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Lattices of minimal covolume in real special linear groups

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by

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2019

The dissertation of François Thilmany is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

À Marie-Louise et Maurice De Coninck

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ABSTRACT OF THE DISSERTATION

Lattices of minimal covolume in real special linear groups

by

François Thilmany

Doctor of Philosophy in Mathematics

University of California San Diego, 2019

Professor Alireza Salehi Golsefidy, Chair

The objective of the dissertation is to determine the lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$, for $n \geq 3$. Relying on Margulis' arithmeticity, Prasad's volume formula, and work of Borel and Prasad, the problem will be translated in number theoretical terms. A careful analysis of the number theoretical bounds involved then leads to the identification of the lattices of minimal covolume. The answer turns out to be the simplest one: $\mathrm{SL}_n(\mathbb{Z})$ is, up to automorphism, the unique lattice of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. In particular, lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$ are non-uniform when $n \geq 3$, contrasting with Siegel's result for $\mathrm{SL}_2(\mathbb{R})$. This answers for $\mathrm{SL}_n(\mathbb{R})$ the question of Lubotzky: is a lattice of minimal covolume typically uniform or not?

Chapter 1

Introduction

A brief history

The study of lattices of minimal covolume in SL_n (or for that matter, in Lie groups in general) originated with Siegel's work [Sie45] on $\mathrm{SL}_2(\mathbb{R})$, which in turn can be traced back to Hurwitz's work [Hur92]. Siegel showed that in $\mathrm{SL}_2(\mathbb{R})$, lattices of minimal covolume are given by the conjugates of the $(2,3,7)$-triangle group (the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ of a $(2,3,7)$-triangle tiling). He used the action of $\mathrm{SL}_2(\mathbb{R})$ on the hyperbolic plane to relate the covolume of the lattice to the area of its Poincaré fundamental domain, which tiles the plane. The result then follows from the fact that the $(2,3,7)$-triangle is the (ideal) polygon with smallest area which tiles the hyperbolic plane. In particular, lattices of minimal covolume in $\mathrm{SL}_2(\mathbb{R})$ are arithmetic and uniform. Siegel raised the question of determining which lattices attain minimum covolume in groups of isometries of higher-dimensional hyperbolic spaces.

For $\mathrm{SL}_2(\mathbb{C})$, which acts on hyperbolic 3-space, the minimum among non-uniform lattices was established by Meyerhoff [Mey85]; among all lattices in $\mathrm{SL}_2(\mathbb{C})$, the minimum was exhibited more recently by Gehring, Marshall and Martin [GM09, MM12] using geometric methods, and is attained by a (unique up to conjugacy) uniform lattice.

Using the action of $\mathrm{SL}_2\big(\mathbb{F}_q(\!(t)\!)\big)$ on its Bruhat-Tits tree and Bass-Serre theory, Lubotzky established the analogous result [Lub90] for $\mathrm{SL}_2\big(\mathbb{F}_q(\!(t)\!)\big)$, where this time $\mathrm{SL}_2\big(\mathbb{F}_q[t^{-1}]\big)$ attains the smallest covolume. Lubotzky observed that in this case, as opposed to the $(2,3,7)$-triangle group in $\mathrm{SL}_2(\mathbb{R})$, the lattice of minimal covolume is not uniform; he then asked whether, for a lattice of minimal covolume in a semi-simple Lie group, the typical situation is to be uniform, or not.

Progress has been made on this question. By proving a quantitative version of the Kazhdan–Margulis theorem, Salehi Golsefidy showed [SG09] that for most Chevalley groups G of rank at least 2, $G\big(\mathbb{F}_q[t^{-1}]\big)$ is the unique (up to automorphism) lattice of minimal covolume in $G\big(\mathbb{F}_q(\!(t)\!)\big)$. Using Prasad's formula, Salehi Golsefidy also obtained [SG13] that for most simply connected simple groups over $\mathbb{F}_q(\!(t)\!)$, a lattice of minimal covolume will be non-uniform (provided Weil's conjecture on Tamagawa numbers holds).

On the other side of the picture, when the rank is 1, Belolipetsky and Emery [Bel04, BE12] determined the lattices of minimal covolume among arithmetic lattices in $\mathrm{SO}(n,1)(\mathbb{R})$ for $n \geq 5$, and showed that they are non-uniform. For $\mathrm{SU}(n,1)(\mathbb{R})$, Emery and Stover [ES14] determined the lattices of minimal covolume among the non-uniform arithmetic ones, but to the best of the author's knowledge, this has not been compared to the uniform arithmetic ones in this case. Unfortunately, in the rank 1 case, it is not known whether a lattice of minimal covolume is necessarily arithmetic. A summary of the advances made in this case (along with references) can be found in [Bel14].

The most recent results concerning lattices of minimal covolume are due to Emery and Kim [EK18], who determined the lattices of minimal covolume in $\mathrm{Sp}(n,1)(\mathbb{R})$ for $n \geq 2$, both among uniform lattices and among non-uniform lattices. For $n \geq 3$, lattices of minimal covolume in $\mathrm{Sp}(n,1)(\mathbb{R})$ are again non-uniform.

The above results give a partial answer to the question of Lubotzky. In this document, we work out the case of $\mathrm{SL}_n(\mathbb{R})$. We show that for $n \geq 3$, up to automorphism, the non-uniform lattice $\mathrm{SL}_n(\mathbb{Z})$ is the unique lattice of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. This is in sharp contrast with Siegel's lattice in $\mathrm{SL}_2(\mathbb{R})$.

Some open questions and further directions

Describing the lattices of minimal covolume in an arbitrary (real) Lie group is likely an unreasonably difficult question, namely because of the groups of rank 1. A more reasonable task would be to identify the lattices of minimal covolume in all (remaining) split real Lie groups, although this might be very tedious for groups of small rank.

In the mean time, Lubotzky's question remains open. The evidence stemming from the present work leads to the following conjecture (which is a particular case of Lubotzky's question).

Conjecture. *There exists a number $N \in \mathbb{N}$ with the following property. Let G be any real (semi)simple Lie group of real rank at least N. If Γ is a (irreducible) lattice of minimal covolume in G, then Γ is not uniform.*

As was reported above, arithmeticity holds for all known lattices of minimal covolume in simple real Lie groups of rank 1. An illustration of how control on the covolume might force a lattice to be arithmetic can be seen in the work of Klingler [Kli03], who proved the arithmeticity of fake projective planes (see also Yeung [Yeu04]). This justifies the following very hard conjecture.

Conjecture (Folklore)**.** *Let G be a simple real Lie group. Then any lattice of minimal covolume in G is arithmetic.*

Somehow, p-adic Lie groups have been mostly left out from the study of lattices of minimal covolume. The only p-adic Lie groups for which a lattice of minimal covolume is known are $\mathrm{SL}_2(K)$ for K a finite extension of $\mathbb{Q}_p$ ($p \neq 2$), for which Lubotzky and Weigel [LW99] classified all lattices of minimal covolume using their action on the Bruhat-Tits tree of $\mathrm{SL}_2(K)$ and Bass-Serre theory, and $\mathrm{SL}_3(\mathbb{Q}_2)$, in which Allcock and Kato [AK13] found the two (up to conjugacy) lattices of minimal covolume using geometric arguments also coming from the action on the Bruhat-Tits building. A good point to resume would thus be to study lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{Q}_p)$.

It should be noted that in semisimple p-adic Lie groups, lattices are always uniform (see for example [Tam65]). Thus Lubotzky's question has an immediate (unenlightening) answer in this case.

Chapter 2

Background material

2.1 Affine algebraic groups

In this section, we recall what affine algebraic groups are, and briefly describe their structure and properties. We refer the reader to [Bor91], [PR94, ch. 2], [Spr98] or [Spr79] for proofs, examples and more on this topic and on the topic of section 2.2.

Let k be a field (which we will later on assume to be perfect) and $\overline{k}$ denote its algebraic closure.

2.1.1 Definition (Algebraic group)**.** An *algebraic group over* k (or simply a k*-group*) consists of the datum (G, m, i, e), where G is an algebraic variety defined over k, $m : G \times G \to G$ and $i : G \to G$ are morphisms of varieties (called *multiplication* or *product*, and *inversion*), and $e : \{*\} \to G$ is a distinguished point of G (called *identity*), subject to the usual group axioms:

(i) Identity: $m \circ (e \times \mathrm{id}) \circ q_1 = \mathrm{id} = m \circ (\mathrm{id} \times e) \circ q_2$ as morphisms $G \to G$ (here q_1 and q_2 are the isomorphisms $G \to \{*\} \times G$ and $G \to G \times \{*\}$).

(ii) Associativity: $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$ as morphisms $G \times G \times G \to G$

(iii) Inverse: $m \circ (\mathrm{id} \times i) \circ \delta = e \circ p = m \circ (i \times \mathrm{id}) \circ \delta$ as morphisms $G \to G$ (here $\delta : G \to G \times G$ is the diagonal embedding $g \mapsto (g, g)$ and p the unique morphism $G \to \{*\}$).

In other words, G is a group object in the category of algebraic varieties over k. Almost always, m, i and e will be dropped from the notation: we will say G is an algebraic group, $m(g, h)$ will be abbreviated gh, $i(g)$ abbreviated g^{-1}, the identity element will be denoted e or id regardless of the group, etc.

When the underlying variety G is affine, we say that G is an *affine algebraic group*. In virtue of the anti-equivalence of the categories of affine k-varieties and of reduced k-algebras of finite type, (G, m, i, e) is an algebraic group if and only if $(k[G], m^*, i^*, e^*)$ is a Hopf algebra. Here and in the following, $k[G]$ denotes the ring of k-regular functions on G, $m^* : k[G] \to k[G] \otimes_k k[G]$ and $i^* : k[G] \to k[G]$ denote the ring morphisms induced by m and i, and $e^* : k[G] \to k$ is the evaluation map at e. We leave as an exercise for the interested reader to determine $k[G]$, m^*, i^* and e^* in the examples below.

2.1.2 Remark (Affine group schemes)**.** In the definition above, we can replace k by an arbitrary ring R, G by a scheme over $\operatorname{Spec} R$ and require m, i and e to be morphism of schemes over $\operatorname{Spec} R$, to obtain the definition of a *group R-scheme*. The group scheme is called *affine* (resp. *smooth*) if the underlying scheme G is so.

2.1.3 Examples.

(i) If G and H are affine algebraic groups, then the product variety $G \times H$ is an algebraic group for componentwise multiplication and inversion.

(ii) If H is a closed subvariety of G which is a subgroup with respect to the operations m and i of G, then H endowed with the restrictions of m and i is an algebraic group, called a *closed subgroup of G*.

(iii) Any finite group can be seen as an (affine) algebraic k-group of dimension 0. Such group, unless trivial, will not be connected.

(iv) The affine line can be identified with k and endowed with the corresponding additive group structure. The resulting affine algebraic group is called the *additive group* and is denoted G_a. Combined with the previous example, we see that affine n-space k^n is an algebraic group whose product coincides with the usual componentwise addition.

(v) The hyperbola $\{(x, y) \in k^2 \mid xy = 1\}$, endowed with the morphisms $m : ((x, y), (x', y')) \mapsto (xx', yy')$ and $i : (x, y) \mapsto (y, x)$, is an algebraic group, called the *multiplicative group* and denoted G_m.

(vi) The special and general linear groups. Let $\mathrm{M}_n(k)$ denote the set of $n \times n$ matrices with coefficients in k. $\mathrm{M}_n(k)$ has a canonical structure of k-variety, obtained by identifying it with affine n^2-space k^{n^2} componentwise. Moreover, as multiplication of matrices is polynomial in terms of the entries, we see that the multiplication map $m : \mathrm{M}_n(k) \times \mathrm{M}_n(k) \to \mathrm{M}_n(k)$ is a morphism of k-varieties. For the same reason, the determinant $\det : \mathrm{M}_n(k) \to k$ is a morphism of varieties, and in consequence, the set $\mathrm{SL}_n(k) = \{g \in \mathrm{M}_n(k) \mid \det g = 1\}$ is a closed subvariety of $\mathrm{M}_n(k)$. Of course, m restricts to a morphism $\mathrm{SL}_n(k) \times \mathrm{SL}_n(k) \to \mathrm{SL}_n(k)$, and in virtue of Cramer's rule, $g \mapsto g^{-1}$ defines a morphism of varieties $i : \mathrm{SL}_n(k) \to \mathrm{SL}_n(k)$. Altogether, $(\mathrm{SL}_n(k), m, i, \mathrm{id})$ is an affine algebraic group, called the *$(n \times n)$ special linear group over k.*

Similarly, the group $\mathrm{GL}_n(k)$ of invertible $n \times n$ matrices is an algebraic group. To see this, embed $\mathrm{GL}_n(k)$ in $\mathrm{SL}_{n+1}(k)$ via the morphism of varieties $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & \det(g) \end{smallmatrix}\right)$. It is then clear that its image is a closed subgroup of $\mathrm{SL}_n(k)$.

(vii) Extension of scalars. Let l be an extension of k and G be an algebraic group over k. Then the variety G_l obtained from G by extending scalars to l is again an algebraic group (over l now), with the corresponding morphisms m_l and i_l. This follows immediately from the functoriality of the extension of scalars. In terms of rings of regular functions, we have $l[G_l] \cong l \otimes_k k[G]$, and m_l^*, i_l^* are induced from m^* and i^* in the obvious way. In particular, if G is given as a closed subvariety of affine space k^n, we see that G_l is simply the closed subvariety of l^n given by the same equations, seen over l.

If G and H are two k-groups, we will say that *G and H are isomorphic over l*, or that *G and H are l/k-forms of each other*, if the groups G_l and H_l are isomorphic as algebraic l-groups. If there exists some field l such that G and H are isomorphic over l, we simply say that *G and H are k-forms of each other.*

(viii) Special unitary groups. Let l be a separable quadratic extension of k, and let us denote σ the non-trivial automorphism of l fixing k. Let $h : l^n \times l^n \to l$ be a non-degenerate hermitian form (i.e. h is σ-sesquilinear and symmetric, and no non-zero vector is orthogonal to the whole space). The *special unitary group* $\mathrm{SU}_h(k) = \{g \in \mathrm{SL}_n(l) \mid h(gx, gy) = h(x, y)\}$ *associated to h* can be endowed with the structure of an algebraic k-group as follows. Pick a basis of l over k, and use it to identify $\mathrm{M}_n(l)$ with $\mathrm{M}_n(k^2)$ (as vector k-spaces). In view of the properties of h and the fact σ is k-linear, we see that the conditions defining $\mathrm{SU}_h(k)$ can be translated into polynomial equations on $\mathrm{M}_n(k^2)$, which means that $\mathrm{SU}_h(k)$ can be identified with a closed k-subvariety of $\mathrm{M}_n(k^2)$. ($\mathrm{SU}_h(k)$ is however not a l-subvariety of $\mathrm{M}_n(l)$, because $\sigma : l \to l$ is not polynomial over l; hence $\mathrm{SU}_h(k)$ is not an algebraic l-group.) It remains to observe that matrix multiplication in $\mathrm{M}_n(l)$ is given by polynomials when translated to $\mathrm{M}_n(k^2)$, but this is clear once we observe that the product map $l \times l \to l$

induces a bilinear (hence polynomial) map $k^2 \times k^2 \to k^2$.

It is interesting to see what happens to $G = \mathrm{SU}_h(k)$ when extending scalars from k to l. Let J be the matrix of the hermitian form h in the canonical basis (J is invertible and $\sigma(J) = J^t$). In view of the discussion above, we can identify the l-points of G_l with matrices $A \in \mathrm{SL}_n(l \otimes_k l)$ satisfying $AJ\sigma(A)^t = J$, where σ acts componentwise on A and acts (say) on the first tensor component of $l \otimes l$. If $l = k(\alpha)$ and $f \in k[t]$ denotes the minimal polynomial of α over k, we see that as rings,

$$l \otimes_k l \cong (k[t]/f) \otimes_k l \cong l[t]/f = l[t]/(t-\alpha)(t-\sigma(\alpha)) \cong l \oplus l,$$

with the action of σ translating to switching both components of $l \oplus l$. Thus, we can identify $\mathrm{SL}_n(l \otimes_k l) \cong \mathrm{SL}_n(l \oplus l) \cong \mathrm{SL}_n(l) \times \mathrm{SL}_n(l)$ and the action of σ again translates to switching both components of $\mathrm{SL}_n(l) \times \mathrm{SL}_n(l)$. Using this identification, we are now looking for matrices $A = (A_1, A_2) \in \mathrm{SL}_n(l) \times \mathrm{SL}_n(l)$ satisfying $A(J, J^t)\sigma(A)^t = (J, J^t)$; in other words, A_1 and A_2 satisfy $A_1 J A_2^t = J$ and $A_2 J^t A_1^t = J^t$. All solutions to these equations are given by $(A_1, J^t A_1^{-t} J^{-t})$ with $A_1 \in \mathrm{SL}_n(l)$, so that projecting on the first component yields an isomorphism between the underlying groups $G_l \to \mathrm{SL}_n(l)$. By examining the above argument more carefully, one can show that this morphism is in fact an isomorphism of algebraic l-groups. In short, SU_h is a l/k-form of SL_n.

2.1.4 Remark (The functor of points)**.** Although most of it will be hidden behind the notation, it is important to emphasize that an algebraic k-group G has more structure than just the underlining group G and its Zariski topology. The extra structure is captured by the structure sheaf of G, and m, i are given to be morphisms of ringed spaces. If G is affine, the ring $k[G]$ of regular functions and the maps m^*, i^* capture this structure.

This is especially crucial if the base field is not algebraically closed. For example, consider the Hopf $\mathbb{Q}$-algebra $\mathbb{Q}[T, T^{-1}]$ with coproduct $T \mapsto T \otimes T$ and antipode $T \mapsto T^{-1}$ (this is the Hopf algebra of the multiplicative group G_m, see 2.1.3 (v)). The two quotient Hopf $\mathbb{Q}$-algebras $\mathbb{Q}[T, T^{-1}]/(T^2 - 1)$ and $\mathbb{Q}[T, T^{-1}]/(T^4 - 1)$ correspond to two algebraic $\mathbb{Q}$-groups that we denote μ_2 and μ_4. It is readily seen that each of these have an underlying group consisting of 2 elements (with the discrete topology); hence both underlying groups $\mu_2(\mathbb{Q})$ and $\mu_4(\mathbb{Q})$ are isomorphic. Nevertheless, μ_2 and μ_4 are not isomorphic as algebraic groups. This is immediate from the fact that their Hopf algebras are not isomorphic, but can also be seen by computing the $\mathbb{Q}[i]$-points (introduced below): $\mu_2(\mathbb{Q}[i])$ has two elements, and $\mu_4(\mathbb{Q}[i])$ is cyclic of order 4.

With this in mind, we introduce the *functor of points.* Given an affine algebraic k-group G and a k-algebra A, one can form the *group* $G(A)$ *of* A*-points of* G by setting $G(A) = \mathrm{Hom}_{k\text{-alg}}(k[G], A)$. The group structure comes from the Hopf algebra structure of $k[G]$: if $g, h \in G(A)$, then gh is the element of $\mathrm{Hom}_{k\text{-alg}}(k[G], A)$ given by the composition

$$k[G] \xrightarrow{m^*} k[G] \otimes_k k[G] \xrightarrow{g \otimes h} A \otimes_k A \xrightarrow{\text{multiplication}} A$$

and similarly, $g^{-1} \in \mathrm{Hom}_{k\text{-alg}}(k[G], A)$ is given by

$$k[G] \xrightarrow{i^*} k[G] \xrightarrow{g} A.$$

When G is given as a closed subvariety of affine space k^n, this precisely amounts to looking for solutions in A^n of the equations defining G. The group operations in these coordinates are in turn given by the same polynomials as for G over k, but seen over A. Given a morphism $f : A \to B$ of k-algebras, one obtains a morphism of groups $G(A) \to G(B)$ by sending $g \mapsto g \circ f$. One readily checks that this is well behaved with respect to composition, hence $G(-)$ is a *representable* functor from the category of k-algebras to the category of groups. In virtue of Yoneda's lemma, one can recover $k[G]$, hence the affine algebraic group G, from the knowledge of this functor.

This construction should be reminiscent of extension of scalars (see 2.1.3 (vii)): indeed, the group $G_l(l)$ of l-points of G_l is nothing but the group $G(l)$ of l-points of G. More generally, the functor of points of G_l is simply the functor of points of G restricted to the category of l-algebras, which can be canonically seen as a subcategory of the category of k-algebras. Thus, in what follows, we will simply drop the extension l from the notation G_l.

Affine algebraic groups are also called *linear algebraic groups* because of the following important theorem.

2.1.5 Theorem (Embedding in GL_n). *Let G be an affine algebraic k-group. There exists a positive integer n and an embedding $G \to \mathrm{GL}_n(k)$ of G onto a closed subgroup of $\mathrm{GL}_n(k)$.*

Jordan decomposition

In this subsection, and therefore in any future statement relying on it, we will assume out of simplicity that all fields involved are perfect. This is not a restriction, as the fields that we will be concerned with later on are of characteristic zero.

2.1.6 Definition (Semisimple and unipotent elements). Let G be an affine algebraic k-group. An element $g \in G$ is called *semisimple* (resp. *unipotent*) if the image of g under some (hence any) embedding $\rho : G \to \mathrm{GL}_n(k)$ is a diagonalizable matrix (resp. is a unipotent matrix, i.e. has 1 as only eigenvalue).

2.1.7 Theorem (Jordan decomposition). *Any element $g \in G$ can be written uniquely as $g = g_s g_u$ with $g_s \in G$ is semisimple, $g_u \in G$ is unipotent, and g_s and g_u commute. Moreover, this decomposition is preserved by morphisms of algebraic groups: if $f : G \to H$ is such a morphism,* *then* $f(g)_s = f(g_s)$ *and* $f(g)_u = f(g_u)$. *An element* $g \in G$ *is semisimple (resp. unipotent) precisely when* $g = g_s$ *(resp.* $g = g_u$*).*

2.1.8 Definition (Unipotent, reductive, semisimple and simple groups)**.**

(i) An affine algebraic k-group G is called *unipotent* if every element of G is unipotent.

(ii) The *(solvable) radical* $\mathrm{R}(G)$ of G is the largest connected solvable (closed) normal subgroup of G.

(iii) The *unipotent radical* $\mathrm{R}_u(G)$ of G is the largest connected unipotent (closed) normal subgroup of G. It coincides with the set of unipotent elements of the radical $\mathrm{R}(G)$.

(iv) A connected affine k-group G is called *reductive* if $\mathrm{R}_u(G)$ is trivial. If further $\mathrm{R}(G)$ is trivial, then G is called *semisimple*.

(v) A connected affine k-group G is called *(almost) simple* if G has no non-trivial proper closed normal k-subgroups. A simple k-group is necessarily semisimple. If G remains simple over any extension l of k, we say that G is *absolutely (almost) simple*.

