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CHAINS IN CR GEOMETRY AS GEODESICS OF A KROPINA METRIC

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ABSTRACT. With the help of a generalization of the Fermat principle in general relativity, we show that chains in CR geometry are geodesics of a certain Kropina metric constructed from the CR structure. We study the projective equivalence of Kropina metrics and show that if the kernel distributions of the corresponding 1-forms are non-integrable then two projectively equivalent metrics are trivially projectively equivalent. As an application, we show that sufficiently many chains determine the CR structure up to conjugacy, generalizing and reproving the main result of [7]. The correspondence between geodesics of the Kropina metric and chains allows us to use the methods of metric geometry and the calculus of variations to study chains. We use these methods to re-prove the result of [15] that locally any two points of a strictly pseudoconvex CR manifolds can be joined by a chain. Finally, we generalize this result to the global setting by showing that any two points of a connected compact strictly pseudoconvex CR manifold which admits a pseudo-Einstein contact form with positive Tanaka-Webster scalar curvature can be joined by a chain.

1. INTRODUCTION

A CR structure on a $2n+1$ dimensional manifold M is a pair (H, J) where H is a contact distribution and J is a complex structure on H satisfying a certain integrability condition. See Section 4.1 for details.

The *chains* for a CR geometry are a family of curves on M which are canonically constructed from the CR structure. See, for instance, the book [14] and the references therein. For any point of M and any direction not contained in H there exists precisely one chain through this point and tangent to this direction.

One of several equivalent definitions of chains goes through the Fefferman metric introduced in [11], which is an indefinite metric on a circle bundle over M. The conformal class of the Fefferman metric is canonically constructed from the CR structure on M. The infinitesimal generator K of the circle action is a null Killing vector field for this metric. Chains are then defined to

be the projections to M of null geodesics for this pseudo-Riemannian metric to M. See Definition 4.1.

On the other hand, the *Kropina metric* is a function on TM given by

$$F(x, \xi) = \frac{g(\xi, \xi)}{\omega(\xi)}$$

where g is a metric and ω is a nonvanishing 1-form on M. In this paper we allow metrics g of all signatures; in fact, we will also allow certain degenerate metrics: as it will become clear below, in order to define geodesics it is sufficient that the restriction of g to $\ker \omega$ is non-degenerate.

Kropina metrics are popular objects in Finsler geometry, despite the fact that they are not strictly speaking Finsler metrics even when g is positive definite. Indeed F is undefined for $\xi \in \ker \omega$.

Our first result is the following theorem.

Theorem 1.1. *Chains are geodesics of a certain Kropina metric.*

By a *geodesic* of a Kropina metric $F = g/\omega$ we mean any smooth regular solution $\gamma(t)$ of the Euler-Lagrange equation for the Lagrangian F satisfying the additional property that $F(\gamma(t), \gamma'(t))$ is defined (i.e., $\omega(\gamma'(t)) \neq 0$) for all t.

In Section 4.3, we give a formula for this Kropina metric. When M is the boundary of a strictly pseudoconvex domain and ρ is Fefferman's defining function for M then we can express this metric as $F = (\partial \bar{\partial} \rho)/\text{Im}(\partial \rho)$ where we regard $\partial \bar{\partial} \rho$ as a symmetric 2-tensor restricted to M and $\text{Im}(\partial \rho)$ as a one-form restricted to M.

In the above theorem and throughout this paper, with the exception of Section 5, we consider geodesics without preferred parameterization. Clearly, $F(x, \xi) = -F(x, -\xi)$, so that for a geodesic $\gamma(t)$ the curve $\gamma(C - t)$ for an appropriate constant C is also a geodesic. For any point $(x, \xi) \in TM$ such that $\omega(\xi) \neq 0$, there exists a local geodesic of g/ω, unique up to re-pameterization, which starts at x tangent to ξ.

To formulate the next result and apply it to CR geometry, we only need to know that F has the form g/θ, where θ is a contact form for the underlying contact structure on the CR manifold M. The metric g is defined up to the transformation $g \mapsto g + \theta \cdot \beta$ with a closed 1-form β. Clearly this transformation corresponds to an addition of the closed form β to the Kropina metric, and does not change its geodesics.

The Kropina metric also has a relation to a different topic in CR geometry: We consider the energy functional $E(\gamma)$, the integral of the Kropina metric over a curve γ transversal to the contact distribution in the CR manifold M. Suppose M bounds an asymptotic complex hyperbolic domain Ω and γ bounds a minimal surface Σ in Ω. In [8], $E(\gamma)$ was shown to appear as the log term coefficient in the area renormalization expansion of Σ for $\dim M = 3$.

The next result concerns the projective equivalence of Kropina metrics. According to the classical definition, two geometric structures (Riemannian, Finsler, or affine connections) are projectively equivalent if they have the same geodesics. In the case of Kropina metrics, we will modify this definition. The reason for this modification is that Kropina geodesics are not defined precisely along directions lying in the kernel of ω. Thus if two Kropina metrics are projectively equivalent according to the classical definition, then their 1-forms coincide up to scale, which is an extremely strong additional condition. Note that Theorem 1.4 below shows that for certain Kropina metrics we can not reconstruct the kernel of ω by the geodesic equation.

We call a set of curves on the manifold M *sufficiently big* if for any point $p \in M$ the set of tangent vectors at p for these curves contains a nonempty open subset of $T_p M$. We call two Kropina metrics *projectively equivalent*, if there exists a sufficiently big set of curves which are geodesics for both metrics.

Our second result is the following theorem:

Theorem 1.2. *Suppose two Kropina metrics $F = g/\omega$, $\widehat{F} = \widehat{g}/\widehat{\omega}$ are projectively equivalent and satisfy the following conditions:*

- $\ker \omega$ *is non-integrable i.e., $\omega \wedge d\omega \neq 0$ at almost every point;*
- $g, \widehat{g}$ *are non-degenerate on $\ker \omega$, $\ker \widehat{\omega}$ respectively.*

Then, $\omega = \alpha \widehat{\omega}$ for a certain non-vanishing function α and there exist a constant $c \neq 0$ and a closed 1-form β such that $\widehat{F} = cF + \beta$.

Combining Theorem 1.1 and Theorem 1.2, we obtain the following generalization of the results of [7, 4]:

Corollary 1.3. *Suppose two non-degenerate CR structures (H, J) and $(\widehat{H}, \widehat{J})$ have the same sufficiently big family of chains. Then these two CR structures coincide or are conjugate: that is, $H = \widehat{H}$ and either $J = \widehat{J}$ or $J = -\widehat{J}$.*

Note that [7] assumes that *all* chains of one structure are chains of the other structure. As explained above, this implies that the corresponding 1-forms are proportional. The latter assumption essentially simplifies the proof. Similarly, [4] requires that the corresponding contact distributions coincide.

Theorem 1.2 describes all pairs of projectively equivalent Kropina metrics such that for at least one of them the kernel distribution of the corresponding 1-form is non-integrable. Let us now consider the remaining case, i.e., when for both 1-forms the kernel distributions are integrable. In this case without loss of generality we can assume that the 1-forms are closed. The next theorem shows that then the geodesics are geodesics of a certain affine connection.

Theorem 1.4. *If the 1-form ω is closed, then for any Kropina metric $F = g/\omega$ there exists an affine connection $\nabla = (\Gamma_{ij}{}^k)$ such that each geodesic of F is a geodesic of ∇.*

The precise formula for the connection is in Theorem 3.9. It is torsion-free.

In particular, in dimension 2, all Kropina metrics are projectively equivalent to affine connections. This is actually known and was one of the motivations for introducing Kropina metrics, see [18].

Note that the question when two affine connections are projectively equivalent is well-understood, see e.g. [21].

Chern and Moser [9, p. 222] told us to think of chains as the CR versions of geodesics. If their analogy is a good one, then any two sufficiently nearby points ought to be connected by a chain, and if M were compact and connected, then any two points at all ought to be connected by a chain. The first assertion does hold for strictly pseudoconvex CR manifolds: nearby points are connected by chains. See [14], p. 185, and the original references therein, including [15, 17]. Surprisingly, the second assertion is false, even if the compact manifold is locally CR equivalent to the standard model, S^3 with its canonical strictly pseudoconvex structure. This example, the Burns-Shnider counterexample, is detailed in [14, p. 185]. See also the original reference, [3].

The fact that chains are geodesics of a Kropina metric allows us to employ variational methods and techniques of metric geometry to investigate chains. We will use these methods to reprove and generalize the famous result of [15, 17] on local chain connectivity.

Theorem 1.5. *Let $F = g/\omega$ be a Kropina metric on M with g positive definite. Then, the following statements hold:*

(A) *If at $p \in M$ we have $\omega \wedge d\omega \neq 0$, then there exists a neighborhood U of p such that for any $q \in U$ one can join p to q by a length minimizing Kropina geodesic which does not leave U. This neighborhood U can be chosen to be arbitrary small.*

(B) *Suppose that M is compact and assume that the set of the points $p \in M$ such that $\omega \wedge d\omega \neq 0$ is connected and everywhere dense in M. Then, any two points of M can be joined by a length-minimizing Kropina geodesic γ such that $F(\gamma(t), \gamma'(t)) > 0$ at each point.*

We will see that for strictly pseudoconvex CR manifolds the corresponding metric g can be made positive definite locally. This follows from Lemma 3.1 applied to the Kropina metric associated to a strictly pseudoconvex CR manifolds. Moreover, on a strictly pseudoconvex CR manifold admitting a pseudo-Einstein contact form with positive Tanaka-Webster scalar curvature, the metric g can be made positive definite globally (see (4.6)). On strictly pseudoconvex CR manifolds we have by definition that $\omega \wedge d\omega \neq 0$ at all points. Therefore, we obtain:

Corollary 1.6. *Let M be a strictly pseudoconvex CR manifold. Then the following statements hold:*

(A) *Each point $p \in M$ has a neighborhood U such that any $q \in U$ is connected to p by a chain lying in U.*

(B) *If M is connected, compact and admits a pseudo-Einstein contact form with positive Tanaka-Webster scalar curvature, then, any two points of M can be joined by a chain.*

The Burns-Shnider example mentioned above is a CR structure on a compact manifold $S^{2n} \times S^1$ such that not every two points can be connected by a chain. It follows from Corollary 1.6 that we cannot find a pseudo-Einstein contact form with positive scalar curvature for this CR structure. We discuss this fact in detail in Section 4.4.

