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Skein algebras and quantum groups

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Curriculum Vitæ

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Publications

1	*Inverse spectral problems for collections of leading principal submatrices of tridiagonal matrices,* with Charles Johnson, Linear Algebra and its Applications Vol. 489 (2016). https://doi.org/10.1016/j.laa.2015.10.004
2	*Triangular decomposition of SL_3 skein algebras,* (2020) preprint. https://arxiv.org/abs/2008.09419

Abstract

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Skein modules of 3-manifolds are situated at the intersection of low-dimensional topology and representation theory. The skein module of a thickened surface has a natural algebra structure induced by the superposition of skeins. In this thesis, we study connections between quantum groups and these skein algebras by focusing on the SL_3 skein algebra of an oriented punctured surface Σ, which may have punctured boundary components. Generalizing a construction of Lê, we associate an SL_3 *stated* skein algebra to any such Σ. These algebras admit natural algebra morphisms, called *splitting maps,* associated to the splitting of surfaces along ideal arcs. We give an explicit basis for the SL_3 stated skein algebra, which is an extension of the Sikora-Westbury basis for the ordinary SL_3 skein algebra. Using this basis, we show that the splitting maps are injective and describe their images. Applying the splitting maps to a triangulable surface, we obtain a triangular decomposition in which we embed the skein algebra in a domain that has an explicit presentation described in terms of the quantum group $\mathcal{O}_q(SL_3)$. The ingredients we collect along the way allow for a skein-theoretic method of recovering the fact that Kuperberg's webs describe a full subcategory of the representation category of $U_q(\mathfrak{sl}_3)$.

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Chapter 1

Introduction

1.1 Background

A *skein* is a low-dimensional topological object like a knot or a ribbon graph, which can have decorations encoding some representation theoretic data. Skein theory originated as a diagrammatic way of computing knot polynomials by using a rule, known as a skein relation, to relate the polynomial of one knot to the polynomial of other knots which differ only locally. Conway helped to popularize skein theory when he showed how to compute the Alexander polynomial of a knot by recursively writing a knot diagram as a formal linear combination of diagrams of simpler knots [Con70]. Although the method of skein relations was known to Alexander, the conventional ways to define and study the Alexander polynomial had used the more sophisticated tools of algebraic topology. Shortly after the discovery of the Jones polynomial, which arose from Jones's work on von Neumann algebras [Jon85], the Kauffman bracket provided a definition of the Jones polynomial from two simple skein relations [Kau87]. The Jones polynomial gave rise to the research area now known as quantum topology, in which skein theory plays a central role.

After it was observed that the Jones polynomial arises from skein relations, and that these skein relations arise from the representation theory of a quantum group, $U_q(\mathfrak{sl}_2)$, a research program was set in motion to study the connections between link invariants and quantum groups. An early success in this program was the work of Reshetikhin and Turaev [RT90] in which it was shown that any quantum group $U_q(\mathfrak{g})$, and any labeling of strands by irreducible representations, gives rise to an invariant of framed tangles and ribbon graphs. When this invariant is restricted to link diagrams and one uses the simplest quantum group $U_q(\mathfrak{sl}_2)$ and its 2-dimensional irreducible representation, the invariant recovers the Kauffman bracket and the Jones polynomial. Shortly after the work of Reshetikhin and Turaev, Kuperberg set in motion a related research program concerned with using categories defined purely in terms of skein relations to describe the representation categories of quantum groups [Kup96].

Since skein theory had been successful in describing already known topological invariants and algebraic objects, it is natural to try to construct new algebraic objects by using skeins. Przytycki and Turaev independently introduced the notion of the skein module of a 3-manifold, which is built from linear combinations of skeins in the 3-manifold, subject to local skein relations [Prz91, Tur88].

In this thesis we are focused on the special case of the skein module of a thickened surface. In the case of a thickened surface $\Sigma \times (-1, 1)$, there is a natural product of skeins given by stacking skeins in the interval direction, giving the skein module the structure of an algebra, called the *skein algebra* of the surface. A second advantage to using thickened surfaces is that skeins can be represented as diagrams on the surface, and skein diagrams are susceptible to tools such as confluence theory, the notion using skein relations to "simplify" a diagram, for finding bases of the skein algebras.

Although the skein algebra of a surface has a natural algebra structure, this structure is difficult to study explicitly since the skein algebra of a surface is not naturally endowed with a presentation by generators and relations. An important breakthrough in the study of the Kauffman bracket skein algebras was the quantum trace map of Bonahon and Wong [BW11], which provided a deep connection between skein theory and the geometry of surfaces. Given a punctured surface with an ideal triangulation, the quantum trace map embeds the Kauffman bracket skein algebra of the surface into a quantum torus, which is a much simpler algebra defined by generators and relations expressed in terms of the data of the triangulation. The definition of the quantum trace map is inspired by geometric ideas and, consequently, one of the difficult steps in its construction is checking that its definition respects the skein relations. In an effort to simplify this step, Lê defined a finer version of the skein algebra, which he called the Kauffman bracket *stated* skein algebra in [Lê18]. The stated skein algebras allow for skeins to have endpoints, labeled by states, on the boundaries of the surfaces so that these algebras admit algebra maps, called splitting maps, associated with splitting the surfaces along ideal arcs.

A followup work by Costantino-Lê [CL19] on Kauffman bracket stated skein algebras suggest that, just as linear skein relations arise from quantum groups, the algebraic structure of skein algebras of surfaces can also be studied via quantum groups. Conversely, the skein algebras can be used to give diagrammatic definitions of quantum groups themselves, complementing Kuperberg's diagrammatic descriptions of the representation categories of quantum groups.

The quantum groups studied in [CL19] are $U_q(\mathfrak{sl}_2)$ and its restricted Hopf dual $\mathcal{O}_q(SL_2)$. The Kauffman bracket skein algebra can also be referred to as the SL_2 skein algebra. Replacing SL_2 by another Lie group G yields analogous quantum groups $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, whose presentations by generators and relations can be extracted from the Dynkin diagram corresponding to G. Similarly, there is a notion of a G skein algebra for other Lie groups. Due to a construction of Walker [Wal06] and of Johnson-Freyd [JF19], these skein algebras can be abstractly constructed by imposing local skein relations corresponding to elements in the kernel of the Reshetikhin-Turaev functor and have been studied from this perspective in [BZBJ18, Coo19, GJS19]. However, it is an open question in general to find explicit diagrammatic descriptions of this kernel. This question has been answered for the case of rank 2 Lie groups, or types $A_1, A_2, B_2/C_2$, and G_2 by Kuperberg in [Kup96] and in the case of SL_n, or type A_n, by Cautis-Kamnitzer-Morrison and Sikora in [CKM14, Sik05]. Very recently the question has been answered in the case of Sp_{2n}, or type C_n, by Bodish-Elias-Rose-Tatham in [BERT21]. Historically, the geometry and combinatorics of these explicit diagrammatic descriptions have been a source of interesting constructions, such as Khovanov homology in the SL_2 case and foams more generally. Consequently, we are motivated to study skein algebras by studying the combinatorics of explicit skein relations.

1.2 Main results

In this thesis we are interested in studying the SL_3 skein algebra of a punctured surface and the SL_3 stated skein algebra of a punctured bordered surface. Although the theory of SL_3 skein algebras is expected to parallel the theory of SL_2 skein algebras, many of the techniques in the SL_2 case rely on the geometry and combinatorics of curves on surfaces. In the SL_3 case, skeins are oriented trivalent ribbon graphs, called webs, subject to skein relations which are more complicated than in the SL_2 case. In pioneering work of Jaeger and Kuperberg [Jae92, Kup94, Kup96], webs were first studied in the plane and then their natural extension to thickened surfaces were investigated by Sikora and Westbury [SW07]. Recently, SL_3 webs on surfaces have been studied in further detail in [FS20, Hig20, DS20a, DS20b, IY21, Kim21].

Our definition of the SL_3 skein algebra is built from a version of Kuperberg's webs, and we allow our coefficients to come from any commutative ring $\mathcal{R}$ containing an invertible element $q^{1/3}$. We introduce a definition of the SL_3 stated skein algebra of a punctured bordered surface Σ, denoted $\mathcal{S}_q^{SL_3}(\Sigma)$, by allowing webs to have endpoints on the boundary of Σ labeled by states from the set $\{-, 0, +\}$ and introducing skein relations along the boundary. These extra skein relations make it so that we can define a splitting map Δ_c, which is an algebra map associated to splitting the surface along an ideal arc c on Σ. The splitting map has a simple diagrammatic definition given by a state-sum, analogous to the splitting map in [Lê18].

The most powerful feature of our presentation of $\mathcal{S}_q^{SL_3}(\Sigma)$ by explicit skein relations is that the skein relations are confluent, meaning that each relation can be interpreted as a reduction rule allowing us to replace a single diagram by a linear combination of simpler diagrams such that all possible reductions of a web to an irreducible form agree up to isotopy of the web diagram on the surface. After this observation, we apply the Diamond Lemma for graphs on surfaces as developed in [SW07] to extract a basis for $\mathcal{S}_q^{SL_3}(\Sigma)$ consisting of isotopy classes of irreducible diagrams on Σ. In [SW07], Sikora and Westbury describe a basis for the ordinary SL_3 skein algebra consisting of irreducible diagrams on the surface, and our notion of the complexity of a diagram is compatible with theirs.

Theorem 1.1 *For any surface Σ, the SL_3 stated skein algebra $\mathcal{S}_q^{SL_3}(\Sigma)$ is a free $\mathcal{R}$-module with a canonical basis which is an extension of the Sikora-Westbury canonical basis of the SL_3 ordinary skein algebra. Furthermore, any element of the stated skein algebra can be written in the basis by repeatedly applying reduction rules.*

We use our basis of $\mathcal{S}_q^{SL_3}(\Sigma)$ to examine the kernel and image of the splitting map Δ_c and obtain the following.

Theorem 1.2 *Suppose that $\bar{\Sigma} = \Sigma/(a = b)$, where a and b are boundary arcs of Σ and* *their common image under the gluing map is an ideal arc c on $\bar{\Sigma}$. Then we have the following exact sequence of $\mathcal{R}$-modules.*

$$0 \to \mathcal{S}_q^{SL_3}(\bar{\Sigma}) \overset{\Delta_c}{\to} \mathcal{S}_q^{SL_3}(\Sigma) \overset{\Delta_a - \tau \circ_b \Delta}{\to} \mathcal{S}_q^{SL_3}(\Sigma) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}),$$

where Δ_a and $\tau \circ_b \Delta$ are certain coactions of the stated skein algebra of the bigon, $\mathcal{S}_q^{SL_3}(\mathfrak{B})$, on $\mathcal{S}_q^{SL_3}(\Sigma)$.

Our proof requires the basis of the stated skein algebra only in the case that the boundary of $\bar{\Sigma}$ is empty. In the case that this boundary is nonempty, the theorem can be proven using diagrammatic operations associated to the Hopf algebra structure of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$, which appear to generalize easily to other skein algebras.

The exact sequence in Thereom 1.2 corresponds to the splitting map Δ_c associated to a single ideal arc c. Given an ideal triangulation of a surface, we can apply splitting maps along all of the edges of the triangulation, yielding an exact sequence which is called the triangular decomposition of the skein algebra.

Corollary 1.3 *Suppose Σ has an ideal triangulation consisting of a set of interior edges $\mathcal{E}$, which separates Σ into n triangular faces. Denote by $\mathfrak{B}$ and $\mathfrak{T}$ the ideal bigon and the ideal triangle, respectively. Then we have the following exact sequence of $\mathcal{R}$-modules*

$$0 \to \mathcal{S}_q^{SL_3}(\Sigma) \overset{\Delta}{\to} \bigotimes_{i=1}^{n} \mathcal{S}_q^{SL_3}(\mathfrak{T}_i) \overset{\Delta^R - \tau \circ {}^L\Delta}{\to} (\bigotimes_{i=1}^{n} \mathcal{S}_q^{SL_3}(\mathfrak{T}_i)) \otimes (\bigotimes_{e \in \mathcal{E}} \mathcal{S}_q^{SL_3}(\mathfrak{B})).$$

This triangular decomposition tells us that the skein algebra of the surface embeds into a tensor product of stated skein algebras of triangles. The exactness of the sequence can be used to prove results on triangles and then extend them globally. We can study the skein algebra by studying the stated skein algebras of the building block surfaces: the monogon, the bigon, and the triangle.

Theorem 1.4 *The stated skein algebras of the monogon $\mathfrak{M}$, bigon $\mathfrak{B}$, and triangle $\mathfrak{T}$ are the following.*

i) $\mathcal{S}_q^{SL_3}(\mathfrak{M}) \cong \mathcal{R}$.

ii) $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$ *as Hopf algebras.*

iii) $\mathcal{S}_q^{SL_3}(\mathfrak{T}) \cong \mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$, *the braided tensor square of* $\mathcal{O}_q(SL_3)$.

Thus, we obtain a skein-theoretic definition of the quantum group $\mathcal{O}_q(SL_3)$ along with diagrammatic definitions of its structure maps: the counit, coproduct, and antipode. Our bialgebra structure of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ was recently used by Kim in [Kim21] to help construct an SL_3 quantum trace map, carrying out a strategy proposed by Douglas in [Dou21] to use the well-defined counit of the bigon to check that the quantum trace map respects the web relations.

By using our skein-theoretic definition of $\mathcal{O}_q(SL_3)$ we obtain a skein-theoretic proof of the following fact. Suppose that C is the full subcategory of $\mathcal{O}_q(SL_3)$-comodules monoidally generated by the standard $\mathcal{O}_q(SL_3)$-comodule V and its dual. Denote by Web^{SL_3} the category of Kuperberg's webs, which we view as *unstated* webs in the bigon $\mathfrak{B}$ modulo Kuperberg's web relations.

Theorem 1.5 *The Reshetikhin-Turaev functor* $\mathrm{Web}^{SL_3} \to C$ *is an equivalence of braided monoidal categories. Furthermore, its inverse can be described by a diagrammatic algorithm using the basis for the stated skein algebra.*

Our proof holds over any commutative ring $\mathcal{R}$ with any choice of invertible element $q^{1/3} \in \mathcal{R}$. Whenever the pairing between $U_q(\mathfrak{sl}_3)$ and $\mathcal{O}_q(SL_3)$ is nondegenerate, we can replace $\mathcal{O}_q(SL_3)$-comodules in Theorem 1.5 with $U_q(\mathfrak{sl}_3)$-modules and recover the theorem first proven by Kuperberg in [Kup96] for the case $\mathcal{R} = \mathbb{C}(q)$, and then proven more generally by Elias in [Eli15].

We also use the embedding afforded to us by the triangular decomposition to prove the following fact about both ordinary and stated SL_3 skein algebras of surfaces with at least one puncture.

Theorem 1.6 *Suppose that Σ is a surface with at least one puncture and that $\mathcal{R}$ has no zero divisors. Then $\mathcal{S}_q^{SL_3}(\Sigma)$ has no zero divisors.*

When q is a root of unity of odd order N, there is a well-known Hopf algebra embedding $\mathcal{O}_1(SL_3) \hookrightarrow Z(\mathcal{O}_q(SL_3))$ defined on standard generators by $X_{ij} \mapsto X_{ij}^N$. This map is dual to Lusztig's Frobenius map. Using the fact that $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$ and applying our triangular decomposition, we obtain the following.

Theorem 1.7 *Suppose that $\mathcal{R}$ has no zero divisors and $q^{1/3}$ is a root of unity of order N coprime to 6. If Σ has at least one puncture, there exists an embedding*

$$\mathcal{S}_1^{SL_3}(\Sigma) \hookrightarrow Z(\mathcal{S}_q^{SL_3}(\Sigma)),$$

commuting with the splitting maps. In the case $\Sigma = \mathfrak{B}$ this map agrees with the dual of Lusztig's Frobenius map.

For the case of SL_2, the corresponding embedding can be described topologically in terms of threading links through Chebyshev polynomials. That map was first constructed by Bonahon and Wong in [BW16], where it was called the Chebyshev-Frobenius homomorphism. We prove the existence of our embedding by using the triangular decomposition, which was a technique developed by Korinman and Quesney in [KQ19] in the SL_2 case. The map has also recently been extended to the case of ordinary and stated SL_2 skein modules for general 3-manifolds in [Lê15, LP19, BL20].

Chapter 2

Preliminaries

The goal of this preliminary chapter is to introduce the reader to skein theory, quantum groups, and SL_2 skein algebras of surfaces. We collect definitions and some main theorems about SL_2 skein algebras of surfaces in Section 2.8. The first part of this chapter is dedicated to motivating these definitions by investigating the relationship between basic representation theory and invariants of tangle diagrams.

The only original result in this preliminary chapter might be in Section 2.7, where we give a topological definition of the Hopf pairing between $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$. This definition is inspired by Bigelow's diagrammatic construction of $U_q(\mathfrak{sl}_2)$ in [Big14] and Korinman's observation in [Kor19] that the construction fits into the framework of Lê's stated skein algebras [Lê18].

2.1 Quantum groups

As mentioned in the introduction, the authors of [RT90] show that algebras which admit certain extra structures allow us to construct isotopy invariants of framed links, tangles, graphs, and braids with ingredients coming from the representation categories of the algebras. These algebras are referred to as *ribbon Hopf algebras*, and they are important examples of algebras known as *quantum groups*.

Suppose H is an $\mathcal{R}$-algebra for some commutative unital ring $\mathcal{R}$ of coefficients. By virtue of H being an algebra, it already admits a certain amount of structure. We can take tensor products of H to obtain algebras $H \otimes H$ and $H \otimes H \otimes H$. The product structure in H can be viewed as a *multiplication* map

$$m : H \otimes H \to H.$$

The *associativity* of m tells us that

$$m \circ (m \otimes \mathrm{id}_H) = m \circ (\mathrm{id}_H \otimes m)$$

on $H \otimes H \otimes H$. The unit element $1 \in H$ can be viewed as the image of $1 \in \mathcal{R}$ under the *unit* map

$$\eta : \mathcal{R} \to H.$$

The property that $1h = h = h1$ for all $h \in H$ can be expressed by saying that

$$m \circ (\eta \otimes \mathrm{id}_H) = \mathrm{id}_H = m \circ (\mathrm{id}_H \otimes \eta)$$

when $\mathcal{R} \otimes H$ and $H \otimes \mathcal{R}$ are both naturally identified with H.

The algebra H has an associated category of representations, H-mod, whose structure is related to the structure of H. The objects of this category are representations.

Definition 2.1 *A representation of H, or a (left) H-module, is a free $\mathcal{R}$-module V of finite rank equipped with an $\mathcal{R}$-algebra map $\rho_V : H \to End(V)$. We say that an element* *h of H acts on an element v of V by*

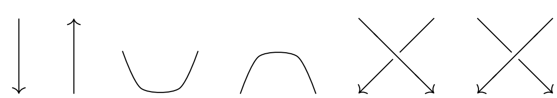
$$h.v = \rho_V(h)(v).$$

The morphisms in the representation category H-mod are maps between representations commuting with the action of H.

Definition 2.2 *If V and W are H-modules, then an $\mathcal{R}$-linear map $f : V \to W$ is a homomorphism of H-modules, also called an intertwiner, if it satisfies*

$$f(\rho_V(h)v) = \rho_W(h)f(v).$$

We now motivate the extra structure that we would like H to admit by describing the flavor of the Reshetikhin-Turaev operator invariant of a framed link. We consider an oriented link diagram in the plane, isotoped so that it is made up of elementary tangles which can be stacked both horizontally or vertically.



In addition to vertical strands, the elementary tangles are called *cups*, *caps*, and *crossings*. We label each component of the link by some H-module and imagine assigning an element of $\mathcal{R}$ to the link by assigning H-module homomorphisms to the elementary tangles which make up the link.

We identify the empty space of the link diagram with the label of the ground ring $\mathcal{R}$. In order to view $\mathcal{R}$ as an H-module, we need a way for H to act on $\mathcal{R}$. This motivates the requirement that H admits a *counit* $\varepsilon : H \to R$ so that $h.1_R = \varepsilon(h)$.

In order to identify a sequence of endpoints of parallel strands labeled by H-modules with a single H-module, we ask that the category of H-modules admits a tensor structure. This motivates the requirement that H admits a *comultiplication* $\Delta : H \to H \otimes H$ so that H acts on a tensor product $V \otimes W$ by $h.(v \otimes w) = \Delta(h)(v \otimes w)$.

Motivated both by the fact that parallel strands can have opposite orientations and that we need to associate H-module homomorphisms to cups and caps, we ask that the category of H-modules be closed under taking duals of modules. For this, we ask that H admits an *antipode* $S : H \to H$.

Definition 2.3 *An algebra H with algebra structure maps η, m is a Hopf algebra if it admits $\mathcal{R}$-algebra maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathcal{R}$ satisfying*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta,$$

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta,$$

and if it admits an $\mathcal{R}$-linear map $S : H \to H$ satisfying

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta.$$

For computations, there is a convenient notation for the map Δ, called Sweedler's sigma notation. We write

$$\Delta(h) = \sum_{(h)} h' \otimes h''.$$

Using this notation, the left side of the antipode axiom can be written

$$\sum_{(h)} S(h')h'' = \varepsilon(h)1_H.$$

The axioms for Δ and ε are the axioms for a bialgebra structure, and are dual to the axioms for an algebra structure. We actually have that a map S satisfying the above is an $\mathcal{R}$-algebra antihomomorphism, meaning that it satisfies $S(h_1h_2) = S(h_2)S(h_1)$ for all $h_1, h_2 \in H$. Furthermore, the existence of S guarantees that we can define an action of H on the dual of a module V^* by

$$(h.f)(v) = f(S(h).v)$$

for $v \in V$ and $f \in V^*$.

We next briefly discuss how H-module homomorphisms are assigned to elementary tangles. Vertical strands labeled by V are assigned id_V and id_{V^*}, depending on the orientation of the strands. The crossings of strands labeled by V and by W are assigned maps $c_{V,W} : V \otimes W \to W \otimes V$ satisfying some compatibility so that the braid relations hold. For some Hopf algebras, the flip maps $\tau_{V,W}$ defined by $v \otimes w \to w \otimes v$ are homomorphisms of H-modules, and others Hopf algebras only admit more complicated maps. However, some Hopf algebras might not even admit such H-module maps $c_{V,W}$.

In order to guarantee that such maps $c_{V,W}$ exist, we are interested in braided Hopf algebras (sometimes called quasi-triangular Hopf algebras). The maps $c_{V,W}$ actually arise from the action of a special element called a universal R-matrix, $R \in H \otimes H$ (or in a topological completion of this tensor product), on $V \otimes W$ composed with the flip map $\tau_{V,W}$. In the case that $c_{V,W} = \tau_{V,W}$, then $R = 1 \otimes 1$.

If a link component is labeled by V, then one of the oriented cap diagrams should be associated to a map $V^* \otimes V \to \mathcal{R}$. It is an easy exercise to compute with Sweedler's notation using the Hopf algebra axioms to show that the standard evaluation map

$$f \otimes v \mapsto f(v)$$

defines an H-module homomorphism. However, the axioms do not imply that the map $V \otimes V^* \to \mathcal{R}$ given by evaluation is an H-module homomorphism, and so we will have to require some extra structure.

Similarly, if $\{v_i\}$ and $\{f^j\}$ denote dual bases for V and V^*, the standard coevaluation map $\mathcal{R} \to V \otimes V^*$ defined by

$$1 \mapsto \sum_i v_i \otimes f^i$$

is a H-module homomorphism for any Hopf algebra H while the coevaluation map $\mathcal{R} \to V^* \otimes V$ is not guaranteed to be. Thus, we have a candidate for one of our cup morphisms, but require extra structure for the other.

A ribbon Hopf algebra is a braided Hopf algebra which has further structure which guarantees that morphisms $V \otimes V^* \to \mathcal{R}$ and $\mathcal{R} \to V^* \otimes V$ exist, and that the braiding and duality structures are compatible so as to give rise to an isotopy invariant. This added ribbon structure arises from a ribbon element $v \in H$ whose action on a module W is diagrammatically depicted by a full twist of a ribbon.

If H is a ribbon Hopf algebra, then [RT90] gives the details for how to assign morphisms to all of the elementary tangles. Assigning a link to the composition of such morphisms produces an H-module map $\mathcal{R} \to \mathcal{R}$ assigned to the link, which can be identified with an element of $\mathcal{R}$. This element is invariant under framed isotopies of the link.

Now that we are interested in ribbon Hopf algebras, we should take a moment to observe that there do exist examples of algebras which admit ribbon Hopf algebra structures. Among these algebras are the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ (and dual to these algebras are so-called co-ribbon Hopf algebras, like the rings of coordinate functions $\mathcal{O}(G)$ on Lie groups G). These algebras are attractive to us because of the rich combinatorics of their well-studied representation categories. However, their ribbon Hopf algebra structures are unfortunately too symmetric, meaning that we do not obtain interesting link invariants from these particular algebras.

The issue with $U(\mathfrak{g})$ is that the coproduct $\Delta(x) = 1 \otimes x + x \otimes 1$ is co-commutative, which causes the braiding maps to just be the flip maps (and on the dual side, $\mathcal{O}(G)$ is a commutative algebra, causing the same issue). Our desire to have algebras which behave like $U(\mathfrak{g})$ and $\mathcal{O}(G)$ but which yield nontrivial link invariants motivates 1-parameter quantizations of these algebras which we denote by $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, whose study are central to the theory of quantum groups.

2.2 Examples from classical Lie theory: $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$

In classical Lie theory, the easiest non trivial Lie group is the Lie group SL_2, which is the group of 2-by-2 matrices over $\mathbb{C}$ with determinant 1 and is a 3-dimensional manifold. A study of its representation theory might take place in the following three steps. In studying the representation theory of SL_2, a first realization is that group theory is harder than linear algebra, which motivates a focus on the representation theory of its corresponding Lie algebra $\mathfrak{sl}_2$, which is a 3-dimensional vector space over $\mathbb{C}$ equipped with a non-associative Lie bracket. A second realization is that non-associative algebras are annoying to work with, which motivates the introduction and study of the universal enveloping algebra $U(\mathfrak{sl}_2)$, which is an associative algebra over $\mathbb{C}$ but is infinite dimensional. The final realization is that the representation theory of $U(\mathfrak{sl}_2)$ is pretty easy. Then one works to prove that the representation theories of the three objects SL_2, $\mathfrak{sl}_2$, and $U(\mathfrak{sl}_2)$ are in correspondence with one another.

There is a second story coming from the theory of algebraic groups. The algebraic group SL_2 is the group of 2-by-2 matrices over $\mathbb{C}$ with determinant 1 and is a 3-dimensional variety. In this setting, the natural replacement for the group SL_2 by an algebra is the algebra of coordinate functions on SL_2, which is denoted $\mathcal{O}(SL_2)$. If we let $\mathfrak{X}(\mathcal{O}(SL_2))$ denote the set of algebra homomorphisms $\mathcal{O}(SL_2) \to \mathbb{C}$, we recover the algebraic group $SL_2 \cong \mathfrak{X}(\mathcal{O}(SL_2))$.

Since both of the algebras $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$ encode essential information about the group SL_2, it is natural to ask how these two algebras are related. Before addressing this question, we recall the definitions of these algebras.

Here, we define the algebras $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$ by generators and relations.

2.2.1 $U(\mathfrak{sl}_2)$

Definition 2.4 *The universal enveloping algebra $U(\mathfrak{sl}_2)$ is the quotient of the free algebra generated by x, y, h subject to the following relations:*

$$[h, x] = 2x$$

$$[h, y] = -2y$$

$$[x, y] = h,$$

where $[-, -]$ denotes the commutator bracket.

The Hopf algebra structure maps are given as follows.

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\epsilon(a) = 0$$

$$S(a) = -a$$

for all generators a in $\{x, y, h\}$.

The standard 2-dimensional representation $\rho_V : U(\mathfrak{sl}_2) \to End(V)$ *is given by*

$$\rho_V(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho_V(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho_V(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

2.2.2 $\mathcal{O}(SL_2)$

Definition 2.5 *The algebra of coordinate functions* $\mathcal{O}(SL_2)$ *is the quotient of the free algebra generated by* $X_{11}, X_{12}, X_{21}, X_{22}$ *subject to the following relations*

$$X_{ij}X_{kl} = X_{kl}X_{ij}$$

for all $1 \le i, j, k, l \le 2$, *and the relation*

$$X_{11}X_{22} - X_{12}X_{21} = 1.$$

In other words, $\mathcal{O}(SL_2)$ *is the commutative algebra of 2-by-2 matrix coordinates subject to the relation that the determinant is equal to 1.*

The Hopf algebra structure maps are given as follows.

$$\Delta(X_{ij}) = \sum_{r=1}^{2} X_{ir} \otimes X_{rj}$$

$$\varepsilon(X_{ij}) = \delta_{ij}$$

$$S(\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}) = \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix}.$$

The matrix notation for the definition of the antipode S represents four separate equations giving the values of $S(X_{ij})$. It is easy to remember the definitions of ε, Δ, and S by comparing ε with the 2-by-2 identity matrix, Δ with the matrix product formula, and S with the matrix inverse formula for SL_2.

$\mathcal{O}(SL_2)$ has a standard 2-dimensional comodule, or corepresentation, with basis $\{v_1, v_2\}$. The coaction is a linear map $\Delta_V : V \to V \otimes \mathcal{O}(SL_2)$ defined by

$$v_i \mapsto v_1 \otimes X_{1i} + v_2 \otimes X_{2i}.$$

It is easy to check that this V satisfies the following general definition of a comodule.

Definition 2.6 *If H has a coalgebra strcuture Δ, ϵ then a (right) comodule of H is a free $\mathcal{R}$-module of finite rank W equipped with an $\mathcal{R}$-linear map $\Delta_W : W \to W \otimes H$ satisfying the following:*

$$(id_W \otimes \varepsilon) \circ \Delta_W = id_W$$

$$(id_W \otimes \Delta) \circ \Delta_W = (\Delta_W \otimes id_H) \circ \Delta_W.$$

We now give an answer to the question about the relationship between $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$. There is a Hopf pairing between $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$ in the form of a bilinear map

$$\langle -, - \rangle : \mathcal{O}(SL_2) \otimes U(\mathfrak{sl}_2) \to \mathbb{C}$$

defined on generators by

$$\langle X_{ij}, x \rangle = (\rho_V(x))_{ij},$$

the i, j entry of the image of x under the standard representation. The extension of the definition from pairs of generators to pairs of arbitrary elements can be computed using the following defining properties of a Hopf pairing.

$$\begin{aligned}
\langle XY, x \rangle &= \sum_{(x)} \langle X, x' \rangle \langle Y, x'' \rangle \\
\langle X, xy \rangle &= \sum_{(X)} \langle X', x \rangle \langle X'', y \rangle \\
\langle S(X), x \rangle &= \langle X, S(x) \rangle \\
\langle 1, x \rangle &= \varepsilon(x) \\
\langle X, 1 \rangle &= \varepsilon(X).
\end{aligned}$$

The Hopf pairing relates the left action of $U(\mathfrak{sl}_2)$ on the standard representation to the right coaction of $\mathcal{O}(SL_2)$ by the following

$$x.v = (\mathrm{id} \otimes \langle -, x \rangle)\Delta_V(v).$$

An analogous formula defines a left action of $U(\mathfrak{sl}_2)$ on any right comodule of $\mathcal{O}(SL_2)$.

The Hopf pairing is nondegenerate, meaning that for any nonzero X in $\mathcal{O}(SL_2)$ and for any nonzero x in $U(\mathfrak{sl}_2)$, both of the maps $\langle X, - \rangle$ and $\langle -, x \rangle$ are not identically zero.

For any $U(\mathfrak{sl}_2)$-modules V and W, the tensor flip map $\tau : V \otimes W \to W \otimes V$ defined by $\tau(v \otimes w) = w \otimes v$ is a map which commutes with the action of $U(\mathfrak{sl}_2)$. The analogous fact is true if V and W are $\mathcal{O}(SL_2)$-comodules. Consequently, there is a natural action of the symmetric group S_n on a tensor power $V^{\otimes n}$ of modules or comodules V.

2.3 The quantum groups $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$

The Hopf algebras $U(\mathfrak{sl}_2)$ and $\mathcal{O}(SL_2)$ admit q-deformations $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$. These quantized Hopf algebras have essentially the same representation theory as their classical counterparts, but with the symmetric group action replaced by a nontrivial braid group action. We give their presentations here, and the rest of the preliminary chapter is concerned with obtaining skein theoretic definitions of these quantum groups.

2.3.1 $U_q(\mathfrak{sl}_2)$

Definition 2.7 *The quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is the quotient of the free algebra generated by E, F, K, K^{-1} subject to the following relations:*

$$KE = q^2 EK \qquad KK^{-1} = 1 = K^{-1}K$$

$$KF = q^{-2} FK \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Hopf algebra structure maps are defined on generators as follows:

$$\Delta(K) = K \otimes K \qquad \varepsilon(K) = 1 \qquad S(K) = K^{-1}$$
$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1} \qquad \varepsilon(K^{-1}) = 1 \qquad S(K^{-1}) = K$$
$$\Delta(E) = E \otimes 1 + K \otimes E \qquad \varepsilon(E) = 0 \qquad S(E) = -K^{-1}E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F \qquad \varepsilon(F) = 0 \qquad S(F) = -FK.$$

There is a standard 2-dimensional representation V with $\rho_V : U_q(\mathfrak{sl}_2) \to End(V)$ given by

$$\rho_V(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \qquad \rho_V(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho_V(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have chosen the definition of $U_q(\mathfrak{sl}_2)$ as presented in [BG02], but there are several other conventions common on the literature, usually only involving different conventions for the coproduct and antipode. Our choice of convention for $U_q(\mathfrak{sl}_2)$ also fixes a convention for the definition of $\mathcal{O}_q(SL_2)$ as follows.

