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AN ELASTIC EXPLANATION FOR THE OBSERVED BUCKLING MODE OF REINFORCING BARS IN CONCRETE

BY

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An Elastic Explanation For the Observed Buckling Mode of Reinforcing Bars in Concrete

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Contents

Chapter 1

Introduction

1.1 Description of a Concrete Column

Consider a cylindrical concrete column similar to those commonly used to support modern highway bridges in areas with seismic activity (Figure 1.1). The column is reinforced along its length by steel *longitudinal bars.* Around

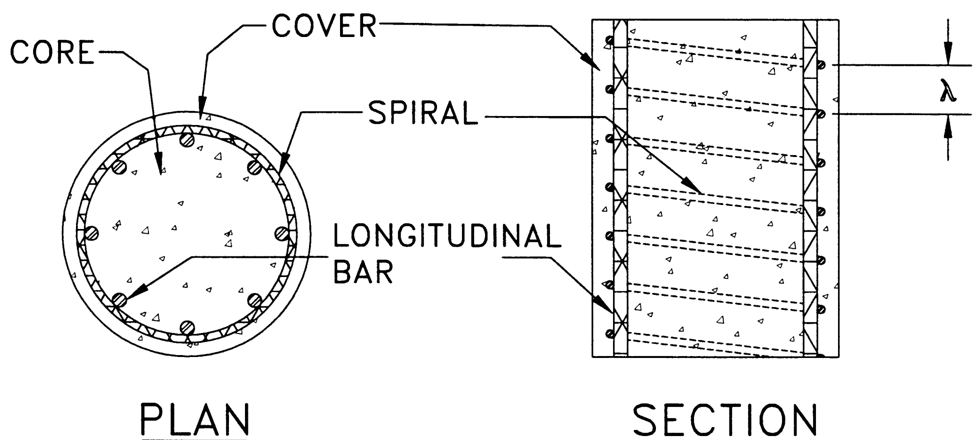


Figure 1.1: Typical seismic resistant cylindrical reinforced concrete column

these longitudinal bars is wound a steel *spiral* at a spacing, λ. The concrete within the spiral is referred to as the *core* and the concrete outside of the spiral is referred to as the *cover.* Combined seismic and gravity loads concentrate deformations in an inelastic *hinge zone* at the column's base (Figure 1.2).

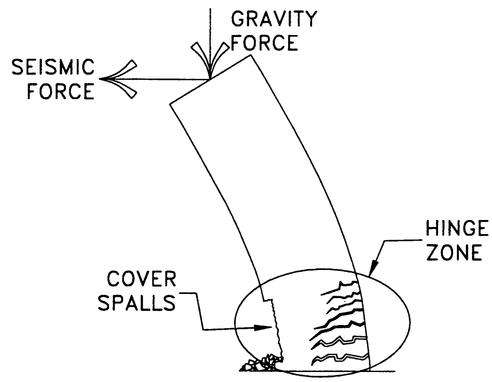


Figure 1.2: Column under the combined action of gravity and seismic forces

1.2 Observed Phenomenon

In the most common laboratory test for theses columns the following loadings are applied: (1) The column is loaded with a constant axial load to represent gravity. (2) In order to simulate the action of an earthquake, the column is deformed side-to-side by the action of lateral forces. The hinge zone experiences the following sequence of events:

1. The cover spalls to a height of 2–3 column diameters before the longitudinal bar buckling is observed. (Figure 1.3a)

2. The longitudinal bar buckles. As the buckled bar emerges from the concrete, the buckle is observed to have a length from 3–6 spiral spacings. Note that this is much less than the spalling length. (Figure 1.3b)

3. The emergence of the buckle seems to be an immediate precursor to the fracture of the buckled bar. As the column is deformed in the other direction, the buckled bar is put into a state of tension. Due to the damage incurred during buckling, the bar fractures. This signals the end of the columns ability to safely support gravity loads. (Figure 1.3c)

1.3 Overview

The nature of the phenomenon is that two states of deformation are known: the initial state and a finitely deformed state. The initial state is observed when the column is constructed. It is seen that a straight bar is installed.

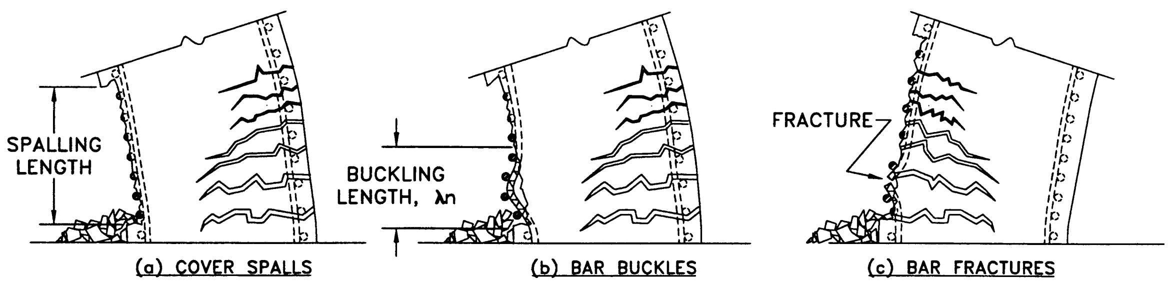


Figure 1.3: Observed phenomenon

Next, the bar is observed in a buckled configuration as it emerges from the concrete cover. The buckling is seen to occur over multiple spiral spacings but not over the entire spalling length.

Since the intermediate levels of deformation are hidden, the objective is to explain how the deformed state is reached from the initial state. To make matters interesting, the deformation observed cannot be scaled-up from a small deformation theory. It will be shown that the buckling must begin over a single spacing. However, when finite levels of deformation are taken into account, the buckling mode can make a transition from the single spacing mode to a multiple spacing mode similar to that observed in the laboratory.

Linear Elastic Stress–Strain Relation

It is the object of this paper to consider the behavior of a concrete reinforcing bar as it buckles with finite amplitude. The reader should be forewarned that reinforcing bars of typical dimensions buckle in the plastic range of material response. Allow me to set aside this material non–linearity, and consider only the kinematic non–linearity of large deformation. In order to do this, a reinforcing bar of typical dimensions cannot be considered. Instead, consider a much more slender bar made of a very high strength steel. As a result the new bar buckles elastically.

The assumption is made that the state of stress in the bar is a linear function of only the state of strain. Working with the Green–St.Venant strain tensor denoted, $\mathbf{E}$, and the Second Piola–Kirchhoff stress tensor denoted, $\mathbf{S}$. The relation between them is written using the fourth rank tensor, $\mathbf{C}$.

$$\mathbf{S} = \mathbf{C} \cdot \mathbf{E} \qquad \mathbf{C}\text{, a matrix of constants}$$

Organization of the Report

chapter 2 The model and coordinate system are described.

chapter 3 The equilibrium equation based on the Principle of Minimum Potential Energy is presented.

chapter 4 The equation for potential energy is derived.

chapter 5 Equilibrium is considered for a deformation characterized by the amplitude of the buckle alone. The equilibrium equation is used to solve for the axial load on the bar. The analysis is carried our first for the small deformation case and then for the finite deformation case.

Section 6 A finite element analysis is performed for the finite deformation case.

Chapter 2

Model Description

Consider a simplified model of one longitudinal bar (Figure 2.1). The bar contacts the core over continuous intervals on one side and contacts the spiral at discrete points on the other side. The spiral spacing is λ. The contact surfaces convey only normal tractions to the bar. When the bar is straight, its axis is aligned with the x-axis. When the bar is deformed, it is assumed to deform only in the x, z plane.

2.1 States of Deformation

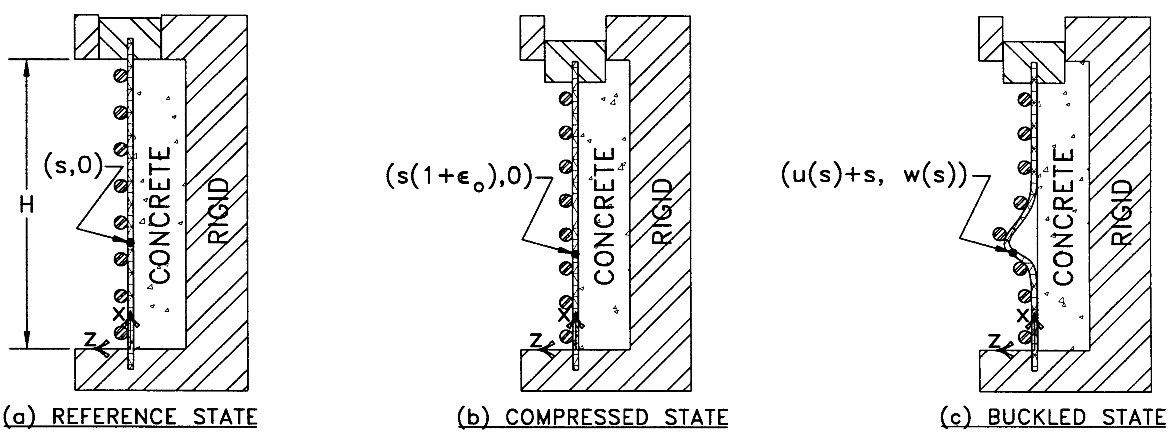


Figure 2.1: States of deformation

Three equilibrium states are shown in Figure 4. In the remainder of this work they will be referred to as defined below.

Reference State – This is the unstressed configuration. The bar is straight with length, H.

Compressed State – The straight bar compressed uniformly with infinitesimal axial strain, ϵ_o. There is no transverse displacement.

Buckled State – The bar with a buckle of arbitrary shape.

2.2 Material and Spatial Coordinates

Consider again Figure 2.1. Let S be the set of material points lying on the centerline of the bar. The points are identified by their x-coordinate in the reference state so that $S = [0, H]$.

Consider a point $s \in S$ as the bar deforms. In the compressed state, the bar is uniformly compressed with infinitesimal axial strain ϵ_o, so s lies on the x-axis with spatial coordinates $\{x(s), z(s)\} = \{s(1+\epsilon_o), 0\}$.

In the buckled state, the point will be moved an amount $u(s)$ in the x-direction and an amount $w(s)$ in the z-direction. So that the spatial coordinates will be $\{x(s), z(s)\} = \{s + u(s), w(s)\}$.

Chapter 3

General Equilibrium Statement

Q: How will we determine the behavior of the bar as it buckles and after it buckles?

In order to answer that question, the question is reformulated to an equivalent one:

Q': What buckled configurations satisfy equilibrium?

The *Principal of Minimum Potential Energy* is used to consider the equilibrium of the buckled configuration characterized by the deformation functions $u(s)$ and $w(s)$. The Principle of Minimum Potential Energy states that *among all the configurations satisfying the prescribed geometric constraints, the state of equilibrium causes the potential energy to be stationary* [1]. This is written

$$\delta U = 0\,. \tag{3.1}$$

Where the the potential energy of the system is denoted, U. And the *variation of the potential energy, δU*, is the change in the value of U when an infinitesimal displacement is imposed that satisfies the prescribed geometric constraints. This infinitesimal displacement is known as an *admissible variation*, denoted

$$[\,u(s)\,,\,w(s)\,] \to [\,u(s)+\delta u(s)\,,\,w(s)+\delta w(s)\,]\;.$$

Chapter 4

Potential Energy

There are three separate force systems acting on the bar. Each is derived from its own potential energy function. The three potential energy functions are: (1) the potential of the axial load, U_{load}; (2) the potential of the strain energy, U_{strain}; and (3) the potential of the spiral, U_{spiral}. The potential energy of the system can be written as the sum of these terms,

$$U = U_{load} + U_{strain} + U_{spiral} . \tag{4.1}$$

In the following sections, the terms of the potential energy equation are shown to be a functionals of the centerline deformations, $u(s)$ and $w(s)$. Therefore, each term of the potential energy equation will be quantifiable for any deformation $u(s)$ and $w(s)$. The three potential functionals will be denoted with a superscript carrot, $(\hat{\cdot})$.

4.1 Axial Load Potential, $\mathbf{U_{load}}$

By assumption the axial load, P, does not depend on the deformation – it is an externally applied force of constant magnitude. Therefore, if P moves downward through a distance ΔH (Figure 4.1) the change of potential is $-P\,\Delta H$. Where both P and ΔH are positive quantities. Identifying the reference state with $U_{load} = 0$, write

$$U_{load} = -P\,\Delta H . \tag{4.2}$$

The quantity ΔH is the amount that the two ends of the bar approach each other,

$$\Delta H = -\int_0^H u'(s)ds . \tag{4.3}$$

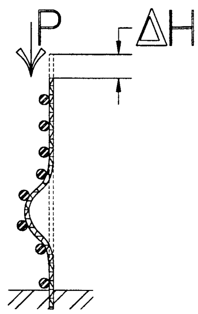


Figure 4.1: Potential energy of axial load

Combining this with (4.2), we can write the axial load, U_{load}, as a functional of the displacement:

$$U_{load} = -P \; \Delta\hat{H}\,[u'(s)] \tag{4.4}$$

4.2 Strain Energy of the Bar, $\mathbf{U_{strain}}$

If the stresses in a body, **S**, are only a function of the strains, **E**, the strain energy employed in deforming in a body, B, can be written

$$U_{strain} = \frac{1}{2}\iiint_B \mathbf{S}\cdot\mathbf{E}\,dV\,. \tag{4.5}$$

where:

S, Second Piola – Kirchhoff stress tensor;
E, Green – St. Venant strain tensor;

The material of the bar is assumed to remain linearly elastic. Using the elasticity tensor, **C**:

$$\mathbf{S} = \mathbf{C}\cdot\mathbf{E} \qquad \mathbf{C}\text{, a matrix of constants}$$

Then equation (4.5) becomes

$$U_{strain} = \frac{1}{2}\iiint_B (\mathbf{C}\cdot\mathbf{E})\cdot\mathbf{E}\,dV\,. \tag{4.6}$$

