REPORT NO.
UCB/SEMM-92/04

STRUCTURAL ENGINEERING,
MECHANICS AND MATERIALS

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Report to Sponsor:
National Science Foundation

March 1992

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A report on research supported by
the National Science Foundation under
Grant No. MSM-8922077

Report No. UCB/SEMM-92/04
Structural Engineering, Mechanics and Materials Program
Department of Civil Engineering
University of California
Berkeley, California, U.S.A.

March 1992

# Abstract

A new method for efficient discretization of random fields (i.e., their representation in terms of random variables) is introduced. The efficiency of the discretization is measured by the number of random variables required to represent the field with a specified level of accuracy. The method is based on principles of optimal linear estimation theory. It represents the field as a linear function of nodal random variables and a set of shape functions, which are determined by minimizing an error variance. Further efficiency is achieved by spectral decomposition of the nodal covariance matrix. The new method is found to be more efficient than other existing discretization methods and more practical than a series expansion method employing the Karhunen-Loève theorem. The method is particularly useful for stochastic finite element studies involving random media, where there is a need to reduce the number of random variables to reduce the amount of required computations.

KEY WORDS: correlation; discretization; eigenvalues; estimation; finite elements; optimization; random fields; random variables.

# Acknowledgement

Support from the National Science Foundation under Grant No. MSM-8922077 is gratefully acknowledged.

# Contents

# 1 Introduction

Many applications in civil engineering require representation of uncertain media as random fields. Problems of continuum mechanics for materials with random spatial variability, and geotechnical problems dealing with stability of slopes or flow of fluids through an uncertain ground medium are common examples. A recent interesting application is to micro-electro-mechanical systems, which consist of structural members with sizes not much greater than the material grain size, in which the random orientation or properties of the material grains have important influences on the response of the system (Mirfendereski and Der Kiureghian 1992). The advent of the stochastic finite element (SFE) method has facilitated the use of random field models in civil engineering applications. It has also highlighted the need for a more accurate and efficient representation of random fields.

In most SFE applications it is necessary to represent a continuous-parameter random field in terms of a vector of random variables. This process is known as "discretization" of the random field. A random field discretization method is characterized by its efficiency, i.e., its ability to accurately represent the random field with as few random variables as possible. This characteristic is particularly important in finite-element reliability analysis by first- and second-order reliability methods (FORM and SORM) (Liu and Der Kiureghian 1991), where the number of repeated finite element solutions is directly proportional to the number of random variables.

Several methods for discretization of random fields have been proposed in the past. These include the midpoint (MP) method, the spatial averaging (SA) method, the shape function (SF) method and the series expansion (SE) method. After a critical review of

these existing methods, a new discretization method is introduced in this paper, which is based on the fundamental concepts of optimum linear estimation theory. It is shown that, for a given level of accuracy, the new method is more efficient than the MP, SA and SF methods, and that it is almost as efficient as the SE method. The SE method is more efficient than the proposed method only when the *exact* eigenfunctions of the covariance kernel are known, a condition that is seldom satisfied in practice. In addition to its efficiency, the proposed method is simple and readily implementable in finite element codes.

Most existing methods of random field representation are limited to Gaussian random fields. Hence, a second item discussed in this paper is the extension of these methods to represent non-Gaussian random fields. This is accomplished in this paper for non-Gaussian random fields that are transformable to Gaussian fields through nonlinear marginal transformations.

# 2 Review of Existing Methods

Let $v(\mathbf{x})$, $\mathbf{x} \in \Omega$, denote a multi-dimensional Gaussian random field (not necessarily homogeneous) defined within the domain $\Omega$. The field is completely described by its mean function $\mu(\mathbf{x})$, variance function $\sigma^2(\mathbf{x})$, and the autocorrelation coefficient function $\rho(\mathbf{x}, \mathbf{x}')$, hereafter called correlation function. Also let $\mathbf{v}$ denote a vector of Gaussian random variables with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{vv}}$, through which the random field is to be described. All existing methods of discretization relate $v(\mathbf{x})$ to $\mathbf{v}$ through a linear transformation. Thus, assuming $\mathbf{v}$ is Gaussian preserves the Gaussianity of $v(\mathbf{x})$. We also assume that the field is discretized through a grid of random field "elements" and "nodal points," as shown in Fig. 1.

The simplest method of discretization is the "midpoint" (MP) method (Der Kiureghian and Ke 1988). In this, the field within the domain $\Omega_e$ of an element is described by a single random variable representing the value of the field at a central point of the element, e.g., the centroid $\mathbf{x}_c$ (see Fig. 1). The field value within the element is assumed to be a constant, i.e.,

$$\hat{v}(\mathbf{x}) = v(\mathbf{x}_c) \,, \qquad \mathbf{x} \in \Omega_e \tag{1}$$

The random variables $v(\mathbf{x}_c)$ for all elements form the vector $\mathbf{v}$. A realization of the field so defined is a stepwise function with discontinuities along the element boundaries. The statistics $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_{\mathbf{vv}}$ of $\mathbf{v}$ are readily available from the mean, variance and autocorrelation coefficient functions of the field evaluated at the element centroidal points.

The "spatial averaging" (SA) method proposed by Vanmarcke and Grigoriu (1983) describes the field within each element in terms of the spatial average of the field over the

element, i.e.,

$$\hat{v}(\mathbf{x}) = \frac{\int_{\Omega_e} v(\mathbf{x})\, d\Omega}{\int_{\Omega_e} d\Omega} = \bar{v}_e \ , \qquad \mathbf{x} \in \Omega_e \tag{2}$$

The average values $\bar{v}_e$ now form the vector $\mathbf{v}$. A realization of the field so defined is also a stepwise function with discontinuities along the element boundaries. However, one would expect a better fit because of the averaging process. The statistics $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_{\mathbf{vv}}$ of $\mathbf{v}$ in this case are given in terms of integrals of the moment functions of the random field (Vanmarcke 1983). In general, the variance of the spatial average variable over an element is smaller than the local variance of the random field.

The "shape function" (SF) method describes the random field within an element in terms of a set of nodal values and corresponding shape functions (Liu et al. 1986), i.e.,

$$\hat{v}(\mathbf{x}) = \sum_{i=1}^{n} N_i(\mathbf{x}) v(\mathbf{x}_i) \ , \qquad \mathbf{x} \in \Omega_e \tag{3}$$

where $n$ = the number of nodes of the element, $\mathbf{x}_i$ = the coordinates of the $i$-th node, and $N_i(\mathbf{x})$ = the $i$-th shape function, typically a polynomial having the property $N_i(\mathbf{x}_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta. The nodal values $v(\mathbf{x}_i)$ for the entire field form the vector $\mathbf{v}$ in this case, i.e., $\mathbf{v} = \{v(\mathbf{x}_i)\}$. A realization of the discretized field is a continuous function, and this clearly is an advantage over the MP and SA methods.

Another method of random field discretization is the "series expansion" (SE) method used by Lawrence (1987), and Spanos and Ghanem (1989). The latter used the Karhunen-Loève theorem to express the field in terms of its spectral decomposition. In a normalized form, it can be written as

$$v(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x}) \sum_{i=1}^{\infty} \zeta_i \sqrt{\lambda_i} f_i(\mathbf{x}) \ , \qquad \mathbf{x} \in \Omega \tag{4}$$

where $\zeta_i$ = independent standard normal variates (zero mean, unit variance, zero correlation) and $\lambda_i$, $f_i(\mathbf{x})$ = eigenvalues and eigenfunctions of the correlation kernel obtained from the integral equation

$$\int_\Omega \rho(\mathbf{x},\mathbf{x}')f_i(\mathbf{x}')\,d\mathbf{x}' = \lambda_i f_i(\mathbf{x}) \tag{5}$$

in which the eigenfunctions are normalized to satisfy

$$\int_\Omega f_i(\mathbf{x})f_j(\mathbf{x})\,d\mathbf{x} = \delta_{ij} \tag{6}$$

The field, thus, is represented by the infinite set of random variables $\zeta_i$, $i = 1, 2, \ldots$. However, usually only a few of the terms with the largest eigenvalues are important and, assuming the eigenvalues are ordered with decreasing magnitude, the field can be approximated by

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x})\sum_{i=1}^{r}\zeta_i\sqrt{\lambda_i}f_i(\mathbf{x})\,, \qquad \mathbf{x}\,\epsilon\,\Omega \tag{7}$$

where $r$ = the number of terms included in the series. The random variables $\zeta_i$, $i = 1, \ldots, r$, now form the vector $\mathbf{v}$. Note that this representation theoretically does not require a discretization of the domain $\Omega$.