If k has characteristic 0, then any affine algebraic k-group G admits a *Levi decomposition*, meaning that there exists a k-subgroup M, called a *Levi subgroup of* G, whose identity component is reductive and is such that G is the semidirect product of M by the unipotent radical $\mathrm{R}_u(G)$. Moreover, any two Levi k-subgroups of G are conjugate under $\mathrm{R}_u(G)(k)$; in particular they are k-isomorphic. We may thus call M the *reductive part of* G, and the semisimple k-group $[M, M]$ is called the *semisimple part of* G.

The Lie algebra of an algebraic group

2.1.9 Definition (Lie algebra, adjoint map). Let G be an affine algebraic group defined over k. The tangent space of G at the point e is called the *Lie algebra of* G and is denoted $\mathrm{Lie}(G)$ or $\mathfrak{g}$. It can be identified with the space of left-invariant derivations of $k[G]$ and, as such, endowed with the usual bracket for derivations. With this operation, $\mathrm{Lie}(G)$ is a Lie k-algebra, and $\mathrm{Lie}(-)$ defines a functor from the category of algebraic k-groups to the category of Lie k-algebras.

The group G acts on itself via conjugation: if $g \in G$, we denote $c_g : G \to G : h \mapsto ghg^{-1}$. The differential $dc_g : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of the Lie algebra $\mathfrak{g}$, called the *adjoint* of g and denoted $\mathrm{Ad}(g)$. The assignment $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ in turn defines a morphism of algebraic groups, called the *adjoint map*. The derivative $d\,\mathrm{Ad}$ of the adjoint map is a morphism of Lie algebras $\mathfrak{g} \to \mathrm{Lie}(\mathrm{GL}(\mathfrak{g}))$. The Lie algebra $\mathrm{Lie}(\mathrm{GL}(\mathfrak{g}))$ can be canonically identified with $\mathfrak{gl}(\mathfrak{g})$, and under this identification, $d\,\mathrm{Ad}$ corresponds to $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : x \mapsto [x, -]$, the adjoint map of the Lie algebra $\mathfrak{g}$.

Tori and characters

2.1.10 Definition (Torus). An algebraic k-group T is called a *torus* if T is isomorphic to $(\mathrm{G}_m)^r$ over the algebraic closure $\overline{k}$ of k. The integer r (which is also the dimension of T) is called the *absolute rank* of T. The *rank of* T *over* k (or *k-rank*) is the largest integer r' such that there is an embedding $(\mathrm{G}_m)^{r'} \to T$ defined over k. If T is isomorphic to $(\mathrm{G}_m)^r$ over k, we say T is a *split torus.*; this happens precisely when $r = r'$. On the other hand, when $r' = 0$, we say that T is *k-anisotropic.*

Over an algebraically closed field, tori can be characterized as those connected affine algebraic groups which are abelian and have trivial unipotent radical (i.e. consist only of semisimple elements).

2.1.11 Example. The closed $\mathbb{R}$-subgroup S of $\mathrm{GL}_n(\mathbb{R})$ consisting of diagonal matrices is a $\mathbb{R}$-split torus of rank n.

The set of matrices $\left\{\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})\right\}$ defines a closed subgroup T of $\mathrm{GL}_2(\mathbb{R})$. As is readily seen, this subgroup can be diagonalized over $\mathbb{C}$:

$$\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x-iy & 0 \\ 0 & x+iy \end{smallmatrix}\right),$$

and is isomorphic to $(\mathrm{G}_m)^2$ over $\mathbb{C}$. Thus, T is a torus. However, the $\mathbb{R}$-points of T are isomorphic (as an abstract group) to $\mathbb{C}^\times$, whereas the $\mathbb{R}$-points of $(\mathrm{G}_m)^2$ are $(\mathbb{R}^\times)^2$. The torus T is thus not split over $\mathbb{R}$.

Nevertheless, T does contain a $\mathbb{R}$-split torus of rank 1, namely the closed subgroup C of T consisting of diagonal matrices. This shows that the rank of T over $\mathbb{R}$ is precisely 1. The kernel T_1 of the determinant $\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right) \mapsto x^2 + y^2$ is a closed connected subgroup of T, hence a torus, but it is $\mathbb{R}$-anisotropic. The product map $C \times T_1 \to T$ is a surjective morphism of $\mathbb{R}$-groups, with kernel μ_2.

2.1.12 Definition (Characters and cocharacters)**.** Let G be an algebraic k-group. A *k-character of G* is a morphism of algebraic groups $G \to \mathrm{G}_m$ defined over k. The set $X_k(G)$ of k-characters of G forms an abelian group under pointwise multiplication, called the *group of characters of G*. The group $X_{\overline{k}}(G)$ of characters of G defined over $\overline{k}$ is abbreviated $X(G)$.

A *k-cocharacter of G* (or sometimes a *one-parameter multiplicative subgroup of G*) is a morphism of algebraic k-groups $\mathrm{G}_m \to G$. If G is an abelian group, the set $X_k^\vee$ of k-cocharacters forms an abelian group under pointwise multiplication, called the *group of cocharacters*. As above, we abbreviate $X_{\overline{k}}^\vee$ by $X^\vee$.

There is a pairing $\langle\ ,\ \rangle : X_k(G) \times X_k^\vee(G) \to \mathbb{Z}$ defined as follows. Given a k-character χ and a k-cocharacter λ, the composition $\chi \circ \lambda$ is a k-endomorphism of the group G_m. There is thus an integer n such that $\chi \circ \lambda$ is the map $x \mapsto x^n$; we set $\langle \chi, \lambda \rangle = n$. When G is abelian, this pairing is bilinear.

2.1.13 Examples. (i) $\det : \mathrm{GL}_n \to \mathrm{G}_m$ is a character. One can show that in fact $X_k(\mathrm{GL}_n)$ is generated by det.

(ii) If T is a torus of k-rank r', the groups $X_k(T)$ and $X_k^\vee(T)$ are free abelian groups of rank r' and the pairing $\langle\ ,\ \rangle$ defined above is a perfect pairing.

The following theorem is crucial to understand the structure of simple groups, and allows us to define the rank for an arbitrary affine algebraic group.

2.1.14 Theorem. *Let G be a connected affine algebraic k-group. All maximal k-split tori of G are conjugates under $G(k)$. In particular, they all have the same rank.*

2.1.15 Definition (Rank)**.** Let G be a connected affine algebraic k-group. The rank of any maximal k-split torus of G is called the *rank of G over k* (or simply *k-rank* of G). The *absolute rank of G* is the rank of G over the algebraic closure $\overline{k}$ of k (note that over $\overline{k}$, all tori split). If the k-rank equals the absolute rank (i.e. if some maximal torus contained in G splits over k), we say that *G splits over k* or that *G is k-split.* On the other hand, if the k-rank of G is 0, we say that G is *k-anisotropic*, and G is called *k-isotropic* if its k-rank is at least 1.

Borel and parabolic subgroups

2.1.16 Definition. Let G be a connected affine algebraic k-group. A k-subgroup P is called a *parabolic subgroup* if the quotient variety G/P is complete. If B is a parabolic subgroup of G which is solvable, then B is called a *Borel subgroup.*

Parabolic subgroups always exist (for example, G itself is a parabolic k-subgroup, but it may be the only one). However, Borel subgroups need not exist (or rather, be defined over k) in general. When G is k-split (in particular, if k is algebraically closed), Borel k-subgroups exist. If Borel k-subgroups do exist, the group G is called *quasisplit* over k. In this case, one shows that the Borel k-subgroups are precisely the minimal parabolic subgroups, and are also precisely the maximal connected solvable subgroups of G. It is easy to see that if a closed subgroup P contains a parabolic subgroup of G, then P itself is parabolic. Thus, if G admits a Borel k-subgroup, the parabolic k-subgroups of G are characterized as those k-subgroups of G which contain a Borel subgroup.

2.1.17 Theorem (Conjugacy of minimal parabolics)**.** *The minimal parabolic k-subgroups of G are conjugate under $G(k)$. In particular, if Borel k-subgroups exist, they are all conjugate under $G(k)$.*

2.2 The structure of absolutely simple groups

In this section, G is an absolutely simple group defined over a perfect field k.

Let S be a maximal k-split torus of G. We let S act on $\mathfrak{g}$ via the adjoint map. Since S, hence $\mathrm{Ad}(S)$ consists of commuting semisimple elements, $\mathfrak{g}$ splits as a (finite) direct sum of eigenspaces for $\mathrm{Ad}(S)$, that we index by their weights $\alpha \in \Phi_k(S) \cup \{0\} \subset X_k(S)$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi_k(S)} \mathfrak{g}_\alpha \quad \text{where } \mathfrak{g}_\alpha \neq 0 \text{ and } \mathrm{Ad}(s)(x) = \alpha(s) \cdot x \text{ for any } s \in S,\, x \in \mathfrak{g}_\alpha.$$

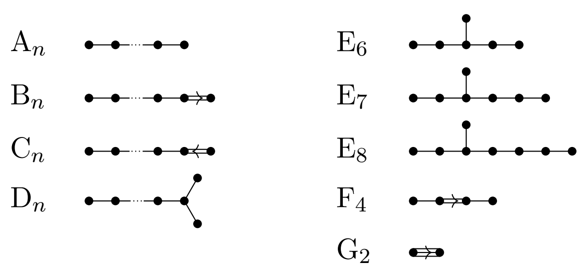
One shows that $\mathfrak{g}_0$ is the Lie algebra of the centralizer $\mathrm{Z}_G(S)$ of S in G.

2.2.1 Definition (Roots)**.** The set $\Phi_k(S)$ of non-zero weights of $\mathrm{Ad}(S)$ is called the *set of k-roots* of G with respect to S. When the torus S is fixed, we write Φ_k instead of $\Phi_k(S)$ (this is acceptable in view of 2.1.14). When $k = \overline{k}$, we drop k from the notation and write $\Phi(S)$ or Φ.

2.2.2 Theorem. *The set Φ_k is an irreducible root system in (its span in) the vector space $X_k \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover precisely, there exists a set $\Phi_k^\vee \subset X_k^\vee$ of k-cocharacters called* coroots*, and a bijection $\Phi_k \to \Phi_k^\vee : \alpha \mapsto \alpha^\vee$ with the property that $(X_k, \Phi_k, X_k^\vee, \Phi_k^\vee)$ is a root datum. (In particular, one has $\langle \alpha, \alpha^\vee \rangle = 2$.) This root system (resp. root datum) is reduced if G is k-split.*

For the definition, properties and classification of root systems, we refer the reader to [Bou07a]. As an illustration, we reproduce the list of all Dynkin diagrams of irreducible (reduced) root systems in table 2.2.3. The Dynkin diagram of a root system has a vertex representing each element of a basis of the root system, and two vertices α and β are connected by an edge, a double edge pointing to the shortest vector, or a triple edge pointing to the shortest vector depending whether $\langle \alpha, \beta^\vee \rangle \cdot \langle \beta, \alpha^\vee \rangle = 1$, 2 or 3.

Table 2.2.3. Dynkin diagrams of the irreducible reduced root systems.



Root groups, Weyl group and Bruhat decomposition

For each $\alpha \in \Phi_k$, there exists a unique unipotent k-subgroup U_α of G, called the *root group associated to* α, whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ ($= \mathfrak{g}_\alpha$ if 2α is not a root). The root groups are normalized by S and their commutators are subject to the rule $[U_\alpha, U_\beta] \subset U_{\alpha+\beta}$ (by convention, if $\chi \in X_k$ is not a root, we set $U_\chi = \{e\}$).

The normalizer $\mathrm{N}_G(S)$ acts on $\mathfrak{g}$ via the adjoint action, and (equivalently) on the set of root groups via conjugation, with its subgroup $\mathrm{Z}_G(S)$ acting trivially. The induced map

$\mathrm{N}_G(S)(k)/\mathrm{Z}_G(S)(k) \to W_k$ is an isomorphism onto the Weyl group W_k of the root system Φ_k. In particular, we can find representatives in $\mathrm{N}_G(S)(k)$ for the action of the Weyl group.

There is a W_k-equivariant bijection between bases of the root system Φ_k and minimal parabolic k-subgroups of G containing S: given a basis $\alpha_1, \ldots, \alpha_r$, the subgroup of G generated by $\mathrm{Z}_G(S)$ and the root groups $U_{\alpha_1}, \ldots, U_{\alpha_r}$ is a minimal k-parabolic containing S. Conversely, if P is a minimal k-parabolic containing S, its unipotent radical $\mathrm{R}_u(P)$ is normalized by S, hence $\mathrm{Lie}(\mathrm{R}_u(P))$ decomposes as a sum $\bigoplus_{\alpha\in\Phi_k^+} \mathfrak{g}_\alpha$ of root spaces. The index set Φ_k^+ defines a set of positive roots in Φ_k, or equivalently, a basis. In consequence, W_k acts simply transitively on the set of minimal parabolics containing S.

Let P_0 be a minimal parabolic k-subgroup containing S and let Δ be the associated basis for Φ_k. There is correspondence between parabolic subsets I of Φ_k containing Δ and parabolic subgroups P containing P_0. To a parabolic subset I, one associates the group $P = P_0 W_{k,I} P_0$, where $W_{k,I}$ is the subgroup of W_k generated by the reflections along the roots $-I \cap \Delta$. Conversely, given a parabolic P containing P_0, its Lie algebra decomposes as $\mathrm{Lie}(P) = \bigoplus_{\alpha\in I} \mathfrak{g}_\alpha$ for some parabolic subset I containing Δ. In particular, one has the *Bruhat decomposition*: $G(k) = P_0(k)W_kP_0(k)$, where, in fact, the sets $P_0(k)wP_0(k)$ are disjoint for different $w \in W_k$.

2.2.4 Example. Let $G = \mathrm{SL}_n(k)$. The k-subgroup S consisting of diagonal matrices is a maximal k-split torus. As S is maximal among all tori (S is its own centralizer), we note that $\mathrm{SL}_n(k)$ is split. The characters

$$\chi_i : a = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \mapsto a_i$$

for $i = 1, \ldots, n-1$ form a basis of the free abelian group $X_k(S)$. The cocharacters

$$\lambda_i : x \mapsto \mathrm{diag}(1, \ldots, 1, x, x^{-1}, 1, \ldots, 1) \quad \text{(with } x \text{ in the } i\text{th position, } i = 1, \ldots, n-1)$$

form a basis of $X_k^\vee$.

The Lie algebra of $\mathrm{SL}_n(k)$ can be identified with $\mathfrak{sl}_n(k)$, the Lie algebra of matrices with trace zero. If we denote by e_{ij} the ijth elementary (nilpotent) matrix (which belongs to $\mathfrak{sl}_n(k)$ as soon as $i \neq j$), we see that $\mathrm{Ad}(a)(e_{ij}) = \chi_i(a)\chi_j(a)^{-1}$. In consequence, the set of roots is precisely $\Phi_k(S) = \{\chi_i\chi_j^{-1} \mid i \neq j\}$. One can show that the set of coroots is $\Phi_k^\vee(S) = \{(\lambda_i\lambda_{i+1}\ldots\lambda_j)^{\pm 1} \mid i < j\}$ and check that $\Phi_k(S)$ is the root system of type A_{n-1}.

For $i \neq j$, Let $u_{ij}(x)$ denote the elementary unipotent matrix with x as its ijth entry. Then u_{ij} defines a isomorphism $\mathrm{G}_a \to U_{ij} : x \to u_{ij}(x)$ onto the root group associated to the root $\alpha_{ij} = \chi_i\chi_j^{-1}$. The parabolic subgroups containing S can then be obtained as the groups generated by S and some appropriate subset of the root groups $\{U_{ij} \mid i \neq j\}$. For example, the following subgroups are Borel subgroups containing S:

$$\left\{\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \in \mathrm{SL}_n(k)\right\}, \quad \left\{\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix} \in \mathrm{SL}_n(k)\right\}, \quad \left\{\begin{pmatrix} * & 0 & * & \cdots & * \\ * & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \in \mathrm{SL}_n(k)\right\}.$$

The spherical building

2.2.5 Definition (Spherical building)**.** The *spherical building of G over k* is the abstract simplicial complex $\mathcal{B} = \mathcal{B}(G/k)$ constructed as follows. The simplices of $\mathcal{B}$ are the proper parabolic k-subgroups of G. A simplex P is a facet of a simplex P' if and only if $P' \subset P$. The maximal simplices of $\mathcal{B}$ are called *chambers*, and the facets of codimension 1 of a chamber are called *panels.* The group acts on $\mathcal{B}$ by conjugation, and in virtue of 2.1.17, this action is transitive on the chambers. In particular, all chambers have the same dimension d. The *rank of $\mathcal{B}$* is the integer $d + 1$; it equals the k-rank of G.

Thus, the vertices of the building $\mathcal{B}$ are the maximal proper parabolic k-subgroups, and a set $\{P_1, \ldots, P_m\}$ of simplices determines a simplex of $\mathcal{B}$ if and only if the intersection

$P_1 \cap \cdots \cap P_m$ (contains, hence) is a parabolic subgroup. Conversely, the vertices of a given simplex P are uniquely determined: they are exactly the maximal proper parabolics containing P. Since parabolic subgroups are self-normalizing, the stabilizer of any simplex P is precisely P. As a consequence, if an element $g \in G(k)$ stabilizes a simplex P, it fixes P pointwise.

The simplicial complex $\mathcal{B}(G/k)$ defined above is a *building*, in the sense that there exists a collection of subcomplexes called *apartments* such that

(i) each apartment A is a Coxeter complex,

(ii) $\mathcal{B}$ is the union of all its apartments,

(iii) any two simplices P and Q of $\mathcal{B}$ are contained in a common apartment, and

(iv) if P and Q are contained in two apartments A and A', there exists an isomorphism $A \to A'$ fixing P and Q pointwise.

The spherical building $\mathcal{B}(G/k)$ has a canonical system of apartments, indexed by the set of maximal k-split tori of G: to each k-split torus T of G, one associates the subcomplex A consisting of all the k-parabolics containing T. In view of the relation between parabolics containing T and parabolic subsets of Φ_k described previously, this apartment A is a Coxeter complex of the same type as the root system Φ_k. In particular, the Coxeter group associated to A is precisely the Weyl group W_k of Φ_k. The normalizer $\mathrm{N}_G(T)$ of T stabilizes the apartment A, the centralizer $\mathrm{Z}_G(T)$ of T fixes it, and the quotient $\mathrm{N}_G(T)(k)/\mathrm{Z}_G(T)(k)$ acts on A through W_k.

Towards a classification

2.2.6 Definition (Isogeny)**.** Let H_1, H_2 be affine algebraic k-groups. A k-*isogeny* is a k-morphism $\phi : H_1 \to H_2$ which is smooth, surjective and has finite central kernel. Isogenies induce a poset structure on the set of affine algebraic k-groups (up to isomorphism) in the obvious way, and connected components of this poset are called *isogeny classes*. When there is an isogeny between H_1 and H_2 (in any direction), we say that they are *isogenous*.

2.2.7 Theorem (Classification: the split case)**.** *If the group G is k-split (in particular, if k is algebraically closed), the absolutely simple group G is uniquely determined by its root datum $(X_k, \Phi_k, X_k^\vee, \Phi_k^\vee)$. Conversely, for each root datum $(X_0, \Phi_0, X_0^\vee, \Phi_0^\vee)$ with Φ_0 reduced and irreducible, there exists an absolutely simple split k-group G and a torus in G with this quadruple as its root datum. The isogeny class of G is uniquely determined by the type of the root system Φ_k. Moreover, given a reduced irreducible root system Φ_0, there is an order-preserving bijection between the isogeny class associated to Φ_0 and the poset of subgroups of the fundamental group of Φ_0.*

2.2.8 Definition (Simply connected and adjoint groups)**.** G is said to be *simply connected* (resp. *of adjoint form*) if G is maximal (resp. minimal) in its isogeny class. In virtue of the last part of theorem 2.2.7, each isogeny class has a unique (up to isomorphism) maximal (resp. minimal) element G, and every group H in this isogeny class is a quotient of G (resp. has G as a quotient) by a finite central subgroup.

2.2.9 Example. Since its coroot lattice equals its cocharacter lattice (see 2.2.4), $\mathrm{SL}_n(k)$ is simply connected. Hence any split k-group of type A_{n-1} is a quotient of $\mathrm{SL}_n(k)$ by a subgroup of its center, C, which is isomorphic to the group μ_n of n-th roots of unity. On the other hand, $\mathrm{PGL}_n(k)$ is of adjoint form, and the quotient of any split k-group of type A_{n-1} by its center will be isomorphic to $\mathrm{PGL}_n(k)$.

If the field k is not algebraically closed and the group G is not split, the root datum alone is not sufficient to pin down the structure of G. Two additional ingredients that we briefly describe below (see [Tit66]) are required: the knowledge of the anisotropic kernel of G and the Tits index of G.

As before, we fix a maximal k-split torus S of G. The *anisotropic kernel of G over k (with respect to S)* is the derived subgroup $\mathcal{D}\mathrm{Z}_G(S)$ of the centralizer $\mathrm{Z}_G(S)$ of S. It is an anisotropic, semisimple k-subgroup of G.

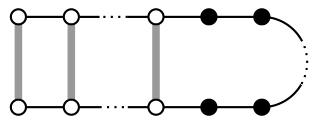
Let T be a maximal torus of G containing S which is defined over k (such a torus exists by a theorem of Grothendieck). Pick a basis for the absolute root system $\Phi(T)$ of G and let Δ denote the Dynkin diagram of Φ with respect to this basis (so that vertices of Δ correspond to simple roots). Let Δ_0 denote the subdiagram of Δ consisting of those simple roots which vanish on S, and let Δ_k be the Dynkin diagram associated to the basis of $\Phi_k(S)$ obtained by restricting the vertices of Δ to S. The *-*action* of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on Δ is constructed as follows: Γ permutes the conjugacy classes of maximal parabolic $\overline{k}$-subgroups of G, and these classes are in canonical bijection with the vertices of Δ. When the *-action is trivial (resp. non-trivial), the group G is said of *inner* (resp. *outer*) *type*, and G is called an *inner* (resp. *outer*) *form* of the *split form* of G, the split k-group with the same root datum as $G_{\overline{k}}$.

The *Tits index of G over k* is the data consisting of Δ, Δ_0 and the *-action of Γ on Δ. The index is often represented directly on the Dynkin diagram Δ by adorning it as follows: vertices belonging to the same orbits of Γ are placed close to each other, and the orbits whose elements do not belong to Δ_0 (i.e. whose restriction to S yield an element of Δ_k) are circled.

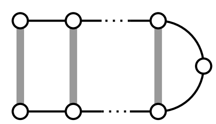
Note that the group G is quasisplit over k precisely when Δ_0 is empty, i.e. when all orbits are circled, or equivalently, when the anisotropic kernel of G is trivial.

