2N+2 \\ \xi\in B(k^{-\frac{1}{4}})}} \left|\frac{D^\alpha e^{-kR_{2N+5}}(\xi)}{\alpha!}\right| \left|\frac{v}{\sqrt{k}}\right|^{2N+2} \\
&\leq C_N k^{-\frac{N+1}{2}}.
\end{aligned} \tag{4.69}
$$

Combining (4.68) and (4.69), we have

$$
\left|e^{-kR\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}\right| \leq C_N k^{-\frac{N+1}{2}}. \tag{4.70}
$$

Applying our estimate directly to the integral,

$$\begin{aligned}
&\left|\int_B \chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2}\left(\sum_{j=0}^{N}\tfrac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)\left(e^{-kR\left(\frac{v}{\sqrt{k}}\right)}-\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}\right)\Omega\left(\tfrac{v}{\sqrt{k}}\right)dV\right|\\
&\leq C_N k^{-\frac{N+1}{2}}\int_B\left|\chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{-\frac{|v|^2}{2}}\left(\sum_{j=0}^{N}\tfrac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)\Omega\left(\tfrac{v}{\sqrt{k}}\right)e^{u\bar{v}-\frac{|v|^2}{2}}\right|dV\\
&\leq C_N k^{-\frac{N+1}{2}}\left(\int_B \chi_k\left(\tfrac{v}{\sqrt{k}}\right)\left|f\left(\tfrac{v}{\sqrt{k}}\right)\right|^2 e^{-|v|^2}dV\right)^{\frac{1}{2}}\\
&\leq C_N\|f\|_{L^2(B,k\varphi)}k^{-\frac{N+1}{2}}.
\end{aligned}$$

The result follows. □

Lemma 4.3.2 (Remainder of determinant)**.** *The following estimate holds*

$$\begin{aligned}
&\left|\int_B \chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2}\left(\sum_{j=0}^{N}\tfrac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}(\Omega-\Omega_{2N+1})\left(\tfrac{v}{\sqrt{k}}\right)dV\right|\\
&\qquad=\|f\|_{L^2(B,k\varphi)}O\left(k^{-\frac{N+1}{2}}\right).
\end{aligned} \tag{4.71}$$

Proof. We first observe the following estimate

$$\left|(\Omega-\Omega_{2N+1})\left(\frac{v}{\sqrt{k}}\right)\right| \leq \sup_{\substack{|\alpha|=2N+2\\ |\xi|\leq\left|\frac{v}{\sqrt{k}}\right|}}\left|\frac{D^\alpha\Omega(\xi)}{\alpha!}\right|\left|\frac{v}{\sqrt{k}}\right|^{2N+2} \leq C_N k^{-\frac{N+1}{2}}. \tag{4.72}$$

Using the above estimate with a similar manipulation as Lemma 4.3.1 we conclude (4.71). □

Lemma 4.3.3 (Estimate Outside Disk)**.** *The following estimate holds*

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right) f_N\left(\frac{v}{\sqrt{k}}\right)e^{u\bar{v}-|v|^2}\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)dV\\
&\quad\leq \|f\|_{L^2(B,k\varphi)}e^{-\frac{1}{16}k^{\frac{1}{2}}}.
\end{aligned}$$

Proof. First note that since $|u|\leq 1$ and $|v|\geq \frac{1}{2}k^{\frac{1}{4}}$, we have $|u-v|\geq\frac{1}{4}k^{\frac{1}{4}}$. Next we use the identity

$$\overline{\partial}\left(\sum_i e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}d\widehat{\overline{V}^i}\right) = -ne^{\bar{v}(u-v)}dV \tag{4.73}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)F\left(\frac{v}{\sqrt{k}}\right)e^{\bar{v}(u-v)}\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)dV\\
=&-\frac{1}{n}\int_{\mathbb{C}^n}\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)F\left(\frac{v}{\sqrt{k}}\right)\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)\\
&\qquad\cdot\overline{\partial}\left(\sum_i e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}d\widehat{\overline{V}^i}\right)\\
=&-\frac{1}{n}\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)\sum_i\overline{\partial}_i\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)\right)\\
&\qquad\cdot e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}dV
\end{aligned}$$