2.3.2 $\mathcal{O}_q(SL_2)$

Definition 2.8 *The quantized coordinate ring of regular functions on SL_2, denoted $\mathcal{O}_q(SL_2)$, is the quotient of the free algebra generated by $X_{11}, X_{12}, X_{21}, X_{22}$ subject to the following relations:*

$$X_{11}X_{12} = qX_{12}X_{11} \qquad X_{12}X_{21} = X_{21}X_{12}$$

$$X_{21}X_{22} = qX_{22}X_{21} \qquad X_{11}X_{22} - qX_{12}X_{21} = 1$$

$$X_{11}X_{21} = qX_{21}X_{11} \qquad X_{22}X_{11} - q^{-1}X_{21}X_{12} = 1$$

$$X_{12}X_{22} = qX_{22}X_{12}$$

The Hopf algebra structure maps are given as follows:

$$\Delta(X_{ij}) = \sum_{r=1}^{2} X_{ir} \otimes X_{rj}$$

$$\varepsilon(X_{ij}) = \delta_{ij}$$

$$S(\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}) = \begin{pmatrix} X_{22} & -q^{-1}X_{12} \\ -qX_{21} & X_{11} \end{pmatrix}.$$

The element $X_{11}X_{22} - qX_{12}X_{21} = X_{22}X_{11} - q^{-1}X_{21}X_{12}$ is called the *quantum determinant* and we will sometimes denote it by $\det_q$.

As in the classical case, $\mathcal{O}_q(SL_2)$ has a standard 2-dimensional comodule, or corepresentation, with basis $\{v_1, v_2\}$. The coaction is a linear map $\Delta_V : V \to V \otimes \mathcal{O}_q(SL_2)$ defined by

$$v_i \mapsto v_1 \otimes X_{1i} + v_2 \otimes X_{2i}.$$

We have a Hopf pairing between $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$

$$\langle -, - \rangle : \mathcal{O}_q(SL_2) \otimes U_q(\mathfrak{sl}_2) \to \mathbb{C}$$

defined on generators by

$$\langle X_{ij}, x\rangle = (\rho_V(x))_{ij},$$

the i, j entry of the image of x under the standard representation.

For the standard $U_q(\mathfrak{sl}_2)$-module and $\mathcal{O}_q(SL_2)$-comodule V, there is a map

$$R : V \otimes V \to V \otimes V$$

commuting with the action and coaction, defined by

$$\begin{aligned}
v_1 \otimes v_1 &\mapsto q^{1/2} v_1 \otimes v_1 \\
v_1 \otimes v_2 &\mapsto q^{-1/2}(q - q^{-1}) v_1 \otimes v_2 + q^{-1/2} v_2 \otimes v_1 \\
v_2 \otimes v_1 &\mapsto q^{-1/2} v_1 \otimes v_2 \\
v_2 \otimes v_2 &\mapsto q^{1/2} v_2 \otimes v_2.
\end{aligned}$$

The map R is called an R-matrix and encodes essential data of the quantum groups. For example, the first five defining relations of $\mathcal{O}_q(SL_2)$ can be obtained from the following equations, involving matrix entries of R:

$$\sum_{1\le k,l\le 2} R^{kl}_{ij} X_{km} X_{ln} = \sum_{1\le k,l\le 2} R^{mn}_{kl} X_{ik} X_{jl},$$

which hold for all $1 \le i, j, m, n \le 2$.

It is easy to check that R satisfies the braid equation, or Yang Baxter equation on $V \otimes V \otimes V$:

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R).$$

Consequently, there is an action of the n-strand braid group, B_n, on the tensor power $V^{\otimes n}$ of the standard $U_q(\mathfrak{sl}_2)$-module and standard $\mathcal{O}_q(SL_2)$-comodule.

The Hopf pairing between $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$ can also be defined using matrix entries of R and R^{-1}, and we illustrate this with a diagrammatic construction of the pairing later in this chapter, in Section 2.7.

2.4 Skein theory for $U(\mathfrak{sl}_2)$ intertwiners

Rumer-Teller-Weyl may have been the first in the literature [WRT32] to use planar diagrams to describe the representation category of $U(\mathfrak{sl}_2)$. These diagrams were again used by Temperley and Lieb in [TL71] and came into prominence after the discovery of the Jones polynomial and Kauffman's skein-theoretic definition of it.

Given a diagram of a framed link, we would like to draw the link as a composition of tangles. By assigning each endpoint of each tangle the standard representation V of $U(\mathfrak{sl}_2)$ we can hope to assign each tangle an interwiner between tensor powers of V. A successful assignment will yield a functor from the category of ribbon tangles to the representation category of $U(\mathfrak{sl}_2)$. When the functor is restricted to a framed link, it will assign an intertwiner from the trivial representation to the trivial representation, which can be identified with a number. This number will be a link invariant.

The standard representation V of $U(\mathfrak{sl}_2)$ is a self dual representation, and so the diagrammatics in this section are simplified by allowing us to use unoriented framed tangles.

In this section we will see that the representation category of $U(\mathfrak{sl}_2)$ has the necessary structure to yield such a link invariant, but this invariant will be quite weak in its ability to distinguish links. However, the desire for link invariants of this same flavor, but which are nontrivial, will motivate the appearance of diagrammatic definitions of the quantum groups $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$ later on in this prelimary chapter.

We will read our link diagrams from right to left and make use of some 1-dimensional Morse theory in the plane. We can isotope any diagram in the plane so that it is a composition of cups, caps, crossings, and horizontal strands. We will call an *elementary diagram* a vertical strip that contains only horizontal strands and possibly one of: a single cup, a single cap, or a single crossing. We will use an experimental approach to decide what intertwiner to assign to each elementary diagram. We will assume such an assignment exists and then use its desired properties to find a candidate for the assignment.

A horizontal strand is clearly an identity elementary tangle, so we must assign the identity intertwiner $V \to V$, and by consideration of the monoidal structure, we must assign the identity map $V^{\otimes k} \to V^{\otimes k}$ to k parallel horizontal strands. To a cap we must assign an intertwiner

$$\text{cap} : V \otimes V \to \mathbb{C}.$$

A weight basis of a representation is a basis consisting of eigenvectors for the action of the element $h \in U(\mathfrak{sl}_2)$. An eigenvalue for h is referred to as a weight. We will define our cap map on a weight basis of $V \otimes V$ induced by our weight basis of V. We rename the basis $\{v_1, v_2\}$ of V to $\{v_+, v_-\}$, for the purpose of incorporating the indices into our diagrams without confusing them with other notation. This notation is also convenient since we can think of $+$ and $-$ as representing the fact that our weight basis satisfies $h.v_+ = v_+$ and $h.v_- = -v_-$.

In order for cap : $V \otimes V \to \mathbb{C}$ to commute with the action of $U(\mathfrak{sl}_2)$ it suffices to require it to commute with the action of the generators h, e, and f.

Recall that h acts on $V \otimes V$ by $h \otimes 1 + 1 \otimes h$. Thus, $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$ is a weight basis of $V \otimes V$, with corresponding weights 2, 0, 0, and -2. Also recall that h acts on $\mathbb{C}$ trivially by $h.1 = \varepsilon(h)1 = 0$.

Since an intertwiner must send a weight vector to a weight vector of the same weight or to the zero vector, we can deduce that $\text{cap}(v_+ \otimes v_+)$ and $\text{cap}(v_- \otimes v_-)$ must both be zero.

Next, we consider the action of the generator e. Recall that e acts on $v_- \otimes v_-$ by $e.(v_- \otimes v_-) = e.v_- \otimes v_- + v_- \otimes e.v_- = v_+ \otimes v_- + v_- \otimes v_+$. So from the requirement that cap must commute with the action of e, we see that

$$\begin{aligned} 0 &= e.\text{cap}(v_- \otimes v_-) \\ &= \text{cap}(e.(v_- \otimes v_-)) \\ &= \text{cap}(v_+ \otimes v_- + v_- \otimes v_+). \end{aligned}$$

Thus, we must have that $\text{cap}(v_+ \otimes v_-) = -\text{cap}(v_- \otimes v_+)$. We then set $\text{cap}(v_+ \otimes v_-) = a$, for an unknown complex number a. The reader can verify that the assignment

$$\begin{aligned} v_+ \otimes v_+ &\mapsto 0 & \qquad v_- \otimes v_+ &\mapsto -a \\ v_- \otimes v_- &\mapsto 0 & \qquad v_+ \otimes v_- &\mapsto a \end{aligned}$$

defines an $U(\mathfrak{sl}_2)$-intertwiner $\text{cap} : V \otimes V \to \mathbb{C}$ for any choice of $a \in \mathbb{C}$. Thus, there is a 1-dimensional space of such intertwiners. Further skein-theoretic considerations will motivate a particular choice of a later on.

The matrix entries of the cap map can be represented diagrammatically by using *stated* diagrams, which are diagrams that have endpoints on a boundary interval labeled by *states* that represent weight vectors. At first, we will consider our boundary interval as just a vertical line on the right side of our diagram. In later sections we will consider multiple boundary intervals and make use of orientations on these intervals.

Whereas a skein relation involving unstated diagrams encodes a relationship between the intertwiners the diagrams represent, a skein relation involving stated diagrams encodes a relationship between the matrix entries of the intertwiners. We represent the matrix entries of our cap map diagrammatically by the following

Note that the endpoints along the boundary are read from top to bottom. This convention is chosen to agree with an algebra structure introduced in the following sections.

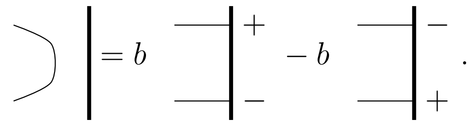
Using similar considerations, we can compute that our cup map

$$\text{cup} : \mathbb{C} \to V \otimes V$$

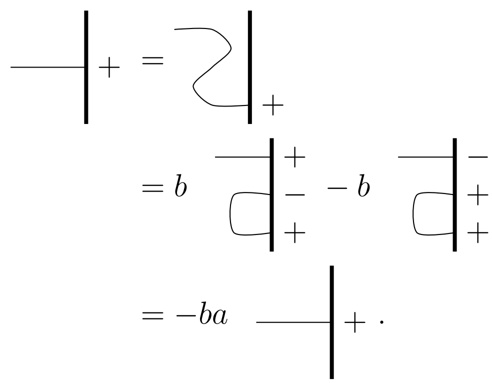
must be of the form

$$1 \mapsto bv_+ \otimes v_- - bv_- \otimes v_+.$$

We represent the definition of our cup map diagrammatically by

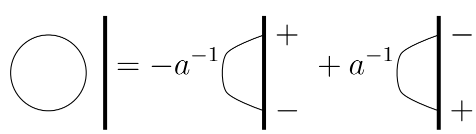


By planar isotopy considerations, and using our diagrammatic definitions of the cup and cap maps, we can compute that



Thus, we are forced to set $b = -a^{-1}$, and in particular a must be chosen so that it is invertible.

We can now compute the value of the unknot



$$= -a^{-1}a + a^{-1}(-a)$$

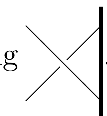
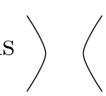
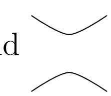
$$= -2.$$

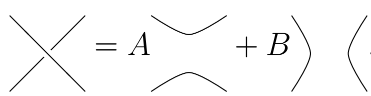
We note that, in our setting, we have observed that the value of the unknot must be

-2, regardless of the choice of the value of a.

Now that we know the definition of cups and caps, we wish to find the definition of a crossing. We are looking for an intertwiner which is an endomorphism of $V^{\otimes 2}$ which satisfies the braid relations. In the literature, such a map is called an R-matrix and we will denote our map

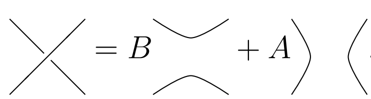
$$R : V \otimes V \to V \otimes V.$$

Diagrammatically, we will draw R as a crossing . From the representation theory of $U(\mathfrak{sl}_2)$, we know that the dimension of the space of intertwiners $\mathrm{Hom}_{U(\mathfrak{sl}_2)}(V \otimes V, V \otimes V)$ is equal to 2. One way to see this is to recall that $V \otimes V$ decomposes as a direct sum of two irreducible representations: one with basis $\{v_+ \otimes v_- - v_- \otimes v_+\}$ (called the exterior power $\bigwedge^2(V)$), and one with basis $\{v_+ \otimes v_+, v_+ \otimes v_- + v_- \otimes v_+, v_- \otimes v_-\}$, (called the symmetric power $\mathrm{Sym}^2(V)$). After this observation, we can apply Schur's lemma. Furthermore, we can compute that the intertwiners that we assign to the two diagrams  and  are linearly independent. These intertwiners thus span this space and so we expect to be able to write our diagram for R as some linear combination of these diagrams:

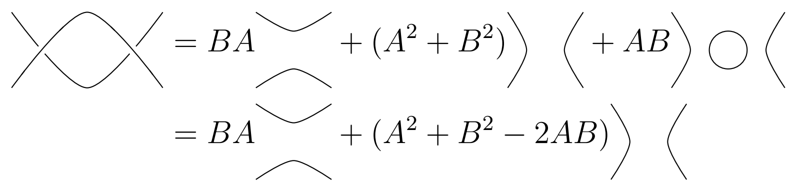


(Even if we didn't know these representation theoretic facts, we could still attempt to write R as such a linear combination.)

Since the inverse of our crossing is a rotation of R, we have also the relation



Now since R must satisfy the second Reidemeister move, we can compute that



and then deduce that $AB = 1$ and $A^2 + B^2 - 2AB = 0$. Substituting $B = A^{-1}$ into the second equation yields $A^2 + A^{-2} - 2 = 0$ which has only $A = \pm 1$ as solutions. We observe that setting $A = 1$ yields the definition of $R : V \otimes V \to V \otimes V$ by

$$R(v \otimes w) = w \otimes v,$$

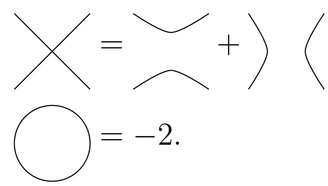
which is the same as the standard tensor flip map $\tau : V \otimes V \to V \otimes V$. It is easy to see algebraically that this definition of R satisfies Reidemeister III:

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R).$$

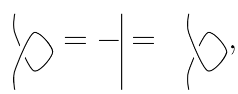
The equation above is called the Yang-Baxter equation.

It is also a diagrammatic observation, due to Kauffman, that Reidemeister II implies Reidemeister III when the skein relations are of the particular form that we have been working with. Diagrammatically, Reidemeister III tells us that we can pass a strand over or under a crossing while Reidemeister II tells us that we can pass portions of strands over or under each other. Since our crossing skein relation locally rewrites a crossing as a linear combination of diagrams without crossings, Reidemeister II implies that we can pass a strand over or under this linear combination.

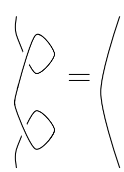
We note that both possible choices of A yield the relation $R = R^{-1}$, which will cause our $U(\mathfrak{sl}_2)$ link invariant to fail to distinguish overcrossings from undercrossings, making it quite weak. This failure will be remedied in the next sections. Since our classical skein theory does not distinguish between an overcrossing and an undercrossing, we can draw our crossing as an intersection of strands. We record our skein relations here.



A consecutive of these relations are the relations



so ribbon Reidemeister I



holds.

We have thus found skein relations that give us invariants of framed links and also describe relations among intertwiners when the diagrams are viewed as their images under a functor from the category of unoriented ribbons to the category of $U(\mathfrak{sl}_2)$ representations.

We stress that we did not carefully check in this section that the functor is actually well-defined. We merely supposed such a functor exists and have found what it should look like. That the functor is well-defined can be seen from the fact that the counit for the algebra $\mathcal{S}^{SL_2}(\mathfrak{B})$ defined later in this section is well-defined. The reader can consult

[RT90] and [CL19] for more details and discussion.

The functor is actually an isomorphism from the category of framed unoriented tangles modulo the skein relations to the full subcategory of $U(\mathfrak{sl}_2)$ having as objects tensor powers of V. The reader can consult [Kup96] for a proof of this fact. The proof there uses Schur-Weyl duality to prove the surjectivity of the functor and a dimension counting argument to prove the injectivity. We will provide a new proof of a similar fact for the case of SL_3 in this thesis.

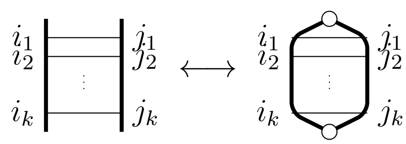
2.5 Skein theoretic definition of $\mathcal{O}(SL_2)$

In the previous section, we obtained skein-theoretic ways to represent $U(\mathfrak{sl}_2)$ intertwiners. In this section, we aim to obtain skein-theoretic ways to represent elements of $\mathcal{O}(SL_2)$. We first give some motivation and explain why we expect the skein theory from the previous section should serve us well in this section.

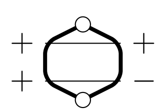
Elements of $\mathcal{O}(SL_2)$ represent elements of the dual space $U(\mathfrak{sl}_2)^*$ by way of the Hopf pairing. Suppose that x is an element of $U(\mathfrak{sl}_2)$ and $X_{i_1j_1}\cdots X_{i_kj_k}$ is a monomial in generators of $\mathcal{O}(SL_2)$. Then $\langle X_{i_1j_1}\cdots X_{i_kj_k}, x\rangle \in \mathbb{C}$ can be computed from the action of x on $V^{\otimes k}$ by taking the coefficient of $v_{i_1}\otimes\cdots\otimes v_{i_k}$ when $x.(v_{j_1}\otimes\cdots\otimes v_{j_k})$ is written in the standard tensor basis.

Similarly, given an intertwiner $f: V^{\otimes k}\to V^{\otimes l}$, a basis element $v = v_{i_1}\otimes\cdots\otimes v_{i_k}$ of the domain and a basis element $w = v_{j_1}\otimes\cdots\otimes v_{j_l}$ of the codomain, the triple ${}_wf_v$ represents an element of $U(\mathfrak{sl}_2)^*$ by defining ${}_wf_v(x)$ to be the coefficient of w in $f(x.v)$. In the special case that $f = \mathrm{id}^{\otimes k}: V^{\otimes k}\to V^{\otimes k}$, then ${}_wf_v$ and $X_{i_1j_1}\cdots X_{i_kj_k}$ represent the same elements of $U(\mathfrak{sl}_2)^*$. Thus, we expect to be able to represent $\mathcal{O}(SL_2)$ from the skein theory of $U(\mathfrak{sl}_2)$ intertwiners and we expect that the skein relations among intertwiners yield relations in $\mathcal{O}(SL_2)$. The reader might see section I.7 of [BG02] for a discussion of algebraically constructing $\mathcal{O}(SL_2)$ from $U(\mathfrak{sl}_2)$ intertwiners.

In this section we will use diagrams that incorporate two boundary arcs instead of one. The rightmost boundary arc will have states representing basis vectors of the domain while the left boundary arc represents the codomain. As we have just discussed, we expect to be able to represent the monomial $X_{i_1j_1}\cdots X_{i_kj_k}$ by a diagram consisting of k parallel horizontal strands with k endpoints on the right boundary arc, labeled by states associated to the indices $j_1,\ldots,j_k$, and k endpoints on the left boundary arc, labeled by states associated to the indices $i_1,\ldots,i_k$. It will be advantageous for us to represent this diagram by drawing it in a bigon $\mathfrak{B}$ which is a disc with two points removed from its boundary, and which carries a distinguished right boundary arc and left boundary arc.



As a reminder, we identify the indices 1 and 2 of X_{ij} with the states $+$ and $-$. For example, the monomial $X_{11}X_{12}$ is represented diagrammatically by

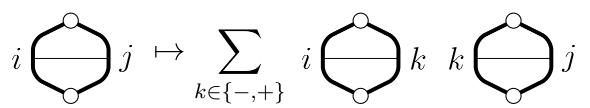


.

We will take the module spanned by planar isotopy classes of stated diagrams in the bigon modulo the classical SL_2 skein relations along with relations along the boundary called stated skein relations, which at the moment still depend on the parameter a that we will soon fix. We give this module a natural product structure given by stacking diagrams vertically in the bigon. So this algebra is an associative unital $\mathbb{C}$-algebra with unit the empty diagram. We will denote this algebra by $\mathcal{S}^{SL_2}(\mathfrak{B})$. Our aim is to show that this algebra is isomorphic to $\mathcal{O}(SL_2)$ as a Hopf algebra.

We begin by finding skein theoretic definitions of the Hopf algebra structure maps so that they line up with the algebraically defined ones associated with $\mathcal{O}(SL_2)$. We begin with the coproduct.

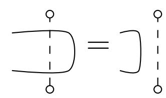
The coproduct $\Delta : \mathcal{O}(SL_2) \to \mathcal{O}(SL_2) \otimes \mathcal{O}(SL_2)$ defined on generators by $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$ motivates us to find a skein theoretic map $\Delta : \mathcal{S}^{SL_2}(\mathfrak{B}) \to \mathcal{S}^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}^{SL_2}(\mathfrak{B})$ which restricts to the assignment



2.5.1 The coproduct

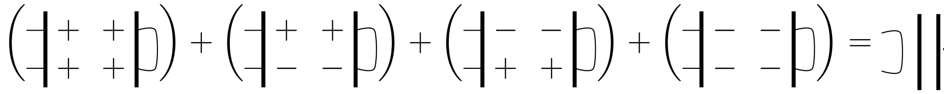
We define our skein theoretic definition of the coproduct on a spanning set consisting of isotopy classes of stated diagrams. Given a stated diagram, we pick an embedded arc from the bottom puncture to the top puncture so that it intersects the strands in the diagram transversely and does not intersect a crossing. We split the bigon into two bigons as above, with the diagram being split as well. The ideal arc becomes two new boundary arcs and we sum over states, so that states on both new endpoints of the same strand match.

In order for this diagrammatic map to be well defined, we must be sure that it respects isotopies of the diagram, isotopies of the ideal arc, and the skein relations. It suffices to check that a cup and cap can slide past the ideal arc

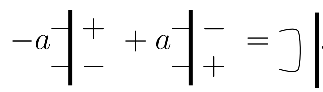


and also its mirror reflection.

So we need the following local relations to be satisfied, and also their mirror reflections:



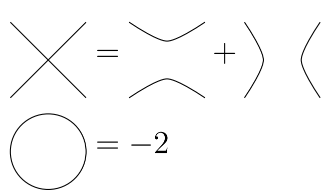
Using the cap relations, this simplifies to



By comparing to our previously deduced cup relation, we see that we must have $a = a^{-1}$ and so $a = \pm 1$ are the only two solutions. The mirror reflection gives the same solution. We will set $a = 1$. The other choice of a would be equivalent to swapping the states $-$ and $+$. Now that we have fixed a, we record the definition of $\mathcal{S}^{SL_2}(\mathfrak{B})$.

Definition 2.9 *The classical stated skein algebra of the bigon, $\mathcal{S}^{SL_2}(\mathfrak{B})$, is the module spanned by planar isotopy classes of stated diagrams in $\mathfrak{B}$ modulo the following skein relations:*

Interior relations:



Boundary relations:

The product structure is given by stacking diagrams vertically.

Now that we have checked that Δ is well-defined, we observe that it is an algebra map. It sends the empty diagram of $\mathcal{S}^{SL_2}(\mathfrak{B})$ to the empty diagram of $\mathcal{S}^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}^{SL_2}(\mathfrak{B})$ and it is easy to see that it commutes with the product structure. The coassociativity of Δ is also clear from the diagrammatic definition.

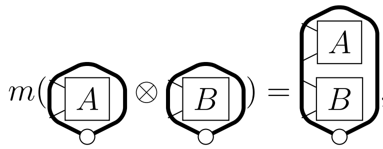
2.5.2 The counit

We next will consider the stated skein algebra of the monogon, $\mathcal{S}^{SL_2}(\mathfrak{M})$, and show that it is 1-dimensional. The motivation for looking at the monogon is that the counit $\varepsilon : \mathcal{S}^{SL_2}(\mathfrak{B}) \to \mathcal{S}^{SL_2}(\mathfrak{M})$ will be realized as the composition of a state inversion map inv_{e_r} and filling in the top puncture.

The monogon is the closed disk with one puncture removed from its boundary circle. It is important that our monogon has a puncture on its boundary. If we considered a disk with no punctures on its boundary the relations will be inconsistent as we could slide an endpoint all the way around the boundary.

Definition 2.10 *The stated skein algebra of the monogon $\mathcal{S}^{SL_2}(\mathfrak{M})$ is the module spanned by planar isotopy classes of stated diagrams in $\mathfrak{M}$ subject to the same local skein relations we used for $\mathcal{S}^{SL_2}(\mathfrak{B})$.*

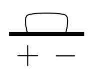
The product is given by stacking along the boundary as follows.



where the strands drawn here can represent any number of strands.

To show that the stated skein algebra of the monogon is 1-dimensional we give an evaluation algorithm, like the one used by Khovanov and Frenkel in [FK97].

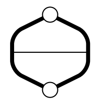
Given a linear combinations of diagrams in the monogon, our algorithm is as follows. We use the crossing relation to write this linear combination as a linear combination of diagrams without crossings. Next, we evaluate each diagram according to the following rules. If the diagram contains an arc with the same state on each endpoint, the diagram evaluates to zero. Otherwise, the diagram evaluates to $(-2)^{\#\text{closed curves}}(-1)^{\#\text{negative arcs}}$ where a negative arc is an arc of the form



$= -1.$

We can check that this algorithm respects the defining relations and thus defines a well-defined linear map from the skein algebra of the monogon to $\mathbb{C}$, which is nonzero since it sends the empty diagram to 1. In the stated skein algebra of the monogon, any diagram is equivalent to a scalar times the empty diagram, and this scalar is unique. Thus, each diagram may be identified with a scalar and we have $\mathcal{S}^{SL_2}(\mathfrak{M}) \cong \mathbb{C}$.

To define the counit, we need a map from the bigon to the monogon. The most natural starting point is to fill in one of the punctures. We decide to fill in the top puncture. This would give us a well-defined algebra map from $\mathcal{S}^{SL_2}(\mathfrak{B})$ to $\mathcal{S}^{SL_2}(\mathfrak{M})$, but we also need the map to satisfy the counit axiom. For example, we will want the counit to satisfy

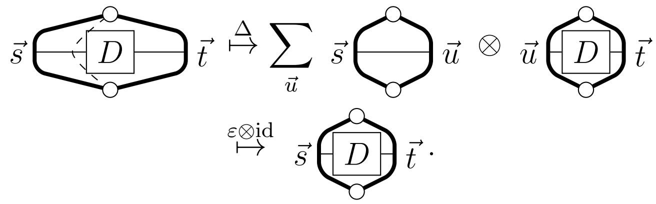
$\varepsilon($ i  j $) = \delta_{ij}.$

After filling in the top puncture and comparing to our boundary skein relations, we are motivated to flip the states along one of the edges, say the right edge e_r and also multiply by a correction factor.

So we introduce a linear map $\mathrm{inv}_{e_r} : \mathcal{S}^{SL_2}(\mathfrak{B}) \to \mathcal{S}^{SL_2}(\mathfrak{B})$ defined on a diagram by multiplying the diagram by (-1) raised to the number of negative states on e_r and changing the signs of all states on e_r to their opposites. We can check that inv_{e_r} respects the skein relations and so can be extended linearly to a well defined map.

We define the counit $\varepsilon : \mathcal{S}^{SL_2}(\mathfrak{B}) \to \mathcal{S}^{SL_2}(\mathfrak{M})$ on diagrams as the composition of inv_{e_r} followed by filling in the top puncture. The counit ε is an algebra map since it respects the product of diagrams.

To see that the counit satisfies the properties $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$, we can check this property on diagrams by using the isotopy invariance of the splitting map Δ as well as the fact that we know that $\varepsilon(\alpha_{ij}) = \delta_{ij}$. For example, to check that $(\varepsilon \otimes \mathrm{id})\Delta(D) = D$ for a stated diagram D, we choose to split the bigon along an ideal arc close to the left boundary arc of the bigon. In the following computation, we denote sequences of states by vectors like $\vec{s}$ and $\vec{t}$ and any strand drawn can represent any number of parallel strands.

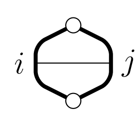


The other counit identity can be checked by choosing to split the bigon along an ideal arc close to the right boundary arc of the bigon.

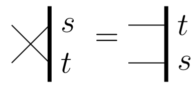
When diagrams are viewed as intertwiners, then we can view the coproduct as the matrix product formula and we can view the counit as extracting the matrix entry of the intertwiner. Thus, we are able to show that the $U(\mathfrak{sl}_2)$ tangle invariant and Reshetikhin-Turaev functor are well defined.

2.5.3 A presentation for $\mathcal{S}^{SL_2}(\mathfrak{B})$

The bialgebra structure for $\mathcal{S}^{SL_2}(\mathfrak{B})$ allows us to easily extract some generators and relations, which will allow us to see that it has the same presentation as $\mathcal{O}(SL_2)$. We first see that the set of stated horizontal strands $\{\alpha_{ij}\}_{i,j\in\{+,-\}}$ is a generating set by splitting any stated diagram D near its right boundary arc using Δ and then applying $(\varepsilon \otimes \mathrm{id})$ to the resulting sum.

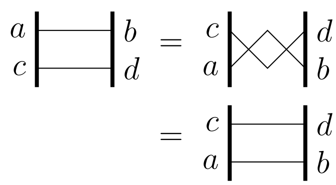
$\alpha_{ij} =$ 

We will observe that these generators commute with each other. First, we observe that the following relations hold along the boundary

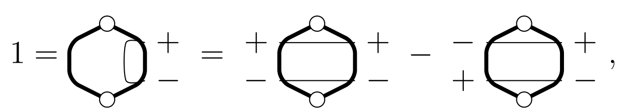


These hold essentially by construction, since we have chosen the crossing to represent the tensor flip map $\tau : V \otimes V \to V \otimes V$, but it is also easy to use the skein relations to see that they hold.

We then observe, using Reidemester II the following computation, showing that the generators α_{ab} and α_{cd} commute with each other for any a, b, c, d in $\{+, -\}$.



Next, we use our cap and cup relations to observe that



and so $\alpha_{++}\alpha_{--} - \alpha_{+-}\alpha_{-+} = 1$.

We thus have a well-defined surjective algebra map $T : \mathcal{O}(SL_2) \to \mathcal{S}^{SL_2}(\mathfrak{B})$ defined on generators by $T(X_{11}) = \alpha_{++}$, $T(X_{12}) = \alpha_{+-}$, $T(X_{21}) = \alpha_{+-}$, and $T(X_{22}) = \alpha_{--}$.

One way to see that T is injective is to see that it takes a basis of $\mathcal{O}(SL_2)$ into a basis for $\mathcal{S}^{SL_2}(\mathfrak{B})$, as in [Lê18, CL19]. Although bases for stated skein algebras will play important roles in this thesis, here we construct an inverse of T without appealing to bases.

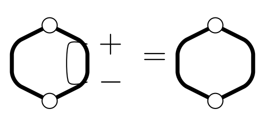
We will construct a candidate for T^{-1} and denote it as T' for now. We will define T' on diagrams and check that T' respects the skein relations. In the last section we defined an evaluation algorithm for stated diagrams in the monogon, which assigned an element of $\mathbb{C}$ to each diagram. Here, we will use an algorithm to assign an element of $\mathcal{O}(SL_2)$ to each stated diagram in the bigon.

Let D be a stated diagram in the bigon. We give an algorithm to write D as a linear combination of monomials in the elements α_{ij} as follows. We use Δ to split D close to its right boundary e_r and then apply $(\varepsilon \otimes \mathrm{id})\Delta(D)$. Once we have written D as a linear combination of monomials in the α_{ij} we obtain $T'(D)$ by defining $T'(\alpha_{ij}) = X_{ij}$.

Next we check that T' respects the skein relations. If a skein relation is applied in the interior of the bigon or along the left boundary e_l, then its application commutes with the application of the splitting map Δ along the right boundary arc of the bigon. Since the counit is well-defined with respect to all skein relations, we see that T' respects these relations. If a skein relation is applied along the right boundary e_r, we are required to check that the algorithm produces equivalent elements of $\mathcal{O}(SL_2)$.

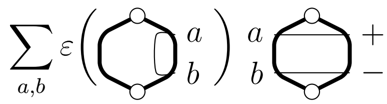
It suffices to check this locally, so we just have to check the following scenarios.

To check one of the cap relations we check that if we apply the algorithm to both sides of the relation,



we obtain equivalent elements in $\mathcal{O}(SL_2)$.

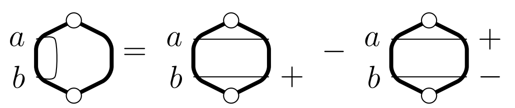
The algorithm will send the right side to the unit $1 \in \mathcal{O}(SL_2)$. Applying the algorithm to the left side of the equation yields



$$= X_{11}X_{22} - X_{21}X_{12},$$

which is equal to 1 in $\mathcal{O}(SL_2)$, as required. The other computations involving the cap relations are similar.

For the cup relation, it suffices to check that the algorithm yields equivalent elements for both sides of the following relation for any states a, b.