2.2.10 Example (Quasisplit special unitary group)**.** Let l be a (separable) quadratic extension of k and recall from 2.1.3 (viii) the special unitary group $\mathrm{SU}_h(k)$ associated to a hermitian form

$h : l^n \times l^n \to l$ $(n \geq 2)$. Let r denote the k-rank of $\mathrm{SU}_h(k)$ (which is also the dimension of a largest totally isotropic subspace of l^n). The Tits index of $\mathrm{SU}_h(k)$ can be pictured as follows:



The *-action permutes the upper and lower branches of the diagram; in particular, $\mathrm{SU}_h(k)$ is an outer form of $\mathrm{SL}_n(k)$. Here, we painted the vertices of Δ_0 in black, vertices whose orbits should be circled are painted white, and instead of circling their orbit, two white vertices which are in the same orbit are connected by a vertical gray bar. There are $n-1$ vertices and r white orbits. When the group is quasisplit (i.e. when the hermitian form h is split), all vertices are white. If n is even, this leaves the rightmost vertex alone in its orbit:



2.2.11 Theorem (Classification: the general case)**.** *The k-group G is uniquely determined by its absolute root datum, its anisotropic kernel and its index.*

2.2.12 Remark. The above theorem reduces the problem of classifying absolutely simple k-groups to that of classifying the possible indices and the possible anisotropic kernels over k. The answer to the former is detailed in [Tit66]. The latter, however, is a much more arduous task and seems out of reach for the moment. (For example, over a given field k, the knowledge of all anisotropic kernels of type ^{1}A already amounts to the knowledge of all central division algebras over k.)

2.3 Galois cohomology

In this section, we briefly recall the basic facts about Galois cohomology that will be needed in the sequel. A good reference is [PR94, §1.3 and ch. 6].

2.3.1 Definition (Group cohomology)**.** Let G be a profinite group. To each G-module A (the action of G on A is required to be continuous, which in this setting simply means that each element of A is fixed by a finite-index subgroup of G), one associates a sequence $\mathrm{H}^n(G, A)$ ($n \in \mathbb{Z}_{\geq 0}$) of abelian groups, called the *cohomology groups of G with coefficients in A*. The cohomology groups $\mathrm{H}^n(G, A)$ have the following defining properties. The group $\mathrm{H}^0(G, A) = A^G$ is the set of fixed points of G in A. Given any exact sequence $0 \to A \to B \to C \to 0$ of G-modules, the higher cohomology groups extend functorially the sequence of fixed points to a long exact sequence

$$0 \to A^G \to B^G \to C^G \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, C) \to \mathrm{H}^2(G, A) \to \mathrm{H}^2(G, B) \to \cdots$$

in a universal way, i.e. any other exact extension on the right of the the sequence $0 \to A^G \to B^G \to C^G$ factors uniquely through the long exact sequence of cohomology groups above. (In particular, the assignment $A \mapsto \mathrm{H}^n(G, A)$ is functorial in A, and so are the connecting homomorphisms $\delta^n : \mathrm{H}^n(G, C) \to \mathrm{H}^{n+1}(G, A)$.)

The cohomology groups measure the failure of exactness of the functor $\mathrm{H}^0(G, -) : A \mapsto A^G$ in the following sense: $\mathrm{H}^0(G, -)$ is exact if and only if $\mathrm{H}^1(G, -) = 0$. More generally, if $0 \to A^G \to B^G \to C^G \to 0$ is not exact, the kernel of the map $\mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B)$ measures the failure of exactness at C^G.

Since A^G can be identified with $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ (where $\mathbb{Z}$ is endowed with the trivial $\mathbb{Z}G$-module structure), one sees that $\mathrm{H}^n(G, A)$ is naturally isomorphic to $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. Hence the cohomology groups $\mathrm{H}^n(G, A)$ can be computed using a projective resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$.

2.3.2 Definition (Galois cohomology)**.** Let l be a Galois extension of a field k, and let A be a $\mathrm{Gal}(l/k)$-module. The cohomology groups $\mathrm{H}^n(\mathrm{Gal}(l/k), A)$ are denoted $\mathrm{H}^n(l/k, A)$ and called the *Galois cohomology groups of* l/k *with coefficients in* A. When $l = k^s$ is the separable closure of k, they are simply denoted $\mathrm{H}^n(k, A)$.

We will also need Galois cohomology with coefficients in a non-commutative algebraic group. Let thus A be a (not necessarily commutative) group on which the profinite group G acts continuously and by automorphisms. We will denote the image of $a \in A$ under $g \in G$ by ${}^g a$. As before, we set $\mathrm{H}^0(G, A) = A^G$. A continuous map $f : G \to A$ is called a 1-*cocycle with values in* A if $f(gh) = f(g)\, {}^g(f(h))$ for any $g, h \in G$. Two 1-cocycles f_1 and f_2 are called *equivalent* if there exists an element $c \in A$ such that $f_1(g) = c^{-1} f_2(g)\, {}^g c$. This defines an equivalence relation on the set $\mathrm{Z}^1(G, A)$ of all 1-cocycles with values in A, and the set of equivalence classes for this relation is called the *first cohomology set of* G *with coefficients in* A and is denoted $\mathrm{H}^1(G, A)$. The class of the trivial cocycle $g \mapsto e$ is distinguished, so that $\mathrm{H}^1(G, A)$ becomes a pointed set.

When A is an abelian group, $\mathrm{H}^1(G, A)$ is also an abelian group and we recover the standard construction of the first cohomology group.

Let B be a normal subgroup of A stable under the action of G, so that G also acts on the group A/B. As in the abelian case, associated to the sequence $1 \to B \to A \to A/B \to 1$ there is a long exact sequence of *pointed sets*, but this time the sequence stops at the sixth term,

$$0 \to B^G \to A^G \to (A/B)^G \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, A/B),$$

unless B happens to be an abelian group, in which case the sequence can be extended by one more term, the standard cohomology group $\mathrm{H}^2(G, B)$.

Typically, A will be the set of l-points of an algebraic group H defined over some field k, endowed with the canonical action of $G = \mathrm{Gal}(l/k)$, in which case we will also use the notations $\mathrm{H}^n(l/k, A)$ and $\mathrm{H}^n(k, A)$ introduced above, or even write $\mathrm{H}^n(l/k, H)$ and $\mathrm{H}^n(k, H)$.

2.3.3 Examples. Here are a few common interpretations of some of the Galois cohomology groups.

(i) By definition, $\mathrm{H}^0(l/k, A)$ is the set of fixed points of $\mathrm{Gal}(l/k)$ in A, i.e. the elements of A *defined over* k. If H is a k-group, then $\mathrm{H}^0(l/k, H)$ is identified with $H(k)$, and a k-morphism $H_1 \to H_2$ induces a map $\mathrm{H}^0(l/k, H_1) \to \mathrm{H}^0(l/k, H_2)$ which corresponds to the the morphism $H_1(k) \to H_2(k)$ between k-points.

(ii) Hilbert's Satz 90 (which was originally due to Kummer) is equivalent to the vanishing of $\mathrm{H}^1(l/k, \mathrm{G}_m)$ for a cyclic extension l of k. Noether generalized this result to $\mathrm{H}^1(l/k, \mathrm{G}_m) = \{1\}$ for any Galois extension l of k.

(iii) Let φ be the k-morphism of k-groups $\mathrm{G}_m \to \mathrm{G}_m$ defined by $x \mapsto x^n$ and let μ_n denote its kernel, so that for any field extension l of k, $\mu_n(l)$ is the group of nth roots of unity in $l^\times$. The exact sequence of algebraic k-groups $1 \to \mu_n \to \mathrm{G}_m \xrightarrow{\varphi} \mathrm{G}_m \to 1$ yields a long exact sequence

$$1 \to \mu_n(k) \to k^\times \xrightarrow{\varphi} k^\times \to \mathrm{H}^1(l/k, \mu_n) \to \mathrm{H}^1(l/k, \mathrm{G}_m) = 1,$$

whose last term vanishes in view of the previous example. In consequence, we read that $\mathrm{H}^1(l/k, \mu_n) \cong k^\times / k^{\times n}$.

(iv) The group $\mathrm{H}^2(l/k, \mathrm{G}_m)$ can be identified with the subgroup $\mathrm{Br}(l/k)$ of the Brauer group $\mathrm{Br}(k)$ of k consisting of those central simple k-algebras which split over l. In particular, $\mathrm{H}^2(k, \mathrm{G}_m) = \mathrm{Br}(k)$ is the full Brauer group.

2.4 Bruhat-Tits theory

In this section, K will be a non-archimedean local field, $\mathcal{O}$ its ring of integers, p the characteristic of its residue field $\mathfrak{f}$, and G will be an absolutely simple simply connected algebraic K-group. We will describe the Bruhat-Tits building of G over K and how its geometry relates to the structure of the locally compact group $G(K)$. The main references are [Tit79] and [PR94, §3.4].

There exists a simplicial complex $\mathcal{A} = \mathcal{A}(G/K)$ endowed with an action of $G(K)$ by simplicial automorphisms called the *Bruhat-Tits* or *affine building of* G *over* K, which has the following properties.

(i) $\mathcal{A}$ is a building (in the sense described after 2.2.5), whose apartments are affine Coxeter complexes. The type of these complexes is obtained by extending the type of the group G. We will denote $\tilde{\Delta}_K$ the *local* or *affine Dynkin diagram* associated to the type of G. It is obtained from the Dynkin diagram Δ_K of G by adding a suitable vertex and connecting it to Δ_K with suitable edges (see table 2.4.1, where the added vertex is painted white).

(ii) There is a labeling of the vertices of $\mathcal{A}$ by vertices of $\tilde{\Delta}_K$, which, restricted to any apartment, is the usual type. Under this labeling, a simplex P in $\mathcal{A}$ corresponds to a unique subset of $\tilde{\Delta}_K$, called its *type*. The action of $G(K)$ on $\mathcal{A}$ is type-preserving.

(iii) The *affine Weyl group* $\tilde{W}_K$ *of* $G(K)$ is the Coxeter group of any of the apartments of $\mathcal{A}$. It is the semidirect product of the (spherical) Weyl group W_K of G by a free abelian group of rank equal to the rank of G.

(iv) Given a maximal K-split torus T of G, the maximal compact subgroup Z_c of $\mathrm{Z}_G(T)(K)$ is the fixator of an apartment A of $\mathcal{A}$. The group $\mathrm{N}_G(T)(K)$ stabilizes A and the quotient

$N_G(T)(K)/Z_c$ acts on A through the affine Weyl group $\tilde{W}_K$. With this identification, $Z_G(T)(K)/Z_c$ corresponds to the translation subgroup Λ of $\tilde{W}_K$.

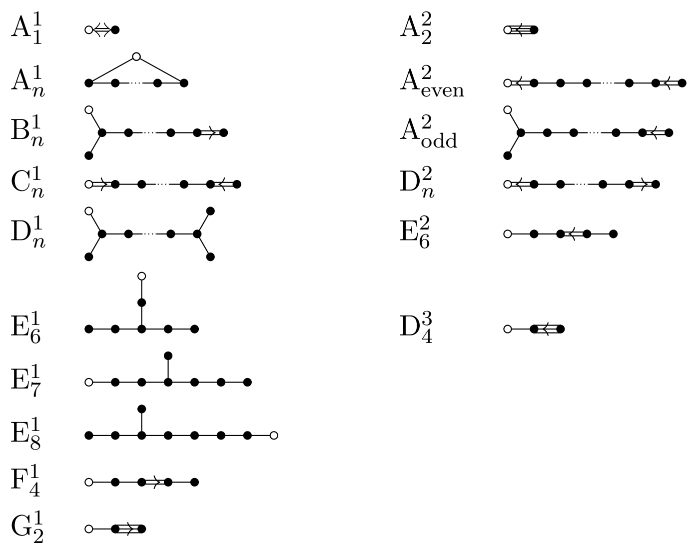
(v) Stabilizers in $G(K)$ of chambers in $\mathcal{A}$ are called *Iwarori subgroups*; they are exactly the normalizers of the maximal pro-p subgroups of $G(K)$. The group $G(K)$ acts transitively on the set of chambers, and equivalently, all Iwahori subgroups are conjugate under $G(K)$.

(vi) Stabilizers in $G(K)$ of simplices in $\mathcal{A}$ are called *parahoric subgroups*; they are precisely the compact open subgroups of $G(K)$ containing an Iwahori subgroup. This correspondence between simplices and parahoric subgroups is bijective and reverses inclusions. We will often denote a simplex and its stabilizer by the same letter (typically P). The *type* of a parahoric subgroup is then the type of the associated simplex.

(vii) The stabilizers in $G(K)$ of vertices in $\mathcal{A}$ are the maximal compact subgroups of $G(K)$.

(viii) The geometric realization of $\mathcal{A}$ is a contractible space.

Table 2.4.1. Irreducible Affine Dynkin diagrams.



2.4.2 Definition (Special and hyperspecial vertices)**.** A vertex $x \in \mathcal{A}$ is called *special* if, with respect to any apartment A containing x, the affine Weyl group $\tilde{W}_K$ is the semidirect product of the stabilizer $\tilde{W}_{K,x}$ of x by the translation subgroup Λ. If so, then $\tilde{W}_{K,x}$ is isomorphic to the (spherical) Weyl group W_K. If G splits over the maximal unramified extension $\widehat{K}$ of K, and x, seen as a vertex of $\mathcal{A}(G/\widehat{K})$, is special, then x is called *hyperspecial*. Hyperspecial vertices are special. If P is the stabilizer in $G(K)$ of a special (resp. hyperspecial) vertex, then P is called a *special* (resp. *hyperspecial*) *parahoric subgroup.*

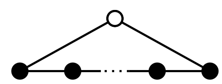
Special vertices can be detected using the affine Dynkin diagram: a vertex P is special if and only if the Weyl group associated to the diagram obtained from $\tilde{\Delta}_K$ by removing the type of P is the Weyl group of the diagram Δ_K. When G is quasi-split over K and splits over $\widehat{K}$, hyperspecial parahoric subgroups exist. When hyperspecial parahorics exist, they are the parahoric subgroups of $G(K)$ of maximal volume.

For each parahoric subgroup P of $G(K)$, there exists a smooth affine group scheme G_P defined over $\mathcal{O}$, called the *Bruhat-Tits group scheme associated to* P, whose group of $\mathcal{O}$-points $G_P(\mathcal{O})$ is isomorphic to P, whose generic fiber $G_P \times_{\mathcal{O}} K$ is isomorphic to G over K and whose special fiber $\overline{G}_P = G_P \times_{\mathcal{O}} \mathfrak{f}$ is called the *residual group at* P. The ring of $\mathcal{O}$-regular functions of G_P can be described as the $\mathcal{O}$-subalgebra of $K[G]$ consisting of the K-regular functions whose values on $\widehat{P}$ lie in $\widehat{\mathcal{O}}$ (here $\widehat{P}$ is the parabolic subgroup of $G(\widehat{K})$ associated to P and $\widehat{\mathcal{O}}$ is the ring of integers of $\widehat{K}$). The reduction homomorphism $P = G_P(\mathcal{O}) \to \overline{G}_P(\mathfrak{f})$ is surjective, and because G is simply connected, $\overline{G}_P$ is a connected $\mathfrak{f}$-group. Moreover, $\overline{G}_P$ admits a Levi decomposition over $\mathfrak{f}$.

The link in $\mathcal{A}$ of a simplex P is a building, which is isomorphic to the spherical building of (the reductive part of) the $\mathfrak{f}$-group $\overline{G}_P$. The group P then acts on the latter through the reduction homomorphism $P \to \overline{G}_P(\mathfrak{f})$. In particular, one can obtain the Dynkin diagram of the semisimple part of $\overline{G}_P$ by removing the type of P from $\tilde{\Delta}_K$. Fix a Borel $\mathfrak{f}$-subgroup $\overline{B}$ of a Levi $\mathfrak{f}$-

subgroup $\overline{M}$ of $\overline{G}_P$. The inverse image I of $\overline{B}(\mathfrak{f}) \cdot \mathrm{R}_u(\overline{G}_P)(\mathfrak{f})$ under the reduction homomorphism $P \to \overline{G}_P(\mathfrak{f})$ is an Iwahori subgroup of $G(K)$. The reduction homomorphism induces a type-preserving, index-preserving bijection between parahoric subgroups Q such that $I \subset Q \subset P$ and parabolic subgroups of $\overline{M}$ containing $\overline{B}$. In particular, $[P : I] = [\overline{M}(\mathfrak{f}) : \overline{B}(\mathfrak{f})]$.

2.4.3 Example ($\mathrm{SL}_n(\mathbb{Q}_p)$)**.** Let $G = \mathrm{SL}_n$ over $K = \mathbb{Q}_p$, so that $\mathcal{O} = \mathbb{Z}_p$, $\mathfrak{f} = \mathbb{F}_p$. Denote by v the p-adic valuation on $\mathbb{Q}_p$. Since G is split of type A_{n-1}, the affine diagram $\tilde{\Delta}$ of G is



Let T denote the maximal (split) torus consisting of diagonal matrices, and let N denote its normalizer in G. Since G is split, T is its own centralizer. The maximal bounded compact subgroup of T is the subgroup T_c of $T(K)$ consisting of diagonal matrices with entries in $\mathcal{O}^\times$; it is precisely the kernel of the valuation map $T(K) \cong (K^\times)^{n-1} \to \mathbb{Z}^{n-1}$. In consequence, $T(K)/T_c \cong \mathbb{Z}^{n-1}$. The affine Weyl group $\tilde{W} = \mathrm{N}(K)/T_c$ then fits in the split exact sequence

$$1 \to T(K)/T_c \to \tilde{W} \to \mathrm{N}(K)/T(K) = W \to 1.$$

Let $P = \mathrm{SL}_n(\mathcal{O})$; one can show that P is a maximal compact subgroup of $G(K)$. Thus, P is a maximal parahoric subgroup. The Bruhat-Tits group scheme associated to P is easy to describe in this case: it is simply SL_n defined over $\mathcal{O}$. Its special fiber $\overline{G}$ is then SL_n defined over $\mathfrak{f}$ (which was expected from the diagram), and the reduction homomorphism is the usual reduction mod p map $\mathrm{SL}_n(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{F}_p)$, which is indeed surjective. Note that in this example, $\overline{G}$ is already reductive (even simple); this is because G is split and P is maximal. Since $\mathrm{SL}_n(\mathfrak{f})$ and $\mathrm{SL}_n(K)$ have the same Weyl group, we see that P is a special parahoric; since G splits over K, P is in fact hyperspecial.

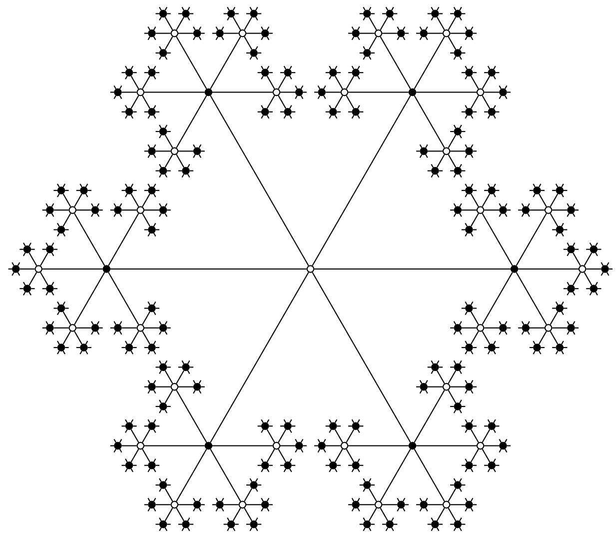
In view of the symmetry of the affine Dynkin diagram, this would also hold if we had chosen P to be any other maximal compact subgroup of $G(K)$, i.e. in this example, all maximal parahorics are hyperspecial. (Note that we do not claim that this symmetry comes from $G(K)$: the group $G(K)$ acts trivially on $\tilde{\Delta}$ since its action on $\mathcal{A}$ is type-preserving. Nonetheless, one can show in this case that the adjoint group $\mathrm{PGL}_n(K)$ does act vertex-transitively on $\mathcal{A}$, i.e. permutes the hyperspecial parahorics. This would imply that, in fact, the Bruhat-Tits group schemes associated to the different maximal parahorics are isomorphic.)

The preimage I under the reduction map of the Borel subgroup $\overline{B}(\mathfrak{f})$ of $\overline{G}$ of upper triangular matrices consists of matrices in $\mathrm{SL}_n(\mathcal{O})$ of the form

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ p\mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ p\mathcal{O} & p\mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p\mathcal{O} & p\mathcal{O} & p\mathcal{O} & \cdots & \mathcal{O} \end{pmatrix},$$

where $\mathcal{O}$, resp. $p\mathcal{O}$ denotes an entry in $\mathcal{O}$, resp. $p\mathcal{O}$. More generally, all the parahorics $Q \subset P$ are obtained as the preimages under the reduction map of parabolic subgroups $\overline{Q}$ of $\overline{G}$, and can be described similarly. Using this description, one can show that the maximal parahoric subgroups containing I are given by the stabilizers P_i in $\mathrm{SL}_n(\mathbb{Q}_p)$ of the lattices $\bigoplus^{n-i} \mathbb{Z}_p \oplus \bigoplus^{i} p\mathbb{Z}_p$ in $\mathbb{Q}_p^n$, for $i = 0, \ldots, n-1$. Notice that $P_0 = P$.

When $n = 2$, i.e. $G(K) = \mathrm{SL}_2(\mathbb{Q}_p)$, the apartments, hence the building have dimension 1. In view of the fact that the geometric realization of building is contractible, it must be a tree. Moreover, the number of edges d at a vertex P of this tree is the number of Iwahori subgroups of $G(K)$ contained in P, which is in turn the number of Borel subgroups $\overline{B}$ of $\overline{P}(\mathbb{F}_p) \cong \mathrm{SL}_2(\mathbb{F}_p)$. Now Borel subgroups are self-normalizing and conjugate, hence $d = [\mathrm{SL}_2(\mathbb{F}_p) : \overline{B}(\mathbb{F}_p)] = p + 1$. In conclusion, the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)$ is a $p+1$-regular tree. As an illustration, we depicted the building of $\mathrm{SL}_2(\mathbb{Q}_5)$ below. The two different types of vertices are painted black and white.



The Bruhat-Tits tree of $\mathrm{SL}_2(\mathbb{Q}_5)$.

2.5 Lattices in locally compact groups

Let G be a locally compact group, so that G can be endowed with a left-invariant Radon measure λ and a right-invariant Radon measure ρ, called (left or right) *Haar measures*; both invariant measures are unique up to a positive scalar (see for example [Bou07b] for the construction and properties of these measures). Left- and right-invariant measures need not coincide in general. In fact, the discrepancy is measured by the *modular function* Δ of G, defined as follows. For any $g \in G$, the measure ${}^g\lambda : A \mapsto \lambda(Ag)$ is obviously regular and left-invariant. In virtue of the uniqueness property, there exists a scalar $\Delta(g)$ such that ${}^g\lambda = \Delta(g)\lambda$. One shows that the modular function Δ is a continuous group homomorphism $G \to \mathbb{R}_{>0}$. The group G is callled *unimodular* if this homomorphism is trivial, which happens precisely when ${}^g\lambda = \lambda$ for all $g \in G$, i.e. when λ is also right-invariant.

2.5.1 Definition (Lattice). (i) A discrete subgroup Γ of G is called a *lattice* if the homogeneous space G/Γ can be endowed with a *finite*, regular, G-invariant measure.