Provided the exact eigenfunctions of the autocorrelation coefficient kernel are available, owing to its spectral representation nature, the SE method is expected to be the most efficient method for discretizing a random field, i.e., it requires the smallest number of random variables to describe the field within a given level of accuracy. However, the integral eigenvalue problem in Eq. 5 seldom has an exact solution (particularly for multi-dimensional fields) and an approximation method must be used. One suggested method uses a discrete integration rule to convert the integral problem into a matrix eigenvalue

problem (Mirfendereski 1990). In that case the field is defined only at the integration points and interpolation must be used to define the field values at other points of interest, e.g., Gauss points of a finite element mesh. A second approach is to use a Galerkin-type approximation, where the eigenfunctions $f_i(\mathbf{x})$ are described in terms of a set of basis functions in $\Omega$ and a corresponding set of coefficients, again converting the integral problem into a matrix eigenvalue problem (Spanos and Ghanem 1989). Both approaches require discretization of the domain $\Omega$, and are essentially equivalent to the shape function method, since they employ prescribed approximating functions to describe the field within the domain.

It is possible to use the spectral decomposition method to increase the efficiency of the MP, SA and SF methods. The random vector $\mathbf{v}$ is expressed in terms of its spectral decomposition

$$\mathbf{v} = \boldsymbol{\mu} + \sum_{i=1}^{N} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i \tag{8}$$

where $N$ = the size of $\mathbf{v}$, $\zeta_i$ = independent standard normal variates as described earlier, and $\theta_i$, $\boldsymbol{\phi}_i$ = eigenvalues and eigenvectors of the covariance matrix of $\mathbf{v}$ obtained from

$$\boldsymbol{\Sigma}_{\mathbf{vv}} \boldsymbol{\phi}_i = \theta_i \boldsymbol{\phi}_i \tag{9}$$

in which the eigenvectors are normalized such that $\boldsymbol{\phi}_i^T \boldsymbol{\phi}_j = \delta_{ij}$. Using a subset $r < N$ of the terms in Eq. 8 for the largest eigenvalues, the number of random variables describing the random field can be reduced. This method has been used by Liu et al. (1986) in connection with the SF method. A measure of the error incurred as a result of this truncation is derived later in this paper.

# 3 The Optimal Linear Estimation Method

Let the random field $v(\mathbf{x})$ in $\Omega$ be described by a linear function of a vector of nodal values $\mathbf{v} = \{v(\mathbf{x}_i)\}$ in the form

$$\hat{v}(\mathbf{x}) = a(\mathbf{x}) + \sum_{i=1}^{N} b_i(\mathbf{x}) v(\mathbf{x}_i) = a(\mathbf{x}) + \mathbf{b}^T(\mathbf{x})\mathbf{v} \,, \qquad \mathbf{x} \in \Omega \tag{10}$$

in which $N$ = the number of nodal points in the domain, $a(\mathbf{x})$ = a scalar function of $\mathbf{x}$, and $\mathbf{b}(\mathbf{x}) = \{b_i(\mathbf{x})\}$ = a vector function of $\mathbf{x}$ with elements $b_i(\mathbf{x})$. Employing the basic notions of optimal linear estimation theory (Vanmarcke 1983), we determine the functions $a(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ by minimizing the variance of the error $v(\mathbf{x}) - \hat{v}(\mathbf{x})$, subject to $\hat{v}(\mathbf{x})$ being an unbiased estimator of $v(\mathbf{x})$ in the mean, i.e.,

$$\textit{minimize} \quad Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] \tag{11}$$

$$\textit{subject to} \quad \mathrm{E}[v(\mathbf{x}) - \hat{v}(\mathbf{x})] = 0 \,, \qquad \mathbf{x} \in \Omega \tag{12}$$

The solution is (see Appendix I)

$$a(\mathbf{x}) = \mu(\mathbf{x}) - \mathbf{b}^T(\mathbf{x})\boldsymbol{\mu} \,, \qquad \mathbf{b}(\mathbf{x}) = \boldsymbol{\Sigma}_{\mathbf{vv}}^{-1} \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \tag{13}$$

in which $\boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}}$ = an $N \times 1$ vector containing the covariances of $v(\mathbf{x})$ with the elements of $\mathbf{v}$. Hence, the discretized field according to the "optimal linear estimation" (OLE) method is

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}}^T \boldsymbol{\Sigma}_{\mathbf{vv}}^{-1} (\mathbf{v} - \boldsymbol{\mu}) \,, \qquad \mathbf{x} \in \Omega \tag{14}$$

Since a Gaussian random field is completely defined through its first and second moment functions, the above representation, which matches the first moment and minimizes the

error in the second moment, is optimal among all linear representations of $v(\mathbf{x})$ in terms of the nodal random variables $\mathbf{v}$.

It is easily verified (see Appendix I) that the variance of the error in the OLE method is

$$Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] = \sigma^2(\mathbf{x}) - \mathbf{\Sigma}^T_{v(\mathbf{x})\mathbf{v}} \mathbf{\Sigma}^{-1}_{\mathbf{vv}} \mathbf{\Sigma}_{v(\mathbf{x})\mathbf{v}} \tag{15}$$

The second term in the above is identical to the variance of $\hat{v}(\mathbf{x})$. Thus, the error variance is equal to the difference between the variances of $v(\mathbf{x})$ and $\hat{v}(\mathbf{x})$. Furthermore, since the error variance is always positive, it follows that the estimate $\hat{v}(\mathbf{x})$ always underestimates the variance of the actual random field $v(\mathbf{x})$.

A good insight into the nature of the error in the OLE method can be gained by examining a one-dimensional random field with zero mean, unit variance and correlation function $\rho(x, x')$. When the field within an element is described in terms of the values $v_1 = v(x_1)$ and $v_2 = v(x_2)$ at the two nodal points $x_1$ and $x_2$ of the element, the error variance is

$$Var[v(x) - \hat{v}(x)] = \frac{1 + 2\rho(x, x_1)\rho(x, x_2)\rho(x_1, x_2) - \rho^2(x, x_1) - \rho^2(x, x_2) - \rho^2(x_1, x_2)}{1 - \rho^2(x_1, x_2)} \tag{16}$$

If the field is further assumed to be homogeneous, i.e., $\rho(x, x') = \rho(|x - x'|)$, and $\rho(x, x') > 0$ for $x, x' \in \Omega_e$, the maximum error will occur at the midpoint between the two nodes. Then, letting $l = |x_1 - x_2|$ denote the element size, the maximum error reduces to

$$\underset{x \in \Omega_e}{maximum}\ Var[v(x) - \hat{v}(x)] = 1 - 2\frac{\rho^2(l/2)}{1 + \rho(l)} \tag{17}$$

The error clearly depends on the form of the correlation function and the element size $l$. For a fixed value of $\rho(l)$, the error is larger when $\rho(l/2)$ is smaller. Considering the two

classes of correlation functions in Fig. 2, it follows that a larger error is expected for a non-differentiable random field, i.e., one whose correlation function does not have a zero slope at $x = x'$. Numerical results later in this section verify this conjecture.

In order to make a quantitative assessment of the accuracy of the various random field discretization methods, the following error norm is introduced:

$$E(\Omega) = \underset{\mathbf{x} \in \Omega}{supremum} \frac{Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})]}{Var[v(\mathbf{x})]} \tag{18}$$

This is a rather severe measure requiring accuracy of the representation at all points of the domain $\Omega$. For certain applications, other measures, such as the average of the error variance over a given domain, might be more appropriate. Nevertheless, the above measure is used in the remainder of this paper for comparison purposes.

For numerical analysis in this paper, the following three autocorrelation models of multi-dimensional homogeneous and isotropic random fields are considered:

$$\rho(\mathbf{x}, \mathbf{x}') = exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{a}\right) \quad \textit{Type A correlation function} \tag{19}$$

$$\rho(\mathbf{x}, \mathbf{x}') = exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{a^2}\right) \quad \textit{Type B correlation function} \tag{20}$$

$$\rho(\mathbf{x}, \mathbf{x}') = \frac{\sin\left(\frac{2.2|\mathbf{x}-\mathbf{x}'|}{a}\right)}{\frac{2.2|\mathbf{x}-\mathbf{x}'|}{a}} \quad \textit{Type C correlation function} \tag{21}$$

The parameter $a$ in each of the above models is a measure of the rate of fluctuations of the random field, commonly known as the "correlation length." For $|\mathbf{x} - \mathbf{x}'| = a$, $\rho = e^{-1}$ for each of the above models. Hence, in comparing errors for different methods or correlation functions, it is appropriate to scale the size of each element by $a$. In subsequent analyses, $l$ will denote the size of each element and results will be presented for varying $l/a$. It is

useful to note that the Type A correlation function describes a non-differentiable random field, whereas Types B and C describe differentiable random fields, with the latter having an oscillatory behavior. Types A and B are the most common forms used in practice.