In the case when $a = +$ and $b = +$ we have that applying T' to the left side yields $0 \in \mathcal{O}(SL_2)$ while applying T' to the right side yields $X_{12}X_{11} - X_{11}X_{12}$, which is equal to 0 since these generators commute. The other cases are similar. Thus, T' respects the skein relations and is a well-defined algebra map. Since it satisfies $T'(T(X_{ij})) = X_{ij}$ and $T(T'(\alpha_{ij})) = \alpha_{ij}$ for generating sets, we have that $T' = T^{-1}$ and have proved the following.

Theorem 2.11 *The map T defines a bialgebra isomorphism $\mathcal{S}^{SL_2}(\mathfrak{B}) \cong \mathcal{O}(SL_2)$.*

We remark that these cup and cap relations describe the antipode equations $\eta\varepsilon = m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta$ applied to the generators α_{ij}. We next describe this antipode.

2.5.4 The antipode

Thus far, we have seen that $\mathcal{S}^{SL_2}(\mathfrak{B})$ is isomorphic to $\mathcal{O}(SL_2)$ as bialgebras. Since $\mathcal{O}(SL_2)$ is a Hopf algebra, a standard exercise shows that our bialgebra isomorphism must be a Hopf algebra isomorphism as well. So $\mathcal{S}^{SL_2}(\mathfrak{B})$ is a Hopf algebra and we can deduce that the antipode S takes values on the generator α_{ij} from the values of the antipode of $\mathcal{O}(SL_2)$ on the generators X_{ij}. So we have

$$S(\alpha_{++}) = \alpha_{--} \qquad S(\alpha_{+-}) = -\alpha_{+-}$$
$$S(\alpha_{-+}) = -\alpha_{-+} \qquad S(\alpha_{--}) = \alpha_{++}.$$

We do not have to appeal to the algebra presentation of $\mathcal{S}^{SL_2}(\mathfrak{B})$ by generators and relations in order to define the antipode. Instead, we can give a diagrammatic definition by defining S on an arbitrary diagram.

Suppose D is a stated diagram in the bigon $\mathfrak{B}$. We define

$$S(D) = (\sqrt{-1})^{T(D)} D',$$

where $T(D)$ is the sum of states on the right boundary e_r minus the sum of states on the left boundary e_l of D. Here, D' is the diagram obtained by flipping every state to its negative and reflecting the diagram across the line between the bigon's bottom and top puncture.

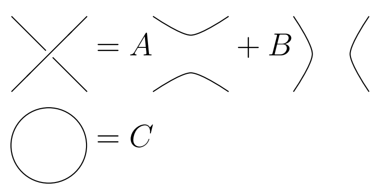
We can check that this diagrammatic definition of S respects the interior skein relations and the boundary skein relations and thus defines a well-defined linear map $S : \mathcal{S}^{SL_2}(\mathfrak{B}) \to \mathcal{S}^{SL_2}(\mathfrak{B})$. We can also observe that S respects the product of diagrams and so is an algebra homomorphism. Recall that the antipode is in general an algebra anti-homomorphism, but since $\mathcal{S}^{SL_2}(\mathfrak{B})$ is a commutative algebra our antipode is also an algebra homomorphism. Since this diagrammatic definition of S agrees on the generators

α_{ij} with the algebraic definition of S given previously, we see that these are the same maps.

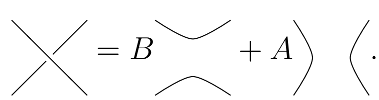
2.6 Skein theoretic definition of $\mathcal{O}_q(SL_2)$

We previously remarked that the skein theory for classical $U(\mathfrak{sl}_2)$ is unable to distinguish overcrossings from undercrossings. We can fix that by finding a deformation of the crossing relation so that it is no longer symmetric. This is how the Kauffman bracket is constructed. We expect that deforming the skein relations will deform the skein algebra as well, so that we expect to find $\mathcal{O}_q(SL_2)$ as the skein algebra of the bigon when we use the q-deformed skein relations.

We set our skein relations equal to



for unknown A, B, C. The following is a standard construction of the Kauffman bracket. We first assume that the crossing relation is preserved by rotation, meaning that we have



By imposing the Reidemeister II equation, we deduce that $B = A^{-1}$, and $C = -A^2 - A^{-2}$ are the unique solutions for B and C. As a consequence Reidemeister III is preserved and so is ribbon Reidemeister I since the following relations hold

$-A^{-3}$  $= \Big| = -A^{3}$  .

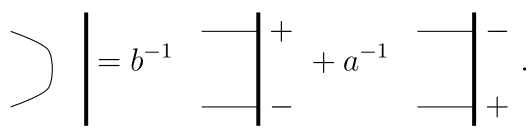
Thus, our skein relations are consistent with isotopies of framed strands. In this section, we set $A = q^{1/2}$ in anticipation of lining up our skein theory with one of the standard definitions of the quantum groups $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(SL_2)$ found in the literature, such as in [BG02].

2.6.1 Boundary arcs require orientations

In this section we find boundary relations involving fractional powers of q which are consistent with the skein relations. We find that we must use more of a 3-dimensional perspective when viewing our diagrams, and this motivates the extra notation of arrows along our boundary arcs.

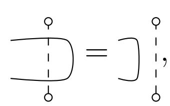
We would like our deformation to break the symmetry of the crossing relation but keep the other features of our skein algebra largely intact. One property we would like to preserve is that our new intertwiners should still be weight preserving (even though we haven't fixed the meaning of intertwiners and weights in our quantized setting). Diagrammatically, this means that we should still use $-$ and $+$ as states, and our boundary stated skein relations should preserve the sums of the states. Thus, in finding our new boundary relations a consideration of weights suggests we should start off with the following cap relations

A consideration of planar isotopy suggests our cup relation would need to take the form

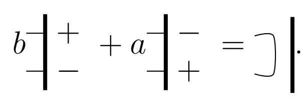


(2.1)

Another property that we would like to preserve from the classical setting is that our coproduct should represent the matrix product formula. So we would like the definition of our coproduct to be unchanged from the classical setting. We now impose the restriction that the coproduct be well-defined. If we want our map to respect the following isotopy



we see we must have the following relation hold:



(2.2)

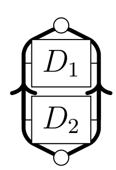
Comparing (2.2) to (2.1), we see that it is impossible to assign values to a and b

unless $a, b \in \{-1, 1\}$ and consequently $q = 1$, which corresponds to our classical skein theory.

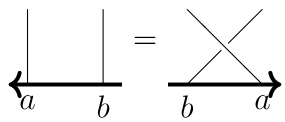
To remedy this issue, we take a more 3-dimensional perspective. Just as the quantization of the crossing relation requires our diagrams to keep track of the heights of strands at a double-point of a crossing, we will also need to keep track of the heights of our endpoints along the boundary. We will now consider a diagram in the bigon as a generic projection of a skein living in $\mathfrak{B} \times (-1, 1)$ such that for each boundary arc e, the endpoints of the skein in $e \times (-1, 1)$ are attached at distinct heights in $(-1, 1)$.

The idea to consider states at different heights seems to first appear in [BW11] and was introduced as a necessary step in defining the quantum trace map by way of an intermediate state-sum. In [Lê18], the state-sum is called the splitting map and its definition motivates the stated skein relations that we will recover in this section.

Diagrammatically, we will record the height order of strands along a boundary arc by giving the arc an orientation so that endpoints are arranged in order of increasing height. In the classical case, we defined the product $D_1 \cdot D_2$ of two stated diagrams by stacking D_1 above D_2 in the bigon and this definition extends the the quantized case as well, if we choose our boundary arcs to be oriented from bottom to top. So the endpoints of D_1 are placed at higher heights than the endpoints of D_2 in $D_1 \cdot D_2$.

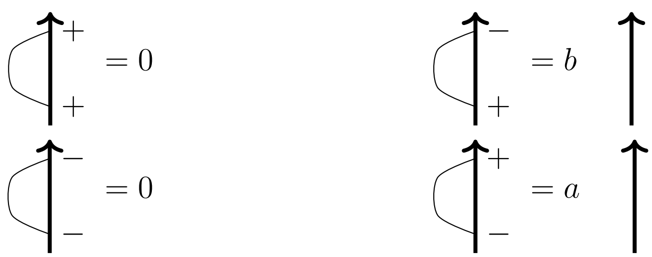


We impose extra skein relations along the boundary representing a 3-dimensional isotopy that slides endpoints of strands horizontally without changing the height order.

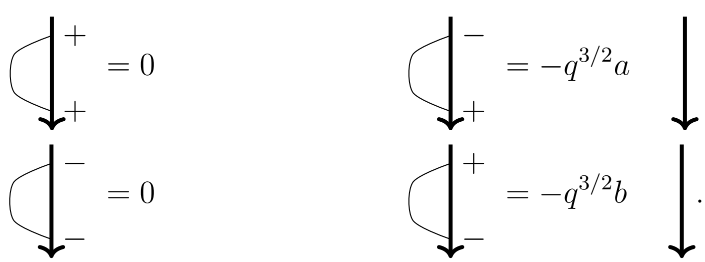


To define the coproduct on a diagram D in the bigon $\mathfrak{B}$, we choose an ideal arc traveling from the bottom puncture of $\mathfrak{B}$ to the top puncture of $\mathfrak{B}$ and split the diagram along this arc and sum over admissible states, analogous to the definition of the coproduct for $\mathcal{S}^{SL_2}(\mathfrak{B})$.

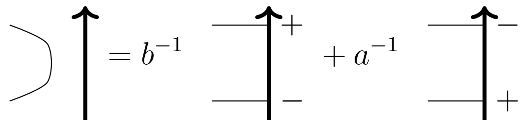
We now search for boundary skein relations which make the coproduct well-defined. Using our new notation involving oriented boundaries, if we temporarily set our cap relations to



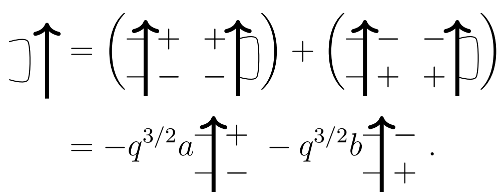
for unknown a and b, then we consequently must have



By consideration of planar isotopy, our cup relation is of the form



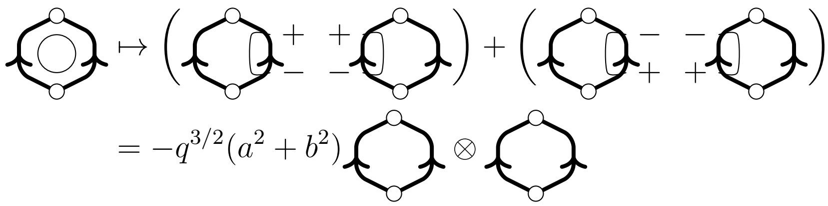
By using the definition of our coproduct, we also can compute a second expression for our cup relation:



The algebraic assumption that $v_- \otimes v_+$ and $v_+ \otimes v_-$ are linearly independent allows us to equate coefficients in both expressions of the cup relation to see that we must have

$$ab = -q^{-3/2}. \tag{2.3}$$

Next, consider a diagram consisting of just a circle bounding a disk inside the bigon. Consider the application of the coproduct by splitting along an arc that cuts through the circle. On the one hand, we could use the circle relation before applying the coproduct, and end up with $-q - q^{-1}$ times the empty diagram. On the other hand, we could apply the coproduct first and obtain the following:



Thus, we obtain a second constraint on a, b of the form

$$a^2 + b^2 = q^{-1/2} + q^{-5/2}. \tag{2.4}$$

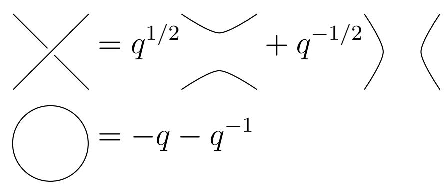
Solving the two equations (2.3) and (2.4), we have narrowed down our search for a and b to just four possibilities: $(a, b) = (\pm q^{-5/4}, \mp q^{-1/4})$ or $(a, b) = (\pm q^{-1/4}, \mp q^{-5/4})$. The fact that there are just four solutions and that each of the solutions yield a well-defined splitting map was originally noted in [Lê18]. Here, we make the choice $a = q^{-5/4}$ and $b = -q^{-1/4}$.

2.6.2 Skein relations for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$

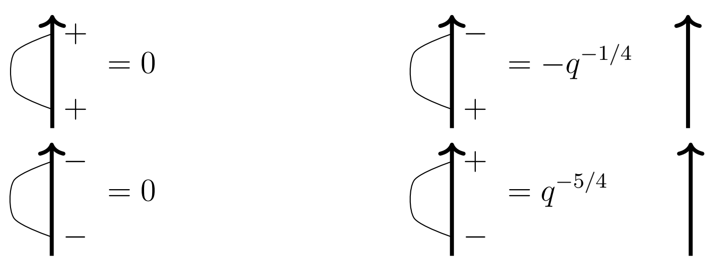
We fix a nonzero element $q^{1/4}$ in $\mathbb{C}$.

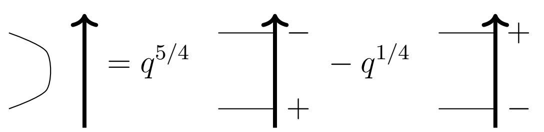
Definition 2.12 *The stated skein algebra of the bigon $\mathcal{S}_q^{SL_2}(\mathfrak{B})$ is the quotient of the free module spanned by isotopy classes of stated skein diagrams in the bigon subject to the following set of skein relations.*

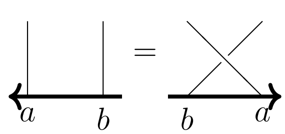
Interior relations:



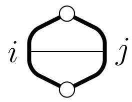
Boundary relations:







We note that by using the skein relations on a diagram, we can get rid of crossings, circles, and arcs which return to the same boundary component, showing that any diagram can be rewritten as a linear combination of polynomials in the generators

$\alpha_{ij} =$ 

as in the classical case.

2.6.3 The coproduct for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$

The coproduct $\Delta : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \to \mathcal{S}_q^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B})$ has the same definition as for $\mathcal{S}^{SL_2}(\mathfrak{B})$, defined on a diagram by splitting the diagram down the middle and summing over admissible states. In particular, the coproduct satisfies

$$\Delta(\alpha_{ij}) = \alpha_{i1} \otimes \alpha_{1j} + \alpha_{i2} \otimes \alpha_{2j}.$$

2.6.4 The counit for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$

As in the classical case, we define the counit $\varepsilon : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \to \mathcal{S}_q^{SL_2}(\mathfrak{M})$ diagrammatically by defining it as the composition of an edge inversion map inv_{e_r} applied to the right edge of $\mathfrak{B}$ followed by the map associated with filling in the top puncture.

We will describe the definition of the edge inversion map inv_{e_r} in the quantized case. In the classical case, the analogue of this map only involved multiplying a diagram by a factor determined by the states on e_r. In the quantized case, we will also need to reverse the height order on e_r. Let D be a stated diagram in $\mathfrak{B}$ so that the right boundary edge of $\mathfrak{B}$, e_r, has height order given by orienting the edge from the bottom to the top. Let $p(e_r)$ be the number of positive states of D on e_r and let $n(e_r)$ be the number of negative states of D on e_r. We define

$$\mathrm{inv}(e_r)(D) = (q^{-5/4})^{p(e_r)}(-q^{-1/4})^{n(e_r)}D'',$$

where D'' is the diagram obtained by switching all states on e_r to their opposite signs and reversing the height order on e_r. This map respects the skein relations and so extends to a

$\mathbb{C}$-linear map $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \to \mathcal{S}_q^{SL_2}(\mathfrak{B})$, although inv_{e_r} does not respect the algebra structure by itself.

To finish the definition of ε, after applying inv_{e_r} to D we fill in the top puncture of $\mathfrak{B}$ so that the two boundary edges e_l and e_r become the single boundary edge of $\mathfrak{M}$ such that each endpoint coming from e_l appears lower than the lowest endpoint coming from e_r. This composition does respect the algebra structure and so we obtain an algebra map $\varepsilon : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \to \mathcal{S}_q^{SL_2}(\mathfrak{M})$.

We next observe that $\mathcal{S}_q^{SL_2}(\mathfrak{M}) \cong \mathbb{C}$. Analogous to the classical case, this fact can be proven by using an evaluation algorithm which respects the skein relations. Suppose D is a stated diagram in the monogon. We first slide the endpoints of D so that the boundary edge is oriented counterclockwise. We then use the crossing relation to write the diagram as a linear combination of diagrams with no crossings. Next, we evaluate each diagram in the linear combination according to the following rules. If the diagram contains an arc with the same state on each endpoint, the diagram evaluates to zero. Otherwise, the diagram evaluates to

$$(-q - q^{-1})^{\#\text{ closed curves}}(q^{-5/4})^{\#\text{ positive arcs}}(-q^{-1/4})^{\#\text{ negative arcs}}.$$

We note that this evaluation algorithm recovers the analogous one from the classical case when $q^{1/4}$ is set to $q^{1/4} = 1$.

Our counit satisfies $\varepsilon(\alpha_{ij}) = \delta_{ij}$, which can be verified by direct computation. Thus, we have described the bialgebra structure for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

2.6.5 The antipode for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$

We next describe the antipode $S : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \to \mathcal{S}_q^{SL_2}(\mathfrak{B})$. Suppose that D is a diagram in the bigon $\mathfrak{B}$ so that both e_l and e_r carry orientations traveling from bottom to top.

We define

$$S(D) = (\sqrt{-q})^{T(D)} D',$$

where $T(D)$ is the sum of states on the right boundary e_r minus the sum of states on the left boundary e_l of D. Here, D' is the diagram obtained by the following process. We take a representative of D in $\mathfrak{B} \times (-1, 1)$, we rotate the representative 180 degrees about the axis traveling through the two punctures of $\mathfrak{B}$, change the framing of the skein to its negative, and then we take a generic projection back onto $\mathfrak{B}$.

We note that if we set $q = 1$, then this definition agrees with the diagrammatic definition of the antipode given for $\mathcal{S}^{SL_2}(\mathfrak{B})$, but here we have been forced to take more of a 3-dimensional perspective.

Since we can check that our definition of S respects the skein relations, we have that it extends to a $\mathbb{C}$-linear map on $\mathcal{S}_q^{SL_2}(\mathfrak{B})$. Since the rotation involved in the definition reverses the height order on the boundary edges, S extends to an algebra anti-homomorphism on $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

We observe that S satisfies

$$S(\alpha_{++}) = \alpha_{--} \qquad S(\alpha_{+-}) = -q^{-1}\alpha_{+-}$$
$$S(\alpha_{-+}) = -q\alpha_{-+} \qquad S(\alpha_{--}) = \alpha_{++},$$

and so it can be checked on generators that S satisfies the definition of the antipode. The diagrammatic definition of the antipode presented here was also given in [CL19] and it is very similar to the definition of a diagrammatic antipode given in [Big14], which will be relevant to our skein theoretic treatment of $U_q(\mathfrak{sl}_2)$.

2.6.6 $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \cong \mathcal{O}_q(SL_2)$

So far we have defined a Hopf algebra structure on $\mathcal{S}_q^{SL_2}(\mathfrak{B})$ and now we observe that $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \cong \mathcal{O}_q(SL_2)$ as Hopf algebras. This is done analogously to the classical case where we define a map $T : \mathcal{O}_q(SL_3) \to \mathcal{S}_q^{SL_3}(\mathfrak{B})$ defined on generators by $T(X_{11}) = \alpha_{++}$, $T(X_{12}) = \alpha_{+-}$, $T(X_{21}) = \alpha_{+-}$, and $T(X_{22}) = \alpha_{--}$.

We observed earlier that the set $\{\alpha_{ij}\}_{i,j\in(-,+)}$ is a generating set for the algebra $\mathcal{S}_q^{SL_2}(\mathfrak{B})$. So if T is well-defined, then it is a surjective algebra homomorphism. We also observe that the values of Δ, ε, and S on the generators α_{ij} are compatible with the corresponding Hopf algebra structure of $\mathcal{O}_q(SL_2)$. Thus, T is a Hopf algebra map by construction.

To check that T is well-defined, it suffices to check that T respects the defining relations of $\mathcal{O}_q(SL_2)$, which is an easy exercise.

To see that T is an isomorphism we provide the definition of its inverse T^{-1}, which is defined on diagrams as in the classical case. Let D be a stated diagram in the bigon. We give an algorithm to write D as a linear combination of monomials in the elements α_{ij} as follows. We use Δ to split D close to its right boundary e_r and then apply $(\varepsilon \otimes \mathrm{id})\Delta(D)$ to write D as a linear combination of monomials in the generators α_{ij}. We obtain $T^{-1}(D)$ by defining $T^{-1}(\alpha_{ij}) = X_{ij}$. By checking that T^{-1} satisfies the skein relations, we have found an inverse for T. We leave this as an exercise, but a similar computation is carried out in the SL_3 case later on.

The fact that $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \cong \mathcal{O}_q(SL_2)$ as algebras was first proven in [Lê18] and the fact that this isomorphism respected the Hopf algebra structure was shown in both [CL19, Kor19].

2.7 Hopf pairing and skein theoretic realization of $U_q(\mathfrak{sl}_2)$

Now that we have a skein theoretic definition of $\mathcal{O}_q(SL_2)$, we turn our attention to $U_q(\mathfrak{sl}_2)$. A diagrammatic definition of $U_q(\mathfrak{sl}_2)$ was already constructed in [Big14], built out of an algebra of strands which pass in front of or behind a vertical pole. In [Kor19], it is shown that this algebra aligns with the stated skein algebra of $\mathfrak{B}_1$, a bigon with one puncture in its interior playing the role of the pole. In both [Big14, Kor19], the $U_q(\mathfrak{sl}_2)$ relations are found in the diagrammatic algebra after taking a quotient by a kernel of diagrammatic maps. In [Big14], the kernel is an intersection of kernels of maps called $\rho^{\otimes n}$ given by threading n strands in replacement of the pole. In [Kor19], the kernel of interest is the kernel of the quantum trace map.

In this section, we see that $U_q(\mathfrak{sl}_2)$ embeds in a quotient of $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ by the kernel of a diagrammatic map, lining up with the constructions of Bigelow and Korinman. Our construction here will use a diagrammatic definition of the Hopf pairing $\langle -, - \rangle : \mathcal{O}_q(SL_2) \otimes U_q(\mathfrak{sl}_2) \to \mathbb{C}$. This construction of $U_q(\mathfrak{sl}_2)$ also lines up algebraically with a construction of $U_q(\mathfrak{sl}_2)$ from matrix coordinates called l-functionals as introduced in [RTF89]; see also [KS97, Section 9.4].

We will define a Hopf pairing $\langle -, - \rangle : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathbb{C}$ on diagrams and we will observe that the pairing between $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$ essentially factors through this pairing. We previously described the Hopf algebra structure of $\mathcal{S}_q^{SL_2}(\mathfrak{B})$ and we will now briefly describe how to adapt this structure to describe $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ as a Hopf algebra.

The surface $\mathfrak{B}_1$ is the once punctured bigon and we think of it as a planar surface isotopic to the unit disk in the plane $\mathbb{R}^2$ with the points $(0, 1)$ and $(0, -1)$ removed from its boundary (called the top and bottom punctures, respectively) and $(0, 0)$ removed from its interior. The stated skein algebra $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ is the quotient of the module freely spanned by isotopy classes of stated skeins in $\mathfrak{B}_1 \times (-1,1)$ by the same local skein relations from $\mathcal{S}_q^{SL_2}(\mathfrak{B})$. The product of skeins is again given by stacking one skein on top of the other and so that its endpoints appear higher in the height order.

The coproduct $\Delta : \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ is defined on diagrams by first introducing a second puncture in a small neighborhood of the first interior puncture and then using the splitting map to split the diagram along an ideal arc that travels from the bottom puncture, between the two interior punctures, and to the top puncture.

The counit $\varepsilon : \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathbb{C}$ is defined by first filling in the interior puncture and then applying the counit of $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

The antipode $S : \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ is defined geometrically in the same way as the antipode for $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

The reader can check that these maps satisfy the Hopf algebra axioms and is invited to compare them to the diagrammatic Hopf algebra defined in [Big14]. A presentation for $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ can be obtained from either [Kor19, CL19], but we won't need an explicit presentation by generators and relations in this section.

We next define the value of the pairing $\langle D, E\rangle$ when D is a stated skein in the thickened bigon $\mathfrak{B} \times (-1,1)$ and E is a stated skein in the thickened punctured bigon $\mathfrak{B}_1 \times (-1,1)$. The value of the pairing is essentially obtained by threading D through E in place of the puncture and then taking the counit. We describe the map explictly here.

Step 1: We start with $D \in (\mathfrak{B} \times (-1,1))$. Let $P \subset \mathfrak{B}$ be the unit circle of radius $1/2$, centered at the origin $(0,0)$ in $\mathfrak{B}$. We isotope the skein D so that it is contained in the following region of $\mathfrak{B} \times (-1,1)$:

$$\Big[(\mathfrak{B} \cap \{x<0\}) \times (1/2,1)\Big] \cup \Big[(\mathfrak{B} \cap \{x>0\}) \times (-1,-1/2)\Big] \cup \Big[P \times (-1,1)\Big].$$

After this isotopy, we rotate D by 180 degrees about the z-axis to obtain a skein D' in

$\mathfrak{B} \times (-1,1)$.

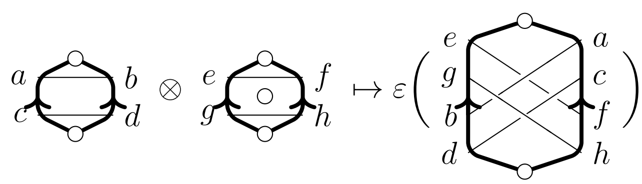
Step 2: We next take $E \in \mathfrak{B}_1 \times (-1,1)$. We isotope E so that it is contained in

$$(\mathfrak{B} \setminus P) \times (-1/2, 1/2)$$

and denote it by E'.

Step 3: We take the union of the skeins D' and E' in $\mathfrak{B} \times (-1,1)$ and take the counit to obtain $\langle D, E\rangle$.

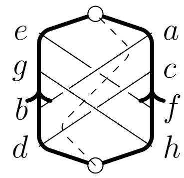
We illustrate the definition of $\langle -, - \rangle : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathbb{C}$ by showing it on a pair of diagrams here.



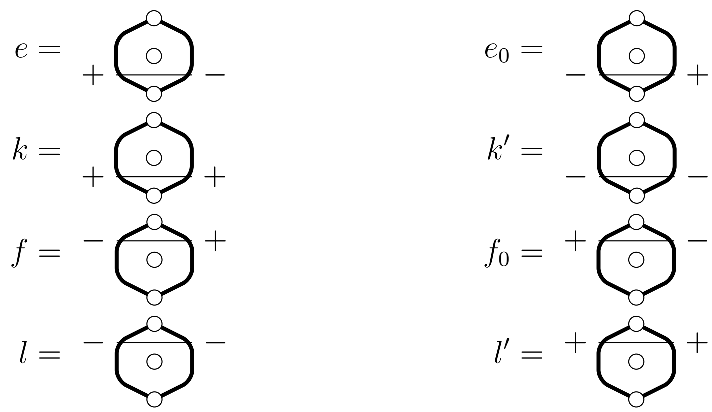
Since this definition is easily seen to respect the skein relations, we have defined a $\mathbb{C}$-linear map $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathbb{C}$. To check that this defines a pairing of Hopf algebras, it suffices to check the compatibility on pairs of diagrams. This can be checked diagrammatically. For example, the axiom

$$\langle AB, x\rangle = \sum_{(x)} \langle A, x'\rangle \langle B, x''\rangle$$

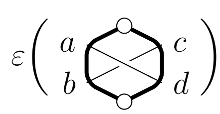
follows from splitting the following diagram along the indicated ideal arc and then using the identity $(\varepsilon \otimes \varepsilon)\Delta = \varepsilon$.



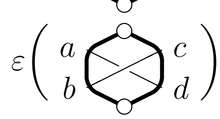
We will use a set of eight generators for $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$, each consists of a single horizontal strand passing above or below the puncture, with all possible states. We call the four generators which pass above the puncture f, f_0, l, l' and the four generators which pass below the puncture e, e_0, k, k'.



To compute the values of the pairing, it will be helpful to perform the following computations



$$= R_{ab}^{cd} = R_{cd}^{ab}$$



$$= (R^{-1})_{ab}^{cd} = (R^{-1})_{cd}^{ab},$$

where R_{ab}^{cd} represent the matrix entries of the co-R-matrix for $\mathcal{O}_q(SL_2)$ having values

$$R_{++}^{++} = q^{1/2} \qquad R_{--}^{--} = q^{1/2}$$
$$R_{-+}^{+-} = q^{-1/2} \qquad R_{+-}^{+-} = q^{-1/2}(q - q^{-1})$$
$$R_{+-}^{-+} = q^{-1/2} \qquad R_{ab}^{cd} = 0 \text{ if not listed above.}$$

Using the matrix entries for R and for R^{-1} we can compute the pairings

$$\langle \alpha_{ij}, e_0 \rangle = 0 = \langle \alpha_{ij}, f_0 \rangle$$

for all states i, j.

We also use the diagrammatic definition of Δ to compute that $\Delta(e_0) = e_0 \otimes k + k' \otimes e_0$ and $\Delta(f_0) = l' \otimes f_0 + f_0 \otimes l$. Thus, the defining properties of the Hopf pairing imply that

$$\langle -, e_0 \rangle = 0 = \langle -, f_0 \rangle.$$

Similarly, the following hold:

$$\langle -, k \rangle = \langle -, l \rangle$$
$$\langle -, k' \rangle = \langle -, l' \rangle.$$

Recall that the pairing $\langle -, - \rangle : \mathcal{S}_q^{SL_2}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathbb{C}$ defines a Hopf algebra map

$$p_r : \mathcal{S}_q^{SL_2}(\mathfrak{B}_1) \to \mathcal{S}_q^{SL_2}(\mathfrak{B})^\circ$$
$$x \mapsto \langle -, x \rangle,$$

where $\mathcal{S}_q^{SL_2}(\mathfrak{B})^\circ$ is the finite Hopf dual of $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

Proposition 2.13 *Suppose q is not a root of unity and let $K = \ker p_r$. Then there is a Hopf algebra embedding*

$$i : U_q(\mathfrak{sl}_2) \hookrightarrow \mathcal{S}_q^{SL_2}(\mathfrak{B}_1)/\ker p_r$$

given by

$$E \mapsto \frac{q}{q - q^{-1}} ek$$
$$F \mapsto \frac{-q^{-1}}{q - q^{-1}} k'f$$
$$K \mapsto k^2$$
$$K^{-1} \mapsto (k')^2.$$

Proof: We first argue that the above assignments define a well-defined Hopf algebra map. To see that it is an algebra map, we can check that the $U_q(\mathfrak{sl}_2)$ relations are respected. This computation will follow after first observing that the relations

$$ke = qek$$
$$lf = q^{-1}f$$
$$ef - fe = (q - q^{-1})(l'k' - lk)$$
$$kk' - qee_0 = 1$$
$$l'l - qf_0f = 1$$

hold inside of $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)$ and then using the facts that $e_0, f_0, k - k', l - l'$ are all in $\ker p_r$.

To see that the map i is a Hopf algebra map, we first observe that since $\ker p_r$ is the kernel of a Hopf algebra map, the quotient $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)/\ker p_r$ has the induced Hopf algebra structure. The fact that i respects the Hopf algebra structure is an easy computation on generators.

To see that our assignment defines an embedding, we appeal to the fact that the non-degenerate Hopf pairing $\mathcal{O}_q(SL_2) \otimes U_q(\mathfrak{sl}_2) \to \mathbb{C}$ defines an embedding

$$q_r : U_q(\mathfrak{sl}_2) \to \mathcal{O}_q(SL_2)^\circ$$
$$x \mapsto \langle -, x \rangle$$

which satisfies

$$q_r = \bar{p}_r \circ i,$$

where $\bar{p_r}$ is the map induced by p_r on $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)/\ker p_r$. This equality can be checked just by computing $\langle \alpha_{ij}, - \rangle$ on the images of each of the generators $i(E), i(F), i(K), i(K^{-1})$ and seeing that they agree with $\langle X_{ij}, - \rangle$ on each of E, F, K, K^{-1}. We conclude that i must be injective. ■

We have previously shown that $\mathcal{O}_q(SL_2) \cong \mathcal{S}_q^{SL_2}(\mathfrak{B})$ for all q. Now for q not a root of unity we have shown that $U_q(\mathfrak{sl}_2)$ is contained in the Hopf algebra $\mathcal{S}_q^{SL_2}(\mathfrak{B}_1)/\ker p_r$. The reader might wonder if there is a skein theoretic construction of classical $U(\mathfrak{sl}_2)$ in the same way that there is a skein theoretic construction of $\mathcal{O}(SL_2)$. However, we have observed that skein algebras at $q = 1$ are commutative algebras, so it might be impossible to obtain the noncommutative algebra $U(\mathfrak{sl}_2)$ in this way.

2.8 SL_2 skein algebras of surfaces

Previously in this chapter, we have described skein algebras built from a few specific surfaces, like $\mathfrak{M}$, $\mathfrak{B}$, and $\mathfrak{B}_1$. Of course, their definitions easily generalize to skein algebras associated to other surfaces. Historically, skein algebras for general surfaces, and skein modules for 3-manifolds, were defined first and then it was later observed that skein algebras of punctured surfaces can be decomposed by studying small surfaces like $\mathfrak{B}$. We give the definition of ordinary and stated SL_2 skein algebras here, which are also referred to in the literature as the Kauffman bracket (stated) skein algebras.