The bar is assumed to be an Euler-Bernoulli beam. Based on the kinematic assumptions of the theory, the only non-vanishing strain component is E_{xx}. This is shown in Appendix A. Equation (4.6) reduces to

$$U_{strain} = \frac{1}{2}\iiint_B C_{xx}\,E_{xx}^2\,dV\,. \tag{4.7}$$

C_{xx} is the constant relating E_{xx} and its work conjugate S_{xx}. Impose the usual assumption that allows an Euler-Bernoulli beam to satisfy zero–traction boundary conditions along its length, $S_{yy} = S_{zz} = 0$. This implies a state of plane stress so that, C_{xx} is the uniaxial elastic modulus, E.

$$U_{strain} = \frac{1}{2} E \iiint_B E_{xx}^2 \, dV \tag{4.8}$$

Also in Appendix A, E_{xx} is derived as a function of the displacements $u'(s)$, $u''(s)$, $w'(s)$ and $w''(s)$.

$$\begin{aligned} E_{xx} &= \hat{E}_{xx}\left[u'(s), u''(s), w'(s), w''(s)\right] \\ &= \frac{1}{2}\left[(1+u'(s))^2 + w'^2(s) - 1\right] \\ &\quad - z\,\frac{w''(s)[1+u'(s)] - w'(s)\; u''(s)}{\sqrt{[1+u'(s)]^2 + w'^2(s)}} \\ &\quad + \frac{1}{2} z^2 \left[\frac{w''(s)[1+u'(s)] - w'(s)\; u''(s)}{[1+u'(s)]^2 + w'^2(s)}\right]^2 \end{aligned} \tag{4.9}$$

z is the material coordinate of the bar through its thickness. This integrates into the section properties, and does not appear when U_{strain} is evaluated. Therefore, based on the kinematic assumptions of Euler–Bernoulli beams (4.9), we can write U_{strain} (4.8) as a functional of the displacement functions $u'(s)$ and $w'(s)$ and their derivatives.

$$U_{strain} = \hat{U}_{strain}\left[u'(s), u''(s), w'(s), w''(s)\right] \tag{4.10}$$

4.3 Spiral Energy, U_{spiral}

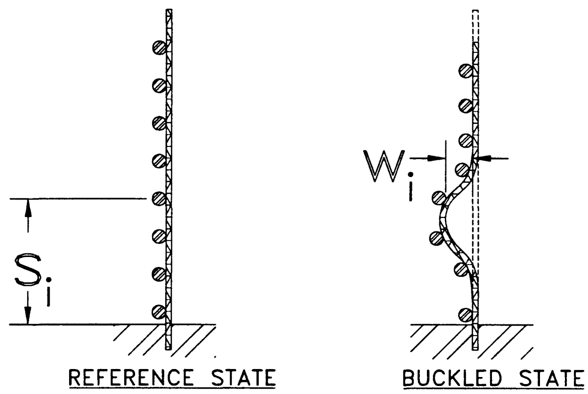


Figure 4.2: Configuration of displaced spiral

Say we have m spirals spacings over the length of our bar. Let s_i be the coordinate of the ith spiral spacing in the reference state. As the bar deforms with transverse displacement $w(s)$, each spiral spacing displaces an amount w_i (Figure 4.2), where:

$$w_i = w(s_i) \qquad i\text{, an integer between 1 and } m \tag{4.11}$$

Let the amount of potential energy stored in deforming one spiral spacing be $U_{one\ spiral}(w_i)$. Then the potential energy stored in deforming the entire spiral:

$$U_{spiral} = \sum_{i=1}^{m} U_{one\ spiral}(w_i) \tag{4.12}$$

In order to determine $U_{one\ spiral}$ as a function of the displacement w_i, the kinematics and equilibrium of the system are investigated. Consider the concrete column under load. As a concrete column compresses, its cross-section dilates; this is known as the Poisson Effect. For a spirally reinforced column, the dilation has the effect of tensioning the spiral. This tension can be sufficiently high to yield the spiral. The Deformation Theory of Plasticity will be employed to consider this behavior [1]. For monotonic deformations we can "replace" the mechanical work done in deforming a plastic material by a strain energy. Thereby, a plastic material is treated as a non–linear elastic material.

Assume that immediately before the onset of buckling, the column has dilated from its original radius, R_o, an amount ΔR to have a radius R. As a result, the spiral goes from its unstressed state to having a uniform tension, T. (Figure 4.3) We will consider the spiral to be rigid-perfectly-plastic so

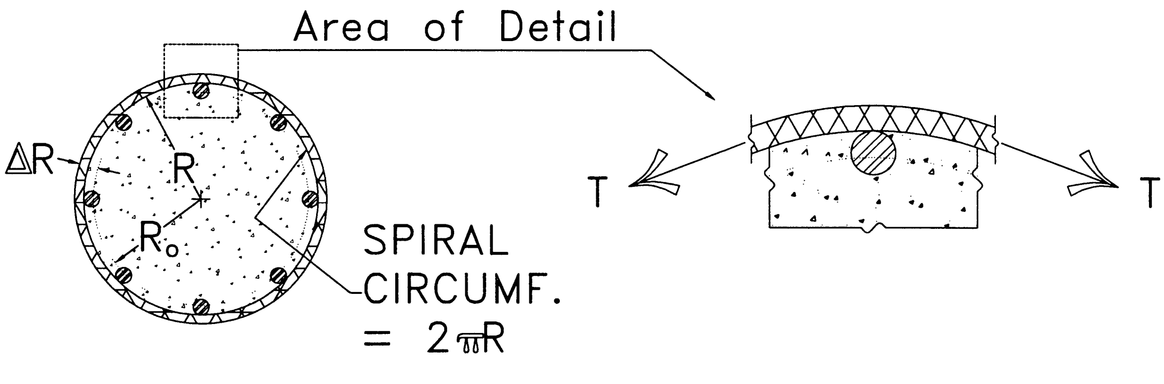


Figure 4.3: Cross section before buckling

that T will be the yield force for the spiral, $T = A_{spiral}F_y$, and will not be a function of the displacement. [1]

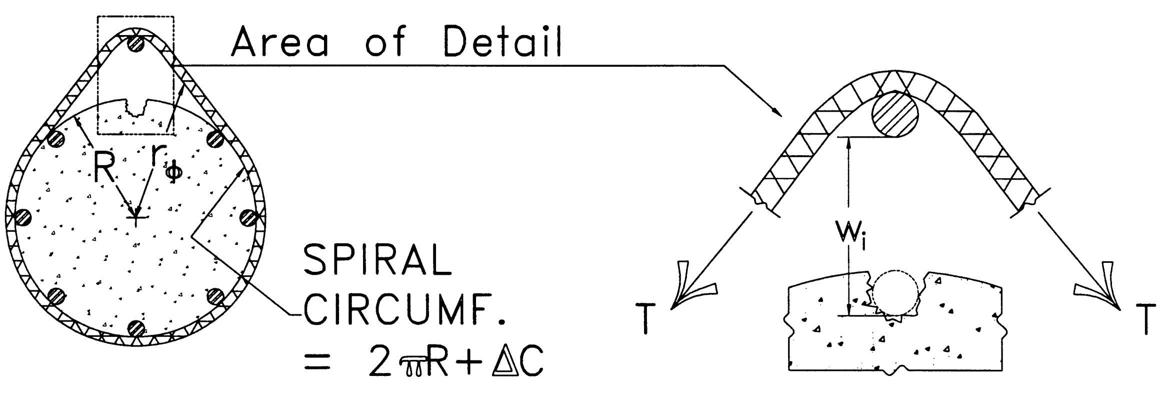


Figure 4.4: Cross section after buckling

Consider what happens at the onset of buckling (Figure 4.4). When the bar moves away from the core a distance, w_i, it exerts a force on the spiral. We make the assumption that as the bar buckles, it moves monotonically away from the core at all points along its length. Based on this assumption

[1]This assumption is not required. The spiral could be modeled as elasto-plastic without affecting the nature of the final result. However, in the interest of simplicity, this model is used.

we can identify the work done by the bar on the spiral, W_T, with a gain of potential of the spiral.

$$U_{one\ spiral}(w_i) \;=\; W_T(w_i) \tag{4.13}$$

The displacement w_i, increases the circumference of the spiral an amount $\Delta C(w_i)$ through the constant force, T. Therefore the work done by the bar on the spiral is

$$W_T(w_i) = T\ \Delta C(w_i)\,.$$

Combining this with (4.13) yields

$$U_{one\ spiral}(w_i) \;=\; T\ \Delta C(w_i)\,.$$

Putting this together with equation (4.12) gives

$$U_{spiral} \;=\; T\ \sum_{i=1}^{m} \Delta C(w_i)\,. \tag{4.14}$$

Now the function $\Delta C(w_i)$ will be approximated by assuming the deformed shape of the spiral, $r_\phi(w_i)$ (Figure 4.4b):

$$r_\phi(w_i) = R + w_i \left(\frac{2\,\phi}{\pi}\right)^2 \qquad\qquad \phi \in [0, \frac{\pi}{2}]$$

The change in circumference as a function of w_i becomes

$$\begin{aligned}\Delta C(w_i) \;&=\; 2\int_0^{\frac{\pi}{2}} \sqrt{r_\phi^2(w_i) \;+\; \left[\frac{\partial}{\partial\phi} r_\phi(w_i)\right]^2}\ d\phi \;-\; \pi R \\ &=\; 2\int_0^{\frac{\pi}{2}} \sqrt{\left[R + w_i\left(\frac{2\,\phi}{\pi}\right)^2\right]^2 \;+\; \frac{64\,w_i^2\,\phi^2}{\pi^4}}\ d\phi \;-\; \pi R\,. \end{aligned} \tag{4.15}$$

Because it will be useful for considering the small deformation case, expand (4.15) in w_i and retain up to terms of the second order.

$$\begin{aligned}\Delta C(w_i) \;&=\; 2\int_0^{\frac{\pi}{2}} \left(R + \frac{4\phi^2 w_i}{\pi^2} + \frac{32\phi^2 w_i^2}{\pi^4 R} \;+\; O(w_i^3)\right)\ d\phi \;-\; \pi R \\ &=\; \frac{\pi}{3} w_i + \frac{8}{3\pi R} w_i^2 \;+\; O(w_i^3) \end{aligned} \tag{4.16}$$

Taken together, equations (4.11), (4.14) and (4.15) give us the ability to write the energy employed in deforming the spiral as a functional of the transverse displacement.[2]

$$U_{spiral} \;=\; \hat{U}_{spiral}\left[w(s)\right] \tag{4.17}$$

[2]Equation (4.16) will be used in place of (4.15) whenever w_i is assumed small.

Chapter 5

Equilibrium Based on Variation of One Parameter

Equation (1) describes an exact statement of equilibrium. However, in order to solve for the exact solution we must consider all possible deformations of the system that satisfy the geometric constraints. For complicated systems it is generally impossible to consider all possible deformations. The best that can be done is an approximation of the deformation.

We proceed by guessing a form of the displacement that can be characterized by finitely many parameters. Then we vary the parameters infinitesimally in the potential energy functional. In this approximate form, The Principle of Stationary Potential Energy defines equilibrium states as a set of parameters for which infinitesimal variations cause no change in the potential energy.

Taking the simplest case, we consider variations in *one* deformation parameter, the amplitude, α. We define an equilibrium state as a state for which the potential energy, U, is stationary for infinitesimal changes, $\alpha \to \alpha + \delta\alpha$. For this, the variation of the potential energy becomes the differential, [1]

$$\delta U = \delta\alpha \, \frac{\partial}{\partial \alpha} U = 0 \, .$$

Since $\delta\alpha \neq 0$ we get a 1-dimensional equilibrium equation which is simply a partial derivative of the potential energy with respect to α,

$$\frac{\partial}{\partial \alpha} U = 0 \, . \tag{5.1}$$

[1]This is the linear part, terms of higher order in $\delta\alpha$ are neglected since $\delta\alpha$ is infinitesimal.

In the rest of this chapter, this equilibrium equation will allow us to determine the axial load, P, that when applied to the system equilibrates a buckle of amplitude, α.

5.1 Assumed Deformation

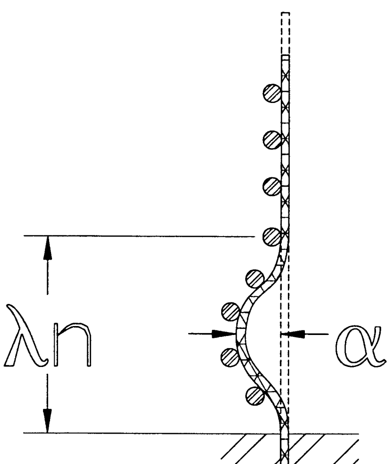


Figure 5.1: Assumed deformation

We assume that for any buckling length, n, the deformation of the bar can be parameterized by the amplitude of the buckle, α. (Figure 5.1) In order to do this, the lateral displacement is assumed sinusoidal over the buckled length and zero everywhere else.

$$w(\alpha, n, s) = \begin{cases} \alpha\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi s}{\lambda n}\right) & \text{if } s < \lambda n \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Taking the derivative with respect to s, we get $w'(s)$ and $w''(s)$.

$$w'(\alpha, n, s) = \begin{cases} \alpha\frac{\pi}{\lambda n}\sin\frac{2\pi s}{\lambda n} & \text{if } s < \lambda n \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

$$w''(\alpha, n, s) = \begin{cases} \alpha\frac{2\pi^2}{(\lambda n)^2}\cos\frac{2\pi s}{\lambda n} & \text{if } s < \lambda n \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

In order to make the potential energy a function of α and n, we will need to make u' and u'' functions of α and n. To do this, we assume that the arc length does not change between the compressed state and the buckled state [2]

$$1 + \epsilon_o = \sqrt{[1 + u'(s)]^2 + w'^2(\alpha, n, s)} \tag{5.5}$$

Recall from Section 2 that ϵ_o is the infinitesimal axial strain in the straight compressed bar,

$$\epsilon_o = -\frac{P}{E\,A_{bar}} \qquad A_{bar}\text{ , cross-sectional area of bar}.$$

Solving (5.5) for u' as a function of ϵ_o, α and s,

$$u'(\epsilon_o, \alpha, n, s) = \sqrt{(1 + \epsilon_o)^2 - w'^2(\alpha, n, s)} - 1 \tag{5.6}$$