The error in the OLE method depends on the form of the autocorrelation function, the normalized element size $l/a$, the variance function $\sigma^2(\mathbf{x})$, and the number of nodal points included in estimating the random field within an element. Figure 3 shows the errors for the OLE method with the above three correlation models when information from increasingly larger grids around an element are used to describe the field within the element. It is seen in Figs. 3b and 3c that for Types B and C correlation models inclusion of information from nodes beyond a $5 \times 5$ grid is unnecessary when $l/a < 1$. This is due to loss of correlation, i.e., nodal points outside the $5 \times 5$ grid are essentially statistically independent of the field values within the element and information gained from them does not improve the fit within the element. One may use this property to reduce the sizes of the vector $\mathbf{\Sigma}_{v(\mathbf{x})\mathbf{v}}$ and matrix $\mathbf{\Sigma}_{\mathbf{vv}}$ in computing the field values for each element from Eq. 14. For the Type A correlation model in Fig. 3a, nodal points outside of the $1 \times 1$ grid are non-informative. This is a result of the one-step memory property of the exponential correlation function.

Comparing Fig. 3a with Figs. 3b and 3c, it is apparent that, for the same level of accuracy, the Type A correlation function requires a much smaller element size than Types B and C. This is a result of the non-differentiable nature of the Type A random field as discussed earlier. Unfortunately, the Type A correlation function has been used widely in civil engineering applications (often without justification) in spite of its undesirable

properties.

The above analysis shows that the OLE method, with a $5 \times 5$ grid around each element, entails negligible error for an element of size $l/a \leq 0.5$ and correlation models B and C. A somewhat smaller element size is necessary when the grid around an element is smaller than $5 \times 5$, e.g., at the boundaries of a domain. For the Type A correlation model, a much smaller element size is necessary to achieve the same level of accuracy.

# 4 Comparison with MP, SA and SF Methods

First consider the one-dimensional random field with zero mean, unit variance, and correlation function $\rho(x, x')$. Using the centroidal value of an element to describe the field within the element, the OLE method is found to give

$$\hat{v}(x) = \rho(x, x_c)v(x_c), \qquad x \in \Omega_e \tag{22}$$

which is a function of $x$. If the spatial average over the element is used to describe the field within the element, the OLE method gives

$$\hat{v}(x) = \frac{\int_{\Omega_e} d\xi \int_{\Omega_e} \rho(x, \eta)\, d\eta}{\int_{\Omega_e} \int_{\Omega_e} \rho(\xi, \eta)\, d\xi\, d\eta}\, \bar{v}_e, \qquad x \in \Omega_e \tag{23}$$

which is also a function of $x$. These compare with $\hat{v}(x) = v(x_c)$ for the MP method and $\hat{v}(x) = \bar{v}_e$ for the SA method, both of which are constants and obviously inferior to the OLE estimate.

If the field within the element is described in terms of the two nodal values $v_1 = v(x_1)$ and $v_2 = v(x_2)$, the OLE method gives

$$\hat{v}(x) = \frac{\rho(x, x_1) - \rho(x, x_2)\rho(x_1, x_2)}{1 - \rho^2(x_1, x_2)} v_1 + \frac{\rho(x, x_2) - \rho(x, x_1)\rho(x_1, x_2)}{1 - \rho^2(x_1, x_2)} v_2 \tag{24}$$

The SF method would give this optimum solution only if the shape functions were selected to be

$$N_1(x) = \frac{\rho(x, x_1) - \rho(x, x_2)\rho(x_1, x_2)}{1 - \rho^2(x_1, x_2)}, \quad N_2(x) = \frac{\rho(x, x_2) - \rho(x, x_1)\rho(x_1, x_2)}{1 - \rho^2(x_1, x_2)} \tag{25}$$

For any other set of shape functions, e.g., the usually assumed polynomials, the SF method would produce a suboptimal representation of the field. Hence, the OLE method is always

superior to the MP, SA and SF methods. It is clear that the OLE method constructs the random field by employing shape functions that take into account the correlation structure of the random field. This is the key to the superior accuracy of this method.

To provide numerical estimates of the relative accuracy, Fig. 4 compares the error norms $E(\Omega_e)$ for the MP, SA, SF and OLE methods for two-dimensional random fields having the correlation functions defined in Eqs. 19-21. For the OLE method both a $1 \times 1$ and a $5 \times 5$ grid around the element are considered. For the SF method, the bilinear shape functions

$$N_i(\mathbf{x}) = (1 - \frac{|x - x_i|}{l})(1 - \frac{|y - y_i|}{l}) \tag{26}$$

are employed, where $\mathbf{x} = (x, y)$, $\mathbf{x}_i = (x_i, y_i)$ = the coordinates of node $i$, and $l$ = the element size. It is seen in Fig. 4 that the MP and SA methods entail very large errors, and that the SF method, although significantly more accurate than the MP and SA methods, is inferior to the OLE method for correlation models B and C even with the $1 \times 1$ grid. Fig. 5 compares the autocorrelation function $\rho((0,0),(x,y))$ along the diagonal $x = y$ for $l/a = 0.5$. Again the OLE method gives the best result in all cases.

# 5 The Expansion OLE Method

The efficiency of the OLE method can be improved by employing the spectral decomposition in Eq. 8. This method is denoted herein as the "expansion OLE" (EOLE) method.

Retaining the first $r < N$ terms in Eq. 8 and substituting in Eq. 14 yields

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \mathbf{\Sigma}_{v(\mathbf{x})\mathbf{v}}^T \mathbf{\Sigma}_{\mathbf{vv}}^{-1} \sum_{i=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i, \qquad \mathbf{x} \in \Omega \tag{27}$$

The above representation, however, may not be optimal, since minimization of the error variance is carried out before the truncation. To derive the optimal solution, we first substitute the truncated series in Eq. 10, obtaining

$$\hat{v}(\mathbf{x}) = a(\mathbf{x}) + \mathbf{b}^T(\mathbf{x})(\boldsymbol{\mu} + \sum_{i=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i), \qquad \mathbf{x} \in \Omega \tag{28}$$

Solutions of $a(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ are now sought according to Eqs. 11 and 12. It is shown in Appendix I that the optimum solution of the estimator is

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{r} \frac{\zeta_i}{\sqrt{\theta_i}} \boldsymbol{\phi}_i^T \mathbf{\Sigma}_{v(\mathbf{x})\mathbf{v}} \tag{29}$$

It turns out that the forms in Eqs. 27 and 29 are identical. To show this, we note from Eq. 9 that $\theta_i \mathbf{\Sigma}_{\mathbf{vv}}^{-1} \boldsymbol{\phi}_i = \boldsymbol{\phi}_i$. Substituting this result in Eq. 27, yields the preceding equation.

The variance of the error for the EOLE method is given by (see Appendix I)

$$Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] = \sigma^2(\mathbf{x}) - \sum_{i=1}^{r} \frac{1}{\theta_i} (\boldsymbol{\phi}_i^T \mathbf{\Sigma}_{v(\mathbf{x})\mathbf{v}})^2 \tag{30}$$

As before, the second term is identical to the variance of $\hat{v}(\mathbf{x})$, and hence the EOLE estimate always underestimates the true variance. Furthermore, since the terms in the series are all positive, it follows that the truncation of each term further reduces the

variance of the estimate, i.e., there is no oscillatory behavior resulting from the truncation. The added error due to the truncation of each term is $(\boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}})^2/\theta_i$. This value should be compared with $\sigma^2(\mathbf{x})$ to determine the incremental effect of truncating each term.