Our paper is organised as follows. In Section 2, we recall the Fermat principle from general relativity and generalize it to the case when the Killing vector field is null; this will give us the proof of Theorem 1.1. In Section 3, we examine some properties of the set of 2-jets of solutions to the Euler-Lagrange equation for the Kropina metric and prove Theorem 1.2 and Theorem 1.4. In Section 4, we describe the Kropina metric associated to the Fefferman metric, and apply our results to prove Theorem 1.1 and Corollary 1.3. We also analyse the Burns-Shnider example in detail and show where the positivity of the scalar curvature fails. In Section 5, we describe the indicatrix of the Kropina metric and prove Theorem 1.5. All objects in our papers are assumed to be sufficiently smooth; C^2-smoothness is enough for all proofs related to Kropina metrics.

2. Fermat Principle and Proof of Theorem 1.1

We start by recalling the Fermat Principle. For simplicity we restrict to a local version.

Consider a Lorentzian metric $\widetilde{g}$ on $\widetilde{M}$ admitting a time-like Killing vector field K and a local space-like hypersurface M. Consider the (Randers) Finsler metric on M given by the formula

$$F(x,\xi) = \sqrt{\widetilde{g}(\xi,\xi) + \widetilde{g}(K,\xi)^2} + \widetilde{g}(K,\xi)$$

for $\xi \in T_xM$. The projection of a null geodesic $\gamma(t)$ to M is the curve $\phi(\tau(t)) \circ \gamma(t)$ where ϕ denotes the flow of K and the 'time function' $\tau(t)$ is such that for each t the point $\phi(\tau(t)) \circ \gamma(t)$ lies on M. If we work locally, in a small neighborhood of a point of M, no ambiguity appears. Then we have:

Fermat Principle (see e.g. [5, 22]): The projected null geodesics $\phi(\tau(t)) \circ \gamma(t)$ are geodesics of the metric F, and vice versa.

Let us now generalize this statement to the case when the Killing vector field is null.

Let $(\widetilde{M}, \widetilde{g})$ be an $(n + 1)$-dimensional Lorentzian or pseudo-Riemannian manifold with a nonvanishing null Killing vector field K. Let M be a hypersurface transverse to K. Using the flow generated by K, we form a local coordinate system $(x^0, x^1, \ldots, x^n)$ around a point $p \in M$ such that $M = \{x^0 = 0\}$ and $K = \partial/\partial x^0$. Then, $\widetilde{g}$ is written in the form

$$\widetilde{g} = g_{ij}dx^i dx^j + 2\omega_i dx^i dx^0$$
$$= \begin{pmatrix} 0 & \omega_j \\ \omega_i & g_{ij} \end{pmatrix},$$

where the components g_{ij}, ω_i are functions of $(x^1, \ldots, x^n)$ and are independent of x^0 since $\widetilde{g}$ is invariant under the flow of K (Hereafter, the Latin indices $i, j, k, \ldots$ run from 1 to n and we adopt the Einstein's summation convention.) We view $\omega := \omega_i dx^i$ as a 1-form on M and g as a (perhaps degenerate) metric on M. Clearly $\omega = (K \lrcorner \widetilde{g})|_{TM}$. The *Kropina metric* F on M associated to $\widetilde{g}$ is defined by

$$(2.1) \qquad F(x,\xi) := \frac{\widetilde{g}(\xi,\xi)}{\widetilde{g}(K,\xi)} = \frac{g_{ij}\xi^i\xi^j}{\omega_l\xi^l}$$

for $\xi \in T_xM \setminus \ker\omega$. Note that F depends only on the conformal class of $\widetilde{g}$. If we consider another hypersurface M' defined by $x^0 = f(x^1, \ldots, x^n)$ and identify it with M by the flow of K, then the Kropina metric changes by adding of the exact form $-df$ and so has the same geodesics.

Now let $\pi(x^0, x^i) = (x^i)$ be the (local) projection $\widetilde{M} \to M$ along the integral curves of K. Then the correspondence between null geodesics of $\widetilde{g}$ and geodesics of F is as follows:

Theorem 2.1 (cf. [6, Theorem 7.8]). *If $\widetilde{\gamma}(t)$ is a null geodesic of $\widetilde{g}$ with $\widetilde{g}(K, \dot{\widetilde{\gamma}}(0)) \neq 0$, then $\gamma(t) := \pi(\widetilde{\gamma}(t))$ is a geodesic of F. Conversely, if $\gamma(t)$*

is a geodesic for F then there exists a null geodesic $\widetilde{\gamma}(t)$ for $\widetilde{g}$ with $\pi(\widetilde{\gamma}(t)) = \gamma(t)$ and which is uniquely determined by the choice of $\widetilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$.

Proof. We work in the local coordinate system defined above. Suppose $\widetilde{\gamma}(t) = (x^0(t), x^i(t))$ is a null geodesic of $\widetilde{g}$ with $\widetilde{g}(K, \dot{\widetilde{\gamma}}(0)) = \omega_l \dot{x}^l(0) \neq 0$. Let

$$L := g_{ij}\xi^i\xi^j + 2\omega_l\xi^l\xi^0$$

be the Lagrangian for the geodesic of $\widetilde{g}$. The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \xi^0}\right) - \frac{\partial L}{\partial x^0} = 0$$

implies that $\omega_l \dot{x}^l$ is constant, so we set $a := (\omega_l \dot{x}^l)^{-1} \in \mathbb{R}$. The nullity condition gives

$$2\dot{x}^0 = -ag_{ij}\dot{x}^i\dot{x}^j.$$

Then we compute

$$\frac{\partial L}{\partial x^k} = (\partial_k g_{ij})\dot{x}^i\dot{x}^j + 2(\partial_k \omega_l)\dot{x}^l\dot{x}^0 = (\partial_k g_{ij})\dot{x}^i\dot{x}^j - a(\partial_k \omega_l)g_{ij}\dot{x}^l\dot{x}^i\dot{x}^j,$$

$$\frac{\partial L}{\partial \xi^k} = 2g_{kj}\dot{x}^j + 2\omega_k\dot{x}^0 = 2g_{kj}\dot{x}^j - a\omega_k g_{ij}\dot{x}^i\dot{x}^j.$$

On the other hand, we have

$$\frac{\partial F}{\partial x^k} = (\omega_l\dot{x}^l)^{-1}(\partial_k g_{ij})\dot{x}^i\dot{x}^j - (\omega_l\dot{x}^l)^{-2}(\partial_k\omega_l)\dot{x}^l g_{ij}\dot{x}^i\dot{x}^j$$

$$= a(\partial_k g_{ij})\dot{x}^i\dot{x}^j - a^2(\partial_k\omega_l)g_{ij}\dot{x}^l\dot{x}^i\dot{x}^j,$$

$$\frac{\partial F}{\partial \xi^k} = 2(\omega_l\dot{x}^l)^{-1}g_{kj}\dot{x}^j - (\omega_l\dot{x}^l)^{-2}\omega_k g_{ij}\dot{x}^i\dot{x}^j$$

$$= 2ag_{kj}\dot{x}^j - a^2\omega_k g_{ij}\dot{x}^i\dot{x}^j.$$

Hence we obtain

$$\frac{\partial F}{\partial x^k} = a\frac{\partial L}{\partial x^k}, \quad \frac{\partial F}{\partial \xi^k} = a\frac{\partial L}{\partial \xi^k}.$$

Since a is constant, the Euler-Lagrange equation for L implies that for F.

Conversely, suppose $\gamma(t) = (x^i(t))$ is a geodesic of F. By definition, we have $\omega_l\dot{x}^l(0) \neq 0$ and by changing the parametrization we may assume that $\omega_l\dot{x}^l$ is constant. For any point $\widetilde{y} \in \pi^{-1}(\gamma(t))$, there exists a unique null tangent vector at $\widetilde{y}$ which projects to $\dot{\gamma}(t)$, and this assignment defines a null vector field along the fiber $\pi^{-1}(\gamma)$. Then, for each choice of $\widetilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$ the integral curve of this vector field projects to γ, and by the above computations, satisfies the Euler-Lagrange equation for L. Thus we have completed the proof. $\square$

We can also perform the inverse construction: Suppose that $F(x, \xi) = g_{ij}\xi^i\xi^j/\omega_l\xi^l$ is a Kropina metric on M such that g is non-degenerate on $\ker\omega$. Then $\widetilde{g} := g_{ij}dx^idx^j + 2\omega_idx^idx^0$ defines a pseudo-Riemannian metric

on $M \times \mathbb{R}$ with $\partial/\partial x^0$ being a null Killing vector field, and the associated Kropina metric is given by F. Thus, by Theorem 2.1, we have the following

Corollary 2.2. *Let $F(x,\xi) = g_{ij}\xi^i\xi^j/\omega_l\xi^l$ be a Kropina metric on M such that g is non-degenerate on $\ker \omega$. For any $p \in M$ and $\xi \in T_pM \backslash \ker \omega$, there exists a unique geodesic $\gamma(t)$ of F with $\gamma(0) = p$, $\dot{\gamma}(0) = \xi$.*

PROOF OF THEOREM 1.1. Apply Theorem 2.1 to the Fefferman metric (see Section 4) to obtain that chains are geodesics of the associated Kropina metric. From the construction of the Fefferman metric described in Section 4 we also see that the one-form ω forming the denominator of the Kropina metric is a contact form for the contact distribution H. $\square$

3. PROJECTIVE EQUIVALENCE OF KROPINA METRICS AND PROOF OF THEOREMS 1.2 AND 1.4

Let $F(x,\xi) = g(\xi,\xi)/\omega(\xi)$ be a Kropina metric on M. For most of this section we assume that

- $H := \ker \omega$ is non-integrable at every point;
- g is non-degenerate on H.

Our first goal is to show that a sufficiently big family of geodesics determines F up to transformations of the form $\widehat{F} = cF + \beta$, where c is a constant and β is a closed 1-form. We begin by reducing to the case where g is a pseudo-Riemannian metric.