Iterating the above integration by parts $2N$ times we obtain

$$\begin{aligned}
=&\frac{(-1)^{2N+1}}{n^{2N+1}}\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)\sum_{\substack{I=(i_1,\ldots,i_{2N+1})\\|I|=2N+1}}\partial_{\bar{I}}\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)\\
&\qquad\cdot\frac{e^{\bar{v}(u-v)}}{(u-v)^I}dV.
\end{aligned} \tag{4.74}$$

Recalling that the degree of $a_m^{r,s}$ and $c_j^{p,q}$ are $2m$ and $2j$ respectively, we will always take the derivative on $1-\chi_k$. Therefore, we only consider the integral on the annulus $\frac{1}{2}k^{\frac{1}{4}}\le|v|\le k^{\frac{1}{4}}$. The above integral is bounded above by

$$\left(\int_{\frac{1}{2}k^{\frac{1}{4}}\leq|v|\leq k^{\frac{1}{4}}}\left|F\left(\frac{v}{\sqrt{k}}\right)\right|^2 e^{-|v|^2}dV\times\right.$$

$$\left.\int_{\frac{1}{2}k^{\frac{1}{4}}\leq|v|\leq k^{\frac{1}{4}}}\left|\sum_{\substack{I=(i_1,\ldots,i_{2N+1})\\|I|=2N+1}}\partial_{\bar{I}}\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right)\frac{e^{\frac{|u|^2-|u-v|^2}{2}}}{(u-v)^I}\right|^2 dV\right)^{\frac{1}{2}}$$

$$\leq C_N\|f\|_{L^2(B,k\varphi)}e^{-\frac{1}{16}k^{\frac{1}{2}}}.$$

□

With the results above, we prove the following.

Proof of Proposition 4.3.1. By our construction, we have

$$F\left(\frac{u}{\sqrt{k}}\right)=\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)e^{u\cdot\bar{v}-|v|^2}\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\bar{v})a_m(r,s)}{\sqrt{k^t}}\right)dV \tag{4.75}$$

for $N\geq 0$. We note that the N is independent of the polynomial F. Since

$$\left|\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\bar{v})a_m(r,s)}{\sqrt{k^t}}\right)-\left(\sum_{j=0}^{N}\frac{c_j(u,\bar{v})}{\sqrt{k}}\right)\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}\Omega_{2N+1}\right|\leq C_N.$$

We have

$$\begin{aligned}F\left(\frac{u}{\sqrt{k}}\right)=&\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)e^{u\cdot\bar{v}-|v|^2}\left(\sum_{j=0}^{N}\frac{c_j(u,\bar{v})}{\sqrt{k}}\right)\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1}\Omega_{2N+1}dV\\&+O(k^{-\frac{N+1}{2}})\|F\|_{L^2(B,k\varphi)}.\end{aligned} \tag{4.76}$$

We then split the above to two pieces

$$
\begin{aligned}
&F\left(\frac{u}{\sqrt{k}}\right) \\
&= \int_{\mathbb{C}^n} \left(1 - \chi_k\left(\frac{v}{\sqrt{k}}\right)\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1} \Omega_{2N+1} dV \\
&\quad + \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} \chi_k\left(\frac{v}{\sqrt{k}}\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1} \Omega_{2N+1} dV.
\end{aligned}
$$

The first integral is bounded above by $\|f\|_{L^2(B,k\varphi)} e^{-\left(\frac{1}{4}k^{\frac{1}{4}-\varepsilon}\right)^2}$ from Lemma (4.3.3). For the second integral, by applying Lemmas (4.3.2) and (4.3.1), we have

$$
\begin{aligned}
&\int_B \chi_k\left(\tfrac{v}{\sqrt{k}}\right) F\left(\tfrac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \tfrac{c_j(u,\bar{v})}{\sqrt{k^j}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_{2N+1} \Omega_{2N+1}\left(\tfrac{v}{\sqrt{k}}\right) dV \\
&= \left\langle \chi_k\left(\tfrac{\cdot}{\sqrt{k}}\right) F\left(\tfrac{\cdot}{\sqrt{k}}\right), e^{\bar{u}\cdot(\cdot)} \left(\sum_{j}^{N} \tfrac{c_j(\bar{u},\cdot)}{\sqrt{k^j}}\right) \right\rangle_{L^2(B(\sqrt{k}),k\varphi(\frac{\cdot}{\sqrt{k}}))} + \|F\|_{L^2(B,k\varphi)} O\left(k^{-\frac{N+1}{2}}\right).
\end{aligned}
$$

We can extend to arbitrary $f \in H^0(B)$ by considering the uniform convergence of the Taylor series since $|u| \leq 1$ and $|v| \leq \sqrt{k}$. The result follows. □

4.4 Computation of the coefficients

We next explicitly compute the coefficient c_1 and c_2 of K^{loc} under Bochner coordinates (the coefficients c_0 was computed in Lemma 4.2.1.