The EOLE method requires decisions at two levels: The grid size that controls the size $N \times N$ of the matrix eigenvalue problem in Eq. 9, and the number $r$ of the terms retained in the series in Eq. 29, which is equivalent to the number of random variables representing the random field. At the expense of a larger eigenvalue problem, a finer grid usually results in a smaller error for the same $r$. As an example, consider a two-dimensional random field of size $L \times L$ having the Type B correlation function with correlation length $a = 0.5L$. Two different uniform grids with $N = 5 \times 5 = 25$ and $N = 7 \times 7 = 49$ nodes are considered. Fig. 6 shows the error norm $E(\Omega)$ as a function of $r$ for the two grids. It can be seen that for the same $r$ the $7 \times 7$ grid provides a more accurate representation of the field than the $5 \times 5$ grid. Therefore, it is advantageous to use as fine a grid as possible, if efficiency of the representation is desired.

# 6 Comparison with SE Method

As suggested earlier, the series expansion (SE) method is expected to be the most efficient method for discretization of a random field (i.e., requiring the smallest number of random variables to represent the field within a given level of accuracy), provided the exact eigenvalues and eigenfunctions are available. Unfortunately, for most correlation models, and particularly for multi-dimensional random fields, exact solutions of the requisite integral eigenvalue problem are not available. As mentioned earlier, approximate methods that convert the integral problem into a matrix eigenvalue problem are essentially equivalent to the SF method coupled with a discrete eigenvalue decomposition as in Eq. 9. Such approximate implementations of the SE method are inferior to the EOLE method, since unlike the latter they do not employ optimal shape functions.

To examine the relative efficiency of the SE and EOLE methods, consider a one-dimensional homogeneous random field of zero mean, unit variance, and Type A correlation function, i.e., $\rho(x,x') = exp(-|x-x'|/a)$. For this correlation function the exact eigenvalues and eigenfunctions are given in (Spanos and Ghanem 1989). For the EOLE method, the field of length $L$ is discretized into $N = 20$ nodes with elements of size $l/L = l/19$. Thus, the EOLE method requires the solution of a $20 \times 20$ eigenvalue problem. Figure 7 shows the error $E(\Omega)$ of the two methods as a function of $r$ for $L/a = 0.5$ and 1. Surprisingly, the EOLE method gives a more accurate representation (according to the considered error measure) for values of $r$ up to 16. To better understand the nature of the error, the error variances are plotted in Fig. 8 for $r = 2$ and $r = 4$, and $L/a = 0.5$. It is observed that the EOLE error is more or less uniform over the domain, whereas the

SE error is larger at the two boundaries of the domain and exceeds the EOLE error there. In the interior of domain, the SE error is smaller than the EOLE error. The larger error of the SE method at the boundaries is a characteristic of the spectral decomposition method. This problem can be circumvented by determining the eigenfunctions of the correlation kernel for a domain that extends beyond the boundaries of the domain of interest. In that case, the SE method would be more efficient than the EOLE method.

It is interesting to note that for the Type A correlation function the error is very large for small $r$ for both the SE and the EOLE methods. This finding contradicts previous studies with the SE method with this correlation function, where values of $r$ as small as 2 were reported to provide sufficient accuracy (Spanos and Ghanem 1989).

For the Types B and C correlation functions, unfortunately exact solutions of the eigenfunctions, even for the one-dimensional case, are not available. For the reasons described earlier, any approximations with the SE method with these correlation functions will produce results that are inferior to the EOLE method. Hence, no comparison between the approximate SE and the EOLE methods for these correlation functions is made.

In summary, if the exact eigenvalues and eigenfunctions of the integral problem in Eq. 5 are available, then the SE method is the most efficient method for discretizing the random field, provided the larger errors at the boundaries are excluded. However, when these exact solutions are not available, as is the case in most practical situations, then the EOLE method is the most efficient method for discretizing a random field.

# 7 Non-Gaussian Random Fields

With the exception of the midpoint method, all discretization methods described in the preceding section are strictly applicable to Gaussian processes only. This is because under the linear transformations involved only the Gaussian distribution remains closed. Many quantities needing random field modeling in SFEM are non-Gaussian, and hence there is a need to extend these methods.

One class of non-Gaussian random fields for which the extension is straightforward is the class of translation processes, where a non-Gaussian random field is defined as a nonlinear function of a Gaussian field (Grigoriu 1984). Let

$$w(\mathbf{x}) = g(v(\mathbf{x})) \tag{31}$$

define the non-Gaussian random field $w(\mathbf{x})$ in terms of the Gaussian field $v(\mathbf{x})$, where $g(.)$ = a nonlinear function. Discretization of $w(\mathbf{x})$ is achieved by simply replacing $v(\mathbf{x})$ by its discretized form, i.e.,

$$\hat{w}(\mathbf{x}) = g(\hat{v}(\mathbf{x})) \tag{32}$$

For example, using the EOLE method,

$$\hat{w}(\mathbf{x}) = g(\mu(\mathbf{x}) + \sum_{i=1}^{r} \frac{\zeta_i}{\sqrt{\theta_i}} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}}) \tag{33}$$

Unfortunately, it is not possible to ascertain that the discretized field $\hat{w}(\mathbf{x})$ so achieved is optimal in the sense of having minimum error variance.

One special class of translation processes that can produce any desired marginal distribution for $w(\mathbf{x})$ is defined by the "Nataf" multivariate distribution of Liu and Der

Kiureghian(1986). For this case, the $g$ function takes the form

$$g(.) = F^{-1}[\Phi(.)] \tag{34}$$

where $\Phi(.)$ = the standard normal cumulative probability function and $F^{-1}[.]$ = the inverse of the desired marginal cumulative distribution function of $w(\mathbf{x})$. It is easily verified then that $v(\mathbf{x})$ has a zero mean, unit variance, and a correlation function $\rho(\mathbf{x}, \mathbf{x}')$ that satisfies the integral equation

$$\rho_{ww}(\mathbf{x}, \mathbf{x}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{w(\mathbf{x}) - \mu_w(\mathbf{x})}{\sigma_w(\mathbf{x})}\right] \left[\frac{w(\mathbf{x}) - \mu_w(\mathbf{x})}{\sigma_w(\mathbf{x})}\right] \phi_2[v, v', \rho(\mathbf{x}, \mathbf{x}')]\, dv\, dv' \tag{35}$$

in which $\mu_w(\mathbf{x})$, $\sigma_w(\mathbf{x})$, $\rho_{ww}(\mathbf{x}, \mathbf{x}')$ = the mean, standard deviation and correlation functions of $w(\mathbf{x})$, $v = \Phi[F(w(\mathbf{x}))]$, $v' = \Phi[F(w(\mathbf{x}'))]$, and $\phi_2(v, v', \rho)$ = the bivariate standard normal density function with correlation coefficient $\rho$. Liu and Der Kiureghian (1986) have shown that in general $|\rho_{ww}(\mathbf{x}, \mathbf{x}')| \leq |\rho(\mathbf{x}, \mathbf{x}')|$ and for most cases $\rho_{ww}(\mathbf{x}, \mathbf{x}') \approx \rho(\mathbf{x}, \mathbf{x}')$. Empirical formulae relating $\rho(\mathbf{x}, \mathbf{x}')$ to $\rho_{ww}(\mathbf{x}, \mathbf{x}')$ for common marginal distributions can be found in the above reference.

# 8 Conclusions

A new method for discretization of random fields, denoted the OLE method, is introduced, which is based on the fundamental concepts of optimal linear estimation theory. In comparison with the existing midpoint, spatial average and shape function methods, the new method is more efficient in the sense that it requires a smaller number of random variables to represent the random field within a given level of accuracy. The superior efficiency of the method lies in its use of optimal shape functions, which are derived based on the prescribed correlation function of the field such that the error variance is minimal. The efficiency of the OLE method is greatly improved by use of the spectral decomposition of the nodal covariance matrix, and the method is called expansion OLE, or EOLE. In comparison to the existing series expansion method, the EOLE method is more efficient when the former employs an approximate solution of the requisite integral eigenvalue problem, which is usually the case.

Other useful findings in the paper include: (1) numerical estimates of the error in different random field discretization methods as a function of element size for selected correlation functions; (2) an understanding of why non-differentiable random fields are more difficult to discretize; (3) the finding that the EOLE method is more efficient (i.e., it requires a smaller number of random variables to represent the field) for a finer grid, at the expense of a larger eigenvalue problem; and (4) the finding that the series expansion method produces large errors at the boundaries of the domain due to truncation effects, which has not been identified before.

The proposed OLE and EOLE methods are useful for many engineering applications

dealing with random media. They are more accurate and efficient than the existing midpoint, spatial average and shape function methods, and are more practical than the series expansion method. The EOLE method is particularly useful for stochastic finite element analysis, where there is a great need to represent a random field with as few random variable as possible.

