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Factorizations of diffeomorphisms of compact surfaces with boundary

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Factorizations of Diffeomorphisms of Compact Surfaces with Boundary

by

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requirements for the degree of
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Abstract

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University of California, Berkeley

Professor Robion Kirby, Chair

We study diffeomorphisms of compact, oriented surfaces, developing methods of distinguishing those which have positive factorizations into Dehn twists from those which satisfy the weaker condition of right veering. We use these to construct open book decompositions of Stein-fillable 3-manifolds whose monodromies have no positive factorization.

Contents

List of Figures

Acknowledgments

Firstly, I would like to express my gratitude to my advisor, Rob Kirby, who was responsible for my coming to Berkeley, and without whose continued support over the past years this thesis would not have been possible.

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Chapter 1

Introduction

Let Σ be a compact, orientable surface with nonempty boundary. The (restricted) mapping class group of Σ, denoted Γ_Σ, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. The goal of this paper is to study the monoid $Dehn^+(\Sigma) \subset \Gamma_\Sigma$ of products of right Dehn twists. In particular, we are able to show:

Theorem 1.0.1. *There exist open book decompositions which support Stein fillable contact structures but whose monodromies cannot be factorized into positive Dehn twists.*

This result follows from much more general results which we develop concerning $Dehn^+(\Sigma)$. We develop necessary conditions on the images of properly embedded arcs under these diffeomorphisms, and also necessary conditions for elements of the set of curves which can appear as twists in some positive factorization of a given $\varphi \in \Gamma_\Sigma$. The first of these can be thought of as a refinement of the *right veering* condition, which dates at least as far back as Thurston's proof of the left orderability of the braid group, and which was introduced into the study of contact structures by Honda, Kazez, and Matic in [HKM]. In that paper, it is shown that the monoid $Veer(\Sigma) \subset \Gamma_\Sigma$ of right-veering diffeomorphisms (see Section 2 for definitions) strictly contains $Dehn^+(\Sigma)$. Our methods allow one to easily distinguish certain elements of $Veer(\Sigma) \setminus Dehn^+(\Sigma)$.

Central to much of current research in contact geometry is the relation between the monodromy of an open book decomposition and geometric properties of the contact structure, such as tightness and fillability. The starting point is the remarkable theorem of Giroux [Gi], demonstrating a one-to-one correspondence between oriented contact structures on a 3-manifold M up to isotopy and open book decompositions of M up to positive stabilization. It has been shown by Giroux [Gi], Loi and Peirgallini [LP], and Akbulut and Ozbagci [AO], that any open book with monodromy which can be factorized into positive Dehn twists supports a Stein-fillable contact structure, that every Stein-fillable contact structure is supported by *some* open book with monodromy which can be factored into positive twists, and in [HKM] that a contact structure is tight if and only if the monodromy of each supporting open book is right-veering. The question of whether *each* open book which supports a Stein-fillable contact structure must be positive, answered in the negative by our Theorem 1.0.1, has been a long-standing open question.

Section 2 introduces some conventions and definitions, and provides motivation for what is to come. Section 3 introduces the concept of a *right position* for the monodromy image of a properly embedded arc in Σ, and compatibility conditions for right positions on collections of arc images under a given monodromy. More importantly, we show that a monodromy can have a positive factorization only if the images of any collection of nonintersecting properly embedded arcs in Σ admit right positions which are pairwise compatible. We use this to construct simple examples of open books whose monodromies, though right veering (see Section 2 for a precise definition), have no positive factorization.

In Section 4 we use these results to address the question of under what conditions various isotopy classes of curves on a surface can be shown not to appear as Dehn twists in any positive factorization of a given monodromy. We obtain various necessary conditions under certain assumptions on the monodromy.

Finally, in Section 5, we construct explicit examples of open book decompositions which support Stein fillable contact structures yet whose monodromies have no positive factorizations. For the construction we demonstrate a method of modifying a certain mapping class group relation (the lantern relation) into an 'immersed' configuration, which we then use to modify certain open books with positively factored monodromies (which therefore support Stein fillable contact manifolds) into stabilization-equivalent open books (which support the same contact structures) whose monodromies now have non-trivial negative twisting. We then apply the methods developed in the previous sections to show that in fact this negative twisting is essential.

It has come to our attention that Baker, Etnyre, and Van Horn-Morris [BEV] have recently constructed similar examples of non-positive open books, all of which use the same surface, $\Sigma_{2,1}$, involved in our construction. Their argument for non-positivity, however, is specific to this surface, hinging on the non-triviality of the first homology group of the mapping class group of the surface (with its boundary capped off), which does not hold for surfaces of higher genus.

Chapter 2

Preliminaries

Throughout, Σ denotes a compact, orientable surface with nonempty boundary. The (restricted) mapping class group of Σ, denoted Γ_Σ, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. In general we will not distinguish between a diffeomorphism and its isotopy class.

Let $SCC(\Sigma)$ be the set of simple closed curves on Σ. Given $\alpha \in SCC(\Sigma)$ we may define a self-diffeomorphism D_α of Σ which is supported near α as follows. Let a neighborhood N of α be identified by oriented coordinate charts with the annulus $\{a \in \mathbb{C} \mid 1 \leq ||a|| \leq 2\}$. Then D_α is the map $a \mapsto e^{-i2\pi(||a||-1)}a$ on N, and the identity on $\Sigma \setminus \alpha$. We call D_α the *positive Dehn twist* about α. The inverse operation, D_α^{-1}, is a *negative* Dehn twist. We denote the mapping class of D_α by τ_α. It can easily be seen that τ_α depends only on the isotopy class of α.

We call $\varphi \in \Gamma_\Sigma$ *positive* if it can be factored as a product of positive Dehn twists.

An open book decomposition (Σ, φ), where $\varphi \in \Gamma_\Sigma$, for a 3-manifold M with binding K is a homeomorphism between $((\Sigma \times [0,1])/\sim_\varphi, (\partial\Sigma \times [0,1])/\sim_\varphi)$ and (M, K). The equivalence relation $\sim_\varphi$ is generated by $(x, 0) \sim_\varphi (\varphi(x), 1)$ for $x \in \Sigma$ and $(y, t) \sim_\varphi (y, t')$ for $y \in \partial\Sigma$.

Definition 2.0.2. For a surface Σ and positive mapping class φ, we define the *positive extension* $p.e.(\varphi) \subset SCC(\Sigma)$ as the set of all $\alpha \in SCC(\Sigma)$ such that τ_α appears in some positive factorization of φ.

A recurring theme of this paper is to use properly embedded arcs $\gamma_i \hookrightarrow \Sigma$ to understand restrictions on $p.e.(\varphi)$ which can be derived from the geometric information of the monodromy images $\varphi(\gamma_i)$ (relative to the arcs themselves). Our general method is as follows: Suppose that P is some property of pairs $(\varphi(\gamma), \gamma)$ (which we abbreviate by referring to P as a property of the image $\varphi(\gamma)$) which holds for the case φ is the identity, and is preserved by positive Dehn twists. Suppose then that $\alpha \in p.e.(\varphi)$; then there is a positive factorization of φ in which τ_α is a twist. Using the well known braid relation, we can assume τ_α is the *final* twist, and so the monodromy given by $\tau_\alpha^{-1} \circ \varphi$ is also positive. It follows that P holds for $(\tau_\alpha^{-1} \circ \varphi)(\gamma)$ as well.

As a motivating example, consider a pair of arcs $\gamma, \gamma' : [0,1] \hookrightarrow \Sigma$ which share an endpoint $\gamma(0) = \gamma'(0) = x \in \partial\Sigma$, isotoped to minimize intersection. Following [HKM], we say γ' is 'to the right' of γ, denoted $\gamma' \geq \gamma$, if either the pair is isotopic, or if the tangent vectors $(\dot{\gamma'}(0), \dot{\gamma}(0))$ define the orientation of Σ at x. The property of being 'to the right' of γ (at x) is then a property of images $\varphi(\gamma)$ which satisfies the conditions of the previous paragraph. We conclude that, if $\varphi(\gamma)$ is to the right of γ, then $\alpha \in p.e.(\varphi)$ only if $(\tau_\alpha^{-1} \circ \varphi)(\gamma)$ is to the right of γ.

A closely related definition which will be made use of throughout the paper is the following (due again to Honda, Kazez and Matic in [HKM]):

Definition 2.0.3. Let φ be a mapping class in Γ_Σ, $\gamma \hookrightarrow \Sigma$ a properly embedded arc with endpoint $x \in \partial\Sigma$. Then φ is *right veering* if, for each such γ and x, the image $\varphi(\gamma)$ is to the right of γ at x.

We denote the set of isotopy classes of right veering diffeomorphisms as $Veer(\Sigma) \subset \Gamma_\Sigma$.

Our aim in this paper is to determine conditions which must be satisfied by *pairs* of images in a positive factorization. As an initial step, we consider what it should means for two arcs to have 'the same' images under φ. An obvious candidate is the following:

Definition 2.0.4. Let γ be a properly embedded arc in Σ, $\varphi \in Veer(\Sigma)$. Define the *step-down* $\mathcal{C}_{\varphi(\gamma)}$ as the embedded multi-curve on Σ obtained by slicing $\varphi(\gamma)$ along γ, and re-attaching the endpoints as in Figure 2.1.

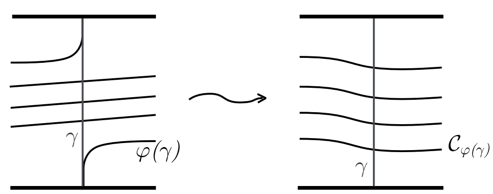


Figure 2.1: Construction of step-down

The stepdown thus allows us to relate distinct arcs to a common object, a multicurve. We want to say that a pair of arcs has in some sense 'the same' image if they have the same stepdown:

Definition 2.0.5. Let γ_1, γ_2 be properly embedded arcs in Σ, and $\varphi \in Veer(\Sigma)$. We say the pair γ_1, γ_2 is *parallel* under φ if $\mathcal{C}_{\varphi(\gamma_1)}$ is isotopic to $\mathcal{C}_{\varphi(\gamma_2)}$.

As it stands, however, this condition is far too strict to be of any practical use. Our approach then is to 'localize' these ideas by decomposing the monodromy images of arcs into segments which we may then compare across distinct arc images. As we shall see in the following section, this approach allows one under certain circumstances to derive useful information concerning positivity of a given monodromy beyond that given by right veering, purely from the geometric information of its arc images.

Chapter 3

Right position

In this section, we develop the idea of a *right position* for the diffeomorphic image of a properly embedded arc in Σ, and develop consistency conditions which allow us to prove:

Theorem 3.0.6. *Let Σ be a compact surface with boundary, $\varphi \in \Gamma_\Sigma$ admit a positive factorization into Dehn twists, and $\{\gamma_i\}_{i=1}^n, n \geq 1$, be a collection of non-intersecting, properly embedded arcs in Σ. Then $\{(\varphi, \gamma_i)\}_{i=1}^n$ admit consistent right positions.*

In particular, following the argument of the previous section, it follows that, for $\alpha \in SCC(\Sigma)$, $\alpha \in p.e.(\varphi)$ only if for each pair of disjoint arcs γ_1 and γ_2, the pairs $(\tau_\alpha^{-1} \circ \varphi, \gamma_1)$ and $(\tau_\alpha^{-1} \circ \varphi, \gamma_2)$ admit consistent right positions. As we shall see, this allows simple construction of examples of right veering monodromies with no positive factorization.

Definition 3.0.7. Suppose $\gamma \hookrightarrow \Sigma$ is a properly embedded arc with boundary $\{c, c'\}$, and $\varphi \in Veer(\Sigma)$, with $\varphi(\gamma)$ isotoped to minimize $\gamma \cap \varphi(\gamma)$. Let A be a subset of the positively oriented interior intersections in $\gamma \cap \varphi(\gamma)$ (sign conventions are illustrated in Figure 3.1). Then the set $\{c, c'\} \cup A$ is a *right position* $\mathcal{P} = \mathcal{P}(\varphi, \gamma)$ for the pair (φ, γ)

Note that we are thinking of the set I of positively oriented interior intersections in $\gamma \cap \varphi(\gamma)$ as depending only on the data of γ and the isotopy class φ, and thus independent of isotopy of γ and $\varphi(\gamma)$ as long as intersection minimality holds. In particular, $|I|$ clearly depends only on this data. We may label elements of I with indices $1, \dots, |I|$ increasing along γ from distinguished endpoint c. Right positions for the pair (φ, γ) are thus in 1-1 correspondence with subsets of the set of these indices.

Associated to a right position is the set $H(\mathcal{P}) := \{[v, v'] \subset \varphi(\gamma) \mid v, v' \in \mathcal{P}\}$. We denote these segments by $h_{v,v'}$, or, if only a single endpoint is required, simply use h_v to denote a subarc starting from v and extending as long as the context requires. In this case direction of extension along $\varphi(\gamma)$ will be clear from context.

Right position can thus be thought of as a way of distinguishing 'horizontal' segments $H(\mathcal{P})$, separated by the points $\mathcal{P}$ (Figure 3.2). Note that for any right-veering φ, and any properly embedded arc γ, (φ, γ) has a trivial right position consisting of the points $\partial(\gamma)$,

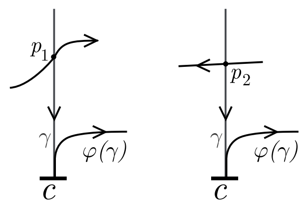


Figure 3.1: p_1 is a positive, p_2 a negative, point of $\gamma \cap \varphi(\gamma)$

and so there is a single horizontal segment $h_{c,c'} = \varphi(\gamma)$. Of course if I is nonempty, there are $2^{|I|} - 1$ non-trivial right positions. Intuitively, the points in $\mathcal{P}$ play a role analogous to the initial (boundary) points of the arc, allowing us to localize the global 'right'ness of an arc image though the decomposition into horizontal segments (this will be made more precise further on in the paper).

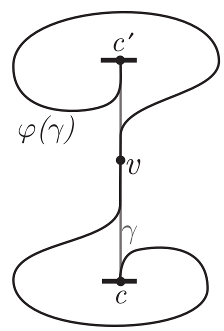


Figure 3.2: $\mathcal{P} = \{c, v, c'\}$ is a right position of (φ, γ). There are three distinct horizontal segments: $h_{c,v}, h_{v,c'}$, and $h_{c,c'}$

We are interested in using right position to compare *pairs* of arcs; the idea being that, if φ is positive, we expect to be able to view any pair of horizontal segments as either unrelated, or as belonging to the 'same' horizontal segment.

To get started, we need to be able to compare horizontal segments. To that end, we have:

Definition 3.0.8. Suppose γ_i, $i = 1, \ldots, 4$ are properly embedded arcs in a surface Σ. A *rectangular region* in Σ is the image of an immersion of the standard disc $[0,1] \times [0,1]$, such that the image of each side is a subset of one of the γ_i, and corners map to intersections $\gamma_i \cap \gamma_j$.

Definition 3.0.9. Let γ_1, γ_2 be distinct, properly embedded arcs in Σ, $\varphi \in Veer(\Sigma)$, and $\mathcal{P}_i = \mathcal{P}_i(\varphi, \gamma_i)$ a right position for $i = 1, 2$. We say that $h_v \in H(\mathcal{P}_1)$ and $h_w \in H(\mathcal{P}_2)$ are *initially parallel (along B)* if there is a rectangular region B in Σ bounded by $\gamma_1, h_w, \gamma_2, h_v$, such that, if each of h_v and h_w is oriented *away* from its respective vertical point, then the orientations give an orientation of ∂B (see Figure 3.3).

Remark 3.0.10. It is helpful to think of the rectangular region B along which two segments are initially parallel as the image under the covering map $\rho : \widetilde{\Sigma} \to \Sigma$ of an *embedded* disc in the universal cover. As such, we will draw these regions as embedded discs whenever doing so does not result in any loss of essential information.

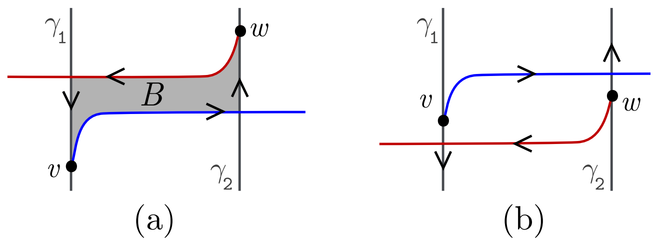


Figure 3.3: (a) h_v and h_w are initially parallel along B. (b) h_v and h_w are *not* initially parallel.

So, to say that a pair of horizontal segments is 'unrelated' is to say that they are not initially parallel. It is left to clarify what we expect of initially parallel segments h_v and h_w. The idea is to extend the notion of being parallel up to a second pair of vertical points v' w', and thus to the horizontal segments $h_{v,v'}$ $h_{w,w'}$, such that the pair is 'completed'; i.e. if we switch the orientations so that $h_{v'}$ and $h_{w'}$ are taken as extending in the opposite direction, then $h_{v'}$ and $h_{w'}$ are themselves initially parallel, and the rectangular regions complete one another to form a 'singular annulus' (Figure 3.4) (notice that there is no restriction on intersections of the regions beyond those given by the properties of the boundary arcs). To be precise:

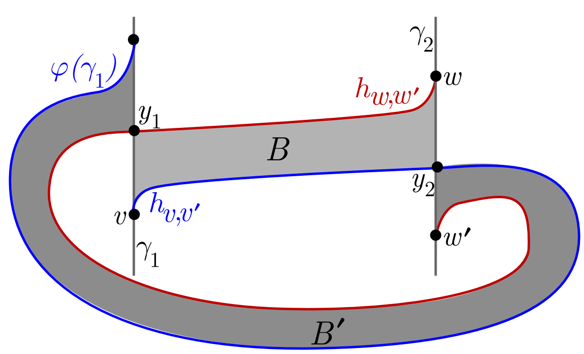


Figure 3.4: Horizontal segments $h_{v,v'}, h_{w,w'}$ complete one another, forming the boundary of a 'singular annulus'.

Definition 3.0.11. Let $\gamma_i, \mathcal{P}_i, i = 1, 2$ be as in the previous definition. Suppose horizontal segments h_v and h_w are initially parallel along B, with corners v, w, y_1, y_2, where $y_i \in \gamma_i$.

Then the pair h_v, h_w are *completed* if there are points $v' \in \mathcal{P}_1, w' \in \mathcal{P}_2$, which we call the *endpoints*, such that the arcs $\gamma_1, \gamma_2, h_{v,v'}, h_{w,w'}$ bound a second rectangular region B', with corners v', w', y_1, y_2 (Figure 3.5). We say the pair h_v, h_w are *completed along* B'. Note that, turning the picture upside-down, the arcs $h_{v',v}$ and $h_{w',w}$ are initially parallel along B', and completed along B.

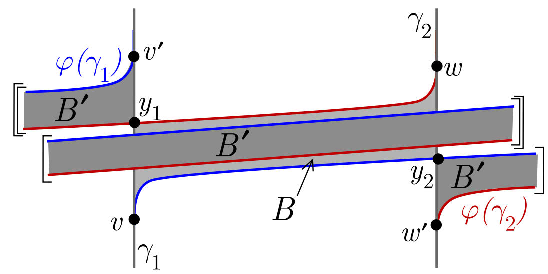


Figure 3.5: The horizontal segments h_v, h_w are completed by points v', w'. Strands which terminate in like brackets are meant to be identified along arcs which are not drawn; a similar convention will be used throughout the paper.

As an intuition-building observation, and justification of the term 'singular annulus', note that if initially parallel h_v and h_w are completed then the resolution of the points y_1 and y_2 as in Figure 3.6 transforms the 'singular' $B \cup B'$ into an immersed annulus.

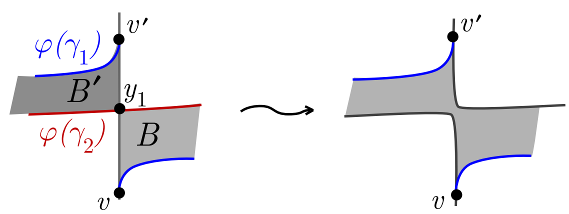


Figure 3.6: resolution of y_1

We have come to the key definition of this section:

Definition 3.0.12. Let $\{\gamma_i\}_{i=1}^n, n \geq 1$, be a collection of non-intersecting, properly embedded arcs in Σ, $\varphi \in Veer(\Sigma)$, and $\mathcal{P}_i = \mathcal{P}_i(\varphi, \gamma_i)$ a right position for each i. We say the $\mathcal{P}_i$ are *consistent* if for all $j \neq k$, each pair of horizontal segments $h \in H(\mathcal{P}_j)$ and $g \in H(\mathcal{P}_k)$ is completed if initially parallel. For the case $n = 2$, we say (φ, γ_1) and (φ, γ_2) are a *right pair* if they admit consistent right positions $\mathcal{P}_1$ and $\mathcal{P}_2$.

Example 3.0.13. The pair (φ, γ_1), (φ, γ_2) in Figure 3.7(a), with the right positions indicated, is a right pair, with two completed pairs of horizontal arcs, as indicated in (b). It is straightforward to verify that there are no other initially parallel segments.

The pair in Figure 3.8, however, is not a right pair: Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are consistent right positions. Then h_{c_1} and h_{c_2} are initially parallel along B, so there must be $v \in \mathcal{P}_1$ and $w \in \mathcal{P}_2$ satisfying the compatibility conditions. The only candidates are c_1' and c_2', and these do not give completed horizontal segments. By Theorem 3.0.6, this is sufficient to conclude that the mapping class has no positive factorization.