(ii) A measurable subset $F \subset G$ is called a *coarse fundamental domain for* Γ *(on the right)* if $F\Gamma = G$. If in addition $\rho(F \cap F\gamma) = 0$ when $\gamma \in \Gamma - \{e\}$, then F is called a *fundamental domain*. (If further, $F \cap F\gamma = \emptyset$ when $\gamma \in \Gamma - \{e\}$, then F is sometimes called a *strict fundamental domain*.)

One shows that Γ is a lattice in G if and only if there exists a (coarse) fundamental domain F for Γ such that $\rho(F)$ is finite. If this is the case, the group G is necessarily unimodular and a finite invariant measure on G/Γ can be induced from an invariant measure on G via the local homeomorphism $G \to G/\Gamma$ (and reciprocally). Having fixed a bi-invariant measure μ on G and the induced measure on G/Γ (which we will abusively also denote using the same symbol), the measure of G/Γ equals the measure of any fundamental domain F for Γ.

(iii) The *covolume* of a lattice Γ in G is the quantity $\mu(G/\Gamma) = \mu(F)$ just described.

(iv) A lattice Γ for which the quotient G/Γ is compact is called a *uniform* or *cocompact lattice.*

In fact, if Γ is a discrete subgroup for which the quotient G/Γ is compact, one can pick a compact rough fundamental domain K for Γ. The regularity of the measure ρ then automatically implies that $\rho(K)$ is finite, hence that Γ is a lattice.

2.5.2 Examples.

(i) If G is compact, lattices in G are precisely finite subgroups.

(ii) If G is discrete, lattices in G are precisely subgroups of finite index.

(iii) $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$. The quotient $\mathbb{R}^n/\mathbb{Z}^n$ is the n-torus; it is compact and has measure 1 for the Lebesgue measure. Any lattice Λ in $\mathbb{R}^n$ is the image of $\mathbb{Z}^n$ under some element

g of $\mathrm{GL}_n(\mathbb{R})$, the group of automorphisms of $\mathbb{R}^n$. The covolume of $g\mathbb{Z}^n$ for the Lebesgue measure is $\det g$. In consequence, $\mathrm{SL}_n(\mathbb{R})$ acts transitively on the set of lattices in $\mathbb{R}^n$ with a given covolume.

(iv) Let $N_n(R)$ denote the group of $n \times n$ upper unitriangular matrices with coefficients in a ring R. The discrete Heisenberg group $N_3(\mathbb{Z})$ is a lattice in the continuous Heisenberg group $N_3(\mathbb{R})$. Since $\mathbb{Z}$ is discrete in $\mathbb{R}$, it is clear that $N_3(\mathbb{Z})$ is a discrete subgroup. To check that $N_3(\mathbb{Z})$ is a lattice, it suffices to show that the quotient $N_3(\mathbb{R})/N_3(\mathbb{Z})$ is compact. One readily checks that the map

$$N_3(\mathbb{R})/N_3(\mathbb{Z}) \to (\mathbb{R}/\mathbb{Z})^3 : \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \cdot N_3(\mathbb{Z}) \mapsto (a + \mathbb{Z}, b + \mathbb{Z}, c + \mathbb{Z})$$

is a homeomorphism. In fact, precomposing this map with the quotient map $N_3(\mathbb{R}) \to N_3(\mathbb{R})/N_3(\mathbb{Z})$ sends the canonical Haar measure μ on $N_3(\mathbb{R})$ (given by $d\mu = da\,db\,dc$ in the coordinates above) to the Lebesgue measure. This shows that the covolume of $N_3(\mathbb{Z})$ for the measure μ is 1.

This example immediately generalizes as follows. Let R be a locally compact topological ring and let S be a subring of R which is a lattice as an additive subgroup of R. Then $N_n(S)$ is a lattice in the locally compact group $N_n(R)$, and the quotient $N_n(R)/N_n(S)$ is homeomorphic to $(R/S)^{n(n-1)/2}$ in a way which preserves the canonical measures.

(v) Recall that $\mathrm{SL}_2(\mathbb{R})$ acts isometrically on the upper half plane $\mathcal{H}$ via Möbius transformations. Let T be a tiling of $\mathcal{H}$ by (possibly ideal) polygons, and suppose that the stabilizer Γ of T acts transitively on the tiles of T. Then Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, which is uniform if and only if the tiles of T are (pre)compact. If moreover $\Gamma/\{\pm 1\}$ acts simply transitively on the tiles of T (which can be achieved by subdividing the tiling), then for the appropriate normalization of the Haar measure, the covolume of Γ equals the area of a tile of T.

(vi) $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$. Again, it is clear that $\mathrm{SL}_n(\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}_n(\mathbb{R})$. Constructing a coarse fundamental domain with finite volume is the subject of reduction theory, and goes back to the work of Siegel [Sie39]. We will briefly describe here the construction of these Siegel sets. See [PR94, ch. 4] for more details.

The Iwasawa decomposition for $\mathrm{SL}_n(\mathbb{R})$ yields that the product map $K \times A \times N \to \mathrm{SL}_n(\mathbb{R}) : (k, a, n) \mapsto kan$ is a homeomorphism, where $K = \mathrm{SO}_n(\mathbb{R})$, A is the subgroup of diagonal matrices in $\mathrm{SL}_n(\mathbb{R})$ with positive entries, and N is the subgroup of upper unitriangular matrices. The *Siegel set* of parameters t, r is $\Sigma_{t,r} = KA_tN_r$, where $A_t = \{a \in A \mid a_i/a_{i+1} \leq t \text{ for } 1 \leq i \leq n-1\}$ and $N_r = \{n \in N \mid |n_{ij}| \leq r \text{ for } 1 \leq i < j \leq n\}$.

Using a Gram-Schmidt process, one shows that if $t \geq 2/\sqrt{3}$ and $r \geq 1/2$, the set $\Sigma_{t,r}$ is a coarse fundamental domain for $\mathrm{SL}_n(\mathbb{Z})$. To prove that $\mathrm{SL}_n(\mathbb{Z})$ is a lattice, it then remains to show that $\Sigma_{2/\sqrt{3},1/2}$ has finite measure. This can be done easily by writing an appropriate Haar measure in the coordinates given by the Iwasawa decomposition. Determining the covolume of $\mathrm{SL}_n(\mathbb{Z})$ (for an appropriate measure) is more complicated, and is also due to Siegel [Sie45]. We will come back to this matter with more advanced tools (cf. chapter 3).

The covolume of lattices in nilpotent groups is usually not bounded from below (we have already observed this for $\mathbb{R}^n$). Moreover, lattices in nilpotent groups are always cocompact. The situation for simple groups is rather different. Borel and Harder [BH78] showed that in every simple Lie group, there is always a uniform and a nonuniform lattice. Concerning the covolume, we have the following result due to Wang [Wan72].

2.5.3 Theorem. *Let G be a simple $\mathbb{Q}$-group not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ (endowed* *with a fixed Haar measure) and c be a positive constant. There are only finitely many conjugacy classes of lattices in $G(\mathbb{R})$ with covolume less than c. In particular, there exists a lattice of minimal covolume in $G(\mathbb{R})$.*

2.6 Arithmetic groups

In this section, k will be a number field (that is, a finite extension of $\mathbb{Q}$), and $\mathcal{O}_k$ will denote its ring of integers. V (resp. V_∞, V_f) will denote the set of places (resp. archimedean places, non-archimedean places) of k. For $v \in V$, we write k_v for the completion of k at the place v, and we abbreviate $k_\infty = \prod_{v\in V_\infty} k_v$. The main reference is [PR94, ch. 4].

2.6.1 Definition (Integer structures)**.** Let G be an algebraic k-group. An *integer structure*, or *$\mathcal{O}_k$-structure* for G is simply the datum of a k-embedding $\iota : G \to \mathrm{SL}_n$. Given an $\mathcal{O}_k$-structure for G, the *$\mathcal{O}_k$-points $G(\mathcal{O}_k)$ of G* are defined by $\iota^{-1}(\iota(G(k)) \cap \mathrm{SL}_n(\mathcal{O}_k))$.

Borel and Harish-Chandra proved the following far-reaching generalization of the fact that $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$ (cf. 2.5.2 (vi)).

2.6.2 Theorem (Borel, Harish-Chandra)**.** *Let G be a k-group endowed with a $\mathcal{O}_k$-structure. If G has no k-characters (in particular, if G is semisimple), then $G(\mathcal{O}_k)$ is a lattice when embedded diagonally in the locally compact group $G(k_\infty) = \prod_{v\in V_\infty} G(k_v)$.*

Moreover, Godement's criterion tells us precisely when this lattice is uniform:

2.6.3 Theorem (Godement's criterion)**.** *In the setting of theorem 2.6.2, the lattice $G(\mathcal{O}_k)$ is uniform if and only if G has no k-cocharacters (i.e. G is k-anisotropic).*

Borel and Harish-Chandra's theorem motivate the following definition.

2.6.4 Definition (Arithmetic group)**.** We keep the setting of theorem 2.6.2. Any subgroup Γ of $G(k_\infty)$ commensurable to the lattice $G(\mathcal{O}_k)$ is called an *arithmetic subgroup of* G or an *arithmetic lattice in* $G(k_\infty)$, or simply an *arithmetic group* when the embedding of Γ in $G(k_\infty)$ is either clear or irrelevant. More generally, any subgroup commensurable to the image of an arithmetic group Γ under a surjective morphism (of locally compact groups) with compact kernel $G(k_\infty) \to H$ will also be called an *arithmetic group*, or an *arithmetic lattice in* H.

2.6.5 Examples. (i) $\mathrm{SL}_n(\mathbb{Z})$ is an arithmetic subgroup of $\mathrm{SL}_n(\mathbb{R})$. Since $\mathrm{SL}_n(\mathbb{R})$ is isotropic, $\mathrm{SL}_n(\mathbb{Z})$ is a nonuniform lattice.

(ii) $\mathrm{SL}_n(\mathbb{Z}[i])$ is an arithmetic subgroup of $\mathrm{SL}_n(\mathbb{C})$. Similarly, $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}])$ is an arithmetic subgroup of $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ via the embedding $g \mapsto (\sigma_1(g), \sigma_2(g))$, where σ_1, σ_2 are the two embeddings $\mathbb{Q}[\sqrt{2}] \to \mathbb{R}$. Both these lattices are nonuniform.

(iii) Let $k = \mathbb{Q}(\alpha)$ be a totally real quadratic extension of $\mathbb{Q}$ and suppose (without loss of generality) that $\alpha^2 \in \mathbb{Q}$. Let l be the quadratic extension of k obtained by adjoining a square root of α to k, and denote by $^-$ its non-trivial automorphism fixing k. Consider the hermitian form $h : l^n \times l^n \to l$ defined by $h(x, y) = x_1\overline{y}_1 + \cdots + x_{n-1}\overline{y}_{n-1} + x_n\overline{y}_n$ and the associated special unitary group SU_h defined over k. Using the description of SU_h made in example 2.1.3 (viii), it is easy to construct a k-embedding $\mathrm{SU}_h \to \mathrm{SL}_{2n}$ and thus endow $G = \mathrm{SU}_h$ with an $\mathcal{O}_k$-structure. By Borel and Harish-Chandra's theorem, $\Gamma = G(\mathcal{O}_k)$ is a lattice in the locally compact group $G(k_{v_0}) \times G(k_{v_1})$, where v_0, v_1 are the two embeddings $k \to \mathbb{R}$. Let us say that $v_0(\alpha)$ is positive (hence $v_1(\alpha)$ is negative).

Observe that since $v_0(\alpha)$ is positive, the embedding $v_0 : k \to \mathbb{R}$ extends to two embeddings $l \to \mathbb{R}$. In particular, k_{v_0} contains l, and thus, as in example 2.1.3 (viii), we see that SU_h is isomorphic to SL_n over k_{v_0} (i.e. splits over k_{v_0}) . On the other hand, the embedding

$v_1 : k \to \mathbb{R}$ extends to two conjugate embeddings $l \to \mathbb{C}$, under which $^-$ is induced by complex conjugation. Consequently, given that h (seen over k_{v_1}) is positive definite, we see that $\mathrm{SU}_h(k_{v_1})$ is the usual compact real special unitary group. In conclusion, $G(k_{v_0}) \times G(k_{v_1}) \to G(k_{v_0}) \cong \mathrm{SL}_n(\mathbb{R})$ is a surjection with compact kernel $G(k_{v_1})$, and the image of Γ under this morphism is thus an arithmetic lattice in $\mathrm{SL}_n(\mathbb{R})$. It is a uniform lattice by Godement's criterion: the group SU_h must be k-anisotropic, since when extending scalars to k_{v_1}, it is anisotropic.

(iv) Let D be a central division algebra over $\mathbb{Q}$ of degree n, which splits over $\mathbb{R}$, i.e. $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_n(\mathbb{R})$ as $\mathbb{R}$-algebras. Let $\mathcal{O}$ be an order in D, i.e. a subring of D for which the map $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} \to D$ is an isomorphism. Let us pick an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_n(\mathbb{R})$ and identify D and $\mathcal{O}$ with their images in $\mathrm{M}_n(\mathbb{R})$. (With this convention, $\mathcal{O}$ is a lattice in the additive group of $\mathrm{M}_n(\mathbb{R})$.) One can show that $\mathcal{O}_1 = \mathcal{O} \cap \mathrm{SL}_n(\mathbb{R})$ is lattice in $\mathrm{SL}_n(\mathbb{R})$. This lattice is arithmetic; indeed, it is possible to endow $D^\times$ with the structure of an algebraic $\mathbb{Q}$-group with a $\mathbb{Z}$-structure in such a way the following holds. The set $D_1^\times$ of elements of D of reduced norm 1 form a closed $\mathbb{Q}$-subgroup, that we denote $\mathrm{SL}_1(D)$. Under the identification above, $\mathrm{SL}_1(D)$ corresponds to $D \cap \mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_1(D)(\mathbb{R})$ corresponds to $\mathrm{SL}_n(\mathbb{R})$, and $\mathrm{SL}_1(D)(\mathbb{Z})$ is commensurable to $\mathcal{O}_1$. Since $\mathrm{SL}_1(D)$ is $\mathbb{Q}$-anisotropic, $\mathcal{O}$ is a uniform lattice.

In general, given a number field k and a central division algebra D over k of degree d, one constructs for any $n \geq 1$ a k-group $\mathrm{SL}_n(D)$ whose k-points will form the group of elements of reduced norm 1 in $\mathrm{M}_n(D)$. The group $\mathrm{SL}_n(D)$ is an inner k-form of SL_{nd}, and it is anisotropic if and only if $n = 1$.

In the 1970's, relying on his superrigidity results, Margulis proved the following spectacular partial converse to Borel and Harish-Chandra's theorem.

2.6.6 Theorem (Margulis' arithmeticity)**.** *Let G be a simple $\mathbb{Q}$-group whose rank over $\mathbb{R}$ is at least 2, and let Γ be a lattice in $G(\mathbb{R})$. Then Γ is arithmetic.*

Although the statement above will be sufficient for our needs, there are more general versions of the arithmeticity theorem available, namely for semisimple groups of real rank ≥ 2, over arbitrary number fields, for the S-arithmetic setting, etc. We refer the reader to [Mar91] for further background, proofs and these variants of the arithmeticity theorem (see namely chapter IX).

The arithmeticity theorem will be the cornerstone of our approach to classify lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. It allows us to describe (up to commensurability) a lattice Γ in $\mathrm{SL}_n(\mathbb{R})$ in terms of some arithmetic data. The covolume of Γ can then be computed from this arithmetic data using Prasad's celebrated volume formula.

Chapter 3

Prasad's volume formula

In this chapter, we introduce the volume formula due to Prasad. We refer the reader to [Pra89] for the proof.

As in section 2.6, k will be a number field (that is, a finite extension of $\mathbb{Q}$), and $\mathcal{O}_k$ will denote its ring of integers. We denote by D_k the absolute value of the discriminant of k. V (resp. V_∞, V_f) will denote the set of places (resp. archimedean places, non-archimedean places) of k. For $v \in V$, we write k_v for the completion of k at the place v, and we abbreviate $k_\infty = \prod_{v \in V_\infty} k_v$. For $v \in V_f$, $\mathcal{O}_v$ denotes the ring of integers of k_v.

3.1 Tamagawa numbers

3.1.1 Definition (Adeles)**.** The *ring of adeles of* k is the restricted product

$$\mathbb{A}_k = \prod_{v \in V}{}' k_v$$

with respect to the family $\{\mathcal{O}_v\}_{v\in V_f}$ of distinguished subrings of the fields $\{k_v\}_{v\in V_f}$. This means that elements of $\mathbb{A}_k$ are tuples $(x_v)_{v\in V}$ such that $x_v \in \mathcal{O}_v$ for all but finitely many $v \in V_f$. The operations in $\mathbb{A}_k$ are performed componentwise. $\mathbb{A}_k$ is endowed with the restricted product topology; that is to say

$$\left\{ \prod_{v\in V} U_v \;\middle|\; U_v \text{ is open in } k_v\text{, and } U_v = \mathcal{O}_v \text{ for all but finitely many } v \in V_f \right\}$$

forms a basis of open sets for the topology on $\mathbb{A}_k$. With this, $\mathbb{A}_k$ becomes a locally compact ring.

The number field k embeds diagonally into $\mathbb{A}_k$ (in this way, $\mathbb{A}_k$ is a k-algebra), and each completion k_v embeds (as a topological ring) in $\mathbb{A}_k$ in the vth component. We will almost always omit these embeddings from the notation and identify k and k_v with their images in $\mathbb{A}_k$.

Let dx_v denote the Haar measure on k_v such that $\mathcal{O}_v$ has measure 1 for dx_v if $v \in V_f$, $dx_v = dx$ is the Lebesgue measure if $k_v = \mathbb{R}$, and $dx_v = idz \wedge d\overline{z} = 2dx \wedge dy$ if $k_v = \mathbb{C}$. The product measure $dx_{\mathbb{A}} = \prod_{v\in V} dx_v$ is a Haar mesure for the additive group of $\mathbb{A}_k$. The following proposition is a consequence of Minkowsky's geometry of numbers.

3.1.2 Proposition. *The image of k is a lattice in the additive group $\mathbb{A}_k$, whose covolume for the measure dx_V defined above is $D_k^{1/2}$.*

Let now G be an absolutely simple algebraic k-group. The $\mathbb{A}_k$-points $G(\mathbb{A}_k)$ of G form a locally compact group with the topology induced by $\mathbb{A}_k$. We may identify $G(\mathbb{A}_k)$ with the restricted product $\prod'_{v\in V} G(k_v)$ as follows. Pick a k-embedding of $\iota : G \to \mathrm{SL}_n$, and for $v \in V_f$ set $G(\mathcal{O}_v) = \iota^{-1}(G(k_v) \cap \mathrm{SL}_n(\mathcal{O}_v))$. Of course, the groups $G(\mathcal{O}_v)$ depend on the choice of ι, but it turns out that a different choice for ι will only alter finitely many of the $G(\mathcal{O}_v)$. Thus, the restricted product $\prod'_{v\in V} G(k_v)$ with respect to the family $\{G(\mathcal{O}_v)\}_{v\in V_f}$ is defined unequivocally.

The group $G(k)$ embeds in $G(\mathbb{A}_k)$ as a discrete subgroup, and in the identification just described this embedding corresponds to the diagonal embedding.

Let ω be an invariant exterior form on G defined over k, and denote by ω_v the Haar measure on $G(k_v)$ induced by ω after extending scalars to k_v. The product measure $\omega_{\mathbb{A}} = \prod_{v\in V} \omega_v$ is a Haar measure on $G(\mathbb{A}_k)$ which is independent of the choice of ω. Indeed, ω is unique up to a scalar $a \in k$, and in virtue of the product formula, the measure induced by $a\omega$ is

$$\prod_{v\in V} |a|_v \omega_v = \prod_{v\in V} |a|_v \cdot \prod_{v\in V} \omega_v = \omega_{\mathbb{A}}.$$

3.1.3 Definition (Tamagawa measure and Tamagawa number)**.** The measure $D_k^{-\frac{1}{2}\dim G}\omega_{\mathbb{A}}$ is called the *Tamagawa measure* on $G(\mathbb{A}_k)$. The *Tamagawa number of G over k* is the quantity

$$\tau_k(G) = D_k^{-\frac{1}{2}\dim G}\omega_{\mathbb{A}}(G(\mathbb{A}_k)/G(k)).$$

Using Borel and Harish-Chandra's theorem (2.6.2), one can show that the Tamagawa number of G is finite. Weil conjectured that in fact $\tau_k(G) = 1$ when G is simply connected. This conjecture was proved following a proposal of Jacquet and Langlands [JL70] by the combined work of Lai [Lai80] and Kottwitz [Kot88]. Many cases of the conjecture had already been verified in previous works of Demazure, Lai, Langlands, Mars, Ono, Tamagawa and Weil (see Ono's appendix to [Wei82]).

3.2 The volume formula

3.2.1 Definition (Principal arithmetic subgroup)**.** A collection of parahoric subgroups $\{P_v\}_{v\in V_f}$ of the groups $\{G(k_v)\}_{v\in V_f}$ is called *coherent* if $\prod_{v\ \in V_\infty} G(k_v) \times \prod_{v\in V_f} P_v$ is an open subgroup of $G(\mathbb{A}_k)$. This amounts to say that, after picking an embedding $\iota : G \to \mathrm{SL}_n$, the parahoric P_v

agrees with $G(\mathcal{O}_v)$ for all but finitely many $v \in V_f$. The *principal arithmetic subgroup determined by a coherent collection of parahorics* $\{P_v\}_{v\in V_f}$ is the subgroup

$$\Lambda = G(k) \cap \Big(\prod_{v\in V_\infty} G(k_v) \times \prod_{v\in V_f} P_v \Big)$$

of $G(k)$. It is a lattice when embedded diagonally in $\prod_{v\ \in V_\infty} G(k_v)$.

3.2.2 Theorem (Strong approximation)**.** *Suppose that the k-group G is simply connected and that G is isotropic over k_v for some $v \in V_\infty$. Then $G(k) \cdot \prod_{v\in V_\infty} G(k_v)$ is dense in $G(\mathbb{A}_k)$. In consequence, if $\{P_v\}_{v\in V_f}$ is a coherent collection of parahorics, then*

$$G(k) \cdot \Big(\prod_{v\in V_\infty} G(k_v) \times \prod_{v\in V_f} P_v \Big) = G(\mathbb{A}_k).$$

For the rest of this chapter, we will assume that G is an absolutely simple simply connected k-group and that $G(k_v)$ is isotropic for at least one $v \in V_\infty$. Recall that $G(k_\infty) = \prod_{v\ \in V_\infty} G(k_v)$. By strong approximation, there is an isomorphism

$$G(\mathbb{A}_k)/G(k) \cong \Big(G(k_\infty) \times \prod_{v\in V_f} P_v \Big) \Big/ \Big(G(k) \cap \Big(G(k_\infty) \times \prod_{v\in V_f} P_v \Big) \Big)$$

which composed with the projection onto the V_∞-components yields a fibration

$$G(\mathbb{A}_k)/G(k) \to G(k_\infty)/\Lambda$$

whose fibers are translates of $\prod_{v\in V_f} P_v$. In consequence, we get the following expression for the covolume of Λ:

$$\omega_\infty(G(k_\infty)/\Lambda) = \omega_{\mathbb{A}}(G(\mathbb{A}_k)/G(k)) \cdot \prod_{v\in V_f} \omega_v(P_v)^{-1} = D_k^{\frac{1}{2}\dim G} \cdot \prod_{v\in V_f} \omega_v(P_v)^{-1},$$

where $\omega_\infty = \prod_{v\in V_\infty} \omega_v$ and we have used the fact that $\tau_k(G) = 1$.

