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Between Binomial and Trinomial Option Pricing Models

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ABSTRACT

This paper shows that the binomial option pricing model, suitably parameterized, is a special case of the explicit finite difference method.

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On the Relation Between Binomial and Trinomial Option Pricing Models

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Abstract

This paper shows that the binomial option pricing model, suitably parameterized, is a special case of the explicit finite difference method.

To prepare for writing the sequel volume of my new book *Derivatives: A PowerPlus Picture Book*, I recently reviewed the work on trinomial option pricing since Boyle's 1988 *JFQA* paper. I found myself attracted to the Kamrad and Ritchken (1991) trinomial model because it seemed to be the "natural" generalization of the binomial model described by Cox, Ross and Rubinstein (1979).

In that model, as is quite well known, the underlying asset price moves by return x over each period of elapsed time h, where x equals either u or d, while cash earns return r for sure. The resulting corresponding binomial tree is designed to emulate continuous-time risk-neutral geometric Brownian motion with annualized logarithmic mean $\mu \equiv \log(r/d) - \frac{1}{2}\sigma^2$ and variance σ^2, where r is the annualized riskless return (discrete) and d is the annualized payout return (discrete) of the asset.

The idea is to choose a parameterization for (r, u, d, and p) in terms of (r, μ and σ) so that for a fixed total time t, $r \to r^h$, $E(\log x) \to \mu h$ and $Var(\log x) \to \sigma^2 h$ as $h \to 0$ where the expectation and variance are measured using the probability p of an up move. Cox/Ross/Rubinstein proposed the following parameterization:[1]

$$r = r^h \qquad u = e^{\sigma\sqrt{h}} \qquad d = 1/u \qquad p = (1/2) + (1/2)(\mu/\sigma)\sqrt{h}$$

An alternative parameterization suggested by Jarrow and Rudd (1983) is:

$$r = r^h \qquad u = e^{\mu h + \sigma\sqrt{h}} \qquad d = e^{\mu h - \sigma\sqrt{h}} \qquad p = 1/2$$

In the trinomial model, x can take one of three possible values (U, 1, D) over a single period of elapsed time H with corresponding probabilities (P_U, $(1 - P_D - P_U)$, P_D). In the Kamrad/Ritchken parameterization,

$$R = r^H \qquad U = e^{\lambda\sigma\sqrt{H}} \qquad D = 1/U$$

$$P_U = (1/2\lambda^2) + (1/2\lambda)(\mu/\sigma)\sqrt{H} \qquad P_D = (1/2\lambda^2) - (1/2\lambda)(\mu/\sigma)\sqrt{H}$$

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[1] For both the Cox/Ross/Rubinstein and Jarrow/Rudd parameterizations, p so defined is only approximately equal to the risk-neutral probability p that (in the absence of payouts) satisfies $pu + (1 - p)d = r$.

Science traditionally tries to unify alternative theories and methodologies that deal with the same questions. So it is natural to ask how alternative parameterizations of trinomial trees are related to each other. To my surprise, and it does not appear to have been previously noted, simple inspection shows that the Kamrad/Ritchken trinomial model has exactly the same parameterization (except for a negligible difference in the riskless return[2]) that Brennan and Schwartz (1978) prove is the solution to the explicit finite difference method when the coefficients of the finite difference equation are generated under a logarithmic transformation. By this I mean that the portion of the rectangular grid in the finite difference model that overlaps the triangular tree produced by the Kamrad/Ritchken trinomial model has exactly the same nodes, and therefore will produce exactly the same current option value.

Several years ago, when I studied the Brennan/Schwartz paper carefully, it was again natural to ask the question of how binomial trees were related to the trinomial tree produced by explicit finite differences. I had a sense that if the world worked elegantly, a binomial tree ought to be a special case of a trinomial tree. Trinomial trees ought to be composed of binomial trees, somewhat like molecules are composed of atoms. That is, one could hope that the trinomial tree could be parameterized so that it could be interpreted more fundamentally as a binomial tree in which every other period were simply skipped:
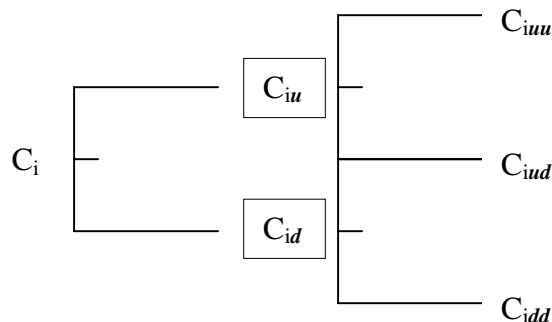
For example, at the node with option value C_i, instead of valuing this by first working backward from two nodes into the future $(C_{iuu}, C_{iud}, C_{idd})$ to one node into the future (C_{iu}, C_{id}), then to C_i, this intermediate step is simply skipped and C_i is determined directly from $(C_{iuu}, C_{iud}, C_{idd})$.