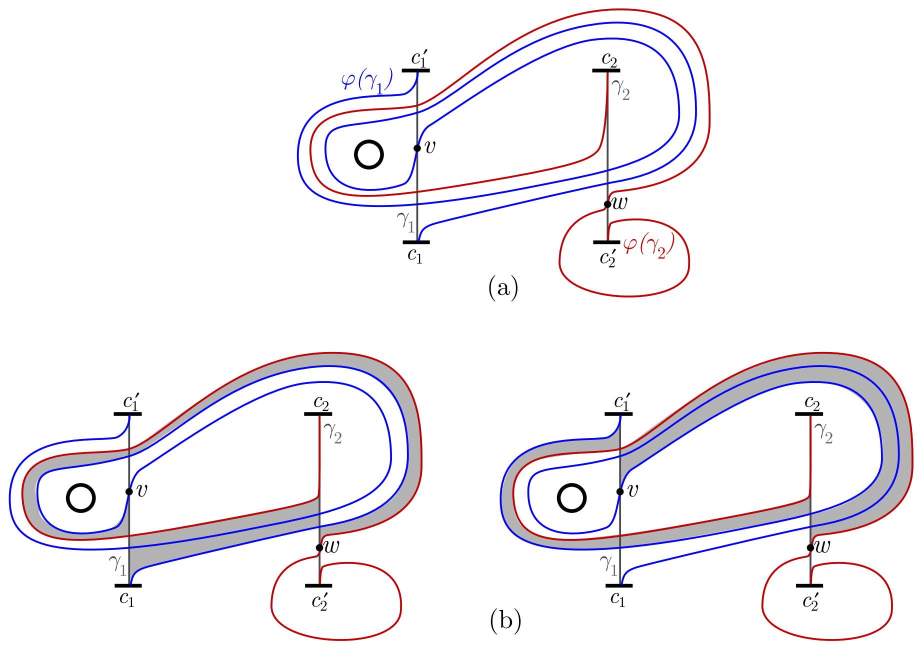


Figure 3.7: (a): The pair (φ, γ_1), (φ, γ_2) is a right pair. The pairs of completing discs are shaded in (b).

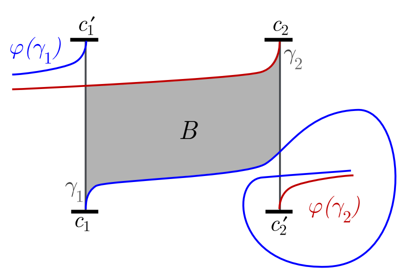


Figure 3.8: The pair (φ, γ_1), (φ, γ_2) is *not* a right pair.

3.1 The right position associated to a factorization

Suppose we have a surface Σ, a mapping class $\varphi \in \Gamma_\Sigma$ given as a factorization ω of positive Dehn twists, and a set $\{\gamma_i\}_{i=1}^n$ of pairwise non-intersecting, properly embedded arcs in Σ. While each such arc admits at least one, and possibly many, right positions, the goal of this subsection is to give an algorithm which associates to ω a unique right position, denoted $\mathcal{P}_\omega(\gamma_i)$ for each i, such that the set $\{\mathcal{P}_\omega(\gamma_i)\}_{i=1}^n$ is consistent (Definition 3.0.12). Note that existence of such an algorithm proves Theorem 3.0.6.

We begin with the case of a single arc γ, and construct $\mathcal{P}_\omega(\gamma)$ by induction on the number of twists m in ω.

Base step: $\mathbf{m = 1}$ Suppose $\varphi = \tau_\alpha$, for some $\alpha \in SCC(\Sigma)$. We begin by isotoping α so as to minimize $\alpha \cap \gamma$. Label $\alpha \cap \gamma = \{x_1, \ldots, x_p\}$, with indices increasing along γ. We then define the right position $\mathcal{P}(\tau_\alpha, \gamma)$ associated to the factorization τ_α to be the points $\{c = v_0, v_1, \ldots, v_{p-1}, c' = v_p\}$, where $\{c, c'\} = \partial(\gamma)$, and, for $0 < i < p$, v_i lies in the connected component of $\gamma \setminus support(\tau_\alpha)$ between x_i and x_{i+1} (Figure 3.9).

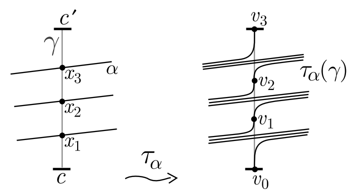


Figure 3.9: The base case: the right position $\mathcal{P}(\tau_\alpha, \gamma)$ associated to a single Dehn twist.

As an aside, note that, in this case, if $\overline{h}_{v_i,v_j}$ denotes the closed curve obtained from the horizontal segment h_{v_i,v_j} by adding the segment along γ from v_i to v_j, then $\overline{h}_{v_i,v_{i+1}} \simeq \alpha$ for $0 \leq i < p$. Compare this with the *stepdown* of Definition 2.0.4.

Now, this construction has as input only the isotopy class of α, so we have:

Lemma 3.1.1. *The right position $\mathcal{P}_{\tau_\alpha}(\gamma)$ defined in **base step** depends only on the isotopy class of α.*

3.1.1 Description of the inductive step

For the inductive step, we demonstrate an algorithm which has as input a surface Σ, mapping class $\varphi \in Veer(\Sigma)$, properly embedded arc $\gamma \hookrightarrow \Sigma$, right position $\mathcal{P}(\varphi, \gamma)$, and $\alpha \in SCC(\Sigma)$. The output then will be a unique right position $\mathcal{P}^\alpha = \mathcal{P}^\alpha(\tau_\alpha \circ \varphi, \gamma)$. A key attribute of the algorithm will be that, while α is given as a particular representative of its isotopy class $[\alpha]$, the resulting right position $\mathcal{P}^\alpha$ will be independent of choice of representative.

To keep track of things in an isotopy-independent way, we consider *triangular regions* in Σ:

Definition 3.1.2. Suppose γ and γ' are properly embedded arcs in a surface Σ such that $\partial(\gamma) = \partial(\gamma')$, isotoped to minimize intersection. Then for $\alpha \in SCC(\Sigma)$, a *triangular region* T (of the triple $(\gamma, \gamma', \alpha)$) is the image of an immersion $f : \Delta \looparrowright \Sigma$, where Δ is a 2-simplex with vertices t_1, t_2, t_3 and edges t_1t_2, t_2t_3, t_3t_1, such that $f(t_1t_2) \subset \gamma$, $f(t_2t_3) \subset \gamma'$, and $f(t_3t_1) \subset \alpha$.

Definition 3.1.3. A triangular region T for the ordered triple $(\alpha, \gamma, \gamma')$ is

- *essential* if α can be isotoped relative to $T \cap \alpha$ so as to intersect γ and γ' in a minimal number of points (Figure 3.13(a)).
- *upward* (*downward*) if bounded by α, γ, γ' in clockwise (counterclockwise) order (Figure 3.13(b)).

We begin with a brief description of the algorithm, with an illustrative example in Figure 3.10. Let α be a representative of $[\alpha]$ which minimizes $\alpha \cap \gamma$ and $\alpha \cap \varphi(\gamma)$, so in particular all triangular regions for the triple $(\alpha, \gamma, \varphi(\gamma))$ are essential. Furthermore, chose *support*(τ_α) so as not to intersect any point of $\gamma \cap \varphi(\gamma)$ (Figure 3.10(a)). Consider then the image $\tau_\alpha(\varphi(\gamma))$, which differs from $\varphi(\gamma)$ only in *support*(τ_α) (Figure 3.10(b)). Now, the only bigons bounded by the arcs $\tau_\alpha(\varphi(\gamma))$ and γ contain vertices which were in upward triangles in the original configuration. In particular, there is an isotopy of $\tau_\alpha(\varphi(\gamma))$ over (possibly some subset of) these bigons which minimizes $\gamma \cap \tau_\alpha(\varphi(\gamma))$ (the issue of exactly when a proper subset of the bigons suffices is taken up below, in Section 3.1.2). There is thus an inclusion map i of those points of $\mathcal{P}$ which are *not* in upward triangles into the set of positive intersections of $\gamma \cap \tau_\alpha(\varphi(\gamma))$, whose image is therefore a right position (Figure 3.10(c)).

Note however that, as described, this resulting right position is *not* independent of the choice of α - indeed, if T_1 is any downward triangle, we may isotope the given α over T_1 (Figure 3.10(d)) to obtain $\alpha' \in [\alpha]$ such that $\alpha' \cap \gamma$ and $\alpha' \cap \varphi(\gamma)$ are also minimal. This new α' thus satisfies the conditions of the algorithm, yet the algorithm of the previous paragraph determines a distinct right position for $\gamma, \tau'_\alpha(\varphi(\gamma))$ (Figure 3.10(f)). Motivated by this observation, we refine the algorithm by adding *all* points of $\gamma \cap \tau_\alpha(\varphi(\gamma))$ which would be obtained by running the algorithm for any allowable isotopy of α (Figure 3.10(g)). As we shall see, this simply involves adding a single point for each downward triangle in the original configuration.

3.1.2 Details of the inductive step

This subsection gives the technical details necessary to make the algorithm work as advertised, and extracts and proves the various properties we will require the algorithm and its result to have.

We begin with some results concerning triangular regions. Let γ, γ' be properly embedded, non-isotopic arcs in Σ, with $\partial(\gamma) = \partial(\gamma')$, isotoped to minimize intersection. Our motivating example, of course, is the case $\gamma' = \varphi(\gamma)$, so we orient the arcs as in Figure 3.1. Let $\mathcal{G}$ denote their union. A *vertex* of $\mathcal{G}$ is thus an intersection point $\gamma \cap \gamma'$. Let $\alpha \in SCC(\Sigma)$ be isotoped so as not to intersect any vertex of $\mathcal{G}$. A *bigon* in this setup is

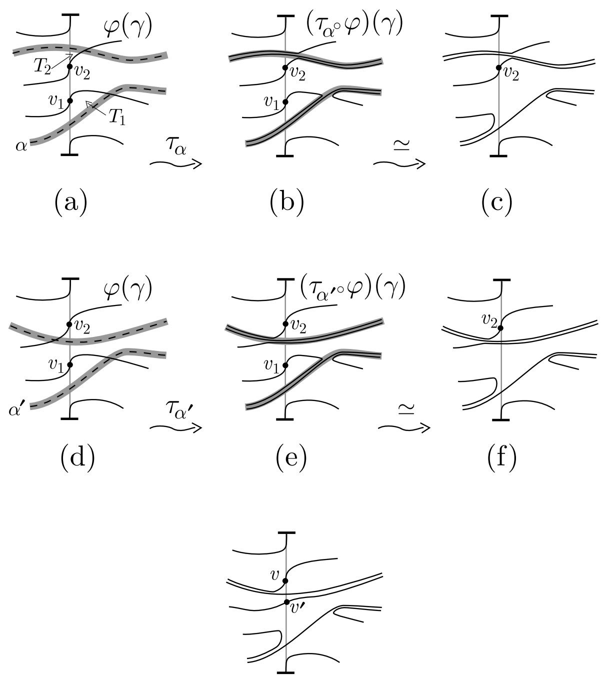


Figure 3.10: Top row: (a) The setup for the above discussion. Triangular region T_1 is upward, while T_2 is downward. The support of the twist τ_α is shaded. (b) The result of τ_α. There is a single bigon, in which v_1 is a vertex. (c) The result of isotoping $\tau_\alpha(\varphi(\gamma))$ over this bigon so as to minimize intersection with γ. Bottom row: Same as the top, but with α'. Note that while the images $\tau_\alpha(\varphi(\gamma))$, $\tau_{\alpha'}(\varphi(\gamma))$ are of course isotopic, the right positions of (c) and (f) differ. Finally, (g) indicates the isotopy-independent right position (i.e. the points v and v', along with $\partial\gamma$) which our algorithm is meant to pick out.

an immersed disc B bounded by the pair $(\alpha,\ \gamma)$ or by $(\alpha,\ \gamma')$. Similarly, a *bigon chain* is a set $\{B_i\}_{i=1}^n$ of bigons bounded exclusively by one of the pairs $(\alpha, \gamma), (\alpha, \gamma')$, for which the union $\bigcup\{\alpha \cap \partial(B_i)\}_{i=1}^n$ is a connected segment of α (Figure 3.11(a)). Finally, we say points $p, p' \in \alpha \cap \mathcal{G}$ are *bigon-related* if they are vertices in a common bigon chain.

Suppose then that T is a triangular region in this setup, with vertex $v \in \gamma \cap \gamma'$. We define the *bigon collection associated to* T, denoted $\mathcal{B}_T$, as the set of points $\{p \in \alpha \cap \mathcal{G} \mid p$ is bigon-related to a vertex of $T\}$ (Figure 3.11(b)). Note that, if a vertex of T is not a vertex in any bigon, the vertex is still included in $\mathcal{B}_T$, so that, for example, the bigon collection associated to an essential triangle T is just the vertices of T.

Now, the point v divides each of γ, γ' into two segments, which we label $\gamma_+, \gamma_-, \gamma'_+, \gamma'_-$ in accordance with the orientation. For each bigon collection $\mathcal{B}$ associated to a triangle with vertex v, we define $\sigma_v(\mathcal{B}) := (|\mathcal{B} \cap \gamma_+|, |\mathcal{B} \cap \gamma_-|, |\mathcal{B} \cap \gamma'_+|, |\mathcal{B} \cap \gamma'_-|) \in (\mathbb{Z}_2)^4$, where intersections numbers are taken mod 2 (Figure 3.11(b)).

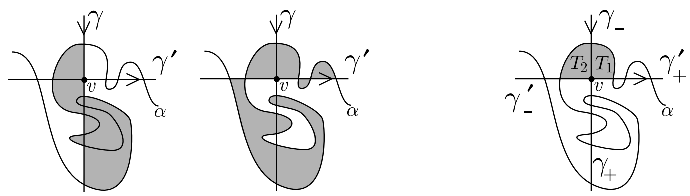


Figure 3.11: (a) The shaded regions are distinct maximal bigon chains. (b) As each chain from (a) has a vertex in each of the triangular regions T_1, T_2, the bigon collection $\mathcal{B}_{T_1} = \mathcal{B}_{T_2}$ includes each intersection point of each chain. In this example, $\sigma_v(\mathcal{B}_{T_1}) = (0,1,1,0)$, so the collection is downward. Note that T_1 is an essential downward triangular region (Definition 3.1.3), while T_2 is non-essential and upward.

Finally, we label a bigon collection $\mathcal{B}$ as *upward* (with respect to v) if $\sigma_v(\mathcal{B}) \in \{(1,0,1,0), (0,1,0,1)\}$, *downward* if $\sigma_v(\mathcal{B}) \in \{(1,0,0,1), (0,1,1,0)\}$, *non-essential* otherwise (Figure 3.11(b)). Compare with Definition 3.1.3,

Lemma 3.1.4. *Let $\alpha \in [\alpha]$ be such that $\alpha \cap \gamma$ and $\alpha \cap \varphi(\gamma)$ are minimal, and $v \in \varphi(\gamma) \cap \gamma$. Then the number of essential triangles with vertex v in the triple $(\gamma, \varphi(\gamma), \alpha)$ depends only on the isotopy class $[\alpha]$.*

Proof. Let α be an arbitrary representative of $[\alpha]$ (in particular $\alpha \cap \gamma$ and $\alpha \cap \varphi(\gamma)$ are not necessarily minimal), and v a vertex. We define an equivalence relation on the set of triangular regions with vertex v by $T_1 \sim T_2 \Leftrightarrow \mathcal{B}_{T_1} = \mathcal{B}_{T_2}$. In particular, there is a 1-1 correspondence between $\{$triangular regions with vertex $v\}/\sim$ and the set of bigon collections associated to triangular regions with vertex v.

Now, if α' is another representative of the isotopy class $[\alpha]$, we may break the isotopy into a sequence $\alpha = \alpha_1 \simeq \cdots \simeq \alpha_n = \alpha'$, where each isotopy $\alpha_i \simeq \alpha_{i+1}$ is either a bigon birth, a bigon death, or does not affect either of $|\alpha \cap \gamma|$ and $|\alpha \cup \varphi(\gamma)|$. Note that, if $\mathcal{B}$ is upward or downward, an isotopy $\alpha_i \simeq \alpha_{i+1}$ which does not cross v does not affect $\sigma_v(\mathcal{B})$, while an isotopy $\alpha_i \simeq \alpha_{i+1}$ which crosses v changes $\sigma_v(\mathcal{B})$ by addition with $(1,1,1,1)$. In either case, the classification is preserved; i.e. we can keep track of an essential upward (downward) bigon collection through each isotopy. Furthermore, any two distinct bigon collections will have distinct images under each such isotopy (a bigon birth/death cannot cause two maximal bigon chains to merge). We therefore have an integer $a(v)$, defined as the number of essential bigon collections $\mathcal{B}$ associated to triangular regions with vertex v, which depends only on $[\alpha]$

Finally, if we take α' to be a representative of $[\alpha]$ which intersects γ and $\varphi(\gamma)$ minimally, then $\{$triangular regions with vertex $v\}/\sim$ is by definition just the set of essential triangular regions with vertex v, and has size $a(v)$.

□

Lemma 3.1.5. *Let α be a fixed representative of the isotopy class $[\alpha]$ which has minimal intersection with $\gamma \cap \varphi(\gamma)$, and $v \in \gamma \cap \varphi(\gamma)$ a vertex of an upward triangle T. Then v is not a vertex of any downward triangular region T' for α.*

Proof. Consider the neighborhood of v as labeled in Figure 3.12. As T is upward, it must be T_1 or T_3, while T' must be T_2 or T_4. Without loss of generality then suppose $T = T_1$ is a triangular region. But then neither of T_2, T_4 can be triangular regions without creating a bigon, violating minimality. □

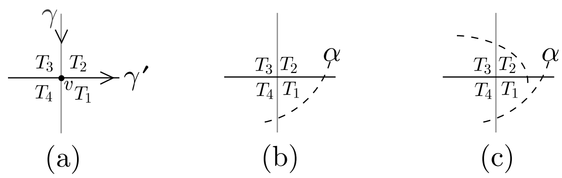


Figure 3.12: (a) A neighborhood of v. (b) The upward triangle T_1. (c) As α has no self-intersection, T_2 cannot be a triangular region without creating a bigon.

It follows from Lemmas 3.1.4 and 3.1.5 that we may unambiguously refer to a vertex of an essential triangular region as downward, upward, or neither, in accordance with any essential triangular region of which it is a vertex. In particular, if α has minimal intersection with $\gamma, \varphi(\gamma)$, then all triangular regions are essential. Henceforth we will drop the adjective 'essential'. Figure 3.13(b) and (c) give examples of the various types of vertices.

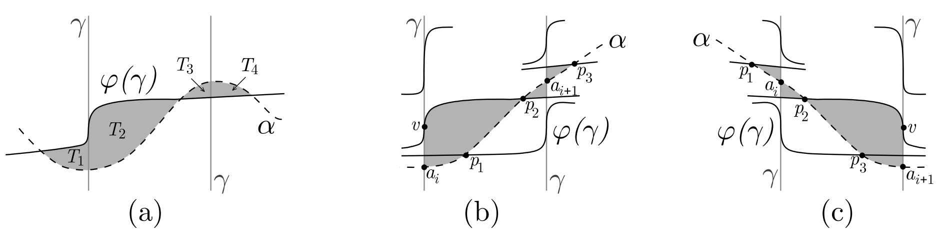


Figure 3.13: (a) If the shaded region is a maximal bigon chain, then T_1 and T_4 are essential triangular regions for $(\alpha, \gamma, \gamma')$, T_2 and T_3 are not. (b) Upward triangular regions and vertices. (c) Downward triangular regions and vertices.

Definition 3.1.6. Suppose T is a triangular region with vertex $v \in \gamma \cap \varphi(\gamma)$. Isotope α over T to obtain another triangular region T' of the same type (upward/downward) as T with vertex v, as in Figure 3.14. Note that if T contains sub-regions with vertex v then the process involves isotoping the innermost region first, and proceeding to T itself. We call such an isotopy a *shift over v*.

We require a final pair of definitions before we bring the pieces together:

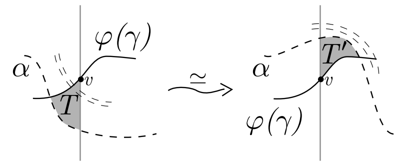


Figure 3.14: A shift over a vertex v.

Definition 3.1.7. Let γ be a properly embedded arc in $\Sigma, \varphi \in \Gamma_\Sigma$. Suppose $\gamma, \varphi(\gamma)$, and α are isotoped to minimize intersection. Let $\alpha \in SCC(\Sigma)$ be a fixed representative of its isotopy class which has minimal intersection with γ and $\varphi(\gamma)$. Choose $support(D_\alpha)$ to be disjoint from $\gamma \cap \varphi(\gamma)$. Then there is an obvious inclusion $i_\alpha : (\gamma \cap \varphi(\gamma)) \hookrightarrow (\gamma \cap D_\alpha(\varphi(\gamma)))$ (recall D_α refers to a specific Dehn twist, while τ_α is its mapping class). Note that $D_\alpha(\varphi(\gamma))$ will not in general have minimal intersection with γ.

Definition 3.1.8. Let γ and φ be as in the previous definition, and $v \in \gamma \cap \varphi(\gamma)$ a positively oriented intersection point. If there is a representative $\alpha \in SCC(\Sigma)$ of the isotopy class $[\alpha]$ such that the image $D_\alpha(\varphi(\gamma))$ which differs from $\varphi(\gamma)$ only in a support neighborhood of α can be isotoped to minimally intersect γ while fixing a neighborhood of $i_\alpha(v)$, we say v and $i_\alpha(v)$ are *fixable under* the mapping class τ_α. Similarly, for a factorization $\omega = \tau_{\alpha_n} \cdots \tau_{\alpha_1}$ of φ, we say $v \in \varphi(\gamma) \cap \gamma$ is *fixable under* ω if v is fixable under each successive τ_{α_i}.

Observation 3.1.9. Using this terminology, we have the following characterization of the right position $\mathcal{P}_\omega(\gamma)$ which the reader may or may not find helpful: The right position $\mathcal{P}_\omega(\gamma)$ is a maximal subset of the positive intersections in $\gamma \cap \varphi(\gamma)$ such that each point of $\mathcal{P}_\omega(\gamma)$ is fixable under ω, and is in fact the unique such position up to a choice involved in the coarsening process (described in detail in the proof of Lemma 3.1.12).

As a simple example, Figure 3.16 illustrates distinct right positions of a pair (φ, γ) associated to distinct factorizations of a mapping class φ.