Taking the derivative with respect to s gives $u''(s)$

$$u''(\epsilon_o, \alpha, n, s) = -\frac{w'(\alpha, n, s)\; w''(\alpha, n, s)}{\sqrt{(1 + \epsilon_o)^2 - w'^2(\alpha, n, s)}} \tag{5.7}$$

The deformations (5.2), (5.3), (5.4), (5.6) and (5.7) are used to compose the functionals of the last chapter (4.4), (4.10) and (4.17). In this way, the functionals become *functions* of ϵ_o, α, n and s. These functions will be denoted with a superscript tilde, $(\tilde{\cdot})$.

$$\left.\begin{aligned}
U_{load} &= -P\;\Delta\tilde{H}(\epsilon_o, \alpha, n) &&= -P\;\Delta\hat{H}\,[u'(s)] \circ u'(\epsilon_o, \alpha, n, s) \\
U_{strain} &= \tilde{U}_{strain}(\epsilon_o, \alpha, n) &&= \hat{U}_{strain}\,[u'(s), u''(s), w'(s), w''(s)] \\
& && \quad \circ \{u'(\epsilon_o, \alpha, n, s), u''(\epsilon_o, \alpha, n, s), \\
& && \quad\quad w'(\alpha, n, s), w''(\alpha, n, s)\} \\
U_{spiral} &= \tilde{U}_{spiral}(\alpha, n) &&= \hat{U}_{spiral}\,[w(s)] \circ w(\alpha, n, s)
\end{aligned}\right\} \tag{5.8}$$

5.2 Why the Shape of the Buckle Cannot Be Explained By a Small Deformation Solution

In this section it will be shown that the buckle begins over one spacing. In order to do this all of the terms of the equilibrium are evaluated for a

[2] Since we will consider finite deformation from the compressed state, the assumption that arc length doesn't change between these two states may seem implausible. However, finite element analysis presented in chapter 6 verifies the validity of the assumption that the end load on the bar completely determines its arclength.

deformation where ϵ_o and α are assumed small.

$$|\epsilon_o| \ll 1 \qquad \alpha^2 \ll 1$$

Equations (5.8) are expanded in ϵ_o and α. Terms up to linear in ϵ_o and up to α^2 are retained in the expansion. Note that all of the potential energy functions vanish at $\epsilon_o = \alpha = 0$,

$$P\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} = \left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} = \left[\tilde{U}_{spiral}(\alpha,n)\right]_{\alpha=0} = 0\,.$$

The other terms of the expansion are,

$$\left.\begin{aligned}
U_{load} &\approx -P\epsilon_o\left[\tfrac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} - P\alpha\left[\tfrac{\partial}{\partial\alpha}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} \\
&\quad - P\alpha\epsilon_o\left[\tfrac{1}{2}\tfrac{\partial^2}{\partial\alpha\,\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} - P\alpha^2\left[\tfrac{1}{2}\tfrac{\partial^2}{\partial^2\alpha}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} \\
U_{strain} &\approx \epsilon_o\left[\tfrac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} + \alpha\left[\tfrac{\partial}{\partial\alpha}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} \\
&\quad + \alpha\epsilon_o\left[\tfrac{1}{2}\tfrac{\partial^2}{\partial\alpha\,\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} + \alpha^2\left[\tfrac{1}{2}\tfrac{\partial^2}{\partial^2\alpha}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} \\
U_{spiral} &\approx \alpha\left[\tfrac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n)\right]_{\alpha=0} + \alpha^2\left[\tfrac{1}{2}\tfrac{\partial^2}{\partial^2\alpha}\tilde{U}_{spiral}(\alpha,n)\right]_{\alpha=0}
\end{aligned}\right\} \tag{5.9}$$

The terms in equations (5.9) are evaluated in the following sections. We will proceed one equation at a time.

5.2.1 ΔH For Small Deformations

Substituting the assumed deformation (5.6) into (4.3) gives an expression for $\Delta\tilde{H}$.

$$\Delta\tilde{H}(\epsilon_o,\alpha,n) = H - \int_0^H \sqrt{(1+\epsilon_o)^2 - w'^2(\alpha,n,s)}\,ds \tag{5.10}$$

Where $w'(\alpha,n,s)$ is given by equation (5.3). Consider the first term of (5.9a) noting that $w'(\alpha,n,s) \to 0$ as $\alpha \to 0$. So that

$$\begin{aligned}
\left[\frac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=\alpha=0} &= -\int_0^H \left[\frac{(1+\epsilon_o)}{\sqrt{(1+\epsilon_o)^2 - w'^2(\alpha,n,s)}}\right]_{\epsilon_o=\alpha=0} ds \\
&= -\int_0^H ds = -H\,.
\end{aligned} \tag{5.11}$$

The second and third terms of (5.9a) are trivial since $w'(\alpha, n, s) \to 0$ as $\alpha \to 0$.

$$\begin{aligned}\left[\frac{\partial}{\partial\alpha}\Delta\tilde{H}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \int_0^H \left[\frac{w'(\alpha, n, s)\,\frac{\partial}{\partial\alpha}w'(\alpha, n, s)}{\sqrt{(1+\epsilon_o)^2 - w'^2(\alpha, n, s)}}\right]_{\epsilon_o=\alpha=0} ds \\ &= 0\end{aligned}$$

$$\begin{aligned}\left[\frac{\partial^2}{\partial\alpha\,\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \int_0^H \left[\frac{(1+\epsilon_o)w'(\alpha, n, s)\,\frac{\partial}{\partial\alpha}w'(\alpha, n, s)}{[(1+\epsilon_o)^2 - w'^2(\alpha, n, s)]^{\frac{3}{2}}}\right]_{\epsilon_o=\alpha=0} ds \\ &= 0\end{aligned}$$

The fourth term of (5.9a) is determined by taking the second derivative of the integrand of (5.10) with respect to α. Only terms where $w'(\alpha, n, s)$ doesn't appear in the numerator survive:

$$\begin{aligned}\left[\frac{1}{2}\frac{\partial^2}{\partial^2\alpha}\Delta\tilde{H}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \frac{1}{2}\int_0^H \left[\frac{\left[\frac{\partial}{\partial\alpha}w'(\alpha, n, s)\right]^2 + w'(\alpha, n, s)\,\frac{\partial^2}{\partial\alpha^2}w'(\alpha, n, s)}{\sqrt{(1+\epsilon_o)^2 - w'^2(\alpha, n, s)}}\right. \\ &\quad \left. - \frac{w'(\alpha, n, s)\left[\frac{\partial}{\partial\alpha}w'(\alpha, n, s)\right]^2}{[(1+\epsilon_o)^2 - w'^2(\alpha, n, s)]^{\frac{3}{2}}}\right]_{\epsilon_o=\alpha=0} ds \\ &= \frac{1}{2}\int_0^H \left[\frac{\partial}{\partial\alpha}w'(\alpha, n, s)\right]^2 = \frac{1}{2}\int_0^{\lambda n}\left(\frac{\pi}{\lambda n}\right)^2 \sin^2\frac{2\pi s}{\lambda n}\,ds \\ &= \frac{\pi^2}{4\lambda n} \qquad (5.12)\end{aligned}$$

5.2.2 $\mathbf{U_{strain}}$ For Small Deformations

Recall the expression for U_{strain} (4.8).

$$U_{strain} = \frac{1}{2}E\iiint_B E_{xx}^2\,dV$$

Substituting the assumed deformations u' and u'' (5.6, 5.7) into the expression for E_{xx} (4.9) yields

$$\begin{aligned}\hat{E}_{xx}(w', w'', \epsilon_o) &= E_{xx}(u', u'', w', w'') \circ (u', u'') \\ &= \epsilon_o + \frac{1}{2}\epsilon_o^2 - \frac{z\,w''(1+\epsilon_o)}{\sqrt{(1+\epsilon_o)^2 - w'^2}} + \frac{1}{2}\left[\frac{z^2\,w''^2}{(1+\epsilon_o)^2 - w'^2}\right]. \qquad (5.13)\end{aligned}$$

In preparation for the evaluation of the terms of Equation 5.9b, now note the following relations that will prove useful. Since $w'(\alpha, n, s) \to 0$ and $w''(\alpha, n, s) \to 0$ as $\alpha \to 0$ from (5.13) we can conclude,

$$[E_{xx}]_{\epsilon_o=\alpha=0} = 0 \quad , \quad \left[\frac{\partial}{\partial \epsilon_o} E_{xx}\right]_{\epsilon_o=\alpha=0} = 0 \quad , \quad \left[\frac{\partial}{\partial \alpha} E_{xx}\right]_{\epsilon_o=\alpha=0} = -z \frac{\partial}{\partial \alpha} w'' \,.$$

In light of these facts, the first three terms appearing in (5.9b) vanish.

$$\left[\frac{\partial}{\partial \epsilon_o} \tilde{U}_{strain}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} = E \iiint_B \left[E_{xx} \frac{\partial}{\partial \epsilon_o} E_{xx}\right]_{\epsilon_o=\alpha=0} dV = 0$$

$$\left[\frac{\partial}{\partial \alpha} \tilde{U}_{strain}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} = E \iiint_B \left[E_{xx} \frac{\partial}{\partial \alpha} E_{xx}\right]_{\epsilon_o=\alpha=0} dV = 0$$

$$\begin{aligned}\left[\frac{1}{2} \frac{\partial^2}{\partial \alpha \, \partial \epsilon_o} \tilde{U}_{strain}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \frac{1}{2} E \iiint_B \left\{ \left[E_{xx} \frac{\partial^2}{\partial \alpha \, \partial \epsilon_o} E_{xx}\right]_{\epsilon_o=\alpha=0} \right. \\ &+ \left. \left[\frac{\partial}{\partial \alpha} E_{xx} \frac{\partial}{\partial \epsilon_o} E_{xx}\right]_{\epsilon_o=\alpha=0} \right\} dV = 0\end{aligned}$$

The derivative appearing in the fourth term becomes,

$$\begin{aligned}\left[\frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \tilde{U}_{strain}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \frac{1}{2} E \iiint_B \left\{ \left[E_{xx} \frac{\partial^2}{\partial \alpha^2} E_{xx}\right]_{\epsilon_o=\alpha=0} \right. \\ &+ \left. \left[\frac{\partial}{\partial \alpha} E_{xx}\right]^2_{\epsilon_o=\alpha=0} \right\} dV \\ &= \frac{1}{2} E \iiint_B z^2 \left[\frac{\partial}{\partial \alpha} w''(\alpha, n, s)\right]^2 dV \,.\end{aligned}$$

Integrating over the cross section and substituting the assumed deformation (5.4) yields,

$$\begin{aligned}\left[\frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \tilde{U}_{strain}(\epsilon_o, \alpha, n)\right]_{\epsilon_o=\alpha=0} &= \frac{1}{2} E I \int_0^H \left[\frac{\partial}{\partial \alpha} w''(\alpha, n, s)\right]^2 ds \\ &= 2 E I \left(\frac{\pi}{\lambda n}\right)^4 \int_0^{\lambda n} \cos^2 \frac{2\pi s}{\lambda n} \, ds \\ &= \frac{\pi^4 E I}{(\lambda n)^3} \,. \end{aligned} \tag{5.14}$$

Where I is the second moment of area,

$$I = \iint z^2 \, dA \,.$$

5.2.3 $U_{\mathbf{spiral}}$ For Small Deformations

In Section 4.3, we began the analysis of U_{spiral} for small deformations. Making use of those results we can get an approximate expression for U_{spiral} by substituting equation (4.16) into equation (4.14):

$$U_{spiral} \approx T \sum_{i=1}^{m} \frac{\pi}{3} w_i + \frac{8}{3\pi R} w_i^2$$

Since the spiral spacings are spaced evenly over the length of the buckle,

$$w_i = \begin{cases} \alpha \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi i}{n}\right) & \text{if } i < n \\ 0 & \text{otherwise}. \end{cases}$$

Combining the above two expressions yields

$$U_{spiral} \approx \alpha \frac{T\pi}{6} \sum_{i=1}^{n-1} \left(1 - \cos\frac{2\pi i}{n}\right) + \alpha^2 \frac{2T}{3\pi R} \sum_{i=1}^{n-1} \left(1 - \cos\frac{2\pi i}{n}\right)^2 . \tag{5.15}$$

These are the first two terms of the expansion of U_{spiral} in α.

5.2.4 Equilibrium in Small Deformation

By substituting (5.11), (5.12), (5.14) and (5.15), equations (5.9) are rewritten,

$$\left.\begin{array}{lcl} U_{load} & \approx & PH\epsilon_o - P\alpha^2 \frac{\pi^2}{4\lambda n} \\ U_{strain} & \approx & \alpha^2 \frac{\pi^4 EI}{(\lambda n)^3} \\ U_{spiral} & \approx & \alpha \frac{T\pi}{6} \sum_{i=1}^{n-1} \left(1 - \cos\frac{2\pi i}{n}\right) \ + \ \alpha^2 \frac{2T}{3\pi R} \sum_{i=1}^{n-1} \left(1 - \cos\frac{2\pi i}{n}\right)^2 \end{array}\right\} \tag{5.16}$$

The growth properties of U_{strain} and U_{spiral} will be of central importance in considering equilibrium for the small deformation case. Notice that the lowest order term in U_{strain} is a square term in α, whereas, U_{spiral} contains a linear term. Therefore, when α is small, the energy employed in deforming the spiral will dominate the energy employed in deforming the bar.

For the special case when $n = 1$ the dominance of U_{spiral} no longer holds. Notice that when $n = 1$ the summations appearing in the expression for U_{spiral} vanish. This case corresponds to buckling over a single spiral spacing, which leaves the spiral undeformed.