In my recent review, I was again surprised to discover that despite the lapse of 20 years since the Brennan/Schwartz and Cox/Ross/Rubinstein articles and despite the extensive literature now extant on trinomial models, the following theorem has apparently never been published:

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[2] Kamrad/Ritchken assume that $R = r^H$, while the explicit finite difference model sets $R = 1 + (\log r)H$. In practice, this difference is negligible. For example, if $h = .001$ and $r = 1.1$, then the former estimate sets $R = 1.000095310$ and the latter estimate sets $R = 1.000095315$.

<u>Theorem</u>: The explicit finite difference method under a logarithmic transformation (= Kamrad/Ritchken trinomial model) is formally equivalent to the binomial method, where only every other period is examined and the binomial tree is parameterized as follows:

$$r = [1 + 2(\log r)h]^{1/2} \qquad u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2} \qquad d = 1/u$$

$$p = \tfrac{1}{2}\{\ 1/(2(\sigma^2 h - \mu^2 h^2)) + [1/(\sigma^2 h - \mu^2 h^2)^{1/2}](\mu/\sigma)\sqrt{(2h)}\ \}^{1/2}$$

<u>Proof</u>: Brennan/Schwartz show that the explicit finite difference approach is equivalent to solving with backwards recursion:

$$[1 + (\log r)H]C_i = P_U C_{iU} + (1 - P_U - P_D)C_{iM} + P_D C_{iD} \qquad (I)$$

Skipping one step in the binomial method gives us the parallel equation:

$$r^2 C_i = p^2 C_{iuu} + 2p(1\text{-}p)C_{iud} + (1\text{-}p)^2 C_{idd} \qquad (II)$$

Consistency between these two methods requires that equations (I) and (II) be the same. In particular, that:

$$C_{iuu} = C_{iU} \qquad C_{iM} = C_{iud} \qquad C_{iD} = C_{idd} \quad (1)$$
$$r^2 = 1 + (\log r)H \qquad (2)$$
$$p^2 = P_U \qquad (3)$$
$$(1 - p)^2 = P_D \qquad (4)$$

Of course, if equations (1) and (2) hold, then the option value C_i will be the same in equations (I) and (II). In addition, if (1) and (2) hold, then it immediately follows that

$$2p(1 - p) = (1 - P_U - P_D)$$

We naturally require the trinomial time interval H to be twice the binomial interval h, so that $H = 2h$, and the trinomial spacing $(U, 1, D)$ to be the square of the binomial spacing (u, d), so that $U = u^2, D = d^2$ and $1 = ud$. This gives us one of our conditions:

$$d = 1/u$$

It also insures that at the expiration date of the option, the boundary conditions have identical values for the two models. So, in particular, if i were one trinomial period or two binomial periods prior to expiration, equations (1) will hold. Clearly, we are free to require equation (2). This gives us another of our conditions:

$$r = [1 + 2(\log r)h]^{1/2}$$

We are also free to require equation (3). A little algebra shows, this gives us another of our conditions:

$$p = \tfrac{1}{2}\{\ 1/(2(\sigma^2 h - \mu^2 h^2)) + [1/(\sigma^2 h - \mu^2 h^2)^{1/2}](\mu/\sigma)\surd(2h)\ \}^{1/2}$$

It then merely remains to be shown, that under these conditions, equation (4) must also hold. Observe that in the trinomial model, defining $a \equiv (1/2\lambda^2)$ and $b \equiv (1/2\lambda)(\mu/\sigma)\surd H$, we can alternatively write $P_U = a + b$ and $P_D = a - b$, so that $p^2 = (a + b)$. Our proof is done if we can show that $(1 - p)^2 = a - b$ so that $(1 - p)^2 = P_D$. For arbitrary parameterization of u this will not be true. However, we will now show that $(1 - p)^2 = a - b$ if and only if $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$. First, we show if $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$, then $(1 - p)^2 = a - b$.