We are now ready to precisely describe the inductive step. Let $\mathcal{P} = \mathcal{P}(\varphi, \gamma)$ be a right position, and let $\alpha \in SCC(\Sigma)$. We construct the right position $\mathcal{P}^\alpha = \mathcal{P}^\alpha(\tau_\alpha \circ \varphi, \gamma)$ in two steps. Firstly, we coarsen $\mathcal{P}$ by removing all points which are not fixable by τ_α. This new right position, being fixable under τ_α, defines a right position for $\gamma, \tau_\alpha(\varphi(\gamma))$. Secondly we refine this new position by adding points so as to have each new fixable point of $\gamma \cap \tau_\alpha(\varphi(\gamma))$ accounted for.

3.1.3 The coarsening $\mathcal{P} \rightsquigarrow \mathcal{P}'$

The coarsening process takes the given right position $\mathcal{P}$ and a curve $\alpha \in SCC(\Sigma)$, and returns a right position $\mathcal{P}' = \mathcal{P}'(\varphi, \gamma) \subset \mathcal{P}$. The intuitive idea is that we will remove a minimal subset of $\mathcal{P}$ such that the remaining points are fixable by τ_α, thus allowing an

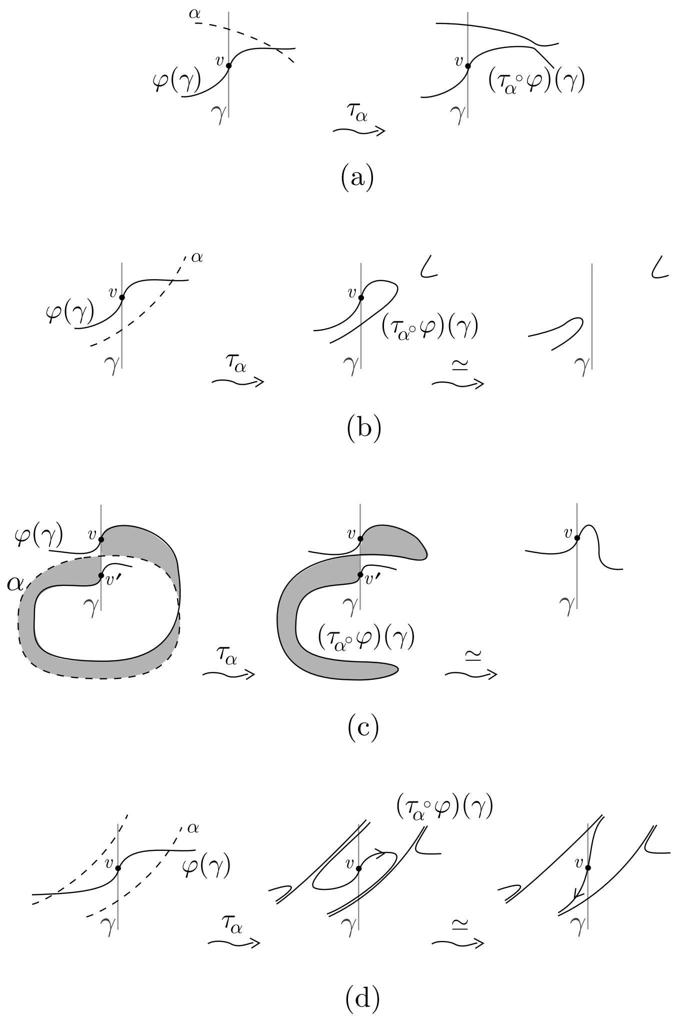


Figure 3.15: (a) A downward point is fixable. (b) An upward point is *not* fixable, unless (c) there is another upward triangle sharing the other two vertices. Note that, in this case, while either of the points v, v' is individually fixable, the pair is not simultaneously fixable. Finally, (d) illustrates the reason for demanding that a *neighborhood* of the point be fixed - though the intersection-minimizing isotopy may be done without removing the intersection point v, there is no fixable neighborhood, and so v is not fixable.

identification with points in $\gamma \cap \tau_\alpha(\varphi(\gamma))$. Of course we must do so in a way which depends only on the isotopy class of α.

We begin by choosing a representative α of the isotopy class $[\alpha]$ which has minimal intersection with γ and $\varphi(\gamma)$. Choose *support*(D_α) to be disjoint from $\mathcal{P}$. Now, the only bigons bounded by the arcs $D_\alpha(\varphi(\gamma)), \gamma$ correspond to upward triangular regions of the original triple $(\gamma, \varphi(\gamma), \alpha)$. Thus, any isotopy which minimizes $\tau_\alpha(\varphi(\gamma)) \cap \gamma$ must isotope

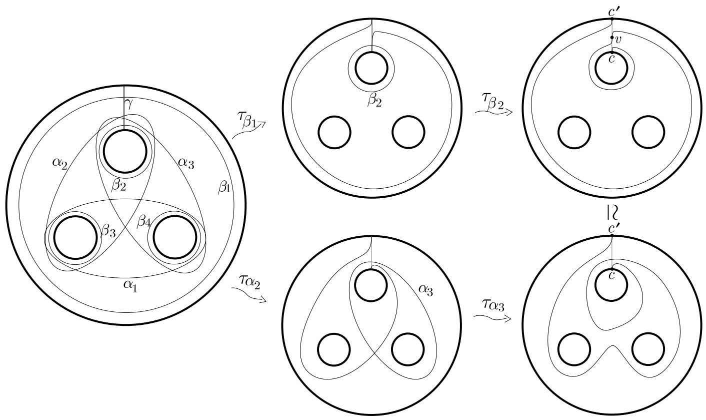


Figure 3.16: On the left are the curves of the well-known lantern relation on $\Sigma_{0,4}$ - setting $\omega_1 = \tau_{\beta_4}\tau_{\beta_3}\tau_{\beta_2}\tau_{\beta_1}, \omega_2 = \tau_{\alpha_3}\tau_{\alpha_2}\tau_{\alpha_1}$, the lantern relation tells us that ω_1, ω_2 are factorizations of a common $\varphi \in \Gamma_\Sigma$. Clearly, for the arc γ indicated, the intersection point $v \in \varphi(\gamma) \cap \gamma$ in the upper right figure is fixable under ω_1. However, as is clear from the lower sequence of figures, v is *not* fixable under the factorization ω_2. Thus the right position $\mathcal{P}(\omega_1, \gamma)$ contains the point v as well as the endpoints of γ, while $\mathcal{P}(\omega_2, \gamma)$ contains only the endpoints of γ.

$D_\alpha(\varphi(\gamma))$ over some subset of the upward triangular regions. The coarsening must therefore consist of a removal of a subset of the upward points of $\mathcal{P}$. We want to determine which of these points are fixable under τ_α.

Lemma 3.1.10. *Suppose that $v \in \mathcal{P}$ is a vertex of an upward triangular region T. Then if v is fixable under τ_α, then there is another upward triangular region T' such that T, T' share the vertices of T other than v (Figure 3.15(c)).*

Proof. Let a be the vertex of T at $\alpha \cap \gamma$ and p the remaining vertex (Figure 3.17(a)). As v is upward, there is a bigon in the image, corresponding to T, bounded by $(\tau_\alpha(\varphi(\gamma))$ and γ, with vertices v and a (Figure 3.17(b)). Therefore, if v is to be fixed through an isotopy of $\tau_\alpha(\varphi(\gamma))$ which minimizes $\tau_\alpha(\varphi(\gamma)) \cap \gamma$, then a must be a vertex of a second bigon also bounded by $\tau_\alpha(\varphi(\gamma))$ and γ, with vertices a and v' for some $v' \in \tau_\alpha(\varphi(\gamma)) \cap \gamma$ (Figure 3.17(c)). By minimality of intersections in the original configuration, this new bigon then corresponds to a second upward triangle T' with vertices a, v', p' in the original configuration (Figure 3.17(d)). As v is fixable, the bigons must cancel, and we have $p = p'$, as desired (Figure 3.15(c) indicates the configuration). Note that, while each of v, v' is fixable, the pair is not *simultaneously* fixable. □

Also,

Lemma 3.1.11. *If triangular regions T, T' share exactly two vertices, then these vertices are the pair along α.*

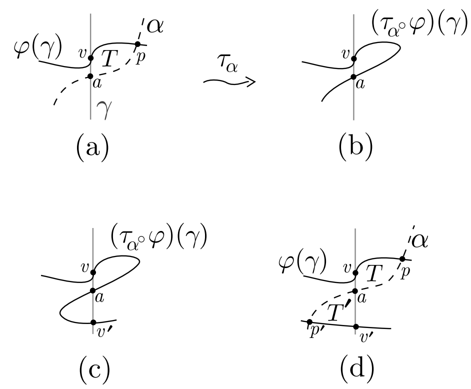


Figure 3.17: Figures for Lemma 3.1.10.

Proof. Let the triples (v_1, p_1, a_1), (v_2, p_2, a_2) be the vertices of upward triangles T_1, T_2, where $v_i \in \gamma \cap \varphi(\gamma)$, $p_i \in \alpha \cap \varphi(\gamma)$, and $a_i \in \alpha \cap \gamma$, and suppose T_1, T_2 have exactly two vertices in common. Essential triangular regions can have no edges in common, so they can have two vertices in common only if the vertices lie along a closed curve given by the union of the two edges defined by the pair of vertices. In particular, as α is the only closed curve in the construction, the points in common must be $p_1 = p_2$ and $a_1 = a_2$. □

Motivated by Lemmas 3.1.10 and 3.1.11, we introduce an equivalence relation $\sim$ on the set of upward triangular regions, such that $T \sim T'$ if and only if they share exactly two vertices. Now, any such equivalence class Ω comprises of m triangular regions, each of which has a unique vertex in $\gamma \cap \varphi(\gamma)$. These vertices (and thus the elements of Ω) may be ordered along $\varphi(\gamma)$ from the distinguished endpoint c. We then define a new right position $\mathcal{P}'$ for the pair (γ, φ) by removing the last (in the sense of the previous sentence) point of $\mathcal{P}$ from each equivalence class in which each triangular region has a vertex in $\mathcal{P}$.

Lemma 3.1.12. *Let $\mathcal{P}' \subset \mathcal{P}$ be as defined in the previous paragraph. Then $\mathcal{P}'$ satisfies the following properties:*

1. *Each point of $\mathcal{P}'$ is fixable under τ_α*
2. *$\mathcal{P}'$ is a maximal subset of $\mathcal{P}$ for which property (1) holds.*
3. *$\mathcal{P}'$ depends only on the isotopy class of α.*

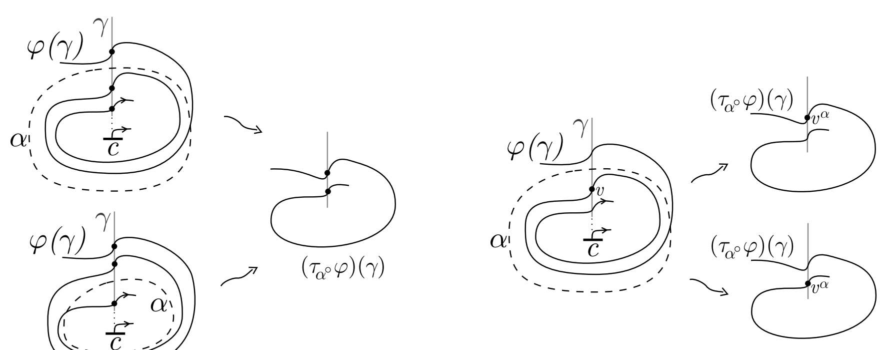
Proof. Let Ω be an equivalence class of upward triangles. We distinguish two subcases, depending on whether each element of Ω has a vertex in $\mathcal{P}$ or not. To emphasize the distinction, we call a point $p \in \gamma \cap \varphi(\gamma)$ a *vertical point* if p is in $\mathcal{P}$, non-vertical otherwise.

1. (Each element of Ω has a vertex in $\mathcal{P}$) Let $T \in \Omega$. Using Lemma 3.1.11, the equivalence class of T is formed of triangles sharing the points along α (a typical equivalence class

of three triangles is shown in the left side of Figure 3.18(a) - note that for either choice of α, exactly two of the three regions are embedded). Suppose then that Ω contains m triangular regions. As noted in the proof of Lemma 3.1.10, at most $m-1$ vertical points from the vertices of elements of Ω are simultaneously fixable under τ_α, and conversely, *any* set of $m-1$ vertical points from these is fixable. In particular, if each triangle in the class has a vertex in $\mathcal{P}$, the removal of a single vertical point is a necessary and sufficient refinement of the vertical points involved in the equivalence class so as to satisfy properties (1) and (2). Clearly the resulting set $\mathcal{P}'$ obtained by repeating this refinement on each equivalence class is independent (as an unordered set) of the actual choice of which vertical point to remove, and so property (3) is satisfied.

2. (Ω contains some element which does not have a vertex in $\mathcal{P}$) If there is a triangle in the class which does not have a vertex in $\mathcal{P}$, then all of the above goes through for the neighborhood of the non-vertical point p of $\varphi(\gamma) \cap \gamma$; i.e., the twist τ_α, along with any isotopy of the image $\tau_\alpha(\gamma)$ necessary to minimize intersections can be done relative to a neighborhood of each vertical point, so that there is no coarsening necessary. Again, property (3) follows immediately.

□



(a) (b)

Figure 3.18: (a) For the case that each element of an equivalence class has a vertical vertex, α can be isotoped so as to fix all but any single vertical point - the result is independent of the choice of which subset to fix. (b) An upward equivalence class of triangles whose set of vertices contain non-consecutive (along γ), non-vertical points of $\gamma \cap \varphi(\gamma)$ (here separated by the vertical point v). There is therefore a choice of vertical point in the image. Our convention will be to choose the lower figure.

Now, let α be a specific representative of the isotopy class $[\alpha]$ which has minimal intersection with γ and $\varphi(\gamma)$. We again consider the effect of the Dehn twist D_α restricted to a supporting neighborhood of the twist chosen so as not to intersect any point of $\gamma \cap \varphi(\gamma)$. As $D_\alpha(\varphi(\gamma))$ will not (in the presence of upward triangular regions) have minimal intersection with γ, the image of the inclusion $i_\alpha(\mathcal{P}) \subset (\gamma \cap D_\alpha(\varphi(\gamma)))$ (Definition 3.1.7) will not in general be a right position. Indeed, i_α is no longer well-defined as a map into $\tau_\alpha(\varphi(\gamma))$ once this has been isotoped to minimize intersections with γ.

One may get around this by fixing conventions: if Ω is an equivalence class of m upward triangles, let $\{x_1, \ldots, x_m\}$ be the collection of vertices in $\gamma \cap \varphi(\gamma)$, indexed in order along $\varphi(\gamma)$ from c. By Lemma 3.1.12, the image of the $\{x_1\}$ under i_α will be a collection of m points in $\gamma \cap D_\alpha(\varphi(\gamma))$. Exactly one of these intersections must be removed by the intersection-minimizing isotopy. The resulting set of $m-1$ simultaneously fixable points in $\gamma \cap \tau_\alpha(\varphi(\gamma))$ is independent of this choice; we label these points $\{y_1, \ldots, y_{m-1}\}$, ordered along $\tau_\alpha(\varphi(\gamma))$ from c. Then, if $x_i \in \mathcal{P}$ for each i, set $i_\alpha(x_i) = y_i$ for $i < m$. Otherwise, let $k := max\{i \mid x_i \notin \mathcal{P}\}$, and set $i_\alpha(x_i) = y_i$ for $i < k$, and $i_\alpha(x_i) = y_{i-1}$ for $i > k$ (Figure 3.18).

With these conventions:

Lemma 3.1.13. *For any $\alpha \in [\alpha]$, the image $i_\alpha(\mathcal{P}')$ is a right position for $(\tau_\alpha \circ \varphi, \gamma)$. Moreover, $i_\alpha(\mathcal{P}')$ is* simultaneously *fixable under τ_α.*

Proof. We again consider the effect of the Dehn twist τ_α restricted to a supporting neighborhood of the twist chosen so as not to intersect any point of $\gamma \cap \varphi(\gamma)$. Observe then that in the image, the only bigons bounded by the curves γ and $D_\alpha(\varphi(\gamma))$ correspond to essential upward triangles in the original configuration. Each singleton equivalence class contributes a single, isolated bigon, while classes with multiple elements contribute canceling pairs of bigons. In particular, $\gamma \cap \tau_\alpha(\varphi(\gamma))$ may be minimized by an isotopy of $\tau_\alpha(\varphi(\gamma))$ over one bigon from each class. This isotopy fixes a neighborhood of each of the points $\mathcal{P}'$.

□

3.1.4 The refinement $i_\alpha(\mathcal{P}') \rightsquigarrow \mathcal{P}^\alpha$

For the final step, we wish to refine a given $i_\alpha(\mathcal{P}')$ by adding points to obtain the desired right position $\mathcal{P}^\alpha$ for $(\tau_\alpha \circ \varphi, \gamma)$ so that $\mathcal{P}^\alpha$ depends only on $[\alpha]$.

So, let α be a given representative of the isotopy class $[\alpha]$ such that the sets $\alpha \cap \gamma, \alpha \cap \varphi(\gamma)$ are minimal. By Lemma 3.1.13 we have a canonical identification of $\mathcal{P}'$ as a subset $i_\alpha(\mathcal{P}')$ of the positive intersections of $\tau_\alpha(\varphi(\gamma)) \cap \gamma$, and thus as a right position for $(\tau_\alpha \circ \varphi, \gamma)$.

As we are after an isotopy-independent result, we shall require that $\mathcal{P}^\alpha$ include each point which is fixed by *any* representative of $[\alpha]$ which satisfies the intersection-minimality conditions. So, let $v \in \mathcal{P}'$ be a downward point. Then for each vertical downward triangular region T with vertex v, let a be the vertex $\alpha \cap \gamma$. We label the positive intersection of $\gamma, \tau_\alpha(\varphi(\gamma))$ corresponding to a by $d_{a,v} \in (\tau_\alpha \circ \varphi)(\gamma) \cap \gamma$ (Figure 3.19), and call the set of all such points V_v. We refer to $i_\alpha(v) \cup V_v \subset \mathcal{P}^\alpha$ as *the refinement set corresponding to v.* Note that, for distinct v, v', the sets $V_v, V_{v'}$ are not necessarily disjoint, but as we only require that each such point be accounted for, this does not affect the construction. We now define the right position $\mathcal{P}^\alpha$ of $(\tau_\alpha \circ \varphi, \gamma)$ as the union of all the refinement sets of each point, $\mathcal{P}^\alpha := i_\alpha(\mathcal{P}') \cup \{V_v | v \in \mathcal{P}'\}$.

Finally,

Lemma 3.1.14. *The right position $\mathcal{P}^\alpha$ of $(\tau_\alpha \circ \varphi, \gamma)$ given by the refinement process is exactly the set $\bigcup_{\alpha\in[\alpha]} i_\alpha(\mathcal{P}')$. In particular, $\mathcal{P}^\alpha$ depends only on the isotopy class of α.*

Proof. We will show that $\mathcal{P}^\alpha = \bigcup_{\alpha\in[\alpha]} i_\alpha(\mathcal{P}')$ by showing that each of these sets are equivalent to the set $W := \{v \in \gamma \cap (\tau_\alpha \circ \varphi)(\gamma) \mid v \text{ is fixable under } \tau_\alpha \text{ with preimage } v' \in \mathcal{P}\}$. Now, by construction, for any $\alpha \in [\alpha]$ or $v' \in \mathcal{P}$, each element of $i_\alpha(\mathcal{P}')$ or $V_{v'}$ is in W, so we only need to show that, for arbitrary α, each $v \in W$ is in both $\mathcal{P}^\alpha$ and $\bigcup_{\alpha\in[\alpha]} i_\alpha(\mathcal{P}')$.

So, let $v \in W$, with preimage v', and α a fixed representative of $[\alpha]$. There are two cases:

1. (v' is downward) Suppose $v \notin i_\alpha(\mathcal{P}')$. Then there is $\alpha' \in [\alpha]$, given by a shift of α over v, such that $v \in i_{\alpha'}(\mathcal{P}')$. Then v is in the refinement set $V_{v'}$ in the right position $\mathcal{P}^\alpha$, and also in $i_{\alpha'}(\mathcal{P}') \subset \bigcup_{\alpha\in[\alpha]} i_\alpha(\mathcal{P}')$.

2. (v' is not downward) Then $v' \in \mathcal{P}'$, and so $v \in i_\alpha(\mathcal{P}')$ for any α.

□

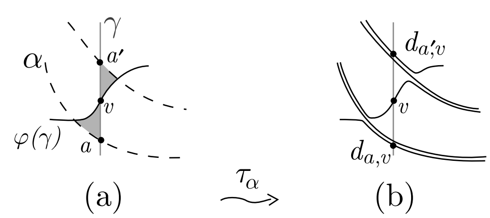


Figure 3.19: Construction of the set V_v for downward $v \in \mathcal{P}'$. Figure (a) is the setup in $\mathcal{P}'$, (b) shows the result of the refinement for $\mathcal{P}^\alpha$ by vertical points $V_v = \{d_{a,v}, d_{a',v}\}$ on $\tau_\alpha(\varphi(\gamma))$.

3.1.5 The associated right position $\mathcal{P}_\omega$

Definition 3.1.15. Let $\varphi \in \Gamma_\Sigma$ be given as a positive factorization ω. The unique right position associated to the pair ω, γ by the algorithm detailed in this section is the *right position associated to* ω, denoted $\mathcal{P}_\omega(\gamma)$.

Finally, we need to extend the associated right position to each pair $\gamma_i, \varphi(\gamma_i)$ for an arbitrary set $\{\gamma_i\}$ of nonintersecting properly embedded arcs. We have:

Lemma 3.1.16. *Let $\{\gamma_i\}$ be a set of pairwise non-intersecting properly embedded arcs in Σ. The algorithm as given above extends to an algorithm which assigns a unique right position $\mathcal{P}_\omega(\gamma_i)$ to each pair $\gamma_i, \varphi(\gamma_i)$.*

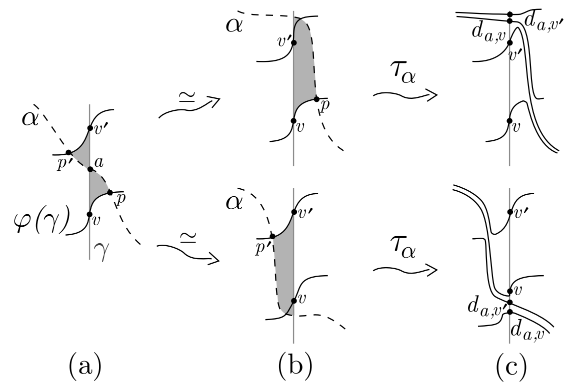


Figure 3.20: $\mathcal{P}^\alpha$ is independent of isotopy of α in the refinement process. (a) is the initial setup, with $p, p' \in \alpha \cap \varphi(\gamma)$, (b) indicates distinct isotopies, and (c) is the resulting $\mathcal{P}^\alpha$ for each choice. While the labeling is different, the right positions are equivalent.