Definition 2.14 *A punctured bordered surface is a pair $(\Sigma', \mathcal{P})$, where Σ' is a smooth compact oriented surface, possibly with boundary, and $\mathcal{P}$ is a collection of finitely many points of Σ'. We require that each boundary component of Σ' contains at least one point of $\mathcal{P}$. We do not require Σ' to be connected. We let $\Sigma = \Sigma' \setminus \mathcal{P}$. To simplify notation, we also refer to the pair $(\Sigma', \mathcal{P})$ simply by Σ. A boundary arc of Σ is a connected component of $\partial\Sigma$.*

For a punctured bordered surface Σ, an (SL_2) skein in $\Sigma \times (-1,1)$ is an embedding of a framed unoriented tangle Γ. We allow Γ to have univalent vertices, called *endpoints*, which must be contained in $\partial\Sigma \times (-1,1)$ such that for each boundary arc b of Σ the vertices contained in $b \times (-1,1)$ have distinct heights. We require the skein to have a vertical framing with respect to the $(-1,1)$ component and we require that strands that terminate in a univalent vertex are transverse to $\partial\Sigma$.

For a skein, Γ a *state* is a function $s : \partial\Gamma \to \{-,+\}$. A *stated skein* is a skein together with a state.

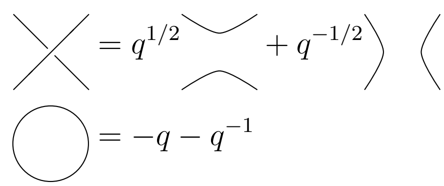
Definition 2.15 *A skein Γ in $\Sigma \times (-1,1)$ is in generic position if the projection $\pi : \Sigma \times (-1,1) \to \Sigma$ restricts to an embedding of Γ except for the possibility of transverse double points in the interior of Σ. Each skein is isotopic to a skein in generic position. A stated diagram D of a generic stated skein Γ is the projection $\pi(\Gamma)$ along with the over/undercrossing information at each double point and the height orders and states of the boundary points of Γ. Skein diagrams are isotopic if they are isotopic through an isotopy of the surface.*

As we have already been doing, we record the local height order of the boundary points of a skein diagram by drawing an arrow along a portion of the boundary arc of Σ.

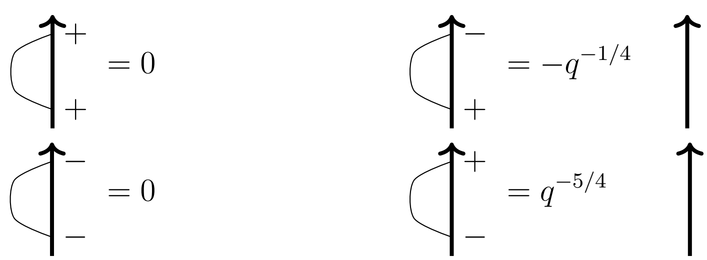
Earlier in this chapter we had been using $\mathbb{C}$ as a coefficient ring, but we do not lose anything by taking a coefficient ring $\mathcal{R}$ be any commutative ring containing an invertible element $q^{1/4}$.

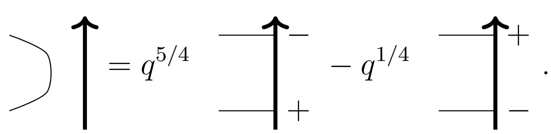
Definition 2.16 *The SL_2 stated skein algebra $\mathcal{S}_q^{SL_2}(\Sigma)$ is the $\mathcal{R}$-module freely spanned by isotopy classes of skeins in $\Sigma \times (-1,1)$ modulo the following relations.*

Interior relations:



Boundary relations:





Definition 2.17 *The ordinary SL_2 skein algebra $\mathring{\mathcal{S}}_q^{SL_2}(\Sigma)$ is the $\mathcal{R}$-module freely spanned by isotopy classes of framed links in $\mathring{\Sigma} \times (-1, 1)$ modulo only the interior relations.*

For both the ordinary and stated skein algebras, the product on skeins is given as follows. The product $\Gamma_1\Gamma_2$ of two stated skeins Γ_1, Γ_2 in $\Sigma \times (-1, 1)$ is given by isotoping Γ_1 so that it is contained in $\Sigma \times (0, 1)$, isotoping Γ_2 so that it is contained in $\Sigma \times (-1, 0)$, and then taking the union of these two stated skeins in $\Sigma \times (-1, 1)$. This gives both

$\mathring{\mathcal{S}}_q^{SL_2}(\Sigma)$ and $\mathcal{S}_q^{SL_2}(\Sigma)$ associative, unital $\mathcal{R}$-algebra structures.

When Σ has empty boundary, it follows from the definitions that we have an equality $\mathring{\mathcal{S}}_q^{SL_2}(\Sigma)$ and $\mathcal{S}_q^{SL_2}(\Sigma)$. When Σ has nonempty boundary then there is still an embedding $\mathring{\mathcal{S}}_q^{SL_2}(\Sigma) \hookrightarrow \mathcal{S}_q^{SL_2}(\Sigma)$ as can be seen from the following characterization of bases of the skein algebras.

Theorem 2.18 *([SW07]) The ordinary skein algebra $\mathring{\mathcal{S}}_q^{SL_2}(\Sigma)$ is a free $\mathcal{R}$-module. A basis for $\mathring{\mathcal{S}}_q^{SL_2}(\Sigma)$ consists of isotopy classes of skein diagrams on Σ containing no crossings and no null-homotopic loops.*

Suppose our surface Σ is given the standard counterclockwise orientation. We then call a boundary arc of Σ whose height order agrees with the orientation a *positively oriented* boundary arc. We call a stated skein diagram on Σ *increasingly stated* if, on each boundary arc, endpoints labeled by $+$ appear at greater heights than endpoints labeled by $-$. Assume we have chosen all of our boundary arcs to have positive orientations.

Theorem 2.19 *([Lê18]) The stated skein algebra $\mathcal{S}_q^{SL_2}(\Sigma)$ is a free $\mathcal{R}$-module. A basis for $\mathcal{S}_q^{SL_2}(\Sigma)$ consists of isotopy classes of stated skein diagrams on Σ which are increasingly stated and contain no crossings, no null-homotopic loops, and no arcs which are homotopic to the boundary.*

That these skein modules are free modules with bases that can be uniformly described for all surfaces does not carry over to the case of skein modules for general 3-manifolds, which can have torsion if the manifold contains non-separating spheres or tori [Prz97].

The advantage to studying stated skein algebras is that they admit natural algebra maps, called splitting morphisms, associated to splitting the surface along an ideal arc.

If Σ is a punctured bordered surface and a and b are two boundary arcs of Σ, we can obtain a new punctured bordered surface $\bar{\Sigma} = \Sigma/(a = b)$ by gluing the arcs a and b

together in the way compatible with the orientation of Σ. It is the reverse of this process that gives us an algebra morphism from $\mathcal{S}_q^{SL_2}(\bar{\Sigma})$ to $\mathcal{S}_q^{SL_2}(\Sigma)$ associated with splitting the surface $\bar{\Sigma}$ along an ideal arc c.

Definition 2.20 *If Σ is a punctured bordered surface, an ideal arc in Σ is a proper embedding $c : (-1,1) \to \mathring{\Sigma}$ such that its endpoints are (not necessarily distinct) points in the set of punctures, $\mathcal{P}$.*

Let $p : \Sigma \to \Sigma/(a = b) =: \bar{\Sigma}$ be the projection map associated to the gluing. Then $c := p(a) = p(b)$ is an ideal arc. The splitting morphism

$$\Delta_c : \mathcal{S}_q^{SL_2}(\bar{\Sigma}) \to \mathcal{S}_q^{SL_2}(\Sigma)$$

is defined on skeins in $\bar{\Sigma} \times (-1,1)$ in the following way.

For a stated skein (Γ, s) in $\bar{\Sigma} \times (-1,1)$ we first isotope it so that Γ intersects $c \times (-1,1)$ transversely in points of distinct heights. By defining p to act trivially on the $(-1,1)$ factor, we can extend it to a map $p : \Sigma \times (-1,1) \to \bar{\Sigma} \times (-1,1)$. We then consider $p^{-1}(\Gamma)$, which is a skein in $\Sigma \times (-1,1)$. Except for the points of $p^{-1}(c \cap \Gamma)$, each boundary point of $p^{-1}(\Gamma)$ inherits a state from Γ.

We will say that s' is an *admissible state* for $p^{-1}(\Gamma)$ if $s'(p^{-1}(x)) = s(x)$ for all $x \in \partial\Gamma$ and if $y, z \in p^{-1}(\Gamma \cap c)$ then $s'(y) = s'(z)$.

We define the splitting morphism on a stated skein (Γ, s) in $\bar{\Sigma} \times (-1,1)$ by

$$\Delta_c(\Gamma, s) = \sum_{\text{admissible } s'} [p^{-1}(\Gamma), s'].$$

When c happens to be the core of the bigon $\mathfrak{B}$, then Δ_c is the same map that we used for the coproduct of $\mathcal{S}_q^{SL_2}(\mathfrak{B})$.

Theorem 2.21 *([Lê18, CL19, KQ19])*

(a) *The map* Δ_c *described above extends linearly to a well-defined algebra morphism* $\Delta_c : \mathcal{S}_q^{SL_2}(\bar{\Sigma}) \to \mathcal{S}_q^{SL_2}(\Sigma)$.

(b) *If* a *and* b *are two ideal arcs with disjoint interiors, then we have*

$$\Delta_a \circ \Delta_b = \Delta_b \circ \Delta_a.$$

(c) *The splitting morphism is injective and its image can be characterized according to the following exact sequence*

$$0 \to \mathcal{S}_q^{SL_2}(\bar{\Sigma}) \xrightarrow{\Delta_c} \mathcal{S}_q^{SL_2}(\Sigma) \xrightarrow{\Delta_a - \tau \circ_b \Delta} \mathcal{S}_q^{SL_2}(\Sigma) \otimes \mathcal{S}_q^{SL_2}(\mathfrak{B}).$$

As was the case over $\mathbb{C}$, we also have over $\mathcal{R}$ that $\mathcal{S}_q^{SL_2}(\mathfrak{M}) \cong \mathcal{R}$ and $\mathcal{S}_q^{SL_2}(\mathfrak{B}) \cong \mathcal{O}_q(SL_2)$. The stated skein algebra of the ideal triangle $\mathfrak{T}$ is isomorphic to the braided tensor square of $\mathcal{O}_q(SL_2)$, denoted by

$$\mathcal{S}_q^{SL_2}(\mathfrak{T}) \cong \mathcal{O}_q(SL_2) \underline{\otimes} \mathcal{O}_q(SL_2).$$

The algebra $\mathcal{S}_q^{SL_2}(\mathfrak{T})$ has an explicit presentation built from the presentation of $\mathcal{O}_q(SL_2)$. It has 8 generators, with 4 generators being the generators $X_{11}, X_{12}, X_{21}, X_{22}$, of one copy of $\mathcal{O}_q(SL_2)$ as well as 4 generators $Y_{11}, Y_{12}, Y_{21}, Y_{22}$ of a second copy of $\mathcal{O}_q(SL_2)$. A complete set of relation is given by imposing the $\mathcal{O}_q(SL_2)$ relations among the generators X_{ij}, imposing the $\mathcal{O}_q(SL_2)$ relations among the Y_{ij}, and then imposing the extra mixed relations

$$Y_{ij} X_{kl} = \sum_{m,n} R_{ik}^{mn} X_{ml} Y_{nj},$$

where R_{ik}^{mn} are the matrix entries of our R-matrix, whose only nonzero entries are $R_{aa}^{aa} =$

$q^{1/2}, R_{ab}^{ba} = q^{-1/2}$, and $R_{12}^{12} = q^{-1/2}(q - q^{-1})$.

The splitting maps allow us to decompose a skein algebra of a surface built from simpler surfaces. In the case that Σ has an ideal triangulation, meaning that Σ can be constructed from a set of n disjoint ideal triangles by identifying some pairs of edges, the composition of splitting maps along the edges of the ideal triangulation gives an algebra embedding

$$\mathcal{S}_q^{SL_2}(\Sigma) \hookrightarrow \bigotimes_{i=1}^{n} \mathcal{S}_q^{SL_2}(\mathfrak{T}_i).$$

An important breakthrough in the theory of the ordinary skein algebra was a construction of Bonahon and Wong in which they embedded the skein algebra of a triangulable surface in a quantum torus, which is an algebra that is much easier to study.

Theorem 2.22 *([BW11]) For a punctured surface Σ with no boundary and an ideal triangulation $\mathcal{E}$, there is an embedding, called the quantum trace map*

$$tr_q : \mathcal{S}_q^{SL_2}(\Sigma) \hookrightarrow \mathcal{Y}_q(\mathcal{E}),$$

from the skein algebra into a quantum torus.

The definition of the quantum trace map is inspired by Checkov and Fock's notion of quantum Teichmüller space [FC99]. The original proof that tr_q respects the skein relations was quite involved. To simplify this construction, Lê defined the stated skein algebra as an intermediate step in the construction of tr_q which made it easier to see that the map was well-defined.

Theorem 2.23 *([Lê18]) The quantum trace map tr_q of Bonahon and Wong factors through the triangular decomposition. So tr_q can be realized as a composition of maps*

$$\mathcal{S}_q^{SL_2}(\Sigma) \hookrightarrow \bigotimes_{i=1}^{n} \mathcal{S}_q^{SL_2}(\mathfrak{T}_i) \to \mathcal{Y}_q(\mathcal{E}).$$

The idea is that the first map is a composition of splitting maps, which are well defined by construction of the stated skein algebra. Then the second map in the composition can be defined on triangles $\mathcal{S}_q^{SL_2}(\mathfrak{T}_i)$ which have explicit presentations.

Although the original motivation for the stated skein algebra was to simplify the construction of tr_q, the stated skein algebra is interesting in its own right and has helped to describe connections between skein algebras and quantum groups. We can view the skein algebra $\mathcal{S}_q^{SL_2}(\Sigma)$ of a surface as an algebra which is a generalization of $\mathcal{O}_q(SL_2)$. Splitting maps can be thought of as generalizations of the coproduct of $\mathcal{S}_q^{SL_2}(\mathfrak{B}) = \mathcal{O}_q(SL_2)$. It is an easy exercise to use the axioms for the coproduct Δ and counit ε in any coalgebra C to see that the exact sequence

$$0 \to C \overset{\Delta}{\to} C \otimes C \overset{\Delta\otimes\mathrm{id}-\mathrm{id}\otimes\Delta}{\to} C \otimes C \otimes C$$

holds. We can view the exact sequence in Theorem 2.21 as a generalization of this exact sequence.

2.9 $U_q(\mathfrak{sl}_3)$ and $\mathcal{O}_q(SL_3)$

In the next chapter we transition to the theory of SL_3 skein algebras. In this section we fix notation and record our definitions of the quantum groups $U_q(\mathfrak{sl}_3)$ and $\mathcal{O}_q(SL_3)$, which agree with those in [BG02].

Definition 2.24 *The quantized universal enveloping algebra $U_q(\mathfrak{sl}_3)$ is the quotient of the free algebra generated by $E_1, E_2, F_1, F_2, K_1^{\pm 1}, K_2^{\pm 1}$ subject to the following relations:*

$$
\begin{aligned}
&K_iE_i = q^2E_iK_i && K_iF_i = q^{-2}F_iK_i & \\
&K_iE_j = q^{-1}E_jK_i && K_iF_j = qF_jK_i & (i \neq j) \\
&K_iK_j = K_jK_i && E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}} & \\
&E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 && & (i \neq j) \\
&F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 && & (i \neq j).
\end{aligned}
$$

The Hopf algebra structure maps are defined on generators by

$$
\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i && \varepsilon(K_i) = 1 && S(K_i) = K_i^{-1} \\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i && \varepsilon(E_i) = 0 && S(E_i) = -K_i^{-1}E_i \\
&\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i && \varepsilon(F_i) = 0 && S(F_i) = -F_iK_i
\end{aligned}
$$

The standard 3-dimensional representation $\rho_V : U_q(\mathfrak{sl}_3) \to \mathrm{End}(V)$ *is given by*

$$\rho_V(K_1) = \begin{pmatrix} q & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho_V(E_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \rho_V(F_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho_V(K_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} \qquad \rho_V(E_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \rho_V(F_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 2.25 *The quantized coordinate ring of regular functions on SL_3, denoted $\mathcal{O}_q(SL_3)$, is the quotient of the free algebra generated by $\{X_{ij}\}_{1\le i,j\le 3}$ subject to the following relations:*

$$X_{ij}X_{lm} = \begin{cases} qX_{lm}X_{ij} & (i<l, j=m) \\ qX_{lm}X_{ij} & (i=l, j<m) \\ X_{lm}X_{ij} & (i<l, j>m) \\ X_{lm}X_{ij} + (q-q^{-1})X_{im}X_{lj} & (i<l, j<m). \end{cases}$$

$$\sum_{\sigma\in S_3} (-q)^{l(\sigma)} X_{\sigma(1)1}X_{\sigma(2)2}X_{\sigma(3)3} = 1,$$

where $l(\sigma)$ denotes the length of the permutation, which is the length of the shortest word expressing σ in simple transpositions $(i, i+1)$.

The Hopf algebra structure maps are given as follows:

$$\Delta(X_{ij}) = \sum_{r=1}^{3} X_{ir} \otimes X_{rj} \qquad \varepsilon(X_{ij}) = \delta_{ij} \qquad S(X_{ij}) = (-q)^{i-j} A[j|i],$$

where $A[j|i]$ *denotes the quantum minor of the matrix* $(A)_{ij} = X_{ij}$ *after deleting row* j *and column* i.

The standard 3-dimensional comodule V *is defined by*

$$v_i \mapsto \sum_{j=1}^{3} v_j \otimes X_{ji}.$$

There is a Hopf pairing $\mathcal{O}_q(SL_3) \otimes U_q(\mathfrak{sl}_3) \to \mathbb{C}$ *defined on generators by*

$$\langle X_{ij}, x \rangle = (\rho_V(x))_{ij}.$$

Chapter 3

SL_3 skein algebras of surfaces

We now turn towards the study of SL_3 skein algebras, which are built from Kuperberg's webs. The relations given by Kuperberg in [Kup96] supply us with the set of skein relations we will impose in the interior of our surface. We introduce stated skein relations along the boundary, which are compatible with Kuperberg's skein relations. The most powerful feature of these stated skein relations is that, together with Kuperberg's relations, they form a *confluent* set of relations. This makes it possible to apply the methods of [SW07] to construct a basis for our stated skein algebra which extends the basis of the ordinary skein algebra.

Definition 3.1 *A punctured bordered surface is a pair $(\Sigma', \mathcal{P})$, where Σ' is a smooth compact oriented surface, possibly with boundary, and $\mathcal{P}$ is a collection of finitely many points of Σ'. We require that each boundary component of Σ' contains at least one point of $\mathcal{P}$. We do not require Σ' to be connected. We let $\Sigma = \Sigma' \setminus \mathcal{P}$. To simplify notation, we also refer to the pair $(\Sigma', \mathcal{P})$ simply by Σ. A boundary arc of Σ is a connected component of $\partial\Sigma$.*

For a punctured bordered surface Σ, a *web* in $\Sigma \times (-1, 1)$ is an embedding of a directed ribbon graph Γ such that each interior vertex of Γ in $\mathring{\Sigma} \times (-1,1)$ is a trivalent sink or a trivalent source. We allow Γ to have univalent vertices, called *endpoints*, contained in $\partial\Sigma \times (-1,1)$ such that for each boundary arc b of Σ the vertices contained in $b \times (-1,1)$ have distinct heights. We require the web to have a vertical framing with respect to the $(-1,1)$ component and we require that strands that terminate in a univalent vertex are transverse to $\partial\Sigma$.

We consider isotopies of webs in the class of webs. In particular, our isotopies must preserve the height orders of boundary points of webs for each boundary arc of Σ.

For a web Γ a *state* is a function $s : \partial\Gamma \to \{-,0,+\}$. A *stated web* is a web together with a state. We will make use of the order $- < 0 < +$ on the set $\{-,0,+\}$. For notational purposes, it will be convenient to sometimes add states together. By identifying the state $-$ with the integer -1 and the state $+$ with the integer 1, we partially define an addition on the set $\{-,0,+\}$ whenever the answer is contained in the set as well.

Definition 3.2 *A web Γ in $\Sigma \times (-1,1)$ is in generic position if the projection $\pi : \Sigma \times (-1,1) \to \Sigma$ restricts to an embedding of Γ except for the possibility of transverse double points in the interior of Σ. Each web is isotopic to a web in generic position. A stated diagram D of a generic stated web Γ is the projection $\pi(\Gamma)$ along with the over/undercrossing information at each double point and the height orders and states of the boundary points of Γ. Web diagrams are isotopic if they are isotopic through an isotopy of the surface.*

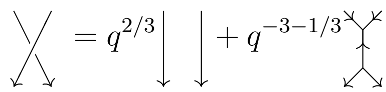
As in [Lê18] it will be convenient for us to record the local height order of the boundary points of a web diagram by drawing an arrow along a portion of the boundary arc of Σ.

Let $\mathcal{R}$ be a unital commutative ring containing an invertible element $q^{1/3}$. The quantum integer $[n]$ denotes the Laurent polynomial defined by $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

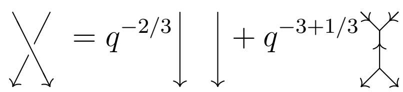
3.1 The skein relations

Definition 3.3 *The SL_3 stated skein algebra $\mathcal{S}_q^{SL_3}(\Sigma)$ is the $\mathcal{R}$-module freely spanned by isotopy classes of webs in $\Sigma \times (-1, 1)$ modulo the following relations.*

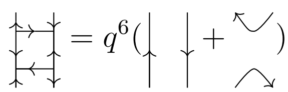
Interior relations:



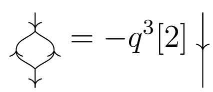
(I1a)



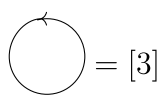
(I1b)



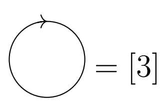
(I2)



(I3)

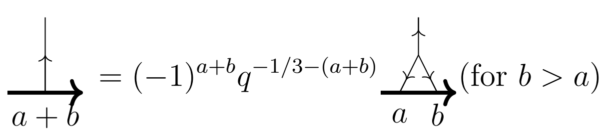


(I4a)

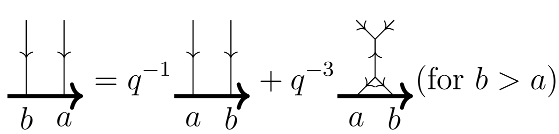


(I4b)

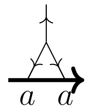
Boundary relations:



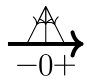
(B1)



(B2)



$= 0 \qquad$ (for any $a \in \{-, 0, +\}$) (B3)



$= q^{-2}$  (B4)

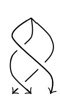
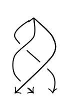
The interior relations above hold for local diagrams contained in an embedded disk in Σ. The boundary relations hold for local diagrams in a neighborhood of a point of $\partial\Sigma$. The thicker line denotes a portion of a boundary arc while the thin lines belong to a web. The arrow along the boundary arc indicates the height order of that boundary arc. For example, in the diagram on the right side of relation (B1), the endpoint with the state b has a greater height than the endpoint with the state a.

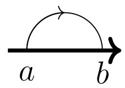
The module defined above admits a natural multiplication where the product $\Gamma_1\Gamma_2$ of two stated webs Γ_1, Γ_2 in $\Sigma \times (-1, 1)$ is given by isotoping Γ_1 so that it is contained in $\Sigma \times (0, 1)$, isotoping Γ_2 so that it is contained in $\Sigma \times (-1, 0)$, and then taking the union of these two stated webs in $\Sigma \times (-1, 1)$. This gives $\mathcal{S}_q^{SL_3}(\Sigma)$ an associative, unital $\mathcal{R}$-algebra structure.

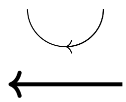
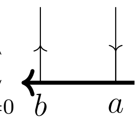
3.2 Consequences of the defining relations

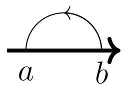
Proposition 3.4 *The following relations are consequences of the defining relations.*

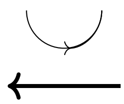
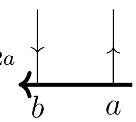
$q^{-8/3}$  $=$  $= q^{8/3}$  (a)

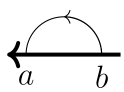
$-q^{-4}$  $=$  $= -q^{4}$  (b)

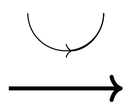
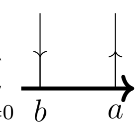
 $= -q^{-4/3}\delta_{a+b,0}$  (c)

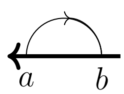
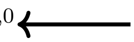
 $= -q^{-4/3}\sum_{a+b=0}$  (d)

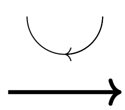
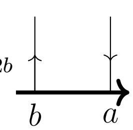
 $= -q^{-4/3}q^{2a}\delta_{a+b,0}$  (e)

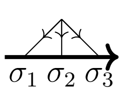
 $= -q^{-4/3}\sum_{a+b=0} q^{2a}$  (f)

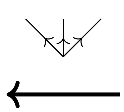
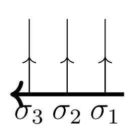
 $= -q^{4/3}\delta_{a+b,0}$  (g)

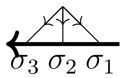
 $= -q^{4/3}\sum_{a+b=0}$  (h)

 $= -q^{4/3}q^{2b}\delta_{a+b,0}$  (i)

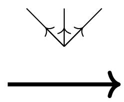
 $= -q^{4/3}\sum_{a+b=0} q^{2b}$  (j)

 $= \begin{cases} q^{-2}(-q)^{l(\sigma)} \text{ if } \sigma = (\sigma_1,\sigma_2,\sigma_3) \in S_3 \\ 0 \text{ if } (\sigma_1,\sigma_2,\sigma_3) \notin S_3 \end{cases}$ (same for sinks) (k)

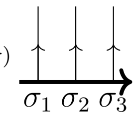
 $= q^{-2}\sum_{\sigma\in S^3}(-q)^{l(\sigma)}$  (same for sinks) (l)



$$= \begin{cases} -q^2(-q)^{l(\sigma)} \text{ if } \sigma = (\sigma_1, \sigma_2, \sigma_3) \in S_3 \\ 0 \text{ if } (\sigma_1, \sigma_2, \sigma_3) \notin S_3 \end{cases} \qquad \text{(same for sinks)} \qquad \text{(m)}$$



$$= -q^2 \sum_{\sigma \in S^3} (-q)^{l(\sigma)}$$



$$\text{(same for sinks)} \qquad \text{(n)}$$

In the notation above, we consider the permutation $(-, 0, +)$ to be the identity permutation and $l(\sigma)$ denotes the length of the permutation σ.

Proof: Relations (a) and (b) follow from the defining interior relations.

The relations involving boundary orientations pointing to the right can be checked by reducing both sides according to the algorithm given by the Diamond Lemma described in Theorem 3.9.

The relations involving boundary orientations pointing to the left can be derived from those involving orientations pointing to the right by sliding the boundary points horizontally to reverse the height order and using the twisting relations (a) and (b). ■

3.3 The splitting map

As in [Lê18], our stated skein algebras of punctured bordered surfaces satisfy a compatibility with the gluing and splitting of surfaces. If Σ is a punctured bordered surface and a and b are two boundary arcs of Σ, we can obtain a new punctured bordered surface $\bar{\Sigma} = \Sigma/(a = b)$ by gluing the arcs a and b together in the way compatible with the orientation of Σ. It is the reverse of this process that gives us an algebra morphism from $\mathcal{S}_q^{SL_3}(\bar{\Sigma})$ to $\mathcal{S}_q^{SL_3}(\Sigma)$ associated with splitting the surface $\bar{\Sigma}$ along an ideal arc c.

Definition 3.5 *If Σ is a punctured bordered surface, an ideal arc in Σ is a proper embedding $c : (0, 1) \to \mathring{\Sigma}$ such that its endpoints are (not necessarily distinct) points in the* *set of punctures,* $\mathcal{P}$.

Let $p : \Sigma \to \Sigma/(a = b) =: \bar{\Sigma}$ be the projection map associated to the gluing. Then $c := p(a) = p(b)$ is an ideal arc. We will define a splitting morphism

$$\Delta_c : \mathcal{S}_q^{SL_3}(\bar{\Sigma}) \to \mathcal{S}_q^{SL_3}(\Sigma)$$

by defining it on stated webs in $\bar{\Sigma} \times (-1, 1)$ and then checking that it is well-defined on $\mathcal{S}_q^{SL_3}(\bar{\Sigma})$.

For a stated web (Γ, s) in $\bar{\Sigma} \times (-1, 1)$ we first isotope it so that Γ intersects $c \times (-1, 1)$ transversely in points of distinct heights. By defining p to act trivially on the $(-1, 1)$ factor, we can extend it to a map $p : \Sigma \times (-1, 1) \to \bar{\Sigma} \times (-1, 1)$. We then consider $p^{-1}(\Gamma)$, which is a web in $\Sigma \times (-1, 1)$. Except for the points of $p^{-1}(c \cap \Gamma)$, each boundary point of $p^{-1}(\Gamma)$ inherits a state from Γ.

We will say that s' is an *admissible state* for $p^{-1}(\Gamma)$ if $s'(p^{-1}(x)) = s(x)$ for all $x \in \partial\Gamma$ and if $y, z \in p^{-1}(\Gamma \cap c)$ then $s'(y) = s'(z)$.

We define the splitting morphism on a stated web (Γ, s) in $\bar{\Sigma} \times (-1, 1)$ by

$$\Delta_c(\Gamma, s) = \sum_{\text{admissible } s'} [p^{-1}(\Gamma), s'].$$

Theorem 3.6 *(a) The map* Δ_c *described above extends linearly to a well-defined algebra morphism* $\Delta_c : \mathcal{S}_q^{SL_3}(\bar{\Sigma}) \to \mathcal{S}_q^{SL_3}(\Sigma)$.

(b) If a *and* b *are two ideal arcs with disjoint interiors, then we have*

$$\Delta_a \circ \Delta_b = \Delta_b \circ \Delta_a.$$

As in [Lê18], the map Δ_c is injective, but we will postpone a discussion of this fact until Theorem 4.8.

Proof: If Δ_c is well-defined, then the fact that it is an algebra morphism and that it satisfies the property given in part (b) of the Theorem 3.6 follows from the definition of the splitting morphism.

To check that it is well-defined, we first check that the effect of passing cups, caps, vertices, and crossings past the ideal arc c commutes with the application of Δ_c. This will tell us that the splitting morphism is well-defined with respect to isotopies of diagrams. Cups and caps can slide past the arc because of relations (c)-(j) from above. To slide a vertex past the arc, we can first rotate the vertex, using the fact that cups and caps can slide past the arc, until it appears as in relations (k)-(n). Since crossings can be rewritten as a linear combination of cups, caps and vertices, this allows us to pass a crossing past the arc.

If strands intersecting $c \times (-1, 1)$ are isotoped vertically so as to alter their height order, then on a diagram this has the effect of a Reidemeister 2 move. Since crossings can slide past c, we can isotope the disk containing the Reidemeister 2 move on the diagram past c and then perform the move. This tells us that the splitting map is well-defined on isotopy classes of webs.

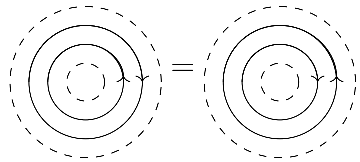
To check that the splitting morphism respects the defining relations of $\mathcal{S}_q^{SL_3}(\Sigma)$ we observe that if c cuts through a disk or half disk appearing in one of the defining relations, we can isotope the diagram away from c first and then apply the relation. ■

3.4 A basis for the stated skein algebra

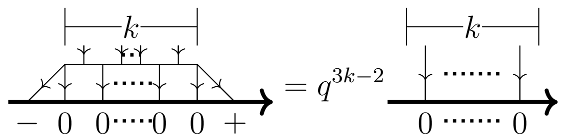
If a module is defined as a quotient of a free module by a list of relations, and if each relation can be interpreted as a reduction rule that permits the replacement of one element by a linear combination of simpler elements, then the module is a good candidate for an attempted application of the Diamond Lemma to produce a basis. As explained in [SW07], the Diamond Lemma can accommodate modules built out of diagrams on surfaces and it has been successful in producing bases for webs on surfaces for the cases of Kuperberg's webs of type A_1, A_2, B_2, and G_2. In [Lê18], Le organized the new boundary relations into reduction rules that are compatible with the reduction rules coming from the Kauffman bracket skein algebra and then applied the Diamond Lemma to find a basis. In this section, we will do the same for the SL_3 case.

We first summarize our goal. To apply the Diamond Lemma, we need to realize our skein module as a quotient of a free module by reduction rules that are terminal and locally confluent. The defining relations from Section 3.1 provide a starting point for a list of reduction rules. We will introduce a measure of complexity that allows us to say that the diagrams in the right side of each defining relation are simpler than the diagram on the left side. Using a reduction rule on a diagram D replaces that diagram with a linear combination of simpler diagrams. We call any linear combination of diagrams obtained by applying a sequence of reduction rules to D a *descendant* of D, and we call the diagrams appearing in the linear combination *descendant diagrams* of D. If there exists no infinite chain of descendant diagrams for D, then D can be written as a linear combination of irreducible diagrams by repeatedly applying reduction rules to the diagram and to its descendants. If no diagram admits an infinite chain of descendant diagrams, then the reduction rules are called *terminal* and this property implies that irreducible diagrams span our module. Sometimes more than one reduction rule will apply to a diagram. If there is always a common descendant for any two ways of reducing a diagram, then the reduction rules are called *locally confluent.* If the set of reduction rules are terminal and locally confluent, then the set of irreducible diagrams forms a basis for our module, by [SW07, Theorem 2.3].