To compute c_1 and c_2 we require preliminary terms of the Kähler potential as well as the coefficients $a_m^{r,s}$.

Proposition 4.4.1 (Expansion of Kähler potential)**.** *We have the following series expansion* *of the potential φ under Bochner coordinates is given by*

$$\begin{aligned}\varphi(z) &= |z|^2 - \frac{\mathrm{Rm}_{i\bar{j}k\bar{l}}(0)}{4} z^i z^k \overline{z}^j \overline{z}^l + O(|z|^5) \\ &= |z|^2 + R(z, \bar{z}).\end{aligned}$$

Lemma 4.4.1 (Properties of $e^{-kR}\Omega$ expansion)**.** *The expansion up to $\frac{1}{k}$ for $\sum_m \sum_{p,q} \frac{a_m^{p,q} v^p \overline{v}^q}{\sqrt{k^m}}$ is*

$$e^{-kR\left(\frac{v}{\sqrt{k}}\right)}\Omega\left(\frac{v}{\sqrt{k}}\right) = 1 - \frac{1}{k}\left(\mathrm{Ric}_{k\bar{l}}\, v^k \overline{v}^l - \frac{1}{4}\mathrm{Rm}_{i\bar{j}k\bar{l}}(0) v^i v^k \overline{v}^j \overline{v}^l\right), \tag{4.77}$$

where the numbers a_j^{pq} for $j = 0, 1, 2$ are given by

$$a_0^{p,q} = \begin{cases} 1 & \text{if } |p| = |q| = j = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.78}$$

$$a_1^{p,q} = 0 \quad \text{for all } p, q, \tag{4.79}$$

and lastly

$$a_2^{p,q} = \begin{cases} -\sum_{k,l} \mathrm{Ric}_{k\bar{l}} & \text{if } |p| = |q| = 1 \\ \frac{1}{4}\sum_{k,l} \mathrm{Ric}_{i\bar{j}k\bar{l}}(0) & \text{if } |p| = |q| = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.80}$$

Proof. We expand each quantity of the product on the left hand side of 4.77. First, for the exponential term we have

$$e^{-kR\left(\frac{v}{\sqrt{k}}\right)} = 1 + \frac{\mathrm{Rm}_{i\bar{j}k\bar{l}}(0)}{4k} v^i v^k \overline{v}^j \overline{v}^l + o(k^{-\frac{3}{2}}). \tag{4.81}$$

And the determinant quantity becomes

$$\begin{aligned}\Omega\left(\frac{v}{\sqrt{k}}\right) &= \det\left(\delta_{i\bar{j}} + \frac{1}{k}\frac{\partial^4\varphi_{i\bar{j}}}{\partial z^k\partial\overline{z}^l}(0)v^k\overline{v}^l + O\left(k^{-\frac{3}{2}}\right)\right)\\ &= 1 - \frac{1}{k}\operatorname{Ric}_{k\bar{l}} v^k\overline{v}^l + O\left(k^{-\frac{3}{2}}\right).\end{aligned}$$

The result follows. □

The computation of c_1 is now immediate.

Corollary 4.4.1. *For all $p, q \in \mathbb{Z}_+$ we have $c_1^{p,q} = 0$.*

Proof. By comparing coefficients in (4.28), we see that there is no contribution from a_1 for the $\frac{1}{\sqrt{k}}$ term, hence

$$\frac{c_1^{p,q}}{\sqrt{k}}u^{p+l-q} = 0.$$

for any l. The result follows. □

4.4.1 Computing the coefficient c_2

By applying equation (4.28) to $f(z) = 1$ we obtain c_2^{00}:

$$\begin{aligned}1 = \int_{\mathbb{C}^n} e^{u\cdot\bar{v}-|v|^2}\left(1 + \frac{c_2^{00}}{k} + \frac{c_2^{(i),(j)}}{k}u^i\overline{v}^j + \frac{c_2^{(i,k)(j,l)}}{k}u^iu^k\overline{v}^j\overline{v}^l\right)&\\ \cdot\left(1 - \tfrac{1}{k}\left(\operatorname{Ric}_{k\bar{l}} v^k\overline{v}^l - \tfrac{1}{4}\operatorname{Rm}_{i\bar{j}k\bar{l}}(0)v^iv^k\overline{v}^j\overline{v}^l\right)\right) dV.&\end{aligned} \tag{4.82}$$