Proof. This follows directly from independence of the algorithm from the choice of representative of the isotopy class of α, and the fact that we may always find some representative of α which intersects each of $\{\gamma_i\}, \{\varphi(\gamma_i)\}$ minimally. □

3.2 Consistency of the associated right positions

The goal of this subsection is to show that any two right positions $\mathcal{P}_\omega(\gamma_1), \mathcal{P}_\omega(\gamma_2)$ associated to a positive factorization ω are consistent (Definition 3.0.12). Theorem 3.0.6 will follow from this result coupled with Lemma 3.1.16.

Our method to show consistency will be to use the right positions to 'localize' the situation. Recall from the previous section (in particular Lemma 3.1.14) that, if $\mathcal{P} \rightsquigarrow \mathcal{P}^\alpha$ denotes the inductive step of the algorithm, then, using the notation of that section, given a point $z \in \mathcal{P}^\alpha$, there is a representative α of $[\alpha]$ such that z is in the image of the inclusion map i_α. In particular, this identification of z as $i_\alpha(v)$, for some $v \in \mathcal{P}$, will hold for any isotopy of α which does not involve a shift over v. Thus, having fixed α up to this constraint, we may consider v as having properties analogous to the endpoints of γ; i.e. it is fixed by τ_α, and α cannot be isotoped so as to cross it. We refer to v as the *preimage* of z.

Suppose then that we have a *pair* of nonintersecting properly embedded arcs on a surface with a given monodromy, with right positions $\mathcal{P}_1$ and $\mathcal{P}_2$. Then, for a Dehn twist τ_α, the inductive step of the algorithm gives a pair $\mathcal{P}_1^\alpha, \mathcal{P}_2^\alpha$. Consider a *pair* of points, one from each of $\mathcal{P}_i^\alpha$. We need to show that for each such pair, the horizontal segments originating at the pair of points are completed if initially parallel. We call a pair satisfying this condition *consistent*.

Now, if there is an isotopy of α such that each point in a given pair of points is in the image of the associated inclusion map i_α, we call the pair a *simultaneous image*. The idea is that the horizontal segments originating at such a pair have well-defined preimages in $\mathcal{P}_1$ and $\mathcal{P}_2$, so we only have to understand the 'local picture' consisting of the images under τ_α of each such pair.

In keeping with analogy of fixable points as boundary points, we generalize the notion of 'to the right' (as presented in [HKM]) to arcs with common endpoint on the interior of a properly embedded arc γ:

Definition 3.2.1. Let γ be a properly embedded arc in Σ, $\eta, \eta' : [0,1] \hookrightarrow \Sigma$ embedded arcs in Σ with common endpoint $\eta(0) = \eta'(0) = x \in \gamma$, and $\eta(0), \eta'(0) \in \partial\Sigma$, isotoped to minimize intersection, each to the same side of γ (i.e. if we fix a lift $\tilde{\gamma}$ in the universal cover $\tilde{\Sigma}$ of Σ, then $\tilde{\eta}, \tilde{\eta'}$ lie in the same connected component of $\tilde{\Sigma} \setminus \tilde{\gamma}$). We say η' is *to the right* of η (at x), denoted $\eta' \geq \eta$, if either the pair is isotopic, or if the tangent vectors $(\dot{\eta'}(0), \dot{\eta}(0))$ define the orientation of Σ at x (Figure 3.21(a)). Similarly we say η is *to the left* of η', denoted $\eta' \leq \eta$. Note that $\eta \geq \eta'$ and $\eta' \geq \eta$ are mutually exclusive situations.

We also need a slightly stronger notion of comparative rightness:

Definition 3.2.2. Let γ, η, x be as in the previous definition, and suppose $\Delta : [0,1] \hookrightarrow \Sigma$ is an embedded arc in Σ with endpoints $\Delta(0) = x$, and $\Delta(1) = y \in \gamma'$, for γ' another properly embedded arc in Σ. Then if there is a triangular region for the triple (η, Δ, γ') in which x and y are vertices, we say η and Δ are *parallel along* η'. If $\eta \geq \Delta$ ($\eta \leq \Delta$) and η and Δ are *not* parallel along η', then we say η is *strictly to the right (left)* of Δ (Figure 3.21(b)).

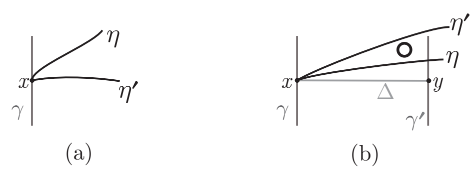


Figure 3.21: (a)Arc η' is to the right of η. (b) Arc η is parallel along Δ, while η' is strictly to the left of Δ.

Observation 3.2.3. Note that, for a pair of horizontal segments h_v and h_w, the property of being initially parallel along a rectangular region B implies that each is parallel along the 'diagonal' arc Δ with endpoints v and w. Conversely, if h_v and h_w are each parallel along an arc η with endpoints v and w, there is a rectangular region along which the segments are initially parallel (given by the union of the triangular regions of Definition 3.2.2).

As for the utility of these definitions, let $\mathcal{P}_i \rightsquigarrow \mathcal{P}_i^\alpha$ denote the inductive step of the algorithm, and consider a fixable pair $v \in \mathcal{P}_1, w \in \mathcal{P}_2$. If the images $\tau_\alpha(h_v)$ and $\tau_\alpha(h_w)$ are initially parallel along rectangular region B with diagonal Δ, then we may compare Δ with the preimages h_v and h_w. Clearly, for either of these horizontal segments, the property of being strictly to the right of Δ is preserved by τ_α, and so incompatible with the images being initially parallel along B. Moreover, if one of the pair, say h_v, is to the left of Δ, then if h_w is to the right of Δ, it must be *strictly* to the right of Δ, which as above gives a contradiction. We summarize:

Lemma 3.2.4. *Let $v \in \mathcal{P}_1(\varphi, \gamma_1)$ and $w \in \mathcal{P}_2(\varphi, \gamma_2)$ be fixable under τ_α. Fix $\alpha \in [\alpha]$, and let $u = i_\alpha(v) \in \mathcal{P}_1^\alpha$ and $z = i_\alpha(w) \in \mathcal{P}_2^\alpha$. Suppose that h_u and h_z are initially parallel along B, with diagonal Δ. Then:*

1. *If $h_v \geq \Delta$, then h_v is parallel along Δ.*
2. *$h_v \geq \Delta \Leftrightarrow h_w \geq \Delta$ (and so $h_v \leq \Delta \Leftrightarrow h_w \leq \Delta$)*

We will break the proof that the algorithm preserves consistency into three lemmas. To begin with, we have

Lemma 3.2.5. *Let $\mathcal{P}_i$, $i = 1, 2$ be consistent, and let $u \in \mathcal{P}_1^\alpha$ and $z \in \mathcal{P}_2^\alpha$ be a simultaneous image. Then if h_u and h_z are initially parallel, they are completed.*

Proof. Suppose h_u and h_z are initially parallel along rectangular region B with diagonal Δ. As usual, we label the remaining two corners of B by $y_i \in \gamma_i$. Let $\alpha \in [\alpha]$ be fixed such that u and z are in the image of i_α. Denote the preimages by $v \in \mathcal{P}_1$ and $w \in \mathcal{P}_2$, so $i_\alpha(v) = u$ and $i_\alpha(w) = z$. Using Lemma 3.2.4, there are two cases:

1. $h_v, h_w \leq \Delta$. We again fix a neighborhood $\nu(\alpha)$ of α, and consider the isotopy $D_\alpha(h_v) \rightsquigarrow h_{i_\alpha(v)}$ which minimizes intersections of $t_\alpha(h_v)$ with γ and Δ. We let p be the initial (along h_v from v) point of $\alpha \cap h_v$, and p' the first point of $\partial(\nu(\alpha)) \cap h_v$ along h_v from p (Figure 3.22). Now, the only bigons bounded by $D_\alpha(h_v)$ and γ or Δ correspond to triangular regions in the preimage. We distinguish two subcases:

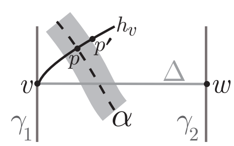


Figure 3.22: Notation for Lemma 3.2.5. The shaded region is a neighborhood of α.

   (a) The points v is the vertex of a downward triangular region T in the triple $(\alpha, \gamma_1, \Delta)$ (Figure 3.23(a)). Then p' (considered now as a point of $D_\alpha(h_v)$) is *not* in a bigon bounded by $D_\alpha(h_v)$ and γ_1 (else γ_1 and h_v bound a bigon), so our isotopy may be done so as to fix p'. Thus, for the image $\tau_\alpha(h_v)$ to lie along Δ, we find that there is a triangular region T' of $(\gamma_2, \Delta, \alpha)$ which intersects T in the point $\alpha \cap \Delta$. As $h_w \leq \Delta$, it follows that an initial segment of h_w is within T', and thus w is also downward. We may then assume that α runs from γ_1 to γ_2 within a neighborhood of Δ, intersecting Δ exactly once. There are then $u' \in \mathcal{P}_1^\alpha, z' \in \mathcal{P}_2^\alpha$ corresponding to the vertices $\alpha \cap \gamma_i$ of the associated triangles (Figure 3.24). The segments h_u and h_z are compatible, initially parallel along Δ with completing region along the remainder of α and endpoints u' and z'.

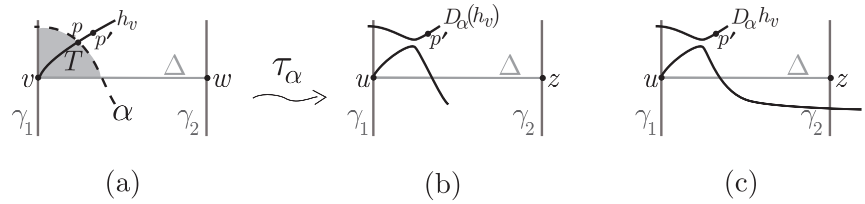


Figure 3.23: Figures for Lemma 3.2.5

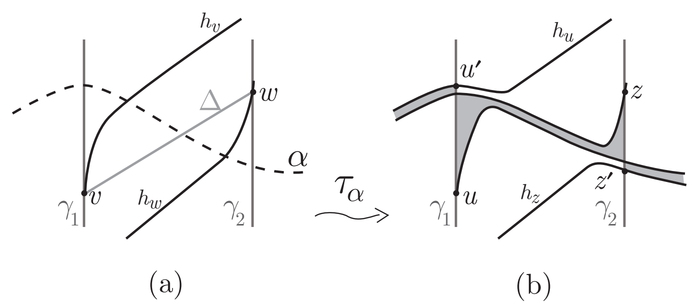


Figure 3.24: (a) If $h_v \geq \Delta$ (at v), then the same is true of h_w (at w), so the pair is initially parallel along Δ. (b) The segments $h_u = \tau_\alpha(h_v)$ and $h_z = \tau_\alpha(h_w)$ are compatible, with endpoints u', z', given by the refinement procedure, corresponding to the points $\alpha \cap \gamma_i$

(b) The points v and p are not vertices in a downward triangular region (Figure 3.25(a)). By part (a), w is also not in such a downward triangular region. Now, h_u is assumed parallel along Δ, so we consider the respective triangular region T in $((\tau_\alpha \circ \varphi)(\gamma_1), \Delta, \gamma_2)$ (Figure 3.25(c)). As the point w is not downward, all arcs of $\alpha \cap T$ connect h_u to Δ or γ_2. In particular, $h_v = \tau_\alpha^{-1}(h_u)$ is parallel along $\tau_\alpha^{-1}(\Delta)$. The same holds for the complementary triangular region T' in $((\tau_\alpha \circ \varphi)(\gamma_2), \Delta, \gamma_1)$. Now, $B = T \cup T'$, so we find the preimages h_v and h_w initially parallel along a region which we refer to as $\tau^{-1}(B)$ (Figure 3.26 gives a typical situation). The result then follows from Lemma 3.2.7.

2. $h_v, h_w \geq \Delta$ By Lemma 3.2.4, h_v and h_w are each parallel along Δ. The pair is thus initially parallel along the same region B, and so completed by points v' and w'. Again, the result follows from Lemma 3.2.7.

□

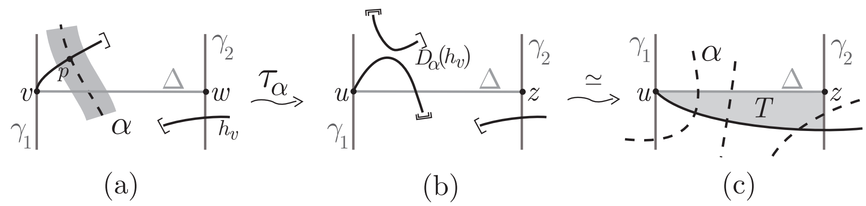


Figure 3.25: (a) The points v and p are not vertices in a downward triangular region. (b) The image under D_α. (c) The triangular region T in $((\tau_\alpha \circ \varphi)(\gamma_1), \Delta, \gamma_2)$, and some possible components of $\alpha \cap T$.

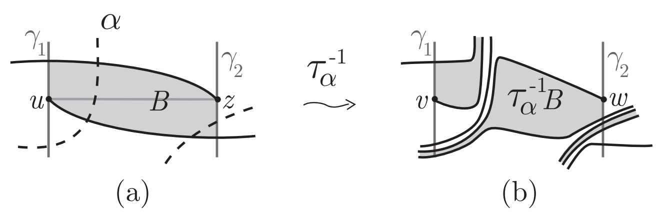


Figure 3.26: (a) Some possible components of $\alpha \cap B$. (b) Segments h_v and h_w are initially parallel along $\tau^{-1}(B)$

We must also consider the case of non-simultaneous image pairs:

Lemma 3.2.6. *Let $\mathcal{P}_i$, $i = 1, 2$ be consistent, and let $u \in \mathcal{P}_1^\alpha$ and $z \in \mathcal{P}_2^\alpha$ not be a simultaneous image. Then if h_u and h_z are initially parallel, they are completed.*

Proof. As discussed at the beginning of this subsection, we may assume, by shifting α, that any given point in $\mathcal{P}_2^\alpha$ is the image under i_α of some $w \in \mathcal{P}_2$. We therefore begin by fixing $\alpha \in [\alpha]$ up to this property (i.e. up to isotopies which do not involve a shift over w). By assumption, this α does *not* fix u, so there is a point $v \in \mathcal{P}_1$ such that u is in the refinement set V_v. In particular, v is downward, and w is in the interior of the associated downward triangular region T (else we could use T to isotope α so as to fix each of u and z simultaneously)(Figure 3.27(a)). The point w is thus also downward.

We wish to adapt the proof of Lemma 3.2.5 to this situation. We again suppose h_u and h_z are initially parallel along B with diagonal Δ, and label the remaining two corners of B by $y_i \in \gamma_i$. However, we no longer have a preimage of h_u. To get around this, we define a new arc σ which starts at the corner $x = \gamma_1 \cap \alpha$ of T, follows α along the length of the edge of T, then continues along h_v. Note that, while σ may have self intersections and/or intersections with h_w (along its coincidence with α), its image under τ_α (fixing the endpoints) coincides with $h_u \in H(\mathcal{P}_1^\alpha)$ (Figure 3.27(b)). We proceed to analyze cases as in Lemma 3.2.5:

1. $\sigma \leq \Delta$: Using Lemma 3.2.4, we observe that $h_w \leq \Delta$ (Figure 3.28(a)). The proof follows exactly as in Lemma 3.2.5, with v and h_v replaced by x and σ.

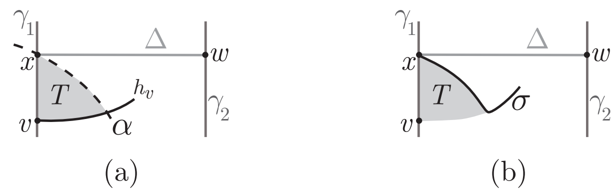


Figure 3.27: (a) The setup of Lemma 3.2.6. Note that in Σ, the point w actually lies in the interior of T. To simplify the figures, we keep to our convention of drawing the picture in the universal cover. (b) The construction of the arc σ with basepoint x.

2. $\sigma \geq \Delta$: By construction, we have $h_v \geq \Delta$, and thus, using Lemma 3.2.4, parallel along Δ. Though we may no longer conclude that $h_w \geq \Delta$, there must be an arc Δ', obtained from Δ by sliding the endpoint x along γ_1 to v, such that h_v and h_w are parallel along Δ (Figure 3.28(b)). Again, the result follows from Lemma 3.2.7.

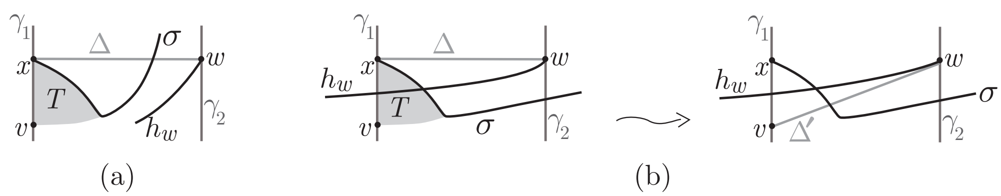


Figure 3.28: (a) Each of the segments σ and h_w is to the left of Δ. (b) Construction of the modified diagonal Δ' in the case that h_w is to the left, σ to the right of Δ.

□

The previous two lemmas thus reduce the problem to a check that, under certain conditions, initially parallel images of initially parallel and completed segments are themselves completed. To be precise:

Lemma 3.2.7. *Let $\mathcal{P}_i$ be consistent right positions for (φ, γ_i), $i = 1, 2$. Suppose $v \in \mathcal{P}_1$ and $w \in \mathcal{P}_2$ are fixable under τ_α, and horizontal segments h_v and h_w initially parallel along a rectangular region B with diagonal Δ, and completed along a region B' to points v' and w'. Then,*

1. *If neither of v and w are downward (Definition 3.1.3), and $h_{i_\alpha(v)}$ and $h_{i_\alpha(w)}$ are initially parallel along $\tau_\alpha(\Delta)$, then $h_{i_\alpha(v)}$ and $h_{i_\alpha(w)}$ are completed.*

2. *Suppose at least one of v and w is downward, and let V_v and V_w denote the refinement classes from the refinement step of the algorithm. Then for any $u \in V_v$ and $z \in V_w$, the horizontal segments h_u and h_w are completed if initially parallel along (a parallel translate of) Δ (here a 'parallel translate' of Δ just refers to an isotopy which allows the endpoints to move along the arcs γ_i).*

Proof. For claim (1), note that, if neither initial point is downward, then $\alpha \cap B$ consists of arcs which either connect h_v to h_w, or 'cut off' the corners y_1 and y_2. We may assume α is shifted over the corners to get rid of arcs which cut off corners, and so y_1 and y_2 are fixable. In analogy to Lemma 3.2.5 1(b), we denote the region along which the images h_u and h_z are initially parallel by $\tau_\alpha(B)$. We wish to understand the effect of τ_α on the completing region B'. Note that, for any allowable $\alpha \in [\alpha]$, α intersects B' in arcs as in Figure 3.29(a). In particular, if neither y_1 nor y_2 is a vertex of an upward triangular region of the triple $(\alpha, \gamma_1, \varphi(\gamma_2)$ (for y_1) or $(\alpha, \gamma_2, \varphi(\gamma_1))$ (for y_2), then $h_{i_\alpha(v)}$ and $h_{i_\alpha(w)}$ are completed by points $u' \in V_{v'}$ and $z' \in V_{w'}$ (Figure 3.29(b)).

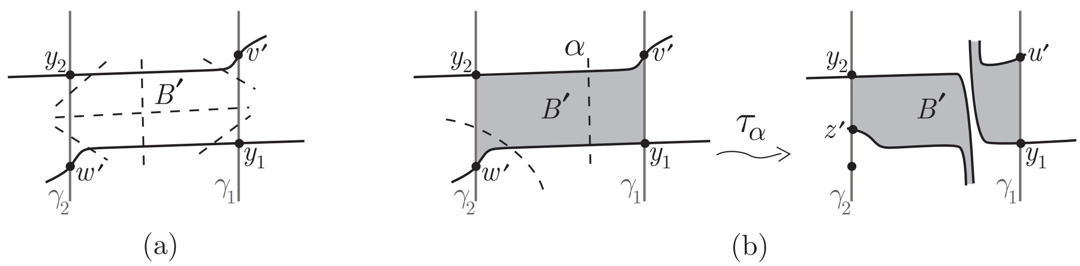


Figure 3.29: (a) Possible intersections $\alpha \cap B'$. (b) If the corners y_i are not cut off, the completing disc is preserved.

Suppose then that y_2 is a vertex of an upward triangular region of the triple $(\alpha, \gamma_1, \varphi(\gamma_2))$. Denote the vertex $\alpha \cap h_v$ by p. Following the argument of Lemma 3.1.12, the image $\tau_\alpha(h_v)$ can be initially parallel along $\tau_\alpha(\Delta)$ only if there are a pair of triangular regions T_1 and T_2 in the triple $(\gamma_2, \varphi(\gamma_1), \alpha)$ with two vertices in common (one of them p). By Lemma 3.1.11, this second of the points in common, which we denote p', is $\alpha \cap \gamma_2$. We then have the situation illustrated in Figure 3.30(a). The edges of T_1 and T_2 along γ_2 form a connected subarc η of γ_2; we distinguish cases depending on whether the endpoint w' is contained in this subarc:

1. $w' \notin \eta$ (Figure 3.30(b)). Then v' itself is a point in an upward equivalence class of triangles, as defined in Lemma 3.1.12. But then, as described in the coarsening algorithm, we may assume the isotopy $D_\alpha(h_v) \rightsquigarrow h_u$ fixes $h_{v,v'}$ (Figure 3.30(c-d)). Thus h_u and h_z are completed by $i_\alpha(v')$ and $i_\alpha(w')$.