From this and using a clever renormalization of the measures, Prasad was able to obtain a concise, much more practical formula for the covolume of Λ. Before describing the formula, we fix the measure on $G(k_\infty)$ as follows.

We fix a left-invariant exterior form ω_{qs} defined over k on the quasi-split inner k-form $\mathcal{G}$ of G. As before, ω_{qs} induces for each $v \in V_\infty$ an invariant form on $\mathcal{G}(k_v)$, and in turn on any maximal compact subgroup of $\mathcal{G}(\mathbb{C})$ through their common Lie algebra. For each $v \in V_\infty$, we choose $c_v \in k_v$ such that the corresponding maximal compact subgroup has measure 1 for the Haar measure determined in this way by $c_v\omega_{qs}$. Let $\varphi : G \to \mathcal{G}$ be an isomorphism defined over some Galois extension K of k, such that $\varphi^{-1} \circ {}^\gamma\varphi$ is an inner automorphism of G for all γ in the Galois group of K over k. Then φ induces an invariant form $\omega^* = \varphi^*(\omega_{qs})$ on G, defined over k. Once again, ω^* induces for each $v \in V_\infty$ a form on $G(k_v)$ and then a form on any maximal compact subgroup of $G(\mathbb{C})$ through their Lie algebras. It turns out [Pra89, §3.5] that the volume of any such maximal compact subgroup for the Haar measure determined in this way by $c_v\omega^*$ is again 1. We denote by μ_v the Haar measure determined on $G(k_v)$ by $c_v\omega^*$, and by μ_∞ the product measure $\prod_{v\in V_\infty} \mu_v$.

3.2.3 Theorem (Prasad's volume formula)**.** *For the measure μ_∞ described above and Λ the principal arithmetic subgroup associated to the collection of parahorics $\{P_v\}_{v\in V_f}$, we have*

$$\mu_\infty(G(k_\infty)/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_l/D_k^{[l:k]})^{\frac{1}{2}\mathfrak{s}(\mathcal{G})}\left(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]}\prod_{v\in V_f} e(P_v).$$

Here,

- *l is the splitting field of the quasi-split inner k-form $\mathcal{G}$ of G,*
- *r is the absolute rank of G,*
- *$\mathfrak{s}(\mathcal{G}) = 0$ if $\mathcal{G}$ is split, otherwise $\mathfrak{s}(\mathcal{G})$ is a positive integer which depends only on the relative*

root system of $\mathcal{G}$,

- *the m_i's are the exponents of the compact form of $\mathcal{G}$, and*
- *the local factor $e(P_v) = \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathcal{M}}_v)/2}}{\# \overline{M}_v(\mathfrak{f}_v)}$ is the inverse of the volume of P_v for a particular measure. It can be computed from the $\mathfrak{f}_v$-groups $\overline{M}_v$ and $\overline{\mathcal{M}}_v$ which are defined as follows. Let G_v be the smooth affine $\mathcal{O}_v$-group scheme associated to the parahoric P_v of $G(k_v)$, and let $\overline{G}_v$ denote its special fiber, i.e. the $\mathfrak{f}_v$-group $G_v \times_{\mathcal{O}_v} \mathfrak{f}_v$. We pick $\overline{M}_v$ to be a Levi $\mathfrak{f}_v$-subgroup of $\overline{G}_v$ (which exist, see [Tit79]), meaning that $\overline{M}_v$ is a connected reductive $\mathfrak{f}_v$-subgroup such that $\overline{G}_v = \overline{M}_v \cdot \mathrm{R}_u(\overline{G}_v)$. The group $\overline{\mathcal{M}}_v$ is defined in the analogous way after fixing a coherent collection of parahoric subgroups $\mathcal{P}_v$ of $\mathcal{G}(k_v)$, which are chosen hyperspecial when possible, and otherwise $\mathcal{P}_v$ is a specific choice of special parahoric.*

We refer to [Pra89] for the unexplained notation.

3.2.4 Example (Covolume of $\mathrm{SL}_n(\mathbb{Z})$)**.** Let $k = \mathbb{Q}$ (hence $V_\infty = \{\infty\}$, $V_f = \{p \in \mathbb{N} \mid p \text{ is prime}\}$) and let $G = \mathrm{SL}_n$. The parahorics $P_v = \mathrm{SL}_n(\mathbb{Z}_v)$ for $v \in V_f$ form a coherent family, and the principal arithmetic subgroup associated to it is $\mathrm{SL}_n(\mathbb{Q}) \cap \prod_{v \in V_f} \mathrm{SL}_n(\mathbb{Z}_v) = \mathrm{SL}_n(\mathbb{Z})$. We will write μ_0 for the measure μ_∞ constructed above. Since SL_n is split over $\mathbb{Q}$, Prasad's formula reads

$$\mu_0(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \left(\prod_{i=1}^{r} \frac{i!}{(2\pi)^{i+1}} \right) \cdot \prod_{v \in V_f} e(P_v)$$

and it remains to compute the local factors $e(P_v)$. We have $\overline{M}_v = \overline{\mathcal{M}}_v = \mathrm{SL}_n$ defined over the finite field $\mathfrak{f}_v$ with q_v elements. Using the formula for the order of $\mathrm{SL}_n(\mathfrak{f}_v)$, we find

$$e(P_v) = \frac{(q_v - 1) q_v^{n^2-1}}{\prod_{i=0}^{n-1} (q_v^n - q_v^i)} = \prod_{i=2}^{n} \frac{1}{1 - q_v^{-i}},$$

and thus $\prod_{v \in V_f} e(P_v) = \prod_{i=2}^{n} \zeta(i)$. Altogether,

$$\mu_0(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \left(\prod_{i=1}^{r} \frac{i!}{(2\pi)^{i+1}} \right) \cdot \prod_{i=2}^{n} \zeta(i).$$

Chapter 4

Lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$

This chapter is dedicated to the proof of the main result of this thesis:

Theorem. *Let $n \geq 3$ and let Γ be a lattice of minimal covolume for some (any) Haar measure in $\mathrm{SL}_n(\mathbb{R})$. Then $\sigma(\Gamma) = \mathrm{SL}_n(\mathbb{Z})$ for some (algebraic) automorphism σ of $\mathrm{SL}_n(\mathbb{R})$.*

Before outlining the strategy of the proof, we (re)introduce some notation.

- $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively denote the sets of strictly positive natural, rational, real and complex numbers. For p a place or a prime, $\mathbb{Q}_p$ denotes the field of p-adic numbers and $\mathbb{Z}_p$ its ring of p-adic integers. $\mathbb{F}_p$ denotes the finite field with p elements.

- In what is to follow, we will fix a number field k of degree m, and V, V_∞ and V_f will always denote the set of places, archimedean places and non-archimedean places of k. We will always normalize each non-archimedean place v so that $\operatorname{im} v = \mathbb{Z}$.

- For $v \in V$, k_v will denote the v-adic completion of k. For $v \in V_f$, $\widehat{k}_v$ is the maximal unramified extension of k_v, $\mathfrak{f}_v$ denotes the residue field of k at v and $q_v = \#\mathfrak{f}_v$ is the cardinality of the latter.

- $\mathbb{A}_k$ denotes the ring of adeles of k, and the adeles of $\mathbb{Q}$ will be abbreviated $\mathbb{A}$.
- When working with the adele points $G(\mathbb{A}_k)$ (or variations of them, e.g. finite adeles) of an algebraic group G, we will freely identify $G(k)$ with its image in $G(\mathbb{A}_k)$ under the diagonal embedding, and vice-versa.
- For l a finite extension of k, we denote D_l the absolute value of the discriminant of l (over $\mathbb{Q}$) and $\mathfrak{d}_{l/k}$ the relative discriminant of l over k; h_l is the class number of l. The units of l will be denoted by U_l, and the subgroup of roots of unity in l by $\mu(l)$.
- G will be a simply connected absolutely almost simple group (of type A_r) defined over k. We denote $r = n - 1$ its absolute rank, and for $v \in V_f$, r_v is its rank over $\widehat{k}_v$.
- $\mathcal{G}$ denotes the quasi-split inner k-form of G, l will denote its splitting field.
- SU_n denotes the special unitary group defined over $\mathbb{R}$ associated to the positive-definite hermitian form $x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$ on $\mathbb{C}^n$. Its group $\mathrm{SU}_n(\mathbb{R})$ of real points is the usual special unitary group, the unique compact connected simply connected almost simple Lie group of type A_{n-1}.
- ζ denotes Riemann's zeta function.
- For $n \in \mathbb{Z}$, we set $\tilde{n} = 1$ or 2 if n is respectively odd or even.
- For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the ceiling of x, that is the smallest integer n such that $n \geq x$.
- V_n will denote the quantity $\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$.

4.1 A brief outline of the proof

The argument relies in an indispensable way on the important work of Prasad [Pra89] and Borel and Prasad [BP89] (there will be multiple references to results contained in these two articles). We will proceed as follows.

We start with a lattice Γ of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. Using Margulis' arithmeticity theorem and Rohlfs' maximality criterion, we find a number field k, an archimedean place v_0 and a simply connected absolutely almost simple k-group G for which Γ is identified with the normalizer of a principal arithmetic subgroup Λ in $G(k_{v_0})$. The latter means that there is a collection of parahoric subgroups $\{P_v\}_{v \in V_f}$ such that Λ consists precisely of the elements of $G(k)$ whose image in $G(k_v)$ lies in P_v for all $v \in V_f$. This allows us to express the covolume of Γ as $\mu(G(k_{v_0})/\Gamma) = [\Gamma : \Lambda]^{-1}\, \mu(G(k_{v_0})/\Lambda)$.

The factor $\mu(G(k_{v_0})/\Lambda)$ can be computed using Prasad's volume formula [Pra89], and the result depends on the arithmetics of k and of the parahorics P_v, as well as on the quasi-split inner form of G.

On the other hand, the index $[\Gamma : \Lambda]$ can be controlled using techniques developed by Rohlfs [Roh79], and Borel and Prasad [BP89]. The bound depends namely on the first Galois cohomology group of the center of G and on its action on the types of the parahorics P_v.

Once we have an estimate on the covolume of Γ, we can compare it to the covolume of $\mathrm{SL}_n(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{R})$. We argue that for the former not to exceed the latter, it must be that k is $\mathbb{Q}$, G is an inner form of SL_n, and all the parahorics are hyperspecial. This is carried out in sections 4.4-4.6.

Finally, using local-global techniques, we conclude that Γ must be the image of $\mathrm{SL}_n(\mathbb{Z})$ under some automorphism of $\mathrm{SL}_n(\mathbb{R})$.

4.2 The setting

Let Γ be a lattice of minimal covolume for μ_0 (see 3.2.4) in $\mathrm{SL}_n(\mathbb{R})$; in particular, Γ is a maximal lattice. By Margulis' arithmeticity theorem [Mar91, ch. IX §1.5] (see 2.6.6) and Rohlfs' maximality criterion [BP89, prop. 1.4] combined, there is a number field k, a place $v_0 \in V_\infty$, a simply connected absolutely simple group G defined over k, and a parahoric subgroup P_v of $G(k_v)$ for each $v \in V_f$, such that:

(i) $k_{v_0} = \mathbb{R}$

(ii) there is an isomorphism $\iota : \mathrm{SL}_n \to G$ defined over k_{v_0} (in particular, $\mathrm{SL}_n(\mathbb{R}) \cong G(k_{v_0})$)

(iii) $G(k_v)$ is compact for any archimedean place $v \neq v_0$

(iv) the collection $\{P_v\}_{v \in V_f}$ is coherent

(v) $\iota(\Gamma)$ is the normalizer of the lattice $\Lambda = G(k) \cap \iota(\Gamma)$ in $G(k_{v_0})$, and $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$ is the principal arithmetic subgroup determined by the collection $\{P_v\}_{v \in V_f}$.

This already imposes the signature of k and of the splitting field l of the quasi-split inner form $\mathcal{G}$ of G. Indeed, we have $k_v \cong \mathbb{R}$ for $v \in V_\infty - \{v_0\}$, otherwise $G(k_v) \cong \mathrm{SL}_n(\mathbb{C})$ is not compact; hence k is totally real. Note that in fact, for each $v \in V_\infty - \{v_0\}$, $G(k_v)$ is isomorphic to $\mathrm{SU}_n(\mathbb{R})$, the unique compact connected simply connected almost simple Lie group of type A_{n-1}.

Recall that since G is of type A, either $l = k$ or l is a quadratic extension of k. Regardless, if $v \in V_\infty - \{v_0\}$, it may not be that l embeds into k_v: indeed, if this happens, then $\mathcal{G}$ splits over k_v, and thus G would be an inner k_v-form of SL_n. This prohibits $G(k_v)$ from being compact, as inner k_v-forms of SL_n are isotropic when $n \geq 3$. Thus, in the former case, when G is an inner k-form, it must be that $V_\infty - \{v_0\}$ is empty, i.e. $l = k = \mathbb{Q}$. In the latter case, when G is an outer k-form, for each $v \in V_\infty - \{v_0\}$ the real embedding $k \to k_v$ extends to two (conjugate) complex embeddings of l. On the other hand, G, hence $\mathcal{G}$, splits over k_{v_0}, thus l embeds in k_{v_0}. Combined, we see in this case that the signature of l is $(2, m-1)$.

On G, we pick a left-invariant exterior form ω of highest degree which is defined over k. The form ω induces a left-invariant form on $G(k_{v_0})$, also to be denoted ω, which in turn induces a left-invariant form on $\mathrm{SU}_n(\mathbb{R})$ through their common Lie algebra. Let $c \in \mathbb{R}$ be such that $\mathrm{SU}_n(\mathbb{R})$ has volume 1 for the Haar measure determined in this way by $c\omega$; we denote μ the Haar measure determined by $c\omega$ on $G(k_{v_0})$. By construction, μ agrees with the measure induced from μ_0 through the isomorphism ι. In what follows, we will freely identify $\mathrm{SL}_n(\mathbb{R})$ with $G(k_{v_0})$, Γ with its image $\iota(\Gamma)$ and μ_0 with μ. With this, we have

$$\mu_0(\mathrm{SL}_n(\mathbb{R})/\Gamma) = \mu(G(k_{v_0})/\Gamma) = [\Gamma : \Lambda]^{-1}\ \mu(G(k_{v_0})/\Lambda).$$

If F is a fundamental domain for Λ in $G(k_{v_0})$, then $F_\infty = F \times \prod_{v \in V_\infty - \{v_0\}} G(k_v)$ is a fundamental domain for Λ in G_∞. Therefore, in view of the normalization of the measures μ_v (see the paragraph preceding 3.2.3),

$$\mu_\infty(G_\infty/\Lambda) = \mu_\infty(F_\infty) = \mu_{v_0}(F) \cdot \prod_{v \in V_\infty - \{v_0\}} \mu_v(G(k_v)) = \mu_{v_0}(F) = \mu(G(k_{v_0})/\Lambda).$$

Using Prasad's volume formula (3.2.3), we can compute

$$\mu(G(k_{v_0})/\Lambda) = \mu_\infty(G_\infty/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_l/D_k^{[l:k]})^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \left(\prod_{i=1}^{r} \frac{i!}{(2\pi)^{i+1}}\right)^{[k:\mathbb{Q}]} \prod_{v \in V_f} e(P_v). \qquad (4.1)$$

We recall that l is the splitting field of the quasi-split inner k-form $\mathcal{G}$ of G, $r = n-1$ is the absolute rank of G, $\mathfrak{s}(\mathcal{G}) = 0$ if $\mathcal{G}$ is split, otherwise $\mathfrak{s}(\mathcal{G}) = \frac{1}{2}r(r+3)$ if r is even or $\mathfrak{s}(\mathcal{G}) = \frac{1}{2}(r-1)(r+2)$ if r is odd, and $e(P_v)$ are the local factors associated to the collection $\{P_v\}_{v \in V_f}$.

4.3 An upper bound on the index

For the convenience of the reader, we briefly recollect the upper bound on the index $[\Gamma : \Lambda]$ developed by Borel and Prasad. The complete exposition, proofs and references are to be found in [BP89, §2 & §5] (in the present setting, $\mathcal{S} = \{v_0\}$, $G' = G$, $\Gamma' = \Gamma$, etc.).

For each place $v \in V_f$, we fix a maximal k_v-split torus T_v of G; we also fix an Iwahori subgroup I_v of $G(k_v)$ such that the chamber in the affine building of $G(k_v)$ fixed by I_v is contained in the apartment corresponding to T_v. We denote by Δ_v the basis determined by I_v of the affine root system of $G(k_v)$ relative to T_v.

The group $\mathrm{Aut}(G(k_v))$, hence also the adjoint group $\overline{G}(k_v)$, acts on Δ_v; we denote by $\xi_v : \overline{G}(k_v) \to \mathrm{Aut}(\Delta_v)$ the corresponding morphism. Let Ξ_v be the image of ξ_v.

Let C be the center of G and $\varphi : G \to \overline{G}$ the natural central isogeny, so that there is an exact sequence of algebraic groups

$$1 \to C \to G \xrightarrow{\varphi} \overline{G} \to 1.$$

This sequence gives rise to long exact sequences (of pointed sets), which we store in the following commutative diagram ($v \in V$).

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & C(k) & \longrightarrow & G(k) & \xrightarrow{\varphi} & \overline{G}(k) & \xrightarrow{\delta} & \mathrm{H}^1(k, C) & \longrightarrow & \mathrm{H}^1(k, G) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & C(k_v) & \longrightarrow & G(k_v) & \xrightarrow{\varphi} & \overline{G}(k_v) & \xrightarrow{\delta_v} & \mathrm{H}^1(k_v, C) & \longrightarrow & \mathrm{H}^1(k_v, G) \end{array} \tag{4.2}$$

When $v \in V_f$, we have that $\mathrm{H}^1(k_v, G) = 1$ by a result of Kneser [Kne65] and thus δ_v induces an isomorphism

$$\overline{G}(k_v)/\varphi(G(k_v)) \cong \mathrm{H}^1(k_v, C).$$

Recall that ξ_v is trivial on $\varphi(G(k_v))$. Thus ξ_v induces a map $\mathrm{H}^1(k_v, C) \to \Xi_v$, which we abusively denote by ξ_v as well.

Let $\Delta = \prod_{v \in V_f} \Delta_v$, $\Xi = \bigoplus_{v \in V_f} \Xi_v$ and $\Theta = \prod_{v \in V_f} \Theta_v$, where $\Theta_v \subset \Delta_v$ is the type of the parahoric P_v associated to Λ. Ξ acts on Δ componentwise, and we denote by Ξ_{Θ_v} the stabilizer of Θ_v in Ξ_v and Ξ_Θ the stabilizer of Θ in Ξ. The morphisms ξ_v induce a map

$$\xi : \mathrm{H}^1(k, C) \to \Xi : c \mapsto \xi(c) = (\xi_v(c_v))_{v \in V_f}$$

where c_v denotes the image of c in $\mathrm{H}^1(k_v, C)$. With this, we define

$$\mathrm{H}^1(k, C)_\Theta = \{c \in \mathrm{H}^1(k, C) \mid \xi(c) \in \Xi_\Theta\}$$

$$\mathrm{H}^1(k, C)'_\Theta = \{c \in \mathrm{H}^1(k, C)_\Theta \mid c_{v_0} = 1\}$$

$$\mathrm{H}^1(k, C)_\xi = \{c \in \mathrm{H}^1(k, C) \mid \xi(c) = 1\}.$$

Borel and Prasad [BP89, prop. 2.9] use the exact sequence due to Rohlfs

$$1 \to C(k_{v_0})/(C(k) \cap \Lambda) \to \Gamma/\Lambda \to \delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta \to 1.$$

Since $k_{v_0} = \mathbb{R}$, $C(k_{v_0}) = \{1\}$ or $\{1, -1\}$ depending whether n is odd or even. In particular, it follows that $C(k_{v_0}) = C(k) \cap \Lambda$ and $\Gamma/\Lambda \cong \delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta$. Also, it is clear that the kernel of ξ restricted to $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta$ is contained in $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi$, implying that $\# \left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)'_\Theta\right) \leq \# \left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \cdot \prod_{v \in V_f} \# \Xi_{\Theta_v}$, and in turn,

$$[\Gamma : \Lambda] \leq \# \left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \cdot \prod_{v \in V_f} \# \Xi_{\Theta_v} \leq \# \mathrm{H}^1(k, C)_\xi \cdot \prod_{v \in V_f} \# \Xi_{\Theta_v}. \tag{4.3}$$

In the next two subsections, we try to control the size of $\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi$. We distinguish the case where G is an inner k-form of SL_n from the case G is an outer k-form. For the former, we follow the argument of [BP89, prop. 5.1]. In the latter, we will adapt to our setting a refinement of the bounds of Borel and Prasad due to Mohammadi and Salehi Golsefidy [MSG12, §4]. Except for minor modifications, all the material in this section can be found in these two sources.

4.3.1 The inner case

Although in the inner case we have already established that $k = \mathbb{Q}$, we will discuss it for an arbitrary (totally real) field k, as this will be useful to treat the outer case as well. Let us thus assume G is an inner k-form, i.e. (by the classification) G is isomorphic to $\mathrm{SL}_{n'}\,\mathfrak{D}$ for some central division algebra $\mathfrak{D}$ over k of index $d = n/n'$. Similarly, over k_v, G is isomorphic to $\mathrm{SL}_{n_v}\,\mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. The center C of G is isomorphic to μ_n, the kernel of the map $\mathrm{GL}_1 \to \mathrm{GL}_1 : x \mapsto x^n$, and thus for any field extension K of k, $\mathrm{H}^1(K, C)$ may (and will in this paragraph) be identified with $K^\times/K^{\times n}$ (where $K^{\times n} = \{x^n \mid x \in K^\times\}$). With this identification, the canonical map $\mathrm{H}^1(k, C) \to \mathrm{H}^1(k_v, C)$ corresponds to the canonical map $k^\times/k^{\times n} \to k_v^\times/k_v^{\times n}$.