Lemma 3.1. *In a neighborhood of any point $p \in M$, there exists a smooth function f such that $g + \omega \cdot df$ is a pseudo-Riemannian metric. If g is positive definite on H, then $g + \omega \cdot df$ is also positive definite for a certain f.*

Proof. Since $g|_H$ is non-degenerate, we have the orthogonal decomposition $T_pM = \mathbb{R}X \oplus H_p$, where X is a basis of the one-dimensional kernel of $g : T_pM \to H_p^*$. If we choose a function f so that $\ker (df)_p = H_p$ and $g(X,X) + \omega(X)df_p(X) > 0$, then $g + \omega \cdot df$ is non-degenerate near p. In particular, when $g|_H$ is positive definite, $g + \omega \cdot df$ is a Riemannian metric. $\square$

Next, we give an algebraic description of the space of the 1st and the 2nd derivatives of solutions to the unparameterized geodesic equations — i.e., solutions to the Euler-Lagrange equations for F. To this end, let J^2M denote the space of 2nd jets of curves on M. J^2M forms a fiber bundle over M with fiber $\mathbb{R}^{2n}$. A local coordinate system $(x^1, \ldots, x^n)$ around a point $p \in M$ induces fiber coordinates ξ^i, η^j, $i,j = 1, \ldots, n$ on J^2M with ξ^i standing for $\dot{x}^i$ and η^j for $\ddot{x}^j$ with $x^i(t)$ being a curve germ in M. There is a canonical projection $J^2M \to TM$ which just keeps the first order Taylor information, and so sends (x^i, ξ^j, η^k) to (x^i, ξ^j). This projection gives J^2M the structure of an affine bundle over TM. Indeed, if $(\widetilde{x}^1, \ldots, \widetilde{x}^n)$ is another

system of coordinates around p, by taking the second derivative of a curve expressed in terms of $\widetilde{x}^i$ instead of x^i we compute the affine relation

$$(3.1) \qquad \widetilde{\eta}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}(p)\eta^j + \frac{\partial^2 \widetilde{x}^i}{\partial x^j \partial x^k}(p)\xi^j \xi^k.$$

relating the fiber coordinates η and $\widetilde{\eta}$. Finally, if $c(t)$ is curve germ passing through p at $t = 0$ then $(j^2 c)(0)$ denotes its second order Taylor expansion, or "2nd jet" and is a well-defined object, independent of coordinates, whose expression in our coordinates is $\xi^i = \dot{x}^i(0)$ and $\eta^j = \ddot{x}^j(0)$.

Let

$$\mathcal{E}_p = \{(j^2 c)(0) : c \text{ a solution to the EL equations for } F \text{ having } c(0) = p\} \subset J_p^2 M.$$

We will give an algebraic description of this subvariety. Due to the homogeneity of F, this variety is given by a system of algebraic equations which are homogeneous in the velocities ξ. The Euler-Lagrange equations are

$$(3.2) \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \xi^k}\right) - \frac{\partial F}{\partial x^k} = 0.$$

The left-hand side is computed as

$$
\begin{aligned}
(3.3) \qquad & \left(\omega_l \dot{x}^l\right)^{-1}(2 g_{kj}\ddot{x}^j + 2(\partial_m g_{kj})\dot{x}^m \dot{x}^j - (\partial_k g_{ij})\dot{x}^i \dot{x}^j) \\
& + \left(\omega_l \dot{x}^l\right)^{-2}(-2 g_{ki}\omega_j \dot{x}^i \ddot{x}^j - 2\omega_k g_{ij}\dot{x}^i \ddot{x}^j \\
& \qquad - \omega_k(\partial_m g_{ij})\dot{x}^m \dot{x}^i \dot{x}^j - 2 g_{kj}(\partial_m \omega_l)\dot{x}^j \dot{x}^m \dot{x}^l \\
& \qquad - g_{ij}(\partial_m \omega_k)\dot{x}^i \dot{x}^j \dot{x}^m + g_{ij}(\partial_k \omega_l)\dot{x}^i \dot{x}^j \dot{x}^l) \\
& + \left(\omega_l \dot{x}^l\right)^{-3}\left(2 g_{il}\omega_k \omega_j \dot{x}^i \dot{x}^l \ddot{x}^j + 2 g_{ij}\omega_k(\partial_m \omega_l)\dot{x}^i \dot{x}^j \dot{x}^m \dot{x}^l\right).
\end{aligned}
$$

Replace $\dot{x}^i$, $\ddot{x}^i$ in (3.3) by ξ^i, η^i respectively, and evaluate all coefficients at p. The result is

$$(\omega_l \xi^l)^{-1} 2(A_{kj}(\xi)\eta^j - b_k(\xi)),$$

where

$$
\begin{aligned}
(3.4) \qquad & A_{kj}(\xi) = g_{kj} - (\omega_l \xi^l)^{-1}(g_{ki}\omega_j \xi^i + \omega_k g_{ij}\xi^i) + (\omega_l \xi^l)^{-2} g_{il}\omega_k \omega_j \xi^i \xi^l, \\
& 2 b_k(\xi) = -2(\partial_m g_{kj})\xi^m \xi^j + (\partial_k g_{ij})\xi^i \xi^j \\
& \qquad + (\omega_l \xi^l)^{-1}\left(\omega_k(\partial_m g_{ij})\xi^m \xi^i \xi^j + 2 g_{kj}(\partial_m \omega_l)\xi^j \xi^m \xi^l\right. \\
& \qquad\qquad \left. + g_{ij}(\partial_m \omega_k)\xi^i \xi^j \xi^m - g_{ij}(\partial_k \omega_l)\xi^i \xi^j \xi^l\right) \\
& \qquad - 2(\omega_l \xi^l)^{-2} g_{ij}\omega_k(\partial_m \omega_l)\xi^i \xi^j \xi^m \xi^l.
\end{aligned}
$$

Lemma 3.2. *If $\xi \notin H$, then $\ker(A_{kj}(\xi)) = \mathbb{R}\xi$.*

Proof. One can readily check by hand that $A_{kj}(\xi)\xi^j = 0$. Since $T_p M = H \oplus \mathbb{R}\xi$, the lemma is proved once we show that if $v \in H$ satisfies $A_{kj}v^j = 0$ then $v = 0$. So let $v \in H$ satisfy $A_{kj}v^j = 0$. Choose any other $w \in H$. A glance at the expression for A shows that $A_{kj}w^k v^j = g_{kj}w^k v^j$. We thus have that v is g-perpendicular to all of H. The non-degeneracy of g restricted to H implies that $v = 0$. $\qquad \square$

Remark 3.3. The fact that $A_{kj}(\xi)\xi^j = 0$ follows directly from the parameterization invariance of $\int F$. Indeed, whenever $\gamma(t)$ is an extremal and $\lambda(t)$ is a reparameterization of the time interval, $\gamma(\lambda(t))$ is an extremal. Differentiating, we find that if ξ^i, η^j are the 2-jets of a solution, then so is $\tilde{\xi}^i = \alpha\xi^i, \tilde{\eta}^j = \alpha^2\eta^j + \beta\xi^i$ where $\alpha = \dot{\lambda}$, $\beta = \ddot{\lambda}$. Now ξ^i, η^j satisfy $A_{kj}(\xi)\eta^j - b_k(\xi) = 0$, as does $\tilde{\xi}^i, \tilde{\eta}^j$. A bit of algebra, combined with the fact that A and b are homogeneous of degree 0, 2, in ξ yields $A_{kj}\xi^j = 0$.

Proposition 3.4. *In terms of the fiber coordinates (ξ, η) we have*

$$\mathcal{E}_p = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \omega_l\xi^l \neq 0, A_{kj}(\xi)\eta^j = b_k(\xi)\}.$$

Proof. Denote the right-hand side by $\mathcal{E}'$. We have $\mathcal{E}_p \subset \mathcal{E}'$ by the above computation of the Euler-Lagrange equation. Conversely, suppose that $(\xi, \eta) \in \mathcal{E}'$. By Corollary 2.2, there exists a geodesic $x(t)$ of F such that $x(0) = p$, $\dot{x}(0) = \xi$. Then $(\xi, \ddot{x}(0)) \in \mathcal{E}'$, so by Lemma 3.2, we have $\eta = \ddot{x}(0) + c\xi$ for some constant c. If we take a reparametrization $x'(t) = x(\varphi(t))$ such that $\dot{\varphi}(0) = 1$, $\ddot{\varphi}(0) = c$, then $(\dot{x}'(0), \ddot{x}'(0)) = (\xi, \eta)$, so we have $(\xi, \eta) \in \mathcal{E}_p$.   □

We now proceed to show that the 2nd derivatives of solutions blow up as their initial conditions approach $H = \ker\omega$. To this end, write $pr: J_p^2 M \to T_p M$ for the canonical projection and set

$$\ell_\xi := \mathcal{E}_p \cap pr^{-1}(\xi).$$

By Lemma 3.2, for $\xi \notin H$ we have that ℓ_ξ is an affine line whose direction is ξ. The line ℓ_ξ lives in $J_p^2 M$ which is diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$. We now imagine ξ varying so that $\xi \to \xi_0$ We will say that $\ell_\xi \to \infty$ as $\xi \to \xi_0$, if, as $\xi \to \xi_0$, the line ℓ_ξ leaves every compact set of $J_p^2 M$.

Lemma 3.5. *There is a Zariski dense open subset $H \backslash \mathcal{N}$ of H such that for all ξ_0 in this set we have $\ell_\xi \to \infty$ whenever $\xi \to \xi_0$.*

Before giving the proof we describe the Zariski dense subset of the lemma. According to Lemma 3.1, we may assume that g is non-degenerate. We set

$$\mathcal{N} := \{\xi \in T_p M \mid |\xi|_g^2 = 0 \text{ or } (\xi \lrcorner d\omega)|_H = 0\}.$$

Since H is non-integrable, $H \backslash \mathcal{N}$ is a Zariski open dense subset of H.

Proof. We will make use of the formula

$$\pi(v) = v - \frac{\omega(v)}{|\omega|_g^2}\omega_g$$

for the g-orthogonal projection $\pi : T_p M \to H$. (Note that $|\omega|_g^2 \neq 0$ since if $|\omega|_g^2 = 0$, we have $\omega_g := (\omega^i) \in H = \ker\omega$ and $g(\omega_g, v) = 0$ for all $v \in H$, contradicting to the assumption that $g|_H$ is non-degenerate.)