Collecting the $\frac{1}{k}$ terms, we obtain

$$c_2^{00} = \int_{\mathbb{C}^n}\operatorname{Ric}_{i\bar{j}} v^i\overline{v}^j e^{u\overline{v}-|v|^2}dV - \frac{1}{4}\int_{\mathbb{C}^n}\operatorname{Rm}_{i\bar{j}k\bar{l}} v^iv^k\overline{v}^j\overline{v}^l e^{u\overline{v}-|v|^2}dV. \tag{4.83}$$

The first integral on the right hand side is nonzero when $i = j$. The left side is nonzero when $i = j, k = l$ and $i = l, j = k$. We therefore obtain

$$c_2^{00} = \frac{\rho}{2}. \tag{4.84}$$

Next to obtain the $c_2^{(i)(j)}$ coefficient, we apply equation (4.28) with $f = v^\alpha$ to obtain

$$\sum_i c_2^{(i)(\alpha)} u^i = \int_{\mathbb{C}^n} \left(\mathrm{Ric}_{k\bar{l}}\, v^k v^\alpha \overline{v}^l - \frac{1}{4} \mathrm{Rm}_{i\bar{j}k\bar{l}}\, v^i v^k v^\alpha \overline{v}^j \overline{v}^l \right) e^{u \cdot v - |v|^2} dV \tag{4.85}$$

The first term on the right hand side is nonzero when $\alpha = l$, hence the only relevant term after integration is

$$\int_{\mathbb{C}^n} \mathrm{Ric}_{i\overline{\alpha}}\, v^i |v^\alpha|^2 e^{u \cdot v - |v|^2} dV = \sum_i \mathrm{Ric}_{i\overline{\alpha}}\, u^i.$$

The second term splits into four cases:

1. $\alpha = j, i = l$.
2. $\alpha = j, k = l$.
3. $\alpha = l, i = j$.
4. $\alpha = l, k = j$.

In each case, after integration, we obtain the term

$$\int_{\mathbb{C}^n} \mathrm{Rm}_{i\overline{\alpha},k,\overline{i}}\, v^k |v^i|^2 |v^\alpha|^2 e^{u \cdot v - |v|^2} dV = \sum_k \mathrm{Ric}_{k\overline{\alpha}}\, u^k.$$

and similar computations for the other cases, hence

$$c_2^{(i)(\alpha)} = 0. \tag{4.86}$$

Next to obtain the $c_2^{(ik)(jl)}$ coefficient, we apply equation (4.28) with $f = v^\alpha v^\beta$ to obtain

$$2\sum_{i,k} c_2^{(ik)(jl)} u^i u^k = \int_{\mathbb{C}^n} \left(\mathrm{Ric}_{k\bar{l}}\, v^k v^\alpha v^\beta \overline{v}^l - \frac{1}{4}\,\mathrm{Rm}_{i\bar{j}k\bar{l}}\, v^i v^k v^\alpha v^\beta \overline{v}^j \overline{v}^l \right) e^{u\cdot v - |v|^2} dV. \qquad (4.87)$$

For the first term on the right hand side, it is not possible to sum over two variables, hence is an irrelevant term. The second term has two cases:

1. $j = \alpha, l = \beta$
2. $l = \alpha, j = \beta$

Hence we have

$$c_2^{(ik)(\alpha\beta)} = -\frac{1}{4} R_{i\overline{\alpha}k\overline{\beta}}. \qquad (4.88)$$

Note that the result matches with [13] except for the emergence of non-analytic terms, however the computations in Lu and Shiffman were done for the lifted Szegö kernel.