The action of $\mathrm{H}^1(k_v, C)$ on Δ_v can be described as follows: Δ_v is a cycle of length n_v, on which $\overline{G}(k_v)$ acts by rotations, i.e. Ξ_v can be identified with $\mathbb{Z}/n_v\mathbb{Z}$. The action of $\mathrm{H}^1(k_v, C)$ is then given by the morphism

$$k_v^\times/k_v^{\times n} \to \mathbb{Z}/n_v\mathbb{Z} : x \mapsto v(x) \mod n_v.$$

From this description, we see that $x \in k_v^\times/k_v^{\times n}$ acts trivially on Δ_v precisely when $v(x) \in n_v\mathbb{Z}$; in particular, if G splits over k_v, x acts trivially if and only if $v(x) \in n\mathbb{Z}$. We can form the exact sequence

$$1 \to k_n/k^{\times n} \to \mathrm{H}^1(k, C)_\xi \xrightarrow{(v)_{v \in V_f}} \bigoplus_{v \in V_f} \mathbb{Z}/n\mathbb{Z},$$

where $k_n = \{x \in k^\times \mid v(x) \in n\mathbb{Z} \text{ for all } v \in V_f\}$. By the above, the image of $\mathrm{H}^1(k,C)_\xi$ lies in the subgroup $\bigoplus_{v\in V_f} n_v\mathbb{Z}/n\mathbb{Z}$. Let T be the set of places $v \in V_f$ where G does not split over k_v, i.e. for which $n_v \neq n$. Then the exact sequence yields

$$\#\mathrm{H}^1(k,C)_\xi \leq \#(k_n/k^{\times n}) \cdot \prod_{v\in T} d_v.$$

The proof of [BP89, prop. 0.12] shows that $\#(k_n/k^{\times n}) \leq h_k \tilde{n} n^{[k:\mathbb{Q}]-1}$, where $\tilde{n} = 1$ or 2 if n is respectively odd or even. In the case $k = \mathbb{Q}$, which will be of interest later, it is indeed clear that $\#(\mathbb{Q}_n/\mathbb{Q}^{\times n}) = \tilde{n}$.

4.3.2 The outer case

Second, we assume G is an outer k-form. The centers of G and of the quasi-split inner form $\mathcal{G}$ of G are k-isomorphic, hence there is an exact sequence

$$1 \to C \to \mathrm{R}_{l/k}(\mu_n) \xrightarrow{N} \mu_n \to 1, \tag{4.4}$$

where μ_n denotes the kernel of the map $\mathrm{GL}_1 \to \mathrm{GL}_1 : x \mapsto x^n$ as above, $\mathrm{R}_{l/k}$ denotes the restriction of scalars from l to k, and N is (induced by) the norm map of l/k. The long exact sequence associated to it yields

$$1 \to \mu_n(k)/N(\mu_n(l)) \to \mathrm{H}^1(k,C) \to l_0/l^{\times n} \to 1 \tag{4.5}$$

where $l_0/l^{\times n}$ denotes the kernel of the norm map $N : l^\times/l^{\times n} \to k^\times/k^{\times n}$. The Hasse principle for simply connected groups allows us to write

$$\begin{array}{ccccc} \overline{G}(k) & \xrightarrow{\delta} & \mathrm{H}^1(k,C) & \longrightarrow & \mathrm{H}^1(k,G) \\ \downarrow & & \downarrow & & \downarrow\wr \\ \prod_{v\in V_\infty} \overline{G}(k_v) & \xrightarrow{(\delta_v)_v} & \prod_{v\in V_\infty} \mathrm{H}^1(k_v,C) & \longrightarrow & \prod_{v\in V_\infty} \mathrm{H}^1(k_v,G). \end{array} \tag{4.6}$$

If n is odd, we can make the following simplifications: $\mu_n(k) = \{1\}$ and thus $\mathrm{H}^1(k, C) \cong l_0/l^{\times n}$ in (4.5); using the analogous sequence for k_v, we also have $\mathrm{H}^1(k_v, C) \cong \{1\}$ for $v \in V_\infty$. Thus, in (4.6), we read that δ is surjective and conclude $\delta(\overline{G}(k)) \cong l_0/l^{\times n}$.

If n is even, a weaker conclusion holds provided l has at least one complex place, i.e. if $V_\infty \neq \{v_0\}$. Indeed, if $v_1 \in V_\infty - \{v_0\}$, so that $l \otimes_k k_{v_1} = \mathbb{C}$, then $(l \otimes_k k_{v_1})^\times / (l \otimes_k k_{v_1})^{\times n} = \{1\}$ and the long exact sequences associated to (4.4) read

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{H}^1(k, C) & \longrightarrow & l_0/l^{\times n} & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \xrightarrow{\sim} & \mathrm{H}^1(k_{v_1}, C) & \longrightarrow & 1 & \longrightarrow & 1. \end{array} \tag{4.7}$$

The first row splits, and thus we may identify $\mathrm{H}^1(k, C) \cong \{\pm 1\} \oplus l_0/l^{\times n}$; then $l_0/l^{\times n}$ is precisely the kernel of the canonical map $\mathrm{H}^1(k, C) \to \mathrm{H}^1(k_{v_1}, C)$. Now since the adjoint map $G(k_{v_1}) \to \overline{G}(k_{v_1})$ is surjective (recall that $G(k_{v_1}) \cong \mathrm{SU}_n(\mathbb{R})$), we have in (4.2) that the image of $\overline{G}(k)$ in $\mathrm{H}^1(k_{v_1}, C)$ is trivial, hence $\delta(\overline{G}(k)) \subset l_0/l^{\times n}$.

If n is even and $V_\infty = \{v_0\}$, then $k = \mathbb{Q}$. We have, for each $v \in V_f$,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{\mu(k)}{N(\mu(l))} & \longrightarrow & \mathrm{H}^1(k, C) & \longrightarrow & l_0/l^{\times n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \frac{\mu(k_v)}{N(\mu_n(l \otimes k_v))} & \longrightarrow & \mathrm{H}^1(k_v, C) & \longrightarrow & \frac{\ker(N: l \otimes k_v \to k_v)}{(l \otimes k_v)^{\times n}} & \longrightarrow & 1. \end{array}$$

We observe that $\mu(k)/N(\mu(l))$ ($\cong \{\pm 1\}$) acts trivially on Δ_v for every $v \in V_f$ (see for example [MSG12, §4]), hence the action factors through $l_0/l^{\times n}$. Thus $\#\mathrm{H}^1(k, C)_\xi = 2 \cdot \# l_\xi / l^{\times n}$, where $l_\xi / l^{\times n} = \{x \in l_0/l^{\times n} \mid \xi(x) = 1\}$, so that the bound we establish below will hold with an extra factor $\tilde{n}$ in the case $k = \mathbb{Q}$.

It remains to understand the action of $l_0/l^{\times n}$ on Δ. Let $x \in l$ and let

$$(x) = \prod_{\mathfrak{P}} \mathfrak{P}^{i_\mathfrak{P}} \overline{\mathfrak{P}}^{i_{\overline{\mathfrak{P}}}} \cdot \prod_{\mathfrak{p}'} \mathfrak{p}'^{i_{\mathfrak{p}'}} \cdot \prod_{\mathfrak{P}''} \mathfrak{P}''^{i_{\mathfrak{P}''}}$$

be the unique factorization of the fractional ideal of l generated by x, where $(\mathfrak{P}, \overline{\mathfrak{P}})$ (resp. $\mathfrak{p}'$, $\mathfrak{P}''$) runs over the set of primes of l that lie over primes of k that split over l (resp. over inert primes of k, over ramified primes of k). When $x \in l_0$, $N(x) \in k^{\times n}$ and thus n divides $i_{\mathfrak{P}} + i_{\overline{\mathfrak{P}}}$, $2i_{\mathfrak{p}'}$ and $i_{\mathfrak{P}''}$.

Observe that $v \in V_f$ splits over l if and only if l embeds into k_v, that is, if and only if ($\mathcal{G}$ splits over k_v and) G is an inner k_v-form of SL_n. In particular, at such a place v, G is isomorphic to $\mathrm{SL}_{n_v} \mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. In [MSG12, §4], it is shown that when v splits as $\mathfrak{P}\overline{\mathfrak{P}}$ over l, the action of $x \in l_0$ is analogous to the inner case described in 4.3.1, hence x acts trivially on Δ_v if and only if n divides $d_v i_{\mathfrak{P}}$ (and thus n also divides $d_v i_{\overline{\mathfrak{P}}}$), i.e. $v_{\mathfrak{P}}(x) = 0 \mod n_v$ (and $v_{\overline{\mathfrak{P}}}(x) = 0 \mod n_v$). When v is inert, say v corresponds to $\mathfrak{p}'$, then x acts trivially on Δ_v if and only if n divides $i_{\mathfrak{p}'}$ [MSG12, §4].

Let T be the set of places $v \in V_f$ such that v splits over l and G is not split over k_v, and let T^l be a subset of the finite places of l consisting of precisely one extension of each $v \in T$, so that restriction to k defines a bijection from T^l to T. By the discussion above, we can form an exact sequence

$$1 \to (l_n \cap l_0)/l^{\times n} \to l_\xi/l^{\times n} \xrightarrow{(w)_{w\in T^l}} \bigoplus_{w\in T^l} \mathbb{Z}/n\mathbb{Z},$$

where $l_n = \{x \in l^\times \mid w(x) \in n\mathbb{Z}$ for each normalized finite place w of $l\}$ and $l_\xi/l^{\times n} = \{x \in l_0/l^{\times n} \mid \xi(x) = 1\}$. Moreover, the image of $l_\xi/l^{\times n}$ lies in the subgroup $\bigoplus_{w\in T^l} n_v\mathbb{Z}/n\mathbb{Z}$. Thus, if we assume $k \neq \mathbb{Q}$ (so that we may identify $\delta(\overline{G}(k))$ with a subgroup of $l_0/l^{\times n}$),

$$\#\left(\delta(\overline{G}(k)) \cap \mathrm{H}^1(k, C)_\xi\right) \leq \#\left(l_\xi/l^{\times n}\right) \leq \#\left((l_n \cap l_0)/l^{\times n}\right) \cdot \prod_{v\in T} d_v.$$

We get the concrete bound on the index

$$[\Gamma : \Lambda] \leq h_l \tilde{n}^m n \cdot \prod_{v\in T} d_v \cdot \prod_{v\in V_f} \#\Xi_{\Theta_v}$$

by combining this with (4.3) and lemma 4.8.1. If $k = \mathbb{Q}$, we have instead

$$[\Gamma : \Lambda] \le h_l \tilde{n}^2 n \cdot \prod_{v \in T} d_v \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}.$$

4.4 The field k is $\mathbb{Q}$

We set $m = [k : \mathbb{Q}]$ and as before, $n = r + 1$. The purpose of this section is to show that $k = \mathbb{Q}$, i.e. $m = 1$.

We start by recalling that if P_v is special (in particular, if it is hyperspecial), i.e. Θ_v consists of a single special (resp. hyperspecial) vertex of Δ_v, then Ξ_{Θ_v} is trivial. Regardless of the type Θ_v, we have $\#\Xi_{\Theta_v} \le \tilde{n}$ unless G is an inner k_v-form of SL_n (say $G \cong \mathrm{SL}_{n_v}(\mathfrak{D}_v)$), in which case $\#\Xi_{\Theta_v} \le \#\Delta_v = n_v$, where $n_v - 1$ is the rank of G over k_v. (For example, this can be seen explicitly on all the possible relative local Dynkin diagrams Δ_v for $G(k_v)$, enumerated in [Tit79, §4] or [MSG12, §2]. In the inner case, the Dynkin diagram is a cycle on which the adjoint group acts as rotations.)

By a result of Kneser [Kne65], G is quasi-split over the maximal unramified extension $\widehat{k}_v$ of k_v for any $v \in V_f$. This means that over $\widehat{k}_v$, G is isomorphic to $\mathcal{G}$. The quasi-split k-forms of simply connected absolutely almost simple groups of type A_{n-1} are well understood [Tit66]: either $\mathcal{G} \cong \mathrm{SL}_n$, or $\mathcal{G} \cong \mathrm{SU}_{n,l}$, the special unitary group associated to the split hermitian form on l^n, where l is a quadratic extension of k equipped with the canonical involution (incidentally, l is the splitting field of $\mathrm{SU}_{n,l}$, in accordance with the notation introduced). Thus, over $\widehat{k}_v$, only these two possibilities arise for G. (Nonetheless, $\mathcal{G}$ might split over $\widehat{k}_v$; in fact, it does so except at finitely many places.) In particular, the rank r_v of G over $\widehat{k}_v$ is either r, or the ceiling of $r/2$.

4.4.1 The inner case

The case where G is an inner k-form of SL_n (i.e. when $l = k$) has been treated in section 4.2. We observed that if G is an inner k_v-form of SL_n for some $v \in V_\infty$, then $G(k_v)$ cannot be compact. This forced $V_\infty = \{v_0\}$ and thus $k = \mathbb{Q}$.

4.4.2 The outer case

Here we settle the case where G is an outer k-form of SL_n, i.e. when $[l : k] = 2$. We observed in section 4.2 that l has two real embeddings (extending $k \to k_{v_0}$) and $m - 1$ pairs of conjugate complex embeddings. Suppose that $m > 1$.

Let T be the finite set of places $v \in V_f$ such that v splits over l and G is not split over k_v. Then, according to section 4.3.2, we have

$$[\Gamma : \Lambda] \leq h_l \tilde{n}^m n \cdot \prod_{v \in T} d_v \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}$$

where $\tilde{n} = 1$ or 2 if n is odd or even, and h_l denotes the class number of l. Combined with (4.1), we find (abbreviating $V_n = \prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$)

$$\mu(G(k_{v_0})/\Gamma) \geq \tilde{n}^{-m} n^{-1} h_l^{-1} D_k^{\frac{n^2-1}{2}} (D_l/D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} V_n^m \cdot \prod_{v \in T} d_v^{-1} \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}^{-1} \cdot \prod_{v \in V_f} e(P_v).$$

We use [Pra89, prop. 2.10, rem. 2.11] and the observations made at the begining of section 4.4 to study the local factors of the right-hand side.

(i) If $v \in T$, then we use $e(P_v) \geq (q_v - 1) q_v^{(n^2 - n^2 d_v^{-1} - 2)/2}$ to obtain $d_v^{-1} \cdot \#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq n^{-1} \cdot (q_v - 1) q_v^{n^2/4 - 1} > 1$ when $n \geq 4$. When $n = 3$, then $d_v = 3$ and we also have $d_v^{-1} \cdot \#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq n^{-1} \cdot (q_v - 1) q_v^{n^2/3 - 1} > 1$ (lemma 4.8.2).

(ii) If $v \notin T$ but P_v is special, then $\#\Xi_{\Theta_v} = 1$ and $e(P_v) > 1$, thus $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) > 1$.

(iii) If $v \notin T$, P_v is not special and G is not split over k_v, then we use that $e(P_v) \geq (q_v+1)^{-1} q_v^{r_v+1}$ to obtain $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) \geq \tilde{n}^{-1} \cdot (q_v+1)^{-1} q_v^{\lceil (n-1)/2 \rceil + 1} > 1$ (lemma 4.8.3).

(iv) If $v \notin T$, P_v is not special but G splits over k_v, then P_v is properly contained in a hyperspecial parahoric H_v. There is a canonical surjection $H_v \to \mathrm{SL}_n(\mathfrak{f}_v)$, under which the image of P_v is the proper parabolic subgroup $\overline{P}_v$ of $\mathrm{SL}_n(\mathfrak{f}_v)$ whose type consists of the vertices belonging to the type of P_v in the Dynkin diagram obtained by removing the vertex corresponding to H_v in the affine Dynkin diagram of $G(k_v)$. In particular, it follows that $[H_v : P_v] = [\mathrm{SL}_n(\mathfrak{f}_v) : \overline{P}_v]$ and we may compute using lemma 4.8.14

$$e(P_v) = [H_v : P_v] \cdot e(H_v) > [H_v : P_v] > q^{n-1}.$$

Hence $\#\Xi_{\Theta_v}^{-1} \cdot e(P_v) > n^{-1} q^{n-1} > 1$.

Multiplying all the factors together, we have that

$$\prod_{v \in T} d_v^{-1} \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}^{-1} \cdot \prod_{v \in V_f} e(P_v) > 1$$

and we can thus write

$$\mu(G(k_{v_0})/\Gamma) > \tilde{n}^{-m} n^{-1} h_l^{-1} D_k^{\frac{n^2-1}{2}} (D_l/D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} V_n^m. \tag{4.8}$$

Recall that D_l/D_k^2 is the norm of the relative discriminant $\mathfrak{d}_{l/k}$ of l over k; in particular, D_l/D_k^2 is a positive integer. Note also that $\mathfrak{s}(\mathcal{G}) \geq 5$ if $n \geq 3$. We combine this with two number-theoretical bounds: from the results in [BP89, §6], we use that

$$h_l^{-1} D_l \geq \frac{1}{100} \left(\frac{12}{\pi} \right)^{2m};$$

from Minkowski's geometry of numbers (see for example [Sam70, §4.3]), we recall (k is totally real)

$$D_k^{\frac{1}{2}} \geq \frac{m^m}{m!}.$$

Altogether, we obtain

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m} D_k^{\frac{n^2-5}{2}}(D_l/D_k^2)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})-1} V_n^m n^{-1} \tag{4.9}$$
$$\geq \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m}\left(\frac{m^m}{m!}\right)^{n^2-5} V_n^m n^{-1}.$$

We consider the function $M : \mathbb{N}\times\mathbb{N}\to\mathbb{R}$ defined by

$$M(m,n) = \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m}\left(\frac{m^m}{m!}\right)^{n^2-5}\left(\prod_{i=1}^{n-1}\frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}.$$

(As V_n appears once as a factor in the covolume of $\mathrm{SL}_n(\mathbb{Z})$, we dropped its exponent above by one.) The function M is strictly increasing in both variables, provided $m \geq 2$ and $n \geq 6$ (lemma 4.8.4). In consequence, if $m \geq 2$, $n \geq 9$,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{M(m,n)}{\prod_{i=2}^{n}\zeta(i)} > \frac{M(2,9)}{\prod_{i=2}^{\infty}\zeta(i)} > 1,$$

(cf. lemma 4.8.13) and Γ is not of minimal covolume.

In a similar manner, we would like to show that m cannot be large. To this end, Odlyzko's bounds on discriminants [Odl76, table 4] are well-suited. We have

$$D_k^{\frac{1}{2}} > A^m \cdot E, \text{ with } A = 29.534^{\frac{1}{2}} \text{ and } E = e^{-4.13335}.$$

Combining with (4.9), we obtain

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m}(A^m E)^{n^2-5} V_n^m n^{-1}.$$

We consider the function $M' : \mathbb{N}\times\mathbb{N}\to\mathbb{R}$ defined by

$$M'(m,n) = \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m}(A^m E)^{n^2-5}\left(\prod_{i=1}^{n-1}\frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}.$$

M' is also strictly increasing in both variables, provided $m \geq 4$ and $n \geq 4$ (lemma 4.8.6). This means that if $m \geq 6$, $n \geq 4$,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{M'(m,n)}{\prod_{i=2}^{n}\zeta(i)} > \frac{M'(6,4)}{\prod_{i=2}^{\infty}\zeta(i)} > 1,$$

(cf. table 4.8.7 and lemma 4.8.13) and Γ is not of minimal covolume.

We may thus restrict our attention to the range $4 \leq n \leq 8$ and $2 \leq m \leq 5$ (we will treat the case $n = 3$ with a separate argument at the end of this section). By further sharpening our estimates on the discriminant, we will show that all these values are excluded as well, forcing $m = 1$.

From the bound (4.9) and the estimate $\mu(G(k_{v_0})/\Gamma) \leq \mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) < 2.3 \cdot V_n$ (4.8.13), we deduce an upper bound on the discriminant of k:

$$\begin{aligned} D_k &< \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} (D_l/D_k^2)^{1-\frac{1}{2}\mathfrak{s}(\mathcal{G})} V_n^{1-m} n\right)^{\frac{2}{n^2-5}} \\ &\leq \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} V_n^{1-m} n\right)^{\frac{2}{n^2-5}} =: C(m,n). \end{aligned} \tag{4.10}$$

As can be seen by comparing the values of C (table 4.8.8) with the smallest discriminants (table 4.8.9), this bound already rules out $n \geq 7$. We use these two tables to obtain information about D_k. A lower bound on D_k in turn will give us a bound on the relative discriminant: using (4.9) again,

$$D_l/D_k^2 < \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} D_k^{\frac{5-n^2}{2}} V_n^{1-m} n\right)^{\frac{2}{\mathfrak{s}(\mathcal{G})-2}}. \tag{4.11}$$

We proceed to rule out all values of m. In what follows, unless specified otherwise, any bound on D_k is obtained using (4.10), (4.8.8) or (4.8.9), and any upper bound on D_l/D_k^2 using (4.11). Claims made on the existence of a field l satisfying certain conditions are always made with the underlying assumption that l is a quadratic extension of k of signature $(2, m-1)$.

$\underline{m=5}$ gives $14641 \leq D_k \leq 15627$ (and $n=4$). A quick look in the online database of number fields [JR14] shows[1] that there is only one such field (with $D_k = 14641$). Now for l, Odlyzko's bound [Odl76, table 4] reads

$$D_l > (29.534)^2 \cdot (14.616)^8 \cdot e^{-8.2667} \geq 4.66756 \cdot 10^8$$

and in particular, we compute that $D_l/D_k^2 \geq 2.177$ (hence $D_l/D_k^2 \geq 3$). On the other hand, (4.11) yields

$$D_l/D_k^2 < 1.271,$$

ruling out this case.

$\underline{m=4}$ gives $725 \leq D_k \leq 1741$ (and $n=4$). A quick look in the database [JR14] shows that there are three fields satisfying this requirement, with discriminants respectively 725, 1125, 1600.

(i) If $D_k = 1600$, then $D_l/D_k^2 < 1.365$, hence $D_l = D_k^2 = 2560000$. But, as observed in the database, there are no fields l of signature $(2,3)$ with $D_l \leq 3950000$.

Unfortunately, the database has no complete records for fields with signature $(2,3)$ and discriminants past 3950000. We will thus need to refine our bounds to be able to treat the two other possible values for D_k. First, we go back to our bound on the class number h_l: as in [BP89, §6], we use Zimmert's bound $R_l \geq 0.04 \cdot e^{2 \cdot 0.46 + (m-1) \cdot 0.1}$ on the regulator of l along with the Brauer-Siegel theorem (with $s=2$) to deduce

$$h_l \leq 100 \cdot e^{-0.82 - 0.1 \cdot m} \cdot (2\pi)^{-2m} \cdot \zeta(2)^{2m} \cdot D_l \leq 29.523 \cdot \left(\frac{\pi}{12}\right)^8 \cdot D_l.$$

[1] The database [JR14] provides a certificate of completeness for certain queries. All allusions made here refer to searches that are proven complete. However, it is important to note that in [JR14], class numbers are computed assuming the generalized Riemann hypothesis (the rest of the data being unconditional). The class numbers referred to in this paper were therefore all verified using PARI/GP's `bnfcertify` command. A PARI/GP script of this process is available on the author's page (`math.ucsd.edu/~fthilman/`).

Using this, we may rewrite the bound (4.11) as

$$D_l/D_k^2 < \left(67.9029\tilde{n}^4\left(\frac{\pi}{12}\right)^8 D_k^{\frac{5-n^2}{2}} V_n^{-3} n\right)^{\frac{2}{\mathfrak{s}(\mathcal{G})-2}}.$$

(ii) If $D_k = 1125$, then our new bound yields $D_l/D_k^2 \leq 2$, hence $D_l \leq 2D_k^2 = 2531250$ and this is ruled out by the database.