Since the assertion of the lemma is independent of coordinates, it suffices to establish the lemma in one choice of coordinate systems. We will work in the normal (=geodesic) coordinate system of g centered at p. Recall that

these coordinates are specified by the condition that $\partial_k g_{ij} = 0$ at p for all k, i, j. Let ∇ be the Levi-Civita connection of g. We define

$$(\nabla \omega)(\xi_1, \xi_2) := (\nabla_i \omega_j)\xi_1^i \xi_2^j,$$

$$(d\omega)_g(\xi)^i := \frac{1}{2}(\nabla_j \omega^i - \nabla^i \omega_j)\xi^j.$$

Then $g((d\omega)_g(\xi_1), \xi_2) = d\omega(\xi_1, \xi_2)$. In these coordinates the endomorphism $A = (A_k{}^j)$ and the vector $b = (b^i)$ are written as

(3.5)
$$A\eta = \eta - \frac{\omega(\eta)}{\omega(\xi)}\xi - \frac{g(\xi, \eta)}{\omega(\xi)}\omega_g + \frac{|\xi|_g^2 \omega(\eta)}{\omega(\xi)^2}\omega_g,$$

$$b = \frac{(\nabla\omega)(\xi, \xi)}{\omega(\xi)}\xi + \frac{|\xi|_g^2}{\omega(\xi)}(d\omega)_g(\xi) - \frac{|\xi|_g^2 (\nabla\omega)(\xi, \xi)}{\omega(\xi)^2}\omega_g.$$

Since $g(\xi, b) = 0$, we have

$$g(\xi, \pi(b)) = -\frac{\omega(\xi)\omega(b)}{|\omega|_g^2}.$$

Then, it follows that

$$A\pi(b) = \pi(b) - \frac{g(\xi, \pi(b))}{\omega(\xi)}\omega_g$$

$$= \pi(b) + \frac{\omega(b)}{|\omega|_g^2}\omega_g$$

$$= b.$$

Hence the solution space to $A(\xi)\eta = b(\xi)$ is $\pi(b(\xi)) + \mathbb{R}\xi$. In other words

$$\ell_\xi = \pi(b(\xi)) + \mathbb{R}\xi$$

as a subspace of the affine $\mathbb{R}^n$ coordinatized by η.

Now choose any Euclidean structure $\mathring{g}$ on this $\mathbb{R}^n$. There is a unique point $\eta_* \in \ell_\xi$ closest to the origin relative to this metric: $|\eta_*|_{\mathring{g}} = \text{dist}_{\mathring{g}}(0, \ell_\xi)$. We call this point the *optimal solution*. Our proof will be complete by showing that $|\eta_*|_{\mathring{g}} \to \infty$ as $\xi \to \xi_0 \in H \backslash \mathcal{N}$.

Since

$$\pi(b) = \frac{|\xi|_g^2}{\omega(\xi)}\pi((d\omega)_g(\xi)) + \frac{(\nabla\omega)(\xi, \xi)}{\omega(\xi)}\pi(\xi)$$

$$= \frac{|\xi|_g^2}{\omega(\xi)}\pi((d\omega)_g(\xi)) + \frac{(\nabla\omega)(\xi, \xi)}{\omega(\xi)}\xi - \frac{(\nabla\omega)(\xi, \xi)}{|\omega|_g^2}\omega_g,$$

we have

$$\eta_* = \pi(b) - \frac{\mathring{g}(\xi, \pi(b))}{|\xi|_{\mathring{g}}^2}\xi$$

$$(3.6) \qquad = \frac{|\xi|_g^2}{\omega(\xi)}\left(\pi(d\omega)_g(\xi) - \frac{\mathring{g}(\xi, \pi(d\omega)_g(\xi))}{|\xi|_{\mathring{g}}^2}\xi\right)$$

$$- \frac{(\nabla\omega)(\xi,\xi)}{|\omega|_g^2}\left(\omega_g - \frac{\mathring{g}(\xi,\omega_g)}{|\xi|_{\mathring{g}}^2}\xi\right).$$

The second term is bounded as $\xi \to \xi_0$. To see that the first term diverges, we will show that

$$\pi(d\omega)_g(\xi_0) - \frac{\mathring{g}(\xi_0, \pi(d\omega)_g(\xi_0))}{|\xi_0|_{\mathring{g}}^2}\xi_0 \neq 0.$$

Suppose the equality holds. Then, if

$$c := \frac{\mathring{g}(\xi_0, \pi(d\omega)_g(\xi_0))}{|\xi_0|_{\mathring{g}}^2} = 0,$$

we have $\pi(d\omega)_g(\xi_0) = 0$. Taking the inner product of $\pi(d\omega)_g(\xi_0)$ with any $v \in H$ yields $d\omega(v, \xi_0) = 0$ which contradicts the assumption $\xi_0 \notin \mathcal{N}$. On the other hand, if $c \neq 0$, we have $\pi(d\omega)_g(\xi_0) = c\xi_0$ and taking g-inner products of both sides of this equation with ξ_0 yields $d\omega(\xi_0, \xi_0) = cg(\xi_0, \xi_0)$,or $0 = |\xi_0|_g^2$ which again contradicts $\xi_0 \notin \mathcal{N}$. Thus we have $|\eta_*|_{\mathring{g}} \to \infty$. $\qquad \square$

We note that the optimal solution η_* is represented by a rational function of ξ in any coordinate system $(x^1, \ldots, x^n)$, and it is regular on $T_pM \backslash H$.

Corollary 3.6. *Under the assumptions of this section, there is no affine connection ∇ on M whose geodesics agree with the F-geodesics.*

Proof. Let Γ^i_{jk} be the Christoffel symbols of an affine connection ∇ defined near p so that the ∇-geodesic equations read $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$. Let $\mathcal{F}_p$ be the space of 2nd jets of unparameterized geodesics through p. It will suffice to show that $\mathcal{F}_p \neq \mathcal{E}_p$ where $\mathcal{E}_p$ is the space of 2nd jets of unparameterized Kropina geodesics, as before. In parallel to what we did just above, $\ell_\xi^\nabla = pr^{-1}(\xi) \cap \mathcal{F}_p$ be the affine line in J^2M_p corresponding to all the (unparameterized) ∇-geodesics whose first derivative at p is ξ. Then $\tilde{\ell}_\xi$ consists of the affine line parameterized as $\eta(\lambda) = \eta_0(\xi) + \lambda\xi$ where $\eta_0^i(\xi) = -\Gamma^i_{jk}\xi^j\xi^k$, and where the parameter λ of the line arises by reparameterizing ∇-geodesics and can be identified with the second derivative of the time reparameterization. Since $\eta_0(\xi) \to \infty$ if and only if $\xi \to \infty$ we see that the lines $\tilde{\ell}_\xi$ cannot go to infinity as $\xi \to \xi_0 \in H \backslash \mathcal{N}$. It follows that these two sets of lines disagree in some neighborhood of H and hence that $\mathcal{E}_p \neq \mathcal{F}_p$. $\qquad \square$

Proposition 3.7. *Let $F = g/\omega$, $\widehat{F} = \widehat{g}/\widehat{\omega}$ be two Kropina metrics on M such that*

- $\ker \omega$ *is non-integrable at every point;*
- $g, \widehat{g}$ *are non-degenerate on* $\ker \omega$, $\ker \widehat{\omega}$ *respectively.*

If F and $\widehat{F}$ are projectively equivalent, then $\ker \omega = \ker \widehat{\omega}$. In particular, $\ker \widehat{\omega}$ is also non-integrable at every point.

Proof. By Lemma 3.1, we may assume that $g, \widehat{g}$ are pseudo-Riemannian. Suppose that there exists a point $p \in M$ with $\ker \omega_p \neq \ker \widehat{\omega}_p$. We consider the subsets $\mathcal{E}, \widehat{\mathcal{E}} \subset \mathcal{A}$ defined from solutions to the Euler-Lagrange equations of $F, \widehat{F}$. Choose a Euclidean structure $\mathring{g}$ on T_pM and a coordinate system $(x^1, \ldots, x^n)$, and let $\eta_*, \widehat{\eta}_*$ be the optimal solutions. By the assumption, there is a nonempty open subset $U \subset \mathbb{R}^n$ such that $\mathcal{E} \cap (U \times \mathbb{R}^n) = \widehat{\mathcal{E}} \cap (U \times \mathbb{R}^n)$. Since $\eta_*(\xi) = \widehat{\eta}_*(\xi)$ on U and both are rational functions of ξ, they coincide globally. In particular, their poles must coincide. But we know that the closure of the locus of poles of η_* is $\ker \omega$. Then, the closure of the locus of poles of $\widehat{\eta}_*$ is $\ker \omega$. Hence the two kernels are equal, and in particular the distribution $\ker \widehat{\omega}$ is also non-integrable. $\square$

Next we will show that g and $\widehat{g}$ are conformally related on $\ker \omega = \ker \widehat{\omega}$. We consider $\eta_*, \widehat{\eta}_*$ in normal coordinates of $g, \widehat{g}$, and identify them with elements of T_pM. Then, by (3.1) we have

$$\widehat{\eta}_* = \eta_* + O(1)$$

as $\xi \to \xi_0 \in \ker \omega \backslash (\mathcal{N} \cup \widehat{\mathcal{N}})$. It follows from (3.6) that

$$
\begin{aligned}
(3.7) \quad & |\xi_0|_{\widehat{g}}^2 \widehat{\pi}(d\omega)_{\widehat{g}}(\xi_0) - \frac{|\xi_0|_{\widehat{g}}^2}{|\xi_0|_{\mathring{g}}^2} \widehat{g}(\xi_0, \widehat{\pi}(d\omega)_{\widehat{g}}(\xi_0))\xi_0 \\
& = |\xi_0|_g^2 \pi(d\omega)_g(\xi_0) - \frac{|\xi_0|_g^2}{|\xi_0|_{\mathring{g}}^2} \mathring{g}(\xi_0, \pi(d\omega)_g(\xi_0))\xi_0.
\end{aligned}
$$