# Appendix I: Derivation of Solutions

In the following, we show the solution of Eqs. 11 and 12 for the EOLE method. A similar approach is used to find the solutions for the OLE method.

Setting the expectation of Eq. 28 equal to $\mu(\mathbf{x})$ and noting that $\mathrm{E}[\zeta_i] = 0$, gives the first equality in Eq. 13. Substituting this result in Eq. 28 yields

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i \ , \qquad \mathbf{x} \in \boldsymbol{\Omega} \tag{36}$$

The variance of the error is

$$\begin{aligned} Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] &= \mathrm{E}\left[ \left( v(\mathbf{x}) - \mu(\mathbf{x}) - \mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i \right)^2 \right] \\ &= \sigma^2(\mathbf{x}) - 2\mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \sqrt{\theta_i} \mathrm{E}[(v(\mathbf{x}) - \mu(\mathbf{x}))\zeta_i] \boldsymbol{\phi}_i + \mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \theta_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \mathbf{b}(\mathbf{x}) \end{aligned} \tag{37}$$

where use has been made of the orthogonality relation $\mathrm{E}[\zeta_i \zeta_j] = \delta_{ij}$. The expectation in the second term on the right-hand side can be obtained as follows:

$$\begin{aligned} \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} &= \mathrm{E}[(v(\mathbf{x}) - \mu(\mathbf{x}))(\mathbf{v} - \boldsymbol{\mu})] \\ &= \sum_{j=1}^{N} \sqrt{\theta_j} \mathrm{E}[(v(\mathbf{x}) - \mu(\mathbf{x}))\zeta_j] \boldsymbol{\phi}_j \end{aligned} \tag{38}$$

Premultiplying both sides by $\boldsymbol{\phi}_i^T$ and using the orthogonality relation $\boldsymbol{\phi}_j^T \boldsymbol{\phi}_i = \delta_{ij}$, gives

$$\sqrt{\theta_i} \mathrm{E}[(v(\mathbf{x}) - \mu(\mathbf{x}))\zeta_i] = \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \tag{39}$$

Substituting this result in Eq. 37, the variance of the error is

$$Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] = \sigma^2(\mathbf{x}) - 2\mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_i + \mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \theta_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \mathbf{b}(\mathbf{x}) \tag{40}$$

Setting the derivative with respect to $\mathbf{b}(\mathbf{x})$ equal to zero, produces

$$\sum_{i=1}^{r} \theta_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \mathbf{b}(\mathbf{x}) = \sum_{i=1}^{r} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_i \tag{41}$$

from which the optimum solution of $\mathbf{b}(\mathbf{x})$ is obtained

$$\mathbf{b}(\mathbf{x}) = \boldsymbol{\Sigma}_{rr}^{-1} \sum_{i=1}^{r} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_i \tag{42}$$

in which

$$\boldsymbol{\Sigma}_{rr} = \sum_{i=1}^{r} \theta_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \tag{43}$$

Substituting Eq. 42 in Eq. 36,

$$\begin{aligned} \hat{v}(\mathbf{x}) &= \mu(\mathbf{x}) + (\boldsymbol{\Sigma}_{rr}^{-1} \sum_{j=1}^{r} \boldsymbol{\phi}_j^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_j)^T \sum_{i=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_i \\ &= \mu(\mathbf{x}) + \sum_{i=1}^{r} \sum_{j=1}^{r} \zeta_i \sqrt{\theta_i} \boldsymbol{\phi}_j^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_j^T \boldsymbol{\Sigma}_{rr}^{-1} \boldsymbol{\phi}_i \end{aligned} \tag{44}$$

where use is made of the symmetry of $\boldsymbol{\Sigma}_{rr}$. From the orthonormality of the eigenvectors, it is easy to verify that $\theta_j \boldsymbol{\phi}_j^T \boldsymbol{\Sigma}_{rr}^{-1} \boldsymbol{\phi}_i = \delta_{ij}$. Using this relation in the preceding equation, we finally obtain

$$\hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{r} \frac{\zeta_i}{\sqrt{\theta_i}} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \tag{45}$$

This is the result shown in Eq. 29.

To determine the variance of the error, we first substitute the identity in Eq. 42 for the last $\mathbf{b}(\mathbf{x})$ in Eq. 40. Noting that $\boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}}$ is a scalar and using the identity $\theta_i \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{rr}^{-1} \boldsymbol{\phi}_j = \delta_{ij}$, the expression for the variance reduces to

$$Var[v(\mathbf{x}) - \hat{v}(\mathbf{x})] = \sigma^2(\mathbf{x}) - \mathbf{b}^T(\mathbf{x}) \sum_{i=1}^{r} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{v(\mathbf{x})\mathbf{v}} \boldsymbol{\phi}_i \tag{46}$$

Using the same procedure as in deriving the expression in Eq. 45, one obtains the result shown in Eq. 30.

# References


Der Kiureghian, A., and Liu, P-L. (1986). "Structural reliability under incomplete probability information ." *J. Engrg. Mech.*, ASCE, 112, (85-104).

Der Kiureghian, A., and Ke, J-B. (1988). "The stochastic finite element method in structural reliability." *Probabilistic Engineering Mechanics*, 3, (83-91).

Grigoriu, M. (1984). "Crossing of non-Gaussian translation processes." *J. Engrg. Mech.*, ASCE, 110, (610-620).

Lawrence, M. (1987). "Basis random variables in finite element analysis ." *Int. J. of Numerical Methods in Engrg.*, 24, (1849-1863).

Liu, P-L. and Der Kiureghian, A. (1991). "Finite element reliability of geometrically nonlinear uncertain structures." *J. Engrg. Mech.*, ASCE, 117, (1806-1825).

Liu, W. K., Belytschko, T., and Mani, A. (1986). "Random field finite elements." *Int. J. Numerical Methods in Engrg.*, 23, (1831-1845).

Mirfendereski, D. (1990). "On series representation of random fields." CE299 Report, Department of Civil Engineering, Structural Engineering, Mechanics and Materials Program, University of California, Berkeley, CA.

Mirfendereski, D., and Der Kiureghian, A. (1992). "Random response of multicrystalline structures." *Proc., 10th Engrg. Mech. Conf.*, College Station, Texas.

Spanos, P. D., and Ghanem, R. (1989). "Stochastic finite element expansion for random media." *J. Engrg. Mech., ASCE*, 115, (1035-1053).

Vanmarche, E. H. (1983). , *Random Fields : Analysis and Synthesis.* The MIT Press, Cambridge, Mass.

Varmarcke, E. H., and Grigoriu, M. (1983). "Stochastic finite element analysis of simple beams." *J. Engrg. Mech., ASCE,* 109, (1203-1214).