4.5 Local to Global

Let $K(x, y)$ be the global Bergman kernel, as a section of $L \otimes \bar{L}$. The norm as a section of this bundle is the *Bergman function* $B(x)$, which in the standard local frame is

$$B(x) = |K(x, x)|_h = |\tilde{K}(x, x)| e^{-\varphi(x)}, \qquad (4.89)$$

where $\tilde{K}(x, x)$ is the coefficient function of the Bergman kernel with respect to this local trivialization. We also have an extremal characterization of the Bergman function given by

$$B(x) = \sup_{\|s\|_{L^2} \leq 1} |s(x)|_h^2, \tag{4.90}$$

where $s \in H^0(M, L)$. Henceforth we will denote $B(x)$ to be the Bergman function associated to $L^{\otimes k}$.

Lemma 4.5.1 (Uniform upper bound on Bergman function)**.** *There exists $C \in \mathbb{R}$ dependent on M which is uniform over all $k \in \mathbb{Z}_+$ and $x \in M$ such that*

$$B_k(x) \leq Ck^n. \tag{4.91}$$

Proof. We use the extremal characterization of the Bergman function

$$B_k(z) = \sum |\tilde{s}(z)|^2 e^{-k\varphi(z)}, \tag{4.92}$$

where s is a section of $L^{\otimes k}$ with $\|s\|_{L^2} = 1$. Fix a point $z \in M$ and choose local trivialization of L such that $z = 0$ and the Taylor expansion of φ under Bochner coordinates is given by (cf. Proposition 4.4.1)

$$\varphi(w) = |w|^2 + O(|w|^4). \tag{4.93}$$

For such coordinates set $dV_E := w^1 \wedge \bar{w}^1 \wedge \cdots w^n \wedge \bar{w}^n$. Then we compute the following estimate, comparing the volume form to the Euclidean volume form for sufficiently large k

and utilizing the subharmonicity of $|\tilde{s}|^2$,

$$\begin{aligned}
1 &\geq \int_{B\left(\frac{1}{\sqrt{k}}\right)} |\tilde{s}(w)|^2 e^{-k\varphi(w)} dV(w) \\
&\geq \frac{1}{C} \int_{B\left(\frac{1}{\sqrt{k}}\right)} |\tilde{s}(w)|^2 e^{-k|w|^2} dV_E(w) \\
&\geq \frac{1}{C} |\tilde{s}(0)|^2 \int_{B\left(\frac{1}{\sqrt{k}}\right)} e^{-k|w|^2} dV_E(w) \\
&= \frac{1}{C'} |\tilde{s}(0)|^2 \frac{1}{k^n}.
\end{aligned}$$

By considering the remainder term of the Taylor expansion of φ, we obtain a uniform $C > 0$ and $C' > 0$ is another uniform constant. Since $\varphi(0) = 0$ we observe that

$$|\tilde{s}(0)|^2 e^{-k\varphi(0)} \leq C' k^n. \tag{4.94}$$

Taking the supremum over all such s yields the desired result. □

Let $K(x, y) = K_y(x)$ be the global Bergman kernel of $H^0(M, L^{\otimes k})$. We view $K_y(x)$ as a section of $L^k \otimes \overline{L}_y^k$. It is defined by the global reproducing property

$$f(y) = \langle f, K_y \rangle_{L^2} \tag{4.95}$$

for any element $f \in H^0(M, L^{\otimes k})$. To be precise, for some local trivialization s, we have

$$\tilde{f}(y)s(y) = \langle f, K_y \rangle_{L^2} s(y) = \int_M (f, K_y)_h \frac{\omega^n}{n!} s(y) \tag{4.96}$$

In other words, the integral kernel reproduces the coefficient function. Using the local reproducing property on the global Bergman kernel, we show that the local Bergman kernel is equivalent to the global Bergman kernel up to a small error. The proof is essentially the same as [2].

Theorem 4.5.1 (Local to Global)**.** *The following equality relates the truncated local Bergman kernel K_N^{loc} to the global Berman kernel K_k.*

$$K\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) = K_N^{loc}\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) + O\left(k^{-\frac{N+1}{2}}\right). \tag{4.97}$$

Proof. Fix $u, v \in B$. We apply the local reproducing property to the global Bergman kernel

$$\tilde{K}\left(\frac{v}{\sqrt{k}}, \frac{u}{\sqrt{k}}\right) = \left\langle \chi_k(\cdot)\tilde{K}\left(\cdot, \frac{u}{\sqrt{k}}\right), K_N^{loc}\left(\cdot, \frac{v}{\sqrt{k}}\right)\right\rangle_{L^2(B,k\varphi)} + O\left(k^{n-\frac{N+1}{2}}\right)\|\tilde{K}\|_{L^2(B,k\varphi)}. \tag{4.98}$$