In anticipation of issues regarding local confluence, we need to introduce the following redundant relations:



(S)



(C_k)

Relation (S) allows one to switch two circles of opposite orientations whenever the two circles bound an annulus. We see from [SW07] that relation (S) will be necessary for our list of reduction rules to be confluent, as none of the left sides of the defining relations are applicable to the diagrams in (S) unless they happen to bound a disk. We borrow notation from [FS20] to say that two circles that bound an annulus on the surface and are oriented inconsistently with the boundary of the annulus form a *British highway.* For example, the two circles on the left side of the relation (S) form a British highway. The fact that we are using oriented surfaces allows us to declare the right side of (S) to be the more reduced side. The relation (S) will serve as a reduction rule that will decrease the number of British highways on any connected component that is not a torus. The torus provides an exception since parallel nontrivial circles will bound two distinct annuli. See the remark after Theorem 2 regarding this exception.

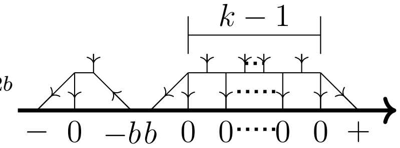
Proposition 3.7 *i) The relations (S) hold in $S_q^{SL_3}(\Sigma)$ for any annulus embedded in Σ.*

ii) The relations (C_k) hold in $\mathcal{S}_q^{SL_3}(\Sigma)$ for all $k \geq 0$.

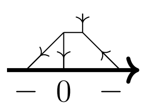
Proof: i) (S) represents an isotopy of webs in the thickened surface $\Sigma \times (0,1)$, so the relation holds in $S_q^{SL_3}(\Sigma)$.

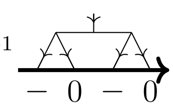
ii) We will proceed by induction on k. (C_0) is the same as (B4), so the statement is true for $k = 0$.

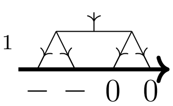
If $k > 0$ we can apply the relation (j) to the horizontal bar to the right of the top left strand to yield

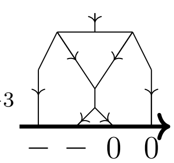
$$-q^{4/3} \sum_{b \in \{-,0,+\}} q^{-2b}$$ 

When $b = 0$ the right connected component of the diagram is zero by relation (B3). When $b = +$ we compute that the left portion of the diagram reduces to



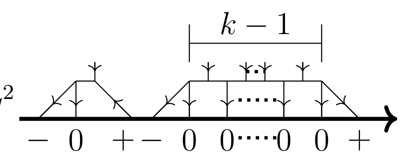
$= -q^{-1/3+1}$ 

$= -q^{-1/3+1}(q^{-1}$ 

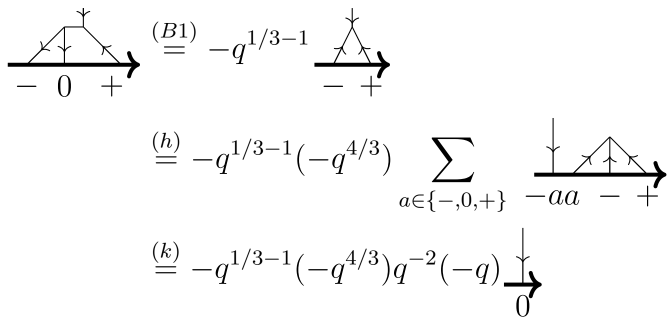
$+ q^{-3}$ 

$)$

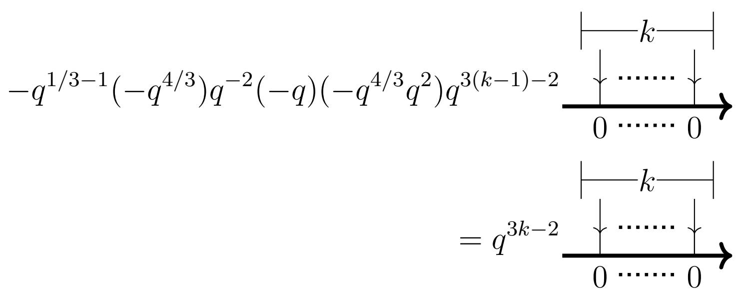
Both of the last terms reduce to 0 using (B3) after applying (I2) to the second diagram. When $b = -$ we are interested in computing

$$-q^{4/3}q^2$$ 

The right part of the diagram can be reduced by induction now while the left part of the diagram can be computed in the following manner:



This all reduces to



which concludes the proof by induction.

■

For the rest of this section, we will assume any boundary arcs in our diagram have an orientation that matches the one appearing in the pictures of the defining boundary relations and that this orientation dictates the height order.

A univalent endpoint of a web diagram is a *bad endpoint* if the strand attached to the endpoint is oriented out of the boundary. For example, the endpoint in the picture on the left of relation (B1) is a bad endpoint while the two endpoints on the right of the relation are good. We say that a pair of two good endpoints on the same boundary arc with states b and a are a *bad pair* if $b > a$ but the endpoint with state b is lower in the height order than the endpoint with state a. For example, the two endpoints on the left of (B2) form a bad pair, while the two endpoints in each diagram of the right side of the relation form a good pair. In the following, by the term *vertices* we mean only trivalent vertices of the web.

Definition 3.8 *We define the complexity of a stated web diagram to be the tuple (#crossings, #bad endpoints, #bad pairs, #vertices, #connected components, #British highways) in* $\mathbb{Z}^6_{\geq 0}$.

We use the lexigocraphic ordering on $\mathbb{Z}^6_{\geq 0}$ and note that each defining relation, each relation (C_k), and each relation (S) involve a single diagram on the left side of the equation while the right side of the equation contains only diagrams of strictly lower complexity than the one on the left side of the equation.

We say that a diagram contains a *reducible feature* if the left side of one of the relations (I1a)-(I4b), (B1)-(B4), (C_k), or (S) applies. If a diagram contains no reducible feature, we call such a diagram an *irreducible diagram.*

Theorem 3.9 *The set of isotopy classes of irreducible diagrams on* Σ *forms a basis for* $S_q^{SL_3}(\Sigma)$.

Remark 3.10 *If* Σ *has a connected component that is a torus, we modify our notion of an irreducible diagram. By omitting the reduction rule (S) on any torus, the proof below can be modified to show that the remaining reduction rules will produce a basis consisting of the set of irreducible diagrams up to isotopy and circle flip moves (S) on any torus.*

Proof: We will apply the Diamond Lemma in much the same setup as [Lê18]. First, we claim that module freely spanned by isotopy classes of web diagrams with our chosen boundary orientations modulo the defining relations along with (C_k) and (S) yields a module isomorphic to $S_q^{SL_3}(\Sigma)$. To do this, one observes that ribbon Reidemeister moves

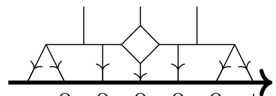
RI, RII, and RIII and the fact that a strand can pass over or under a vertex all follow from the defining interior relations, as shown in [Kup94]. The fact that (C_k) and (S) are redundant relations completes this part of the argument.

Next, we must verify that given a diagram D, the process of iteratively applying the left sides of our relations to D and to its descendants always terminates in a linear combination of irreducible diagrams. This is guaranteed by the fact that our reduction rules involve replacing a diagram by a linear combination of diagrams of strictly lower complexity in our lexicographic ordering, as in Theorem 2.2 of [SW07]. Thus, the set of isotopy classes of irreducible diagrams span $\mathcal{S}_q^{SL_3}(\Sigma)$.

To show that each diagram can be uniquely written as a linear combination of irreducible diagrams, we must show the local confluence of our relations. This is the reason that we had to include the redundant relations (C_k) and (S). We must check that if more than one relation is applicable to a diagram then we can reach a common descendant regardless of which relation we choose to apply. We use the same notion of the *support* of a relation as [Lê18]. If two relations are applicable to a diagram, but their support is disjoint, then the applications of these relations commute with each other, and thus immediately reach a common descendant.

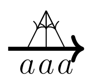
We must find local confluence for relations whose supports overlap nontrivially. If the two relations are both interior relations or (S), then we see by [SW07] that they are locally confluent.

There is one possible way for a the support of an interior relation to intersect the support of a boundary relation: a square could be connected to the top of the relations (C_k) for some $k \geq 2$. The following diagram shows an example of an overlap of (C_4) and (I2).



Such a situation will terminate at 0 no matter which relation (C_k) or (I2) is applied first, as each resulting diagram will provide an opportunity to apply (B3).

Finally, we consider the cases of overlapping supports of the defining boundary relations and the additional relations (C_k). A first easy case is an overlap of (B3) with (B3), which must be of the following form:



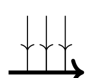
$(a \in \{-, 0, +\})$

Applying (B3) to either the left triangle or the right triangle in the above diagram yields zero.

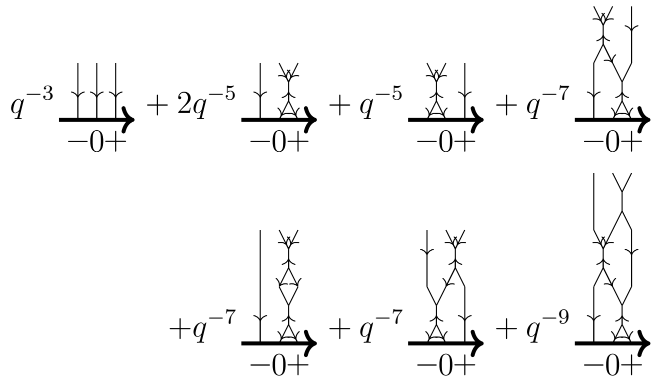
We see that the only other supports that can overlap are those of (B2) with any of (B2), (B3), (B4), and (C_k).

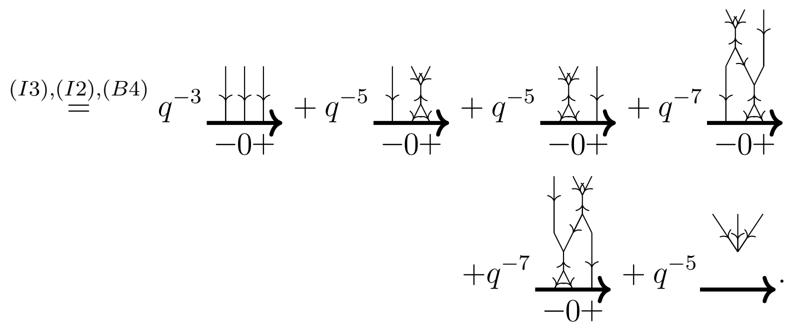
<u>(B2) and (B2)</u>:

If (B2) overlaps with (B2): the overlap must be of the following form.



If we first apply (B2) to the right two endpoints, and then we continue to apply (B2) until there are no longer any bad pairs we obtain:

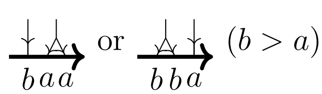




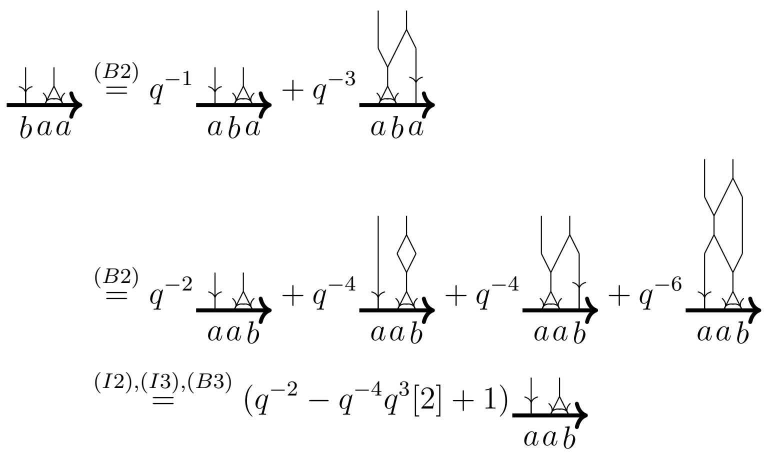
If, instead, we first apply (B2) to the left two endpoints, and then we continue to apply (B2) until there are no longer any bad pairs, we obtain the same linear combination but with the diagrams reflected in a vertical line (but with the state locations and boundary orientation remaining the same). By noting the coefficients in our last equation are symmetric with respect to this reflection, we see that we obtain the same answer in both cases.

(B2) and (B3):

If (B2) overlaps with (B3): the overlap must take one of the following forms.



Both cases are handled symmetrically, so we will focus on the left case. If we apply (B3) first, we obtain zero. So we must show that if we instead apply (B2) first we eventually obtain zero. We do this by computing:



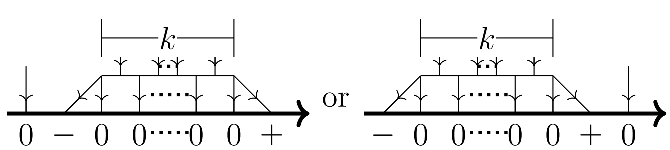
$= 0,$

resolving this case.

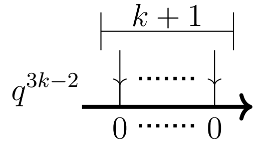
Since (B4) is the same as (C_0) the last overlap we need to check is an overlap between (B2) and (C_k) for any $k \geq 0$.

<u>(B2) and (C_k):</u>

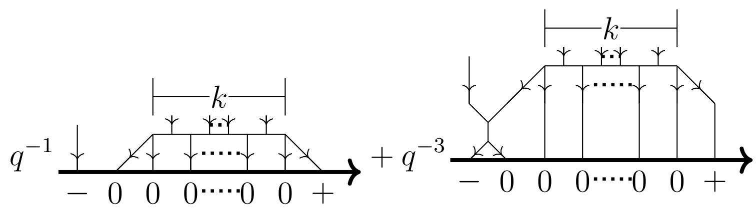
There are four cases for such an overlap. Consider first the following two cases:



These two cases are handled symmetrically, so we will focus on the left case. If we apply (C_k) first, we obtain

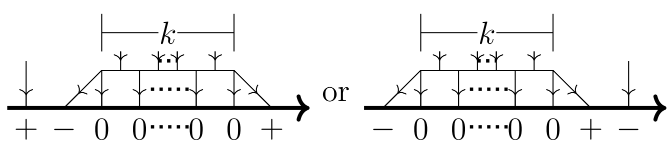


If we apply (B2) first, we obtain



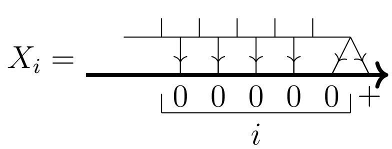
The first term in this linear combination becomes zero after applying (B3). The diagram in the second term is isotopic to the diagram appearing on the left side of (C_{k+1}). After application of (C_{k+1}) we obtain confluence in this case.

The other two possible overlaps between (B2) and (C_k) are of the following forms:



Since these two cases are handled symmetrically, we will focus on the left case.

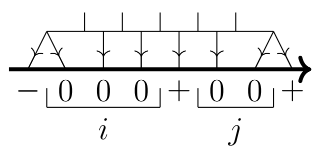
We introduce some notation to simplify this computation. We will use symbols placed next to each other to represent certain diagrams appearing next to each other. We represent the diagrams in the left case above by $\downarrow_+ \cdot C_k$. We denote by 0_i the diagram involving i parallel strands that terminate in good endpoints with states labeled 0. We also denote by X_i the diagram



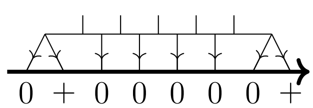
By applying the relation (C_k) to $\downarrow_+ \cdot C_k$ we obtain $q^{3k-2} \downarrow_+ \cdot 0_k$. Consider the effect of using relation (B2) on $q^{3k-2} \downarrow_+ \cdot 0_k$ to get rid of bad pairs, using (B3) at each opportunity. The reduced result is of the form

$$q^{3k-2}\sum_{l=0}^{k} q^{-l}q^{-3(k-l)}0_l \cdot X_{k-l} = \sum_{l=0}^{k} q^{2l-2}0_l \cdot X_{k-l}.$$

We now check that we reach the same reduced result if we instead apply (B2) first to $\downarrow_+ \cdot C_k$. We introduce another piece of notation. The diagram $A_{i,j}$ has i 0-states on the left of the $+$-state and j 0-states on the right.

$A_{i,j} =$ 

We also note that diagrams of the following form



$= 0$

are zero, as can be shown by induction on the number of zero states appearing between the two $+$ states. The inductive hypothesis can be applied after applying (B2) once to improve the order of the states and then applying (I2) to remove the square that forms.

If we apply relation (B2) to $\downarrow_+ \cdot C_k$ then one of the resulting terms will become zero as it is of the form above. We are then left with

$$\downarrow_+ \cdot C_k = q^{-3}A_{0,k+1}.$$

Now consider the diagram $A_{l,m}$ for some $l, m \geq 0$. We have $A_{l,0} = 0$ by relation (B3). For $m > 0$ we can apply relation (B2) followed by (I2) and, ignoring the term with the zero diagram as above, we see that

$$A_{l,m} = q^{-1}A_{l+1,m-1} + q^3 C_l \cdot X_{m-1}.$$

A repeated application of this equation yields

$$\begin{aligned} q^{-3}A_{0,k+1} &= q^{-3}q^3 \sum_{i=0}^{k} q^{-i} C_i \cdot X_{k-i} \\ &\overset{(C_i)}{=} \sum_{i=0}^{k} q^{-i} q^{3i-2} 0_i \cdot X_{k-i} \\ &= \sum_{i=0}^{k} q^{2i-2} 0_i \cdot X_{k-i}. \end{aligned}$$

Thus, we have reached local confluence in this last case. The Diamond Lemma now gives us the result. ■

3.5 The ordinary skein algebra embeds in the stated skein algebra

We define the *ordinary skein algebra* $\mathring{\mathcal{S}}_q^{SL_3}(\Sigma)$ as the module freely spanned by closed webs contained in the interior of Σ modulo the interior relations (I1a)-(I4b) only.

Corollary 3.11 *There is an algebra embedding*

$$\mathring{\mathcal{S}}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma)$$

induced by the inclusion map on diagrams.

Proof: Using the reduction rules (I1a)-(I4b) and (S), the Diamond Lemma applies to give a basis for $\mathring{\mathcal{S}}_q^{SL_3}(\Sigma)$. This set of basis diagrams is a subset of basis diagrams of $\mathcal{S}_q^{SL_3}(\Sigma)$, thus the inclusion induces an injective map. ■

Chapter 4

Triangular decomposition of SL_3 skein algebras

In this chapter we study the building block surfaces $\mathfrak{M}, \mathfrak{B}$, and $\mathfrak{T}$, and use the splitting map to decompose the the skein algebra in terms of these blocks.

4.1 Bialgebra and comodule structure associated to the bigon

The surface made by removing one point from the boundary of a closed disk is called the monogon and will be denoted $\mathfrak{M}$. The surface obtained by removing two points from the boundary of a closed disk is called the bigon and will be denoted $\mathfrak{B}$.

Figure 4.1: Bigon $\mathfrak{B}$ on left and monogon $\mathfrak{M}$ on right.

Proposition 4.1 *We have that*

$$\mathcal{S}_q^{SL_3}(\mathfrak{M}) \cong \mathcal{R}.$$

Proof: We show that $\mathcal{S}_q^{SL_3}(\mathfrak{M})$ is spanned by the empty diagram. The fact that the empty diagram is nonzero follows from the fact that it is irreducible and is thus a basis element.

Consider a web diagram W in $\mathcal{S}_q^{SL_3}(\mathfrak{M})$. We can use relations (I1a) and (I1b) to inductively write W as a linear combination of crossingless diagrams. We can use relations (l) or (m) to get rid of vertices near the boundary. If there are strands between a vertex and the boundary we can apply relations (d) or (f) to create room for the vertex to slide over to the boundary without introducing crossings.

So by induction we can write W as a linear combination of diagrams with no crossings and no vertices. After applying relations (I4a) and (I4b) to get rid of circles, these diagrams only have arcs connected to the single boundary arc. By applying relations (g) and (i), these diagrams become scalar multiples of the empty diagram. ■

We recall that in [Kup94], Kuperberg used an Euler characteristic argument to show that the module spanned by closed webs in the plane is 1-dimensional. We remark that by Proposition 4.1 along with Corollary 3.11, we obtain an alternate proof that Kuperberg's relations are enough to reduce any closed web in the plane to a scalar multiple of the empty web, and that this reduction can be performed algorithmically by iteratively applying the left sides of the interior relations. We also observe that Proposition 4.1 and the algorithm produced by the Diamond Lemma imply that any stated web in $\mathfrak{M}$ can be reduced to a scalar multiple of the empty diagram by iteratively applying just the left sides of the defining relations and (C_k).

We next describe the bialgebra structure of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. For a counit, we will construct an algebra morphism $\varepsilon : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{S}_q^{SL_3}(\mathfrak{M}) \cong \mathcal{R}$. As in [Lê18] we will use an edge inversion map.

Definition 4.2 *If b is a boundary arc of Σ with the orientation given in the defining relations of $\mathcal{S}_q^{SL_3}(\Sigma)$ we define the inversion along b, $inv_b : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma)$ to be the $\mathcal{R}$-module homomorphism defined on web diagrams by reversing the height order of b, switching the states to their negatives, and multiplying by scalars $C_s^{\uparrow}$ and $C_s^{\downarrow}$ for each endpoint on b. Here, we use $C_s^{\downarrow} = -q^{-4/3}$ for each good endpoint on b with any state s and we use $C_t^{\uparrow} = -q^{-4/3}q^{-2t}$ for each bad endpoint on b with a state $t \in \{-, 0, +\}$.*

Proposition 4.3 *The map inv_b defined above is a well-defined $\mathcal{R}$-module automorphism.*

Proof: We must check that the map respects the defining boundary relations. To do so, we apply the map to both sides of a boundary relation and then reduce the results using the Diamond Lemma algorithm to see that we obtain the same answers in each case. Thus, the map is well-defined. Alternatively, it is easier to use the relations in 3.2 to check that inv_b respects the relations (c),(e),(h),(j),(k), and (n). We then observe that these relations imply relations (B1)-(B4). To check that it is an automorphism, one needs to check that the obvious candidate for its inverse is well-defined in the same way.

■

We define $\varepsilon : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{S}_q^{SL_3}(\mathfrak{M})$ to be the map given by the result of inverting the the right boundary arc e_r of the bigon with inv_{e_r} and then filling in the puncture. The map is well-defined since it is a composition of well-defined maps. The fact that it is an algebra morphism is an easy diagrammatic observation, and can be seen in the same way as in [CL19].

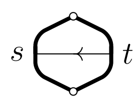
The comultiplication $\Delta : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{S}_q^{SL_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B})$ is given by the splitting morphism Δ_c for an ideal arc c that travels from the bottom puncture to the top puncture. By Theorem 3.6, Δ is an algebra morphism and satisfies the coassociativity property.

Figure 4.2: Generator α_{st} on left and generator β_{st} on right.

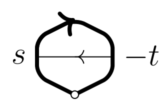
To check that ε satisfies the counit property, we only need to check on generators. To find a nice set of generators, we use the method in the proof of Proposition 4.1 to see that any web in the bigon can be written as a linear combination of webs which have no crossing, no vertex, and no circle. Any trivial arcs that start and end on the same boundary arc can be replaced by scalars, and we are left with a linear combination of webs containing only parallel and antiparallel strands with one endpoint on each boundary arc. Thus, $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ has a generating set consisting of diagrams, each of which contain a single strand traveling from one boundary arc of the diagram to the other. We denote such diagrams α_{st} and β_{st} depending on the strand orientation and states.

We use our diagrammatic definition of ε to compute that

$$\varepsilon(\alpha_{st}) = \varepsilon(\quad)$$



$$= -q^{-4/3}q^{-2t}(\quad)$$



$$\overset{(i)}{=} -q^{-4/3}q^{-2t}(-q^{4/3}q^{2t}\delta_{s-t,0})$$

$$= \delta_{st}$$

We similarly compute that $\varepsilon(\beta_{st}) = \delta_{st}$.

By the definition of Δ, we compute that

$$\Delta(\alpha_{st}) = \sum_{l\in\{-,0,+\}} \alpha_{sl} \otimes \alpha_{lt}.$$

Similarly,

$$\Delta(\beta_{st}) = \sum_{l\in\{-,0,+\}} \beta_{sl} \otimes \beta_{lt}.$$

These equations allow us to verify that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(\alpha_{st}) = \alpha_{st} = (\mathrm{id} \otimes \varepsilon) \circ \Delta(\alpha_{st}).$$

The same equations hold for β_{st} and we have proven the following proposition.

Proposition 4.4 *The algebra $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ has a natural biaglebra structure given by the maps Δ and ε defined above.*

The ingredients here are now the same as in [CL19] and so we obtain an analogue of their Proposition 4.1

Proposition 4.5 *Suppose b is a boundary arc of Σ. The map defined by splitting Σ along an ideal arc isotopic to b so as to split off a bigon $\mathfrak{B}$ whose right edge is b gives an $\mathcal{R}$-algebra homomorphism*

$$\Delta_b : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}).$$

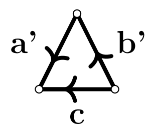
This endows $\mathcal{S}_q^{SL_3}(\Sigma)$ with a right comodule-algebra structure over $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. Similarly, the map ${}_b\Delta$ defined by splitting off from Σ a bigon $\mathfrak{B}$ whose left edge is b gives an $\mathcal{R}$-algebra homomorphism

$${}_b\Delta : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\Sigma).$$

This endows $\mathcal{S}_q^{SL_3}(\Sigma)$ with a left comodule-algebra structure over $\mathcal{S}_q^{SL_3}(\mathfrak{B})$.

4.2 Gluing or cutting along a triangle

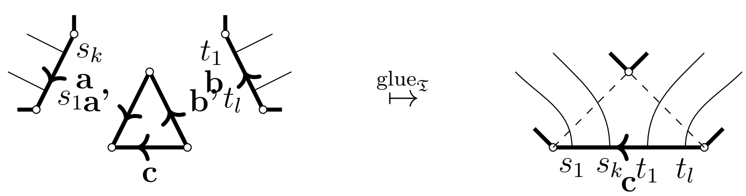
Consider a punctured bordered surface Σ with two distinct boundary arcs a and b. Also consider an ideal triangle $\mathfrak{T}$, which is a disk with three points removed from its boundary. We will denote the punctured bordered surface $\Sigma \# \mathfrak{T}$ obtained by gluing Σ to $\mathfrak{T}$ along a and b. We label the edges of $\mathfrak{T}$ as in the following diagram.



There is a well-defined $\mathcal{R}$-module homomorphism:

$$\text{glue}_{\mathfrak{T}} : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma \# \mathfrak{T})$$

defined on diagrams by continuing the strands with endpoints on a or b until they reach c. The map is depicted in the following diagram.

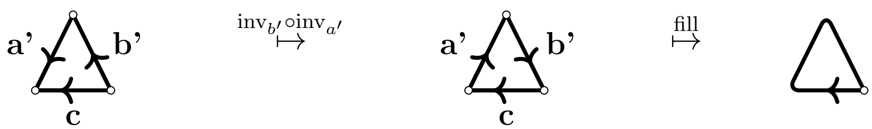


The map $\text{glue}_{\mathfrak{T}}$ was introduced in [CL19] for the SL_2 case. In general, $\text{glue}_{\mathfrak{T}}$ does not respect the algebra structure, but it gives rise to an algebra structure that is called a *self braided tensor product* in [CL19]. In Section 4.5, we describe a special case of this structure, called the *braided tensor product.* In this section, we are interested in $\text{glue}_{\mathfrak{T}}$ because it is an $\mathcal{R}$-linear isomorphism. We will show this by constructing a natural inverse.

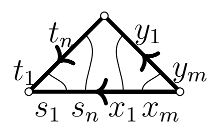
The triangle $\mathfrak{T}$ admits an analogue of the bigon's counit. We define

$$\varepsilon_{\mathfrak{T}} : \mathcal{S}_q^{SL_3}(\mathfrak{T}) \to \mathcal{S}_q^{SL_3}(\mathfrak{M})$$

as the map obtained by applying $inv_{b'} \circ inv_{a'}$ and then filling in the punctures between c and a' and between a' and b' as in the following figure.



Since $\varepsilon_{\mathfrak{T}}$ is defined as a composition of well-defined $\mathcal{R}$-linear maps it is an $\mathcal{R}$-linear map. What makes $\varepsilon_{\mathfrak{T}}$ an analogue of ε is that if $\varepsilon_{\mathfrak{T}}$ is applied to a diagram W of the following form (with any choice of strand orientations):

$W =$ 

the result is

$$\varepsilon_{\mathfrak{T}}(W) = (\prod_{i=1}^{n} \delta_{s_i,t_i})(\prod_{j=1}^{m} \delta_{x_j,y_j}).$$

We next define an $\mathcal{R}$-linear map

$$\mathrm{cut}_{\mathfrak{T}} : \mathcal{S}_q^{SL_3}(\Sigma \# \mathfrak{T}) \to \mathcal{S}_q^{SL_3}(\Sigma).$$

Recall the notation of the projection $p : \Sigma \sqcup \mathfrak{T} \to \Sigma \# \mathfrak{T}$ associated to gluing Σ to the triangle along a and b. If $a'' = p(a') = p(a)$ and $b'' = p(b') = p(b)$, we define $\mathrm{cut}_{\mathfrak{T}}$ by

$$\mathrm{cut}_{\mathfrak{T}} = (\varepsilon_{\mathfrak{T}} \otimes \mathrm{id}) \circ (\Delta_{b''} \circ \Delta_{a''}).$$

Since $(\Delta_{b''} \circ \Delta_{a''})$ cuts out a triangle, we view it as a linear map $\mathcal{S}_q^{SL_3}(\Sigma\#\mathfrak{T}) \to S_q^{SL_3}(\mathfrak{T}) \otimes S_q^{SL_3}(\Sigma)$, so the composition above makes sense.

Proposition 4.6 *The $\mathcal{R}$-linear maps $glue_{\mathfrak{T}}$ and $cut_{\mathfrak{T}}$ satisfy*

$$cut_{\mathfrak{T}} \circ glue_{\mathfrak{T}} = id_{\mathcal{S}_q^{SL_3}(\Sigma)}$$

and

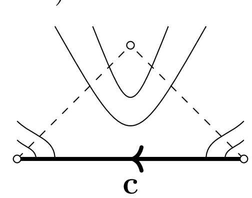
$$glue_{\mathfrak{T}} \circ cut_{\mathfrak{T}} = id_{\mathcal{S}_q^{SL_3}(\Sigma\#\mathfrak{T})}.$$

Proof: We will check each equality on a spanning set for the skein algebra involved. For the case of $S_q^{SL_3}(\Sigma)$ we consider the spanning set consisting of all stated web diagrams. Suppose D is a stated web diagram on Σ. If we examine the diagrams that appear in the triangle cut out by $(\Delta_{b''} \circ \Delta_{a''}) \circ \mathrm{glue}_{\mathfrak{T}}(D)$, we see that they are all of the form W above. Thus, the computation for $\varepsilon_{\mathfrak{T}}(W)$ above shows that

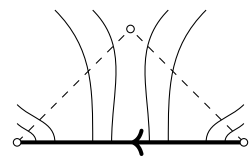
$$(\varepsilon_{\mathfrak{T}} \otimes \mathrm{id})(\Delta_{b''} \circ \Delta_{a''}) \circ \mathrm{glue}_{\mathfrak{T}}(D) = D.$$

This proves the first equality of Proposition 4.6.

For the second equality, we wish to use a smaller spanning set of $\mathcal{S}_q^{SL_3}(\Sigma\#\mathfrak{T})$. Consider a stated web diagram D on $(\Sigma\#\mathfrak{T})$ and examine it in a neighborhood of $p(\mathfrak{T})$. By applying an isotopy we can guarantee that $p(\mathfrak{T})$ contains only arcs, and that any arc that enters the triangle through one of the sides either leaves through the other side or terminates at an endpoint on c. After such an isotopy, we obtain a diagram of the following form (for some choice of strand orientations):



Using relations (f) and (j) we can break up the strands that pass through both a'' and b'' and thus write our diagram D as a linear combination of diagrams of the following form:



So a spanning set consists of diagrams on $\Sigma\#\mathfrak{T}$ that are of the above form in a neighborhood of $p(\mathfrak{T})$. Let E be such a diagram. We see that the triangles that appear in the terms of $(\Delta_{b''} \circ \Delta_{a''})(E)$ are all of the form W above. Again the computation of $\varepsilon_{\mathfrak{T}}(W)$ above allows us to see that

$$\text{glue}_{\mathfrak{T}} \circ (\varepsilon_{\mathfrak{T}} \otimes \text{id}) \circ (\Delta_{b''} \circ \Delta_{a''})(E) = E.$$

This proves the second equality of Proposition 4.6.