Let us denote the restrictions of $\mathring{g}, g, \widehat{g}, d\omega$ to $\ker \omega$ respectively by

$$\mathring{h}_{\alpha\beta}, \quad h_{\alpha\beta}, \quad \widehat{h}_{\alpha\beta}, \quad B_{\alpha\beta},$$

where Greek indices run from 1 to $n-1$. Note that $B_{\alpha\beta}$ is skew symmetric and $(B_{\alpha\beta}) \neq 0$ by non-integrability. Since (3.7) holds for any $\xi_0 \in \ker \omega \backslash (\mathcal{N} \cup \widehat{\mathcal{N}})$, we have

$$
\begin{aligned}
(3.8) \quad & \mathring{h}_{(\sigma\rho} \widehat{h}_{\alpha\beta} \widehat{h}^{\mu\nu} B_{\gamma)\nu} - \widehat{h}_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} \widehat{h}^{\nu\tau} B_{\beta|\tau|} \delta_{\gamma)}{}^{\mu} \\
& = \mathring{h}_{(\sigma\rho} h_{\alpha\beta} h^{\mu\nu} B_{\gamma)\nu} - h_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} h^{\nu\tau} B_{\beta|\tau|} \delta_{\gamma)}{}^{\mu},
\end{aligned}
$$

where $(\cdots)$ denotes the symmetrization over the indices $\sigma, \rho, \alpha, \beta, \gamma$. Taking the trace of μ, γ gives

$$-(n+1)\widehat{h}_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} \widehat{h}^{\nu\tau} B_{\beta)\tau} = -(n+1)h_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} h^{\nu\tau} B_{\beta)\tau}.$$

Thus we have

$$\widehat{h}_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} \widehat{h}^{\nu\tau} B_{\beta|\tau|} \delta_{\gamma)}{}^{\mu} = h_{(\sigma\rho} \mathring{h}_{\alpha|\nu|} h^{\nu\tau} B_{\beta|\tau|} \delta_{\gamma)}{}^{\mu}$$

and hence, by (3.8),

$$(3.9) \qquad (\widehat{h}_{\alpha\beta}\xi^\alpha\xi^\beta)\widehat{h}^{\mu\nu}B_{\gamma\nu}\xi^\gamma = (h_{\alpha\beta}\xi^\alpha\xi^\beta)h^{\mu\nu}B_{\gamma\nu}\xi^\gamma$$

for all $\xi \in \ker\omega$.

First, we consider the case $n \geq 4$. Looking at a nonzero row in the matrix $\widehat{h}^{\mu\nu}B_{\gamma\nu}$, we have

$$(\widehat{h}_{\alpha\beta}\xi^\alpha\xi^\beta)a_\gamma\xi^\gamma = (h_{\alpha\beta}\xi^\alpha\xi^\beta)b_\gamma\xi^\gamma$$

for some $(a_\gamma) \neq 0$ and $(b_\gamma) \neq 0$. We regard this as an equality in the polynomial ring $\mathbb{R}[\xi^1, \ldots, \xi^{n-1}]$. Since $a_\gamma\xi^\gamma$ is irreducible, it divides either $h_{\alpha\beta}\xi^\alpha\xi^\beta$ or $b_\gamma\xi^\gamma$. In the former case, we have

$$h_{\alpha\beta} = a_{(\alpha}c_{\beta)}$$

for some (c_γ). Since $\dim\ker\omega \geq 3$, there exists $\xi_0 \in \ker\omega\backslash\{0\}$ such that $a_\gamma\xi_0^\gamma = c_\gamma\xi_0^\gamma = 0$, and hence $h_{\alpha\beta}\xi_0^\beta = 0$. This is contradiction since $h_{\alpha\beta}$ is non-degenerate. In the latter case, we have $b_\gamma = ca_\gamma$ for some constant c and $\widehat{h}_{\alpha\beta} = ch_{\alpha\beta}$.

Next, let $n = 3$. In this case, since $\dim\ker\omega = 2$, non-integrability condition is equivalent to the contact condition and B is non-degenerate. Then, the equation (3.9) implies that $(\widehat{h}^{-1}B)(h^{-1}B)^{-1}$ is a scalar multiple of identity, since it is represented by a diagonal matrix in any basis of $\ker\omega$. Thus $\widehat{h}$ is a multiple of h.

As a result

$$\widehat{g} = \alpha g + 2\beta \cdot \omega$$

with a function α and a 1-form β.

Thus we complete the proof of Theorem 1.2 if we prove the following proposition:

Proposition 3.8. *Let $F = g/\omega, \widehat{F} = \widehat{g}/\omega$ be two projectively equivalent Kropina metrics on M. Suppose that there exist a function α and a 1-form β such that $\widehat{g} = \alpha g + 2\beta \cdot \omega$. Then α is constant and β is closed.*

Proof. We take local coordinates around a point $p \in M$ and consider the defining equations of $\mathcal{E}$ and $\widehat{\mathcal{E}}$. Since $\widehat{g}_{ij} = \alpha g_{ij} + \beta_i\omega_j + \beta_j\omega_i$, a direct computation using (3.4) gives

$$2(\widehat{A}_{kj}\eta^j - \widehat{b}_k) - 2\alpha(A_{kj}\eta^j - b_k) = 2\omega_l\xi^l(\partial_m\beta_k - \partial_k\beta_m)\xi^m$$
$$+ 2(\partial_m\alpha)g_{kj}\xi^m\xi^j - (\partial_k\alpha)g_{ij}\xi^i\xi^j$$
$$- (\omega_l\xi^l)^{-1}\omega_k(\partial_m\alpha)g_{ij}\xi^m\xi^i\xi^j.$$

By the assumption, the equations $A\eta = b$ and $\widehat{A}\eta = \widehat{b}$ have the same solutions, so the right-hand side of the above equation must vanish for any ξ. Multiplying it by $\omega_l\xi^l$ and taking the limit $\xi \to \xi_0 \in \ker\omega$, we have

$$\omega_k|\xi_0|_g^2\,\xi_0^m(\partial_m\alpha) = 0.$$

Thus, the derivatives of α in the directions of $\ker \omega$ vanish at any point. Since $\ker \omega$ is non-integrable, this implies that α is constant. Then it follows that $\partial_m \beta_k - \partial_k \beta_m = 0$, so β is closed. $\qquad\square$

Finally we consider the case where ω is integrable, and prove Theorem 1.4. In this case, we may assume that ω is closed by multiplying g and ω by a function. We note that in the non-integrable case, geodesics of the Kropina metric are never given by geodesics of an affine connection due to the singularity described in Lemma 3.5. In contrast, we have the following theorem:

Theorem 3.9. *Let g be a pseudo-Riemannian metric and ω a closed 1-form on M. Let ∇ be the affine connection whose Christoffel symbols are given by*

$$\Gamma_{ij}{}^k = \Gamma_{ij}^{g\,k} + \frac{1}{|\omega|_g^2}(\nabla_i^g \omega_j)\omega^k,$$

where ∇^g is the Levi-Civita connection of g and $\Gamma_{ij}^{g\,k}$ are its Christoffel symbols. Then, any geodesic of the Kropina metric $F = g/\omega$ is a geodesic of ∇.

Proof. Let $\gamma(t)$ be a geodesic of ∇. We will show that γ satisfies the Euler-Lagrange equation (3.2) for F. Take an arbitrary point $p \in \gamma$. For simplicity, we let $p = \gamma(0)$. Take normal coordinates $(x^1, \dots, x^n)$ of g centered at p. Then, a curve $(x^i(t))$ satisfies (3.2) at p if and only if $(\xi, \eta) = (\dot{x}^i(0), \ddot{x}^i(0))$ satisfies $A\eta = b$ with A, b given by (3.5). We recall that one of the solutions is given by $\eta = \pi(b)$, and using $d\omega = 0$, we have

$$\pi(b) = \frac{(\nabla^g \omega)(\xi, \xi)}{\omega(\xi)}\xi - \frac{(\nabla^g \omega)(\xi, \xi)}{|\omega|_g^2}\omega_g.$$

Since $\xi \in \ker A$,

$$-\frac{(\nabla^g \omega)(\xi, \xi)}{|\omega|_g^2}\omega_g$$

is also a solution. On the other hand, the geodesic equation for γ implies

$$\ddot{\gamma}(0) + \frac{(\nabla^g \omega)(\dot{\gamma}(0), \dot{\gamma}(0))}{|\omega|_g^2}\omega_g = 0.$$

Hence $\gamma(t)$ solves (3.2). $\qquad\square$

4. Kropina metric for chains on CR manifolds

4.1. CR manifolds. Let M be a C^∞-manifold of dimension $2n+1$. A *CR structure* (H, J) is a corank-1 distribution $H \subset TM$ together with a complex structure $J \in \Gamma(\mathrm{End}(H))$. We assume that the CR structure satisfies the integrability condition that $\Gamma(T^{1,0}M)$ is closed under Lie bracket, where

$T^{1,0}M \subset \mathbb{C}H$ is the eigenspace of J corresponding to the eigenvalue $i = \sqrt{-1}$. For a choice of a 1-form θ with $\ker \theta = H$, we define the *Levi form* by

$$h_\theta(X, Y) := d\theta(X, JY), \quad X, Y \in H.$$

We say the CR structure is *non-degenerate* if its Levi form is non-degenerate, and we assume non-degeneracy hereafter. Then H becomes a contact distribution.