By the reproducing property, we obtain from Lemma 4.5.1,

$$\|\tilde{K}_{\frac{u}{\sqrt{k}}}\|^2_{L^2(B,k\varphi)} \leq \|K_{\frac{u}{\sqrt{k}}}\|^2_{L^2} = \tilde{K}\left(\frac{u}{\sqrt{k}}, \frac{u}{\sqrt{k}}\right) = B\left(\frac{u}{\sqrt{k}}\right)e^{k\varphi(\frac{u}{\sqrt{k}})} \leq Ck^n, \tag{4.99}$$

where $K_{\frac{u}{\sqrt{k}}}(w)$ means section with respect to w and coefficient function with respect to u. Thus we have

$$\tilde{K}\left(\frac{v}{\sqrt{k}}, \frac{u}{\sqrt{k}}\right) = \left\langle \chi_k(\cdot)\tilde{K}\left(\cdot, \frac{u}{\sqrt{k}}\right), K_N^{loc}\left(\cdot, \frac{v}{\sqrt{k}}\right)\right\rangle_{L^2(B,k\varphi)} + O\left(k^{2n-\frac{N+1}{2}}\right). \tag{4.100}$$

We next estimate the difference of the local Bergman kernel with the projection of the local kernel.

$$\begin{aligned} g_{k,v}\left(\frac{w}{\sqrt{k}}\right) :=& \chi_k\left(\frac{w}{\sqrt{k}}\right)K_N^{loc}\left(\frac{w}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - \overline{\left\langle \chi_k(\cdot)\tilde{K}\left(\cdot, \frac{w}{\sqrt{k}}\right), K_N^{loc}\left(\cdot, \frac{v}{\sqrt{k}}\right)\right\rangle}_{L^2(B,k\varphi)} \\ =& \chi_k\left(\frac{w}{\sqrt{k}}\right)K_N^{loc}\left(\frac{w}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - \left\langle \chi_k(\cdot)K_N^{loc}\left(\cdot, \frac{v}{\sqrt{k}}\right), K\left(\cdot, \frac{w}{\sqrt{k}}\right)\right\rangle_{L^2}. \end{aligned} \tag{4.101}$$

We can regard $g_{k,v}$ as a global section of $L^{\otimes k}$ because of the cut-off function χ_k for each fixed

v. Since

$$\left\langle \chi_k K^{loc}_{N,\frac{v}{\sqrt{k}}}, K_{\frac{u}{\sqrt{k}}} \right\rangle_{L^2} = \mathcal{P}_{H^0}\left(\chi_k K^{loc}_{N,\frac{v}{\sqrt{k}}}\right), \tag{4.102}$$

where $\mathcal{P}_{H^0}$ is the Bergman projection, $g_{k,v}$ is the L^2-minimal solution to

$$\overline{\partial} g_{k,v} = \overline{\partial}\left(\chi_k K^{loc}_{N,\frac{v}{\sqrt{k}}}\right). \tag{4.103}$$

Now we estimate $\overline{\partial}(\chi_k K^{loc}_{N,\frac{v}{\sqrt{k}}})$ by using Hormänder's L^2-estimate:

$$\begin{aligned}\overline{\partial}\left(\chi_k K^{loc}_{N,\frac{v}{\sqrt{k}}}\right)\Big|_{\frac{w}{\sqrt{k}}} &= \left(\overline{\partial}\left(\chi_k\right) K^{loc}_{N,\frac{v}{\sqrt{k}}} + \chi_k \overline{\partial}\left(K^{loc}_{N,\frac{v}{\sqrt{k}}}\right)\right)\Bigg|_{\frac{w}{\sqrt{k}}} \\ &= \overline{\partial}\left(\chi_k\right) K^{loc}_{N,\frac{v}{\sqrt{k}}}\Bigg|_{\frac{w}{\sqrt{k}}}.\end{aligned}$$