2. $w' \in \eta$ (Figure 3.30(e)). We now have w' a point in an upward equivalence class of triangles. Again, we may assume α is shifted over w' so that a neighborhood of w' is fixed under τ_α (Figure 3.30(f-g)). The segments h_u and h_z are again completed by $i_\alpha(v')$ and $i_\alpha(w')$.

For claim (2), recall that, by shifts of α over points of $\mathcal{P}_2$, we may assume that any given point in $\mathcal{P}_2^\alpha$ is in the image of i_α. In the present case, we may assume $z = i_\alpha(w)$. Now, in addition to the intersections of α with B' as in the previous case, we must deal with the

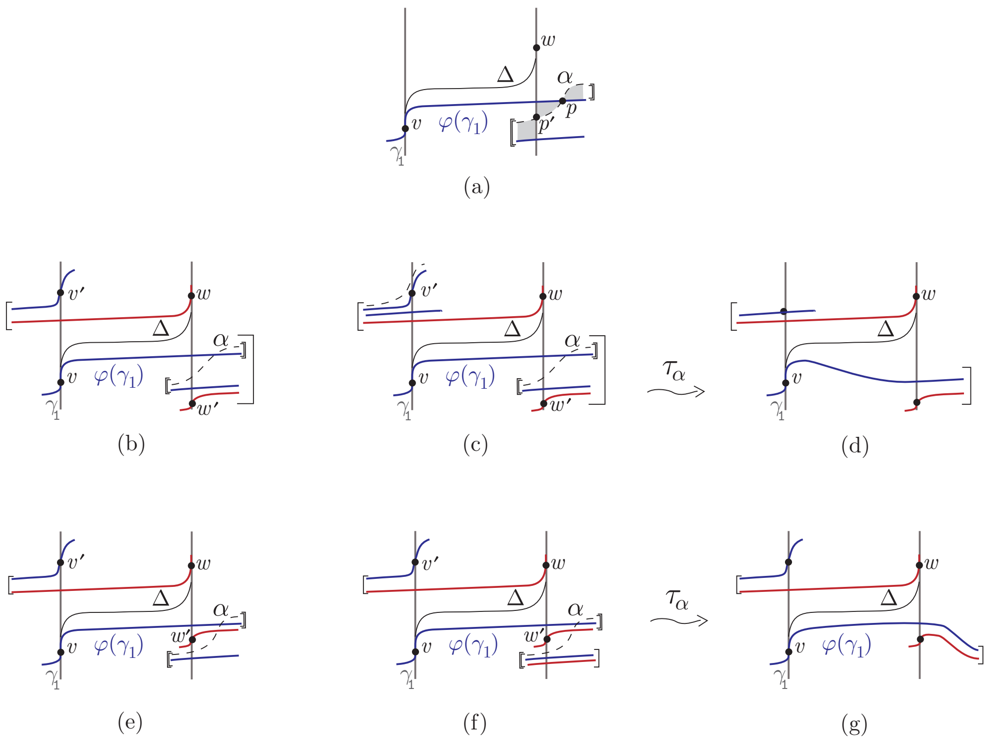


Figure 3.30: (a) If $\alpha \cap h_j$ is a diagonal intersection point p, then $\tau_\alpha(h_j)$ can be initially parallel along $\tau_\alpha(\Delta)$ only if the image $\tau_\alpha(h_j)$ and γ_2 bound canceling bigons, corresponding to the shaded triangular regions. (b) The case that w_m is 'below' x. (c) and (d) indicate the effect of the twist on the completing region B'. Figures (e),(f) and (g) do the same for the case w_m is 'above' x.

case that our initial points v and w are downward. So, let $p \in \alpha \cap \varphi(h_v)$ be a vertex in a downward triangle with vertex v. Referring to Figure 3.31, we call p *isolated* if p is not a vertex in a triangular region of the triple $(\alpha, \gamma_2, \varphi(\gamma_1))$ *non-isolated* otherwise.

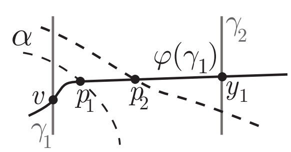


Figure 3.31: Point p_1 is isolated, p_2 is non-isolated.

Note firstly that if $u = i_\alpha(v)$, then each downward triangular region with vertex v is non-isolated (else h_u would be strictly to the right of Δ). We may then assume α is shifted over y_2 so that $h_v \cap \partial B$ is fixed by τ_α. In particular, y_2 itself is fixed, and so we only need check that the images are completed, which follows from the proof of case (1) above.

To deal with arbitrary $u \in V_v$, we must first introduce some notation. Note firstly that (by definition) such a u corresponds to the corner $x \in \alpha \cap \gamma_1$ of a downward triangle T of the triple $(\alpha, \gamma_1, \varphi(\gamma_1))$. We may then, as in the proof of Lemma 3.2.6 define an arc σ with endpoint x, following α along the length of its edge of T, then continuing along h_v. Again, the important property of σ is that its image under τ_α (fixing the endpoints) coincides with $h_u \in H(\mathcal{P}_1^\alpha)$. But then σ is initially parallel with h_w, completed by v' and w', and so again h_u and h_z are completed by the argument of part (1).

□

Finally, we bring all of the above together to prove Theorem 3.0.6.

Proof. (of Theorem 3.0.6) It follows from Lemmas 3.2.5, 3.2.6, and 3.2.7 that the inductive step of the algorithm preserves consistency of right positions. It is left to show that the base step, in which φ is a single twist τ_α, gives consistent right positions to each pair γ_1, γ_2.

So, let $\mathcal{P}_i$ be the right position associated to (γ_i, τ_α), $i = 1, 2$ by the base step. We index the points and associated horizontal segments in increasing order along γ_i from the base-point c_i. Suppose then that $h_{v_j} \in H(\mathcal{P}_1)$ and $h_{w_k} \in H(\mathcal{P}_2)$ are initially parallel horizontal segments, along region B (Figure 3.32(a)). As noted in the construction of the base step, each $h_{v_j,v_{j+1}}$ is just the image of a segment of γ_1 which has a single intersection with α, with endpoints corresponding to each connected component of the fixable points $\gamma_1 \setminus \mathit{support}(\tau_\alpha)$, and similarly for each $h_{w_j,w_{k+1}}$. Thus, if n is the number of points of $\mathcal{P}_1 \cap \partial B$, then h_{v_j} and g_{v_k} are completed by the points v_{j+n+1} and w_{k+n+1}. Figure 3.32(b) illustrates the local picture for the case $n = 1$.

By induction, then, given a surface Σ, and $\varphi \in \Gamma_\Sigma$ with positive factorization ω, the associated right positions $\mathcal{P}_\omega(\gamma)$, $\mathcal{P}_\omega(\gamma')$ are consistent for any nonintersecting properly embedded arcs γ, γ'. □

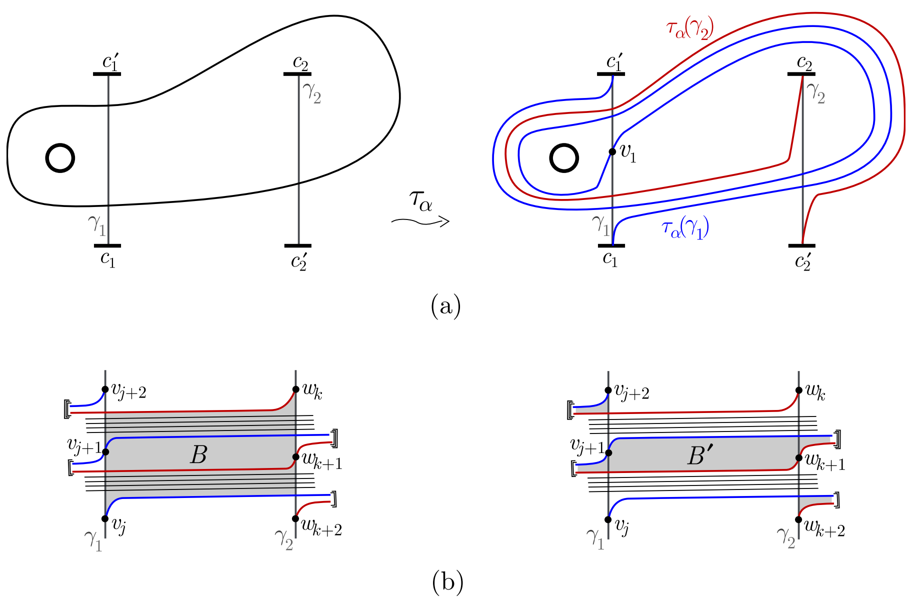


Figure 3.32: (a) An example of the right position associated to a single Dehn twist. (b) Segments h_{v_j} and h_{w_k} are initially parallel along the shaded disk in the leftmost figure, and are completed to v_{j+2}, w_{k+2} along the shaded disk in the rightmost figure

Chapter 4

Restrictions on $p.e.(\varphi)$

In this section, we switch focus back to the entirety of pairs of distinct properly embedded arcs γ_1, γ_2 in a surface Σ, and the images of these arcs under right-veering $\varphi \in \Gamma_\Sigma$. The motivating observation here is that, as the endpoints of each arc are by definition included in any right position, the property of h_0, g_0 being initially parallel is independent of right position, and so is a property of the pair $\varphi(\gamma_1), \varphi(\gamma_2)$. In particular, for *positive* φ, if $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are initially parallel, they must admit consistent right positions $\mathcal{P}_i(\varphi, \gamma_i)$ in which the initial segments are completed (see Example 3.0.13). We are interested in understanding what necessary conditions on $\alpha \in p.e.(\varphi)$ (Definition 2.0.2) can be derived from the information that $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are initially parallel. These are summarized in Lemma 4.0.11 and Theorem 4.0.17.

Throughout the section, we will be considering rectangular regions R in Σ, and classifying various curves and arcs by their intersection with such regions. We need:

Definition 4.0.8. Let R be a rectangular region with distinguished oriented edge e_1, which we call the *base* of R. We label the remaining edges e_2, e_3, e_4 in order, using the orientation on e_1 (Figure 4.1). We classify properly embedded (unoriented) arcs on R by the indices of the edges corresponding to their boundary points, so that an arc with endpoints on e_i and e_j is of *type* $[i, j]$ (on R). We are only concerned with arcs whose endpoints are not on a single edge. An arc on R is then

- *horizontal* if of type $[2, 4]$
- *vertical* if of type $[1, 3]$
- *upward* if of type $[1, 2]$ or $[3, 4]$
- *downward* if of type $[1, 4]$ or $[2, 3]$
- *non-diagonal* if horizontal or vertical

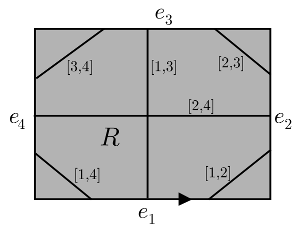


Figure 4.1: Representatives of each of the 6 possible types of arc on R

- *type 1* if not downward

Definition 4.0.9. Let R in Σ be an rectangular region with distinguished base as in Definition 4.0.8. We say $\alpha \in SCC(\Sigma)$ is *type 1 on* R if each arc $\alpha \cap R$ is type 1 on R.

Following Definition 4.0.8, we use a given pair of properly embedded arcs to define a rectangular region D of Σ as follows: Suppose γ_1 and γ_2 are disjoint properly embedded arcs, where $\partial\gamma_i = \{c_i, c_i'\}$, and $\bar{\gamma}_1$ is a given properly embedded arc with endpoints c_1' and c_2. Let $\bar{\gamma}_2$ be a parallel copy of $\bar{\gamma}_1$ with endpoints isotoped along γ_1, γ_2 to c_1 and c_2'. We then define D as the rectangular region bounded by $\gamma_1, \gamma_2, \bar{\gamma}_1, \bar{\gamma}_2$ (the endpoints are labeled so that c_1 and c_2 are diagonally opposite), and base $\bar{\gamma}_2$, oriented away from c_1 (see Figure 4.2). For the remainder of this section, D will always refer to this construction. Of course, for a given pair of arcs, D is unique only up to the choice of $\bar{\gamma}_1$. We also make use of the *diagonal* Δ of D, a representative of the unique isotopy class of arcs with boundary $\{c_1, c_2\}$, interior within D, and such that each of the $\varphi(\gamma_i)$ is parallel along Δ (from its respective endpoint). In particular, the diagonal determines $\bar{\gamma}_1$, and vice versa.

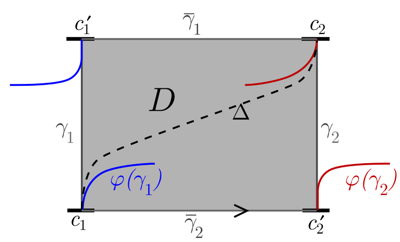


Figure 4.2: disc construction

Definition 4.0.10. We say the pair $\varphi(\gamma_1), \varphi(\gamma_2)$ is *flat* if D can be constructed such that $\varphi(\gamma_i) \cap \bar{\gamma}_j = \emptyset$ for $i, j \in \{1, 2\}$, and *initially parallel (on D)* if there exists D such that $\varphi(\gamma_1), \varphi(\gamma_2), \gamma_1, \gamma_2$ bound a rectangular region $B \hookrightarrow D$ on which c_1 and c_2 are vertices.

As expected, then, the pair of arcs $\varphi(\gamma_1), \varphi(\gamma_2)$ is initially parallel exactly when the horizontal segments h_{c_1} and g_{c_2} originating from the boundary of each pair of right positions are initially parallel. Using this, we immediately obtain:

Lemma 4.0.11. *Let φ be positive, and the pair $\varphi(\gamma_1), \varphi(\gamma_2)$ initially parallel on D. Then $\alpha \in p.e.(\varphi)$ only if α is type 1 on D.*

Proof. Suppose otherwise - then $\alpha \cap D$ has an arc of type $[1,4]$ or $[2,3]$ on D. Now, if $\alpha \in p.e.(\varphi)$, there is some positive factorization of φ in which τ_α is the initial Dehn twist. However, if $\alpha \cap D$ has an arc of type $[1,4]$, then $\tau_\alpha(\gamma_1)$ is strictly to the right of $\varphi(\gamma_1)$ (in the sense of section 2), while if the arc is of type $[2,3]$, then $\tau_\alpha(\gamma_2)$ is strictly to the right of $\varphi(\gamma_2)$, either of which contradicts $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ being initially parallel.

□

To tie this more firmly to the previous section, note that, by Theorem 3.0.6, if φ is positive, there are consistent right positions $\mathcal{P}_1$ and $\mathcal{P}_2$ for any pair $\varphi(\gamma_1), \varphi(\gamma_2)$. Now, if the initial horizontal segments $h_{c_1} \in H(\mathcal{P}_1), h_{c_2} \in H(\mathcal{P}_2)$ are initially parallel, then there are $v \in \mathcal{P}_1$ and $w \in \mathcal{P}_2$ such that v and w are endpoints of the completed segments. Letting B' be the completing disk (Definition 3.0.11), we can define a new rectangular region by extending B' so that its corners lie on $\bar{\gamma}_i$, rather than γ_i :

Definition 4.0.12. A *bounded path* in (Σ, D, φ) is a rectangular region P bounded by $\bar{\gamma}_i$ and $\varphi(\gamma_i), i = 1, 2$, with corners c_1 and c_2 in common with D (Figure 4.3).

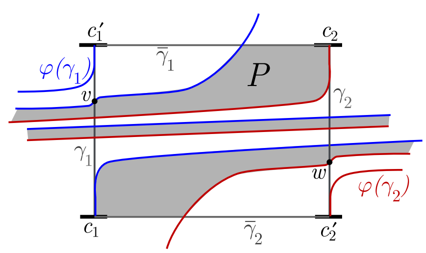


Figure 4.3: P is a bounded path for type 1 $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$.

Lemma 4.0.13. *If $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are initially parallel, for positive φ, then (Σ, D, φ) has a bounded path.*

Proof. This is a restatement of the above discussion. □

Let Σ, D, and φ be given such that the pair $\varphi(\gamma_1), \varphi(\gamma_2)$ is initially parallel and flat (and thus initially parallel) with respect to D. The remainder of this section is an exploration of what necessary conditions this places on elements of $p.e.(\varphi)$.

We start by observing that under these conditions (i.e. initially parallel and flat), the pair of trivial right positions whose vertical sets consist only of the arc boundary points are consistent. Also, the entirety of each image $\varphi(\gamma_i)$ is in the boundary of the bounded path (see Figure 4.4). In particular, such images are *parallel* in the sense of Definition 2.0.5. Now, for $\alpha \in SCC(\Sigma)$, it is clear that α is flat (i.e. $\alpha \cap \bar{\gamma}_1 = \alpha \cap \bar{\gamma}_2 = \emptyset$) if and only if all arcs $\alpha \cap P$ are horizontal on P (Definition 4.0.8), and that, for such curves, $(D, \tau_\alpha^{-1}\varphi)$ is again initially parallel and flat, and so admits consistent right positions. Our interest then lies in non-flat α.

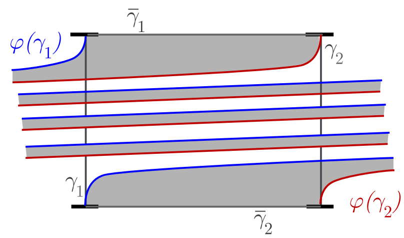


Figure 4.4: The bounded path of flat, initially parallel (D, φ)

Lemma 4.0.14. *Let the pair $\varphi(\gamma_1), \varphi(\gamma_2)$ be flat and initially parallel. Then $\alpha \in p.e.(\varphi)$ only if:*

(1) $\alpha \cap \gamma_1 \neq \emptyset \Leftrightarrow \alpha \cap \gamma_2 \neq \emptyset$

(2) $\alpha \cap \varphi(\gamma_1) \neq \emptyset \Leftrightarrow \alpha \cap \varphi(\gamma_2) \neq \emptyset$

Proof. For statement (1), suppose α is such that $\alpha \cap \gamma_1 \neq \emptyset$ and $\alpha \cap \gamma_2 = \emptyset$. Referring to Figure 4.1, this means that arcs $\alpha \cap D$ cannot be of type $[2, 3], [1, 2]$ or $[2, 4]$ on D, while any arc of type $[1, 4]$ on D would result in $\tau_\alpha^{-1} \circ \varphi$ not being right veering. Similarly, arcs $\alpha \cap P$ cannot be of type $[1, 2]$ or $[1, 3]$ on P, while the right veering conditions eliminates $[3, 4]$ and $[1, 4]$. Putting these together, we find that α may be isotoped such that all intersections $\alpha \cap P, D$ are along subarcs isotopic to one of the ρ, ρ', ρ'' shown in the left side of Figure 4.6, and in particular that there is at least one along ρ.

Now, for such α, the pair $\tau_\alpha^{-1}(\varphi(\gamma_1)), \tau_\alpha^{-1}(\varphi(\gamma_2))$ is initially parallel, and so, if $(\tau_\alpha^{-1} \circ \varphi)$ is positive, must contain a bounded path. Also, arcs along ρ', ρ'' preserve the bounded path, so we may assume all intersections are along ρ. (Figure 4.5). It is immediate then that $(D, \tau_\alpha^{-1}\varphi)$ has no bounded path.

The argument for statement (2) is nearly identical: We suppose $\alpha \cap \varphi(\gamma_1) \neq \emptyset$ and $\alpha \cap \varphi(\gamma_2) = \emptyset$, and find that all intersections $\alpha \cap P$ and $\alpha \cap D$ are isotopic to one of the arcs $\sigma, \sigma', \sigma''$ shown in the right side of Figure 4.6, and in particular that there is at least one along σ. Again, $(D, \tau_\alpha^{-1}\varphi)$ has no bounded path.

Note that this is a slight generalization of the the second example of Example 3.0.13.

□

We have one final application of the bounded path construction. Note that, for α and D as above, an orientation of α gives a derived orientation on each arc in $D \cap \alpha$. Thus, given a pair of upward arcs, we can ask whether their derived orientations agree, independently of the actual orientation of α.

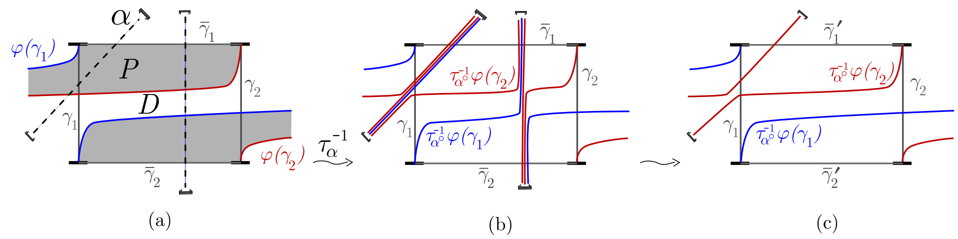


Figure 4.5: Illustration of the terminology of Lemma 4.0.14. (a) A typical α, with a 'cutting pair' of intersections, and a diagonal intersection. (b) is the image under τ_α^{-1}. Notice that, to recover a standard picture of the associated D, we must 'normalize' the picture as in (c), thereby justifying the assertion that 'cutting pairs may be ignored'. It is clear that there is no bounded path.

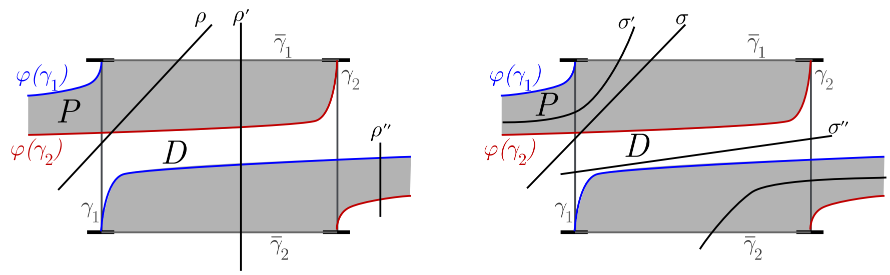


Figure 4.6:

Lemma 4.0.15. *Let $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ be flat and initially parallel. Let α be type 1 on D, such that $\alpha \cap D$ has exactly 2 diagonal arcs s_1 and s_2. Then $\alpha \in p.e.(\varphi)$ only if the derived orientations on s_1 and s_2 agree.*

Proof. Note firstly that, by Lemma 4.0.14, s_1 and s_2 must be as in Figure 4.7(a); i.e. one cuts the upper left corner of D, the other the lower right corner. We show that $(D, \tau_\alpha^{-1}\varphi)$ has a bounded path if and only if the derived orientations on s_1 and s_2 agree, which, by Lemma 4.0.13 gives the result.