The *Reeb vector field* of θ is the vector field T on M characterized by the two conditions:

$$\theta(T) = 1, \quad T \lrcorner \, d\theta = 0.$$

Any local frame for $\mathbb{C}TM$ of the form $\{T, Z_\alpha, Z_{\overline{\alpha}} := \overline{Z_\alpha}\}$ where $\{Z_\alpha\}_{1 \leq \alpha \leq n}$ is a local frame for $T^{1,0}M$ is called an *admissible frame*. The dual coframe $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ is called an *admissible coframe*. Then we have

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \quad \text{with } h_{\alpha\overline{\beta}} := h_\theta(Z_\alpha, Z_{\overline{\beta}}).$$

Since θ is contact the matrix of the Levi form $h_{\alpha\overline{\beta}}$ is invertible and we can use its inverse $h^{\alpha\overline{\beta}}$ to raise indices. If we rescale the contact form as $\widehat{\theta} = e^\Upsilon \theta$ with $\Upsilon \in C^\infty(M)$, the Reeb vector field transforms as

$$e^\Upsilon \widehat{T} = T - i\Upsilon^\alpha Z_\alpha + i\Upsilon^{\overline{\alpha}} Z_{\overline{\alpha}},$$

and accordingly the admissible coframe $\{\widehat{\theta}, \widehat{\theta^\alpha}, \widehat{\theta^{\overline{\alpha}}}\}$ dual to $\{\widehat{T}, Z_\alpha, Z_{\overline{\alpha}}\}$ satisfies

(4.1) $$\widehat{\theta^\alpha} = \theta^\alpha + i\Upsilon^\alpha \theta.$$

A contact form θ defines a canonical linear connection ∇ on TM, called the *Tanaka-Webster* connection. This connection preserves $T^{1,0}M$, satisfies $\nabla T = 0$, $\nabla h_\theta = 0$, and the structure equation

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + A_{\overline{\beta}}{}^\alpha \theta \wedge \theta^{\overline{\beta}},$$

where $\omega_\beta{}^\alpha$ is the connection 1-form, and $A_{\overline{\beta}}{}^\alpha$ is a tensor called *Tanaka-Webster torsion*. Taking the trace of the curvature form $d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha$, we obtain the *Tanaka-Webster Ricci tensor* $R_{\beta\overline{\gamma}}$:

(4.2) $$d\omega_\alpha{}^\alpha = R_{\beta\overline{\gamma}}\theta^\beta \wedge \theta^{\overline{\gamma}} + (\nabla^\alpha A_{\alpha\beta})\theta^\beta \wedge \theta - (\nabla^{\overline{\alpha}} A_{\overline{\gamma}\overline{\alpha}})\theta^{\overline{\gamma}} \wedge \theta.$$

The *Tanaka-Webster scalar curvature* is defined by $R := R_\beta{}^\beta$. There are transformation formulas for these quantities under rescaling $\widehat{\theta} = e^\Upsilon \theta$. For example, R transforms as

$$e^\Upsilon \widehat{R} = R + (n+1)\Delta_b\Upsilon - n(n+1)\Upsilon_\alpha\Upsilon^\alpha,$$

where $\Delta_b := -\nabla_\alpha\nabla^\alpha - \nabla^\alpha\nabla_\alpha$; see e.g., [19].

4.2. **Fefferman's metric and chains.** We will introduce the Fefferman metric by following [19]. The *CR canonical bundle* is the CR version of the canonical line bundle of complex geometry. It is the complex line bundle over M defined by

$$K_M := \{\zeta \in \wedge^{n+1}\mathbb{C}T^*M \mid \overline{Z}\lrcorner\zeta = 0 \text{ for any } Z \in T^{1,0}M\}.$$

Then the *Fefferman space* is defined to be the the circle bundle

$$\mathcal{C} := K_M^*/\mathbb{R}_+,$$

where $K_M^* = K_M \setminus \{\text{zero section}\}$. Fix a contact form θ. Each choice of admissible coframe $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ yields a local section $\zeta_0 = [\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n]$ for $\mathcal{C}$. (Here [.] means the equivalence class under $\mathbb{R}_+$) The corresponding fiber coordinate $s \in \mathbb{R}/2\pi\mathbb{Z}$ is defined by writing $\zeta = e^{is}\zeta_0 \in \mathcal{C}$.

Let $\omega_\alpha{}^\beta$ be the Tanaka-Webster connection 1-form with respect to $\{\theta^\alpha\}$. One can see that

$$\sigma := \frac{1}{n+2}\left(ds - \operatorname{Im}\omega_\alpha{}^\alpha - \frac{R}{2(n+1)}\theta\right)$$

is invariant under changes of $\{\theta^\alpha\}$ and defines a global 1-form on $\mathcal{C}$. For another choice of contact form $\widehat{\theta} = e^\Upsilon\theta$, we have

(4.3) $$\widehat{\sigma} = \sigma + \frac{i}{2}(\Upsilon_\alpha\theta^\alpha - \Upsilon_{\overline{\alpha}}\theta^{\overline{\alpha}}) - \frac{1}{2}\Upsilon_\alpha\Upsilon^\alpha\theta.$$

We define the *Fefferman metric* on $\mathcal{C}$ as the pseudo-Riemannian (Lorentzian if the Levi form $h_{\alpha\overline{\beta}}$ is positive definite) metric

$$G := h_{\alpha\overline{\beta}}\theta^\alpha \cdot \theta^{\overline{\beta}} + 2\theta \cdot \sigma.$$

By using the transformation formulas (4.1), (4.3), one has

$$\widehat{G} = e^\Upsilon G$$

for $\widehat{\theta} = e^\Upsilon\theta$. Thus, the conformal class of the Fefferman metric is independent of θ. We set

$$K := (n+2)S,$$

where S is the infinitesimal generator of the circle action on $\mathcal{C}$. K is a null Killing vector field on $\mathcal{C}$ satisfying $K\lrcorner G = \theta$.

Definition 4.1. A *chain* is a curve $\gamma(t)$ on M which is the projection of a null geodesic $\widetilde{\gamma}(t)$ for the Fefferman metric G which satisfies $G(K, \dot{\widetilde{\gamma}}(0)) \neq 0$.

Note that $\theta(\dot{\gamma}(t)) \neq 0$ follows from the definition. Since null geodesics are conformally invariant, chains are CR invariant curves on M.

4.3. **The Kropina metric for Fefferman's metric.** We apply our construction of the Kropina metric to the Fefferman metric. Let $\zeta_0 = \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$ be the section of $\mathcal{C}$ on an open subset $U \subset M$ defined by an admissible coframe $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$. Identify U with the hypersurface $\{s = 0\} \subset \mathcal{C}$ transverse to K. From definition (2.1), the corresponding Kropina metric is given by

$$(4.4) \qquad F_{\theta, \zeta_0} = \frac{h_{\alpha\overline{\beta}}\, \theta^\alpha \cdot \theta^{\overline{\beta}}}{\theta} - \frac{2}{n+2}\left(\mathrm{Im}\,\omega_\alpha{}^\alpha + \frac{R}{2(n+1)}\theta\right).$$

By Theorem 2.1, we have

Theorem 4.2. *Chains on a non-degenerate CR manifold M are locally the geodesics of the Kropina metric F_{θ, ζ_0} defined by (4.4).*

If we consider the corresponding admissible coframe (4.1) for rescaled contact form $\widehat{\theta} = e^\Upsilon \theta$, we see that $\widehat{\zeta}_0 = \zeta_0$. Thus, the hypersurface does not change and it follows that

$$F_{\widehat{\theta}, \widehat{\zeta}_0} = F_{\theta, \zeta_0}.$$

On the other hand, if we take another admissible coframe $\theta'^\alpha = \theta^\beta V_\beta{}^\alpha$ for the same θ, we have $\zeta_0' = |\det(V_\beta{}^\alpha)|^{-1} \det(V_\beta{}^\alpha)\zeta_0$ and

$$F_{\theta, \zeta_0'} = F_{\theta, \zeta_0} + \frac{1}{n+2}d\,\mathrm{Im}\big(\log \det(V_\beta{}^\alpha)\big).$$

The change corresponds to the fact that we have different hypersurfaces in $\mathcal{C}$. This transformation formula implies that we can define a global Kropina metric for chains if $c_1(T^{1,0}M) = 0$ in $H^2(M; \mathbb{R})$. In particular, if M admits a pseudo-Einstein contact form, we have a global Kropina metric, which is described as follows.

Recall that a contact form θ is called *pseudo-Einstein* if the associated Tanaka-Webster connection satisfies

$$(4.5) \qquad R_{\alpha\overline{\beta}} = \frac{1}{n}Rh_{\alpha\overline{\beta}}, \quad \nabla_\alpha R = in\nabla^\beta A_{\alpha\beta}.$$

When $n \geq 2$, the first equation implies the second, and when $n = 1$ the first equation always holds. If θ is pseudo-Einstein, a new contact form $\widehat{\theta} = e^\Upsilon \theta$ is pseudo-Einstein if and only if Υ is CR pluriharmonic, namely,

$$\nabla_\alpha \nabla_{\overline{\beta}} \Upsilon = \frac{1}{n}(\nabla_\gamma \nabla^\gamma \Upsilon)h_{\alpha\overline{\beta}}, \quad \nabla_\alpha \nabla_\beta \nabla^\beta \Upsilon = -in A_{\alpha\beta}\Upsilon^\beta.$$

Again, when $n \geq 2$ the first equation implies the second, and when $n = 1$ the first equation always holds. When M is embedded in $\mathbb{C}^{n+1}$, Fefferman's defining function ρ, which gives an approximate solution to the complex Monge-Ampère equation [11], defines a pseudo-Einstein contact form $\theta = (\mathrm{Im}\,\partial\rho)|_{TM}$.

Lemma 4.3 ([20, Lemma 4.1]). *Let θ be a pseudo-Einstein contact form. Then, in a neighborhood of any point on M, there exists an admissible coframe such that $\omega_\alpha{}^\alpha = -(i/n)R\,\theta$.*

Proof. By (4.2) and (4.5), we have $d(\omega_\alpha{}^\alpha + (i/n)R\,\theta) = 0$ for any admissible coframe. Then, using the transformation formula $\omega'_\alpha{}^\alpha = \omega_\alpha{}^\alpha - d\log\det(V_\beta{}^\alpha)$ for the change of coframe $\theta'^\alpha = \theta^\beta V_\beta{}^\alpha$, we can construct a coframe with $\omega_\alpha{}^\alpha + (i/n)R\,\theta = 0$. $\square$

We take a local section ζ of $\mathcal{C}$ defined via a coframe given in Lemma 4.3 around each point on M. Then the set $\{F_{\theta,\zeta}\}$ defines a global Kropina metric

$$(4.6) \qquad F_\theta = \frac{h_{\alpha\bar\beta}\,\theta^\alpha \cdot \theta^{\bar\beta}}{\theta} + \frac{R}{n(n+1)}\theta.$$

Note that the right-hand side is now defined invariantly for any admissible coframe $\{\theta^\alpha\}$. If $\widehat\theta = e^\Upsilon\theta$ is another pseudo-Einstein contact form, we have

$$F_{\widehat\theta} = F_\theta - i(\Upsilon_\alpha\theta^\alpha - \Upsilon_{\bar\alpha}\theta^{\bar\alpha}) + \frac{1}{n}(\Delta_b\Upsilon)\theta.$$

The CR pluriharmonicity of Υ implies that the 1-form in the right-hand side is closed ([13, Lemma 3.2]). When $\theta = (\operatorname{Im}\partial\rho)|_{TM}$ with Fefferman's defining function ρ, we can also express the metric as $F_\theta = (\partial\bar\partial\rho)/\theta$, where we regard $\partial\bar\partial\rho$ as a symmetric 2-tensor and restrict it to M.