With these points in mind lets precede with determining the load, P, which will hold the slightly deformed bar in equilibrium. Substituting equations (5.16) into the potential energy equation (4.1) yields,

$$\begin{aligned} U \;\approx\; PH\epsilon_o \;-\; P\alpha^2\tfrac{\pi^2}{4\lambda n} \;+\; \alpha^2\tfrac{\pi^4 EI}{(\lambda n)^3} \\ +\,\alpha\tfrac{T\pi}{6}\textstyle\sum_{i=1}^{n-1}\left(1-\cos\tfrac{2\pi i}{n}\right)+\alpha^2\tfrac{2T}{3\pi R}\textstyle\sum_{i=1}^{n-1}\left(1-\cos\tfrac{2\pi i}{n}\right)^2 \end{aligned} \tag{5.17}$$

Making use of the one-parameter equilibrium equation (5.1),

$$\begin{aligned} \tfrac{\partial}{\partial\alpha}U \;\approx\; -P\alpha\tfrac{\pi^2}{2\lambda n} \;+\; \alpha\tfrac{2\pi^4 EI}{(\lambda n)^3}+\tfrac{T\pi}{6}\textstyle\sum_{i=1}^{n-1}\left(1-\cos\tfrac{2\pi i}{n}\right) \\ +\;\alpha\tfrac{4T}{3\pi R}\textstyle\sum_{i=1}^{n-1}\left(1-\cos\tfrac{2\pi i}{n}\right)^2 \;=\; 0 \end{aligned}$$

Solving for P,

$$P=\frac{4\pi^2 EI}{(\lambda n)^2}+\frac{1}{\alpha}\frac{T\lambda n}{3\pi}\sum_{i=1}^{n-1}\left(1-\cos\frac{2\pi i}{n}\right)+\frac{4T\lambda n}{\pi^3 R}\sum_{i=1}^{n-1}\left(1-\cos\frac{2\pi i}{n}\right)^2 \tag{5.18}$$

The *critical load*, P_{cr} is the minimum value of P in the limit as $\alpha \to 0$ for all possible choices of n .

$$P_{cr}=\min_{n\geq 1}\left(\lim_{\alpha\to 0}P\right)$$

Considering equation (5.18), notice that it is a sum of positive terms. The first term is the resistance to the axial force due to the strain energy. The second and third terms are the resistance to the axial force generated by the spiral. The second term becomes large for small values of α. So for small values of the buckling amplitude, the force generated by the spiral will be large. However, in the case that the buckling mode is over a single spacing, $n = 1$, the summations vanish, the spiral is undisplaced by the buckle.

As a result the single spacing buckling mode will always furnish the critical load.

$$P_{cr}=\frac{4\pi^2 EI}{\lambda^2} \tag{5.19}$$

This is the Euler buckling load for a fixed–fixed column with a length of one spiral spacing.

In addition, equation (5.18) suggests that for larger values of α, this may not be the case. As α becomes finite, the second term may no longer dominate and other buckling modes may be possible. However, to consider this we can no longer make assumptions about the magnitude of α. This will lead us into Section 5.3.

5.2.5 Results Hold for Plastic Material

Before moving onto the finite deformation case, briefly consider the bar deforming as described above but in the plastic range of material response. A famous experimental observation was made by F.R. Shanely [2] concerning inelastic buckling of rods. He observed that during the initial stages of inelastic buckling, all fibers of the bar load monotonically in compression.

This gives us the right to identify the mechanical work done in deforming a fiber of the bar with the area under the stress–strain diagram, W_{plas}. Figure 5.2.

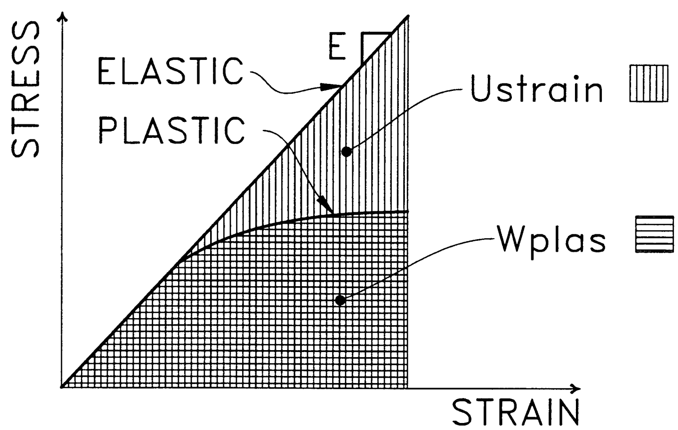


Figure 5.2: Mechanical work and elastic strain energy

In order to consider the plastic small–deformation case, we need only replace the elastic strain energy, U_{strain} with W_{plas}. Note that W_{plas} is bounded above by U_{strain}. Consider our result in equation 5.18. Only the first term is due to U_{strain}. When the plastic response is considered this term cannot be any greater than the result from elasticity. The other two terms are independent of U_{strain} so they remain the same.

So, for the plastic case the result remains intact. The spiral terms preclude buckling for all modes except $n = 1$. And for the case of $n = 1$ the summations vanish and we have a critical load that is bounded above by the elastic case. Therefore, the conclusion is drawn that even for the plastic case, the buckling begins over a single spacing.

We now return to the linear elastic material and proceed to consider finite deformation.

5.3 Finite Deformation Buckling of a Slender Elastic Bar Using One-Parameter Equilibrium

The objective of this section is to investigate what happens as the size of the buckle becomes finite. In the last section, it was shown that for an initially straight bar, the buckle will always begin over one spiral spacing. In this section it will be shown that as the buckle grows in amplitude, buckling over more spacings can become favorable.

Kinematic Assumption – Near Inextensibility

Although no assumption is made about the size of the amplitude, α, we can still make an assumption about the size of ϵ_o. Recall that ϵ_o is the strain in the bar when it is straight and loaded with axial load P. Since the bar is slender, $\epsilon_o \ll 1$.

Reexamine equations (5.8) where the terms in the potential energy equation are functions of ϵ_o, α and n. Now we expand in ϵ_o, keeping only linear terms. Equations (5.8) become, [3]

$$\begin{aligned}
U_{load} &\approx -P\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} - P\,\epsilon_o\left[\frac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}\\
U_{strain} &\approx \left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \epsilon_o\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}\\
U_{spiral} &= \tilde{U}_{spiral}(\alpha,n)
\end{aligned}$$

Substituting $\epsilon_o = -\frac{P}{E\,A_{bar}}$:

$$\left.\begin{aligned}
U_{load} &\approx -P\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + P^2\frac{1}{E\,A_{bar}}\left[\frac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}\\
U_{strain} &\approx \left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} - P\frac{1}{E\,A_{bar}}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}\\
U_{spiral} &= \tilde{U}_{spiral}(\alpha,n)
\end{aligned}\right\} \tag{5.20}$$

[3] $\tilde{U}_{spiral}(\alpha,n)$ is unaffected by the expansion since it is not a function of ϵ_o

The functions (5.20) can now be put into the potential energy equation (4.1),

$$\begin{aligned}
U \;\approx\; & -P\,\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \;+\; P^2\,\tfrac{1}{E\,A_{bar}}\left[\tfrac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \\
& + \left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \;-\; P\,\tfrac{1}{E\,A_{bar}}\left[\tfrac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \\
& + \tilde{U}_{spiral}(\alpha,n)
\end{aligned}$$

Again making use of the one-parameter equilibrium equation (5.1),

$$\begin{aligned}
\tfrac{\partial}{\partial\alpha}U \;\approx\; & P^2\,\tfrac{1}{E\,A_{bar}}\,\tfrac{\partial}{\partial\alpha}\left[\tfrac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \\
& - P\left\{\tfrac{\partial}{\partial\alpha}\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \tfrac{1}{E\,A_{bar}}\tfrac{\partial}{\partial\alpha}\left[\tfrac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}\right\} \\
& + \tfrac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \tfrac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n) \;=\; 0
\end{aligned} \tag{5.21}$$

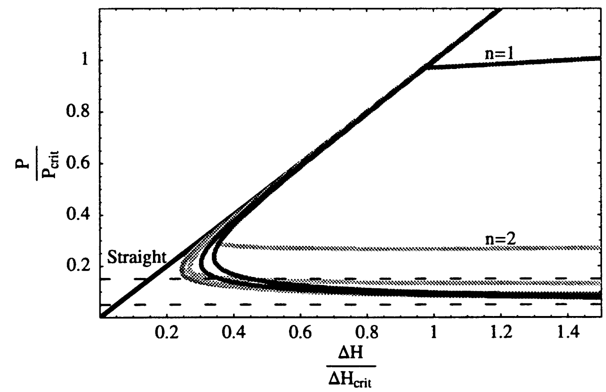
Equation 5.21 is quadratic in P. In the range of deformation we are concerned with it has only one positive root, [4]

$$P = \frac{-B(\alpha,n) + \sqrt{B(\alpha,n)^2 + 4\,A(\alpha,n)\,C(\alpha,n)}}{2\,A(\alpha,n)} \tag{5.22}$$

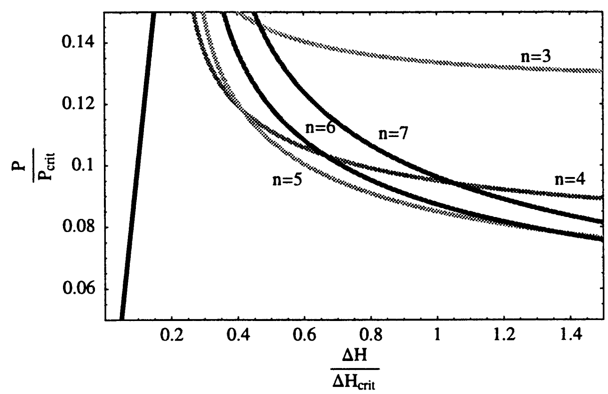
Where:

$$\begin{aligned}
A(\alpha,n) \;&=\; -\frac{1}{E\,A_{bar}}\,\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \\
B(\alpha,n) \;&=\; \frac{\partial}{\partial\alpha}\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0.} + \frac{1}{E\,A_{bar}}\,\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} \\
C(\alpha,n) \;&=\; \frac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \frac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n)
\end{aligned}$$

[4]It is shown in Appendix 2 that the functions $A(\alpha,n)$, $B(\alpha,n)$ and $C(\alpha,n)$ are strictly positive functions when $\alpha < \frac{1}{7}\lambda n$. So we can be sure that there is only one positive root to the quadratic as long as the buckling amplitude is less than one-seventh of the buckling length.



(a) Full view



(b) Blow–up of dashed region

Figure 5.3: Force–displacement curves

Force Displacement Curves

For any value of α and n the integrals $A(\alpha, n)$, $B(\alpha, n)$ and $C(\alpha, n)$ can be computed numerically. Therefore, for any amplitude and buckling length, Equation(5.22) computes the axial load, P, that maintains equilibrium for the system. Once we have determined P and, thus ϵ_o (recall $\epsilon_o = -\frac{P}{E A_{bar}}$), we can evaluate ΔH numerically using Equation(5.10).

This procedure was carried out for a system with the parameters shown in Table 5.1. A plot of P vs. ΔH is shown in Figure 5.3. Each curve represents the locus of solutions for a buckling length, n. The solid line through the origin represents the straight, compressed solution. The axes are normalized

Parameter		*Value*	
Bar Height,	H	10	inches
Bar Diameter,	d_{bar}	0.1	inches
Spiral Spacing,	λ	1.25	inches
Spiral Diameter,	d_{spiral}	0.025	inches
Column Radius,	R_0	12	inches
Elastic Modulus,	E	$29\,10^6$	psi
Spiral Yield Force,	T	150	pounds

Table 5.1: Model parameters

with the buckling load over one spacing, P_{crit}, and the displacement of the straight bar compressed under critical load, $\Delta H_{crit} = \frac{H\,P_{crit}}{E\,A_{bar}}$.

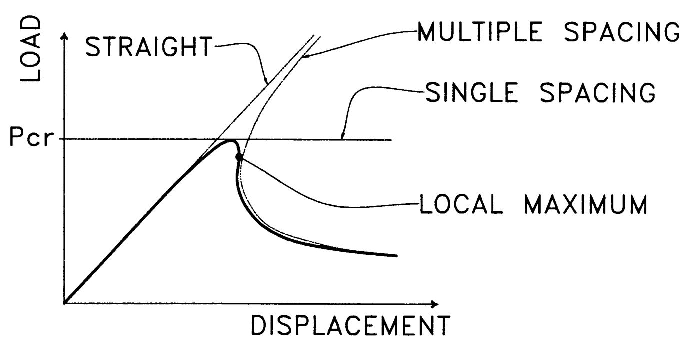


Figure 5.4: Snap buckling phenomenon

The plots reveal a snap buckling behavior [4]. Notice that as we start to compress the bar, the solution will move along the straight solution until it approaches the critical load. Then the bar will begin to buckle over one spacing. Shortly after beginning to buckle, the solution encounters a local maximum for ΔH. Figure 5.4. As ΔH is increased further snapping occurs because no nearby equilibrium state can be found. The solution snaps to the multiple spacing mode that has the smallest corresponding value for the axial load. Figure 5.3 reveals that buckling modes $n = 4$ to $n = 6$ become favorable.