Note that if $\alpha \cap D$ has a horizontal arc, it cannot have a vertical arc, and vice-versa. We distinguish the two cases:

1. $\alpha \cap D$ has no vertical arcs. The boundary of bounded path P must include the initial segments of each arc $\tau_\alpha^{-1}(\varphi(\gamma_i))$ from c_i, $i = 1, 2$. We can then start from either corner c_i and follow the initial segment around α. Figure 4.7 does this for c_1. It is clear then that these initial segments will close up to bound P if the orientations match (Figure 4.7(b)). If, on the other hand, the orientations do not match, the segments do not form the boundary of a bounded path (Figure 4.7(c)).

2. $\alpha \cap D$ has no horizontal strands. If there are vertical strands, we must adjust D such

that the diagonal is given by $\tau_\alpha^{-1}(\Delta)$. Again, as in case (1), there is a bounded path if and only if the derived orientations on s_1 and s_2 agree.

□

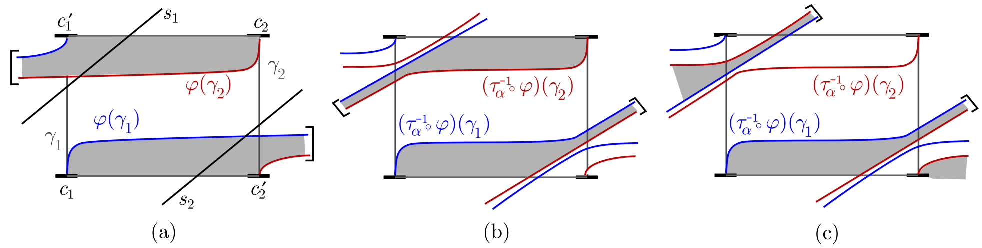


Figure 4.7: (a) $\alpha \cap D$ has exactly 2 diagonal arcs s_1 and s_2. (b) Orientations agree. (c) Orientations disagree.

To sum up, for flat (D, φ), $\alpha \in p.e.(\varphi)$ only if α is type 1 on D, and satisfies the intersection conditions of Lemma 4.0.14. Furthermore, if $\alpha \cap D$ has has only two diagonal arcs, it must satisfy the orientation conditions of Lemma 4.0.15. We now wish to utilize the results of Section 3 to extend these results to the horizontal segments of consistent right positions of the arc/monodromy pairs, and thus to obtain stronger necessary conditions on arbitrary type 1 α for inclusion in $p.e.(\varphi)$.

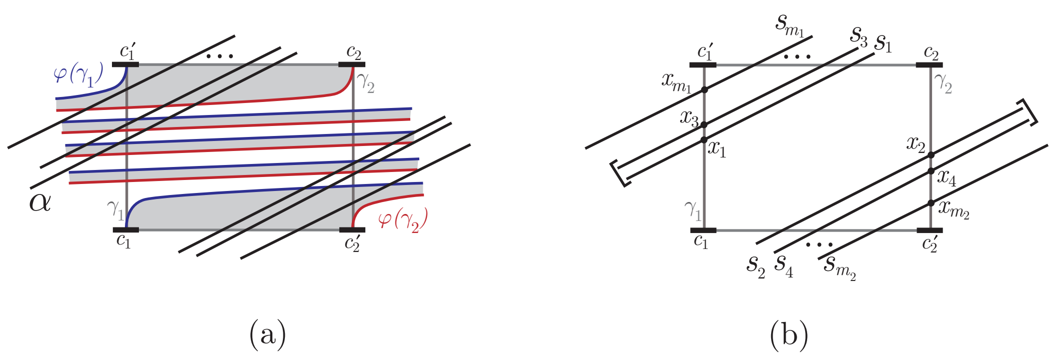


Figure 4.8: (a) The diagonal arcs of $\alpha \cap D$. (b) Labeling of arcs and intersections. The paths $\eta_{2,3}$ and $\eta_{4,1}$ are components of a symmetric multipath.

Given (D, φ), and a type 1 curve α, we label the upward sloping arcs $s_i \in \alpha \cap D$ as in Figure 4.8. A subarc s_i is thus defined only in a neighborhood of the point x_i. Let $a_1, a_2, \ldots a_m$, where $a_1 = 1$, $m = max\{m_1, m_2\}$, be the indices of the x_i with order given by the order in which they are encountered traveling along α to the right from $x_1 \cap \gamma_1$. We associate to α the list $\pi_\alpha = (a_1, a_2, \ldots a_m)$, defined up to cyclic permutation. We decorate these entries with a bar, $\overline{a_j}$, if the derived orientations of s_{a_j} and s_1 do not agree.

Given a pair $x_i \in \alpha \cap \gamma_2$ and $x_j \in \alpha \cap \gamma_1$, the pair splits α into two disjoint arcs. If the orientations of s_i and s_j agree, we label these arcs $\eta_{i,j}$ and $\eta_{j,i}$ (where $\eta_{i,j}$ refers to the arc starting at x_i and initially *exterior* to D), and refer to each such arc as a *path* of the pair (α, D). Accordingly, we call $\eta_{i,j}$ *exterior*, $\eta_{j,i}$ *interior.* We say paths η and η' are *parallel* if, after sliding endpoints along the γ_i so as to coincide, they are isotopic as arcs (relative to the boundary). Then a *multipath* of (α, D) is a collection of more than one pairwise parallel path (Figure 4.8). Note that the components of a multipath are $\eta_{i,j}, \eta_{i+2,j-2}, \ldots, \eta_{i+2r,j-2r}$ for some even i, odd j and r, where $r+1$ is the number of paths in the multipath. Finally, we call a (multi)path *symmetric* if $j+1 = i+2r$. So, for example, any symmetric path is $\eta_{i,i-1}$ or $\eta_{i-1,i}$ for some even i.

Definition 4.0.16. We say α is *nested* with respect to D if its associated π_α can be reduced to the trivial by successive removals of pairs (i, j) of consecutive entries, where $\eta_{i,j}$ is a symmetric path or component of a symmetric multipath. We call α *balanced* (with respect to D) if each x_i is an endpoint in a symmetric path or component of a symmetric multipath..

Note that, in the absence of multipaths, these properties are encoded in π_α. For example, $(1\overline{4}5632)$ is nested but not balanced, (135462) is balanced but not nested, and $(1\overline{4}56\overline{3}2)$ is both nested and balanced. Also note that either definition requires $m_1 + 1 = m_2$. The motivation for these definitions becomes clear with the following:

Theorem 4.0.17. *Let (D, φ) be flat and parallel. Then $\alpha \in p.e.(\varphi)$ only if α is nested and balanced with respect to D.*

We will postpone the proof so as to introduce even more terminology and a couple of helpful lemmas. The basis of the argument is that, if $\alpha \in p.e.(\varphi)$, then $\tau_\alpha^{-1} \circ \varphi$ admits a positive factorization, and so by Theorem 3.0.6 there are consistent right positions $\mathcal{P}_i$ of $((\tau_\alpha^{-1} \circ \varphi), \gamma_i)$. By Lemma 4.0.13 the arcs $\tau_\alpha^{-1}(\varphi(\gamma_i))$ are initially parallel, and so the $\mathcal{P}_i$ must include completing points v_1 and w_1. As it turns out, for any such v_1 and w_1, the horizontal segments h_{v_1} and h_{w_1} are also initially parallel. In fact, for $1 \leq j \leq p$, where p depends on the number of symmetric paths of (α, D), there are points v_j and w_j such that each pair of horizontal segments h_{v_j}, h_{w_j} is initially parallel, and completed by the pair v_{j+1}, w_{j+1}. This in turn will require π_α *balanced* for this sequence to continue to the opposite endpoints of the γ_i, and *nested* for overall consistency of the right positions. The situation for the simplest case $(m_2 = 2)$ is illustrated in Figure 4.9.

For the more general case, we will adopt the convention of drawing the situation as in Figure 4.10, in particular restricting τ_α^{-1} to a neighborhood of α. As our main interest in intersection points $\gamma_i \cap \tau_\alpha^{-1}(\varphi(\gamma_j))$ for $i, j \in \{1, 2\}$,we will keep track of subarcs of $\tau_\alpha^{-1}(\varphi(\gamma_j))$ in a neighborhood of each point $\alpha \cap \gamma_i$. Each neighborhood will of course contain exactly m subarcs (which we will call *strands*), each of which is a subarc of one of the subarcs $[y_k, y_{k+2}]$ of one of the $\tau_\alpha^{-1}(\varphi(\gamma_j))$ (Figure 4.10 makes this clear). The entire (inverse) image then is determined by the 'local pictures' given these neighborhoods along with the information

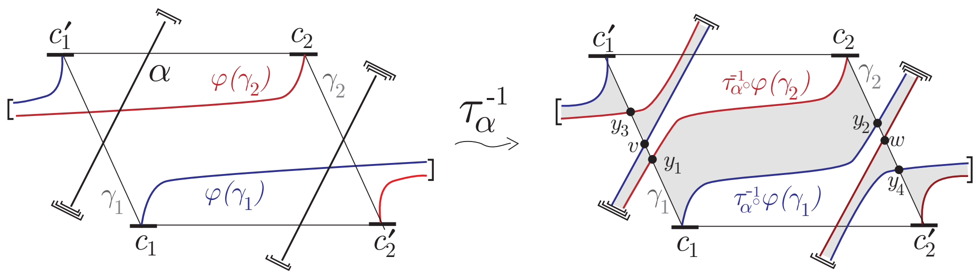


Figure 4.9: The simplest possible nested and balanced α, and the regions involved in the minimal consistent right positions on $\tau_\alpha^{-1}(\varphi(\gamma_i))$.

of how they fit together. We will refer to such a neighborhood as N_j. In particular, as each such subarc has a unique intersection point with a γ_i, we may refer to it by way of 'coordinates', and so refer to the intersection of the subarc $[y_k, y_{k+2}]$ (in the inverse image) with the neighborhood N_j as s_j^i (Figure 4.10 gives an example).

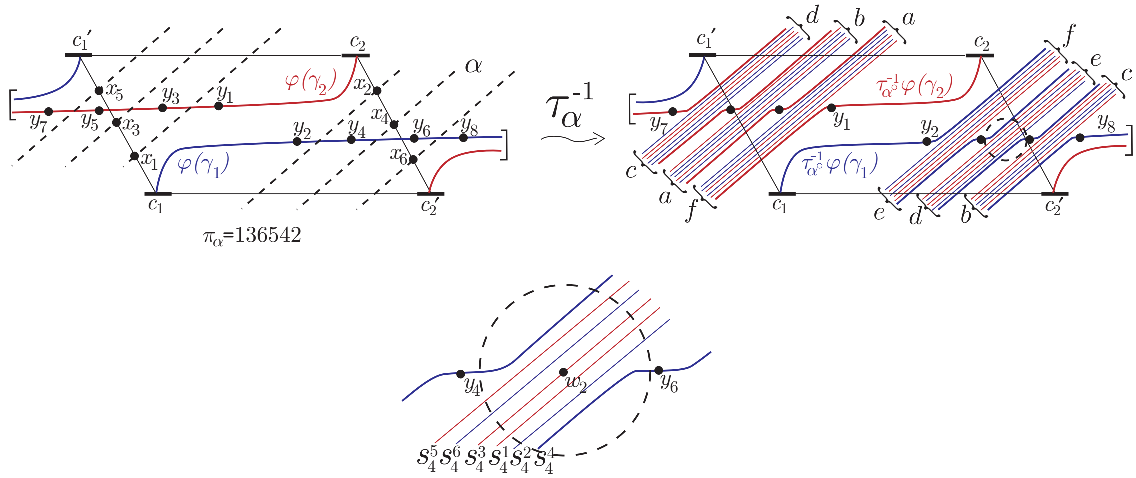


Figure 4.10: Above left, the various points of $\alpha \cap \gamma_i$ and $\alpha \cap \varphi(\gamma_i)$. Above right, the inverse image $\tau_\alpha^{-1}(\varphi(\gamma_i))$, for α with $\pi_\alpha = 136542$ (strands terminating in brackets with like letters are meant to be identified). Below, a detail of a neighborhood N_4.

Throughout the rest of this section, all notation will be as in figures 4.8 and 4.10. We will further assume that $m_1 \geq m_2 - 1$; i.e. that there are at least as many upward intersections $\alpha \cap \gamma_1$ as there are upward intersections $\alpha \cap \gamma_2$.

Our immediate goal is to show that, if $(\tau_\alpha^{-1} \circ \varphi, \gamma_i), i = 1, 2$ admit consistent right positions, then any such pair of positions extends non-trivially along the entire length of each arc, as described above. We begin by exploring restrictions on such right positions.

Suppose y_j and y_k are such that $j < k$, and j and k are both odd. Then we say y_j and y_k are *multipath separated* if $x_j, x_{j+2}, \ldots x_{k-2}$ are endpoints of (the components of) a symmetric (multi)path of α, D. A *multipath index chain* is an ordered subset $I \subset \{1, 3, \ldots, m_1 + 2\}$ such that, for all consecutive entries j and k in I, y_j and y_k are multipath separated. Finally, suppose $\mathcal{P}_i, i = 1, 2$ are right positions of $(\gamma_i, \tau_\alpha^{-1}(\varphi(\gamma_i)))$. We say multipath index chain I is *contained in* the pair $\mathcal{P}_i$ of right positions if, for each $j \in I$, the points y_j and y_{j+1} are corners of an initially parallel region.

Using this terminology, we have

Lemma 4.0.18. *Let (D, φ) be flat and parallel, $\alpha \in p.e.(\varphi)$, and $\mathcal{P}_i, i = 1, 2$ consistent right positions of $(\tau_\alpha^{-1} \circ \varphi, \gamma_i)$. Then the pair $\mathcal{P}_i$ contains a multipath index chain I in which 1 and $m_2 - 1$ are both elements.*

Proof. The proof is by induction. By construction, y_1 and y_2 are corners of an initially parallel region B in $\mathcal{P}_i$, so the trivial chain $\{1\}$ is contained in the pair. For the induction step, we need to show that, if the pair $\mathcal{P}_i$ contains a multipath index chain in which 1 and j are both elements for some $j < m_2 - 1$, then there is k satisfying $j < k \leq m_2 - 1$ such that the pair $\mathcal{P}_i$ contain a multipath index chain in which 1 and k are both elements.

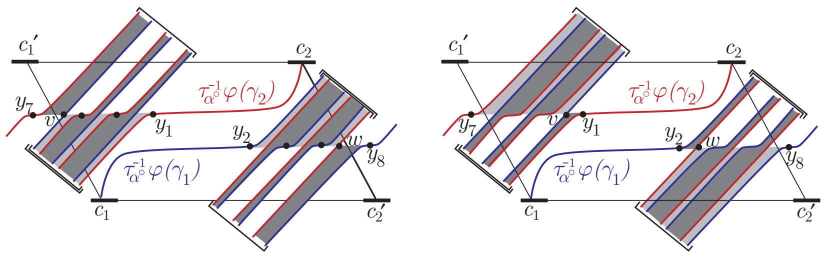
To make the indexing more manageable, we will go through the case $j = 1$, from which the general case will follow by adjusting indices. We therefore need to show that there is $k > 1$ such that y_1 and y_k are multipath separated, and such that y_k and y_{k+1} are corners of an initially parallel region. Our method will be to show that, for any completing points $v \in \mathcal{P}_1$ and $w \in \mathcal{P}_2$ for the initially parallel region B, the horizontal segments h_v and h_w are themselves initially parallel, along a region with corners y_k and y_{k+1}, such that y_1 and y_k are multipath separated.

So, let v and w be completing points. Using the 'coordinate' notation introduced above, recall that v can be uniquely described by the strand s_r^l on which it lies. By symmetry, this also determines w as lying on s_{r+1}^{l-1}. We distinguish 3 cases:

1. Suppose firstly that $r = 1, l = 2$, so v lies in the intersection of the arc $[y_2, y_4]$ with N_1, and thus w lies in the intersection of the arc $[y_1, y_3]$ with N_2. It follows from Lemma 4.0.15 that v and w are completing points if and only if $\eta_{2,1}$ is a symmetric path of α, D, in which case h_v and h_w will be initially parallel along a region with corners y_3 and y_4. Thus $\{1, 3\}$ is a multipath index chain contained in the $\mathcal{P}_i$. For later reference, we note that the strands contained in the completing region for h_{c_1} and h_{c_2} are $\{s_r^l \mid x_r \in \eta_{1,2}, x_l \in \eta_{2,1}\}$, while the strands of the initially parallel region for h_v and h_w are $\{s_r^l \mid x_r \in \eta_{2,1}, x_l \in \eta_{1,2}\}$.

2. We next consider the case $r > 1$. Then v in N_r and w in N_{r+1} define a completing region, which in turn gives a symmetric (exterior) multipath of α, D with $(r+1)/2$ components $\eta_{2,r}, \eta_{4,r-2}, \ldots, \eta_{2r,1}$. But then h_v and h_w are initially parallel along a region with corners y_{2r+1} and y_{2r+2}. Thus $\{1, 2r+1\}$ is a multipath index chain contained in the $\mathcal{P}_i$. Note also that $l = 2$. Figure 4.11(a) illustrates the situation for the case $r = 5$, from which the general situation is clear.

3. Finally, suppose $l > 2$. By case (2), this implies $r = 1$. The configuration is complementary to that of case (2), in that h_v and h_w are now initially parallel along a symmetric (interior) multipath with $(r+1)/2$ components, $\eta_{1,2r}, \eta_{3,2(r-1)}, \ldots, \eta_{2,r}$. Again, h_v and h_w are initially parallel along a region with corners y_{2r+1} and y_{2r+2}, and $\{1, 2r+1\}$ is a multipath index chain contained in the $\mathcal{P}_i$. Again, Figure 4.11(b) illustrates the situation for the case $s = 5$.



(a) (b)

Figure 4.11: (a) The case $r = 5$, so $v \in N_5$ and $w \in N_6$. The completing region for h_{c_1} and h_{c_2} is lightly shaded, while the initially parallel region for h_v and h_w is darkly shaded. The arcs $\tau_\alpha^{-1}(\varphi(\gamma_i))$ are drawn from c_i up to the points y_7 and y_8. (b) The case $s = 5$.

□

We are now ready to prove Theorem 4.0.17. We break the statement into three lemmas, starting with:

Lemma 4.0.19. *Let (D, φ) be flat and parallel. Then $\alpha \in p.e.(\varphi)$ only if α is balanced with respect to D.*

Proof. Suppose α is *not* balanced with respect to D. By Lemma 4.0.14, we may assume that, for $i = 1, 2$, there is at least one upward arc of $\alpha \cap D$ with endpoint γ_i. The idea of the proof is to use Lemma 4.0.18 to reduce the problem to the situation of Lemma 4.0.14.

Now, it follows from the definitions that 1 and $m_2 - 1$ can be connected by a multipath index chain if and only if the subword π' of π_α obtained by removing all entries greater than m_2 is balanced. In particular, it follows from Lemma 4.0.18 that the result is true whenever $m_2 = m_1 + 1$. Without loss of generality, suppose then that $m_1 > m_2$, and π' is balanced.

Again using Lemma 4.0.18, y_{m_2-1} and y_{m_2} are corners of any consistent pair of positions $\mathcal{P}_i$, $i = 1, 2$. However, the only available completing point for h_w is the boundary point c_2', which does not give a completing region (again, this is essentially the argument of Lemma 4.0.14). Figure 4.12 shows the relevant regions.

Thus $(\tau_\alpha^{-1} \circ \varphi, \gamma_i), i = 1, 2$ admit no consistent right positions, and so α cannot be in $p.e.(\varphi)$.

□

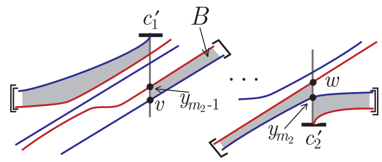


Figure 4.12:

Lemmas 4.0.18 and 4.0.19 thus tells us that, for any consistent and minimal pair of right positions for $(\tau_\alpha^{-1} \circ \varphi, \gamma_i), i = 1, 2$, we may write $\mathcal{P}_1 = \{v_i\}, \mathcal{P}_2 = \{w_i\}$, $i = 1 \dots n$, where $v_1 = c_1, v_n = c_1', w_1 = c_2$ and $w_n = c_2'$, such that, for $1 \leq j < n$, each pair of horizontal segments h_{v_j}, h_{w_j} is initially parallel along a region B_j, and completed along a region B_j' up to v_{j+1} and w_{j+1}. Note that n is determined by the size of a (minimal) multipath index chain I from 1 to $m_1 + 2$

We would now like to understand the implications of the *nested* condition on α for these positions. In particular, we will show that, if α is *not* nested, then the regions of any consistent right positions must overlap, in that there is some v_j in the interior of some B_k $(j \neq k)$. As we shall see, the rigidity of the positions means that this overlapping is somewhat akin to a wrinkle which cannot be ironed out; i.e. we find that it propagates up to the boundary, which then obstructs completion of some initially parallel region.