(iii) If $D_k = 725$, then our new bound yields $D_l/D_k^2 \leq 11$, hence $D_l \leq 11D_k^2 = 5781875$. Selmane [Sel99] has computed all fields of signature $(2,3)$ that possess a proper subfield and have discriminant $D_l \leq 6688609$. It turns out that among those, only the field with discriminant -5781875 can be an extension of k. As observed in the online database, this field has class number 1. Substituting this information in (4.8), we see that the right-hand side exceeds $2.3 \cdot V_n$.

$\underline{m=3}$ gives $49 \leq D_k \leq 194$ (and $n = 4$ or 5). A quick look in the database [JR14] shows that there are four fields satisfying this requirement, with discriminants respectively 49, 81, 148, 169.

(i) If $D_k = 169$, then $D_l/D_k^2 < 1.661$ hence $D_l = D_k^2 = 28561$. There are no fields l with $D_l \leq 28000$.

(ii) If $D_k = 148$, then $D_l/D_k^2 \leq 2$. There are no fields l with $D_l/148^2 = 1$ or 2.

(iii) If $D_k = 81$, then $D_l/D_k^2 \leq 24$. An extensive search in the database shows that this can only be satisfied by one field l, with discriminant $D_l = 81^2 \cdot 17$. It has class number $h_l = 1$, hence we may substitute this information in (4.8) and compute that the right-hand side exceeds $2.3 \cdot V_n$.

(iv) If $D_k = 49$, then $D_l/D_k^2 \leq 155$. An extensive search in the database shows that there are 6 fields l satisfying this condition. They correspond to $D_l/D_k^2 = 13, 29, 41, 64, 97$

or 113, and all have class number 1. Then, in (4.8), the right-hand side again exceeds $2.3 \cdot V_n$ (note that it suffices to check this for the smallest value of D_l/D_k^2).

$\underline{m = 2}$ gives $5 \leq D_k \leq 21$ (and $4 \leq n \leq 6$). It is well known (and can be observed in the database [JR14]) that there are 6 fields satisfying this requirement, with discriminants respectively 5, 8, 12, 13, 17, 21. From (4.11), we see that $D_l/D_k^2 \leq$ 214, 38, 8, 6, 2, 1 respectively.

(i) If $D_k = 21$ or 17, we observe that $D_l \leq 578$. There are no fields with $D_l \leq 578$ that can be extensions of k in these cases.

(ii) If $D_k = 13$, then the database exhibits only one possible field l with $D_l = 13^2 \cdot 3$. This field has trivial class group, and using this information in (4.8), we see that the right-hand side exceeds $2.3 \cdot V_n$.

(iii) If $D_k = 12$, then there are again no fields with $D_l \leq 8D_k^2$.

(iv) If $D_k = 8$, then there are 11 candidates l with $D_l \leq 38 \cdot 8^2$, and all have trivial class group. The one with smallest relative discriminant has $D_l/D_k^2 = 7$. For this field (hence for all of them), the right-hand side of (4.8) is again too large.

(v) If $D_k = 5$, there are 25 candidates l with $D_l \leq 214 \cdot 5^2$, and all have trivial class group. The one with smallest relative discriminant has $D_l = 11$. This field (hence all of them) is one more time excluded by (4.8).

It remains to deal with the case $n = 3$. First, we proceed as above, using lemma 4.8.6, $M'(16, 3) \simeq 4.6751...$, and $\zeta(2) \cdot \zeta(3) < 1.97731$ to see that

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_3(\mathbb{R})/\,\mathrm{SL}_3(\mathbb{Z}))} > \frac{M'(m,3)}{\zeta(2) \cdot \zeta(3)} > 1$$

provided $m \geq 16$. Hence we may restrict our attention to the range $2 \leq m \leq 15$.

Unfortunately, this bound on the degree of k is too large to allow us to work with a number field database. Of course, the reason this bound is large is that the powers of D_k and D_l appearing in (4.8) are very small. In turn, the bound we used for the class number h_l was very greedy in terms of D_l, aggravating the situation. In fact, we can use (4.8) and one of Odlyzko's bounds [Odl76] for D_l to obtain a lower bound on h_l:

$$h_l \geq \frac{D_k^{-1} D_l^{\frac{5}{2}} V_3^{m-1}}{3 \cdot \zeta(2) \cdot \zeta(3)} \geq \frac{D_l^2 V_3^{m-1}}{3 \cdot \zeta(2) \cdot \zeta(3)} > \frac{(25.465^2 \cdot 13.316^{2m-2} \cdot e^{-7.0667})^2 \cdot V_3^{m-1}}{3 \cdot \zeta(2) \cdot \zeta(3)}. \tag{4.12}$$

We record the values of this bound in table 4.8.10 (for small values of m, we used the actual minimum for D_l to obtain this lower bound for h_l).

To solve this issue, we use the following trick. The Hilbert class field L of l has degree $[L : \mathbb{Q}] = 2mh_l$, signature $(2h_l, (m-1)h_l)$ and discriminant $D_L = D_l^{h_l}$. Hence, when the class number is large, we can use Odlyzko's bounds [Odl76] for D_L in order to improve our bounds on D_l. Namely, we have

$$D_l = D_L^{\frac{1}{h_l}} > 60.015^2 \cdot 22.210^{2m-2} \cdot e^{\frac{-80.001}{h_l}}.$$

We record this bound for D_l in table 4.8.11.

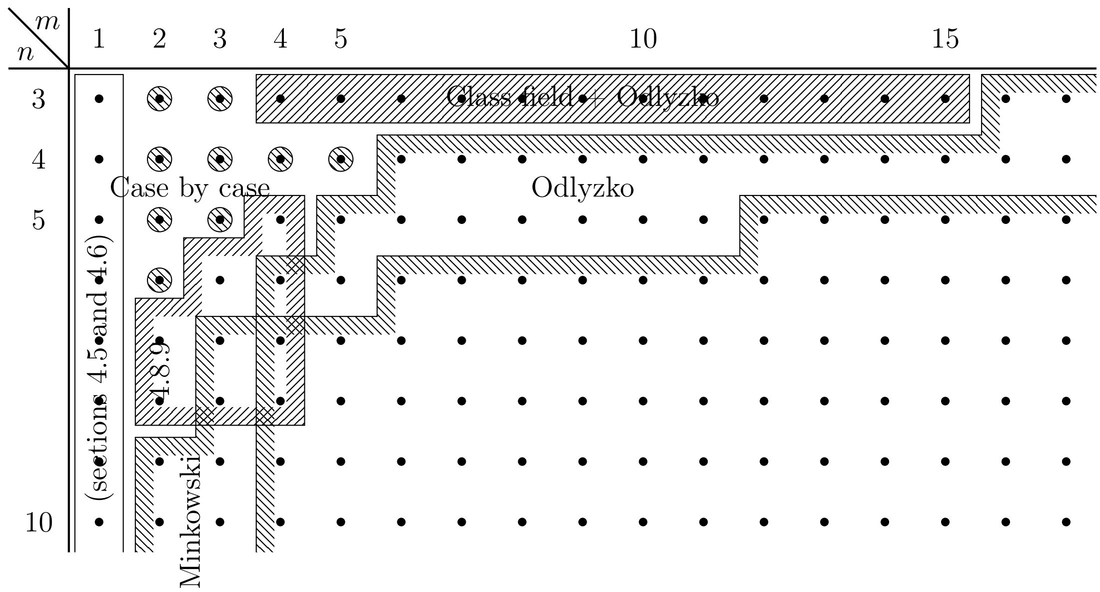
Now using $D_l \geq D_k^2$, we may rewrite (4.9) as

$$\zeta(2) \cdot \zeta(3) \cdot V_3 > \mu(G(k_{v_0})/\Gamma) > \frac{1}{300} \left(\frac{12}{\pi}\right)^{2m} D_l \cdot V_3^m$$

and check that this inequality contradicts the bound in table 4.8.11 as soon as $m \geq 4$. For $m = 3$ and $m = 2$, the bound reads respectively $D_l \leq 4578732$ and $D_l \leq 13643$.

Finally, to treat the remaining two cases, we can use the online database [JR14]. If $m = 3$, we observe that all fields of signature $(2, 2)$ with discriminant $D_l \leq 4578732$ have class number either $h_l = 1$ or $h_l = 2$; this contradicts (4.12) and table 4.8.10. Similarly, if $m = 2$, we observe in the database that all fields of signature $(2,1)$ with discriminant $D_l \leq 13643$ also have class number either $h_l = 1$ or $h_l = 2$. This is again a contradiction to (4.12) and table 4.8.10.

Table 4.4.1. Below is a summary of the various discriminant bounds that were used in this section to exclude a given pair (m, n) from giving rise to a lattice of minimal covolume.



4.5 G is an inner form of SL_n

The purpose of this section is to show that G is an inner k-form of SL_n, i.e. that $\mathcal{G}$ splits over k. Let us thus suppose, for contradiction, that $[l : k] > 1$.

We have shown in section 4.4 that $k = \mathbb{Q}$, so that the bounds (4.8) and (4.9) obtained in 4.4.2 can be adapted as follows: (the extra factor $\tilde{n}$ is due to the correction in the index bound when $k = \mathbb{Q}$, cf. section 4.3.2)

$$\begin{aligned}\mu(G(k_{v_0})/\Gamma) &> \tilde{n}^{-2} n^{-1} h_l^{-1} D_l^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} V_n \\ &\geq \frac{1}{100\tilde{n}^2} \left(\frac{12}{\pi}\right)^2 D_l^{\frac{1}{2}\mathfrak{s}(\mathcal{G})-1} V_n n^{-1}.\end{aligned}$$

First, let us assume that $h_l \neq 1$. Since l is totally real, this implies $D_l \geq 40$. Note that $\mathfrak{s}(\mathcal{G}) \geq \frac{1}{2}(r^2+r-2) = \frac{1}{2}(n^2-n-2)$. Therefore

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{100\tilde{n}^2}\left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} V_n n^{-1}.$$

We consider the function $N : \mathbb{N} \to \mathbb{R}$ defined by

$$N(n) = \frac{1}{100\tilde{n}^2}\left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} n^{-1}.$$

The function N is strictly increasing, provided $n \geq 2$ (lemma 4.8.12). In consequence, if $n \geq 4$, then $N(n) \geq N(4) \simeq 2.30692...$ and thus

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{N(n)}{\prod_{i=2}^{n}\zeta(i)} > \frac{N(4)}{\prod_{i=2}^{\infty}\zeta(i)} > 1,$$

hence Γ is not of minimal covolume. For $n=3$ we notice that $\mathfrak{s}(\mathcal{G}) = 5$, so that

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{300}\left(\frac{12}{\pi}\right)^2 40^{\frac{3}{2}} \cdot V_3 > 12.3035 \cdot V_3$$

and Γ is not of minimal covolume.

Second, if $h_l = 1$, then at least $D_l \geq 5$ and we may consider the function $N' : \mathbb{N} \to \mathbb{R}$ defined by

$$N'(n) = \tilde{n}^{-2} n^{-1} 5^{\frac{1}{4}(n^2-n-2)}.$$

The function N' is strictly increasing (lemma 4.8.12) and $N'(4) \simeq 3.49385...$, thus

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} > \frac{N(n)}{\prod_{i=2}^{n}\zeta(i)} > \frac{N(4)}{\prod_{i=2}^{\infty}\zeta(i)} > 1,$$

and Γ is not of minimal covolume. For $n=3$, we use again that $\mathfrak{s}(\mathcal{G}) = 5$ to see that

$$\mu(G(k_{v_0})/\Gamma) > \frac{1}{3} \cdot 5^{\frac{5}{2}} \cdot V_3 > 18.6338 \cdot V_3$$

and Γ is not of minimal covolume. This forces $l = k$ and G to be an inner form.

4.6 The parahorics P_v are hyperspecial and G splits at all places

So far, we have established that $k = l = \mathbb{Q}$ and G is an inner k-form of SL_n; thus, G is isomorphic to $\mathrm{SL}_{n'}\,\mathfrak{D}$ for some central division algebra $\mathfrak{D}$ over k of index $d = n/n'$. Similarly, over k_v, G is isomorphic to $\mathrm{SL}_{n_v}\,\mathfrak{D}_v$ for some central division algebra $\mathfrak{D}_v$ over k_v of index $d_v = n/n_v$. Recall that T is the finite set of places $v \in V_f$ where G does not split over k_v, and let T' be the finite set of places $v \in V_f$ where P_v is not a hyperspecial parahoric; of course, $T \subset T'$. The goal of this section is to show that T' is empty.

According to section 4.3.1, we have

$$\#\mathrm{H}^1(k, C)_\xi \leq \tilde{n} \cdot \prod_{v \in T} d_v,$$

with $d_v \geq 2$ if $v \in T$. Also, as we noted at the begining of section 4.4,

$$\#\Xi_{\Theta_v} \leq n_v \text{ if } v \in T, \quad \#\Xi_{\Theta_v} \leq r + 1 = n \text{ if } v \in T', \quad \#\Xi_{\Theta_v} = 1 \text{ otherwise.}$$

Combined with (4.1) and (4.3), we obtain

$$\begin{aligned} \mu(G(k_{v_0})/\Gamma) &\geq \tilde{n}^{-1} V_n \cdot \prod_{v \in T} d_v^{-1} \cdot \prod_{v \in T} n_v^{-1} \cdot \prod_{v \in T' - T} n^{-1} \cdot \prod_{v \in V_f} e(P_v) \\ &= \tilde{n}^{-1} V_n \cdot \prod_{v \in T'} n^{-1} \cdot \prod_{v \in V_f} e(P_v). \end{aligned} \tag{4.13}$$

Recall that for any $v \in V_f$, $e(P_v) > 1$. If $v \in T$, then according to [Pra89, remark 2.11], we have

$$e(P_v) \geq (q_v - 1) q_v^{\frac{1}{2}(n^2 - n^2 d_v^{-1} - 2)} \geq (q_v - 1) q_v^{\frac{1}{4}n^2 - 1}.$$

Now if T is not empty, then by looking at the Hasse invariant of $\mathfrak{D}$, it appears that $d_v \geq 2$ for at least two (finite) places. This means that T has at least two elements, and using lemma 4.8.2, we see that if $n \geq 4$,

$$\prod_{v \in T} n^{-1} e(P_v) \geq (n^{-1}(2-1) \cdot 2^{\frac{1}{4}n^2-1}) \cdot (n^{-1}(3-1) \cdot 3^{\frac{1}{4}n^2-1}) \geq 27.$$

If $n = 3$, then actually $d_v = 3$ for at least two (finite) places, and

$$\prod_{v \in T} n^{-1} e(P_v) \geq (n^{-1}(2-1) \cdot 2^{\frac{1}{3}n^2-1}) \cdot (n^{-1}(3-1) \cdot 3^{\frac{1}{3}n^2-1}) = 8.$$

In particular, it is clear from (4.13) that Γ is not of minimal covolume. Hence it must be that T is empty and G splits everywhere.

On the other hand, if $v \in T' - T$, then P_v is properly contained in a hyperspecial parahoric H_v. As discussed previously, there is a canonical surjection $H_v \to \mathrm{SL}_n(\mathfrak{f}_v)$, under which the image of P_v is the proper parabolic subgroup $\overline{P}_v$ of $\mathrm{SL}_n(\mathfrak{f}_v)$ whose type consists of the vertices belonging to the type of P_v in the Dynkin diagram obtained by removing the vertex corresponding to H_v in the affine Dynkin diagram of $G(k_v)$. In particular, it follows that $[H_v : P_v] = [\mathrm{SL}_n(\mathfrak{f}_v) : \overline{P}_v]$ and thus using lemma 4.8.14,

$$e(P_v) = [H_v : P_v] \cdot e(H_v) \geq q_v^{n-1} \cdot e(H_v).$$

Of course, as G splits everywhere, we have that $e(H_v)$ is equal to the corresponding factor $e(\mathrm{SL}_n(\mathbb{Z}_v)) = \prod_{i=2}^{n} \frac{1}{1-q_v^{-i}}$ for $\mathrm{SL}_n(\mathbb{Q}_v)$. In consequence,

$$\frac{\mu(G(k_{v_0})/\Gamma)}{\mu(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}))} \geq \frac{\tilde{n}^{-1} \prod_{v \in T'} n^{-1} \cdot \prod_{v \in V_f} e(P_v)}{\prod_{v \in V_f} e(\mathrm{SL}_n(\mathbb{Z}_v))} \geq \tilde{n}^{-1} \prod_{v \in T'} (n^{-1} q_v^{n-1}) \geq 1$$

with equality only if $n = 4$, $T' = \{2\}$ and $\#\Xi_{\Theta_2} = 4$. Notice however that this bound is rather rough; by examining the types of the parahorics carefully, one obtains much better bounds. For example, to achieve $\#\Xi_{\Theta_v} = n$, P_v must be an Iwahori subgroup, in which case $[H_v : P_v] \geq q_v^{(n^2-n)/2}$ in lemma 4.8.14. This rules out the equality case above and thus T' must be empty as well.

4.7 Conclusion

As we have shown in section 4.6, G splits over k_v for all $v \in V_f$ and thus for all $v \in V$. As before, let $\mathfrak{D}$ be a central division algebra over k $(= \mathbb{Q})$ of degree d such that $G \cong \mathrm{SL}_{n'}(\mathfrak{D})$ over k. Now since G splits at all places, we have for any $v \in V$ that $G(k_v) \cong \mathrm{SL}_n(k_v)$, or in other words, that the group of elements of reduced norm 1 in $\mathrm{M}_{n'}(\mathfrak{D}) \otimes_k k_v$ is isomorphic to $\mathrm{SL}_n(k_v)$. This implies that $\mathrm{M}_{n'}(\mathfrak{D}) \otimes_k k_v \cong \mathrm{M}_n(k_v)$, i.e. $\mathfrak{D}_v = \mathfrak{D} \otimes_k k_v$ splits over k_v. It then follows from the Albert–Brauer–Hasse–Noether theorem that $\mathfrak{D} = k$ and in turn $G(k) \cong \mathrm{SL}_n(k)$ and G is split over k. From hereon, we will thus identify G with SL_n through this isomorphism, to be denoted η.

Since each parahoric P_v is hyperspecial, for each $v \in V_f$ there exists $g_v \in \mathrm{GL}_n(\mathbb{Q}_v)$ such that $g_v P_v g_v^{-1} = \mathrm{SL}_n(\mathbb{Z}_v)$. As the family $\{P_v\}$ is coherent, we may assume that $g_v = 1$ except for finitely many $v \in V_f$. In this way, $g = (1, (g_v)_{v \in V_f})$ determines an element of the adele group $\mathrm{GL}_n(\mathbb{A})$. The class group of GL_n over $\mathbb{Q}$ is trivial [PR94, ch. 8], therefore

$$\mathrm{GL}_n(\mathbb{A}) = (\mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in V_f} \mathrm{GL}_n(\mathbb{Z}_v)) \cdot \mathrm{GL}_n(\mathbb{Q}),$$

and we can write $g = (1, (g'_v h)_{v \in V_f})$ for $g'_v \in \mathrm{GL}_n(\mathbb{Z}_v)$ and $h \in \mathrm{GL}_n(\mathbb{Q})$. In consequence, $h P_v h^{-1} = g'^{-1}_v \mathrm{SL}_n(\mathbb{Z}_v) g'_v = \mathrm{SL}_n(\mathbb{Z}_v)$, and thus

$$h \Lambda h^{-1} = h \,\mathrm{SL}_n(\mathbb{Q}) h^{-1} \cap \prod_{v \in V_f} h P_v h^{-1} = \mathrm{SL}_n(\mathbb{Q}) \cap \prod_{v \in V_f} \mathrm{SL}_n(\mathbb{Z}_v) = \mathrm{SL}_n(\mathbb{Z}).$$

In turn, $h \Gamma h^{-1} = \mathrm{SL}_n(\mathbb{Z})$, as $\mathrm{SL}_n(\mathbb{Z})$ (or equivalently Λ) is its own normalizer in $\mathrm{SL}_n(\mathbb{R})$. One way to obtain this fact is using Rohlfs' exact sequence (see section 4.3). Indeed, clearly $C(k_{v_0}) = C(k) \cap \Lambda$, and on the other hand, since Λ is given by hyperspecial parahorics, we may identify

$$\mathrm{H}^1(k, C)'_\Theta = \{x \in \mathbb{Q}^\times / \mathbb{Q}^{\times n} \mid v(x) \in n\mathbb{Z} \text{ for } v \in V_f, \text{ and } x \in \mathbb{R}^{\times n}\} = \{1\}.$$

Hence Γ/Λ is trivial as claimed.

Finally, retracing our identifications, we find that $\mathrm{SL}_n(\mathbb{Z})$ is the image of Γ under the automorphism $\sigma : \mathrm{SL}_n(\mathbb{R}) \xrightarrow{\iota} G(k_{v_0}) \xrightarrow{\eta} \mathrm{SL}_n(\mathbb{R}) \xrightarrow{c_h} \mathrm{SL}_n(\mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$ (here c_h denotes conjugation by h). This concludes the proof of the

Theorem. *Let $n \geq 3$ and let Γ be a lattice of minimal covolume for some (any) Haar measure in $\mathrm{SL}_n(\mathbb{R})$. Then $\sigma(\Gamma) = \mathrm{SL}_n(\mathbb{Z})$ for some (algebraic) automorphism σ of $\mathrm{SL}_n(\mathbb{R})$.*

4.8 Bounds for sections 4.4 through 4.6

4.8.1 Lemma. *Let k be a totally real number field of degree m and let l be a quadratic extension of k of signature $(2m_1, m_2)$, so that $m = m_1 + m_2$. Let $n \in \mathbb{N}$ and set $l_0 = \{x \in l^\times \mid N_{l/k}(x) \in k^{\times n}\}$ and $l_n = \{x \in l^\times \mid w(x) \in n\mathbb{Z}$ for each normalized finite place w of $l\}$. Then*

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left(\mu(l)/\mu(l)^n\right) \cdot \tilde{n}^{m-1} n^{m_1} \cdot \#\mathcal{C}_n,$$

where $\mu(l)$ is the group of roots of unity of l, $\tilde{n} = 1$ or 2 depending if n is odd or even, and $\mathcal{C}_n$ is the n-torsion subgroup of the class group $\mathcal{C}$ of l.

Moreover, if $N_{l/k}$ is surjective from U_l onto $U_k/\{\pm 1\}$, then

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left(\mu(l)/\mu(l)^n\right) \cdot n^{m_1} \cdot \#\mathcal{C}_n.$$

Proof. According to [BP89, prop. 0.12], there is an exact sequence

$$1 \to U_l/U_l^n \to l_n/l^{\times n} \to \mathcal{C}_n \to 1,$$

where U_l denotes the group of units of the ring of integers of l, and $\mathcal{C}_n$ is the n-torsion subgroup of the class group $\mathcal{C}$ of l. Intersecting with $l_0/l^{\times n}$ yields

$$\#\left((l_n \cap l_0)/l^{\times n}\right) \leq \#\left((U_l \cap l_0)/U_l^n\right) \cdot \#\mathcal{C}_n.$$

Dirichlet's units theorem [Sam70, §4.4] states that U_l is the internal direct product $F_l \times \mu(l)$ of F_l, the free abelian subgroup of U_l (of rank $2m_1 + m_2 - 1$) generated by some system of fundamental units, and $\mu(l)$, the subgroup of roots of unity in $l^\times$. Since $\mu(l) \subset l_0$, we also have that $U_l \cap l_0$ is the internal direct product of $F_l \cap l_0$ and $\mu(l)$. Additionally, it is clear that under this identification, U_l^n corresponds to the subgroup $F_l^n \times \mu(l)^n$ of $(F_l \cap l_0) \times \mu(l)$. In consequence,

$$\#\left((U_l \cap l_0)/U_l^n\right) = \#\left((F_l \cap l_0)/F_l^n\right) \cdot \#\left(\mu(l)/\mu(l)^n\right),$$

and it remains to study $(F_l \cap l_0)/F_l^n$; to this end, we switch to additive notation.