Considering the right hand side of the equality, we have that the second term vanishes due to analyticity. The first term of the right hand side ensures $|w-v| \geq \frac{1}{4}k^{\frac{1}{4}}$. Furthermore, since $K^{loc}_{N,\frac{v}{\sqrt{k}}}(\frac{w}{\sqrt{k}}) = e^{w\bar{v}}\left(1 + O\left(\frac{1}{\sqrt{k}}\right)\right)$ then we observe that

$$|e^{w\bar{v}}|^2 e^{-|w|^2} = e^{2\operatorname{Re} w\bar{v} - |w|^2} = e^{-|w-v|^2+|v|^2} \leq Ce^{-\frac{1}{16}k^{\frac{1}{2}}}. \tag{4.104}$$

Therefore for some constant $\delta > 0$,

$$\left\|\overline{\partial}(\chi_k) K^{loc}_{N,\frac{v}{\sqrt{k}}}\right\|_{L^2(M,L^{\otimes k})} \leq e^{-\delta k^{\frac{1}{4}}}. \tag{4.105}$$

So by the Hormänder's L^2-estimate, the following inequality holds uniformly for $v \in B$,

$$\|g_{k,v}\|_{L^2(M,L^{\otimes k})} \leq Ce^{-\delta k^{\frac{1}{4}}}. \tag{4.106}$$

By the same step as the Lemma 4.5.1 above, for the fixed $u \in B$ we obtain the pointwise estimate

$$\left| g_{k,v}\left(\frac{u}{\sqrt{k}}\right)\right| \leq Ce^{-\delta k^{\frac{1}{4}}}. \tag{4.107}$$

Note the δ here may be smaller than in (4.106). Finally we conclude the estimate, for any $u, v \in B$,

$$\left| K_N^{loc}\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - K\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right)\right| \leq Ck^{2n-\frac{N+1}{2}}. \tag{4.108}$$

The result follows. □

4.5.1 Twisted bundle case

Let $(E, H) \to M$ be a Hermitian holomorphic vector bundle. A twisting of a line bundle $L^{\otimes k}$ by a vector bundle E is simply the tensor product $E \otimes L^{\otimes k}$. Sections $f \in H^0(M, E \otimes L^{\otimes k})$ are now locally viewed as vector valued holomorphic functions. Let $x_0 \in M$, $\{e_i(x)\}$ be a local frame for E at x_0, and $s(x)$ be a local trivialization for L at x_0. Then the reproducing property takes the form

$$f^i(y)s(y) \otimes e_i(y) = \int_M (f, K_x)_{H,h^k} \, dV_g s(y) \otimes e_i(y) \tag{4.109}$$

Locally,

$$\left(f^k, K_x\right)_{H,h^k} = \tilde{f}^k(y)\overline{\tilde{K}}^{j,i}(y,x)H_{k\bar{j}}(y)e^{-k\varphi(y)}e_i(x) \tag{4.110}$$

where $H(e_k, e_j) = H_{k\bar{j}}$.

By considering the Taylor expansion of H, we can repeat the same local construction and obtain similar remainder estimates to extend our result to the twisted bundle case.

4.6 Appendix

Any background notions pertinent to our work will be discussed here. In particular, we discuss the background notions of the Bargmann-Fock space.

4.6.1 Bargmann-Fock space

In this subsection we discuss the Bargmann Fock space, $\mathcal{F}$ is the space of entire functions that satisfy the weighted square integrability condition:

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dV < \infty. \tag{4.111}$$

The space $\mathcal{F}$ is precisely $L^2(\mathbb{C}^n, |z|^2)$, and is thus a closed linear subspace of the space $L^2(\mathbb{C}^n)$ with inner product given by

$$\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} dV, \tag{4.112}$$

and thus is a Hilbert space. In fact, it is a *reproducing kernel Hilbert space* on $\mathbb{C}^n$, with reproducing kernel

$$\mathcal{R}_{\mathbb{C}^n}(u, v) := e^{u \cdot \bar{v}}. \tag{4.113}$$

We first show that this kernel has the reproducing property on $\mathbb{C}$ and then extend this argument to $\mathbb{C}^n$.