Chapter 6

Finite Element Analysis

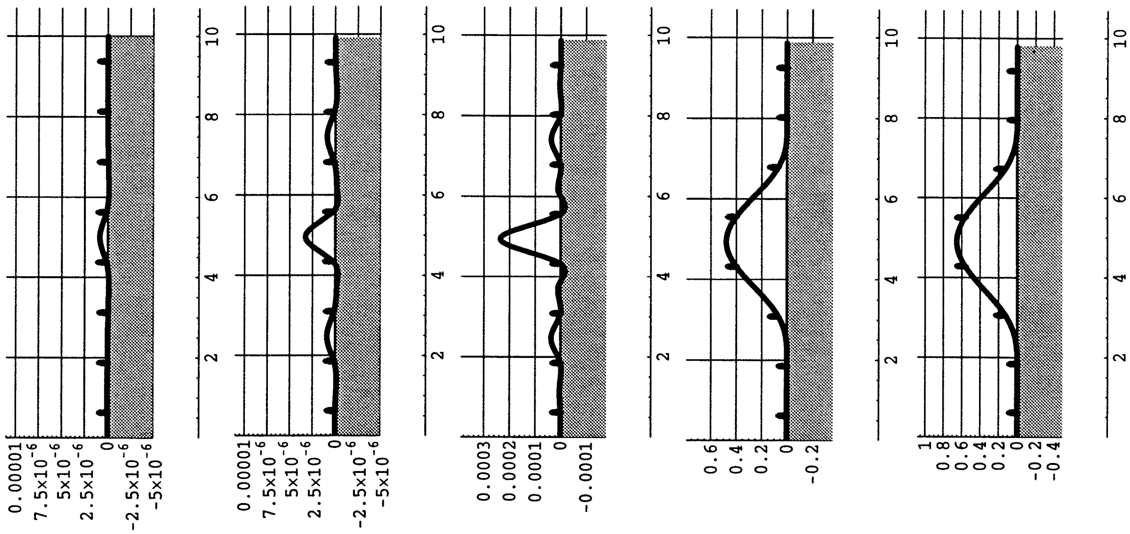
In this chapter, the deformation is no longer assumed to be sinusoidal. Instead we allow the deformation to have a more general form. Whereas in Chapter 5 the deformation was determined by α, n and, ϵ_o it will now be determined by arbitrarily many degrees of freedom. Equilibrium states are solved using a straight–forward non-linear finite element procedure. The results of the finite element analysis are presented first in Section 6.1. In Section 6.2, a full description of the finite element formulation is given.

6.1 Simulated Test Procedure and Results

A computer simulation of the bar buckling was performed. In order to force the bar to buckle the top of the bar was incrementally displaced downwards. The prescribed axial displacement will be denoted $\overline{\Delta H}$. This is similar to a "displacement controlled" test that would be performed in a laboratory. By prescribing the displacement, we can be assured that a solution exists for each step. On the other hand, if the procedure was to increment the end load, P, we would be unable to solve for equilibrium points on decreasing portions of the force-displacement curve.

In order to keep the bar from remaining straight a small *perturbation force* is applied transversely at midspan of the bar. The magnitude of the perturbation force was small $\approx 10^{-6} P_{crit}$. Using symmetry, only half of the bar was modeled, this constrained the bar to buckle over an odd number of spacings. The model of half of the bar was descetized into 8 elements.

The "test specimen" had the same parameters as were used to generate the force–displacement curve in Section 5.3. (Table 5.1) Figure 6.1 shows the displaced shape of the model at several levels of axial displacement. The axial displacement is scaled by the displacement of the straight bar under



$\Delta H = 0$ (a) $\quad 0.5\,\Delta H_{crit}$ (b) $\quad 0.75\,\Delta H_{crit}$ (c) $\quad 0.8\,\Delta H_{crit}$ (d) $\quad 1.3\,\Delta H_{crit}$ (e)

Figure 6.1: Displaced shape from finite element analysis

the critical load, $\Delta H_{crit} = \frac{P_{crit}H}{EA}$.

Figure 6.1(a) : No Axial Displacement

Figure 6.1(a) shows the the bar under the action of the perturbation force alone. The transverse displacement is negligible with a maximum of about 10^{-6} inches.

Figure 6.1(b) : Before The Onset of Buckling

As the top of the bar is moved downward the transverse displacement increases gradually, at first. When the axial displacement reaches 0.5 ΔH_{crit} the transverse displacement has only reached a maximum of $\approx 3 \cdot 10^{-6}$ inches.

Figure 6.1(c) : The Onset of Buckling - Single Spacing Mode

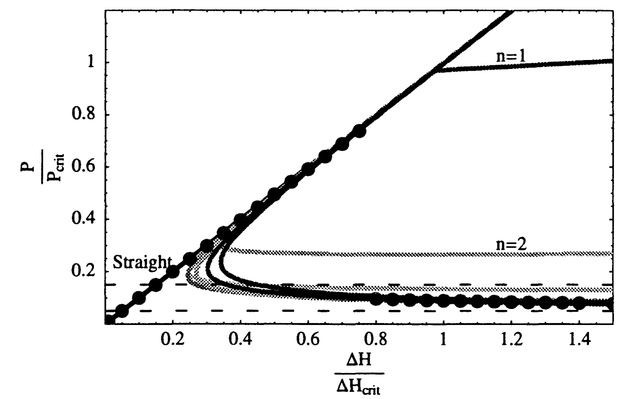
At 0.75 ΔH_{crit} the bar has buckled over a single spacing. The transverse displacement has increased 100 times to $\approx 3 \cdot 10^{-4}$ inches. The reason why the bar does not buckle at ΔH_{crit} is due to the imperfections in the model caused by the perturbation force and the contact model. (The contact model is discussed in the next section)

Figure 6.1(d) : Transition to 5-Spacing Mode

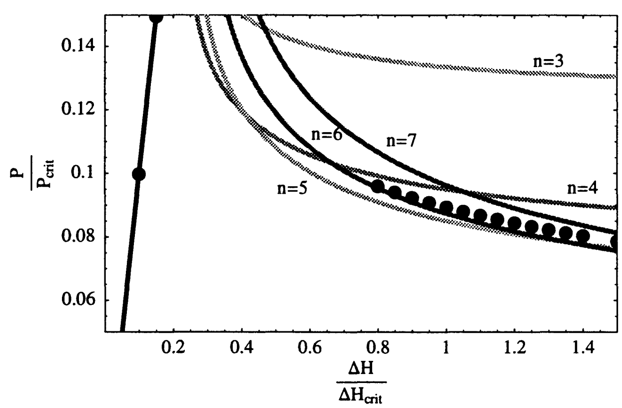
When the axial displacement is incremented slightly more to 0.8 ΔH_{crit} the buckling mode changes to the 5 spacing mode. The transverse displacement increases over one thousand times to ≈ 0.5 inches.

Figure 6.1(e) : Bar Remains in 5-Spacing Mode

As the axial displacement is incremented further the buckling mode remains at 5 spacings and the transverse displacement increases gradually. Even though the axial displacement has increased from 0.8 ΔH_{crit} to 1.3 ΔH_{crit} the lateral displacement has only increased to ≈ 0.65 inches.



(a) Full view



(b) Blow–up of dashed region

Figure 6.2: Axial reaction, P vs. prescribed axial displacement $\overline{\Delta H}$

Force vs. Displacement

Figure 6.2 shows a plot of axial reaction, P, versus axial prescribed axial displacement for the test just discussed. The data from the finite element analysis is plotted as points over the force versus displacement curves generated in Section 5.3. We see that for small loadings the solution travels along the solution for the straight bar. When the displacement nears the critical level, the bar buckles. (Because of the sensitivity of the buckling load to perturbation, P never reaches P_{crit}.) Since the interval where the bar buckles in the single spacing mode is very short it does not appear demonstratively on the plot. However, the plot does reveal good agreement when the the bar has buckled in the multiple spacing mode, even for displacements that are quite large.

Change in Arc Length vs. Axial Reaction

Recall the claim made in Section 5.1. We assumed that the arc length of the deformed bar was the same as the straight bar under the same load. In Figure 6.3, the change in arc length of each solution is plotted against the axial reaction. This is compared with the axial shortening of a straight bar, $\Delta L = P\frac{H}{EA}$, plotted on Figure 6.3 as well. Notice the agreement.

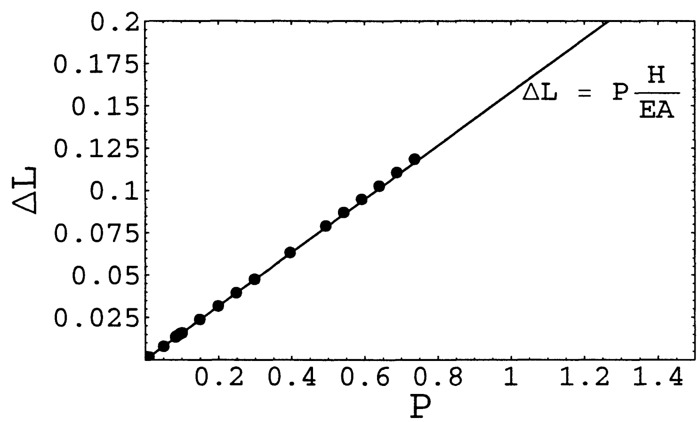


Figure 6.3: Change in arc length, ΔL vs. axial reaction, P

6.2 Finite Element Formulation

A full discussion of the finite element formulation follows in this section.

6.2.1 Penetration of the Bar Into the Core

In Section 5, we considered only buckling shapes that did not penetrate into the core. In this section it is inconvenient to make such a restriction on our displacements. Instead, we will model the penetration of the bar into the concrete core similarly to the classical "beam on and elastic foundation". In the classical problem the energy associated with the beam penetrating into the elastic foundation is [3],

$$U_{found} = \frac{\beta}{2} \int_0^H w^2(s)\, ds$$

Where β is called the *modulus of foundation* and has units of force divided by the square of length. This model assumes that the foundation acts in both directions. In order to have the core act in only one direction a contact function, $\mathcal{C}$, is introduced,

$$U_{core} = \frac{\beta}{2} \int_0^H \mathcal{C}\,[-w(s)]\, w^2(s)\, ds \tag{6.1}$$

The function $\mathcal{C}$, is like a step function except that it is smoothed to be differentiable. A sketch of $\mathcal{C}$ versus its argument, ρ, is shown in Figure 6.4. The

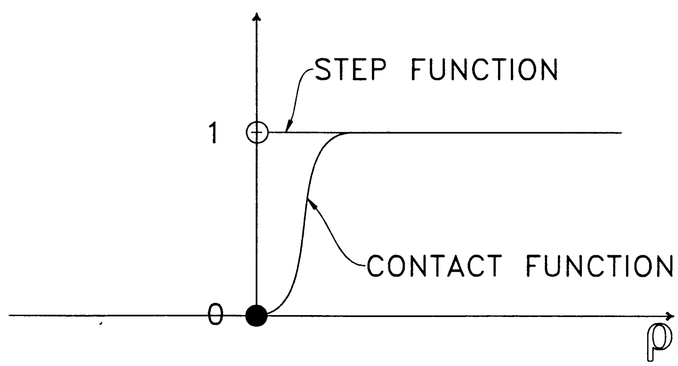


Figure 6.4: Contact function, $\mathcal{C}$

reason why a differentiable function is preferable to the discontinuous step function is that a discontinuous function causes problems when integrated numerically.[1]

Adding the new term, the potential energy (4.1) becomes,

$$U = U_{load} + U_{strain} + U_{spiral} + U_{core}\,. \tag{6.2}$$

One other adjustment needs to be made to consider deformation into the core. The energy associated with deforming the spiral (4.14) needs to reflect

[1] w'' and u'' are also discontinuous functions over the length of the bar. However, we know the points of discontinuity – the nodes. So integration is not problematic. On the other hand, the points where the bar changes from contacted with the core to separated from the core are not known a priori. So they must either be identified or smoothed. In my experience, smoothing is more painless.

that the spiral is deformed only if the bar moves away from the core.

$$U_{spiral} = T \sum_{i=1}^{m} \mathcal{C}(w_i)\, \Delta C(w_i) \tag{6.3}$$

6.2.2 Variation of Load Potential, $\delta \mathbf{U}_{\mathbf{load}}$, Vanishes

In Section 5, the axial load, P, played a central role in the equilibrium equation. (Recall, we used the equilibrium equation to solve for the axial load corresponding to the assumed buckled configuration.) In the finite element formulation, P will play no role at all.

Consider an experimental test where the bar shown in Figure 6.5 is subject to an axial displacement, $\overline{\Delta H}$. The bar is held at that axial displacement until it reaches equilibrium.

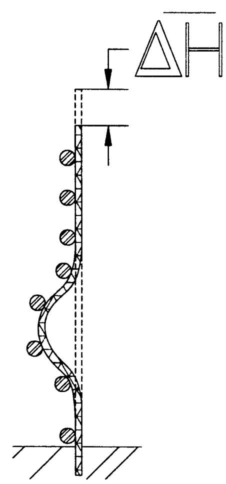


Figure 6.5: Prescribed axial displacement, $\overline{\Delta H}$

In order to determine the equilibrium state, we will consider only deformations that satisfy $u(H) = \overline{\Delta H}$. This is called a *displacement boundary condition.* When we vary the deformation infinitesimally between any two possible deformations, the axial displacement at the top does not move – it is fixed at $\overline{\Delta H}$, so its variation vanishes for any admissible variation.[2]

$$\delta \Delta H = \delta \left(\overline{\Delta H}\right) = 0$$

[2]Recall that we defined an admissible variation as a infinitesimal displacement that satisfies geometric constraints, such as the requirement that $\Delta H = \overline{\Delta H}$.

Therefore, the variation of the load potential also vanishes for any admissible variation.

$$\begin{aligned}\delta(U_{load}) &= \delta(-P\,\Delta H) = -\delta P\,\Delta H - P\,\delta\Delta H \\ &= 0\end{aligned}$$

(Recall from Section 4, $\delta P = 0$.)