■

Corollary 4.7 *Suppose c is a boundary arc of a punctured bordered surface $\bar{\Sigma}$ and that a'' and b'' are ideal arcs with disjoint interiors such that $a'' \cup b'' \cup c$ bound an ideal triangle. Then both $\Delta_{a''}$ and $\Delta_{b''}$ are injective.*

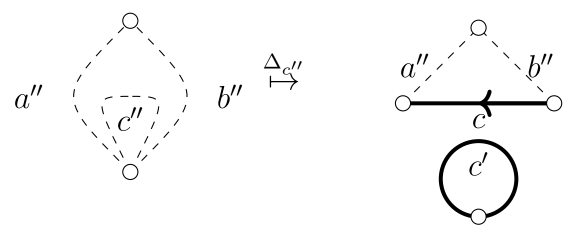
Proof: Let $\mathfrak{T}$ be the ideal triangle that is split off from $\bar{\Sigma}$ if $\Delta_{b''} \circ \Delta_{a''}$ is applied. Then $\bar{\Sigma} = \Sigma\#\mathfrak{T}$ for the punctured bordered surface Σ containing two distinct boundary arcs a and b resulting from the splitting maps. Proposition 4.6 tells us that $\text{cut}_{\mathfrak{T}}$ is injective. By the definition of $\text{cut}_{\mathfrak{T}}$ we see that $\Delta_{b''} \circ \Delta_{a''}$ is injective. Thus, $\Delta_{a''}$ is injective. By Theorem 3.6, we see that $\Delta_{b''} \circ \Delta_{a''} = \Delta_{a''} \circ \Delta_{b''}$. Thus, $\Delta_{b''}$ is injective as well. ■

4.3 The triangular decomposition

We are now able to prove the following addendum to Theorem 3.6.

Theorem 4.8 *Suppose $\bar{\Sigma}$ is a punctured bordered surface and a'' is an ideal arc on $\bar{\Sigma}$. Then the map $\Delta_{a''}$ is injective.*

Proof: Let b'' be an ideal arc isotopic to a'' so that the ideal arcs have disjoint interiors and bound a bigon. Let c'' be an ideal arc that bounds a monogon whose ideal vertex is an endpoint of a'', and such that a'', b'', c'' have disjoint interiors and $a'' \cup b'' \cup c''$ bounds an ideal triangle. The following diagram depicts the map $\Delta_{c''}$.



Consider the application of $\Delta_{c''}$ to the set of basis diagrams described in Theorem 3.9. Each irreducible diagram D can be isotoped so that it does not intersect the monogon bounded by c''. This allows us to observe that $\Delta_{c''}(D)$ is an irreducible diagram on its surface as well, and that the isotopy class of D can be completely determined by the isotopy class of this irreducible representative of $\Delta_{c''}(D)$. Thus, $\Delta_{c''}$ maps a basis to a linearly independent set and we conclude that $\Delta_{c''}$ is injective.

After splitting off the monogon bounded by c'' we are left with a surface Σ that contains a boundary arc c such that $p(c) = c''$. Now the ideal arcs a'', b'' and the boundary arc c satisfy the hypothesis of 4.7. By the corollary, $\Delta_{a''}$ is injective on the image of $\Delta_{c''}$ and thus $\Delta_{a''} \circ \Delta_{c''}$ is an injective map. The fact that these maps commute implies that $\Delta_{a''}$ is injective on $\mathcal{S}_q^{SL_3}(\bar{\Sigma})$ as well. ■

Now that we have determined the splitting morphisms have trivial kernels, we discuss their images.

Suppose Σ is a punctured bordered surface with distinct boundary arcs a and b. Let $\bar{\Sigma} = \Sigma/(a = b)$ and denote by c the common image of a and b under the gluing map. Recall the comodule structure maps associated to the boundary arcs a, and b. We will be interested in $\Delta_a : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B})$ and $\tau \circ_b \Delta : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B})$, where τ only transposes the tensor factors. We are interested in the following result.

Theorem 4.9 *Let $\bar{\Sigma} = \Sigma/(a = b)$ and denote by c the common image of a and b under the gluing map. Then we have*

$$im(\Delta_c) = ker(\Delta_a - \tau \circ_b \Delta).$$

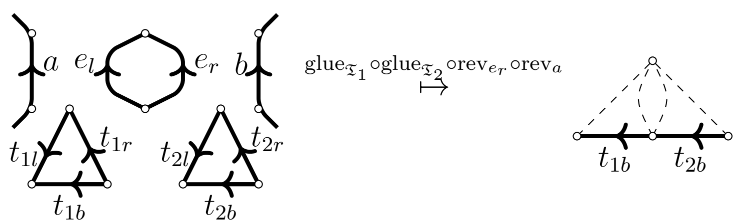
Proof: The inclusion $\mathrm{im}(\Delta_c) \subseteq \ker(\Delta_a - \tau \circ_b \Delta)$ follows by coassociativity of splitting $\bar{\Sigma}$ along c and an ideal arc isotopic to c.

To prove the other inclusion, we assume that $y \in \mathcal{S}_q^{SL_3}(\Sigma)$ satisfies $\Delta_a(y) = \tau \circ_b \Delta(y)$. Our goal is to find some $x \in \mathcal{S}_q^{SL_3}(\bar{\Sigma})$ such that $y = \Delta_c(x)$. The element y is represented by a linear combination of stated web diagrams on Σ. We will find a candidate for x by trying to weld the strands with endpoints on a or on b to each other. This process uses a map similar to the edge inversion maps inv before, but this time with a different choice of scalars associated to the endpoints.

For a boundary arc e with positive orientation, we define the edge reversal map rev_e to be the $\mathcal{R}$-linear automorphism of the stated skein module that reverses the height order on e, flips the states to their negatives and multiplies by the following scalars for each endpoint on e: ${}^{\downarrow}_{s}C = -q^{-4/3}q^{2s}$ for good endpoints with a state s and ${}^{\uparrow}_{s}C = -q^{-4/3}$ for bad endpoints with a state s. We can check that this map is well-defined and an automorphism in the same way that we checked this for inv_e.

Let $z = \Delta_a(y) = \tau \circ_b \Delta(y)$. Denote the left boundary arc of the bigon of $\Sigma \sqcup \mathfrak{B}$ by e_l and the right arc by e_r. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two triangles. We will use the gluing maps $\text{glue}_{\mathfrak{T}}$ defined in Section 4.2. Denote the left, right, and bottom edges of the triangles t_{1l}, t_{2l}, t_{1r}, t_{2r}, and t_{1b}, t_{2b}, respectively. We will consider the result of reversing the arc a, reversing the arc e_r, then gluing to the triangles. To glue to $\mathfrak{T}_2$ we glue b to t_{2r} and glue e_r to t_{2l}. To glue to $\mathfrak{T}_1$, we glue e_l to t_{1r} and glue a to t_{1l}.

We can write the new element as $\text{glue}_{\mathfrak{T}_1} \circ \text{glue}_{\mathfrak{T}_2} \circ \text{rev}_{e_r} \circ \text{rev}_a(z)$. This gluing is depicted in the following diagram.



First, we view z as $z = \tau \circ_b \Delta(y)$. Write y as a linear combination of diagrams D_i. For each i, $\tau \circ_b \Delta(D_i)$ is a linear combination of diagrams D_{ij}. Each D_{ij} has k_i endpoints on e_r, k_i endpoints on b, and the states of corresponding endpoints match. After applying rev_{e_r} to D_{ij} and then gluing to $\mathfrak{T}_2$, we see that there are $2k_i$ endpoints on t_{2b}, and that the endpoints which are k_i-th and $k_i + 1$-st in the height order have opposite states and opposite orientations. The scalars associated with the application of rev_{e_r} guarantee that relations (d) or (f) are applicable and allow us to reduce the number of endpoints on t_{2b}. After applying these relations k_i times for each D_i, we see that we can write $\text{glue}_{\mathfrak{T}_1} \circ \text{glue}_{\mathfrak{T}_2} \circ \text{rev}_{e_r} \circ \text{rev}_a(z)$ as a linear combination of diagrams, where no diagram has an endpoint on t_{2b}. As no reduction rule from our Diamond Lemma algorithm can result in an endpoint appearing on a boundary arc that previously contained no endpoints, we see that when we write $\text{glue}_{\mathfrak{T}_1} \circ \text{glue}_{\mathfrak{T}_2} \circ \text{rev}_{e_r} \circ \text{rev}_a(z)$ as a linear combination of our basis diagrams, each basis diagram that appears in the linear combination has no endpoints on t_{2b}.

Next, we view z as $z = \Delta_a(y)$. In a similar way as the last paragraph, we see that after applying rev_a and gluing to $\mathfrak{T}_1$, we can apply relations (d) and (f) to write $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ as a linear combination of basis diagrams such that no diagram has an endpoint on t_{1b}. By the uniqueness of this linear combination we see that we can write it as a linear combination of basis diagrams so that no diagram appearing in the linear combination has an endpoint on t_{1b} or on t_{2b}.

Now, $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ is a linear combination of basis diagrams on $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2$. Consider the surface $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2 \setminus (t_{1b} \cup t_{2b})$. This is not a punctured bordered surface, but depending on whether the appropriate endpoint of c was a boundary puncture or was an interior puncture, this surface is either a punctured bordered surface missing an interval on its boundary or it is a punctured bordered surface missing a boundary circle. In either case, it is naturally diffeomorphic to the original punctured bordered surface $\bar{\Sigma}$ by replacing this missing boundary interval or boundary circle with a single puncture. There is a linear map defined on the submodule of $\mathcal{S}_q^{SL_3}((\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2)$ spanned by basis diagrams that have no endpoints on t_{1b} or t_{2b} that takes such a basis diagram and embeds it in $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2 \setminus (t_{1b} \cup t_{2b})$. After applying this map to $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ and composing with our diffeomorphism, we obtain our candidate $x \in \mathcal{S}_q^{SL_3}(\bar{\Sigma})$.

To see that x is the correct choice, we consider $\Delta_c(x)$ and then apply the same process to it as we did to y and observe that

$$\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a \circ \Delta_a(y) = \mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a \circ \Delta_a(\Delta_c(x)).$$

The injectivity of the maps involved allow us to conclude that $\Delta_c(x) = y$. ■

We say that a punctured bordered surface is *ideal triangulable* if it can be obtained from a finite collection of disjoint triangles by gluing some pairs of edges together. It is known that a punctured bordered surface is ideal triangulable if it has no connected component that is one of the following: a closed surface, a sphere with fewer than three punctures, a bigon, or a monogon.

If Σ is an ideal triangulable punctured bordered surface, then the images of the glued edges are ideal arcs on Σ with disjoint interiors. These form the set of interior edges $\mathcal{E}$ for the *ideal triangulation* of Σ. Let $p : \sqcup_{i=1}^n \mathfrak{T}_i \to \Sigma$ be the gluing map. If $e \in \mathcal{E}$, then its preimage $p^{-1}(e) = \{e', e''\}$ consists of two triangle edges. The composition Δ of the splitting maps Δ_e for $e \in \mathcal{E}$ gives an algebra embedding

$$\Delta : \mathcal{S}_q^{SL_3}(\Sigma) \to \bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i).$$

The composition ${}^L\Delta$ of all left comodule maps ${}_{e''}\Delta$ gives a map

$${}^L\Delta : \bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i) \to (\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{SL_3}(\mathfrak{B})) \otimes (\bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i)).$$

The composition Δ^R of all right comodule maps $\Delta_{e'}$ gives a map

$$\Delta^R : \bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i) \to (\bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i)) \otimes (\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{SL_3}(\mathfrak{B})).$$

Then Theorem 4.8 and Theorem 4.9 allow us to observe the following corollary.

Corollary 4.10 *If Σ admits an ideal triangulation with a set of interior edges $\mathcal{E}$, then the following sequence is exact:*

$$0 \to \mathcal{S}_q^{SL_3}(\Sigma) \xrightarrow{\Delta} \bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i) \xrightarrow{\Delta^R - \tau\circ {}^L\Delta} (\bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i)) \otimes (\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{SL_3}(\mathfrak{B})).$$

4.4 The stated skein algebra of the bigon is $\mathcal{O}_q(SL_3)$

In [CL19], it was shown that the Kauffman bracket stated skein algebra of the bigon is isomorphic to $\mathcal{O}_q(SL_2)$ as a Hopf algebra (with a suitable renormalization of q). They showed this by defining a bialgebra map between $\mathcal{O}_q(SL_2)$ and the Kauffman bracket stated skein algebra of the bigon. The fact that this map is an isomorphism follows because it maps the canonical basis of the stated skein algebra to a well known basis of $\mathcal{O}_q(SL_2)$. There is an analogous isomorphism between our SL_3 stated skein algebra of the bigon and $\mathcal{O}_q(SL_3)$. However, the proof here will require us to define maps in both directions since it is not otherwise clear that the canonical basis of the SL_3 stated skein algebra of the bigon matches up with a basis of $\mathcal{O}_q(SL_3)$.

We first recall the R-matrix definition of $\mathcal{O}_q(SL_3)$. Consider the free $\mathcal{R}$-module V with basis $\{x_1, x_2, x_3\}$. The standard R-matrix for SL_3 is a linear map

$$R : V \otimes V \to V \otimes V$$

defined by

$$R(x_i \otimes x_j) = q^{-1/3} \begin{cases} q x_i \otimes x_j & \text{(if } i = j) \\ x_j \otimes x_i & \text{(if } i > j) \\ x_j \otimes x_i + (q - q^{-1}) x_i \otimes x_j & \text{(if } i < j). \end{cases}$$

We develop some notation for the matrix entries R_{ij}^{kl} of R. We have that $R(x_i \otimes x_j)$ is uniquely written as

$$R(x_i \otimes x_j) = \sum_{1 \le k,l \le 3} R_{ij}^{kl} x_k \otimes x_l.$$

We define $\mathcal{O}_q(SL_3)$ as the free $\mathcal{R}$-algebra generated by elements $\{X_{ij}\}_{1 \le i,j \le 3}$ modulo the following relations

$$\begin{cases} \sum_{1\le k,l\le 3} R_{ij}^{kl} X_{km} X_{ln} = \sum_{1\le k,l\le 3} R_{kl}^{mn} X_{ik} X_{jl} \quad (\text{for } 1 \le i,j,m,n \le 3) \\ \sum_{\sigma\in S_3} (-q)^{l(\sigma)} X_{\sigma_1 1} X_{\sigma_2 2} X_{\sigma_3 3} = 1. \end{cases}$$

Here, we consider $(\sigma_1, \sigma_2, \sigma_3) = (1,2,3)$ the identity permutation.

The left side of the second equation is called the *quantum determinant*, $\det_q$, of the matrix of generators $(A)_{ij} = X_{ij}$. We will also make use of notation $A[i|j]$ to mean the quantum minor of A after deleting row i and column j.

$\mathcal{O}_q(SL_3)$ has a Hopf algebra structure with structure maps given by

$$\varepsilon(X_{ij}) = \delta_{ij}$$

and

$$\Delta(X_{ij}) = \sum_{k=1}^{3} X_{ik} \otimes X_{kj}.$$

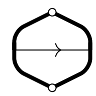
The antipode $S : \mathcal{O}_q(SL_3) \to \mathcal{O}_q(SL_3)$ is defined by

$$S(X_{ij}) = (-q)^{i-j} A[j|i].$$

For the purpose of notation to match up our stated skein algebra with the standard definition of $\mathcal{O}_q(SL_3)$, we define a bijection $t : \{1,2,3\} \to \{-,0,+\}$ given by $t(1) = +$, $t(2) = 0$, $t(3) = -$. Since t reverses the order we've placed on the sets $\{1,2,3\}$ and $\{-,0,+\}$ we will have to take care when we apply relations (k)-(n) to diagrams.

Proposition 4.11 *There is a unique bialgebra morphism* $\phi : \mathcal{O}_q(SL_3) \to \mathcal{S}_q^{SL_3}(\mathfrak{B})$ *de-*

fined by

$\phi(X_{ij}) =$ $t(i)$  $t(j)$

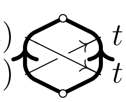
Proof: Since the elements X_{ij} generate $\mathcal{O}_q(SL_3)$, the morphism will be unique if it exists. By construction, such a morphism will preserve the bialgebra structure. To prove that ϕ gives a well-defined algebra morphism we must check that it respects the defining relations of $\mathcal{O}_q(SL_3)$. We must show that the relations

$$\sum_{1\leq k,l\leq 3} R^{kl}_{ij}\phi(X_{km})\phi(X_{ln}) = \sum_{1\leq k,l\leq 3} R^{mn}_{kl}\phi(X_{ik})\phi(X_{jl})$$

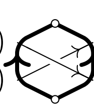
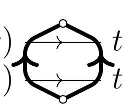
and

$$\sum_{\sigma\in S_3}(-q)^{l(\sigma)}\phi(X_{\sigma_1 1})\phi(X_{\sigma_2 2})\phi(X_{\sigma_3 3}) = 1$$

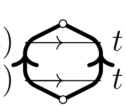
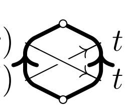
hold in $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. For this, we recall the bialgebra structure of the bigon given in Section 4.1. We consider the result of applying $(\varepsilon\otimes \mathrm{id})\circ\Delta$ to the following diagram in two different ways.

$t(i)$, $t(j)$  $t(m)$, $t(n)$

For the first way, we split the bigon along an ideal arc that stays to the right of the crossing and obtain

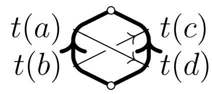
$\sum_{1\leq k,l\leq 3} \varepsilon\Big($ $t(i)$, $t(j)$  $t(k)$, $t(l)$ $\Big)$ $t(k)$, $t(l)$  $t(m)$, $t(n)$

For the second way, we split the bigon along an ideal arc that stays to the left of the crossing and then apply $\mathrm{id}\otimes\varepsilon$.

$\sum_{1\leq k,l\leq 3}$ $t(i)$, $t(j)$  $t(k)$, $t(l)$ $\varepsilon\Big($ $t(k)$, $t(l)$  $t(m)$, $t(n)$ $\Big)$

The bialgebra axiom $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta$ along with the isotopy invariance of the splitting map guarantees that both answers must be the same.

We can use the defining relations to compute that

$\varepsilon\Big($ 

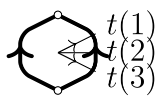
$\Big) = R^{cd}_{ab}$

Equating our two answers shows that the relations

$$\sum_{1\leq k,l\leq 3} R^{kl}_{ij}\phi(X_{km})\phi(X_{ln}) = \sum_{1\leq k,l\leq 3} R^{mn}_{kl}\phi(X_{ik})\phi(X_{jl})$$

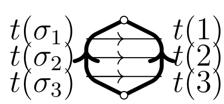
hold in $\mathcal{S}_q^{SL_3}(\mathfrak{B})$.

Next, we consider the following diagram



On one hand, we can evaluate this diagram using relation (k) from Section 3.2 along the right edge of the bigon. On the other hand, we could use relation (l) along the left edge of the bigon.

This gives us the relation

$q^{-2} = q^{-2} \sum_{\sigma \in S_3} (-q)^{l(\sigma)}$ 

Thus, the relation

$$\sum_{\sigma \in S_3} (-q)^{l(\sigma)}\phi(X_{\sigma_1 1})\phi(X_{\sigma_2 2})\phi(X_{\sigma_3 3}) = 1$$

holds in $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. Thus, ϕ is well-defined.

■

To prove that ϕ is an isomorphism, we will construct an inverse function. We will define an algebra morphism $\psi : S_q^{SL_3}(\mathfrak{B}) \to \mathcal{O}_q(SL_3)$ by defining it on diagrams and then checking that it is well-defined.

In order for ψ to be the inverse of ϕ we are forced to define it on the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ from Section 4.1 as

$$\psi(\beta_{t(i)t(j)}) = X_{ij}$$

and

$$\psi(\alpha_{t(i)t(j)}) = (-q)^{j-i} A[4-i|4-j].$$

As was noted in Section 4.1, the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ generate $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. So the values of ψ on these diagrams would determine ψ on $\mathcal{S}_q^{SL_3}(\mathfrak{B})$. However, as we do not a priori have a definition of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ as a quotient of a free algebra by relators, it will be tricky to check that the map is well-defined. Instead, we have a definition of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ as a quotient of a free module and so we will define ψ on any diagram by giving specific directions on how to write the diagram in terms of the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ and then check that this process leads to a well-defined map.

Given a diagram D, we obtain $\psi(D)$ by performing the following algorithm:

- Apply Δ by splitting D near the right boundary arc of $\mathfrak{B}$ so that $\Delta(D)$ is written as

$$\Delta(D) = \sum D_i \otimes E_i,$$

  where the diagrams E_i each contain only parallel and antiparallel strands.

- Apply $(\varepsilon \otimes \mathrm{id})$ to $\Delta(D)$ to write

$$(\varepsilon \otimes \mathrm{id})\Delta(D) = \sum \varepsilon(D_i) E_i.$$

- Obtain
$$\psi(D) = \sum \varepsilon(D_i)\psi(E_i) \in \mathcal{O}_q(SL_3),$$
where $\psi(E_i)$ is determined by the values of $\psi(\alpha_{t(i)t(j)})$ and $\psi(\beta_{t(i)t(j)})$ given above.

Proposition 4.12 *The map $\psi : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{O}_q(SL_3)$ described above is a well-defined algebra homomorphsim.*

Proof: We observe that if ψ is well-defined, then it does respect the natural multiplication of diagrams in $\mathcal{S}_q^{SL_3}(\mathfrak{B})$.

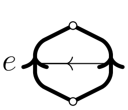
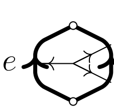
We must check that the process outlined in the bulletpoints above respects the defining relations of the stated skein algebra. We split the relations into three cases: interior relations, boundary relations along the left boundary arc of $\mathfrak{B}$, boundary relations along the right boundary arc of $\mathfrak{B}$.

Consider a relation falling under the first two cases. Such a relation only affects the diagrams D_i during the process. Since ε is well-defined, application of such relations will result in identical representatives in $\mathcal{O}_q(SL_3)$, and so the process respects these relations.

The case of a relation along the right boundary arc of $\mathfrak{B}$ is more difficult since it will change the diagrams E_i and will thus ultimately produce different representatives in $\mathcal{O}_q(SL_3)$. It is our task to show that these representatives are equivalent. We handle each relation separately.

Relation (B1):

To prove that ψ respects relation (B1) it will suffice to check that

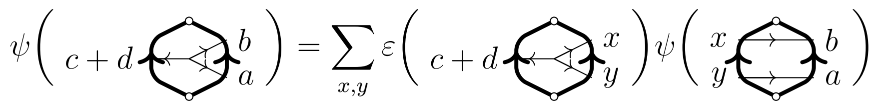
$\psi\Big($ e  $a+b$ $\Big) = (-1)^{a+b} q^{-1/3-(a+b)} \psi\Big($ e  b a $\Big)$

for any states $e, a, b \in \{-, 0, +\}$ with $a < b$.

Fix such e, a, b and let $i = t^{-1}(e)$ and $j = t^{-1}(a+b)$ be the corresponding integers in $\{1, 2, 3\}$. Then by the definition of ψ, the left side of our relation is $(-q)^{j-i}A[4-i|4-j]$.

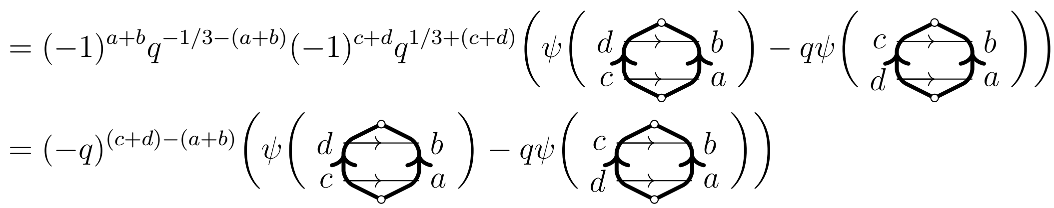
We now compute the right side of the equation. It will be convenient to let c, d be the unique states in $\{-, 0, +\}$ such that $c < d$ and $c + d = e$.

By the definition of ψ, we compute that



We will denote the values of the counit appearing in the above equation as $\varepsilon_{c+d,x,y}$. We use (B3) and (B1) to compute that $\varepsilon_{c+d,x,y} = 0$ unless $\{x, y\} = \{c, d\}$ and we use (B2) to see that $\varepsilon_{c+d,c,d} = -q\varepsilon_{c+d,d,c}$. We also use (B1) to compute that $\varepsilon_{c+d,d,c} = (-1)^{c+d}q^{1/3+(c+d)}$.

The right side of our relation becomes



We check that this formula agrees with

$$(-q)^{t^{-1}(a+b)-t^{-1}(c+d)}(X_{t^{-1}(d)t^{-1}(b)}X_{t^{-1}(c)t^{-1}(a)} - qX_{t^{-1}(c)t^{-1}(b)}X_{t^{-1}(d)t^{-1}(a)})$$

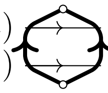
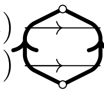
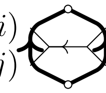
which is

$$(-q)^{(j-i)}A[4-i|4-j],$$

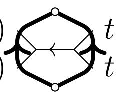
as required.

<u>Relation (B2):</u>

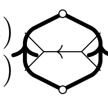
To show that ψ respects relation (B2) it suffices to check that the following relation holds in $\mathcal{O}_q(SL_3)$.

$\psi\Big($ $t(i)$, $t(j)$  $t(m)$, $t(n)$ $\Big) = q^{-1}\psi\Big($ $t(i)$, $t(j)$  $t(n)$, $t(m)$ $\Big) + q^{-3}\psi\Big($ $t(i)$, $t(j)$  $t(n)$, $t(m)$ $\Big)$

for $i, j, m, n \in \{1, 2, 3\}$ such that $n < m$. So we must show that

$\psi\Big($ $t(i)$, $t(j)$  $t(n)$, $t(m)$ $\Big) = q^3 X_{im}X_{jn} - q^2 X_{in}X_{jm}.$

From relation (I1a) and the computations of $\varepsilon(\beta_{st})$ from Section 4.1, we compute that

$\varepsilon\Big($ $t(i)$, $t(j)$  $t(n)$, $t(m)$ $\Big) = q^{3+1/3}R^{kl}_{ij} - q^4\delta_{ik}\delta_{jl}.$

Thus, we must show that

$$\Big(\sum_{k,l} q^{3+1/3}R^{kl}_{ij}X_{kn}X_{lm}\Big) - q^4 X_{in}X_{jm} = q^3 X_{im}X_{jn} - q^2 X_{in}X_{jm}.$$

We apply the identity

$$\sum_{k,l} R^{kl}_{ij}X_{kn}X_{lm} = \sum_{k,l} R^{nm}_{kl}X_{ik}X_{jl}.$$

Since $n < m$, we have that $R^{nm}_{nm} = q^{-1/3}(q - q^{-1})$ and $R^{nm}_{mn} = q^{-1/3}$ are the only nonzero values of R^{nm}_{kl} as k and l vary.

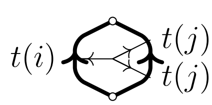
The left side of our equation now becomes

$$
\begin{aligned}
&(\sum_{k,l} q^{3+1/3} R^{kl}_{ij} X_{kn} X_{lm}) - q^4 X_{in} X_{jm} \\
&= (\sum_{k,l} q^{3+1/3} R^{nm}_{kl} X_{ik} X_{jl}) - q^4 X_{in} X_{jm} \\
&= q^3(q - q^{-1}) X_{in} X_{jm} + q^3 X_{im} X_{jn} - q^4 X_{in} X_{jm} \\
&= q^3 X_{im} X_{jn} - q^2 X_{in} X_{jm},
\end{aligned}
$$

as required. So ψ respects (B2).

Relation (B3):

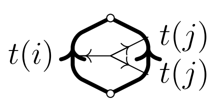
To show that ψ respects (B3) we need to show that

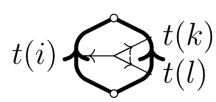
$\psi\Bigg($ 

$\Bigg) = 0$

for any $i, j \in \{1, 2, 3\}$.

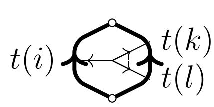
By the definition of ψ, we have

$\psi\Bigg($ 

$\Bigg) = \sum_{k,l} \varepsilon\Bigg($ 

$\Bigg) X_{kj} X_{lj}.$

We compute that

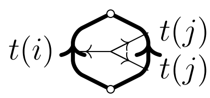
$\varepsilon\Bigg($ 

$\Bigg) = 0$

if $4 - i$ is in $\{k, l\}$ or if $k = l$.

If $l < k$ we have $\varepsilon_{ikl} = -q\varepsilon_{ilk}$. This can be computed by using relations (B2) and (I3).

Thus,

$\psi\Bigg($ 

$\Bigg) = \varepsilon_{ilk}(X_{lj}X_{kj} - qX_{kj}X_{lj})$

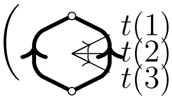
for the unique suitable pair l, k for which ε_{ilk} is nonzero. The result follows from the identity

$$X_{lj}X_{kj} = qX_{kj}X_{lj}$$

which holds in $\mathcal{O}_q(SL_3)$ for $l < k$.

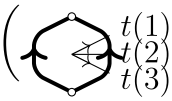
Relation (B4):

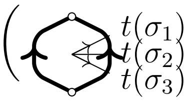
To check that ψ respects relation (B4) it suffices to check

$\psi\Bigg($ 

$\Bigg) = q^{-2}.$

By the definition of ψ, we compute

$\psi\Bigg($ 

$\Bigg) = \sum_{\sigma \in S_3} \varepsilon\Bigg($ 

$\Bigg) X_{\sigma_1 1}X_{\sigma_2 2}X_{\sigma_3 3}.$

We see that this is equal to

$$\begin{aligned} q^{-2}\sum_{\sigma \in S_3}(-q)^{l(\sigma)}X_{\sigma_1 1}X_{\sigma_2 2}X_{\sigma_3 3} &= q^{-2}\mathrm{det}_q \\ &= q^{-2}. \end{aligned}$$

So we see that ψ respects (B4) and, thus, ψ is well-defined. ■

Our previous two propositions allow us to state the following theorem.

Theorem 4.13 *We have that*

$$\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$$

as Hopf algebras.

Proof: In Proposition 4.11 we showed that ϕ is a well-defined map of bialgebras. To show that ϕ is an isomorphism, it suffices to show that ϕ is invertible as a map of $\mathcal{R}$-modules. We claim that ψ is its inverse.

We observe that $\psi \circ \phi(X_{ij}) = X_{ij}$ for all generators X_{ij} of $\mathcal{O}_q(SL_3)$. Since ψ and ϕ are both algebra maps, this implies that

$$\psi \circ \phi = \mathrm{id}_{\mathcal{O}_q(SL_3)}.$$

Similarly, $\phi \circ \psi$ agrees with $\mathrm{id}_{\mathcal{S}_q^{SL_3}(\mathfrak{B})}$ for all generating diagrams α_{st} and β_{st}. Thus,

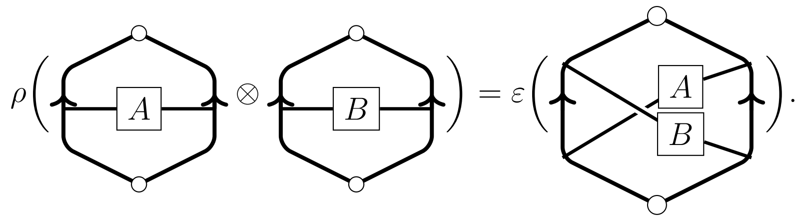
$$\phi \circ \psi = \mathrm{id}_{\mathcal{S}_q^{SL_3}(\mathfrak{B})}.$$

Thus, $\mathcal{O}_q(SL_3)$ and $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ are isomorphic as bialgebras. Since $\mathcal{O}_q(SL_3)$ is a Hopf algebra, then $\mathcal{O}_q(SL_3)$ and $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ are isomorphic as Hopf algebras. ■

4.5 The stated skein algebra of the triangle is a braided tensor square of $\mathcal{O}_q(SL_3)$

The Hopf algebra $\mathcal{O}_q(SL_3)$ is equipped with a cobraiding $\rho : \mathcal{O}_q(SL_3) \otimes \mathcal{O}_q(SL_3) \to \mathcal{R}$. In [CL19] the cobraiding for the SL_2 case was shown to have a simple diagrammatic definition, and an analogous definition will work here as well. This cobraiding will allow us to describe the SL_3 stated skein algebra of the triangle, $\mathfrak{T}$.

We define the cobraiding $\rho : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{R}$ on diagrams by



In the diagrams above, the strands depict a bundle of parallel or antiparallel strands. The diagrammatic definition of the map makes it easy to see that it respects the defining relations of the stated skein algebra, so it is well-defined. The argument that this satisfies the cobraiding axioms is identical to the one in [CL19, Section 3.7], but we do not need to use it in this section.

We recall that a cobraiding is determined by its values on a set of generators and so we see that the map ρ that we have defined diagrammatically satisfies

$$\rho(X_{ij} \otimes X_{kl}) = R^{jl}_{ki},$$

and thus matches up with the standard co-R-matrix.

In the situation that we have two algebras M and N which are both left comodule-algebras over $\mathcal{O}_q(SL_3)$ we can endow the vector space $M \otimes N$ with a left comodule-algebra structure using the cobraiding ρ. We will denote this algebra by $M \underline{\otimes} N$ and call it the *braided tensor product* of the algebras M and N. Using Sweedler's notation, its multiplication is defined as follows:

$$(x \otimes y) \star (z \otimes t) = (x \otimes 1)(\sum_{(z)(y)} \rho(z' \otimes y')(z'' \otimes y''))(1 \otimes t)$$

Equivalently, if we identify M with $M \otimes \{1\}$ and N with $\{1\} \otimes N$, then our product structure is given by

$$xy = \begin{cases} xy & \text{if } x, y \text{ both in } M \text{ or both in } N \\ x \otimes y & \text{if } x \text{ in } M \text{ and } y \text{ in } M \\ \sum_{(x)(y)} \rho(y' \otimes x')(y'' \otimes x'') & \text{if } x \text{ in } N \text{ and } y \text{ in } M \end{cases}$$

Le and Costantino showed in [CL19] that gluing disjoint surfaces along a triangle yields a braided tensor product of stated skein algebras for the SL_2 case. The same is true for the SL_3 case and our Proposition 4.6 takes care of most of the work we need to do to show it.