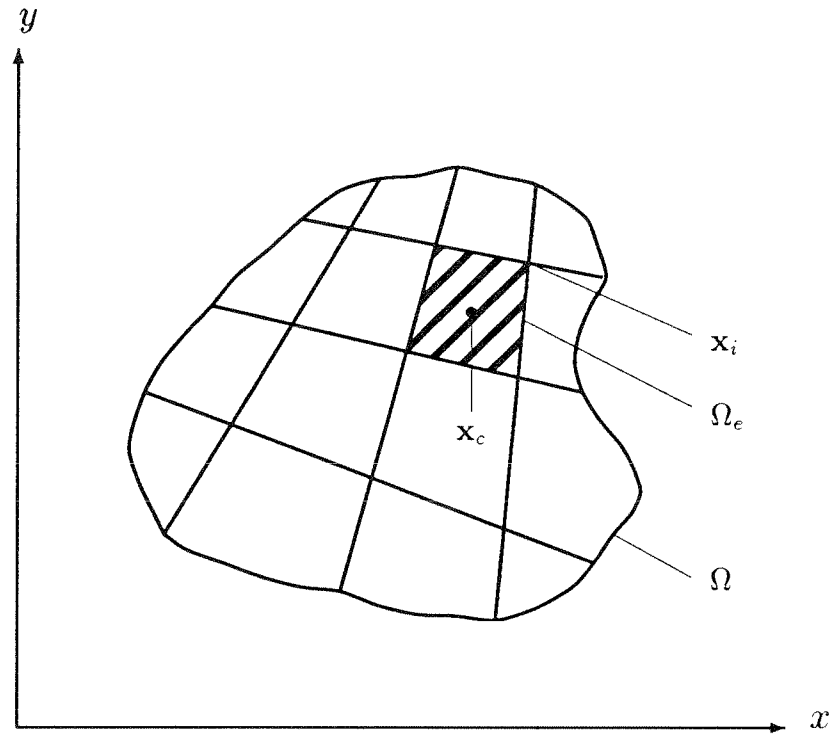


Figure 1. Random Field Mesh

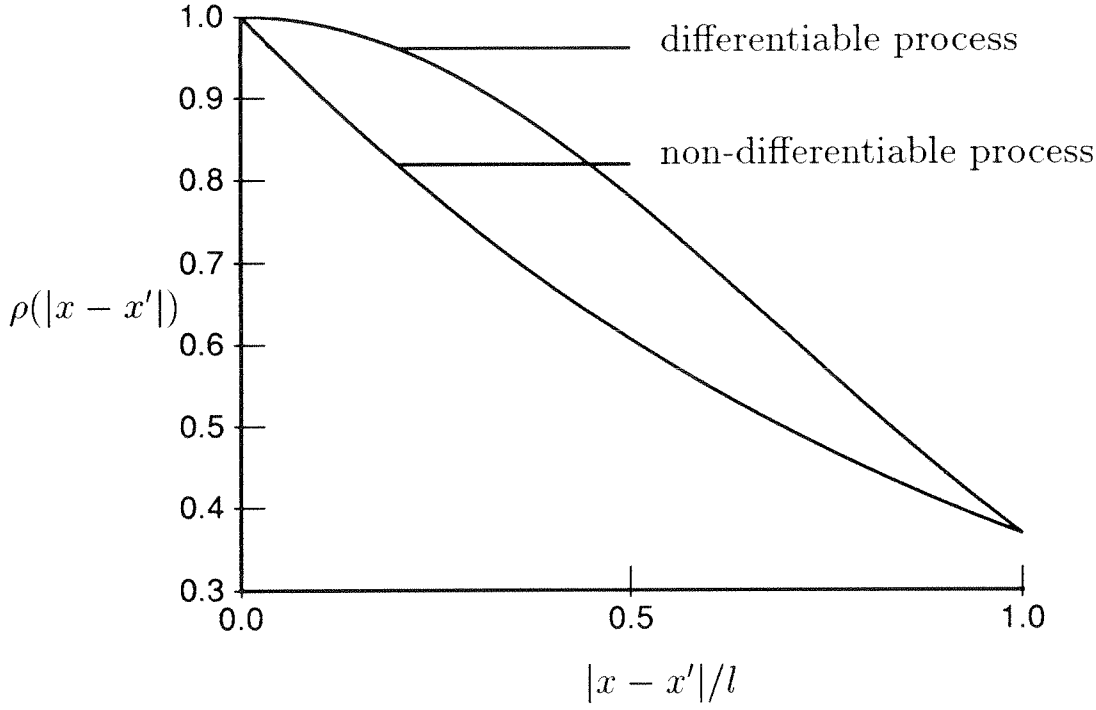


Figure 2. Class of Correlation Functions

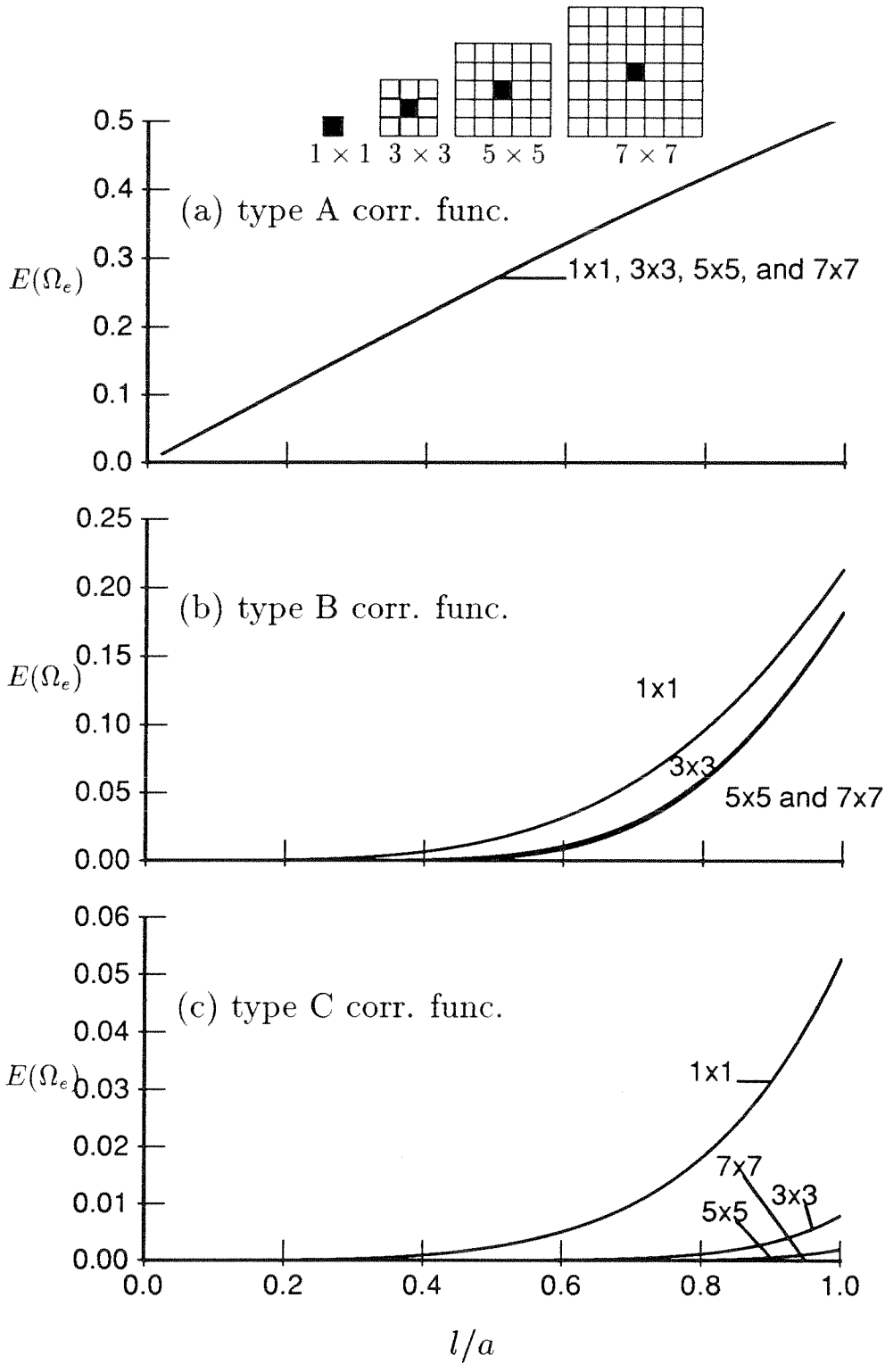


Figure 3. Discretization Errors for OLE Method with Varying Grid and Element Size