We write L for the free abelian group $U_l/\mu(l)$ (canonically isomorphic to F_l) in additive notation, and M for its free subgroup $U_k/\{\pm 1\}$ (of rank $m-1$) consisting of units lying in k. The norm $N_{l/k}$ induces a map $N : L \to M$, and in turn a map $L/nL \to M/nM$ also denoted by N, whose kernel L_0/nL corresponds precisely to $(F_l \cap l_0)/F_l^n$. In other words, the sequence

$$0 \to L_0/nL \to L/nL \xrightarrow{N} M/nM$$

is exact. It is clear that $\#(L/nL) = n^{2m_1+m_2-1}$ and $\#(M/nM) = n^{m-1}$. If N is surjective, then it follows that $\#(L_0/nL) = n^{m_1}$. In any case, we have $2M \subset N(L)$ hence we may write

$$\#\left(\frac{N(L)+nM}{nM}\right) = \#\left(\frac{N(L)+nM}{2M+nM}\right) \cdot \#\left(\frac{2M+nM}{nM}\right).$$

As $2M + nM = \tilde{n}M$, we have $\#\left(\frac{2M+nM}{nM}\right) = \left(\frac{n}{\tilde{n}}\right)^{m-1}$ and the lemma follows. □

4.8.2 Lemma. *The function* $\mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *defined by* $E(n,q) = n^{-1} \cdot (q-1)q^{n^2/4-1}$ *is increasing in both* n *and* q *provided* $n, q \geq 2$. *In consequence,* $n^{-1} \cdot (q-1)q^{n^2/4-1} > 1$ *provided* $n \geq 4$. *Similarly,* $n^{-1} \cdot (q-1)q^{n^2/3-1} > 1$ *provided* $n \geq 3$.

Proof. We compute, for $n, q \geq 2$,

$$\frac{E(n,q+1)}{E(n,q)} = \frac{q(q+1)^{\frac{1}{4}n^2-1}}{(q-1)q^{\frac{1}{4}n^2-1}} = \frac{q^2(q+1)^{\frac{1}{4}n^2}}{(q^2-1)q^{\frac{1}{4}n^2}} > 1.$$

and

$$\frac{E(n+1,q)}{E(n,q)} = \frac{n}{n+1} \cdot q^{\frac{1}{4}(2n+1)} \geq \frac{2}{3} \cdot 2^{\frac{5}{4}} > 1.$$

Thus E is strictly increasing in n and q if $n, q \geq 2$, and $E(4,2) = 2$. The proof of the second inequality is analogous. □

4.8.3 Lemma. *Let $n, q \in \mathbb{N}$ with $q \geq 2$. Then $\tilde{n}^{-1} \cdot (q+1)^{-1} q^{\lceil (n+1)/2 \rceil} > 1$ provided $n \geq 3$.*

Proof. Observe that $E(n,q) = \frac{q^{\lceil (n+1)/2 \rceil}}{(q+1)\tilde{n}}$ is increasing in n and strictly increasing in q, as

$$\frac{E(n+1,q)}{E(n,q)} = \frac{\tilde{n}}{\widetilde{n+1}} q^{2-\tilde{n}} \geq 1$$

and

$$\frac{E(n,q+1)}{E(n,q)} = \frac{(q+1)(q+1)^{\lceil (n+1)/2 \rceil}}{(q+2)q^{\lceil (n+1)/2 \rceil}} = \frac{(q^2+2q+1)(q+1)^{\lceil (n+1)/2 \rceil - 1}}{(q^2+2q)q^{\lceil (n+1)/2 \rceil - 1}} > 1.$$

Finally $E(3,2) = \frac{4}{3}$. □

4.8.4 Lemma. *The function $M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by*

$$M(m,n) = \frac{1}{100\tilde{n}^m} \left(\frac{12}{\pi}\right)^{2m} \left(\frac{m^m}{m!}\right)^{n^2-5} \left(\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}$$

(where $\tilde{n} = 1$ or 2 if n is odd or even) is strictly increasing in both m and n, provided $m \geq 2$ and $n \geq 6$.

Proof. For F a function of two integer variables m and n, we denote $\partial_m F$ (resp. $\partial_n F$) the function defined by $\partial_m F(m,n) = \frac{F(m+1,n)}{F(m,n)}$ (resp. $\partial_n F(m,n) = \frac{F(m,n+1)}{F(m,n)}$). In order to show that M increases in m (resp. in n), we intent to show that $\partial_m M > 1$ (resp. $\partial_n M > 1$).

We have

$$\partial_m M(m,n) = \frac{144}{\pi^2 \tilde{n}} \left(\frac{(m+1)^m}{m^m}\right)^{n^2-5} \cdot \prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$$

$$\partial_n M(m,n) = \left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^m \cdot \frac{n}{n+1} \cdot \left(\frac{m^m}{m!}\right)^{2n+1} \left(\frac{n!}{(2\pi)^{n+1}}\right)^{m-1}$$

and thus

$$\partial_m(\partial_n M)(m,n) = \partial_n(\partial_m M)(m,n) = \frac{\tilde{n}}{\widetilde{n+1}} \cdot \left(\frac{(m+1)^m}{m^m}\right)^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}}.$$

Now if $m \geq 2$ and $n \geq 4$, then $\frac{(m+1)^m}{m^m} \geq \frac{9}{4}$ and we have

$$\partial_m(\partial_n M)(m,n) \geq \frac{1}{2}\left(\frac{9}{4}\right)^{2n+1} \frac{n!}{(2\pi)^{n+1}} = \frac{9}{16\pi} \cdot \left(\frac{81}{16\pi}\right)^n \cdot \frac{n!}{2^n} \geq \frac{9}{16\pi} \cdot \left(\frac{81}{16\pi}\right)^4 > 1.$$

This means that provided $m \geq 2$ and $n \geq 4$, $\partial_m M$ increases in n and $\partial_n M$ increases in m.

Finally, assuming $m \geq 2$ and $n \geq 6$ respectively, we have

$$\partial_m M(m,6) = \frac{144}{2\pi^2}\left(\frac{(m+1)^m}{m^m}\right)^{31} \cdot \prod_{i=1}^{5} \frac{i!}{(2\pi)^{i+1}} \geq \frac{144}{2\pi^2}\left(\frac{9}{4}\right)^{31} \cdot \prod_{i=1}^{5} \frac{i!}{(2\pi)^{i+1}} > 1$$

$$\partial_n M(2,n) = \left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^2 \frac{n}{n+1} \cdot 2^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}} \geq \frac{3}{14} \cdot 2^n \frac{n!}{\pi^{n+1}} \geq \frac{3}{14} \cdot 2^6 \cdot \frac{6!}{\pi^7} > 1$$

hence $\partial_m M(m,n) > 1$ and $\partial_n M(m,n) > 1$ provided $m \geq 2$ and $n \geq 6$, completing the proof. □

Table 4.8.5. The table below contains some values of the function M from lemma 4.8.4.

(n,m)	1	2	3	4	5	6	7	8
2	0.0364756	0.00337012	0.000276781	0.0000215771	1.63315×10^{-6}	1.21281×10^{-7}	8.88761×10^{-9}	6.44933×10^{-10}
3	0.0486342	0.00231876	0.000177084	0.0000166585	1.76356×10^{-6}	2.01469×10^{-7}	2.42731×10^{-8}	3.04153×10^{-9}
4	0.0182378	0.000214239	9.19392×10^{-6}	6.99962×10^{-7}	7.37412×10^{-8}	9.57798×10^{-9}	1.43998×10^{-9}	2.41175×10^{-10}
5	0.0291805	0.000860260	0.000267434	0.000235765	0.000375160	0.000873531	0.00265357	0.00980934
6	0.0121585	0.000715847	0.00162363	0.0185268	0.528020	27.1489	2107.97	221884.
7	0.0208432	0.0374453	11.9823	37981.0	4.41409×10^{8}	1.18530×10^{13}	5.71337×10^{17}	4.24155×10^{22}
8	0.00911891	0.556912	35451.1	4.88495×10^{10}	3.84324×10^{17}	9.29477×10^{24}	4.92580×10^{32}	4.65827×10^{40}
9	0.0162114	685.655	2.23863×10^{11}	3.83726×10^{21}	6.20398×10^{32}	4.26138×10^{44}	8.04066×10^{56}	3.19899×10^{69}
10	0.00729513	306071.	9.29184×10^{17}	3.98641×10^{32}	2.82701×10^{48}	1.22281×10^{65}	1.87055×10^{82}	7.27033×10^{99}
11	0.0132639	1.40574×10^{10}	1.27888×10^{28}	4.91209×10^{48}	5.79785×10^{70}	6.22507×10^{93}	3.12510×10^{117}	4.89869×10^{141}

4.8.6 Lemma. *The function $M' : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by*

$$M'(m,n) = \frac{1}{100\tilde{n}^m}\left(\frac{12}{\pi}\right)^{2m} (A^m E)^{n^2-5} \left(\prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}\right)^{m-1} n^{-1}$$

(where $\tilde{n} = 1$ or 2 if n is odd or even, and $A = 29.534^{\frac{1}{2}}$, $E = e^{-4.13335}$) is strictly increasing in both m and n, provided $m \geq 4$ and $n \geq 4$. Moreover, $M'(m,n)$ is strictly increasing in m provided $n \geq 3$.

Proof. In order to show that M' increases in m (resp. in n), we intend to show that $\partial_m M > 1$ (resp. $\partial_n M > 1$); the notation is as in lemma 4.8.4.

We have

$$\partial_m M'(m,n) = \frac{144}{\pi^2 \tilde{n}} \cdot A^{n^2-5} \cdot \prod_{i=1}^{n-1} \frac{i!}{(2\pi)^{i+1}}$$
$$\partial_n M'(m,n) = \left(\frac{\tilde{n}}{\widetilde{n+1}}\right)^m (A^m E)^{2n+1} \left(\frac{n!}{(2\pi)^{n+1}}\right)^{m-1} \left(\frac{n}{n+1}\right)$$

and thus

$$\partial_m(\partial_n M')(m,n) = \partial_n(\partial_m M')(m,n) = \frac{\tilde{n}}{\widetilde{n+1}} \cdot A^{2n+1} \cdot \frac{n!}{(2\pi)^{n+1}}.$$

As clearly $A^2 > 2\pi$, we have (if $n \geq 3$)

$$\partial_m(\partial_n M')(m,n) > \frac{1}{2} \cdot A \cdot \frac{n!}{2\pi} > 1.$$

This means that $\partial_m M'$ increases in n and $\partial_n M'$ increases in m. Assuming respectively $m \geq 1$ and $n \geq 4$, we have

$$\partial_m M'(m,3) = \frac{144}{\pi^2} \cdot A^4 \cdot \frac{2}{(2\pi)^5} > 1$$
$$\partial_n M'(4,n) \geq \frac{1}{2^4} \cdot \left(A^4 E\right)^{2n+1} \cdot \frac{(n!)^3}{(2\pi)^{3n+3}} \cdot \frac{4}{5}$$
$$\geq \frac{1}{2^4} \cdot \left(A^4 E\right)^9 \cdot \frac{(6!)^3}{(2\pi)^{21}} \cdot \frac{4}{5} > 1$$

hence $\partial_m M'(m,n) > 1$ and $\partial_n M'(m,n) > 1$ provided $m \geq 4$ and $n \geq 4$. Moreover, $\partial_m M'(m,n) > 1$ if $n \geq 3$, completing the proof. □

Table 4.8.7. The table below contains some values of the function M' from lemma 4.8.6.

(n, m)	1	2	3	4	5	6	7	8
2	0.418729	0.0142379	0.000484124	0.0000164615	5.59732×10^{-7}	1.90323×10^{-8}	6.47149×10^{-10}	2.20047×10^{-11}
3	2.80041×10^{-6}	7.27880×10^{-6}	0.0000189190	0.0000491740	0.000127813	0.000332209	0.000863474	0.00224433
4	3.99708×10^{-14}	2.79970×10^{-11}	1.96100×10^{-8}	0.0000137356	0.00962086	6.73878	4720.08	3.30611×10^{6}
5	1.84711×10^{-23}	2.62212×10^{-16}	3.72231×10^{-9}	0.0528412	750123.	1.06486×10^{13}	1.51165×10^{20}	2.14591×10^{27}
6	1.68676×10^{-35}	2.85139×10^{-23}	4.82016×10^{-11}	81.4827	1.37743×10^{14}	2.32849×10^{26}	3.93621×10^{38}	6.65400×10^{50}
7	4.80891×10^{-49}	1.09207×10^{-29}	2.48000×10^{-10}	5.63189×10^{9}	1.27896×10^{29}	2.90442×10^{48}	6.59571×10^{67}	1.49783×10^{87}
8	2.65506×10^{-65}	6.66279×10^{-38}	1.67200×10^{-10}	4.19583×10^{17}	1.05293×10^{45}	2.64229×10^{72}	6.63074×10^{99}	1.66396×10^{127}
9	4.52005×10^{-83}	1.88536×10^{-45}	7.86407×10^{-8}	3.28019×10^{30}	1.36821×10^{68}	5.70695×10^{105}	2.38043×10^{143}	9.92906×10^{180}
10	1.47804×10^{-103}	1.08376×10^{-54}	7.94662×10^{-6}	5.82681×10^{43}	4.27247×10^{92}	3.13276×10^{141}	2.29707×10^{190}	1.68431×10^{239}
11	1.48182×10^{-125}	3.59121×10^{-63}	0.870337	2.10928×10^{62}	5.11187×10^{124}	1.23887×10^{187}	3.00243×10^{249}	7.27644×10^{311}

Table 4.8.8. The table below contains some values of $C(m,n) = \left(230\tilde{n}^m \left(\frac{\pi}{12}\right)^{2m} V_n^{1-m} n\right)^{\frac{2}{n^2-5}}$.

(n, m)	1	2	3	4	5	6	7	8
3	6.87691	125.979	2307.81	42276.9	774473.	1.41876×10^{7}	2.59904×10^{8}	4.76120×10^{9}
4	2.40966	21.6241	194.053	1741.42	15627.4	140239.	1.25850×10^{6}	1.12937×10^{7}
5	1.54762	8.80582	50.1044	285.090	1622.14	9229.86	52517.2	298819.
6	1.40247	6.73460	32.3393	155.292	745.707	3580.86	17195.1	82570.5
7	1.23838	4.82334	18.7864	73.1708	284.992	1110.01	4323.37	16839.0
8	1.20619	4.19700	14.6037	50.8142	176.811	615.221	2140.69	7448.64
9	1.13928	3.44306	10.4054	31.4468	95.0368	287.215	868.006	2623.24

Table 4.8.9. The table below contains the absolute value of the smallest discriminant D_k of a totally real number field of degree m (see for example [Voi08] or [JR14]).

m	1	2	3	4	5	6	7	8
$\min D_k$	1	5	49	725	14641	300125	20134393	282300416

Table 4.8.10. The tables below contains some values of $H(m) = \frac{(A^2 B^{2m-2} E)^2 V_3^{m-1}}{3\cdot\zeta(2)\cdot\zeta(3)}$ for $A = 25.465$, $B = 13.316$, $E = e^{-7.0667}$ if $m \geq 5$, and otherwise $H(m)$ is obtained from (4.12) using the smallest discriminant for the signature $(2, m-1)$ (see [JR14, Sel99]).

m	2	3	4	5	6	7	8	9
$H(m)$	2.603	5.527	26.39	87.71	563.2	3616.4	23222.2	149118.

m	10	11	12	13	14	15
$H(m)$	9.58×10^5	6.15×10^6	3.95×10^7	2.54×10^8	1.63×10^9	1.05×10^{10}

Table 4.8.11. The table below contains some values of $60.015^2 \cdot 22.210^{2m-2} \cdot e^{\frac{-80.001}{H(m)}}$, where $H(m)$ is as in table 4.8.10.

m	2	3	4	5	6	7	8
$D_l >$	8.05×10^{-8}	454.01	2.08×10^{10}	8.57×10^{13}	9.13×10^{16}	5.08×10^{19}	2.55×10^{22}

m	9	10	11	12	13	14	15
$D_l >$	1.26×10^{25}	6.23×10^{27}	3.07×10^{30}	1.52×10^{33}	7.48×10^{35}	3.69×10^{38}	1.82×10^{41}

4.8.12 Lemma. *The function $N : \mathbb{N} \to \mathbb{R}$ defined by*

$$N(n) = \frac{1}{100\tilde{n}^2} \left(\frac{12}{\pi}\right)^2 40^{\frac{1}{4}(n^2-n-6)} n^{-1}.$$

(where $\tilde{n} = 1$ or 2 if n is odd or even) is strictly increasing provided $n \geq 2$. The same holds for

$$N'(n) = \tilde{n}^{-2} n^{-1} 5^{\frac{1}{4}(n^2-n-2)}.$$

Proof. We compute

$$\frac{N(n+1)}{N(n)} = \frac{\tilde{n}^2}{\widetilde{n+1}^2} \cdot 40^{\frac{1}{2}n} \cdot \frac{n}{n+1} \geq \frac{1}{4} \cdot 40 \cdot \frac{2}{3} > 1.$$

The proof for N' is analogous. $\square$

4.8.13 Lemma.

$$\prod_{i=2}^{\infty} \zeta(i) < 2.3$$

Proof. We have

$$\begin{aligned}\ln \prod_{i=9}^{\infty} \zeta(i) &= \sum_{i=9}^{\infty} \ln(1 + (\zeta(i) - 1)) \leq \sum_{i=9}^{\infty} (\zeta(i) - 1) = \sum_{i=9}^{\infty} \sum_{j=2}^{\infty} \frac{1}{j^i} \\ &= \sum_{j=2}^{\infty} \frac{1}{j^9} \sum_{i=0}^{\infty} \frac{1}{j^i} = \sum_{j=2}^{\infty} \frac{1}{j^9} \frac{j}{j-1} \leq 2 \sum_{j=2}^{\infty} \frac{1}{j^9} = 2(\zeta(9) - 1);\end{aligned}$$

hence $\prod_{i=2}^{\infty} \zeta(i) \leq \exp(2\zeta(9) - 2) \cdot \prod_{i=2}^{8} \zeta(i) < 2.3$ □

4.8.14 Lemma. *Let P be a parabolic subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$ and let $n_1, n_2, \ldots, n_{\#\theta+1}$ be integers such that the complement of the type θ of P in the Dynkin diagram of $\mathrm{SL}_n(\mathbb{F}_q)$ consists of k' connected components of respectively $n_1 - 1, n_2 - 1, \ldots, n_{k'} - 1$ vertices and $n_{k'+1} = n_{k'+2} = \cdots = n_{\#\theta+1} = 1$. Then $[\mathrm{SL}_n(\mathbb{F}_q) : P] \geq q^{\frac{1}{2}(n^2 - \sum_{i=1}^{\#\theta+1} n_i^2)}$. In particular, if P is a proper parabolic subgroup, then $[\mathrm{SL}_n(\mathbb{F}_q) : P] \geq q^{n-1}$.*

Proof. Set $k = \#\theta + 1$. Without loss of generality, we may assume that P contains the subgroup B of upper triangular matrices and that elements of P are of the form

$$\left(\begin{array}{cc|c|c|ccc|c} \multicolumn{2}{c|}{\multirow{2}{*}{n_1}} & * & \cdots & * & * & * & * \\ & & * & \cdots & * & * & * & * \\ \hline 0 & 0 & n_2 & \cdots & * & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & & & * \\ 0 & 0 & 0 & \cdots & & n_{k-1} & & * \\ 0 & 0 & 0 & \cdots & & & & * \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n_k \end{array}\right)$$

where n_i indicates a block in $\mathrm{GL}_{n_i}(\mathbb{F}_q)$, $*$ indicates an arbitrary entry in $\mathbb{F}_q$, and the determinant of the whole matrix is 1. Hence

$$\#P = \frac{\prod_{j=1}^{n_1-1}(q^{n_1} - q^j) \cdots \prod_{j=1}^{n_k-1}(q^{n_k} - q^j) \cdot q^{\frac{1}{2}(n^2 - \sum_{i=1}^{k} n_i^2)}}{q-1}$$

and

$$
\begin{aligned}
\frac{\#\,\mathrm{SL}_n(\mathbb{F}_q)}{\#P} &= \frac{\prod_{j=0}^{n-1}(q^n - q^j)}{\prod_{j=0}^{n_1-1}(q^{n_1} - q^j)\cdots\prod_{j=0}^{n_k-1}(q^{n_k} - q^j)\cdot q^{\frac{1}{2}(n^2-\sum_{i=1}^k n_i^2)}} \\
&= \frac{q^{\frac{n(n-1)}{2}}\cdot\prod_{j=1}^{n}(q^j-1)}{q^{\frac{1}{2}(n^2-\sum_{i=1}^k n_i^2)}q^{\frac{1}{2}\sum_{i=1}^k n_i(n_i-1)}\cdot\prod_{j=1}^{n_1}(q^j-1)\cdots\prod_{j=1}^{n_k}(q^j-1)} \\
&= \frac{\prod_{j=1}^{n}(q^j-1)}{\prod_{j=1}^{n_1}(q^j-1)\cdots\prod_{j=1}^{n_k}(q^j-1)} \\
&= q^{\frac{1}{2}(n(n-1)-\sum_{i=1}^k n_i(n_i-1))}\cdot\frac{\prod_{j=1}^{n_1}(q^j-1)\cdot\prod_{j=1}^{n_2}(q^j-q^{-n_1})\cdots\prod_{j=1}^{n_k}(q^j-q^{-\sum_{i=1}^{k-1}n_i})}{\prod_{j=1}^{n_1}(q^j-1)\cdots\prod_{j=1}^{n_k}(q^j-1)}.
\end{aligned}
$$

Of course, $n(n-1) - \sum_{i=1}^k n_i(n_i-1) = (n^2 - \sum_{i=1}^k n_i^2)$. Now the ratio in the right-hand side is clearly greater then 1, as, taken in order, each factor in the numerator is bigger than the corresponding one in the denominator.

Finally, we observe that if P is proper, $k \geq 2$ and $n^2 - \sum_{i=1}^k n_i^2 \geq 2(n-1)$. □

This chapter contains material from *Lattices of minimal covolume in* $\mathrm{SL}_n(\mathbb{R})$, Proc. Lond. Math. Soc., 118 (2019), pp. 78–102. The dissertation author was the primary investigator and author of this paper.

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