Theorem 4.14 *Let Σ_1 and Σ_2 be disjoint punctured bordered surfaces. If a is a boundary arc of Σ_1 and b is a boundary arc of Σ_2, then we have an algebra isomorphism*

$$\mathcal{S}_q^{SL_3}(\Sigma_1) \underline{\otimes} \mathcal{S}_q^{SL_3}(\Sigma_2) \cong \mathcal{S}_q^{SL_3}((\Sigma_1 \sqcup \Sigma_2) \# \mathfrak{T})$$

given by the map $\text{glue}_{\mathfrak{T}}$ defined in Section 4.2.

Proof: By Proposition 4.6, the map

$$\text{glue}_{\mathfrak{T}} : \mathcal{S}_q^{SL_3}(\Sigma_1 \sqcup \Sigma_2) \to \mathcal{S}_q^{SL_3}((\Sigma_1 \sqcup \Sigma_2) \# \mathfrak{T})$$

is an $\mathcal{R}$-module isomorphism. Since $S_q^{SL_3}(\Sigma_1 \sqcup \Sigma_2)$ is naturally isomorphic to $S_q^{SL_3}(\Sigma_1) \otimes S_q^{SL_3}(\Sigma_2)$, we see that the isomorphism claimed in Theorem 4.14 holds on the level of $\mathcal{R}$-modules. To see that it holds on the level of $\mathcal{R}$-algebras we must show that $\text{glue}_{\mathfrak{T}}$ respects the algebra structure.

For this fact, the same diagrammatic proof in [CL19] works here. In each of the following cases:

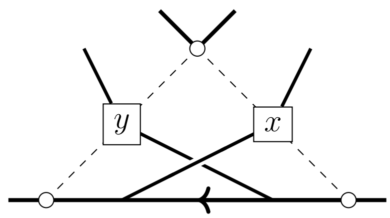
- x, y are both in $S_q^{SL_3}(\Sigma_1)$,

- x, y are both in $S_q^{SL_3}(\Sigma_2)$,
- or x is in $S_q^{SL_3}(\Sigma_1)$ while y is in $S_q^{SL_3}(\Sigma_2)$,

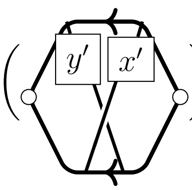
it is clear that $\text{glue}_{\mathfrak{T}}(x)\text{glue}_{\mathfrak{T}}(y) = \text{glue}_{\mathfrak{T}}(xy)$.

In the remaining case, we have that x is in $\mathcal{S}_q^{SL_3}(\Sigma_2)$ and y is in $\mathcal{S}_q^{SL_3}(\Sigma_1)$. We diagrammatically compute that

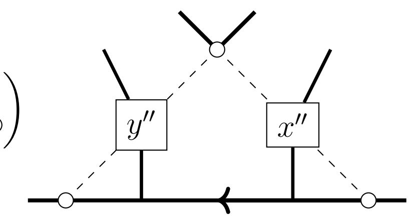
$\text{glue}_{\mathfrak{T}}(x)\text{glue}_{\mathfrak{T}}(y) =$



$= \sum_{(x)(y)} \varepsilon\Big($



$\Big)$



$$= \sum_{(x)(y)} \rho(y' \otimes x')\text{glue}_{\mathfrak{T}}(y'' \otimes x'')$$

$$= \text{glue}_{\mathfrak{T}}\left(\sum_{(x)(y)} \rho(y' \otimes x')(y'' \otimes x'') \right)$$

$$= \text{glue}_{\mathfrak{T}}(xy).$$

This shows that $\text{glue}_{\mathfrak{T}}$ respects the multiplication of $\mathcal{S}_q^{SL_3}(\Sigma_1) \underline{\otimes} \mathcal{S}_q^{SL_3}(\Sigma_2)$ and completes our proof.

■

By applying Theorem 4.14 in the special case where Σ_1 and Σ_2 are both bigons $\mathfrak{B}$ we obtain the following corollary.

Corollary 4.15 *We have that*

$$\mathcal{S}_q^{SL_3}(\mathfrak{T}) \cong \mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3).$$

Chapter 5

Consequences

In the previous chapters, we established two important relationships between skein algebras and quantum groups. The first is the triangular decomposition of $\mathcal{S}_q^{SL_3}(\Sigma)$ for a punctured surface Σ, which allows us to study $\mathcal{S}_q^{SL_3}(\Sigma)$ by using the well-studied quantum group $\mathcal{O}_q(SL_3)$. In this chapter, we will use the triangular decomposition to observe that $\mathcal{S}_q^{SL_3}(\Sigma)$ is a domain and also to construct a Frobenius map F_Σ which embeds the classical skein algebra $\mathcal{S}_1^{SL_3}(\Sigma)$ in the center of the skein algebra $\mathcal{S}_q^{SL_3}(\Sigma)$ when q is root of unity of order N coprime to 6. The fact that the skein algebra is a domain will follow for essentially the same reason that $\mathcal{O}_q(SL_3)$ is a domain. Similarly, the existence of the Frobenius map will follow essentially from a well-known Frobenius map for $\mathcal{O}_q(SL_3)$.

The second important observation we made was the isomorphism $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$, providing a skein-theoretic definition of the quantum group $\mathcal{O}_q(SL_3)$. This observation allows us to use skein theory to study the comodules of the quantum group $\mathcal{O}_q(SL_3)$ (which are $U_q(\mathfrak{sl}_3)$-modules). In this chapter, we use properties of our splitting map to show that Kuperberg's SL_3 web category describes a full subcategory of $\mathcal{O}_q(SL_3)$-comodules (or of $U_q(\mathfrak{sl}_3)$-modules). The properties of our splitting map were proven directly from the skein relations, using the confluence of the stated skein relations. Thus, by extending Kuperberg's original skein relations to a confluent set of stated skein relations, we have obtained a self-contained skein-theoretic proof of the existence, injectivity, and surjectivity of the Reshetikhin-Turaev functor for Kuperberg's web category.

5.1 The SL_3 skein algebra of a punctured surface is a domain

Suppose that our ground ring $\mathcal{R}$ is a domain, which means that if $xy = 0$ for elements $x, y \in \mathcal{R}$, then we must have that $x = 0$ or $y = 0$. Our goal in this section is to show that whenever $\mathcal{R}$ is a domain, $\mathcal{S}_q^{SL_3}(\Sigma)$ is a domain as well. We are able to prove this fact as long as Σ has at least one puncture. We state our main theorem here and then prove it in the rest of the section.

Theorem 5.1 *If $\mathcal{R}$ is a domain and Σ has at least one puncture, then $\mathcal{S}_q^{SL_3}(\Sigma)$ is a domain as well.*

We remark that this theorem also implies that the ordinary skein algebra is a domain since it embeds in the stated skein algebra.

We first prove the theorem for the cases when Σ has no ideal triangulation. A punctured bordered surface Σ is called a *small surface* if it is one of the following: a bigon, a monogon, a sphere with two punctures or a sphere with one puncture.

Proposition 5.2 *If $\mathcal{R}$ is a domain and Σ is a small surface, then $\mathcal{S}_q^{SL_3}(\Sigma)$ is a domain.*

Proof: If Σ is a monogon or a sphere with one puncture, then $\mathcal{S}_q^{SL_3}(\Sigma) \cong \mathcal{R}$ and is a domain.

If Σ is a bigon, then $\mathcal{S}_q^{SL_3}(\Sigma) \cong \mathcal{O}_q(SL_3)$, which is a domain by [BG02, Theorem I.2.10]. The proof there is stated for $\mathcal{R} = k$, a field but their proof works for any domain $\mathcal{R}$.

Finally, if Σ is a sphere with two punctures, then by applying the splitting map associated to an ideal arc traveling from one puncture to the other, we obtain an embedding $\mathcal{S}_q^{SL_3}(\Sigma) \hookrightarrow \mathcal{S}_q^{SL_3}(\mathfrak{B})$ and so our skein algebra is a domain in this case as well. ■

If Σ is a punctured surface that is not a small surface, then Σ has an ideal triangulation and we can apply our triangular decomposition to obtain an embedding

$$\mathcal{S}_q^{SL_3}(\Sigma) \hookrightarrow \bigotimes_i \mathcal{S}_q^{SL_3}(\mathfrak{T}_i),$$

where for each triangle $\mathfrak{T}_i$, we have $\mathcal{S}_q^{SL_3}(\mathfrak{T}_i) \cong \mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$.

So it will suffice to show that $\bigotimes_i(\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3))$ is a domain. Since domains are not necessarily well behaved under tensor products or braided tensor products (recall that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not a domain), our result does not follow immediately from the fact that $\mathcal{O}_q(SL_3)$ is a domain. However, we will still model our proof on the proof in [BG02] by first using properties of $\mathcal{O}_q(M_3)$ and then using a localization.

Recall that the bialgebra $\mathcal{O}_q(M_3)$ has a similar presentation as $\mathcal{O}_q(SL_3)$, generated by elements X_{ij} with the only difference being that the presentation of $\mathcal{O}_q(M_3)$ does not include the relation $\det_q = 1$. We first prove the following.

Proposition 5.3 *If $\mathcal{R}$ is a domain then*

$$\bigotimes_i(\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$$

is a domain.

Proof: We use a compatibly ordered basis of $\mathcal{O}_q(M_3)$ and build it up to a compatibly ordered basis of $\bigotimes_i(\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$.

We define an order on our generators X_{ij} using the lexicographic ordering, meaning $X_{ij} < X_{kl}$ if $i < k$ or if both $i = k$ and $j < l$. Using the defining relations of $\mathcal{O}_q(M_3)$ as reduction rules and a standard Diamond Lemma argument, we have a basis for $\mathcal{O}_q(M_3)$ consisting of monomials of generators appearing in increasing order.

To each basis monomial, we associate a degree $d \in \mathbb{Z}^9_{\geq 0}$ by

$$d(X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}}) = (m_{33}, m_{32}, \ldots, m_{11}),$$

the list of exponents of the generators, listed in reverse order. A basis element is determined uniquely by its degree, and so we have an indexing of our basis by the totally ordered monoid $\mathbb{Z}^9_{\geq 0}$.

To an arbitary nonzero element $x \in \mathcal{O}_q(M_3)$ we can associate a degree $d(x)$ by writing x in the basis and defining $d(x)$ to be the maximum degree among all basis elements appearing with nonzero coefficients.

Suppose m_1 and m_2 are two monomial basis elements. The reduction rules imply that generators q-commute up to terms of smaller degree and so

$$d(m_1 m_2) = d(m_1) + d(m_2).$$

From this we can deduce that $d(xy) = d(x) + d(y)$ for arbitrary nonzero elements x and y and so $\mathcal{O}_q(M_3)$ is a domain.

We next upgrade our compatibly ordered basis of $\mathcal{O}_q(M_3)$ to a compatibly ordered basis of $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$. We will continue to use X_{ij} to refer to the generators in the first factor and use Y_{ij} to refer to the generators in the second factor. Recall that the algebra $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$ is isomorphic as a module to $\mathcal{O}_q(M_3) \otimes \mathcal{O}_q(M_3)$ and thus has a basis $\{m_X m_Y\}$ where m_X is a monomial of generators X_{ij} in increasing order and m_Y is a monomial of generators Y_{ij} appearing in increasing order. We claim that this basis is compatibly ordered as well.

For a basis element $m_X m_Y$ we define its degree $d(m_X m_Y) \in \mathbb{Z}^{18}_{\geq 0}$ to be the concatenation $(d(m_X), d(m_Y))$ where $d(m_X)$ was defined earlier in this proof and $d(m_Y)$ is the corresponding definition using the generators Y_{ij}. We recall that in the braided tensor product $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$ we have that

$$Y_{ij}X_{kl} = q^{-1/3}\begin{cases} qX_{kl}Y_{ij} & (i = k) \\ X_{kl}Y_{ij} + (q - q^{-1})X_{il}Y_{kj} & (i < k) \\ X_{kl}Y_{ij} & (i > k). \end{cases}$$

We note that if $i < k$ then $X_{il} < X_{kl}$. Thus, the generators Y_{ij} and X_{kl} q-commute up to lower order terms. From this we deduce that

$$d(m_{X_1}m_{Y_1}m_{X_2}m_{Y_2}) = d(m_{X_1}m_{Y_1}) + d(m_{X_2}m_{Y_2})$$

and consequently $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$ is a domain.

We then use the tensor product of these bases to get a compatibly ordered basis of $\bigotimes_i(\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ and see that it is a domain. ■

We would like to take the Ore localization of $\bigotimes_i(\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ with respect to the multiplicative set generated by the elements $\det_{X_i}$ and $\det_{Y_i}$. This will be easy to do if we can show that these determinant elements are central. It suffices to show the following.

Proposition 5.4 *The quantum determinant elements* $\det_X$ *and* $\det_Y$ *are central elements of* $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$.

Proof: We will prove that $\det_X$ is central. The argument that $\det_Y$ is central is similar.

It is well known that $\det_X$ commutes with the generators X_{ij} so we need to check that it commutes with the generators Y_{kl}. Recall from the previous proof that the commutativity relations involving Y_{kl} and X_{ij} only depend on the row indices k and i.

We must check that

$$Y_{kl}(\sum_{\sigma}(-q)^{l(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}) = (\sum_{\sigma}(-q)^{l(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)})Y_{kl}.$$

If $k = 3$ then the row index of Y is not smaller than any row indices of the generators X_{ij} and so Y_{3l} can slide past the determinant, picking up one factor of $q^{2/3}$ and two factors of $q^{-1/3}$ along the way. Thus, the relation holds if $k = 3$.

If $k = 2$, then we use the relations to slide Y_{2l} past the generators $X_{i\sigma(i)}$ to get

$$\begin{aligned} Y_{2l}(\sum_{\sigma}(-q)^{l(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}) &= \sum_{\sigma}(-q)^{l(\sigma)}q^{2/3}q^{-1/3}X_{1\sigma(1)}X_{2\sigma(2)}Y_{2l}X_{3\sigma(3)} \\ &= (\sum_{\sigma}(-q)^{l(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)})Y_{2l} \\ &+ \sum_{\sigma}(-q)^{l(\sigma)}(q-q^{-1})X_{1\sigma(1)}X_{2\sigma(2)}X_{2\sigma(3)}Y_{3l} \end{aligned}$$

This last term is zero since $\sum_{\sigma}(-q)^{l(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}X_{2\sigma(3)}$ has a repeated row index and so is zero by properties of quantum determinants. Thus, Y_{2l} commutes with $\det_X$.

When $k = 1$ a similar computation shows that Y_{1l} commutes with $\det_X$. ∎

We can then take an Ore localization of $\bigotimes_i(\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ to obtain the algebra $\bigotimes_i(\mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3))$, where $\mathcal{O}_q(GL_3) = \mathcal{O}_q(M_3)[\det_q^{-1}]$. Since the localization of a domain is a domain, we have that $\bigotimes_i(\mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3))$ is a domain.

The proof of Theorem 5.1 then follows from the following.

Proposition 5.5 *The algebra* $\bigotimes_i(\mathcal{O}_q(SL_3)\underline{\otimes}\mathcal{O}_q(SL_3))$ *embeds in* $\bigotimes_i(\mathcal{O}_q(GL_3)\underline{\otimes}\mathcal{O}_q(GL_3))$.

Proof: Producing an embedding $\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3) \hookrightarrow \mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3)$ will induce the desired embedding since these algebras are free $\mathcal{R}$-modules and so the tensor product of injective maps will be an injective map.

To produce this embedding we follow the construction of an the embedding $\mathcal{O}_q(SL_3) \hookrightarrow \mathcal{O}_q(GL_3)$ from [BG02].

We show that

$$(\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3))[z_X^{\pm 1}, z_Y^{\pm 1}] \cong \mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3).$$

For notation we will denote by X_{ij} and Y_{ij} the generators of $\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$ and by x_{ij} and y_{ij} the generators of $\mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3)$. Define

$$F : (\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3))[z_X^{\pm 1}, z_Y^{\pm 1}] \to \mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3)$$

on generators by

$$\begin{aligned}
X_{i1} &\mapsto x_{i1}\mathrm{det}_x^{-1} \\
X_{ij} &\mapsto x_{ij} && (j \neq 1) \\
Y_{i1} &\mapsto y_{i1}\mathrm{det}_y^{-1} \\
Y_{ij} &\mapsto y_{ij} && (j \neq 1) \\
z_X &\mapsto \mathrm{det}_x \\
z_Y &\mapsto \mathrm{det}_y
\end{aligned}$$

Since $\det_x$ and $\det_y$ are central, we can see that F respects the standard $\mathcal{O}_q(M_3)$ relations. By construction, it satisfies $F(\det_X) = 1 = F(\det_Y)$. It also satisfies the mixed relations involving Y_{ij} and X_{kl}. Thus F is a well-defined algebra map.

We define

$$G : \mathcal{O}_q(GL_3) \underline{\otimes} \mathcal{O}_q(GL_3) \to (\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3))[z_X^{\pm 1}, z_Y^{\pm 1}]$$

on generators by

$$\begin{aligned} x_{i1} &\mapsto X_{i1} z_X \\ x_{ij} &\mapsto X_{ij} && (j \neq 1) \\ y_{i1} &\mapsto Y_{i1} z_Y \\ y_{ij} &\mapsto Y_{ij} && (j \neq 1) \\ \det_x^{-1} &\mapsto z_X^{-1} \\ \det_y^{-1} &\mapsto z_Y^{-1}. \end{aligned}$$

G respects the relations and so is a well-defined algebra map. We can see on generators that $FG = \mathrm{id}$ and $GF = \mathrm{id}$ and so we have an isomorphism. Restricting F to $\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$ produces the desired embedding. ■

5.2 SL_3 analogue of the Chebyshev-Frobenius map

In [BW16], an algebra map called the Chebyshev-Frobenius homomorphism was constructed, which embedded the classical skein algebra $\mathcal{S}_1^{SL_2}(\Sigma)$ into the center of the skein algebra $\mathcal{S}_q^{SL_2}(\Sigma)$ at a root of unity q of odd order. The map is interesting from a topological viewpoint since it has a definition in terms of threading links through Chebyshev polynomials and the fact that it is well-defined follows from "miraculous cancellations" in skein theoretic computations when q is a root of unity. From an algebraic viewpoint, the map is important because it provides a source of central elements which can be used to study the representation theory of $\mathcal{S}_q^{SL_2}(\Sigma)$ at roots of unity. We are interested in finding an analogous map for the case of SL_3.

Throughout this section we assume that $\mathcal{R}$ is a domain and $q^{1/3}$ is a root of unity of order N coprime to 6. We are interested in the relationship between $\mathcal{S}_1^{SL_3}(\Sigma)$ and $\mathcal{S}_q^{SL_3}(\Sigma)$, where the skein algebra $\mathcal{S}_1^{SL_3}(\Sigma)$ is obtained from the definition of $\mathcal{S}_q^{SL_3}(\Sigma)$ by replacing $q^{1/3}$ by 1 in all of the defining skein relations. Our goal in this section is to prove the following.

Theorem 5.6 *Suppose that $\mathcal{R}$ is a domain and $q^{1/3}$ is a root of unity of order N coprime to 6. Then for a punctured bordered surface Σ with at least one puncture per connected component, there exists an embedding*

$$F_\Sigma : \mathcal{S}_1^{SL_3}(\Sigma) \hookrightarrow Z(\mathcal{S}_q^{SL_3}(\Sigma)).$$

The Frobenius map F_Σ will be constructed by starting with the Hopf algebra embedding $\mathcal{O}_1(SL_3) \hookrightarrow \mathcal{O}_q(SL_3)$ constructed in [PW91] and then, in some sense, extending the map to Σ. We follow the strategy of [KQ19] from the SL_2 case.

When we say that $q^{1/3}$ has order N, we mean that $(q^{1/3})^N = 1$ and $(q^{1/3})^k \neq 1$ for $0 < k < N$. Our assumption that N is coprime to 6 guarantees that q and q^2 are also roots of unity of the same order N, which is a hypothesis used in [PW91].

Proposition 5.7 *([PW91]) There is a Hopf algebra map $F_{\mathfrak{B}} : \mathcal{O}_1(SL_3) \to \mathcal{O}_q(SL_3)$*

defined on generators by $F_{\mathfrak{B}}(X_{ij}) = (X_{ij})^N$*. Furthermore, the image of* $F_{\mathfrak{B}}$ *is contained in the center of* $\mathcal{O}_q(SL_3)$*.*

We observe the following.

Lemma 5.8 *The Hopf algebra map* $F_{\mathfrak{B}}$ *is an embedding.*

Proof: The set of monomials

$$\{X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}} \mid m_{11} m_{22} m_{33} = 0\}$$

forms a basis for both $\mathcal{O}_1(SL_3)$ and $\mathcal{O}_q(SL_3)$. Since

$$F_{\mathfrak{B}}(X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}}) = X_{11}^{Nm_{11}} X_{12}^{Nm_{12}} \cdots X_{33}^{Nm_{33}},$$

we see that $F_{\mathfrak{B}}$ maps the basis of $\mathcal{O}_1(SL_3)$ injectively into the basis of $\mathcal{O}_q(SL_3)$. ■

We next extend this map to the case of the braided tensor square. Recall that when $q = 1$, the braided tensor square of $\mathcal{O}_1(SL_3)$ is just the ordinary tensor square.

Proposition 5.9 *There is an algebra embedding*

$$F_{\mathfrak{T}} : \mathcal{O}_1(SL_3) \underline{\otimes} \mathcal{O}_1(SL_3) \hookrightarrow Z(\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3))$$

defined by $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$*.*

Proof: Since $\mathcal{O}_1(SL_3)$ is a free $\mathcal{R}$-module, by setting $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$ we obtain an embedding of $\mathcal{R}$-modules $\mathcal{O}_1(SL_3) \underline{\otimes} \mathcal{O}_1(SL_3) \hookrightarrow \mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$. We need to check that this map respects the algebra structure.

Recall the notation $X_{ij}Y_{kl}$ for generators of $\mathcal{O}_q(SL_3) \underline{\otimes} \mathcal{O}_q(SL_3)$. To see that $F_{\mathfrak{T}}$ respects the algebra structure of the braided tensor product, it will suffice to observe that the images of generators $F_{\mathfrak{X}}(Y_{kl}) = Y_{kl}^N$ commute with the generators X_{ij} in $\mathcal{O}_q(SL_3) \underline{\otimes}$ $\mathcal{O}_q(SL_3)$. A symmetric argument shows the same is true for $F_{\mathfrak{X}}(X_{ij})$ and Y_{kl}.

If $k = i$, we have the relation $Y_{kl}X_{ij} = q^{2/3}X_{ij}Y_{kl}$ and so

$$\begin{aligned} Y_{kl}^N X_{ij} &= (q^{2/3})^N X_{ij} Y_{kl}^N \\ &= X_{ij} Y_{kl}^N, \end{aligned}$$

since $(q^{1/3})^N = 1$.

Similarly, if $k > i$, we have the relation $Y_{kl}X_{ij} = q^{-1/3}X_{ij}Y_{kl}$ and so $Y_{kl}^N X_{ij} = X_{ij}Y_{kl}^N$ in this case as well.

If $k < i$, then we will use the relation $Y_{kl}X_{ij} = q^{-1/3}X_{ij}Y_{kl} + q^{-1/3}(q-q^{-1})X_{kj}Y_{il}$. We will prove the following for $m \geq 1$ by induction:

$$Y_{kl}^m X_{ij} = (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q-q^{-1}) \sum_{n=0}^{m-1} q^{-2n} X_{kj} Y_{il} Y_{kl}^{m-1}.$$

We are given the base case. Now assume the inductive hypothesis. We have

$$\begin{aligned} Y_{kl}^m X_{ij} &= Y_{kl}(q^{-1/3})^{m-1} X_{ij} Y_{kl}^{m-1} + (q^{-1/3})^{m-1}(q-q^{-1}) \sum_{n=0}^{m-2} q^{-2n} Y_{kl} X_{kj} Y_{il} Y_{kl}^{m-2} \\ &= (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q-q^{-1}) X_{kj} Y_{il} Y_{kl}^{m-1} \\ &\quad + (q^{-1/3})^{m-1}(q-q^{-1}) \sum_{n=0}^{m-2} q^{-1/3} q^{-2n+2} X_{kj} Y_{il} Y_{kl}^{m-1} \\ &= (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q-q^{-1}) \sum_{n=0}^{m-1} q^{-2n} X_{kj} Y_{il} Y_{kl}^{m-1}, \end{aligned}$$

as claimed.

When we specialize this formula to the case $m = N$ we obtain

$$Y_{kl}^N X_{ij} = X_{ij} Y_{kl}^N$$

as required since $(q^{-1/3})^N = 1$ and $(q - q^{-1}) \sum_{n=0}^{N-1} q^{-2n} = q(1 - q^{-2N}) = 0$.

■

We next investigate the diagrammatic properties of our maps $F_{\mathfrak{B}}$ and $F_{\mathfrak{T}}$ when we view them as maps on the skein algebras of the bigon $\mathfrak{B}$ and triangle $\mathfrak{T}$.

Proposition 5.10 *When $\mathcal{S}_q^{SL_2}(\mathfrak{B})$ is identified with $\mathcal{O}_q(SL_3)$, $F_{\mathfrak{B}}$ is defined on generating strands by $F_{\mathfrak{B}}(\alpha_{t(i)t(j)}) = \alpha_{t(i)t(j)}^N$ and $F_{\mathfrak{B}}(\beta_{t(i)t(j)}) = \beta_{t(i)t(j)}^N$ for all $i, j \in \{1, 2, 3\}$.*

Proof: Our isomorphism $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{O}_q(SL_3)$ sends $\beta_{t(i)t(j)}$ to X_{ij} and so we already know that $F_{\mathfrak{B}}(\beta_{t(i)(j)}) = \beta_{t(i)t(j)}^N$.

For the strands $\alpha_{t(i)t(j)}$ we will use the antipodes S and the fact that $F_{\mathfrak{B}}$ commutes with the antipodes. For our strands $\alpha_{t(i)t(j)}$, we use the fact $\alpha_{t(i)t(j)} = q^{2j-2i} S(\beta_{t(\bar{j})t(\bar{i})})$, where $\bar{i} = 4 - i$.

We then compute

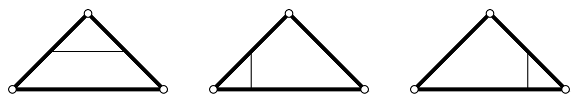
$$\begin{aligned} F_{\mathfrak{B}}(\alpha_{t(i)t(j)}) &= F_{\mathfrak{B}}(S(\beta_{t(\bar{j})t(\bar{i})}) \\ &= S(F_{\mathfrak{B}}(\beta_{t(\bar{j})t(\bar{i})})) \\ &= S((\beta_{t(\bar{j})t(\bar{i})})^N) \\ &= S(\beta_{t(\bar{j})t(\bar{i})})^N \\ &= (q^{2i-2j})^N \alpha_{t(i)t(j)}^N \\ &= \alpha_{t(i)t(j)}^N, \end{aligned}$$

as claimed. ∎

Thus, even though our isomorphism $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$ depends on a choice of right and left boundary arcs of $\mathfrak{B}$, the definition of $F_{\mathfrak{B}}$ is invariant under this choice.

Similarly, even though our isomorphism $\mathcal{S}_q^{SL_3}(\mathfrak{T}) \cong \mathcal{S}_q^{SL_3}(\mathfrak{B}) \underline{\otimes} \mathcal{S}_q^{SL_3}(\mathfrak{B})$ depended on a choice of bottom edge of the triangle, we aim to show that the map $F_{\mathfrak{T}} : \mathcal{S}_1^{SL_3}(\mathfrak{T}) \to \mathcal{S}_q^{SL_3}(\mathfrak{T})$ is invariant under this choice.

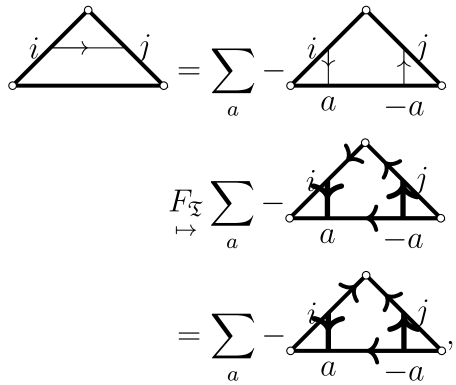
We call a stated arc in the the triangle $\mathfrak{T}$ a *corner arc* if it admits a crossingless diagram and it is not homotopic to a boundary arc. The following illustrates examples of top, left, and right corner arcs.



Proposition 5.11 *The map $F_{\mathfrak{T}}$ sends a stated corner arc to its Nth power.*

Proof: If the arc is a left or right corner arc, then by the definition of $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$ and our diagrammatic interpretation of $F_{\mathfrak{B}}$, we already know that $F_{\mathfrak{T}}$ sends the arc to its Nth power. So we just have to show the same is true for a top corner arc. We will compute this for a top corner arc with one orientation. A similar computation works for the opposite orientation.

We compute the value of $F_{\mathfrak{T}}$ on our arc by first writing it in terms of left and right corner arcs and then applying $F_{\mathfrak{T}}$.



where the thick strands denote N parallel strands. The orientation reversal of the left edge in the last equality is possible since it comes at the expense of a factor of $(q^{2/3})^{N(N-1)/2} = 1$.

We claim the last expression in our computation is the same as the Nth power of our top corner arc. To show this, we will make use of the fact that $F_{\mathfrak{B}}$ is a bialgebra map. Let e_r denote the right boundary arc of the left bigon in a disjoint union $\mathfrak{B} \sqcup \mathfrak{B}$ and denote by e_l the left boundary arc of the right bigon. Recall the maps rev_{e_r} and $\mathrm{glue}_{\mathfrak{T}}$. We have that the Nth power of our top corner arc is the same as

$$\mathrm{glue}_{\mathfrak{T}}\mathrm{rev}_{e_r}\Delta(F_{\mathfrak{B}}(\beta_{ij})) = \mathrm{glue}_{\mathfrak{T}}\mathrm{rev}_{e_r}(F_{\mathfrak{B}} \otimes F_{\mathfrak{B}})\Delta(\beta_{ij}),$$

which is our last expression in our computation above. This part uses the fact that rev_{e_r} multiplies each diagram by $(-1)^N = -1$, since N is odd. ■

Now that we have established diagrammatic interpretations of our maps $F_{\mathfrak{B}}$ and $F_{\mathfrak{T}}$, we can observe that they satisfy a compatibility with our splitting maps. Suppose that a is a boundary arc of a triangle $\mathfrak{T}$.

Lemma 5.12 *Our Frobenius maps F commute with Δ_a and ${}_a\Delta$ in the sense that*

$$(F_{\mathfrak{T}} \otimes F_{\mathfrak{B}})\Delta_a = \Delta_a F_{\mathfrak{T}}$$

and

$$(F_{\mathfrak{T}} \otimes F_{\mathfrak{B}})_a\Delta = {}_a\Delta F_{\mathfrak{T}}.$$

Proof: This follows from the fact that $\Delta \circ F_{\mathfrak{B}} = (F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}) \circ \Delta$ and from an embedding $\mathfrak{B} \sqcup \mathfrak{B} \hookrightarrow \mathfrak{B} \sqcup \mathfrak{T}$. ■

We can now construct a Frobenius map F_Σ for any ideal triangulable Σ.

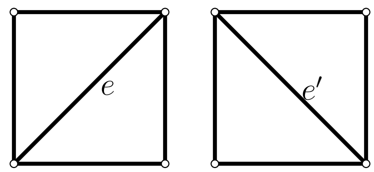
Proposition 5.13 *Suppose Σ has an ideal triangulation with a set of interior edges $\mathcal{E}$. There exists an algebra embedding $F_{\Sigma,\mathcal{E}}$ of $\mathcal{S}_1^{SL_3}(\Sigma)$ into the center $Z(\mathcal{S}_q^{SL_3}(\Sigma))$ defined as the unique algebra map making the left square in the following diagram commute:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}_1^{SL_3}(\Sigma) & \xrightarrow{\Delta} & \bigotimes_i^n \mathcal{S}_1^{SL_3}(\mathfrak{T}_i) & \xrightarrow{\Delta^R - \tau\circ{}^L\Delta} & (\bigotimes_{i=1}^n \mathcal{S}_1^{SL_3}(\mathfrak{T}_i)) \otimes (\bigotimes_{e\in\mathcal{E}} \mathcal{S}_1^{SL_3}(\mathfrak{B})) \\ & & \Big\downarrow F_{\Sigma,\mathcal{E}} & & \Big\downarrow \otimes_i F_{\mathfrak{T}_i} & & \Big\downarrow \otimes_i F_{\mathfrak{T}_i} \otimes_e F_{\mathfrak{B}} \\ 0 & \longrightarrow & \mathcal{S}_q^{SL_3}(\Sigma) & \xrightarrow{\Delta} & \bigotimes_i^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i) & \xrightarrow{\Delta^R - \tau\circ{}^L\Delta} & (\bigotimes_{i=1}^n \mathcal{S}_q^{SL_3}(\mathfrak{T}_i)) \otimes (\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{SL_3}(\mathfrak{B})). \end{array}$$

Proof: The horizontal rows are exact, by our triangular decomposition theorem. The right square commutes by Lemma 5.12. Thus, there exists a unique map of modules $F_{\Sigma,\mathcal{E}}$ as claimed. The map is an injective algebra map because Δ and $F_{\mathfrak{T}_i}$ are injective algebra maps. By the centrality of $F_{\mathfrak{T}_i}$ and the injectivity of Δ, we see that $F_{\Sigma,\mathcal{E}}$ is central. ■

So far we have defined $F_{\Sigma,\mathcal{E}}$ in terms of the ideal triangulation $\mathcal{E}$. We next aim to show that if $\mathcal{E}$ and $\mathcal{E}'$ are two ideal triangulations of Σ, then $F_{\Sigma,\mathcal{E}} = F_{\Sigma,\mathcal{E}'}$. As shown in [KQ19] for the SL_2 case, this will follow from showing that $\Sigma = \mathfrak{Q}$ is an ideal square, then $F_{\mathfrak{Q}}$

is the same map for both triangulations of $\mathfrak{Q}$. We illustrate the two triangulations of $\mathfrak{Q}$ here. One triangulation is $\mathcal{E} = \{e\}$ and the other is $\mathcal{E}' = \{e'\}$.