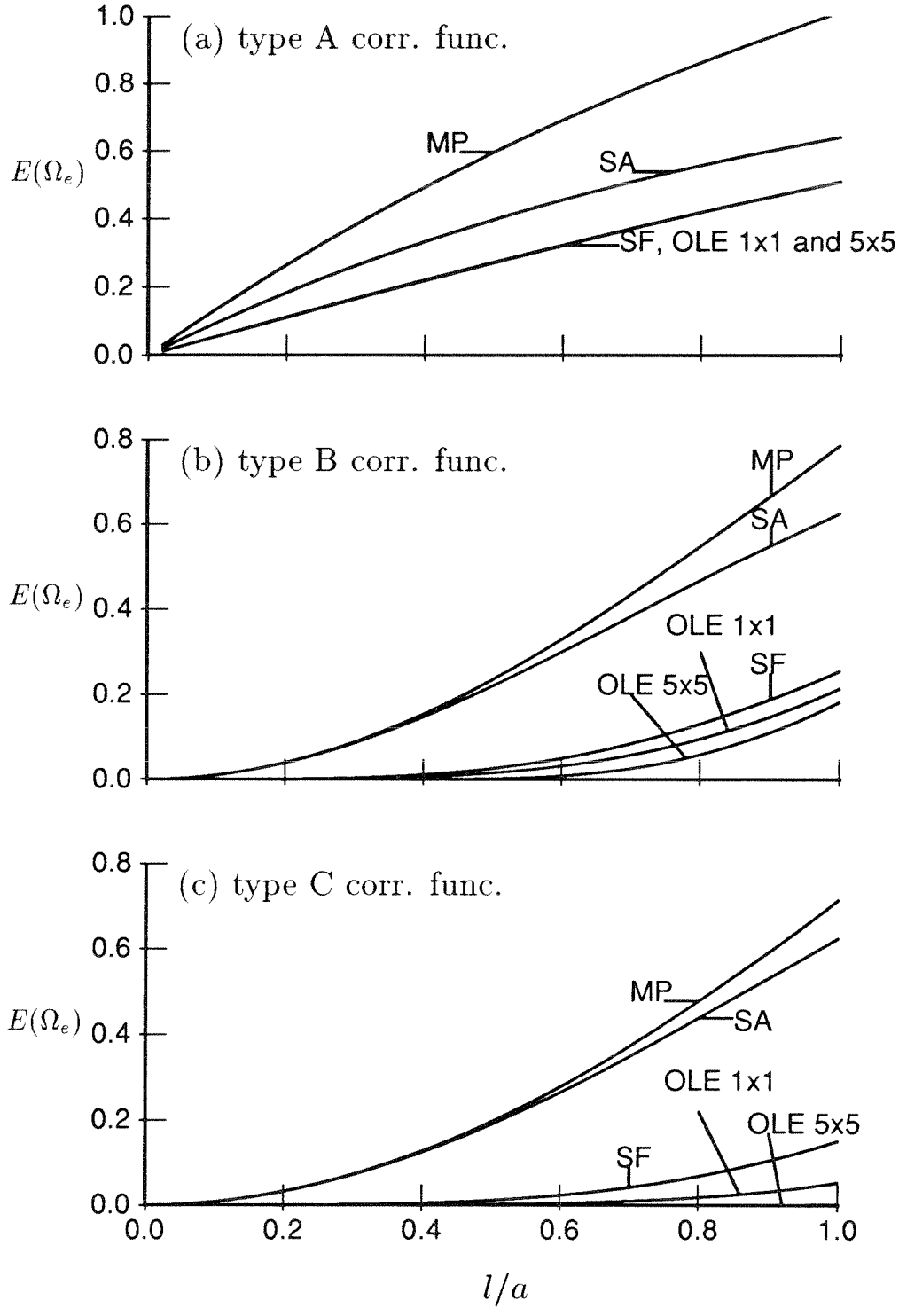


Figure 4. Comparison of Errors for MP, SA, SF, and OLE Methods for Varying Element Size

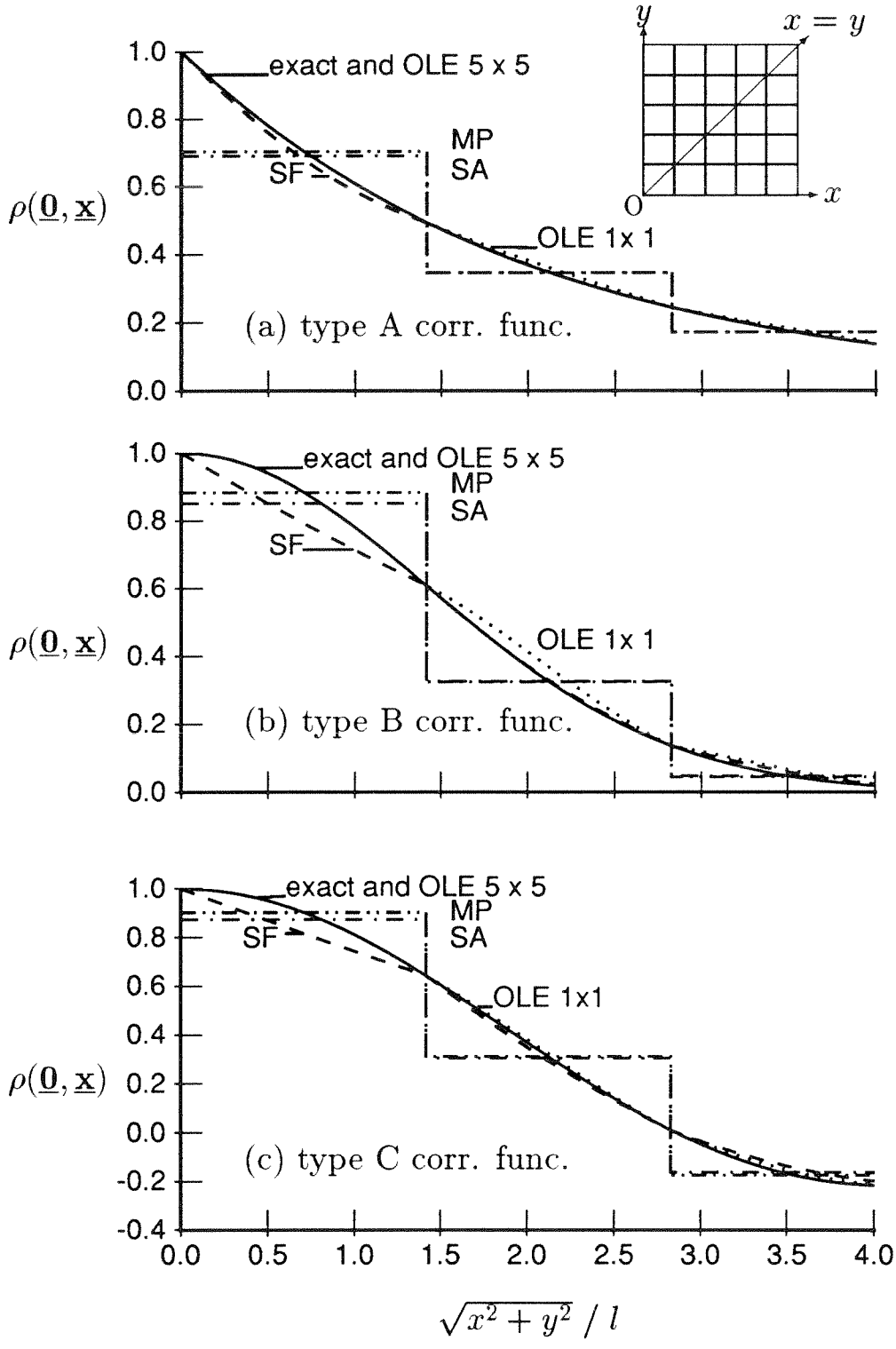


Figure 5. Comparison of Correlation Functions Along the Diagonal x = y for MP, SA, SF, and OLE Methods

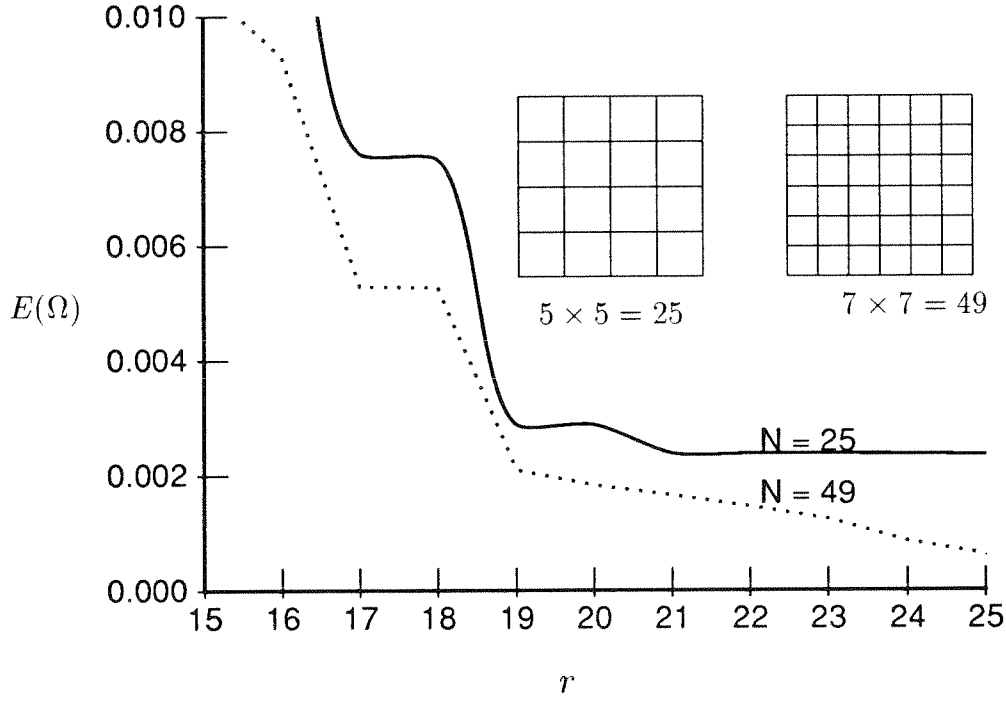


Figure 6. Error in EOLE Method for Different Grid Sizes as a Function of Number of Random Variables $r$