6.2.3 Equilibrium For Finite Elements

In chapter 5, we based our equilibrium on the variation of one parameter, α. In this section, we will base it on the variation of several parameters, $\boldsymbol{\alpha}$. Where $\boldsymbol{\alpha}$ is a vector of real scalars of length p.

$$\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$$

Each entry of $\boldsymbol{\alpha}$ is called a *degree of freedom* and p is know as the *number of degrees of freedom*. In the following sections it will be shown that the deformation and the potential energy can be written as functions of $\boldsymbol{\alpha}$. For now we assume,

$$U = \breve{U}(\boldsymbol{\alpha}, \ldots)\,.$$

In order to determine whether a configuration characterized by $\boldsymbol{\alpha}$ is in equilibrium, we vary all of the degrees of freedom infinitesimally and arbitrarily,

$$\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha} + \boldsymbol{\delta\alpha}\,.$$

The change in the potential energy is the total differential of U with respect to $\boldsymbol{\delta\alpha}$. The Principal of Minimum Potential Energy says that this must vanish.

$$\begin{aligned}\delta U &= \delta\alpha_1 \frac{\partial U}{\partial \alpha_1} + \delta\alpha_2 \frac{\partial U}{\partial \alpha_2} + \cdots + \delta\alpha_p \frac{\partial U}{\partial \alpha_p} = 0 \\ &= \boldsymbol{\delta\alpha} \cdot \nabla_{\boldsymbol{\alpha}} U = 0\end{aligned}$$

Where $\nabla_{\boldsymbol{\alpha}} U$ is the gradient of U with respect to the degrees of freedom, $\boldsymbol{\alpha}$. Since $\boldsymbol{\delta\alpha}$ is an arbitrary vector, the only way that the above equation can be satisfied is if each term of the gradient is zero,

$$\nabla_{\boldsymbol{\alpha}} U = \mathbf{0}\,.$$

This results in p nonlinear equations in $\boldsymbol{\alpha}$ that must be solved to find an equilibrium position. Applying the gradient to the terms of the potential en-

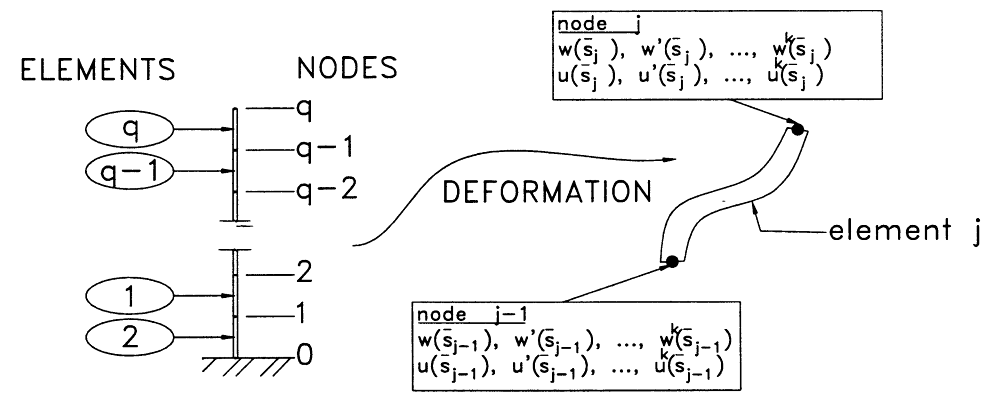


Figure 6.6: Descretization of the bar into finite elements

ergy equation (6.2) and neglecting the gradient of U_{load} [3] yields an expression for equilibrium that will be of use in the following sections.

$$\nabla_{\boldsymbol{\alpha}} U = \nabla_{\boldsymbol{\alpha}} U_{strain} + \nabla_{\boldsymbol{\alpha}} U_{spiral} + \nabla_{\boldsymbol{\alpha}} U_{core} = 0 \tag{6.4}$$

Iterative Solution Using Newton's Method

The above equilibrium equations will be solved iteratively using Newton's Method:

$$\boldsymbol{\alpha}_{i+1} = \boldsymbol{\alpha}_i + \left(\nabla^2_{\boldsymbol{\alpha}} U\right)_i^{-1} \left(\nabla_{\boldsymbol{\alpha}} U\right)_i \tag{6.5}$$

Where $(\nabla_{\boldsymbol{\alpha}} U)_i$ and $(\nabla^2_{\boldsymbol{\alpha}} U)_i$ are the gradient and the Hessian of U evaluated at $\boldsymbol{\alpha}_i$.[4] The algorithm is halted when a tolerance, τ, is reached.

$$\| \left(\nabla_{\boldsymbol{\alpha}} U\right)_i \| < \tau$$

6.2.4 Assumed Deformation

In order to implement the finite element method, the bar is discretized into many pieces called *finite elements.* See Figure 6.6. Lets say we break up the

[3]In Section 6.2.2, we reasoned that $\delta(-P\Delta H) = 0$ because $\delta P = \delta \Delta H = 0$. We could have explained it making using the notation of this section. P and ΔH are <u>not</u> functions of $\boldsymbol{\alpha}$ so $\nabla_{\boldsymbol{\alpha}} P = \nabla_{\boldsymbol{\alpha}} \Delta H = \mathbf{0}$ and, therefore, $\nabla_{\boldsymbol{\alpha}} (-P\Delta H) = \mathbf{0}$

[4]In the finite element literature, the gradient is often referred to as the "residual" and the Hessian as the "tangent".

bar into q finite elements of equal length, H^e:

$$H^e = \frac{H}{q}$$

The material coordinates where the elements meet are known as *nodes.* It is easy to verify that there are $q + 1$ nodes in the model. The nodes are denoted,

$$\bar{s}_j = j\,H^e \qquad\qquad j \in \{0, 1, \ldots, q\}\,.$$

The deformation over the entire model is assumed to be uniquely determined by state of deformation at the nodes. We will characterize the state of deformation at a node by evaluating the value of the deformation functions u and w and their first k derivatives. Therefore, for each deformation function there will be $k + 1$ values for each node, referred to as the *nodal displacements.* We make an additional assumption that the deformation within the element is affected only by the nodal displacements of the adjacent nodes. The deformation between nodes we will interpolated using a polynomial. It can be verified that the $2(k+1)$ nodal displacements associated with a given element can uniquely determine the coefficients a polynomial of degree $2k+1$.

In order to determine what value of k should be used, we must consider the problem we are trying to solve. For our case, the assumed deformation must allow the evaluation of the potential energy equation. In our expressions for the potential energy derived in chapter 4, we encountered up to second derivatives of $u(s)$ and $w(s)$. Therefore, we need to interpolate with a polynomial whose second derivatives do not vanish identically. The minimum choice for k (and therefore the most economical one) is $k = 1$. So the displacements will be interpolated using cubic functions and the nodal displacements will be the value of the function at the node and its first derivative.

The case where $k = 1$ is know as *Hermitian interpolation.* Write the cubic displacements within element j:

$$\begin{aligned} u_j(s^e) &= b_o + b_1 s^e + b_2\,(s^e)^2 + b_3\,(s^e)^3 \\ &= \mathbf{X}(s^e)\cdot\mathbf{b} \qquad\qquad 0 \le s^e < H^e \\ \\ w_j(s^e) &= c_o + c_1 s^e + c_2\,(s^e)^2 + c_3\,(s^e)^3 \\ &= \mathbf{X}(s^e)\cdot\mathbf{c} \qquad\qquad 0 \le s^e < H^e \end{aligned} \tag{6.6}$$

Where s_e is the material coordinate along the length of the element and:

$$\begin{aligned} \mathbf{X}(s^e) &= \left\{1\,,\; s^e\,,\; (s^e)^2\,,\; (s^e)^3\right\} \\ \mathbf{b} &= \{b_o\,,\; b_1\,,\; b_2\,,\; b_3\} \\ \mathbf{c} &= \{c_o\,,\; c_1\,,\; c_2\,,\; c_3\} \end{aligned}$$

Now we want to express the b's and c's in terms of the nodal displacements. Substituting the definitions of the eight nodal displacements associated with element j:

$$\begin{array}{rclcrcl} w_{j-1} & = & w(0) & , & w_j & = & w(H^e) \, , \\ w'_{j-1} & = & w'(0) & , & w'_j & = & w'(H^e) \, , \\ u_{j-1} & = & u(0) & , & u_j & = & u(H^e) \, , \\ u'_{j-1} & = & u'(0) & , & u'_j & = & u'(H^e) \, , \end{array}$$

into equations (6.6) yields:

$$\left\{ \begin{array}{c} u_{j-1} \\ u'_{j-1} \\ u_j \\ u'_j \end{array} \right\} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & H^e & (H^e)^2 & (H^e)^3 \\ 0 & 1 & 2H^e & 3\,(H^e)^3 \end{array} \right] \left\{ \begin{array}{c} b_o \\ b_1 \\ b_2 \\ b_3 \end{array} \right\} \qquad \text{or} \qquad \mathbf{u}_j = \mathbf{A}\,\mathbf{b}$$

and

$$\left\{ \begin{array}{c} w_{j-1} \\ w'_{j-1} \\ w_j \\ w'_j \end{array} \right\} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & H^e & (H^e)^2 & (H^e)^3 \\ 0 & 1 & 2H^e & 3\,(H^e)^3 \end{array} \right] \left\{ \begin{array}{c} c_o \\ c_1 \\ c_2 \\ c_3 \end{array} \right\} \qquad \text{or} \qquad \mathbf{w}_j = \mathbf{A}\,\mathbf{c}$$

Therefore $\mathbf{b} = \mathbf{A}^{-1}\,\mathbf{u}_j$ and $\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{w}_j$. So we can rewrite equations (6.6) as:

$$\begin{array}{rcl} u_j(s^e) & = & \mathbf{N}(s^e) \cdot \mathbf{u}_j \\ w_j(s^e) & = & \mathbf{N}(s^e) \cdot \mathbf{w}_j \end{array} \qquad (6.7)$$

where

$$\mathbf{N}(s^e) = \mathbf{X}(s^e)\,\mathbf{A}^{-1}$$

We define an operation, $elem(s)$, that takes a material coordinate, s, and returns its corresponding element number. And define another operation, $node(s)$, that takes a material coordinate, s, and returns the node number that immediately precedes it. We can define the displacement functions

$$\begin{array}{rcl} u(s) & = & \mathbf{N}(s - \bar{s}_{node(s)}) \cdot \mathbf{u}_{elem(s)} \\ w(s) & = & \mathbf{N}(s - \bar{s}_{node(s)}) \cdot \mathbf{w}_{elem(s)} \end{array} \qquad (6.8)$$

These functions are the piecewise cubic interpolating functions.

6.2.5 Displacement Boundary Conditions

In this section it will be noted that some of the nodal displacements are prescribed or fixed.

Axial Displacement Boundary Condition

In order compress the bar, we prescribe the axial displacement, $\Delta H = \overline{\Delta H}$, so that

$$u_q = \overline{\Delta H} \,.$$

Other Displacement Boundary Conditions

In addition to the axial displacement boundary condition, we have the condition that the bottom of the bar is clamped in place,

$$w_0 = w_0' = u_0 = 0 \ .$$

And the top of the bar cannot displace transversely nor rotate,

$$w_q = w_q' = 0 \ .$$

We now form a list of all of the nodal displacements with boundary conditions. These will be known as the fixed nodal displacements, denoted $\overline{\mathbf{d}}$.

$$\overline{\mathbf{d}} = \left\{ w_0\,,\ w_0'\,,\ u_0\,,\ w_q\,,\ w_q'\,,\ u_q \right\}$$

6.2.6 The Degrees of Freedom, $\boldsymbol{\alpha}$

Consider a list of all the nodal degrees of freedom, $\mathbf{d}$.

$$\mathbf{d} = \left\{ w_0\,,\ w_0'\,,\ u_0\,,\ u_0'\,,\ w_1\,,\ w_1'\,,\ u_1\,,\ u_1', \ldots, w_q\,,\ w_q'\,,\ u_q\,,\ u_q' \right\}$$

The nodal displacements without displacement boundary conditions are variable parameters that we will use characterize the deformation completely. These are the degrees of freedom, $\boldsymbol{\alpha}$, that we used in Section 6.2.3 to formulate equilibrium. Now we identify $\boldsymbol{\alpha}$ with the elements of $\mathbf{d}$ that are not in $\overline{\mathbf{d}}$.

$$\boldsymbol{\alpha} = \left\{ u_0'\,,\ w_1\,,\ w_1'\,,\ u_1\,,\ u_1', \ldots, w_{q-1}\,,\ w_{q-1}'\,,\ u_{q-1}\,,\ u_{q-1}'\,,\ u_q' \right\}$$

6.2.7 Gradient and Hessian of $\mathbf{U_{core}}$, $\mathbf{U_{strain}}$ and $\mathbf{U_{spiral}}$

For the solution of equilibrium equation (6.4) by Newton's Method (6.5), the gradient and Hessian of U_{core}, U_{spiral} and U_{strain} taken with respect to α will be needed.

Gradient and Hessian of $\mathbf{U_{core}}$

Recall equation(6.1),

$$U_{core} = \frac{\beta}{2} \int_0^H \mathcal{C}\left[-w(s)\right] w^2(s)\, ds\,.$$

Break up the integral into elements,

$$U_{core} = \frac{\beta}{2} \sum_{j=1}^{q} \int_0^{H^e} \mathcal{C}\left[-w_j(s^e)\right] w_j^2(s^e)\, ds^e$$

By composing the integrand with the assumed element displacements (6.7), U_{core} becomes a function of the nodal displacements, $\mathbf{d}$. This function is denoted $\breve{U}_{core}(\mathbf{d})$.

$$\breve{U}_{core}(\mathbf{d}) = \frac{\beta}{2} \sum_{j=1}^{q} \int_0^{H^e} dU_{core}\left(\mathbf{w}_j, s^e\right)\, ds^e$$

Where

$$dU_{core}\left(\mathbf{w}_j, s^e\right) = \mathcal{C}\left[-w_j(s^e)\right] w_j^2(s^e) \circ \mathbf{N}(s^e)\left(\mathbf{w}_j\right)$$

Now take the gradient and Hessian of $\breve{U}_{core}(\mathbf{d})$.

$$\nabla_{\boldsymbol{\alpha}} \breve{U}_{core}(\mathbf{d}) = \frac{\beta}{2} \sum_{j=1}^{q} \int_0^{H^e} \nabla_{\boldsymbol{\alpha}} \left[dU_{core}\left(\mathbf{w}_j, s^e\right)\right] ds^e \qquad (6.9)$$

$$\nabla^2_{\boldsymbol{\alpha}} \breve{U}_{core}(\mathbf{d}) = \frac{\beta}{2} \sum_{j=1}^{q} \int_0^{H^e} \nabla^2_{\boldsymbol{\alpha}} \left[dU_{core}\left(\mathbf{w}_j, s^e\right)\right] ds^e \qquad (6.10)$$

The gradient and Hessian of $dU_{core}\left(\mathbf{w}_j, s^e\right)$ can be taken directly with respect to $\boldsymbol{\alpha}$ since the elements of $\boldsymbol{\alpha}$ appear as elements of $\mathbf{w}_j$. [5]

In order to have the integrand of (6.10) to be continuous over the length of an element, we need the contact function, $\mathcal{C}$, to have a continuous second derivative. Require

$$\left.\begin{array}{lllllll} \mathcal{C}(0) & = & 0 & , & \mathcal{C}(\eta) & = & 1 \;, \\ \mathcal{C}'(0) & = & 0 & , & \mathcal{C}'(\eta) & = & 0 \;, \\ \mathcal{C}''(0) & = & 0 & , & \mathcal{C}''(\eta) & = & 0 \;. \end{array}\right\} \qquad (6.11)$$

Where η is called the *contact dimension.* See Figure 6.7. In the results presented in Section 6.1 the contact dimension was taken to be $\eta = 10^{-5}$.