The skein algebra $\mathcal{S}_q^{SL_3}(\mathfrak{Q})$ has a nice generating set consisting of single corner arcs, single horizontal arcs, single vertical arcs, with all possible strand orientations and state labels.

Proposition 5.14 *Suppose that γ is a stated arc in our generating set for $\mathcal{S}_q^{SL_3}(\mathfrak{Q})$. Then both $F_{\mathfrak{Q},\mathcal{E}}$ and $F_{\mathfrak{Q},\mathcal{E}'}$ send γ to γ^N. Thus, the map $F_{\mathfrak{Q}}$ is invariant under change of triangulation of $\mathfrak{Q}$.*

Proof: We must check that for the generator $\gamma \in \mathcal{S}_1^{SL_3}(\mathfrak{Q})$, we have the equalities

$$\Delta_e(\gamma^N) = (F_{\mathfrak{T}} \otimes F_{\mathfrak{T}})\Delta_e$$

and

$$\Delta_{e'}(\gamma^N) = (F_{\mathfrak{T}} \otimes F_{\mathfrak{T}})\Delta_{e'}.$$

If γ can be isotoped so that it does not intersect e then the first equation is obvious. Otherwise, it can be isotoped so that it intersects e exactly once and then the first equality follows from the fact that $F_{\mathfrak{B}}$ is a bialgebra map and from an embedding $\mathfrak{B} \sqcup \mathfrak{B} \hookrightarrow \mathfrak{T} \sqcup \mathfrak{T}$. An analogous argument works for the second equality. ∎

Next we record a compatibility of $F_{\Sigma,\mathcal{E}}$ with a partial splitting of the triangulation. Suppose a and b are two boundary arcs of a punctured bordered surface Σ and let

$\bar{\Sigma} = \Sigma/(a = b)$. Then the common image of a and b on $\bar{\Sigma}$ is an ideal arc we will denote e. Suppose $\bar{\Sigma}$ has an ideal triangulation with set of interior edges $\mathcal{E}$ with $e \in \mathcal{E}$. Then Σ naturally inherits an ideal triangulation with edge set $\mathcal{E} \setminus \{e\}$. We are interested in the relationship between $F_{\bar{\Sigma},\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$.

Proposition 5.15 *We have that $F_{\bar{\Sigma},\mathcal{E}}$ is equal to the unique algebra map making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{S}_1^{SL_3}(\bar{\Sigma}) & \xrightarrow{\Delta_e} & \mathcal{S}_1^{SL_3}(\Sigma) \\ \downarrow{\scriptstyle F_{\bar{\Sigma},\mathcal{E}}} & & \downarrow{\scriptstyle F_{\Sigma,\mathcal{E}\setminus\{e\}}} \\ \mathcal{S}_q^{SL_3}(\bar{\Sigma}) & \xrightarrow{\Delta_e} & \mathcal{S}_q^{SL_3}(\Sigma). \end{array}$$

Proof:

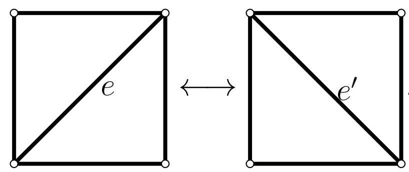
We examine the following diagram:

$$\begin{array}{ccccc} \mathcal{S}_1^{SL_3}(\bar{\Sigma}) & \xrightarrow{\Delta_e} & \mathcal{S}_1^{SL_3}(\Sigma) & \xrightarrow{\Delta_{\mathcal{E}\setminus\{e\}}} & \bigotimes_i^n \mathfrak{T}_i \\ \downarrow{\scriptstyle F_{\bar{\Sigma},\mathcal{E}}} & & \downarrow{\scriptstyle F_{\Sigma,\mathcal{E}\setminus\{e\}}} & & \downarrow{\scriptstyle \otimes_i F_{\mathfrak{T}_i}} \\ \mathcal{S}_q^{SL_3}(\bar{\Sigma}) & \xrightarrow{\Delta_e} & \mathcal{S}_q^{SL_3}(\Sigma) & \xrightarrow{\Delta_{\mathcal{E}\setminus\{e\}}} & \bigotimes_i^n \mathfrak{T}_i. \end{array}$$

The outer rectangle and the right square both commute by the definitions of $F_{\bar{\Sigma},\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$. Thus, the left square commutes. The injectivity of Δ_e and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$ imply the uniqueness of $F_{\bar{\Sigma},\mathcal{E}}$. ■

Corollary 5.16 *Suppose that Σ is a punctured bordered surface with an ideal triangulation with a set of internal edges $\mathcal{E}$. The map $F_{\Sigma,\mathcal{E}}$ does not depend on the triangulation $\mathcal{E}$.*

Proof: Suppose that Σ has a second ideal triangulation $\mathcal{E}'$. Then $\mathcal{E}'$ may be obtained from $\mathcal{E}$ by a finite sequences of edge flips involving an internal edge that borders two distinct faces. Thus, without loss of generality, we can assume that $\mathcal{E}$ and $\mathcal{E}'$ are identical except for a single edge flip in a square $\mathfrak{Q}$:



Let $\Delta_{\mathfrak{Q}} : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma \setminus \mathfrak{Q})$ be the composition of splitting maps associated to cutting the square $\mathfrak{Q}$ out of Σ.

By Proposition 5.14, $F_{\mathfrak{Q}}$ does not depend on its triangulation. A repeated application of Proposition 5.15 implies that since both $F_{\Sigma,\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}'}$ make the following diagram commute:

$$\begin{array}{ccc} \mathcal{S}_1^{SL_3}(\Sigma) & \xrightarrow{\Delta_{\mathfrak{Q}}} & \mathcal{S}_1^{SL_3}(\Sigma \setminus \mathfrak{Q}) \otimes \mathcal{S}_1^{SL_3}(\mathfrak{Q}) \\ \downarrow & & \downarrow{\scriptstyle F_{\Sigma\setminus Q}\otimes F_{\mathfrak{Q}}} \\ \mathcal{S}_q^{SL_3}(\Sigma) & \xrightarrow{\Delta_{\mathfrak{Q}}} & \mathcal{S}_q^{SL_3}(\Sigma \setminus \mathfrak{Q}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{Q}), \end{array}$$

we have the equality $F_{\Sigma,\mathcal{E}} = F_{\Sigma,\mathcal{E}'}$. ■

So far we have shown that Theorem 5.6 is true for any ideal triangulable surface Σ and that the definition of F_Σ in these cases does not depend on the triangulation. We now briefly comment on the surfaces with at least one puncture which do not admit an ideal triangulation.

Proposition 5.17 *The four punctured bordered surfaces which do not admit a triangulation admit a Frobenius map.*

Proof: The four surfaces are the monogon $\mathfrak{M}$, the bigon $\mathfrak{B}$, and the sphere with 2 or 1 punctures. If the surface Σ is the monogon or the sphere with one puncture, then

$S_q^{SL_3}(\Sigma) \cong \mathcal{R}$, which is commutative. So in these cases, F_Σ is determined by the fact that it sends the empty diagram to the empty diagram.

If $\Sigma = \mathfrak{B}$, we have already constructed $F_{\mathfrak{B}}$ as the map from [PW91] from $\mathcal{O}_1(SL_3)$ to $\mathcal{O}_q(SL_3)$.

If Σ is the sphere with 2 punctures, then we let c be an ideal arc connecting the two punctures and we define F_Σ to be the unique map making the following diagram commute:

$$\begin{array}{ccc} \mathcal{S}_1^{SL_3}(\Sigma) & \xrightarrow{\Delta_c} & \mathcal{S}_1^{SL_3}(\mathfrak{B}) \\ \Big\downarrow F_\Sigma & & \Big\downarrow F_{\mathfrak{B}} \\ \mathcal{S}_q^{SL_3}(\Sigma) & \xrightarrow{\Delta_c} & \mathcal{S}_q^{SL_3}(\mathfrak{B}). \end{array}$$

■

In this section, we have defined our Frobenius morphism F_Σ locally, in mostly an algebraic manner, and extended it to the whole surface. We have shown that for a triangulable surface Σ, the map F_Σ does not depend on the triangulation, and so is canonical in some sense. However, there should be a nice global definition of F_Σ that can be given without reference to a triangulation, and one which will generalize to the case of skein algebras of closed surfaces and skein modules of 3-manifolds. We would hope for a description of the image of an arbitrary web with a single connected component. For example, it is certain that a stated arc α should be sent to its Nth framed power. A knot should be threaded through an SL_3 analogue of the Nth Chebyshev polynomial analogous to the SL_2 constructions in [BW16, Lê15, BL20, KQ19]. It is unclear what should be the image of a more complicated web, so it would be interesting to find a nice description for it. These questions are beyond the scope of the current thesis but deserve to be explored in the future.

5.3 Reshetikhin-Turaev functor for Kuperberg's SL_3 web category

Our goal in this section is to prove the following.

Theorem 5.18 *The Reshetikhin-Turaev functor* $RT : Web^{SL_3} \to \mathcal{O}_q(SL_3)$*-comod*$\langle V\rangle$ *is an isomorphism of braided monoidal categories.*

The theorem will follow from an interpretation of the exact sequence associated to our splitting map.

To define relevant categories and functors, it will be convenient to introduce modified versions of $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ in which we allow for one or both boundary arcs of $\mathfrak{B}$ to be designated to contain endpoints of webs without states and in which we do not impose any boundary skein relations along the designated boundary arcs. We can call such a boundary arc an *inactive* boundary arc. In our notation, we will use "_" on the right or left of $\mathfrak{B}$ to indicate an inactive boundary arc, which is one designated to have endpoints which are not labeled by states. For example, $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}_{-})$ denotes the skein algebra of webs in the bigon with endpoints unlabeled by states and subject to only the interior skein relations. The notation $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B})$ denotes the skein algebra of webs in the bigon such that any endpoints on the left boundary arc of $\mathfrak{B}$ are unlabeled by states (but endpoints on the right boundary arc are labeled by states), and which is subject to only the interior skein relations and stated skein relations along the right boundary arc. Similarly, the skein algebra $\mathcal{S}_q^{SL_3}(\mathfrak{B}_{-})$ denotes the skein algebra of webs in the bigon such that any endpoints on the right boundary arc of $\mathfrak{B}$ are unlabeled by states, and which is subject to only the interior skein relations and the stated skein relations along the left boundary arc.

Our theorems involving bases and splitting maps carry over to the situation of inactive boundary arcs. We use these modified versions of skein algebras to define certain categories and functors. First we observe that these new versions of our skein algebras admit a module decomposition in terms of boundary data of webs. Let $\vec{a}$ be a sequence of left and right arrows $\vec{a} = (a_1, ..., a_k)$ for some $k \geq 0$ with each $a_i \in \{\leftarrow, \rightarrow\}$.

In this section, we will identify the state + with the integer 1, the state 0 with the integer 2 and the state − with the integer 3.

Definition 5.19 *For an arrow sequence $\vec{a}$ we define $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ to be the submodule of $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B})$ spanned by webs whose left boundary data, read from top to bottom, agrees with the arrow sequence $\vec{a}$. Similarly, we define $\mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ to be the submodule of $\mathcal{S}_q^{SL_3}(\mathfrak{B}_{-})$ spanned by webs whose left boundary data, read from top to bottom, agrees with the arrow sequence $\vec{a}$. Finally, for two arrow sequences $\vec{a}, \vec{b}$ we define $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ to be the submodule of $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}_{-})$ spanned by webs whose right boundary data agrees with $\vec{b}$ and whose right boundary data agrees with $\vec{a}$.*

Proposition 5.20 *Our algebras are graded with respect to the following decompositions as $\mathcal{R}$-modules:*

i)

$$\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}) = \bigoplus_{\vec{a}} \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}),$$

where the direct sum is over all possible arrow sequences $\vec{a}$.

ii)

$$\mathcal{S}_q^{SL_3}(\mathfrak{B}_{-}) = \bigoplus_{\vec{a}} \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}}),$$

where the direct sum is over all possible arrow sequences $\vec{a}$.

iii)

$$\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}_{-}) = \bigoplus_{\vec{a},\vec{b}} \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}}),$$

where the direct sum is over all possible arrow sequences $\vec{a}, \vec{b}$.

Proof: The proposition follows from the fact that none of our reduction rules coming from our Diamond Lemma algorithm will change the boundary data of a web along an inactive boundary arc, and so the algebras are graded with respect to this data. ■

5.3.1 The category $\mathbf{Web}^{SL_3}$

The first category we will define is Kuperberg's SL_3 web category, modified to our setting. The category Web^{SL_3} is the monoidal $\mathcal{R}$-linear category consisting of the following data:

- An object $\vec{a}$ of Web^{SL_3} is a sequence of arrows $\vec{a} = (a_1, a_2, ..., a_k)$ for some $k \geq 0$ where $a_i \in \{\leftarrow, \rightarrow\}$.
- $\mathrm{Hom}(\vec{a}, \vec{b})$ is the module $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$.
- The composition of morphisms is defined on diagrams $D \in \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and $E \in \mathcal{S}_q^{SL_3}({}_{\vec{c}}\mathfrak{B}_{\vec{b}})$ by horizontally gluing E on the left of D to obtain a diagram $E \circ D$ in $\mathcal{S}_q^{SL_3}({}_{\vec{c}}\mathfrak{B}_{\vec{a}})$.
- The tensor product $\vec{a} \otimes \vec{b}$ of objects $\vec{a}$ and $\vec{b}$ is the concatenation $(\vec{a}, \vec{b})$. The tensor product of morphisms is then given by the product operation in $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}_{-})$.

5.3.2 The category $\mathcal{O}_q(SL_3)$-comod$\langle V \rangle$

We next give the definition of our category $\mathcal{O}_q(SL_3)$-comod$\langle V \rangle$ and then give it a diagrammatic interpretation. The category $\mathcal{O}_q(SL_3)$-comod$\langle V \rangle$ is the full subcategory of right $\mathcal{O}_q(SL_3)$ comodules tensor-generated by the standard rank 3 comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$.

Before giving a precise definition of our category, we fix conventions for the standard comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$. We let $V_{\rightarrow}$ be the free $\mathcal{R}$-module with basis v_1, v_2, v_3 with coaction $V_{\to} \to V_{\to} \otimes \mathcal{O}_q(SL_3)$ given by

$$v_i \mapsto \sum_{j=1}^{3} v_j \otimes X_{ji}.$$

Due to conventions associated to our definition of the stated skein relations and the splitting map, we will use a non-standard weight basis of $V_{\leftarrow}$, meaning that our basis will not be the dual basis of our basis for $V_{\to}$. We let $V_{\leftarrow}$ be the free $\mathcal{R}$-module with basis w_1, w_2, w_3 with coaction $V_{\leftarrow} \to V_{\leftarrow} \otimes \mathcal{O}_q(SL_3)$ given by

$$w_i \mapsto \sum_{j=1}^{3} w_j \otimes q^{2i-2j} S(X_{\bar{i}\bar{j}}),$$

where we use the notation $\bar{k} = 4 - k$.

Given a sequence of arrows $\vec{a} = (a_1, \dots, a_k)$, we denote by $V_{\vec{a}}$ the tensor product

$$V_{\vec{a}} = V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_k}.$$

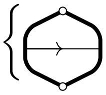
The category $\mathcal{O}_q(SL_3)$-comod$\langle V \rangle$ consists of the following data:

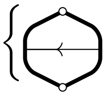
- Objects are the modules $V_{\vec{a}}$, which are finite tensor products of copies of $V_{\to}$ and $V_{\leftarrow}$.
- Morphisms are $\mathcal{R}$-linear maps between objects which commute with the right coaction of $\mathcal{O}_q(SL_3)$. We call the set of morphisms $\mathrm{Hom}_{\mathcal{O}_q(SL_3)}(V_{\vec{a}}, V_{\vec{b}})$.

Recall that the splitting map $\Delta : \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B})$ gives $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ the structure of a right $\mathcal{S}_q^{SL_3}(\mathfrak{B})$ comodule, which is a right $\mathcal{O}_q(SL_3)$ comodule structure when we use the identification $\mathcal{S}_q^{SL_3}(\mathfrak{B}) \cong \mathcal{O}_q(SL_3)$. We find the following diagrammatic interpretation of the objects of our category.

Proposition 5.21 *Given a sequence of arrows $\vec{a}$, we have that $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \cong V_{\vec{a}}$ as $\mathcal{O}_q(SL_3)$ comodules.*

Proof: We first look at the generating cases. If $\vec{a}$ happens to be the empty sequence, then both $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ and $V_{\vec{a}}$ are isomorphic to $\mathcal{R}$ with the trivial comodule structure.

If $\vec{a} = (\rightarrow)$, then $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ has a basis $\left\{ \right.$  $\left. i \right\}_{i=1}^{3}$ and the image of this basis under the splitting map agrees with the coaction on the basis $\{v_i\}_{i=1}^3$ of $V_{\rightarrow}$.

Similarly, if $\vec{a} = (\leftarrow)$ then $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ has a basis $\left\{ \right.$  $\left. i \right\}_{i=1}^{3}$ and the image of this basis under the splitting map agrees with the coaction on the basis $\{w_i\}_{i=1}^3$ of $V_{\leftarrow}$.

If $\vec{a}$ is an arbitrary sequence of arrows, then a basis of $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ consists of a product of basis elements of $\mathcal{S}_q^{SL_3}({}_{\leftarrow}\mathfrak{B})$ and $\mathcal{S}_q^{SL_3}({}_{\rightarrow}\mathfrak{B})$. Since the splitting map is an algebra map, we have that the image of this basis under the splitting map agrees with the coaction on the standard tensor basis of $V_{\vec{a}}$. ■

Next, we provide a diagrammatic interpretation of some of the morphisms of our category.

Proposition 5.22 *Given a diagram E in $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and a diagram D in $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$, we obtain a diagram $E \circ D$ in $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B})$ by gluing horizontally. This gluing defines a linear map $E : \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B})$. The linear map commutes with the coaction.*

Proof: The equation $(E \otimes \mathrm{id})\Delta(D) = \Delta(E \circ D)$ can be seen diagrammatically, so E commutes with the coaction. ■

We now have the ingredients to define our Reshetikhin-Turaev functor.

Proposition 5.23 *We produce a functor $RT : Web^{SL_3} \to \mathcal{O}_q(SL_3)$-comod$\langle V \rangle$ in the following manner. On objects, we define $RT(\vec{a}) = V_{\vec{a}}$, which we have identified with $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$. On morphisms, RT is the identity on the module $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$, which we have previously identified as a submodule of $Hom_{\mathcal{O}_q(SL_3)}(V_{\vec{a}}, V_{\vec{b}})$.*

We will eventually show that RT is an isomorphism of categories. First, we will need a diagrammatic interpretation of $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}})$.

5.3.3 The category split($\mathbf{Web}^{SL_3}$)

The category split(Web^{SL_3}) is the monoidal $\mathcal{R}$-linear category consisting of the following data:

- An object $\vec{a}$ of split(Web^{SL_3}) is a sequence of arrows.
- $\mathrm{Hom}(\vec{a}, \vec{b})$ is the module $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$.
- The composition of morphisms is defined on diagrams $D_1 \otimes D_2 \in \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ and $E_1 \otimes E_2 \in \mathcal{S}_q^{SL_3}({}_{\vec{c}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{b}})$ by gluing E_2 on the left of D_1, taking the counit, and obtaining $\varepsilon(E_2 \circ D_1)E_1 \otimes D_2 \in \mathcal{S}_q^{SL_3}({}_{\vec{c}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$.
- The tensor product $\vec{a} \otimes \vec{b}$ of objects $\vec{a}$ and $\vec{b}$ is the concatenation $(\vec{a}, \vec{b})$. The tensor product of morphisms is then given by the product operation in $\mathcal{S}_q^{SL_3}({}_{-}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{-})$.

5.3.4 The category $\mathcal{R}$-comod$\langle V \rangle$

We now give the definition of the category $\mathcal{R}$-comod$\langle V \rangle$ and then give it a diagrammatic interpretation. The category $\mathcal{R}$-comod$\langle V \rangle$ is the full subcategory of right $\mathcal{R}$-comodules tensor generated by the standard rank 3 comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$. The coaction of $\mathcal{R}$ is the trivial coaction $V_{\vec{a}} \to V_{\vec{a}} \otimes \mathcal{R}$. So it does no harm to think of this category as the full subcategory of $\mathcal{R}$-modules tensor generated by $V_{\rightarrow}$ and $V_{\leftarrow}$.

We record the data of our category $\mathcal{R}$-comod$\langle V \rangle$:

- Objects are the modules $V_{\vec{a}}$, which are finite tensor products of copies of $V_{\rightarrow}$ and $V_{\leftarrow}$.

- Morphisms are $\mathcal{R}$-linear maps between objects which commute with the (trivial) right coaction of $\mathcal{R}$. We call the set of morphisms $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}})$.

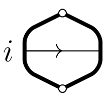
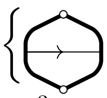
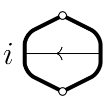
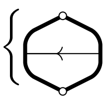
Proposition 5.24 *We have the following:*

(i) $V_{\vec{b}} \cong \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B})$ *as* $\mathcal{R}$*-comodules.*

(ii) $(V_{\vec{a}})^* \cong \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ *as* $\mathcal{R}$*-comodules, with evaluation of* $E \in \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ *and* $D \in \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ *given by gluing horizontally and taking the counit to obtain* $\varepsilon(E \circ D)$*.*

(iii) $Hom_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) \cong \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$*.*

Proof: We already proved (i) holds for $\mathcal{O}_q(SL_3)$-comodules, so it holds for $\mathcal{R}$-comodules as well.

Under the pairing described in (ii), we have that the basis $\Big\{$ i  $\Big\}_{i=1}^{3}$ of $\mathcal{S}_q^{SL_3}(\mathfrak{B}_{\rightarrow})$ and the basis $\Big\{$  i $\Big\}_{i=1}^{3}$ of $\mathcal{S}_q^{SL_3}({}_{\rightarrow}\mathfrak{B})$ are dual bases. Similarly, the bases $\Big\{$ i  $\Big\}_{i=1}^{3}$ and $\Big\{$  i $\Big\}_{i=1}^{3}$ are dual to each other. Thus, for an arbitrary arrow sequence $\vec{a}$, the standard basis of the tensor product $\mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ is dual to the standard basis of the tensor product $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$.

The statement (iii) follows from the property $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) \cong V_{\vec{b}} \otimes (V_{\vec{a}})^*$. ■

We now have the ingredients to prove a category isomorphism.

Proposition 5.25 *The following functor split*(RT) : *split*$(Web^{SL_3}) \rightarrow \mathcal{R}$*-comod*$\langle V \rangle$ *defines an isomorphism of categories.*

- *On objects, split*$(RT)(\vec{a}) = V_{\vec{a}}$*, which is identified with* $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$*.*

- *On morphisms, split*(RT) *is the identity on* $\mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$*.*

5.3.5 Relating the categories

So far we have discussed four $\mathcal{R}$-linear monoidal categories. We next observe that they fit into a commutative diagram of categories.

Proposition 5.26 *The following diagram of categories is commutative:*

$$\begin{array}{ccc} Web^{SL_3} & \xrightarrow{\Delta} & split(Web^{SL_3}) \\ \downarrow RT & & \downarrow split(RT) \\ \mathcal{O}_q^{SL_3}\text{-}comod\langle V\rangle & \xrightarrow{incl} & \mathcal{R}\text{-}comod\langle V\rangle, \end{array}$$

where the functor Δ is defined as

- *On objects, $\Delta(\vec{a}) = \vec{a}$.*
- *On morphisms, $\Delta : \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}}) \to \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}})$ is the splitting map.*

Proof: We first address the functoriality of Δ. It respects the monoidal structure since Δ is an algebra map. We need to check that it respects compositions of diagrams. Suppose that $D \in \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and $E \in \mathcal{S}_q^{SL_3}({}_{\vec{c}}\mathfrak{B}_{\vec{b}})$ are diagrams. We need to check that

$$\Delta(E \circ D) = \Delta(E) \circ \Delta(D).$$

Before we check this with a computation, we introduce some notation. Given an arrow sequence $\vec{b} = (b_1, ..., b_k)$ we let $\mathrm{St}(\vec{b}) = \{1, 2, 3\}^k$ denote the set of sequences of states of the same length as $\vec{b}$. To verify that our equality holds, we choose to cut E very close to its right boundary so that $\Delta(E) = \sum_{v\in \mathrm{St}(\vec{b})} E_v \otimes {}_vE''$ such that each diagram ${}_vE''$ consist of only parallel strands whose left endpoints are labeled with a sequence of states corresponding to the standard basis vector $v \in V_{\vec{b}}$ and each diagram E_v is the same underlying diagram as E but with its right endpoints labeled with states corresponding to v. Similarly, we choose to cut D very close to its left boundary so that $\Delta(D) = \sum_{w \in \mathrm{St}(\vec{b})} D'_w \otimes {}_w D$ such that each diagram D'_w consist of only parallel strands whose right endpoints are labeled with a sequence of states corresponding to the standard basis vector $w \in V_{\vec{b}}$. This allows us to compute that

$$\begin{aligned} \Delta(E) \circ \Delta(D) &= \sum_{v,w \in \mathrm{St}(V_{\vec{b}})} \varepsilon({}_v E'' \circ D'_w) E_v \otimes {}_w D \\ &= \sum_{v,w \in \mathrm{St}(V_{\vec{b}})} \delta_{vw} E_v \otimes {}_w D \\ &= \sum_{v \in \mathrm{St}(V_{\vec{b}})} E_v \otimes {}_v D \\ &= \Delta(E \circ D), \end{aligned}$$

as required.

Next, we check that the diagram commutes. We see that it commutes for objects, so we need to check that it commutes for morphisms. We can check this on a diagram. Let $E \in \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ be a diagram. Then $\mathrm{incl}(RT(E)) : \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B})$ is defined on a diagram $D \in \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ by gluing to obtain the diagram

$$E \circ D \in \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}).$$

On the other hand, $\mathrm{split(RT)}\Delta(E) = \sum_{(E)} E' \otimes E''$ is a morphism $\mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B})$ which sends a diagram $D \in \mathcal{S}_q^{SL_3}({}_{\vec{a}}\mathfrak{B})$ to

$$\sum_{(E)} E'\varepsilon(E'' \circ D) = \sum_{(E \circ D)} (E \circ D)' \otimes \varepsilon((E \circ D)'')$$
$$= E \circ D,$$

by the counit axiom. So the diagram commutes. ■

5.3.6 Proof that RT is an isomorphism

We now will observe that RT is an isomorphism on Hom modules. The following proposition is a consequence of the identifications we have established in this section.

Proposition 5.27 *The following diagram of $\mathcal{R}$-modules commutes*

$$\begin{array}{ccc} \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}}) & \xrightarrow{\Delta_{\vec{b}}\mathfrak{B} - \Delta_{\mathfrak{B}_{\vec{a}}}} & \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}}) \\ \downarrow{\scriptstyle id \otimes id} & & \downarrow{\scriptstyle id \otimes \psi \otimes id} \\ Hom_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) & \xrightarrow{\Delta_{V_{\vec{b}}} \circ (-) - ((-) \otimes id) \circ \Delta_{V_{\vec{a}}}} & Hom_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}} \otimes \mathcal{O}_q(SL_3)), \end{array}$$

where $\psi : \mathcal{S}_q^{SL_3}(\mathfrak{B}) \to \mathcal{O}_q(SL_3)$ is our isomorphism from before and we have used identifications in the bottom row of the form $Hom_{\mathcal{R}}(X, Y) = Y \otimes X^*$, so that the vertical maps make sense.

Corollary 5.28 *The RT functor is an isomorphism of $\mathcal{R}$-linear braided monoidal categories $Web^{SL_3} \to \mathcal{O}_q(SL_3)$-comod$\langle V \rangle$.*

Proof: The functor RT is bijective on objects, so we just need to show that it induces isomorphisms on Hom sets. For that we observe the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}}) & \xrightarrow{\Delta} & \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}}) & \xrightarrow{\Delta_{\vec{b}\mathfrak{B}} - \Delta_{\mathfrak{B}\vec{a}}} & \mathcal{S}_q^{SL_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{SL_3}(\mathfrak{B}_{\vec{a}}) \\
 & & \big\downarrow RT & & \big\downarrow \mathrm{id}\otimes\mathrm{id} & & \big\downarrow \mathrm{id}\otimes\psi\otimes\mathrm{id} \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q(SL_3)}(V_{\vec{a}}, V_{\vec{b}}) & \xrightarrow{\mathrm{incl}} & \mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) & \xrightarrow{\Delta_{V_{\vec{b}}}\circ(-)-((-)\otimes\mathrm{id})\circ\Delta_{V_{\vec{a}}}} & \mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}} \otimes \mathcal{O}_q(SL_3)).
\end{array}
$$

The top row is exact by the splitting theorem and the bottom row is exact by the definition of a morphism of $\mathcal{O}_q(SL_3)$-comod$\langle V\rangle$. The vertical maps in the middle and the right are isomorphisms. Thus, RT is an isomorphism as well, by a special case of the five lemma. ■

Finally, we will observe that since the pairing $\langle -, -\rangle : \mathcal{O}_q(SL_3) \otimes U_q(\mathfrak{sl}_3)$ turns right $\mathcal{O}_q(SL_3)$-comodules into left $U_q(\mathfrak{sl}_3)$-modules, we obtain an embedding of categories

$$\mathrm{Web}^{SL_3} \xrightarrow{\cong} \mathcal{O}_q(SL_3)\text{-comod}\langle V\rangle \hookrightarrow U_q(\mathfrak{sl}_3)\text{-mod}\langle V\rangle.$$

If the pairing is nondegenerate, then the embedding will be an isomorphism. We can see this after observing the following.

Lemma 5.29 *Suppose the pairing $\langle -, -\rangle : \mathcal{O}_q(SL_3) \otimes U_q(\mathfrak{sl}_3) \to \mathcal{R}$ is nondegenerate. If U and W are right $\mathcal{O}_q(SL_3)$-comodules and $T : U \to W$ is an $\mathcal{R}$-linear map which commutes with the induced left $U_q(\mathfrak{sl}_3)$ action on U and W, then T commutes with the $\mathcal{O}_q(SL_3)$ coaction as well.*

Proof: Fix an arbitrary $u \in U$. By assumption, we have that for any $x \in U_q(\mathfrak{sl}_3)$,

$$x.T(u) = T(x.u).$$

We expand both sides of this equation by the definitions of the actions in terms of the pairing.

The left side is

$$x.T(u) = (\mathrm{id} \otimes \langle -, x \rangle)\Delta_W(T(u)).$$

The right side is

$$\begin{aligned} T(x.u) &= T((\mathrm{id} \otimes \langle -, x \rangle)\Delta_U(U)) \\ &= (\mathrm{id} \otimes \langle -, x \rangle)(T \otimes \mathrm{id})\Delta_U(u). \end{aligned}$$

These equations hold for all $x \in U_q(\mathfrak{sl}_3)$ and so by considering bases of V and W, we are able to use the fact that the pairing is nondegenerate to conclude that

$$(T \otimes \mathrm{id})\Delta_U(u) = \Delta_W T(u),$$

and T commutes with the coaction. ■

Corollary 5.30 *Whenever the pairing $\langle -, - \rangle : \mathcal{O}_q(SL_3) \otimes U_q(\mathfrak{sl}_3) \to \mathcal{R}$ is nondegenerate, our Reshetikhin-Turaev functor gives an equivalence of braided monoidal categories $Web^{SL_3} \to U_q(\mathfrak{sl}_3)$-mod$\langle V \rangle$.*

Remark 5.31 *When $\mathcal{R} = \mathbb{C}$ and q is not a root of unity, then the pairing $\langle -, - \rangle$ is nondegenerate, as discussed in [Tak02]. To work at a root of unity, one can replace $U_q(\mathfrak{sl}_3)$ with a form of Lusztig's divided powers algebra studied in [DCL94].*

Using a similar method as in the proof of Theorem 4.9, we can use the reduction rules from Theorem 3.9 to define an explicit algorithm which takes as input a morphism in split(Web^{SL_3}) which commutes with the coaction and gives as output a morphism in Web^{SL_3}. The algorithm gives us a diagrammatic description of the inverse of the Reshetikhin-Turaev functor.

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