By the non-degeneracy of the CR structure, the Kropina metric F_{θ,ζ_0} satisfies the assumption of Theorem 1.2. As an immediate consequence of Corollary 3.6, we have the following theorem:

Theorem 4.4 ([4, Theorem 5.3]). *There is no affine connection on M such that sufficiently big set of its geodesics are chains of a certain CR structure.*

We can also prove that a sufficiently big family of chains determines the CR structure up to conjugacy.

Proof of Corollary 1.3

By Theorem 1.2, the contact distributions coincide. Let θ be a contact form for $H = \widehat H$. Then, the Kropina metrics $F, \widehat F$ defined by (4.4) are related as $\widehat F = cF + \beta$ with a constant c and a closed 1-form β. If we write $F = g/\theta, \widehat F = \widehat g/\theta$, we have

$$\widehat g = cg + \beta \cdot \theta$$

on $TM\backslash H$. By continuity, this also holds on H, so we obtain

$$d\theta(X, \widehat JY) = c\,d\theta(X, JY), \quad X, Y \in H,$$

which implies $\widehat J = cJ$. Since J and $\widehat J$ are complex structures, we have $c = \pm 1$. $\square$

4.4. Pseudo-Einstein contact forms for the Burns-Shnider example.
Corollary 1.6 says that on a connected, compact strictly pseudoconvex CR manifold which admits a pseudo-Einstein contact form with positive Tanaka-Webster scalar curvature, any two points can be joined by a chain. As we mentioned in the introduction, in the Burns-Shnider's example of a compact, spherical, strictly pseudoconvex CR manifold not every two points can be joined by a chain. It follows that the Tanaka-Webster scalar curvature with respect to the canonically constructed pseudo-Einstein contact form fails to be strictly positive.

Let us examine this contact form to see how the positivity fails. We fix a real number $1 \neq r > 0$ and define an action of $\mathbb{Z}$ on the Heisenberg group $H = \mathbb{C}^n \times \mathbb{R}$ by $m \cdot (z, t) := (r^m z, r^{2m} t)$. The example is given by the compact quotient $S^{2n} \times S^1_{(r)} := (H \backslash \{(0,0)\})/\mathbb{Z}$. Let

$$\theta_0 := dt + i \sum_{\alpha=1}^{n} (z^\alpha dz^{\overline{\alpha}} - z^{\overline{\alpha}} dz^\alpha)$$

be the standard contact form on H. We set $\theta := \rho^{-2}\theta_0$, where $\rho := (|z|^4 + t^2)^{1/4}$ is the Heisenberg norm. Then since $\log \rho$ is CR pluriharmonic, θ is pseudo-Einstein, and is invariant under the action of $\mathbb{Z}$, so descends to a pseuo-Einstein contact form on $S^{2n} \times S^1_{(r)}$. We set

$$Z_\alpha := \frac{\partial}{\partial z^\alpha} + i z^{\overline{\alpha}} \frac{\partial}{\partial t}$$

so that $\{Z_\alpha\}$ is a frame for $T^{1,0}H$. The Levi form for θ_0 with respect to this frame is given by $2\delta_{\alpha\overline{\beta}}$. By using the transformation formula in [20], one computes the Tanaka-Webster scalar curvature for θ as

$$R = (n+1)\rho^2 \delta^{\alpha\overline{\beta}} \big((Z_\alpha Z_{\overline{\beta}} + Z_{\overline{\beta}} Z_\alpha) \log \rho - 2n(Z_\alpha \log \rho)(Z_{\overline{\beta}} \log \rho) \big)$$

$$= \frac{n(n+1)|z|^2}{2\rho^2}.$$

This is nonnegative, but vanishes on the circle $\{(0,t) \mid t \in \mathbb{R}^*\}/\mathbb{Z}$. Thus, the 2-tensor g in the associated Kropina metric $F = g/\theta$ (see (4.6)) is not globally positive definite.

On the other hand, observe that $S^{2n} \times S^1_{(r)}$ is spherical and has positive CR Yamabe constant $\lambda = \lambda(n,r)$ less than the one for the standard CR sphere of dimension $2n+1$. By a theorem of Jerison and Lee ([16]), we can find a contact form (called the Yamabe contact form) with Tanaka-Webster scalar curvature equal to λ. From Corollary 1.6 and the chain nonconnectivity of $S^{2n} \times S^1_{(r)}$, this Yamabe contact form cannot be pseudo-Einstein.

5. Indicatrix and Proof of Theorem 1.5

Following the standard usage of Finsler geometry, the *indicatrix* of the Kropina metric $F = g/\omega$ of M at the point $x \in M$ is the closure of the locus

$\{\xi \in T_xM : F(x, \xi) = 1\}$. As we will see shortly, the indicatrix is a conic passing through the zero vector.

Figure 1 shows the relation between the indicatrix and the lightcone of the corresponding "Fefferman metric" $\tilde{g} = g + 2\omega dx^0$. This lightcone at a point $\tilde{x} \in \widetilde{M}$ over x is the quadric $\tilde{g}_{\tilde{x}}(v, v) = 0$ within $T_{\tilde{x}}\widetilde{M}$. (Since K is Killing, this quadric is independent of choice of lift $\tilde{x}$ of x.) Projectivize the light cone to obtain a projective quadric within $\mathbb{P}(T_{\tilde{x}}\widetilde{M})$. View the projectivized quadric in the affine chart $dx^0 = -1/2$, so that its equation becomes $\tilde{g} = g - \omega = 0$ which is the equation for the indicatrix of the Kropina metric.
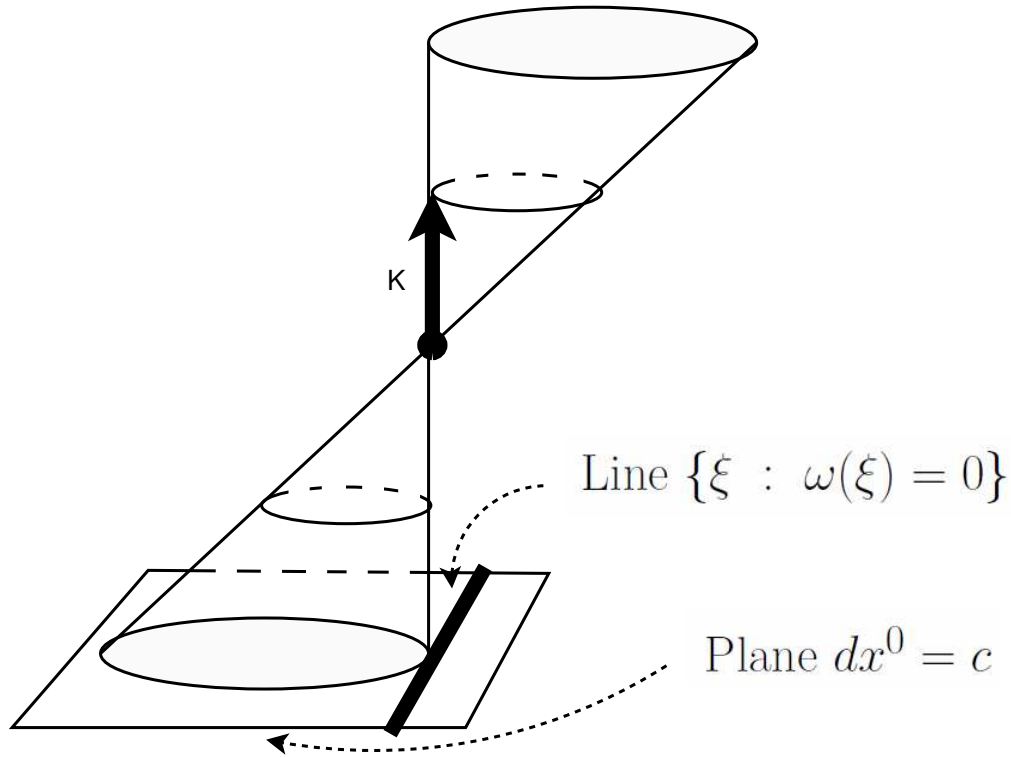

FIGURE 1. The indicatrix corresponds to the lightcone of the Fefferman metric. The fact that 0 lies on the indicatrix corresponds to the fact that the Killing vector K is lightlike.

Direct calculations show that the indicatrix at x is given by

$$g_x(v - W(x), v - W(x)) = g_x(W(x), W(x)),$$

where $W(x)$ is the half of the vector field obtained by raising the indices of ω using g:

$$W^i = \frac{1}{2}g^{ij}\omega_j.$$

If g is positive definite we see that the indicatrix is the g-sphere of radius $|W(x)| = \sqrt{g_x(W(x), W(x))}$ centered at $W(x)$ within $T_x M$.

Let us begin the proof of Theorem 1.5. We assume that g/ω is a Kropina metric on a connected M with g positive definite. We assume that $\omega \wedge d\omega \neq 0$ at every point p of some connected everywhere dense subset of M.

We say that an absolutely continuous curve c on M is *admissible* if $F(c(t), c'(t)) \geq 0$ for almost all t. We allow $F(c(t), c'(t)) = \infty$. The *length* of an admissible curve c is given by $L(c) = \int_c F(c, c')$. This length can be infinity. The length is invariant under orientation-preserving re-parameterizations of the curve.

Define the *distance* $d(p, q)$ of two points $p, q \in M$ to be the infimum of lengths of all admissible curves connecting p to q. In view of the Chow-Rashevsky theorem (see [10] or [12, §1]), the condition $\omega \wedge d\omega \neq 0$ on a connected everywhere dense subset implies that any point p can be joined to any point q by a regular smooth curve such that $0 < F(c(t), c'(t)) \neq \infty$ for all t. This guarantees that the distance function is well defined and finite. This distance clearly satisfies the triangle inequality. Due to the compactness of the indicatrix, the distance is positive for $p \neq q$. However, this distance is not symmetric in p and q.