To make this precise,

Definition 4.0.20. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be consistent right positions. Then if there is some point $v \in \mathcal{P}_1 \cup \mathcal{P}_2$ in the interior of some initially parallel region of the pair, we say $\mathcal{P}_1$ and $\mathcal{P}_2$ *overlap.*

Lemma 4.0.21. *Suppose (D, φ) is flat and parallel, and $\alpha \in SCC(\Sigma)$ balanced with respect to D. Suppose further that $(\tau_\alpha^{-1} \circ \varphi, \gamma_i), i = 1, 2$ is a right pair, with consistent positions $\mathcal{P}_i$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ do not overlap only if α is nested.*

Proof. Following the above discussion, we write $\mathcal{P}_1 = \{v_j\}$, $\mathcal{P}_2 = \{w_j\}$, with B_j and B_j' the initially parallel and completing regions. Now, given symmetric paths (or multipath components) $\eta_{j,j'}$ and $\eta_{k,k'}$ in $\alpha \cap D$, the nested condition implies that either one of these paths is contained in the other, or they are disjoint. We call paths satisfying either condition *nested.* Conversely, existence of any such pair of paths which are *not* nested is a necessary and sufficient condition for α to *not* be nested. Our goal is to show that any such (not nested) pair gives rise to overlapping regions in the right positions $\mathcal{P}_i$.

Suppose then that α is not nested with respect to D, then there are symmetric paths (or multipath components) $\eta_{j,j'}$ and $\eta_{k,k'}$ in $\alpha \cap D$ are such that the entries j, k, j', k' appear in that order in π_α. Consider first the case that $\eta_{j,j'}$ and $\eta_{k,k'}$ are both symmetric *paths*, so $j = j' + 1$, and $k = k' + 1$. In the absence of symmetric multipaths, all initially parallel and completing discs determined by Lemma 4.0.18 are inside a neighborhood of α, so we consider the segments $[y_j, y_{j+2}]$, $[y_{j'}, y_{j'+2}]$, $[y_k, y_{k+2}]$, and $[y_{k'}, y_{k'+2}]$ of the arcs $\tau_\alpha^{-1}(\varphi(\gamma_i))$ (Figure 4.13(a)).

Using Lemma 4.0.18, the points y_j and $y_{j-1} = y_{j'}$ are corners in an initially parallel region of the $\mathcal{P}_i$, and similarly for y_k and $y_{k'}$. The right positions will include the unique completing points for each of these, so that there is a point v_j on $[y_j, y_{j+2}]$ in $N_{j'}$ (i.e. on the strand $s^j_{j'}$), a point w_j on $[y_{j'}, y_{j'+2}]$ in N_j (i.e. on the strand $s^{j'}_j$), and similarly for v_k and w_k. Clearly, v_k is in the interior of the region B_j along which h_{v_j} and h_{w_j} are initially parallel. Note that this also follows from the explicit description of the regions given in the proof of Lemma 4.0.18, case (1).

In the case that one or more of the paths is a component of a symmetric multipath, the situation is essentially the same as above, but there are multiple possibilities for the completing and initially parallel regions, as described in Lemma 4.0.18. Note that, if $\eta_{j,j'}$ is an interior component of a multipath, then each component of the multipath will also be non-nested, so we may assume $\eta_{j,j'}$ and $\eta_{k,k'}$ are outer paths in their respective multipaths. In particular, the points y_j and y_{j-1} are again corners in an initially parallel region, but, again following Lemma 4.0.18, may be completed either in (respectively) N_{j-1} and N_j, or in $N_{j'}$ and $N_{j'+1}$. The relevant configuration is essentially the same in either case, but now does not all happen in a neighborhood of α (Figure 4.13(b)). Again, the regions overlap. $\square$

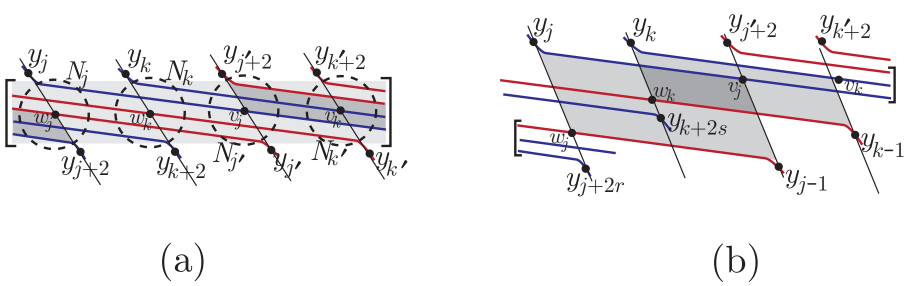


Figure 4.13: (a) The case of non-nested symmetric paths $\eta_{j,j'}$ and $\eta_{k,k'}$. The entire figure takes place in a neighborhood of α (lightly shaded). The dashed circles are the relevant neighborhoods N_i, and the region B_j is shaded. Clearly $v_k \in B_j$. (b) The general case. Here $\eta_{j,j'}$ is in a multipath of r components, and $\eta_{k,k'}$ is in a multipath of s components, for $r > s$. The completing regions of each pair are shaded. Though the path of the regions B_j and B_k will be somewhat more complicated, our interest is in the simple observation that the regions overlap.

Finally, the proof of Theorem 4.0.17 is completed by:

Lemma 4.0.22. *Let (D, φ) be flat and parallel. Then $\alpha \in p.e.(\varphi)$ only if α is nested with respect to D.*

Proof. Note firstly that, if there are at most 2 diagonal arcs in $\alpha \cap D$, this is exactly Lemmas 4.0.14 and 4.0.15. For, by Lemma 4.0.14, π_α of such a curve must be one of $\{(1,2),(1,\overline{2})\}$ (so α is trivially nested), and so by Lemma 4.0.15, we must have $\pi_\alpha = (1,2)$, i.e. α is balanced.

Now, suppose $\mathcal{P}_i$ are minimal consistent right positions for the pair $(\tau_\alpha^{-1} \circ \varphi, \gamma_i), i = 1, 2$. Using Lemma 4.0.18, we see that each $\mathcal{P}_i$ will contain a single interior point for each of the p elements of a (minimal) multipath index chain I from 1 to m_1+2. We label these points $\{v_j\}$ and $\{w_j\}$, for $j = 1, \ldots, p$, so that indices increase along γ_i from c_i. Now, for each $j < p$, the horizontal segments h_{v_j} and h_{w_j} are initially parallel along region B_j with corners y_k and y_{k+1} for k in the index chain I, and completed by v_{j+1} and w_{j+1}. The final segments h_{v_p} and h_{w_p} are completed by the boundary points c_1' and c_2' (Figure 4.9 illustrates the situation for the simplest case of $m = 2$).

We now wish to use Lemma 4.0.21 to show that the 'mixed' pairs h_{v_j} and h_{w_k} are all consistent only if α is nested. In particular, that Lemma implies that, if α is *not* nested, then there are j and k such that v_j is in the interior of B_k. We take $j < k$, and assume k is the largest such index such that the previous sentence is true.

We require:

Observation 4.0.23. Suppose γ_1, γ_2 is a right pair with consistent right positions $\mathcal{P}_1, \mathcal{P}_2$, with vertical points $v \in \mathcal{P}_1, w \in \mathcal{P}_2$. Let Σ' be the surface obtained by removal of a neighborhood of each of these points, and call the new boundary components ∂_v, ∂_w. Then, for $i = 1, 2$, let $\mathcal{P}_i'$ be the restrictions of $\mathcal{P}_i$ to the new arcs γ_i', which follow the original arc from c_i to the respective new boundary component (Figure 4.14 makes this clear). It follows from Definition 3.0.12 that, if neither of v and w are in the interior of any initially parallel region from the pair $\mathcal{P}_1, \mathcal{P}_2$, then $\mathcal{P}_1', \mathcal{P}_2'$ are consistent.

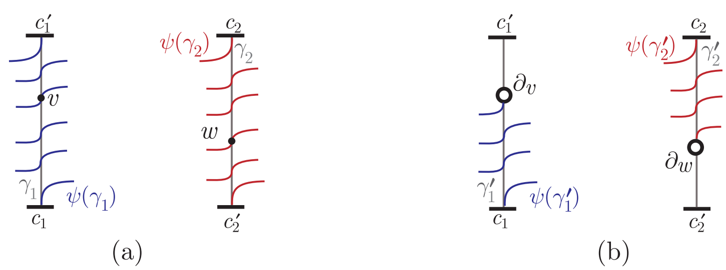


Figure 4.14: Construction of observation 4.0.23: (a) is the original configuration in Σ, (b) is the modified surface Σ'

For our purposes, then, we may assume $k = p$, so v_j is in the interior of the 'final' initially parallel region B_p (Figure 4.15). Thus h_{v_j} is initially parallel with h_{w_p} along a sub-region B of B_p, and so for consistency there must be completing points $v_l \in \mathcal{P}_1$ and $w_l \in \mathcal{P}_2$. Let y be the corner of the B on γ_1. However, the completing region (with corner w_l) terminates on γ_1 outside of B_p, and so cannot contain y, a contradiction.

□

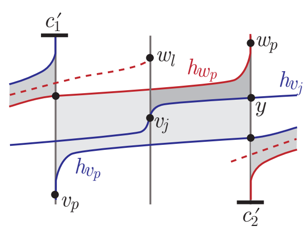


Figure 4.15: The initially parallel region B of the segments h_{v_j} and h_{w_p} lies in the interior of that of the compatible pair h_{v_p}, h_{w_p}. For completion of h_{v_j} and h_{w_p}, there must be endpoint v_l as indicated, but then the corresponding horizontal segment h_{v_l,v_j}, indicated by the dotted line, does not give a completing region.

Chapter 5

Non-positive open books of Stein-fillable contact 3-folds

In this section we prove Theorem 1.0.1 by constructing explicit examples of open book decompositions which support Stein fillable contact structures yet whose monodromies have no positive factorizations. We first introduce a construction, based on a modification of the lantern relation, which allows us to introduce essential left-twisting into a stabilization-equivalence class of open book decompositions. As, by Giroux, elements of such an equivalence class support a common contact manifold, proving the essentiality of this left-twisting (accomplished here using the methods of the previous sections) is sufficient to produce examples of non-positive open books which support Stein-fillable contact structures.

5.1 Immersed lanterns

For our construction, we start out on the surface $\Sigma_{0,4}$ of genus zero with 4 boundary components, and a mapping class with factorization $\tau_{\epsilon_1}\tau_{\epsilon_2}$ as in Figure 5.1(a). It is easy to see that this defines an open book decomposition of $S^1 \times S^2$ with the standard (and unique) Stein fillable contact structure. We use the well known lantern relation to give a mapping class equivalence between the words indicated in Figure 5.1(a) and (b), thus introducing a (non-essential) left twist about the curve α into the positive monodromy. To make room for what is to come, we must enlarge the surface by adding a 1-handle (Figure 5.1(c)) to the support of the lantern relation, so that the associated open book decomposition is of $\#^2(S^1 \times S^2)$, with its (also unique) Stein fillable contact structure (recall that stabilization-equivalence to an open book with positive monodromy is sufficient to show Stein fillability of the supported contact manifold). Note in particular that this operation does not change the property of Stein fillability. We then stabilize by plumbing a positive Hopf band, use this stabilization curve to braid two of the lantern curves into a new configuration in which the lantern relation does not apply, and then destabilize to a book in which, as the lantern is unavailable, the left twist about α can no longer be canceled. The steps are indicated clearly in the remainder of Figure 5.1. As none of the steps (c) through (f) affect the supported contact structure, we have obtained an open book decomposition, supporting a Stein-fillable contact structure, which has no obvious positive factorization. We refer to this construction as an *immersed* lantern relation. To motivate this terminology, observe that the surface and curves of Figure 5.1(f) are obtained from those of Figure 5.1(f) by a self plumbing of the surface. Figure 5.2 summarizes the construction.

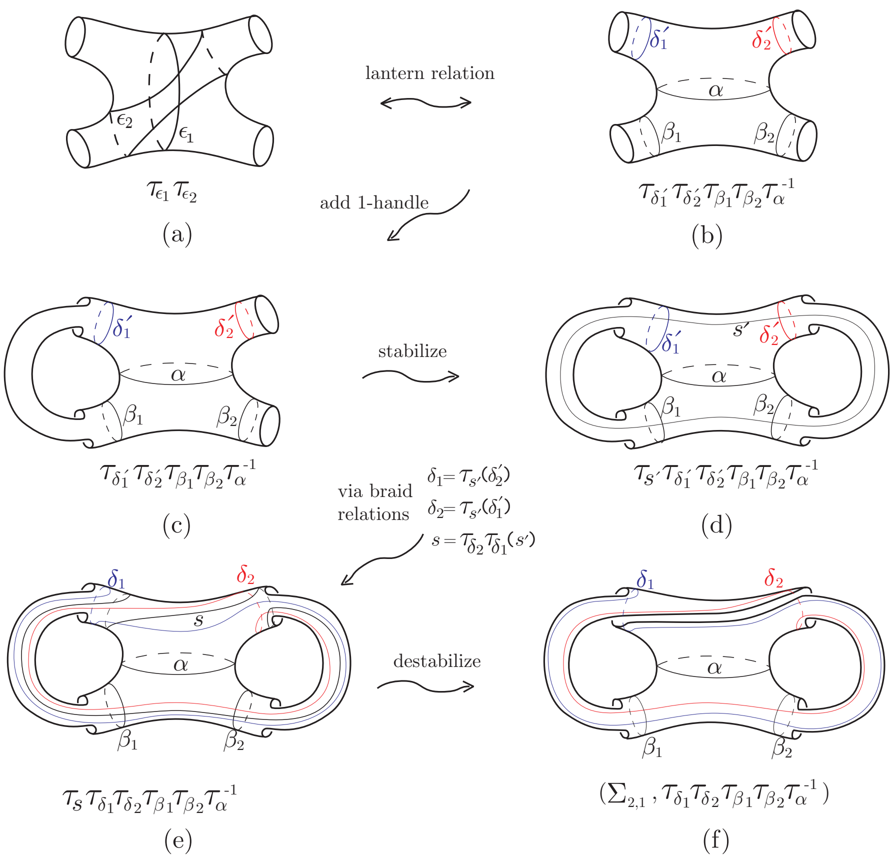


Figure 5.1: $(\#^2(S^1 \times S^2), \xi_{st}) = (\Sigma_{2,1}, \tau_\alpha^{-1}\tau_{\delta_1}\tau_{\delta_2}\tau_{\beta_1}\tau_{\beta_2})$.

The remainder of this chapter uses the results of Section 4 to demonstrate that the left-twisting introduced by the 'immersed lantern' is in fact essential. We also show how this result generalizes to give similar examples of other open books with the same page.

5.2 The positive extension of φ'

Our method of demonstrating essentiality of the left twisting introduced by the immersed lantern is as follows: Let $\varphi = \tau_{\delta_1}\tau_{\delta_2}\tau_{\beta_1}\tau_{\beta_2}\tau_\alpha^{-1}$, where all curves are as in Figure 5.1, and define

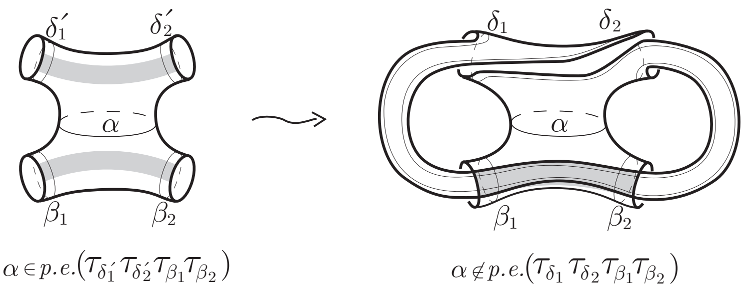


Figure 5.2: Immersing a lantern: To the left, curves of the lantern relation. In particular, a positive twist about each boundary parallel curve is sufficient to cancel a negative twist about the curve α. To the right, the curves in an 'immersed' configuration for which the cancelation no longer holds. Topologically, one obtains the immersed configuration by a self-plumbing of the surface; i.e. an identification of the shaded rectangles via a 90° twist as in the figure. As open book decompositions, one gets from one picture to the other by the steps of Figure 5.1. The effect on the contact manifold is a connect sum with $S^1 \times S^2$, with the standard (Stein-fillable) contact structure.

$\varphi' := \tau_\alpha \circ \varphi = \tau_{\delta_1}\tau_{\delta_2}\tau_{\beta_1}\tau_{\beta_2}$ (note that α has no intersection with any of the other curves, so τ_α commutes with each twist). Suppose that φ has positive factorization $\tau_{\alpha_n}\cdots\tau_{\alpha_1}$. Then we may factorize $\varphi' = \tau_\alpha\tau_{\alpha_n}\cdots\tau_{\alpha_1}$. Our goal is then to derive a contradiction by showing that $\alpha \notin p.e.(\varphi')$. This subsection comprises of two steps. Firstly, we show that α has trivial intersection with each curve in $p.e.(\varphi')$, and secondly use this to conclude that $\alpha \notin p.e.(\varphi')$.

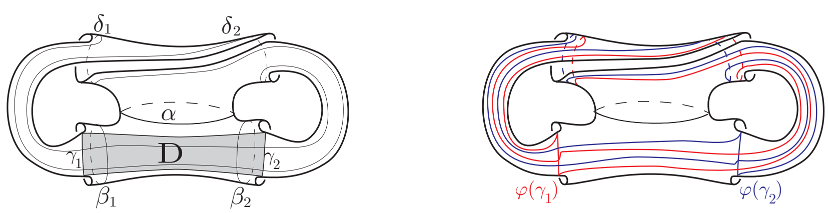


Figure 5.3: To the left, the arcs $\gamma_i, \bar{\gamma}_i$,and the region D. To the right, the images $\varphi'(\gamma_1), \varphi'(\gamma_2)$

For the first step, we wish to show that $p.e.(\varphi')$ lies entirely on $\Sigma \setminus \alpha$ by applying the results of Section 4 to the pair γ_1, γ_2 shown in Figure 5.3. Observe firstly that $\Sigma \setminus \{\gamma_1, \gamma_2\}$ is a pair of pants bounded by β_1, β_2, and α, and so these three curves are the only elements of $SCC(\Sigma)$ which have no intersection with $\{\gamma_1, \gamma_2\}$. The region (D, φ') is flat (Definition 4.0.10), and so by the results of Section 4, and in particular Lemma 4.0.22, each curve $\epsilon \in p.e.(\varphi') \setminus \{\beta_1, \beta_2, \alpha\}$ is type 1, nested, and balanced. This brings us to:

Lemma 5.2.1. *Let Σ, α and D be as in Figure 5.3, and $\epsilon \in SCC(\Sigma)$ be type 1, nested, and balanced on D. Then $\epsilon \cap \alpha = \emptyset$*

The remainder of this section will bring together the necessary information to prove this Lemma. We will assume throughout that $\epsilon \cap D$ has no vertical segments (if this were not the case, then $\epsilon \cap D$ has no horizontal segments, and we would simply exchange γ_i and $\bar{\gamma}_i$ throughout the proof).

So as to simplify the situation, we begin by cutting Σ along $\bar{\gamma}_i, i = 1, 2$, keeping track of the points $\epsilon \cap \bar{\gamma}_i$ (see Figure 5.4). The resulting surface is a pair of pants Σ' with points $\epsilon \cap \bar{\gamma}_i$ labeled as in Figure 5.4, which are connected by m embedded, nonintersecting arcs $\epsilon \cap \Sigma'$. Examples are given in Figure 5.5. Our goal is to show that, for ϵ satisfying our hypotheses, none of these arcs intersect α.

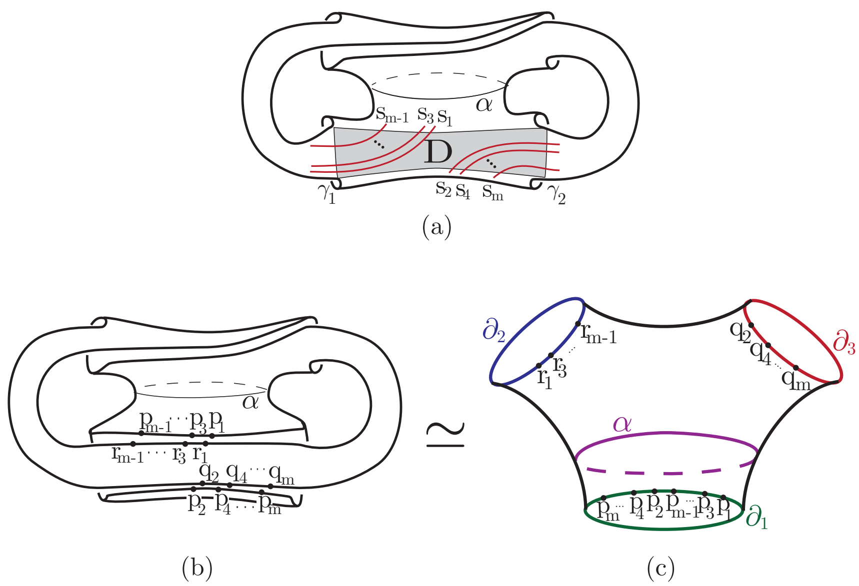


Figure 5.4: (a) indicates the diagonal arcs $\{s_i\} = \epsilon \cap D$ on Σ, while (b) and (c) illustrate the construction of Σ' by cutting Σ along two edges of D, as well as indicating the notation for the various boundary points used in the Lemma.

Recall (Definition 4.0.16) that π_ϵ is nested if and only if it may be reduced to the trivial by successive removal of consecutive pairs a, b of entries, where each such pair is the indices of the endpoints of a component of a symmetric (multi)path of ϵ, D. We will call such pairs *symmetric*, and further distinguish these as *path-symmetric* and *multipath-symmetric*. So for instance each pair $i, i+1$ or $\bar{i}, \overline{i+1}$ (for i odd) determines a symmetric path $\eta_{i,i+1}$, so is path-symmetric, and conversely each path-symmetric pair is of this form. We will also use the observation that each symmetric multipath with an odd number of components contains a path-symmetric component, while each symmetric multipath with an even number of components contains a component $\eta_{i,i-1}$ (again for odd i); we will refer to this second type of path as *off-symmetric*.

Observe that each arc in Σ' corresponds to a pair (i, j) of consecutive entries in π_ϵ

(Definition 4.0.16), where i, j are the indices of the arc endpoints (so, e.g., if there is an arc with endpoints $\{p_4, r_3\}$, then $3, 4$ appear consecutively in π_α in either order). Also recall that endpoints are considered consecutive. We call an arc *(path/multipath-)symmetric* depending on the corresponding pair of entries. As ϵ is nested, $\epsilon \cap \Sigma'$ contains at least two symmetric arcs.