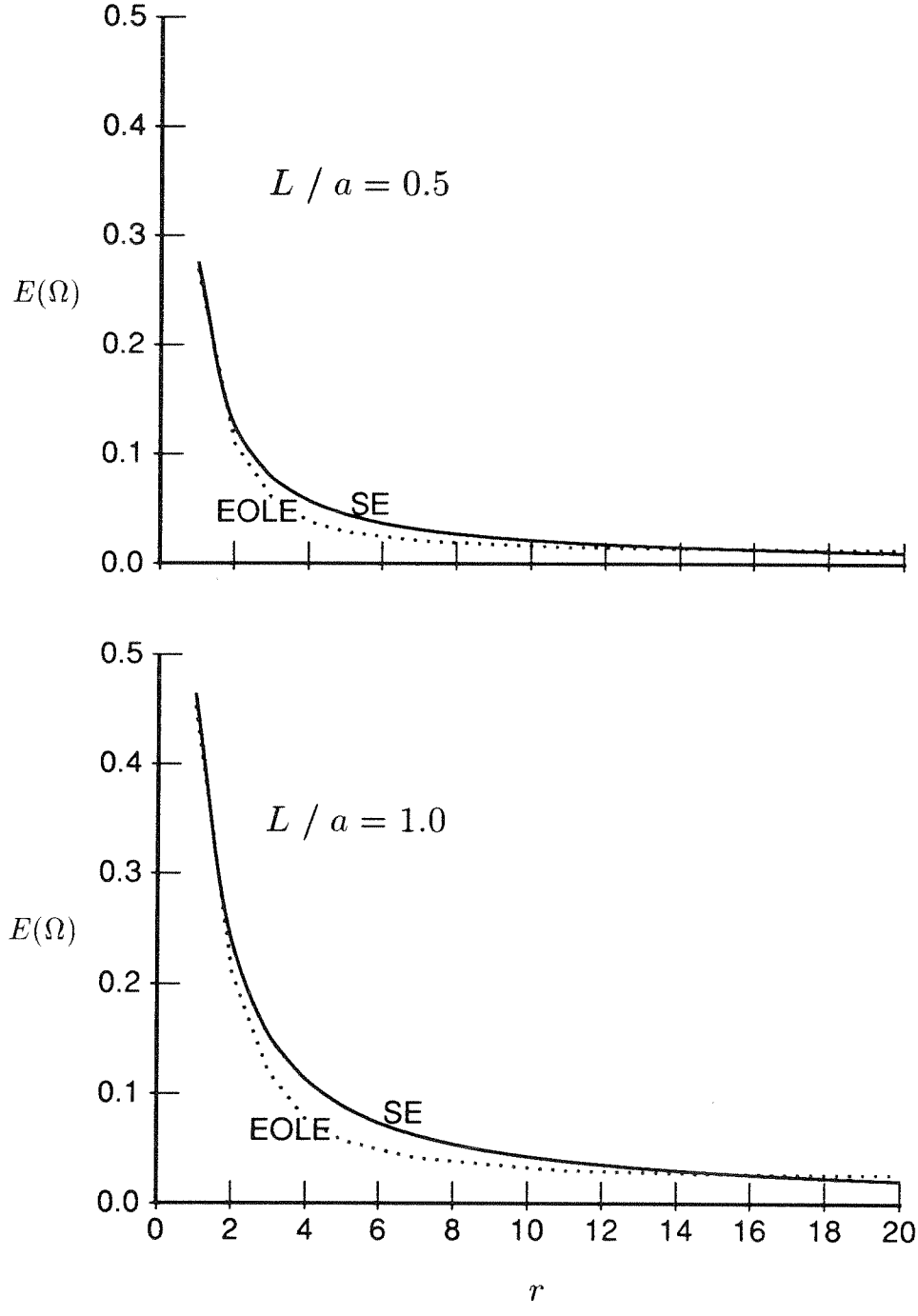


Figure 7. Comparison of Errors for SE and EOLE Methods with the Type A Correlation Function

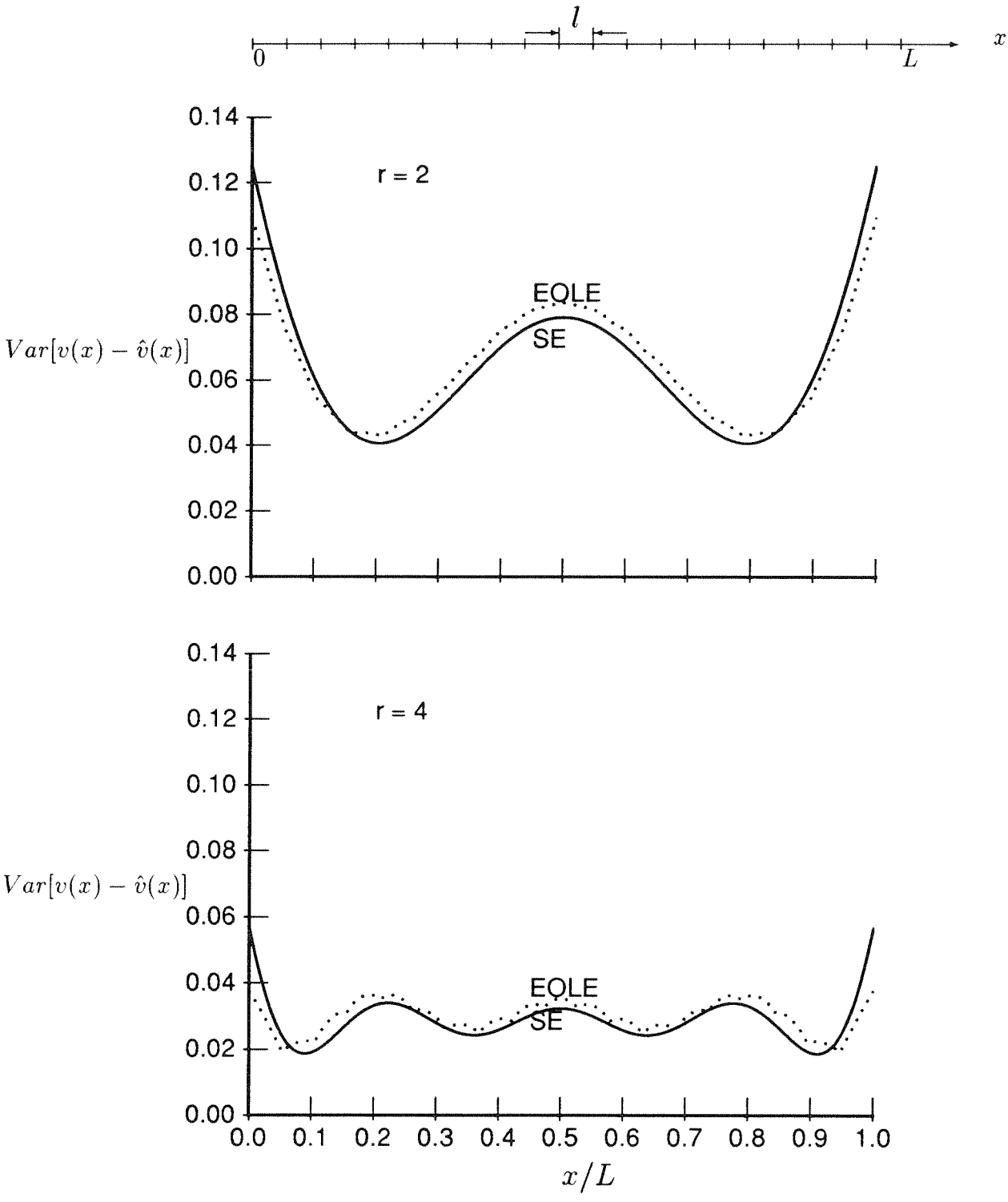


Figure 8. Error Variance for SE and EOLE Methods as a Function of Spatial Coordinate for $L/a = 0.5$