Lemma 4.6.1. *On $\mathbb{C}$, the Bargmann-Fock kernel is given by*

$$\mathcal{R}_{\mathbb{C}}(u, v) := e^{u\bar{v}}. \tag{4.114}$$

Proof. Taking some $f \in H^0(\mathbb{C}^n)$, we consider the inner product against $\mathcal{R}_{\mathbb{C}}$. We convert the resulting integral to polar coordinates and then apply the Cauchy Integral Formula to obtain

$$\begin{aligned}
\langle f(v), \mathcal{R}_{\mathbb{C}} \rangle_{\mathcal{F}} &= \sqrt{-1} \int_{\mathbb{C}} f(v) e^{u\bar{v}-|v|^2} \frac{dv \wedge d\bar{v}}{2\pi} \\
&= -\frac{1}{\pi} \int_0^\infty \int_0^{2\pi} f(u+re^{i\theta}) e^{u(\bar{u}+re^{-i\theta})-|u+re^{i\theta}|^2} \frac{r}{2} d\theta dr \\
&= -\frac{1}{\pi} \int_0^\infty re^{-r^2} \int_0^{2\pi} f(u+re^{i\theta}) e^{-\bar{u}re^{i\theta}} d\theta dr \\
&= -f(u) \int_0^\infty 2re^{-r^2} dr \\
&= f(u).
\end{aligned}$$

The result follows. □

Corollary 4.6.1. *On $\mathbb{C}^n$, the Bargmann-Fock kernel is given by*

$$\mathcal{R}_{\mathbb{C}^n}(u, v) := e^{u \cdot \bar{v}}. \tag{4.115}$$

Proof. Let $u, v \in \mathbb{Z}_+^n$ with $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Observe that

$$e^{u \cdot \bar{v}} = \prod_{i=1}^{n} e^{u_i \bar{v}_i - |v_i|^2}. \tag{4.116}$$

To demonstrate the reproducing property, we consider $f \in H^0(\mathbb{C}^n)$ and decompose the integrand of the resulting inner product agains $\mathcal{R}_{\mathbb{C}^n}$. Applying Lemma 4.6.1 to each dimensional component, we have

$$\begin{aligned}\langle f(v), \mathcal{R}_{\mathbb{C}^n} \rangle_{\mathcal{F}} &= \int_{\mathbb{C}^n} f(v) e^{u\cdot v - |v|^2} dV \\ &= \int_{\mathbb{C}^n} f(v_1, \ldots, v_n) \left(\prod_{i=i}^{n} e^{u_i \bar{v}_i - |v_i|^2} \right) dV \\ &= f(u).\end{aligned}$$

The result follows. □

The following Lemmas demonstrate the Bargmann-Fock kernel on monomials of different variables.

Lemma 4.6.2. *Given some multiindex $m \in \mathbb{Z}_+^n$ the following equality holds.*

$$\int_{\mathbb{C}^n} \bar{v}^m e^{u\cdot\bar{v} - |v|^2} dV = 0. \tag{4.117}$$

Proof. By manipulation and an application of Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathbb{C}^n} \bar{v}^m e^{u\cdot\bar{v} - |v|^2} dV &= \int_{\mathbb{C}^n} \partial_u^{(m)} \left[e^{u\cdot\bar{v} - |v|^2} \right] dV \\ &= \partial_u^{(m)} \left[\int_{\mathbb{C}^n} e^{u\cdot\bar{v} - |v|^2} dV \right] \\ &= 0.\end{aligned}$$

Note that the integral is constant with respect to u, hence the derivative vanishes. The result follows. □

Lemma 4.6.3. *The following equality holds, for $p, q \in \mathbb{Z}_+$ with $p \leq q$.*

$$\int_{\mathbb{C}^n} \bar{v}^p v^q e^{u\cdot\bar{v} - |v|^2} dV = \begin{cases} 0 & \text{if } p > q, \\ \frac{q!}{(q-p)!} u^{q-p} & \text{if } p \leq q. \end{cases} \tag{4.118}$$

Proof. Again by manipulation and an application of Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathbb{C}^n} \bar{v}^p v^q e^{u.\bar{v}-|v|^2} dV &= \int_{\mathbb{C}^n} \partial_u^{(p)} \left[v^q e^{u\cdot\bar{v}-|v|^2} \right] dV \\ &= \partial_u^{(p)} \left[\int_{\mathbb{C}^n} v^q e^{u\cdot\bar{v}-|v|^2} dV \right] \\ &= \partial_u^{(p)} \left[u^q \right],\end{aligned}$$

therefore

$$\int_{\mathbb{C}^n} \bar{v}^p v^q e^{u\cdot\bar{v}-|v|^2} dV = \begin{cases} 0 & \text{if } p > q, \\ \frac{q!}{(q-p)!} u^{q-p} & \text{if } p \leq q. \end{cases}$$

The result follows. □

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