By the r−*ball* $B_r(p)$ around p we understand the set $B_r(p) := \{p \in M : d(p, q) \leq r\}$. For any point p and for small positive r the ball $B_r(p)$ is compact. By a standard argument (see e.g. [2, §2.5] or [1, §6.6]), if $B_r(p)$ is compact, then for any point q in $B_r(p)$ we have the existence of an arc-length parameterized minimal geodesic γ connecting p to q. By definition, this is a curve γ such that for any two times $t_1 \leq t_2$ in its domain we have $t_2 - t_1 = d(\gamma(t_1), \gamma(t_2))$. This condition implies that the length of γ is the distance from p to q and that for any point $\gamma(t) \neq q$ we have $d(p, \gamma(t)) < d(p, q)$.

Note that though [2] generally considers symmetric distance functions (when $d(p, q) = d(q, p)$), symmetry is not used in [2, §2.5] so the existence of an arc-length parameterized minimal geodesic is also established in our situation.

If we knew that arc-length parameterized minimal geodesics were Kropina geodesics as defined in the introduction, we would have proven (A) of the theorem already. But we do not know this yet. According to our original definition in the introduction, a Kropina geodesic must be smooth and satisfy $\omega(\gamma') \neq 0$ everywhere, while an arc-length parameterized minimal geodesic may have points at which γ' is not defined or $\omega(\gamma') = 0$. Clearly, every arc-length parameterized minimal geodesic such that $\gamma'(t)$ exists and satisfies $\omega(\gamma'(t)) \neq 0$ for all t is a Kropina geodesic along which $F(\gamma') \equiv 1 > 0$.

In order to prove that any arc-length parameterized minimal geodesic γ is a Kropina geodesic, observe that for any $\delta > 0$ we can find a smooth (but irreversible: $\tilde{F}(x, -v) \neq \tilde{F}(x, v)$) Finsler metric $\tilde{F}$ such that $\tilde{F} = 1$ agrees
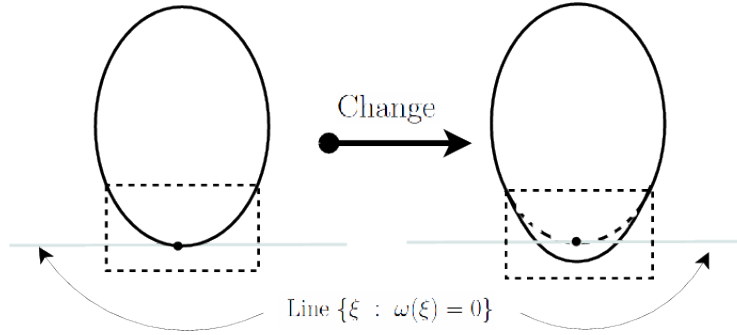
FIGURE 2. Change of the indicatrix of a Kropina metric such that the result is the indicatrix of a Finsler metric. The change is done only in a small box around zero so the Finsler metric coincides with the Kropina metric on vectors with $F(v) = 1$ and $\omega(v) > \delta$ for some small $\delta > 0$.

with F on the locus $\omega(v) > \delta$, and such that $\tilde{F} \leq F$ whenever $\omega(v) > 0$, see figure 2.

By the standard "Gauss-lemma" type theorem of Finsler geometry (see e.g. [1, §6.3]), we have that for any $v \in T_x M$ with $\tilde{F}(v) = 1$ there is a unique $\tilde{F}$-geodesic $\tilde{\gamma}$ through x, tangent to v at $t = 0$, and such that the parameter t is the $\tilde{F}$-arclength. This curve is smooth and for all sufficiently small ε its restriction to $[0, \varepsilon]$ is the unique $\tilde{F}$-shortest curve between x and its endpoint $\tilde{\gamma}(\varepsilon)$. Take ε_0 small enough so that $\omega(\tilde{\gamma}'(t)) > \delta$ for all $t \in [0, \varepsilon_0)$. Since F and $\tilde{F}$ agree on $\{F = 1, \omega > \delta\}$, each curve $\tilde{\gamma}|_{[0,\varepsilon_0)}$ is also the unique arc-length parameterized minimal F-geodesic joining its endpoints. (Use $\tilde{F} \leq F$ and argue by contradiction.) Observe that ε_0 can be universally bounded away from zero by a constant which depends on δ, $\omega(v)$, and the first and the second derivatives of ω and g in some neighborhood of x.

Now consider an arc-length parameterized minimal geodesic γ for F. We call t from its domain of definition *regular*, if $\gamma'(t)$ exists and $\omega(\gamma'(t)) > 0$. By the paragraph above, the regular times form an open set of the interval of definition of the curve. The *singular* times, being by definition the complement of the regular times, form a closed set of measure zero.

The key step in the proof of Theorem 1.5 is the following Lemma:

Lemma 5.1. *Let $c : [0, T] \to M$ be an arc-length parameterized minimal geodesic of F such that all $t \in [0, T)$ are regular. Then, $t = T$ is also regular, that is $\gamma'(T)$ exists and $\omega(\gamma'(T)) > 0$.*

Proof. As in Section 3, for any $t \in [0, T)$ we set $\xi := \gamma'(t)$, $\eta := \gamma''(t)$. The Euler-Lagrange equation for F in normal geodesic coordinates of g around a point $\gamma(t)$ is calculated in Section 3 and is equivalent to the system of

equations given by $A\eta = b$ with

$$A\eta = \eta - \frac{\omega(\eta)}{\omega(\xi)}\xi - \frac{g(\xi,\eta)}{\omega(\xi)}\omega_g + \frac{|\xi|_g^2\omega(\eta)}{\omega(\xi)^2}\omega_g,$$

$$b = \frac{(\nabla\omega)(\xi,\xi)}{\omega(\xi)}\xi + \frac{|\xi|_g^2}{\omega(\xi)}(d\omega)_g(\xi) - \frac{|\xi|_g^2(\nabla\omega)(\xi,\xi)}{\omega(\xi)^2}\omega_g.$$

Recall that these equations are written for arbitrary parameterized geodesics; let us use the assumption that our geodesic is arc-length parameterized, which in regular points means

(5.1) $$|\xi|_g^2 = \omega(\xi).$$

Differentiating this equation in the direction of ξ, we obtain

(5.2) $$2g(\xi,\eta) = \omega(\eta) + (\nabla\omega)(\xi,\xi).$$

By (5.1) and (5.2), the equation $A\eta = b$ is reduced to

$$\eta - \frac{\omega(\eta)}{\omega(\xi)}\xi + \frac{\omega(\eta)}{2\omega(\xi)}\omega_g = \frac{(\nabla\omega)(\xi,\xi)}{\omega(\xi)}\left(\xi - \frac{1}{2}\omega_g\right) + (d\omega)_g(\xi).$$

Taking ω of both sides gives

$$\omega(\eta) = \frac{2(\nabla\omega)(\xi,\xi) + 2d\omega(\xi,\omega_g)}{|\omega|_g^2}\omega(\xi) - (\nabla\omega)(\xi,\xi).$$

Thus we have

$$\frac{d}{dt}\omega(\xi) = \omega(\eta) + (\nabla\omega)(\xi,\xi)$$
$$= \frac{2(\nabla\omega)(\xi,\xi) + 2d\omega(\xi,\omega_g)}{|\omega|_g^2}\omega(\xi)$$
$$\geq -C\omega(\xi)$$

where the constant C is obtained by taking the maximum of the absolute value of the coefficient $(2(\nabla\omega_x)(\xi,\xi) + 2d\omega_x(\xi,\omega_g(x)))/(|\omega(x)|_g^2)$ appearing in the second line, the maximum being taken as x varies over the compact curve $c([0,T])$ and ξ varies over over the compact indicatrix at x. (Recall that $F(\gamma(t),\gamma'(t)) = 1$ for $t < T$.) Hence we obtain for all $t \in [0,T)$

$$\omega(\xi) \geq \omega(\gamma'(0))e^{-Ct} > \omega(\gamma'(0))e^{-CT} =: \delta.$$

As explained above, there exists a universal positive constant ε_0 depending on δ and on behavior of g and ω in some neighborhood of the geodesic such that for each $t \in [0,T)$ the geodesic can be extended for time at least ε_0. In particular, $t = T$ is regular. $\qquad\square$

We return to the proof of Theorem 1.5. We will connect p to a point q by a Kropina geodesic in the interior of the ball $B_r(p)$. If we work under the assumptions of part (A) of the theorem, we take $r > 0$ sufficiently small. In view of the condition $\omega \wedge d\omega_p \neq 0$, the set of interior points of $B_r(p)$ is an open neighborhood of p. If we work under the assumptions of part (B), we

take r sufficiently large so that the set of interior points of $B_r(p)$ coincides with the whole M.

For any interior point q of $B_r(p)$ we have $d(p, q) < r$. Consider an arc-length parameterized minimal geodesic $\gamma : [0, d(p, q)] \to M$ joining p to q. The existence of γ is explained above. All interior points of γ have distance less than $d(p, q)$ to p which implies that they lie in U as we require in the part (A) of Theorem 1.5. Let us show that all points of the geodesic are regular.

Because this geodesic is a locally-Lipschitz curve of finite length, it has a regular point. As explained above, the set of regular times is open in $[0, d(p, q)]$ and so is a union of open intervals of the form $[0, b)$, (a, b), or $(a, d(p, q)]$, with $0 < a < b < d(p, q)$. Observe though that the intervals of the form $[0, b)$ and (a, b) can not occur, since the endpoint b is necessarily a regular point by Lemma 5.1. Thus, the set of regular points is either all of $[0, d(p, q)]$ or is $(0, d(p, q)]$.

It remains to show that only the first possibility occurs, that is, to show that the starting point $t = 0$ is also regular. For this purpose, we consider backward geodesics instead of forward geodesics. (These are geodesics for the Kropina structure $-F = g/(-\omega)$. Clearly, $\gamma(d(p, q) - t)$ is a minimal arc-length parameterized geodesic for the backward distance). All arguments above survive the change from 'forward' to 'backward' geodesic. The change replaces $t = 0$ by $t = d(p, q)$ yielding that $t = 0$ is regular. Since now all points of γ are regular, it is a Kropina geodesic of F. Theorem 1.5 is proved.

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