Since $u^2 = e^{\lambda\sigma\surd(2h)}$, then $(2 \log u)^2 = \lambda^2\sigma^2(2h)$. Therefore, $1/(2\lambda^2) = \tfrac{1}{4}(\sigma^2 h)/(\log u)^2$. Substituting the supposition for u, $1/(2\lambda^2) = \tfrac{1}{4}\sigma^2/[(\sigma^2 - \mu^2 h)]$. Now

$$a - b^2 = [1/(2\lambda^2)] - [1/(2\lambda^2)](\mu/\sigma)^2 h = [1/(2\lambda^2)][1 - (\mu^2 h/\sigma^2)] = [1/(2\lambda^2)][(\sigma^2 - \mu^2 h)/\sigma^2)]$$

Substituting into this the above expression for $1/(2\lambda^2)$, $a - b^2 = \tfrac{1}{4}$ so $a = b^2 + \tfrac{1}{4}$. Since $p^2 = a + b$, then $p = (a + b)^{1/2}$. Substituting for a, $p = (b^2 + b + \tfrac{1}{4})^{1/2} = b + \tfrac{1}{2}$. Therefore, $(1 - p)^2 = (1 - b - \tfrac{1}{2})^2 = (b - \tfrac{1}{2})^2 = b^2 - b + \tfrac{1}{4} = a - b$. The converse of this, if $(1 - p)^2 = a - b$, then $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$, follows since this argument may be reversed.

Comments: Four comments are in order. First, in addition to the two better known parameterizations of the binomial model stated above from Cox/Ross/Rubinstein and Jarrow/Rudd, I have provided here a third which also converges to lognormality in the limit (it must converge because it makes the binomial method equivalent to the explicit finite difference method which is known to converge[3]).

Second, this comparison of binomial and trinomial methods emphasizes Ritchken's (1995) well-made point that the advantage of the trinomial method over the binomial is that it provides another degree of freedom since the move spacing can be set independently of move timing. In the binomial method, the requirement that $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$ ties together the spacing u and timing h in a particular way.

Third, under the above binomial parameterization, $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$, and in the trinomial method $U = e^{\lambda\sigma\surd H}$. Since $U = u^2$ and $H = 2h$, consistency requires that $\lambda = (\surd 2)[1 - (\mu/\sigma)^2 h]^{1/2}$. In practice, for a reasonably small H, say H = 1/50 or smaller, $\lambda \cong \surd 2$. That is, simulation shows there will be little difference in standard option values calculated with the exact or approximate parameterization of λ. But if we set $\lambda = \surd 2$, then $u = e^{\sigma\surd h}$, the exact parameterization of the Cox/Ross/Rubinstein version of the binomial option pricing model.

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[3] It is well known that necessary and sufficient conditions for convergence of the explicit finite difference method are $H \le \sigma^2/\mu^2$ and $\log U \le \sigma^2/|\mu|$. In the consistent binomial model these conditions are $h \le \tfrac{1}{2}(\sigma/\mu)^2$ and $\log u \le \tfrac{1}{2}\sigma^2/|\mu|$. It is easily shown that $u = \exp(\sigma^2 h - \mu^2 h^2)^{1/2}$ is consistent with this. That is, under this parameterization for u, $h \le \tfrac{1}{2}(\sigma/\mu)^2$ if and only if $\log u \le \tfrac{1}{2}\sigma^2/|\mu|$. Therefore, in applying the consistent binomial model, convergence is assured if and only if $h \le \tfrac{1}{2}(\sigma/\mu)^2$.

Fourth, in speed of convergence comparisons between binomial and trinomial methods, if the binomial is given a fair chance by skipping every other period, under the parameterization in the theorem, there can be no difference. For the trinomial method to be superior requires a different relation of the spacing and timing parameters. Preliminary simulation comparing the trinomial method to Black-Scholes values for European near-the-money standard options, suggests that setting $\lambda = \sqrt{2}$ usually gives good results. Testing whether a different parameterization typically improves the rate of convergence of the trinomial method is a task I leave to others who may find that question more interesting than I do.

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