[5]This formulation was developed specifically to be implemented in *Mathematica.* Symbolic manipulators of this type automate the process of taking derivatives of complicated algebraic expressions. Therefore, it is preferable to compose the integrand of the energy symbolically in terms of the nodal displacements. Then gradients and Hessians are taken by the the built-in subroutines. In this way, programming bugs due to “inconsistent tangents” are eliminated. (Hessians are always the gradient of the gradient.)

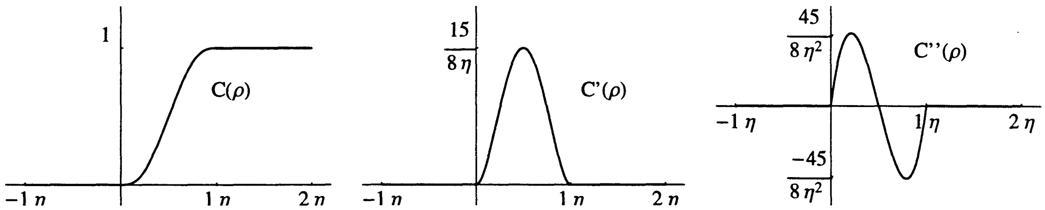


Figure 6.7: Contact function $\mathcal{C}$ with continuous second derivative

If we fit the continuity conditions (6.11) with the coefficients of a fifth degree polynomial then $\mathcal{C}$ is

$$\mathcal{C}(\rho) = \begin{cases} 0 & \text{if } \rho \leq 0 \\ \frac{10}{\eta^3}\rho^3 - \frac{14}{\eta^4}\rho^4 + \frac{6}{\eta^5}\rho^5 & \text{if } 0 < \rho < \eta \\ 1 & \text{if } \rho \geq \eta \end{cases} \tag{6.12}$$

Gradient and Hessian of $\mathbf{U_{strain}}$

Recall form chapter 4 the definition of U_{strain}

$$U_{strain} = \frac{1}{2} E \iiint_B E_{xx}^2 \, dV$$

Where

$$\begin{aligned} E_{xx} &= \hat{E}_{xx}\left[u'(s), u''(s), w'(s), w''(s)\right] \\ &= \frac{1}{2}\left[(1+u'(s))^2 + w'^2(s) - 1\right] \\ &\quad - z\,\frac{w''(s)[1+u'(s)] + w'(s)\; u''(s)}{\sqrt{[1+u'(s)]^2 + w'^2(s)}} \\ &\quad + \frac{1}{2} z^2 \left[\frac{w''(s)[1+u'(s)] + w'(s)\; u''(s)}{[1+u'(s)]^2 + w'^2(s)}\right]^2 \end{aligned} \tag{6.13}$$

Once again we break up the integral into elements,

$$U_{strain} = \frac{1}{2} E \sum_{j=1}^{q} \iiint_B \left\{\hat{E}_{xx}\left[u_j'(s^e), u_j''(s^e), w_j'(s^e), w_j''(s^e)\right]\right\}^2 dV$$

The axial strain in element j can be written as a function of the nodal displacements by composing (6.13) with the element displacement functions (6.7). Therefore, U_{strain} becomes a function of the nodal displacements $\mathbf{d}$. This function will be denoted $\breve{U}_{strain}(\mathbf{d})$.

$$\breve{U}_{strain}(\mathbf{d}) = \frac{1}{2} E \sum_{j=1}^{q} \iiint_B \left\{ \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \right\}^2 dV$$

Where

$$\begin{aligned} \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] &= \hat{E}_{xx} \left[u_j'(s^e), u_j''(s^e), w_j'(s^e), w_j''(s^e) \right] \\ &= \hat{E}_{xx} \left[\mathbf{N}'(s^e) \cdot \mathbf{u}_j, \mathbf{N}''(s^e) \cdot \mathbf{u}_j, \mathbf{N}'(s^e) \cdot \mathbf{w}_j, \mathbf{N}''(s^e) \cdot \mathbf{w}_j \right] \end{aligned}$$

As for $\breve{U}_{core}(\mathbf{d})$, we take the gradient and Hessian of $\breve{U}_{strain}(\mathbf{d})$ with respect to $\boldsymbol{\alpha}$ since the elements of $\boldsymbol{\alpha}$ appear as elements of $\mathbf{u}_j$ and $\mathbf{w}_j$.

$$\nabla_{\boldsymbol{\alpha}} \breve{U}_{strain} = E \sum_{j=1}^{q} \iiint_B \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \nabla_{\boldsymbol{\alpha}} \left\{ \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \right\} dV \quad (6.14)$$

$$\begin{aligned} \nabla^2_{\boldsymbol{\alpha}} \breve{U}_{strain} = E \sum_{j=1}^{q} \iiint_B \Big[& \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \nabla^2_{\boldsymbol{\alpha}} \left\{ \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \right\} \\ & + \nabla_{\boldsymbol{\alpha}} \otimes \nabla_{\boldsymbol{\alpha}} \left\{ \hat{E}_{xx} \left[\mathbf{u}_j, \mathbf{w}_j, s^e \right] \right\} \Big] dV \quad (6.15) \end{aligned}$$

Gradient and Hessian of $\mathbf{U_{spiral}}$

Recall the expression for U_{spiral} (6.3),

$$\begin{aligned} U_{spiral} &= T \sum_{i=1}^{m} \mathcal{C}(w_i) \Delta C(w_i) \\ &= T \sum_{i=1}^{m} \left[\mathcal{C}[w(s)] \Delta C[w(s)] \right]_{s=s_i} \end{aligned}$$

Where the summation is over the spiral spacings (not over the elements), and where s_i and w_i are the material coordinate and the transverse displacement of the ith spiral spacing, respectively. Since we have not broken this expression into elements, we compose it with the global displacements (6.8). Denoting the composed function $\breve{U}_{spiral}(\mathbf{d})$,

$$\begin{aligned} \breve{U}_{spiral}(\mathbf{d}) &= T \sum_{i=1}^{m} \left\{ \mathcal{C}[w(s)] \Delta C[w(s)] \circ \left[\mathbf{N}(s - \bar{s}_{node(s)}) \cdot \mathbf{w}_{elem(s)} \right] \right\}_{s=s_i} \\ &= T \sum_{i=1}^{m} \mathcal{C}[w(s)] \Delta C[w(s)] \circ \left[\mathbf{N}(s_i - \bar{s}_{node(s_i)}) \cdot \mathbf{w}_{elem(s_i)} \right] \end{aligned}$$

Since the elements of $\boldsymbol{\alpha}$ will appear as elements in $\mathbf{w}_{elem(s_i)}$, we can take the gradient and Hessian directly and evaluate the summation term-wise.

$$\nabla_{\boldsymbol{\alpha}} \breve{U}_{spiral}(\mathbf{d}) = T \sum_{i=1}^{m} \nabla_{\boldsymbol{\alpha}} \left\{ \mathcal{C}[w(s)]\, \Delta C[w(s)] \circ \left[\mathbf{N}(s_i - \overline{s}_{node(s_i)}) \cdot \mathbf{w}_{elem(s_i)} \right] \right\} \quad (6.16)$$

$$\nabla^2_{\boldsymbol{\alpha}} \breve{U}_{spiral}(\mathbf{d}) = T \sum_{i=1}^{m} \nabla^2_{\boldsymbol{\alpha}} \left\{ \mathcal{C}[w(s)]\, \Delta C[w(s)] \circ \left[\mathbf{N}(s_i - \overline{s}_{node(s_i)}) \cdot \mathbf{w}_{elem(s_i)} \right] \right\} \quad (6.17)$$

6.2.8 Summary of Finite Element Procedure

The expressions for the gradients and Hessians (6.9), (6.10), (6.14), (6.15), (6.16), (6.17) are employed to solve the equilibrium equation (6.4) iteratively using Newton's method (6.5).

Chapter 7

Conclusion

The work presented in this thesis considers only the behavior of bars which buckle elastically. The explanation it presents for the observed buckling mode of concrete reinforcing bars is promising but far from rigorous. In order to truly understand this phenomenon, a more representative constitutive law should be used. I feel that even for the plastic case, consideration should be given to the snap buckling phenomenon presented here.

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Appendix A

Finite Deformation Euler–Bernoulli Beam

Let $\mathcal{P}$ be a material point, defined by its position in the reference configuration, $\mathbf{P}$ (Figure A.1).

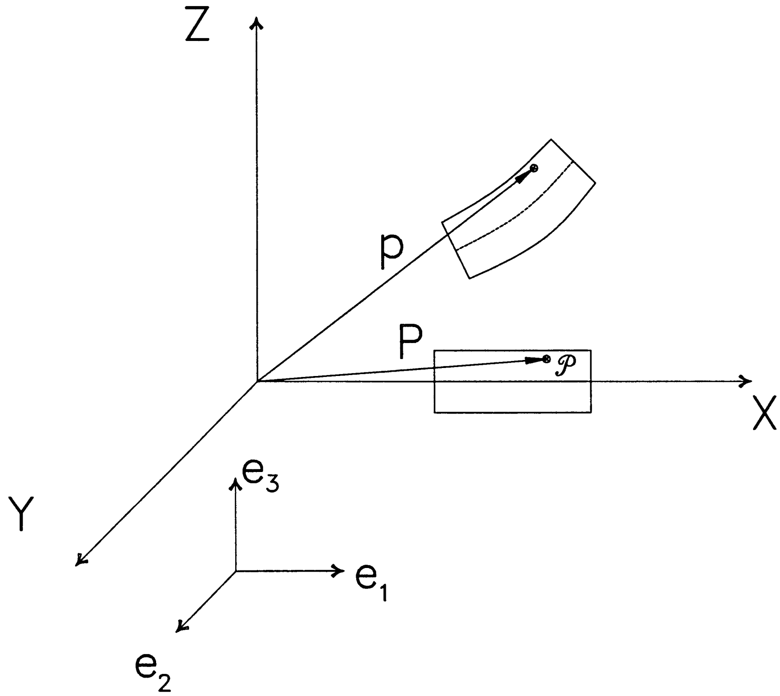

Figure A.1: Reference and deformed configuration

We define a Cartesian basis such that $\mathbf{P} = X\,\mathbf{e_1} + Y\,\mathbf{e_2} + Z\,\mathbf{e_3}$. Consider a deformation that maps $\mathcal{P}$ to a deformed position, $\mathbf{p} = x\,\mathbf{e_1} + y\,\mathbf{e_2} + z\,\mathbf{e_3}$. Assume that plane sections perpendicular to the centerline of the bar before deformation remain plane and unstretched after deformation. (Figure A.2). Also assume no deformation in the Y–direction. The deformation can be

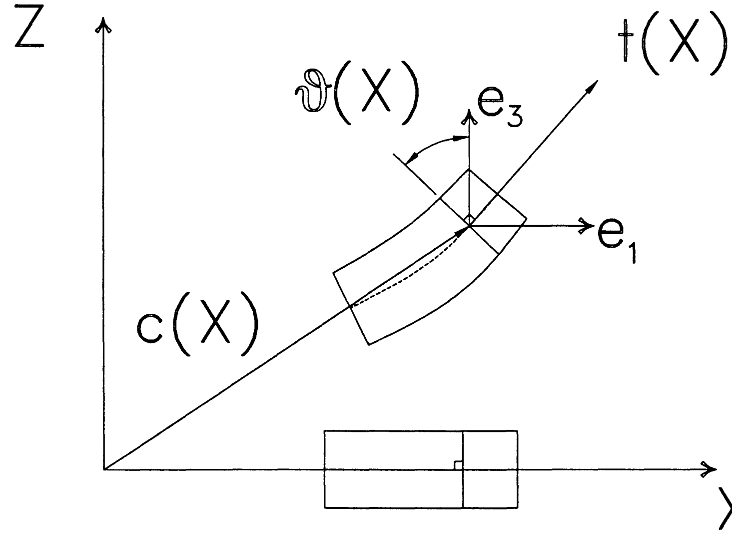

Figure A.2: Cross sections remain plane, unstretched and perpendicular to centerline

written,

$$\left.\begin{aligned} x &= X + u(X) - Z\,\sin\theta(X)\,, \\ y &= Y\,, \\ z &= w(X) + Z\,\cos\theta(X)\,. \end{aligned}\right\} \tag{A.1}$$

Where $u(X)$ and $w(X)$ are the amounts that a point on the centerline ($Y = Z = 0$) moves in the $\mathbf{e_1}$ and $\mathbf{e_3}$ directions respectively. $\theta(X)$ is the angle that the deformed cross–section makes with the $\mathbf{e_3}$ direction. (Figure A.2). Assume that cross–sections perpendicular to the centerline before deformation are perpendicular to the cross section after deformation. This requires that tangent of the deformed centerline, $\mathbf{t}(X)$, makes the same angle $\theta(X)$ with but respect to the $\mathbf{e_1}$ direction (Figure A.3). Let the deformed position of the centerline be defined by the vector $\mathbf{c}(X)$

$$\mathbf{c}(X) = [X + u(X)]\,\mathbf{e_1} + w(X)\mathbf{e_3}$$

Then the tangent to the centerline,

$$\mathbf{t}(X) = [1 + u'(X)]\,\mathbf{e_1} + w'(X)\mathbf{e_3}$$

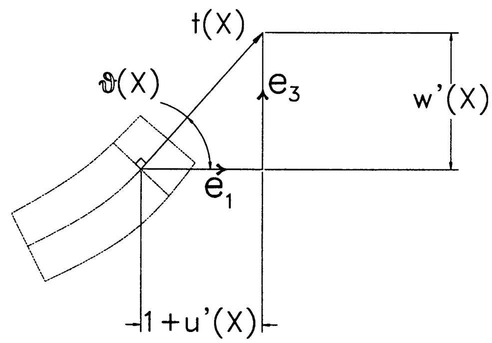

Figure A.3: Relation between θ and $\mathbf{t}(X)$

So that from Figure A.3.