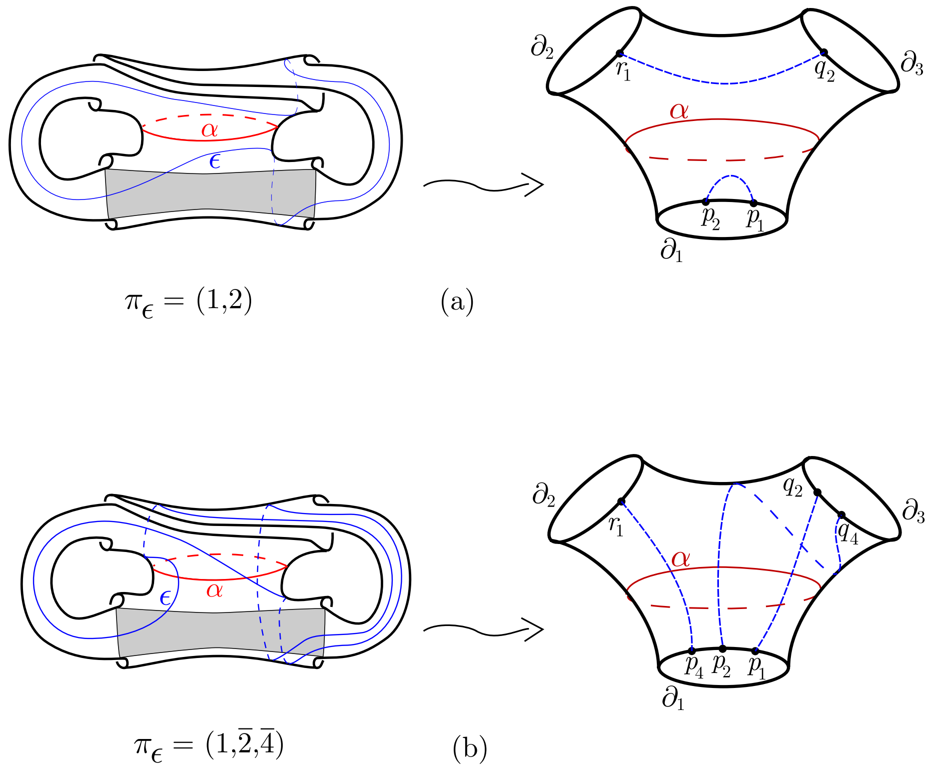


Figure 5.5: (a) A nested, balanced curve ϵ in Σ, and the result of cutting Σ as in Lemma 5.2.1. As demonstrated in the Lemma, such ϵ cannot intersect α. Note that each arc $\epsilon \cap \Sigma$ is symmetric. (b) A curve ϵ which is neither nested nor balanced. None of the arcs are symmetric.

Referring again to Figure 5.4, we classify the arcs by the boundary components on which the endpoints lie. There are six possibilities, denoted $[p, p]$, $[q, q]$, $[r, r]$, $[r, q]$, $[p, r]$, or $[p, q]$. Furthermore, we see that if ϵ is balanced (Definition 4.0.16), then all symmetric arcs are of form $[p, p]$,$[r, q]$, or $[q, r]$.

We pause for a lemma to collect several observations concerning restrictions which the type-1 condition places on these arcs:

Lemma 5.2.2. *Let Σ, Σ' and D be as above, $\epsilon \in SCC(\Sigma)$ type-1 with respect to D (again, Figure 5.4 illustrates the situation). Then*

1. *There are no arcs of form $[r, r]$ or $[q, q]$.*

2. *Each arc of form $[p, p]$ is boundary parallel, and the indices of its endpoints are given by $2i+1$ and $m-2i$ for some non-negative $i \leq m$ (again, m is the number of connected arcs in $\epsilon \cap \Sigma'$).*

3. *If ϵ is balanced and nested with respect to D, there is at least one arc of form $[r, q]$ or $[q, r]$.*

Proof. For claim (1), observe that any boundary-parallel arc on Σ' of form $[q, q]$ either corresponds to a bigon bounded by ϵ and one of the $\overline{\gamma_i}$, contradicting intersection minimality, or contains a downward arc $\epsilon \cap D$, so ϵ is not type 1. Thus there are no boundary-parallel arcs of form $[q, q]$ (or $[r, r]$, by an identical argument). On the other hand, an arc of the form $[q, q]$ which is *not* boundary-parallel separates Σ' into two cylinders, one of which has all of the points $\{r_i\}$ on one boundary component, and some subset of the $\{q_i\}$ on the other. But then there is some boundary-parallel arc of form $[r, r]$, a contradiction. Switching q's and r's in the previous sentence, we find that there are no arcs of form $[q, q]$ or $[r, r]$.

For claim (2), any arc of form $[p, p]$ which is not boundary-parallel separates Σ' into two cylinders, such that one cylinder has all of the $\{q_i\}$ on one boundary component, a subset of the $\{p_i\}$ on the other, while the other cylinder has all of the $\{r_i\}$ on one boundary component, and again a subset of the $\{p_i\}$ on the other. However, the total number of $\{p_i\}$ on the two cylinders is $m-2$, in particular 2 less than the total number of $\{q_i\}$ and $\{r_i\}$, forcing some arc $[q, q]$ or $[r, r]$, contradicting claim (1).

Consider then the arc $[p_k, p_k]$ connecting p_j to p_k. Each such arc separates a disc $D_{j,k} \subset \Sigma'$ whose boundary contains some subset of the $\{p_i\}$ and none of the $\{q_i\}$ or $\{p_i\}$. Now, it is clear that $D_{j,k}$ contains a bigon as in the argument for case (1) unless each arc in $D_{j,k}$ has endpoints with indices of opposite parity. Thus j and k are $2i+1$ and $m-2i$ for some i, as desired.

Finally, for claim (3), note that (1) implies that the number of $[p, p]$ arcs equals the number of $[r, q]$ or $[q, r]$ arcs. But the nested condition implies there are at least 2 symmetric arcs, each of which has one of these forms.

□

We sum up Lemma 5.2.2 in Figure 5.6. In particular, we find that all arcs which intersect α are in one of two sets of parallel arcs, each having the same size. Furthermore, as long as there is some arc intersecting α, the entire picture is determined by (1) the number of such arcs, and (2) the endpoints of, say, the rightmost element of each of the two sets. It remains to show that no such configuration is balanced and nested.

We will further distinguish arcs by the parity of the indices of their endpoints, along with the derived orientation of the corresponding arc s_i in $\epsilon \cap D$. Of course this information is already encoded in π_ϵ. We label an endpoint as type e if its index appears in π_ϵ as even or $\overline{\text{odd}}$, and type o if odd or $\overline{\text{even}}$.

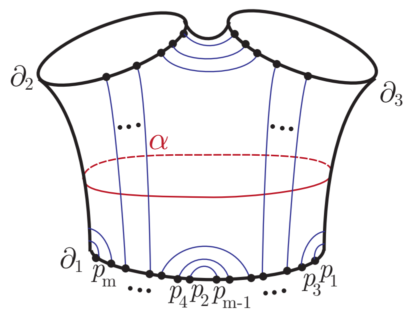


Figure 5.6: Standard form for type-1 arcs on Σ', up to permutation of ∂_2 and ∂_3.

Note that arcs of alternating type, (o, e) or (e, o), are exactly those which have no intersection with α. Furthermore, each symmetric arc is alternating. We call π_ϵ, or any sublist of consecutive entries, *alternating* if consecutive entries alternate type (so, e.g. $(1, \overline{3}, \overline{4}, 2, 5, 6)$ is of type (o, e, o, e, o, e), and thus alternating, while $(3, 7)$ is of type (o, o), therefore not alternating, and thus cannot be a pair of consecutive entries in any alternating word).

We will also refer to a nested word or subword as *totally nested* if it may be written $(a_p \dots a_2 a_1 b_1 b_2 \dots b_p)$ where the pair a_i, b_i is symmetric for each i (recall that we are considering words only up to cyclic permutation).

We have:

Lemma 5.2.3. *Let Σ, α and D be as in Figure 5.3, and $\epsilon \in SCC(\Sigma)$ be type 1, nested, and balanced with respect to D. Then each totally nested subword of π_ϵ is alternating.*

Proof. Let π' be a totally nested subword of π_ϵ. We write $\pi' = (a_p \dots a_2 a_1 b_1 b_2 \dots b_p)$, where a_i and b_i are symmetric for each i, and $p = m/2$.

We consider two cases:

1. The innermost pair $a_1 b_1$ is of form $[p, p]$. Then $\partial_1(\Sigma') \setminus [p_{a_1}, p_{b_1}]$ consists of two components, one a disc R. Note in particular that, as a_1, b_1 is a symmetric pair, R contains exactly one element of each symmetric pair a_i, b_i. Consider then the pair a_2, b_2, and suppose $a_2 \in R$. Now, the arc corresponding to $a_2 a_1$ must terminate on $\partial_2(\Sigma)$ or $\partial_3(\Sigma)$; i.e. on the point q_{a_1} or r_{a_1}. But then as p_{a_2} is in R, the arc must be $[q, r]$ or $[r, q]$, so alternating. By symmetry the same is true of $b_1 b_2$, so $a_2 a_1 b_1 b_2$ is alternating. We continue similarly - one of a_3 and b_3, say a_3, is a index in R, so the arc $a_3 a_2$ is contained in R, so alternating. We continue in this fashion to see that the entire word is alternating.

2. The innermost pair $a_1 b_1$ is of form $[r, q]$ or $[q, r]$ (say $[q, r]$). If the pair is in a multipath, we may choose the path- or off-symmetric element. We first note that the complement in Σ' of the arc $[q_{a_1}, r_{b_1}]$ is an annulus, which we denote Σ'' (see Figure 5.7). We label the new boundary component $\partial_2(\Sigma'')$. Now, if $a_2 a_1 b_1 b_2$ is not alternating, then the arc

a_2a_1 (and similarly b_1b_2) has an endpoint on each boundary component, so the two arcs cut Σ'' into two discs. We will show that the number of points on each disc is odd, and thus cannot be connected by arcs, a contradiction.

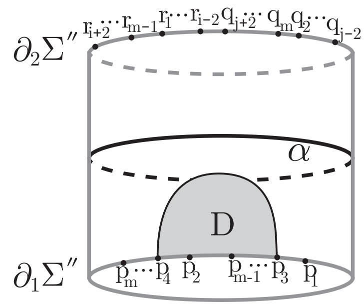
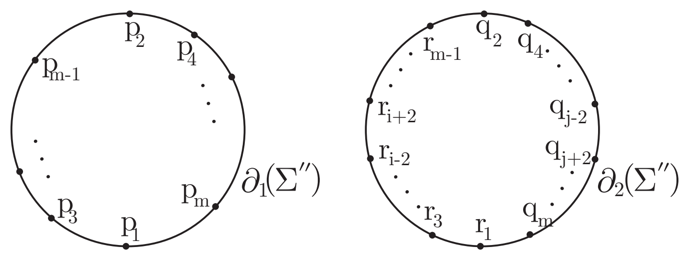


Figure 5.7: The surface Σ'', obtained by removing an initial arc (here $[r_i, q_j]$) from Σ'. The region D is separated by the path-symmetric arc $[p_3, p_4]$, and so will contain an even number of points if and only if $m \equiv 2(\mod 4)$.

Note (Figure 5.8) that any path-symmetric pair of points will divide either boundary component into two arcs each containing an equal number of points. But then the pair q_{a_2} and r_{b_2} separate $\partial_2(\Sigma'')$ into two components, each containing $m/2-2$ points, while p_{a_1} and p_{b_1} separate $\partial_1(\Sigma'')$ into components each containing $m/2-1$ points. Thus each disc has exactly $m-3$ points, which gives a contradiction as desired (recall that m is even).

We have then that the arcs are $[p_{a_2}, p_{a_1}]$, and $[p_{b_1}, p_{b_2}]$. As all arcs of form $[p, p]$ are boundary parallel, we modify Σ'' by removing the discs cut out by these arcs. This removes an even number of path-symmetric pairs from ∂_1. In particular, the difference in numbers of points on the 2 boundary components is still $2(\mod 4)$. We may then continue as above to find that the pair a_3 and b_3 must be of type o and e, giving arcs $[r_{a_3}, q_{a_2}]$ and $[q_{b_2}, r_{b_3}]$, and so on. At each step we find that the total number of boundary points decreases (by a multiple of 4), and each arc is alternating. Thus the totally nested subword is alternating, as desired.

For the case that our pairs are off-symmetric we have essentially the same argument: consider non-alternating $a_2a_1b_1b_2$, where each a_i,b_i is off-symmetric. Now as in the above case, the inner arc connects q_{a_1} to r_{b_1}, so the annulus Σ'' given by cutting along this arc has m points on ∂_1, $m-2$ points on ∂_2. The relevant arcs on Σ'' will again be $[q_{a_2}, p_{a_1}]$ and $[p_{b_1}, r_{b_2}]$. However, as the pairs are now off-symmetric, the pair q_{a_2} and r_{b_2} separates $\partial_2(\Sigma'')$ into components of $m/2-3$ and $m/2-1$ points, while p_{a_1} and p_{b_1} separate $\partial_1(\Sigma'')$ into components each containing $m/2$ and $m/2-2$ points. Again, each disc has an odd number of points, which gives a contradiction. Continuing exactly as in the earlier case, we find that the totally nested subword is alternating.



(a) (b)

Figure 5.8: Points on the boundary of Σ''.

□

It remains to be shown then that each totally nested subword is of the same alternating type $(e \ldots o)$ or $(o \ldots e)$, so that their concatenation is again alternating. We first consider the case of multipath symmetric arcs:

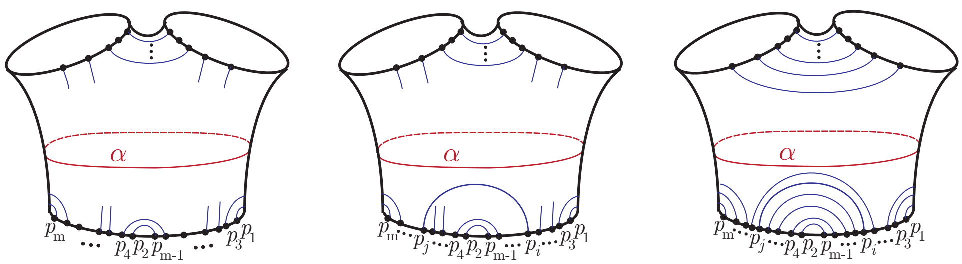
Lemma 5.2.4. *If a collection of totally nested subwords of π_ϵ are such that the innermost pair of each corresponds to an component of a common multipath, then the paths defined by the entire subwords are themselves components of a multipath, and each subword has the same length.*

Proof. This can be easily seen from Figure 5.6. Alternately, suppose our collection has r subwords. There are then r symmetric pairs $\{a_1^j b_1^j\}_{j=1}^r$ such that each is the innermost pair of a nested subword. We will go through the case that each of these arcs is of form $[q, r]$ (the other case is nearly identical). Now, for fixed j, consider the totally nested subword $a_2 a_1 b_1 b_2$. By Lemma 5.2.3, the arcs corresponding to $a_2 a_1$ and $b_1 b_2$ are each of form $[p, p]$, so a_2 and b_2 are determined by the inner pair. In particular, each subword of length ≤ 4 will correspond to the components of a common multipath. The same argument holds for each additional pair a_k, b_k.

As for the length of the subwords, observe that the subwords corresponding to the 'central' path(s) (by which we mean the path- or off-symmetric pair(s)) of the multipath must be at least as long as any other (as any longer path is not symmetric). However, each extension of the central path to succesive a_k, b_k is boundary parallel (in Σ''), which forces all paths to one side to be isotopic. For symmetry, all paths to the other side must also be parallel, so all paths extend to at least the same length as the central path(s).

□

To finish the proof of Lemma 5.2.1, we observe (again using Lemma 5.2.2 and Figure 5.6) that there can be at most 2 (disjoint) symmetric multipaths. If there is only one, then by 5.2.4, π_ϵ is the concatenation of subwords of the same alternating type, so is itself alternating. If there are two, they cannot each be of form $[p, p]$ or of form $[q, r]$ or $[r, q]$, so there is one of each, and it is left to understand how the endpoints connect. We refer back to the 'standard form' of Figure 5.6. As we are trying to connect the endpoints of two alternating multipaths, the situation is as in Figure 5.9(a). However, as there is some arc connecting the symmetric pair p_i, p_j (b), and the ends of our multipaths are themselves symmetric pairs, one of the multipaths must end in the disc region cut out by $[p_i, p_j]$. The endpoints must then connect as in (c), so that ϵ is alternating, and thus $\epsilon \cap \alpha = \emptyset$.



(a) (b) (c)

Figure 5.9: Connecting two symmetric multipaths.

We bring all of this together with:

Theorem 5.2.5. *The monodromy of the open book decomposition* (Σ, φ) *of* $(\#^2(S^1 \times S^2), \xi_{st})$ *shown in Figure 5.1(f), where* ξ_{st} *is the unique Stein-fillable structure, has no factorization into positive Dehn twists.*

Proof. Suppose that φ is positive, so $\tau_\alpha^{-1}\tau_{\delta_1}\tau_{\delta_2}\tau_{\beta_1}\tau_{\beta_2} = \tau_{\alpha_n}\cdots\tau_{\alpha_1}$. Then $\varphi' := \tau_{\delta_1}\tau_{\delta_2}\tau_{\beta_1}\tau_{\beta_2} = \tau_\alpha\tau_{\alpha_n}\cdots\tau_{\alpha_1}$; i.e. $\alpha \in p.e.(\varphi')$. Then, by Lemma 5.2.1, each α_i has trivial intersection with α. But then α must be an element of the step-down (Definition 2.0.4) $\mathcal{C}_{\tau_\alpha\tau_{\alpha_n}\cdots\tau_{\alpha_1}(\gamma)} = \mathcal{C}_{\varphi'}$ for any properly embedded arc γ which intersects α. The choice of γ shown in Figure 5.10 then gives a contradiction.

□

5.3 Further examples

We conclude with some observations allowing immediate generalizations of Theorem 5.2.5.

Observation 5.3.1. Let $\psi \in \Gamma_{\Sigma_{2,1}}$ have a factorization $\psi' \circ \varphi'$, where ψ' is positive, and φ' as in the previous subsection. Then $\psi \circ \tau_\alpha^{-1} = \psi' \circ \varphi' \circ \tau_\alpha^{-1}$, and so, following the notation of Figure 5.1, the open books $(\Sigma_{2,1}, \psi \circ \tau_\alpha^{-1})$ and $(\Sigma_{2,2}, \psi' \circ \tau_{\epsilon_1}\tau_{\epsilon_2}\tau_{s'})$ are stabilization-equivalent, and in particular each support a Stein-fillable contact structure.

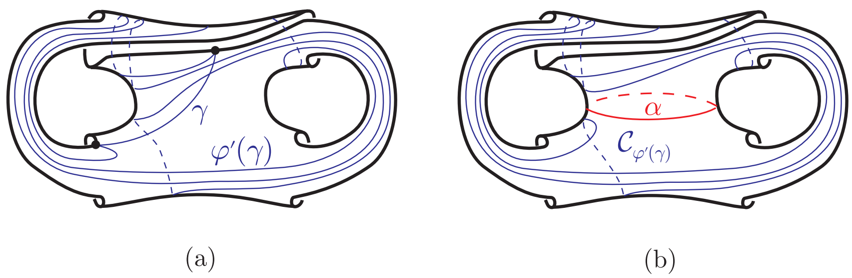


Figure 5.10: (a) The arc γ, and its image $\varphi'(\gamma)$. (b) The stepdown $\mathcal{C}_{\varphi'(\gamma)}$, and the curve α; clearly $\alpha \notin \mathcal{C}_{\varphi'(\gamma)}$

Observation 5.3.2. Let $\psi \in \Gamma_{\Sigma_{2,1}}$ be such that (ψ, D) is flat and parallel (where D is as in Figure 5.4), and $\alpha \notin \mathcal{C}_{\psi(\gamma)}$ (where γ is as in Figure 5.10). Then the proof of Theorem 5.2.5 holds for φ' replaced by ψ; i.e. the monodromy $\psi \circ \tau_\alpha^{-1}$ has no positive factorization.

For example, $\psi := \tau_{\delta_1}^{e_1}\tau_{\delta_2}^{e_2}\tau_{\beta_1}^{e_3}\tau_{\beta_2}^{e_4}$ clearly satisfies these conditions for any positive integers e_i, so that the resulting open book decomposition $(\Sigma_{2,1}, \tau_\alpha^{-1} \circ \psi)$ is stabilization-equivalent to $(\Sigma_{2,2}, \tau_{\delta_1}^{e_1-1}\tau_{\delta_2}^{e_2-1}\tau_{\epsilon_1}\tau_{\epsilon_2}\tau_{s'}\tau_{\beta_1}^{e_3-1}\tau_{\beta_2}^{e_4-1})$, thus supports a Stein-fillable structure, yet its monodromy has no positive factorization. As a simple example, for the case $e_1 = e_2 = e_3 = 1$, we obtain an open book decomposition of $(S^1 \times S^2)\#L(e_4 - 1, e_4 - 2)$ for each positive e_4 (Figure 5.11).

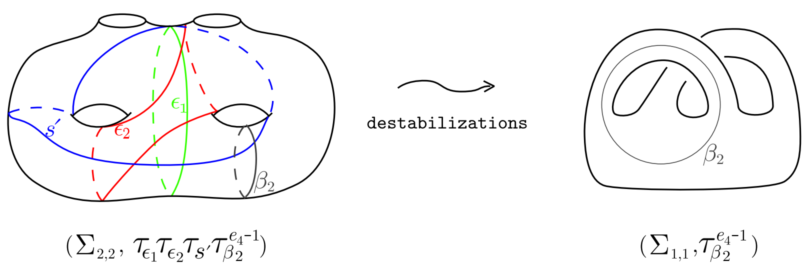


Figure 5.11: $(\Sigma_{2,2}, \tau_{\epsilon_1}\tau_{\epsilon_2}\tau_{s'}\tau_{\beta_2}^{e_4-1}) = ((S^1 \times S^2)\#L(e_4 - 1, e_4 - 2), \xi_{st})$

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