$$\begin{aligned}
\sin\theta(X) &= \frac{w'(X)}{\sqrt{[1+u'(X)]^2+[w'(X)]^2}} \\
\cos\theta(X) &= \frac{1+u'(X)}{\sqrt{[1+u'(X)]^2+[w'(X)]^2}} \\
\tan\theta(X) &= \frac{w'(X)}{1+u'(X)}
\end{aligned}$$

Taking the deformation gradient of (A.1) gives,

$$\mathbf{F} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}$$

Where,

$$\begin{aligned}
\frac{\partial x}{\partial X} &= 1+u'(X) - Z\theta'(X)\cos\theta(X)\,, \\
\frac{\partial x}{\partial Z} &= -\sin\theta(X)\,, \\
\frac{\partial y}{\partial Y} &= 1\,, \\
\frac{\partial z}{\partial X} &= w'(X) - Z\theta'(X)\sin\theta(X)\,, \\
\frac{\partial z}{\partial Z} &= \sin\theta(X)\,, \\
\frac{\partial x}{\partial Y} &= \frac{\partial y}{\partial X} = \frac{\partial y}{\partial Z} = \frac{\partial z}{\partial Y} = 0\,.
\end{aligned}$$

The Green–St.Venant Strain,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$$

For convenience, the functions of X are written without showing this dependence. For example, $u(X)$ will be written as simply u. The components of $\mathbf{E}$ are,

$$\begin{aligned}
E_{XX} &= u' + \frac{1}{2}\left[u'^2 + w'^2\right] - Z\theta'\left[(1+u')\cos\theta + w'\sin\theta\right] + \frac{1}{2}Z^2\theta'^2 \\
&= \frac{1}{2}\left[(1+u')^2 + w'^2 - 1\right] - Z\theta'\sqrt{(1+u')^2 + w'^2} + \frac{1}{2}Z^2\theta'^2 \\
E_{XZ} &= \frac{1}{2}\left[-\sin\theta\,(1+u') + \cos\theta w'\right] \\
&= \frac{1}{2}\left[\frac{-w'\,(1+u')}{\sqrt{(1+u')^2 + w'^2}} + \frac{w'\,(1+u')}{\sqrt{(1+u')^2 + w'^2}}\right] = 0 \\
E_{XZ} &= \frac{1}{2}\left(-1 + \sin^2\theta + \cos^2\theta\right) = 0 \\
E_{XZ} &= E_{YY} = E_{YZ} = 0
\end{aligned}$$

So E_{XX} is the <u>only</u> surviving strain component. Now we express θ' in terms of the other functions of X. Recall

$$\sin\theta = \frac{w'}{\sqrt{(1+u')^2 + (w')^2}}.$$

Taking the derivative yields,

$$\begin{aligned}
\cos\theta\theta' &= \frac{w''}{\sqrt{(1+u')^2 + (w')^2}} - \frac{w'\left[(1+u')u'' + w'w''\right]}{\left[(1+u')^2 + (w')^2\right]^{\frac{3}{2}}} \\
\theta' &= \frac{w''}{1+u'} - \frac{w'\left[(1+u')u'' + w'w''\right]}{(1+u')\left[(1+u')^2 + (w')^2\right]} \\
&= \frac{w''(1+u')^2 + w''(w')^2 - w'u''(1+u') - (w')^2w''}{(1+u')\left[(1+u')^2 + (w')^2\right]} \\
&= \frac{w''(1+u') - w'u''}{(1+u')^2 + (w')^2}
\end{aligned}$$

Putting this together with the expression for E_{XX} gives

$$E_{XX} = \frac{1}{2}\left[(1+u')^2 + w'^2 - 1\right] - Z\frac{w''(1+u') - w'u''}{\sqrt{(1+u')^2 + (w')^2}} + \frac{1}{2}Z^2\left[\frac{w''(1+u') - w'u''}{(1+u')^2 + (w')^2}\right]^2$$

Appendix B

$A(\alpha, n)$, $B(\alpha, n)$, and $C(\alpha, n)$ Are Positive For $\alpha < \dfrac{\lambda n}{\sqrt{5}\pi}$

$\underline{A(\alpha, n)}$

From Section 5.3,

$$\begin{aligned}
A(\alpha, n) &= -\frac{1}{EA}\frac{\partial}{\partial\alpha}\left\{\frac{\partial}{\partial\epsilon_o}\Delta H(\epsilon_o, \alpha, n)\right\}_{\epsilon_o=0} \\
&= -\frac{1}{EA}\frac{\partial}{\partial\alpha}\left\{\frac{\partial}{\partial\epsilon_o}\left[H - \int_0^H \sqrt{(1+\epsilon_o)^2 - w'^2(\alpha, n, s)}ds\right]\right\}_{\epsilon_o=0} \\
&= -\frac{1}{EA}\frac{\partial}{\partial\alpha}\left\{-\int_0^H \frac{(1+\epsilon_o)\,ds}{\sqrt{(1+\epsilon_o)^2 - w'^2(\alpha, n, s)}}\right\}_{\epsilon_o=0} \\
&= \frac{1}{EA}\frac{\partial}{\partial\alpha}\int_0^H \frac{ds}{\sqrt{1 - w'^2(\alpha, n, s)}}
\end{aligned}$$

Substitute the assumed deformation.

$$\begin{aligned}
A(\alpha, n) &= \frac{1}{EA}\frac{\partial}{\partial\alpha}\int_0^H \frac{ds}{\sqrt{1 - \left(\frac{\alpha\pi}{\lambda n}\right)^2 \sin^2\frac{2\pi s}{\lambda n}}} \\
&= \frac{1}{EA}\int_0^H \frac{\alpha\left(\frac{\pi}{\lambda n}\right)^2 \sin^2\frac{2\pi s}{\lambda n}}{\left[1 - \left(\frac{\alpha\pi}{\lambda n}\right)^2 \sin^2\frac{2\pi s}{\lambda n}\right]^{\frac{3}{2}}}
\end{aligned}$$

Note that the integrand, and therefore $A(\alpha, n)$, will be positive provided $\alpha < \frac{\lambda n}{\pi}$.

$\underline{B(\alpha, n)}$

From Section 5.3,

$$B(\alpha,n) = \frac{\partial}{\partial\alpha}\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0}$$

The first term of $B(\alpha, n)$ becomes,

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\left[\Delta\tilde{H}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} &= \frac{\partial}{\partial\alpha}\left[H - \int_0^H \sqrt{(1+\epsilon_o)^2 - w'^2(\alpha,n,s)}ds\right]_{\epsilon_o=0} \\
&= -\frac{\partial}{\partial\alpha}\int_0^H \sqrt{1 - w'^2(\alpha,n,s)}ds \\
&= -\frac{\partial}{\partial\alpha}\int_0^H \sqrt{1 - \left(\frac{\alpha\,\pi}{\lambda\,n}\right)^2 \sin^2\frac{2\,\pi\,s}{\lambda\,n}}ds \\
&= \int_0^H \frac{\alpha\left(\frac{\pi}{\lambda\,n}\right)^2 \sin^2\frac{2\,\pi\,s}{\lambda\,n}\,ds}{\sqrt{1 - \left(\frac{\alpha\,\pi}{\lambda\,n}\right)^2 \sin^2\frac{2\,\pi\,s}{\lambda\,n}}}
\end{aligned}$$

The integrand will be positive for $\alpha < \frac{\lambda n}{\pi}$.

The second term of $B(\alpha, n)$ becomes,

$$\begin{aligned}
\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} &= \frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\frac{1}{2}E\iiint_B E_{xx}^2\,dV\right]_{\epsilon_o=0} \\
&= \frac{1}{A_{bar}}\frac{\partial}{\partial\alpha}\iiint_B\left[E_{xx}\frac{\partial}{\partial\epsilon_o}E_{xx}\,dV\right]_{\epsilon_o=0}
\end{aligned}$$

Where from equation 5.13,

$$E_{xx} = \epsilon_o + \frac{1}{2}\epsilon_o^2 - \frac{z\,w''(1+\epsilon_o)}{\sqrt{(1+\epsilon_o)^2 - w'^2}} + \frac{1}{2}\left[\frac{z^2\,w''^2}{(1+\epsilon_o)^2 - w'^2}\right].$$

Substituting this into the second term gives,

$$\begin{aligned}
\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} = \frac{1}{A_{bar}}\frac{\partial}{\partial\alpha}\iiint_B\Bigg\{ & z\left[\frac{2w''}{\sqrt{1-(w')^2}}\right] + z^2(w'')^2\left[\frac{1-3(w')^2}{[1-(w')^2]^2}\right] \\
& + z^3(w'')^3\left[\frac{1+2(w')^2}{[1-(w')^2]^{\frac{5}{2}}}\right] + z^4(w'')^4\left[\frac{1}{[1-(w')^2]^3}\right]\Bigg\}dV
\end{aligned}$$

Integrating over the symmetric cross-section eliminates all of the terms odd in z

$$\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} = \frac{I_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{\partial}{\partial\alpha}\left[(w'')^2\frac{1-3(w')^2}{[1-(w')^2]^2}\right]ds + \frac{Q_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{\partial}{\partial\alpha}\left[\frac{(w'')^4}{[1-(w')^2]^3}\right]ds$$

Where I and Q are section properties,

$$I_{bar} = \iint_A z^2\,dA \qquad Q_{bar} = \iint_A z^4\,dA$$

Substituting the assumed deformation and taking $\frac{\partial}{\partial\alpha}$,

$$\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\epsilon_o}\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} = \frac{I_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{8\alpha\left(\frac{\pi}{\lambda n}\right)^4\cos^2\left(\frac{2\pi s}{\lambda n}\right)\left[1-5\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]ds}{\left[1-\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]^3} + \frac{Q_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{32\alpha^3\left(\frac{\pi}{\lambda n}\right)^8\cos^4\left(\frac{2\pi s}{\lambda n}\right)\left[2+\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]ds}{\left[1-\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]^4}$$

The denominators of the integrands will be strictly positive for $\alpha < \frac{\lambda n}{\pi}$. The numerator of the first integrand is positive for $\alpha < \frac{\lambda n}{\sqrt{5}\pi} \approx \frac{1}{7}\lambda n$.

<u>$C(\alpha, n)$</u>

From Section 5.3,

$$C(\alpha,n) = \frac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} + \frac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n)$$

The first term of $C(\alpha, n)$ becomes,

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} &= \frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\frac{1}{2}E\iiint_B E_{xx}^2\,dV\right]_{\epsilon_o=0} \\ &= \frac{1}{A_{bar}}\frac{\partial}{\partial\alpha}\iiint_B\left\{z^2\left[\frac{(w'')^2}{1-(w')^2}\right] - z^3\left[\frac{(w'')^3}{[1-(w')^2]^{\frac{3}{2}}}\right] + z^4\left[\frac{(w'')^4}{4\,[1-(w')^2]^4}\right]\right\}dV\end{aligned}$$

Integrating over the cross-section,

$$\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} = \frac{I_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{\partial}{\partial\alpha}\left[(w'')^2\frac{1-(w')^2}{[1-(w')^2]^2}\right]ds + \frac{Q_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{\partial}{\partial\alpha}\left[\frac{(w'')^4}{4\,[1-(w')^2]^4}\right]ds$$

Substituting the assumed deformation and taking $\frac{\partial}{\partial\alpha}$,

$$\frac{1}{E\,A_{bar}}\frac{\partial}{\partial\alpha}\left[\tilde{U}_{strain}(\epsilon_o,\alpha,n)\right]_{\epsilon_o=0} = \frac{I_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{8\alpha\left(\frac{\pi}{\lambda n}\right)^4\cos^2\left(\frac{2\pi s}{\lambda n}\right)ds}{\left[1-\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]^2} + \frac{Q_{bar}}{A_{bar}}\int_0^{\lambda n}\frac{16\alpha^3\left(\frac{\pi}{\lambda n}\right)^8\cos^4\left(\frac{2\pi s}{\lambda n}\right)ds}{\left[1-\left(\frac{\alpha\pi}{\lambda n}\right)^2\sin^2\left(\frac{2\pi s}{\lambda n}\right)\right]^3}$$

The integrands will be strictly positive for $\alpha < \frac{\lambda n}{\pi}$.

The second term of $C(\alpha, n)$ becomes,

$$\frac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n) = \frac{\partial}{\partial\alpha}\left\{T\sum_{i=1}^{m}2\int_0^{\frac{\pi}{2}}\sqrt{\left[R+w_i\left(\frac{2\phi}{\pi}\right)^2\right]^2+\frac{64w_i^2\phi^2}{\pi^4}}\,d\phi\ -\pi R\right\}$$

Only w_i depends on α,

$$w_i = \left[\alpha\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2\pi s}{\lambda n}\right)\right]_{s=s_i} \tag{B.1}$$

$$\frac{\partial w_i}{\partial\alpha} = \frac{1}{2}\left(1-\cos\frac{2\pi s_i}{\lambda n}\right) \tag{B.2}$$

Note that $\frac{\partial w_i}{\partial\alpha}$ is strictly positive.

$$\frac{\partial}{\partial\alpha}\tilde{U}_{spiral}(\alpha,n) = 2T\sum_{i=1}^{m}\int_0^{\frac{\pi}{2}}\frac{\left[R+w_i\left(\frac{2\phi}{\pi}\right)^2\right]\left(\frac{2\phi}{\pi}\right)+\frac{64w_i^2\phi^2}{\pi^4}}{\sqrt{\left[R+w_i\left(\frac{2\phi}{\pi}\right)^2\right]^2+\frac{64w_i^2\phi^2}{\pi^4}}}\frac{\partial w_i}{\partial\alpha}\,d\phi$$

This is positive since all of its terms are positive.